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Credibility of Incentive Equilibrium Strategies in Linear-State Differential Games[1]

G. Martín-Herrán[2] and G. Zaccour[3]

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Abstract. We characterize in this paper the credibility of incentive equilibrium strategies for the class of linear-state differential games. We derive a general condition for credibility and illustrate its use on two differential games taken from the literature of environmental economics and knowledge accumulation. We show that the proposed linear incentive strategies are not always credible. Further, we provide alternative nonlinear credible strategies which suggest that we should not stick only to linear incentive strategies, even in a simple class of differential games such as the linear-state one.

Key Words. Linear-state differential games, cooperation, incentive equilibria, credibility, environmental economics, knowledge acccumulation.

1. Introduction

A major issue in cooperative differential games is the sustainability over time of an agreement reached at the starting date of the game. Schematically, the difficulty stems from the fact that the initial sharing rule agreed upon may become individually irrational in the course of the game (Ref. 1). The literature has attempted to ensure sustainability following two alternative approaches:

(i) Equilibrium Approach. The idea here is to embody the cooperative solution with an equilibrium property so that, by definition, each

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[2]Assistant Professor, Departamento de Economía Aplicada (Matemáticas), Universidad de Valladolid, Valladolid, Spain.
[3]Professor, GERAD and Marketing Department, HEC Montréal, Montréal, Canada.

player will find it individually rational to stick to her part in the coordinated solution. If it happens that the efficient solution is in itself an equilibrium, as in some differential games of special structure (see e.g. Refs. 2–4), then the problem is solved. In the absence of such rare coincidence, one may build a cooperative equilibrium using trigger strategies. These are strategies based on the past actions in the game and they include a threat to punish, credibly and effectively, any player who cheats on the agreement; (see e.g. Refs. 5–6). In two-player differential games, another option is to support the cooperative solution by incentive strategies; see e.g. Refs. 7–11. Informally, incentive strategies are functions which depend on the possible deviation of the other player with respect to the coordinated solution. If this deviation is null, then the incentive strategy will prescribe to the player to also choose the cooperative control. By this, we mean that each player will indeed implement her incentive strategy, and not the coordinated solution, if she observes that the other one has deviated from the coordinated solution. The credibility of incentive strategies is the topic of this paper.

(ii) Time Consistency. A coordinated solution is time-consistent if no player finds it optimal to switch to her noncooperative control at any intermediate instant of time. The test here consists in comparing the coordinated and noncooperative payoffs-to-go along, importantly, the cooperative state trajectory; see e.g. Refs. 12–16. The implicit assumption is that each player is confident that, if the coordinated solution is time-consistent, then her partners will stick to it. A stronger concept in this category is agreeability. The main difference with time-consistency is that the comparison condition must hold along any state trajectory and not only along the cooperative one; see e.g. Refs. 17–20.

The objective of this paper is the characterization of credible incentive strategies for the class of linear-state differential games; see Ref. 21 for an analysis of this class of games. More specifically, we provide a condition to check the credibility of such strategies and further illustrate its use on two examples taken from the economic literature. We show that, in both cases, the proposed linear incentive strategies may lack credibility and provide nonlinear ones which are always credible. The message from this last result is that we should not confine ourselves to linear incentive strategies, even for simple game structures such as the linear-state one.

The rest of the paper is organized as follows. In Section 2, we recall the ingredients of linear-state differential games and derive the coordinated solution. In Section 3, we define incentive equilibrium strategies and provide a characterization formula for assessing their credibility. In Section 4, we discuss two economic applications; in Section 5, we conclude.

2. Linear-State Differential Game

Consider a two-player differential game played on the time interval $[0, \infty)$. The state equation is

$$\dot{x}(t) = g(u_1(t), u_2(t)) + \delta x(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in X \subseteq \mathfrak{R}$ denotes the state, $u_i(t) \in \mathfrak{R}$ the control of player $i = 1, 2$, δ is a constant, and X is the state space.

The payoff functional to be maximized by player i is

$$W_i = \int_0^{\infty} [f_i(u_1(t), u_2(t)) + m_i x(t)] e^{-\rho t} dt, \tag{2}$$

where ρ, $0 < \rho$, $\rho \neq \delta$, is the discount rate and m_i a parameter.

By (1)–(2), we have defined a linear-state differential game with one state variable and where each player has a scalar control. The extension to a multidimensional setting is straightforward.

Linear-state differential games have the feature that Markov perfect equilibrium strategies are degenerate in the sense that they are constant with respect to the state variable[4]. This property follows from the fact that value functions are linear in the state.

In what follows, we shall confine our interest to stationary Markov perfect equilibria, which is standard in autonomous dynamic games played on an infinite horizon. Due to stationarity, equilibrium strategies and value functions do not depend explicitly on t.

2.1. Cooperative Solution. Assume that the players agree to play a cooperative game in which they maximize jointly the aggregate payoff

$$\sum_{i=1}^{2} W_i = \sum_{i=1}^{2} \int_0^{\infty} [f_i(u_1(t), u_2(t)) + m_i x(t)] e^{-\rho t} dt.$$

The next proposition characterizes the solution of the cooperative problem that we wish to sustain by incentive equilibrium strategies.

Proposition 2.1. Denote by (u_1^c, u_2^c) the cooperative solution. Assuming interior solutions, the following system[5] defines implicitly $u_i^c, i = 1, 2$

[4]General classes of games with degenerate Markovian strategies were considered in Refs. 22–23.

[5]From now on, the time argument is omitted when no confusion can arise.

which are constant with respect to the state variable x,

$$\sum_{j=1}^{2} \frac{\partial f_j}{\partial u_i}(u_1, u_2) + \left[\frac{(m_1 + m_2)}{(\rho - \delta)} \right] \frac{\partial g}{\partial u_i}(u_1, u_2) = 0, \quad i = 1, 2. \tag{3}$$

The optimal state trajectory is

$$x^c(t) = \left(x_0 + g(u_1^c, u_2^c)/\delta\right) e^{\delta t} - g(u_1^c, u_2^c)/\delta. \tag{4}$$

Proof. We define the current-value Hamiltonian,

$$H^c(x, u_1, u_2, \lambda^c) = \sum_{j=1}^{2} [f_j(u_1, u_2) + m_j x] + \lambda^c [g(u_1, u_2) + \delta x],$$

where λ^c is the costate variable associated with the state variable x.

Assuming an interior solution, the sufficient conditions for optimality derived from the Pontryagin maximum principle include the relations

$$\frac{\partial H^c}{\partial u_i}(x, u_1, u_2, \lambda^c) = \sum_{j=1}^{2} \frac{\partial f_j}{\partial u_i}(u_1, u_2) + \lambda^c \frac{\partial g}{\partial u_i}(u_1, u_2) = 0, \quad i = 1, 2, \tag{5a}$$

$$\dot{x} = g(u_1, u_2) + \delta x, \quad x(0) = x_0, \tag{5b}$$

$$\begin{aligned} \dot{\lambda}^c &= \rho \lambda^c - \frac{\partial H^c}{\partial x}(x, u_1, u_2, \lambda^c) \\ &= (\rho - \delta)\lambda^c - (m_1 + m_2), \end{aligned} \tag{5c}$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda^c(t) = 0. \tag{5d}$$

It is easy to verify that the costate differential equation satisfying the transversality condition has as the solution

$$\lambda^c = (m_1 + m_2)/(\rho - \delta).$$

The costate variable is constant over time and independent of the state and control variables. This implies that the optimally conditions in (5) are independent of the state. Therefore, the system of equations in (3) defines implicitly the constant u_i^c, $i = 1, 2$.

Inserting the constant optimal controls in the state equation and solving leads to the optimal state trajectory in (4). □

Once the optimal cooperative solution is known, we can compute the part corresponding to each player of the total cooperative payoff.

Corollary 2.1. Player i's optimal payoff under the cooperative strategy is given by

$$W_i(u_1^c, u_2^c) = \frac{(\rho-\delta) f_i(u_1^c, u_2^c) + m_i(\rho x_0 + g(u_1^c, u_2^c))}{(\rho-\delta)\rho}, \tag{6}$$

where the difference $\rho-\delta$ is assumed to be positive.

Proof. Substitute in each player objective functional the controls by u_i^c, $i=1,2$, which are constant, and the state variable by its optimal trajectory given in (4),

$$\begin{aligned} W_i(u_1^c, u_2^c) &= \int_0^\infty [f_i(u_1^c, u_2^c) + m_i x^c(t)] e^{-\rho t} dt \\ &= \int_0^\infty \left[f_i(u_1^c, u_2^c) + m_i \left\{\left(x_0 + g(u_1^c, u_2^c)/\delta\right) e^{\delta t} - g(u_1^c, u_2^c)/\delta\right\}\right] \\ &\quad \times e^{-\rho t} dt. \end{aligned}$$

An easy integration with respect to time leads to

$$f_i(u_1^c, u_2^c)/\rho + m_i \left[\left(x_0 + g(u_1^c, u_2^c)/\delta\right)[1/(\rho-\delta)] - g(u_1^c, u_2^c)/\delta\rho\right],$$

where the assumption $\rho - \delta > 0$ is needed to guarantee a convergent improper integral. Arranging terms, the expression in (6) results. □

We state the following remark to be used later on.

Remark 2.1. If $(\hat{u_1}, \hat{u_2})$ is any pair of constant controls, which do not depend on the state variable x, the associated state trajectory is

$$\hat{x}(t) = \left(x_0 + g(\hat{u}_1, \hat{u}_2)/\delta\right) e^{\delta t} - g(\hat{u}_1, \hat{u}_2)/\delta,$$

and player i's payoff is given by

$$W_i(\hat{u}_1, \hat{u}_2) = [(\rho-\delta) f_i(\hat{u}_1, \hat{u}_2) + m_i(\rho x_0 + g(\hat{u}_1, \hat{u}_2))]/[(\rho-\delta)\rho].$$

3. Incentive Equilibria

In general, the cooperative solution is not an equilibrium. As mentioned in the introduction, one way of substaining this coordinated solution in time is to support it by incentive equilibrium strategies. This section recalls some definitions and characterizes the credibility of the incentive strategies.

Let $(u_1^c, u_2^c) \in \mathbb{R} \times \mathbb{R}$ denote the desired cooperative solution. Denote by

$$\Psi_1 = \{\psi_1 | \psi_1 : \mathbb{R} \longrightarrow \mathbb{R}\}, \quad \Psi_2 = \{\psi_2 | \psi_2 : \mathbb{R} \longrightarrow \mathbb{R}\}$$

the sets of admissible incentive strategies.

Definition 3.1. A strategy pair $(\psi_1 \in \Psi_1, \psi_2 \in \Psi_2)$ is an incentive equilibrium at (u_1^c, u_2^c) if

$$W_1(u_1^c, u_2^c) \geq W_1(u_1^c, \psi_2(u_1)), \qquad \forall u_1 \in \mathbb{R},$$
$$W_2(u_1^c, u_2^c) \geq W_2(\psi_1(u_2), u_2^c), \qquad \forall u_2 \in \mathbb{R},$$
$$\psi_1(u_2^c) = u_1^c, \quad \psi_2(u_1^c) = (u_2^c).$$

To characterize an incentive equilibrium, we need to solve the following pair of optimal control problems where each player assumes that the other is using the incentive strategy:

$$\max_{u_i} W_i = \int_0^\infty [f_i(u_1, u_2) + m_i x] e^{-\rho t}, \tag{7a}$$

$$\text{s.t.} \quad \dot{x} = g(u_1, u_2) + \delta x, \quad x(0) = x_0, \quad \rho > 0, \quad \rho \neq \delta, \tag{7b}$$

$$u_j = \psi_j(u_i), \quad i, j = 1, 2, \quad i \neq j. \tag{7c}$$

The next proposition characterizes the solutions of these optimal control problems.

Proposition 3.1. An interior solution $u_i^*, i = 1, 2$, of the optimal control problem in (7) satisfies the following equation:

$$\begin{aligned}
&\frac{\partial f_i}{\partial u_i}(u_i, \psi_j(u_i)) + \left[\frac{m_i}{(\rho - \delta)}\right] \frac{\partial g}{\partial u_i}(u_i, \psi_j(u_i)) \\
&+ \psi_j'(u_i) \left[\frac{\partial f_i}{\partial u_j}(u_i, \psi_j(u_i)) + [m_i/(\rho - \delta)] \frac{\partial g}{\partial u_j}(u_i, \psi_j(u_i))\right] = 0, \\
&i, j = 1, 2, \quad i \neq j.
\end{aligned} \tag{8}$$

Proof. Introduce the current-value Hamiltonian associated with the optimal control problem for player i in (7),

$$H^i(x, u_i, \lambda^i) = f_i(u_i, \psi_j(u_i)) + m_i x + \lambda^i[g(u_i, \psi_j(u_i)) + \delta x],$$

where λ^i is the costate variable.

Assuming an interior solution, the sufficient conditions for optimality derived from the Pontryagin maximum include

$$\frac{\partial H^i}{\partial u_i}(x, u_i, \lambda^i) = \frac{\partial f_i}{\partial u_i}(u_i, \psi_j(u_i)) + \frac{\partial f_i}{\partial u_j}(u_i, \psi_j(u_i))\psi_j'(u_i) + \lambda^i\left[\frac{\partial g}{\partial u_i}(u_i, \psi_j(u_i)) + \frac{\partial g}{\partial u_j}(u_i, \psi_j(u_i))\psi_j'(u_i)\right] = 0, \tag{9a}$$

$$\dot{x} = g(u_i, \psi_j(u_i)) + \delta x, \quad x(0) = x_0, \tag{9b}$$

$$\dot{\lambda}^i = \rho\lambda^i - \frac{\partial H^i}{\partial x}(x, u_i, \lambda^i) = (\rho - \delta)\lambda^i - m_i, \tag{9c}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda^i(t) = 0, \quad i = 1, 2. \tag{9d}$$

It is easy to verify that the costate differential equation satisfying the transversality condition has as the solution

$$\lambda^i = m_i/(\rho - \delta).$$

The costate variable is constant over time and independent of the state and control variables. This implies that the optimality conditions in (9) are independent of the state. Therefore, the equation in (8) defines implicitly the constant u_i^*. □

To determine an incentive equilibrium, we need to impose

$$u_i^* = u_i^c, \quad i = 1, 2.$$

Therefore, u_i^c must satisfy equation (8) which characterizes u_i^*. Using this fact and equations (3), which define the cooperative solution, the following proposition establishes the necessary conditions which must be satisfied by the incentive equilibrium strategies.

Proposition 3.2. To be an incentive equilibrium at (u_1^c, u_2^c), a strategy pair $(\psi_1 \in \Psi_1, \psi_2 \in \Psi_2)$ must satisfy the following conditions:

$$\psi_1'(u_2^c)\psi_2'(u_1^c) = 1, \tag{10}$$

and

$$\psi_1'(u_2^c) = -\frac{\dfrac{\partial f_2}{\partial u_2}\left(u_1^c, u_2^c\right) + [m_2/(\rho-\delta)]\dfrac{\partial g}{\partial u_2}\left(u_1^c, u_2^c\right)}{\dfrac{\partial f_2}{\partial u_1}\left(u_1^c, u_2^c\right) + [m_2/(\rho-\delta)]\dfrac{\partial g}{\partial u_1}\left(u_1^c, u_2^c\right)}, \tag{11}$$

or

$$\psi_2'\left(u_1^c\right) = -\frac{\dfrac{\partial f_1}{\partial u_1}\left(u_1^c, u_2^c\right) + [m_1/(\rho-\delta)]\dfrac{\partial g}{\partial u_1}\left(u_1^c, u_2^c\right)}{\dfrac{\partial f_1}{\partial u_2}\left(u_1^c, u_2^c\right) + [m_1/(\rho-\delta)]\dfrac{\partial g}{\partial u_2}\left(u_1^c, u_2^c\right)}, \tag{12}$$

where

$$\frac{\partial f_i}{\partial u_j}\left(u_1^c, u_2^c\right) + [m_i/(\rho-\delta)]\frac{\partial g}{\partial u_j}\left(u_1^c, u_2^c\right), \quad i, j = 1, 2, \;\; i \neq j,$$

are assumed to be nonnull.

Proof. Since u_i^c must satisfy equations (8), which characterize u_i^*, and taking into account that $\psi_i(u_j^c) = u_i^c, i, j = 1, 2, i \neq j$, then the following equations can be derived:

$$\begin{aligned}&\frac{\partial f_i}{\partial u_i}\left(u_1^c, u_2^c\right) + [m_i/(\rho-\delta)]\frac{\partial g}{\partial u_i}\left(u_1^c, u_2^c\right)\\ &+\psi_j'\left(u_i^c\right)\left[\frac{\partial f_i}{\partial u_j}\left(u_1^c, u_2^c\right) + [m_i/(\rho-\delta)]\frac{\partial g}{\partial u_j}\left(u_1^c, u_2^c\right)\right] = 0,\\ &i, j = 1, 2, \quad i \neq j.\end{aligned} \tag{13}$$

From equations (3), which characterize the cooperative solution, one has

$$\begin{aligned}&\frac{\partial f_i}{\partial u_i}(u_1^c, u_2^c) + [m_i/(\rho-\delta)]\frac{\partial g}{\partial u_i}(u_1^c, u_2^c)\\ &= -\frac{\partial f_j}{\partial u_i}(u_1^c, u_2^c) - \left[m_j/(\rho-\delta)\right]\frac{\partial g}{\partial u_i}(u_1^c, u_2^c), \quad i, j = 1, 2, \;\; i \neq j.\end{aligned}$$

Replacing these two expressions in the equation (13) for $i = 2, j = 1$, we get

$$\begin{aligned}&\frac{\partial f_1}{\partial u_2}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\frac{\partial g}{\partial u_2}(u_1^c, u_2^c)\\ &+\psi_1'(u_2^c)\left[\frac{\partial f_1}{\partial u_1}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\frac{\partial g}{\partial u_1}(u_1^c, u_2^c)\right] = 0.\end{aligned} \tag{14}$$

From the equation (13), for $i=1$ and $j=2$, we obtain

$$\frac{\partial f_1}{\partial u_1}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\,\frac{\partial g}{\partial u_1}(u_1^c, u_2^c)$$
$$= -\psi_2'(u_1^c)\left[\frac{\partial f_1}{\partial u_2}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\,\frac{\partial g}{\partial u_2}(u_1^c, u_2^c)\right]. \quad (15)$$

Substituting the right-hand side of the last equation in (14) and arranging terms, we have

$$\left[\frac{\partial f_1}{\partial u_2}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\,\frac{\partial g}{\partial u_2}(u_1^c, u_2^c)\right][1-\psi_1'(u_2^c)\psi_2'(u_1^c)] = 0.$$

Since

$$\frac{\partial f_1}{\partial u_2}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\,\frac{\partial g}{\partial u_2}(u_1^c, u_2^c)$$

is assumed to be different from zero, the last equation leads to (10). Moreover, (15) can be rewritten as (12). Finally, assuming

$$\frac{\partial f_2}{\partial u_1}(u_1^c, u_2^c) + [m_2/(\rho-\delta)]\,\frac{\partial g}{\partial u_1}(u_1^c, u_2^c)$$

nonnull, from equation (13) for $i=2, j=1$, the expression (11) can be derived. □

Corollary 3.1. If there exist $i, j \in \{1, 2\}$, $i \neq j$, such that

$$\frac{\partial f_i}{\partial u_j}(u_1^c, u_2^c) + [m_i/(\rho-\delta)]\,\frac{\partial g}{\partial u_j}(u_1^c, u_2^c) = 0,$$

then (u_1^c, u_2^c) is a Nash equilibrium of the noncooperative game.

Proof. Let assume that

$$\frac{\partial f_1}{\partial u_2}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\,\frac{\partial g}{\partial u_2}(u_1^c, u_2^c) = 0.$$

Under this assumption, the equation (3), which has to be satisfied by the cooperative solution, for $i=2$ reads as

$$\frac{\partial f_2}{\partial u_2}(u_1^c, u_2^c) + [m_2/(\rho-\delta)]\,\frac{\partial g}{\partial u_2}(u_1^c, u_2^c) = 0. \quad (16)$$

The equation (13) for $i=1$ and $j=2$ simplifies as follows:

$$\frac{\partial f_1}{\partial u_1}(u_1^c, u_2^c) + [m_1/(\rho-\delta)]\,\frac{\partial g}{\partial u_1}(u_1^c, u_2^c) = 0. \quad (17)$$

It is easy to show that, if $(\check{u}_1, \check{u}_2)$ is a Nash equilibrium of the differential game defined by (1)–(2), then it is characterized completely by the following pair of equations:

$$\frac{\partial f_i}{\partial u_i}(\check{u}_1, \check{u}_2) + [m_i/(\rho - \delta)] \frac{\partial g}{\partial u_i}(\check{u}_1, \check{u}_2) = 0, \quad i = 1, 2.$$

Therefore, equations (16) and (17) establish that the cooperative solution (u_1^c, u_2^c) is a Nash equilibrium of the noncooperative game. □

In the rare event where the condition in the above corollary holds true (i.e., the cooperative solution is a Nash equilibrium), the use of the incentive strategies would not be needed anymore to sustain the cooperative solution.

3.1. Credibility of Incentive Strategies. We turn now to the focal issue of the paper, the credibility of incentive strategies. For an incentive equilibrium to be credible, it must be in the best interest of each player to implement her incentive strategy if the other player deviates from the coordinated solution rather than to play her part of the cooperative solution. A formal definition follows.

Definition 3.2. The incentive equilibrium strategy pair $(\psi_1 \in \Psi_1, \psi_2 \in \Psi_2)$ at (u_1^c, u_2^c) is credible in $U_1 \times U_2 \subseteq R^2$ if the following inequalities are satisfied:

$$W_1(\psi_1(u_2), u_2) \geq W_1(u_1^c, u_2), \quad \forall u_2 \in U_2, \tag{18}$$

$$W_2(u_1, \psi_2(u_1)) \geq W_2(u_1, u_2^c), \quad \forall u_1 \in U_1. \tag{19}$$

Note that the above definition characterizes the credibility of the equilibrium strategies for any possible deviation in the set $U_1 \times U_2$. The next proposition states the credibility conditions for a linear-state differential game.

Proposition 3.3. Consider the differential game defined by (1)–(2) and denote by (u_1^c, u_2^c) its cooperative solution. The incentive equilibrium strategy pair $(\psi_1 \in \Psi_1, \psi_2 \in \Psi_2)$ at (u_1^c, u_2^c) is credible in $U_1 \times U_2 \subseteq R^2$ if the following conditions hold:

$$(\rho - \delta)[f_1(u_1^c, u_2) - f_1(\psi_1(u_2), u_2)]$$
$$+ m_1[g(u_1^c, u_2) - g_1(\psi_1(u_2), u_2)] \leq 0, \quad \forall u_2 \in U_2,$$
$$(\rho - \delta)[f_2(u_1, u_2^c) - f_2(u_1, \psi_2(u_1))]$$
$$+ m_2[g(u_1, u_2^c) - g_1(u_1, \psi_2(u_1))] \leq 0, \quad \forall u_1 \in U_1.$$

Proof. It suffices to compute the expressions of the different payoffs appearing in the inequalities (18) and (19) taking into account the expressions of player i's payoff along a given pair of constant established in Remark 2.1. Straightforward computations lead to the inequalities in the proposition. □

The two inequalities provide conditions which could be tested easily once the forms of the involved functions are at hand. The following two corollaries provide sufficient conditions which are easier to check. Actually, the later ensures the existence of neighborhoods in which the incentive strategies are credible.

Corollary 3.2. If

$$(\rho-\delta)\frac{\partial f_i}{\partial u_i}(u_1^c,u_2^c)+m_i\frac{\partial g}{\partial u_i}(u_1^c,u_2^c)\le 0, \quad i=1,2, \tag{20}$$

then there exists a neighborhood $\mathcal{N}\subseteq R^2$ of (u_1^c,u_2^c) such that the incentive equilibrium strategies $(\psi_1\in\Psi_1,\psi_2\in\Psi_2)$ at (u_1^c,u_2^c) are credible in $\mathcal{N}$.

Proof. Let us define functions $h_i(u_j), i, j=1,2, i\neq j$, such that

$$\begin{aligned}h_i(u_j)=&(\rho-\delta)[f_i(u_i^c,u_j)-f_i(\psi_i(u_j),u_j)]\\&+m_i[g(u_i^c,u_j)-g_i(\psi_i(u_j),u_j)].\end{aligned}$$

The credibility conditions in Proposition 3.3 can be rewritten as

$$h_i(u_j)\le 0, \quad i,j=1,2, \ \ i\neq j, \ \ \forall u_j\in U_j. \tag{21}$$

It is straightforward to deduce that

$$h_i(u_j^c)=0, \quad i,j=1,2, \ \ i\neq j,$$

taking into account that

$$\psi_i(u_j^c)=u_i^c, \quad i,j=1,2, \ \ i\neq j.$$

Taking the derivative, we have that, for $i,j=1,2, \ \ i\neq j$,

$$\begin{aligned}h_i'(u_j)=&(\rho-\delta)\left[\frac{\partial f_i}{\partial u_j}(u_i^c,u_j)-\frac{\partial f_i}{\partial u_i}(\psi_i(u_j),u_j)\psi_i'(u_j)-\frac{\partial f_i}{\partial u_j}(\psi_i(u_j),u_j)\right]\\&+m_i\left[\frac{\partial g}{\partial u_j}(u_i^c,u_j)-\frac{\partial g}{\partial u_i}(\psi_i(u_j),u_j)\psi_i'(u_j)-\frac{\partial g}{\partial u_j}(\psi_i(u_j),u_j)\right].\end{aligned}$$

Moreover,

$$h_i'(u_j^c) = -\psi_i'(u_j^c)\left[(\rho-\delta)\frac{\partial f_i}{\partial u_i}(u_1^c, u_2^c) + m_i\frac{\partial g}{\partial u_i}(u_1^c, u_2^c)\right], \quad i, j = 1, 2, \quad i \neq j.$$

Replacing the expression of $\psi_i'(u_j^c), i, j = 1, 2, i \neq j$, given in (10) or (11), the derivative of the functions $h_i(u_j^c)$ can be rewritten as

$$h_i'(u_j^c) = (\rho-\delta)\frac{\partial f_j}{\partial u_j}(u_1^c, u_2^c) + m_j\frac{\partial g}{\partial u_j}(u_1^c, u_2^c), \quad i, j = 1, 2, \quad i \neq j.$$

The assumptions in (20) imply that

$$h_i'(u_j^c) \leq 0, \quad i, j = 1, 2, \quad i \neq j.$$

Together with $h_i(u_j^c) = 0, \; i, j = 1, 2, i \neq j$, we can deduce by continuity arguments that

$$h_i(u_j) \leq 0, \quad i, j = 1, 2, \quad i \neq j, \quad \forall u_j \in \mathcal{N}_j,$$

where $\mathcal{N}_j$ denotes a neighborhood of u_j^c. Therefore, the credibility conditions in (21) are satisfied in $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ and the incentive equilibrium strategies are credible in this neighborhood. □

Corollary 3.3. If

$$u_i^c - \psi_i(u_j) \geq 0 \quad [\text{resp. } \leq 0],$$
$$(\rho-\delta)\frac{\partial f_i}{\partial u_i}(u_i^c, u_j) + m_i\frac{\partial g}{\partial u_i}(u_i^c, u_j) \leq 0 \quad [\text{resp. } \geq 0],$$

for $i, j = 1, 2, i \neq j$, for all u_j in a neighborhood $\mathcal{M}_j$ of u_j^c, then the incentive equilibrium strategies $(\psi_1 \in \Psi_1, \psi_2 \in \Psi_2)$ at (u_1^c, u_2^c) are credible in $\mathcal{M}_1 \times \mathcal{M}_2$.

Proof. Using a first-order approximation, we can write

$$f_i(u_i^c, u_j) - f_i(\psi_i(u_j), u_j) \approx [u_i^c - \psi_i(u_j)]\frac{\partial f_i}{\partial u_i}(u_i^c, u_j),$$
$$g(u_i^c, u_j) - g(\psi_i(u_j), u_j) \approx [u_i^c - \psi_i(u_j)]\frac{\partial g}{\partial u_i}(u_i^c, u_j).$$

These approximations remain valid for all u_j which belong to a neighborhood $\mathcal{M}_j$ of u_j^c.

Substituting these approximations in the credibility conditions in (21), these conditions simplify as follows:

$$\left(u_i^c - \psi_i(u_j)\right)\left[(\rho-\delta)\frac{\partial f_i}{\partial u_i}(u_i^c, u_j) + m_i \frac{\partial g}{\partial u_i}(u_i^c, u_j)\right] \leq 0,$$
$$i, j = 1, 2, \quad i \neq j, \quad \forall u_j \in \mathcal{M}_j. \qquad \square$$

4. Applications

In this section, we apply the conditions for the credibility of the incentive equilibrium strategies to two differential games, one in environmental economics and the other in knowledge accumulation.

4.1. Pollution Control Problem. Consider two players (countries, regions, etc.) who wish to coordinate their pollution strategies in order to maximize their joint payoff. As in Refs. 10 and 19, we assume that the emissions resulting from production are proportional to production.[6] This assumption allows us to express the revenue from production as a function of the emissions.

Denote by $E_i(t)$ the emissions of country i at time t and by $S(t)$ the stock of pollution. The evolution of the latter is described by the differential equation

$$\dot{S}(t) = E_1(t) + E_2(t) - \delta S(t), \quad S(0) = S_0, \tag{22}$$

where $\delta > 0$ represents the natural absorption rate of pollution and S_0 is the initial stock of pollution. The revenue function is concave increasing and the damage cost linear, specified as follows:

$$R(E_i) = \log(E_i), \quad D_i(S) = \varphi_i S, \quad \varphi_i > 0.$$

Player i's payoff is given by

$$W_i = \int_0^\infty [\log(E_i) - \varphi_i S] e^{-\rho t} dt,$$

where ρ is a positive discount rate.

[6]For the sake of completeness, we present the model in this subsection, but it corresponds to a particular case of that presented in Ref. 19 and shares a lot of features with that studied in Ref. 10.

The cooperative solution (E_1^c, E_2^c) is obtained as the result of the joint optimization problem

$$\max_{E_1,E_2} (W_1+W_2)=\int_0^{\infty}[\log(E_1)+\log(E_2)-(\varphi_1+\varphi_2)\,S]e^{-\rho t}dt,$$

subject to (22).

Applying the same optimality conditions as in Proposition 2.1, it is straightforward to obtain

$$E_i^c=(\delta+\rho)/(\varphi_1+\varphi_2), \quad i=1,2.$$

After an appropriate change of notation, the expression (6) gives player i's cooperative payoff,

$$W_i(E_1^c, E_2^c)=\frac{(\varphi_1+\varphi_2)(\rho+\delta)\log[(\rho+\delta)/(\varphi_1+\varphi_2)]-\varphi_i[\rho(\varphi_1+\varphi_2)S_0+2(\rho+\delta)]}{(\varphi_1+\varphi_2)(\rho+\delta)\rho}.$$

The cooperative solution (E_1^c, E_2^c) is not an equilibrium; therefore, we use the incentive strategies to achieve the cooperative emission levels as an incentive equilibrium.

In this example, the sets of admissible incentive strategies are

$$\Psi_1=\{\psi_1|\psi_1:\mathbb{R}^+\longrightarrow\mathbb{R}^+\}, \quad \Psi_2=\{\psi_2|\psi_2:\mathbb{R}^+\longrightarrow\mathbb{R}^+\}.$$

A strategy pair $(\psi_1\in\Psi_1, \psi_2\in\Psi_2)$ is an incentive equilibrium at (E_1^c, E_2^c) if

$$W_1(E_1^c, E_2^c)\geq W_1(E_1^c, \psi_2(E_1)), \quad \forall E_1\in\mathbb{R}^+,$$
$$W_2(E_1^c, E_2^c)\geq W_2(\psi_1(E_2), E_2^c), \quad \forall E_2\in\mathbb{R}^+,$$
$$\psi_1(E_2^c)=E_1^c, \quad \psi_2(E_1^c)=E_2^c.$$

The solution $E_i^*, i=1,2$, of the optimal control problems (7) characterizing the incentive strategies satisfies

$$1/E_i-[\varphi_i/(p+\delta)][1+\psi_j'(E_i)]=0, \quad i,j=1,2, \ \ i\neq j.$$

Imposing $E_i^*=E_i^c, i=1,2$, we get

$$\psi_i'(E_j^c)=\varphi_i/\varphi_j, \quad i,j=1,2, \ \ i\neq j.$$

Therefore, any strategy pair $(\psi_1\in\Psi_1, \psi_2\in\Psi_2)$ is an incentive equilibrium at (E_1^c, E_2^c) if

$$\psi_i(E_j^c)=E_j^c, \quad \psi_i'(E_j^c)=\varphi_i/\varphi_j, \quad i,j=1,2, \ \ i\neq j. \tag{23}$$

We now check for the credibility of the above incentive strategies. The inequalities in Proposition 3.3 characterizing the credible incentive strategies in $\mathcal{E}=\mathcal{E}_1\times\mathcal{E}_2$ now read as follows:

$$(\rho+\delta)\log\left(E_i^c/\psi_i(E_j)\right)-\varphi_i(E_i^c-\psi_i(E_j))\le 0,\quad i,j=1,2,\ i\ne j,\quad \forall E_j\in\mathcal{E}_j.$$

Define the following functions:

$$h_i(E_j)=(\rho+\delta)\log\left(E_i^c/\psi_i(E_j)\right)-\varphi_i(E_i^c-\psi_i(E_j)),\quad i,j=1,2,\ i\ne j.$$

The credibility conditions are

$$h_i(E_j)\le 0,\quad i,j=1,2,\quad i\ne j,\ \forall E_j\in\mathcal{E}_j.$$

Note that

$$h_i(E_j^c)=0,\quad i,j=1,2,\ i\ne j.$$

Moreover,

$$h_i'(E_j)=\psi_i'(E_j)\left[\varphi_i-(\rho+\delta)/\psi_i(E_j)\right],\quad i,j=1,2,\ i\ne j, \tag{24}$$
$$h_i'(E_j^c)=-\varphi_i<0,\quad i,j=1,2,\ i\ne j. \tag{25}$$

It can be shown easily that the Nash equilibrium of this pollution control differential game is given by $[(\delta+\rho)/\varphi_1,(\delta+\rho)/\varphi_2]$. It is an established result in this class of models that the cooperative emission levels are lower than their noncooperative counterparts. Thus, if a player deviates, then she will choose an emission level greater than the cooperative one. Therefore, we try to establish conditions for which

$$h_i'(E_j)\le 0,\quad i,j=1,2,\quad i\ne j,$$

since in this case the credibility conditions

$$h_i(E_j)\le 0,\quad i,j=1,2,\quad i\ne j,$$

are satisfied for

$$E_i\ge E_i^c,\quad i=1,2.$$

Inequalities (25) guarantee that there exist neighborhoods of $E_i^c, i=1,2,$ where

$$h_i'(E_j)\le 0,\quad i,j=1,2,\ i\ne j,$$

and the credibility conditions are fulfilled.

Since

$$\psi_i'(E_j^c) = \varphi_i/\varphi_j > 0, \quad i, j = 1, 2, \ i \neq j,$$

we assume that

$$\psi_i'(E_j) > 0, \forall E_j \geq 0, \quad i, j = 1, 2, \ i \neq j.$$

Under this assumption, from (24) to have

$$h_i'(E_j) \leq 0, \quad i, j = 1, 2, \ i \neq j,$$

we need to impose

$$\psi_i(E_j) \leq (\rho + \delta)/\varphi_i, \quad i, j = 1, 2, i \neq j, \forall E_j \in \mathcal{E}_j. \tag{26}$$

Then, all functions $\psi_i(E_j), i, j = 1, 2, \ i \neq j$, satisfying conditions (23) and (26) lead to credible incentive strategies in $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2$.

Let us note that, if the incentive strategies are linear, then the condition in (26) cannot be satisfied for all $E_j \geq 0$. It can be shown that linear incentive strategies are credible if and only if $E_j \leq \tilde{E}_j$, where

$$(\tilde{E}_j) = \left[(\rho + \delta)/(\varphi_i + \varphi_j)\right] \left[1 + \varphi_j^2/\varphi_i^2\right].$$

Recalling that the coordinated emission level is given by

$$E_i^c = (\delta + \rho)/(\varphi_1 + \varphi_2), \quad i = 1, 2,$$

then the linear incentive strategies are credible if the deviation from the cooperative solution is at most the quantity $\left[(\rho + \delta)/(\varphi_i + \varphi_j)\right] \varphi_j^2/\varphi_i^2$.

An example of nonlinear functions $\psi_j(E_i), \ i, j = 1, 2, i \neq j$, which lead to credible incentive strategies in $\mathbb{R}^+ \times \mathbb{R}^+$ [they satisfy all the requirements in conditions (23) and (26)] are given by

$$\psi_i(E_j) = A_i - B_i/(E_j - C_i),$$

where

$$A_i = (\rho + \delta)/\varphi_i,$$
$$B_i = \left[(\rho + \delta)/(\varphi_i + \varphi_j)\right]^2 \left(\varphi_j/\varphi_i\right)^3,$$
$$C_i = (\rho + \delta)(\varphi_i - \varphi_j)/\varphi_i^2.$$

This function has an horizontal asymptote in A_i and a vertical asymptote in C_i. It can be proved easily that $C_i \leq E_i^c$. The constant C_i is positive if and only if $\varphi_i > \varphi_j$.

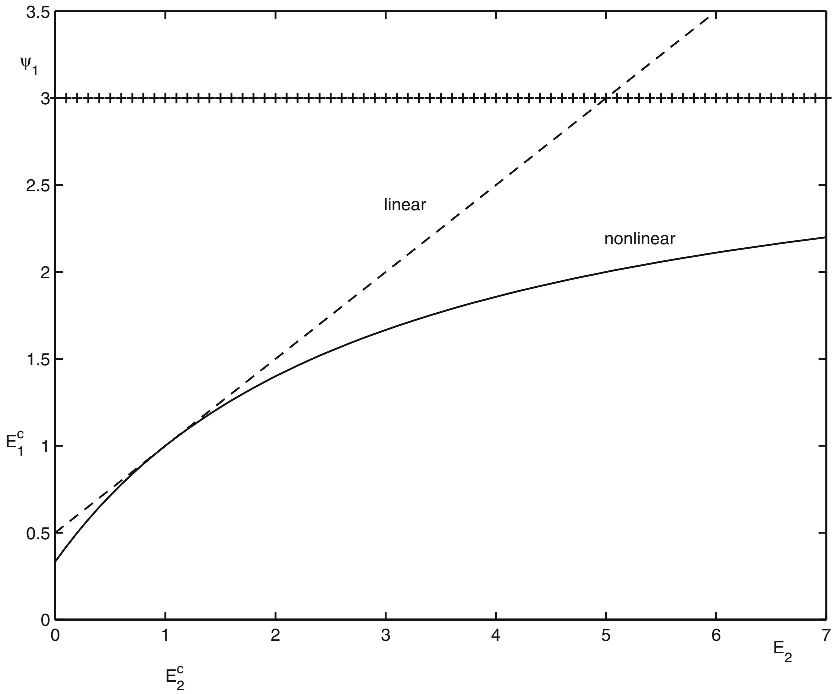

Fig. 1. Environmental economics game: Linear and nonlinear incentive strategies.

Figure 1 shows linear and nonlinear incentive strategies when the values of the parameters are as follows:

$$\rho=0.1, \quad \delta=0.2, \quad \varphi_1=0.1, \quad \varphi_2=0.2.$$

In this case, the cooperative emissions levels are $E_1^c=E_2^c=1$. Conditions (23) and (26) impose

$$\psi_1(1)=1, \quad \psi_2(1)=1, \quad \psi_1'(1)=0.5, \quad \psi_2'(1)=2, \quad \psi_1(E_2)\leq 3, \quad \psi_2(E_1)\leq 1.5.$$

This figure shows the function $\psi_1(E_2)$ as an example of a nonlinear credible incentive strategies (continuous line), as well as the linear incentive strategies (discontinuous line) which are credible only for emission levels lower than $\tilde{E}_2=5$.

4.2. Knowledge Accumulation Game. We analyze now a knowledge accumulation game which corresponds to a variation of the model used in Refs. 24–25. The linear-state statement of the game is borrowed from

Ref. 21, Example 7.1. In this example, knowledge is modeled as a pure public good and there are two individuals who invest in a single stock of knowledge.

Denote by $x(t)$ the stock of knowledge at time t and by $u_i(t)$ the investment level of player i at this time. The evolution of this stock is governed by the following differential equation:

$$\dot{x}(t) = a_1 u_1(t) + a_2 u_2(t) - \alpha x(t), \quad x(0) = x_0, \tag{27}$$

where $\alpha > 0$ represents a constant rate of depreciation and x_0 is a given initial stock of knowledge.

Player i chooses her investment level so as to maximize the following objective functional:

$$\max_{u_i} W_i = \int_0^\infty [m_i x - \beta_i u_i - (1/2)\gamma_i u_i^2] \; e^{-\rho t} dt,$$

subject to (27). The first term in the objective represents the linear utility that player i derives from the stock of knowledge. The other two terms correspond to the cost of investment, which is assumed to be quadratic.

The cooperative (joint maximization) solution is given by (u_1^c, u_2^c) where

$$u_i^c = [(m_1 + m_2)a_i - \beta_i(\rho + \alpha)]/[(\rho + \alpha)\gamma_i].$$

To have positive investments, it is assumed that

$$(m_1 + m_2)a_i - \beta_i(\rho + \alpha) > 0, \quad i = 1, 2.$$

It can be shown easily that the Nash equilibrium $(\hat{u}_1, \hat{u}_2)$ of this accumulation game is given by

$$\hat{u}_i = [m_i a_i - \beta_i(\rho + \alpha)]/(\rho + \alpha)\gamma_i, \quad i = 1, 2,$$

where

$$m_i a_i - \beta_i(\rho + \alpha), \quad \text{for } i = 1, 2,$$

is supposed to be positive.

The noncooperative Nash equilibrium leads to lower investments than those corresponding to the coordinated solution. Therefore, the cooperative solution is not a Nash equilibrium and incentive strategies are implemented to sustain the cooperative investment levels as an incentive equilibrium.

Using the same notation and following the same steps as in the previous example, we derive that any strategy pair $(\psi_1 \in \Psi_1, \psi_2 \in \Psi_2)$ is an incentive equilibrium at (u_1^c, u_2^c) if

$$\psi_i(u_j^c) = u_i^c, \quad \psi_i'(u_j^c) = (m_i/m_j)a_j/a_i, \quad i, j = 1, 2, \ i \neq j. \tag{28}$$

From all these incentive equilibrium strategies, we want to select those which are credible.

The inequalities in Proposition 3.3 characterizing the credible incentive strategies in $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ now read as follows:

$$(u_i^c - \psi_i(u_j))[m_i a_i - (\rho + \alpha)(\beta_i + (1/2)\gamma_i(u_i^c + \psi_i(u_j)))] \leq 0,$$
$$i, j = 1, 2, \ i \neq j, \quad \forall u_j \in \mathcal{U}_j.$$

Since it has been proved that the investment levels corresponding to the Nash equilibrium are lower than those of the cooperative solution, then it can be assumed that, if one player deviates from the agreed solution, she is going to underinvest in the public good. That is, we assume that

$$u_i^c - \psi_i(u_j) > 0, \quad i, j = 1, 2, i \neq j, \forall u_j.$$

Under this assumption, the previous inequalities simplify as follows:

$$m_i a_i - (\rho + \alpha)(\beta_i + (1/2)\gamma_i(u_i^c + \psi_i(u_j))) \leq 0, \quad i, j = 1, 2, \ i \neq j, \ \forall u_j \in \mathcal{U}_j.$$

Replacing the expression of the cooperative investment levels, the last inequalities can be rewritten as

$$\psi_i(u_j) \geq [1/\gamma_i(\rho + \alpha)][(m_i - m_j)a_i - \beta_i(\rho + \alpha)],$$
$$i, j = 1, 2, \ i \neq j, \ \forall u_j \in \mathcal{U}_j. \tag{29}$$

It is straightforward to show that the right-hand side of the last inequality is always lower than u_i^c.

All the functions satisfying the conditions (28) and (29) lead to credible incentive strategies in $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$.

There are some conditions which simplify the characterization of the credible incentive strategies. First, if

$$(m_i - m_j)a_i - \beta_i(\rho + \alpha) < 0, \quad i, j = 1, 2, \ i \neq j, \tag{30}$$

then the lower bounds in (29) are negative. Therefore, under conditions (30), all functions $\psi_i(u_j), i, j = 1, 2, i \neq j$, satisfying conditions (28) lead to credible incentive strategies in $\mathbb{R}^+ \times \mathbb{R}^+$. Note that, if $m_1 = m_2$, then both inequalities in (30) are satisfied. Second, at least for player i with the lowest value of parameter m_i, the corresponding inequality in (30) always holds and all functions $\psi_i \in \Psi_i$ fulfilling the conditions in (28) lead to credible incentive strategies in $\mathcal{U}_j$.

Let us note that, if the incentive strategies are linear, then condition (29) cannot be satisfied for all $u_j > 0$. Indeed, it can shown that linear incentive strategies are credible if and only if $u_j \geq \tilde{u}_j$, where

$$\tilde{u}_j = \left\{ m_i a_j \gamma_i [(m_1 + m_2)a_j - \beta_j(\rho+\alpha)] - 2m_j^2 a_i^2 \gamma_j \right\} / [(\rho+\alpha) m_i a_j \gamma_i \gamma_j].$$

If

$$m_i a_j \gamma_i [(m_1 + m_2)a_j - \beta_j(\rho+\alpha)] - 2m_j^2 a_i^2 \gamma_j < 0, \quad i, j = 1, 2, i \neq j,$$

then $\tilde{u}_j < 0$, $j = 1, 2$, and the linear incentive strategies are credible in $\mathbb{R}^+ \times \mathbb{R}^+$.

An example of nonlinear functions $\psi_j(u_i)$, $i, j = 1, 2, i \neq j$, which lead to credible incentive strategies in $\mathbb{R}^+ \times \mathbb{R}^+$ [they satisfy all the requirements in conditions (28) and (29)] is given by

$$\psi_i(u_j) = A_i + B_i u_j + C_i u_j^2,$$

where

$$A_i = [1/\gamma_i(\rho+\alpha)][(m_i - m_j)a_i - \beta_i(\rho+\alpha)],$$
$$B_i = \frac{4m_j^2 a_i^2 \gamma_j - m_i a_j \gamma_i [(m_1+m_2)a_j - \beta_j(\rho+\alpha)]}{m_j a_i \gamma_i [(m_1+m_2)a_j - \beta_j(\rho+\alpha)]},$$
$$C_i = \frac{(\rho+\alpha)\gamma_j \{m_i a_j \gamma_i [(m_1+m_2)a_j - \beta_j(\rho+\alpha)] - 2m_j^2 a_i^2 \gamma_j\}}{\gamma_i m_j a_i [(m_1+m_2)a_j - \beta_j(\rho+\alpha)]^2}.$$

This function corresponds to a parabola with a maximum at $-B_i/2C_i < 0$ if B_i and C_i are positive. The constants B_i, C_i are positive if and only if

$$2m_j^2 a_i^2 \gamma_j < m_i a_j \gamma_i [(m_1+m_2)a_j - \beta_j(\rho+\alpha)] < 4m_j^2 a_i^2 \gamma_j.$$

Under this assumption, since the minimum of function $\psi_i(u_j)$ is attained at a negative value of u_j, for all $u_j > 0$, this function takes values greater than A_i and condition (29) is fulfilled.

Figure 2 shows linear (discontinuous line) and nonlinear (continuous line) incentive strategies for this differential game where the values of the parameters are

$$\rho = 0.1, \quad \alpha = 0.2, \quad a_1 = a_2 = 1, \quad m_1 = 1.2,$$
$$m_2 = 1, \quad \beta_1 = \beta_2 = 0.5, \quad \gamma_1 = \gamma_2 = 1.$$

For these values, both players choose the same cooperative investment level

$$u_1^c = u_2^c = 6.8333.$$

Inequality (30) applies for the second player; therefore, any function $\psi_2 \in \Psi_2$ satisfying

$$\psi_2(u_1^c) = u_2^c, \quad \psi_2'(u_1^c) = 0.8333,$$

fulfills the credibility conditions. As far as the other player is concerned, any function $\psi_1 \in \Psi_1$ to be credible in $\mathcal{U}_2$ must fulfill the following requirements:

$$\psi_1(u_2^c) = u_1^c, \quad \psi_1'\left(u_2^c\right) = 1.2, \quad \psi_1(u_2) \geq 0.1667, \ \forall u_2 \in \mathcal{U}_2.$$

Figure 2 shows nonlinear incentive strategies credible for all $u_2 > 0$ and linear-strategies which are only credible for investment levels greater than $\tilde{u}_2 = 1.2778$.

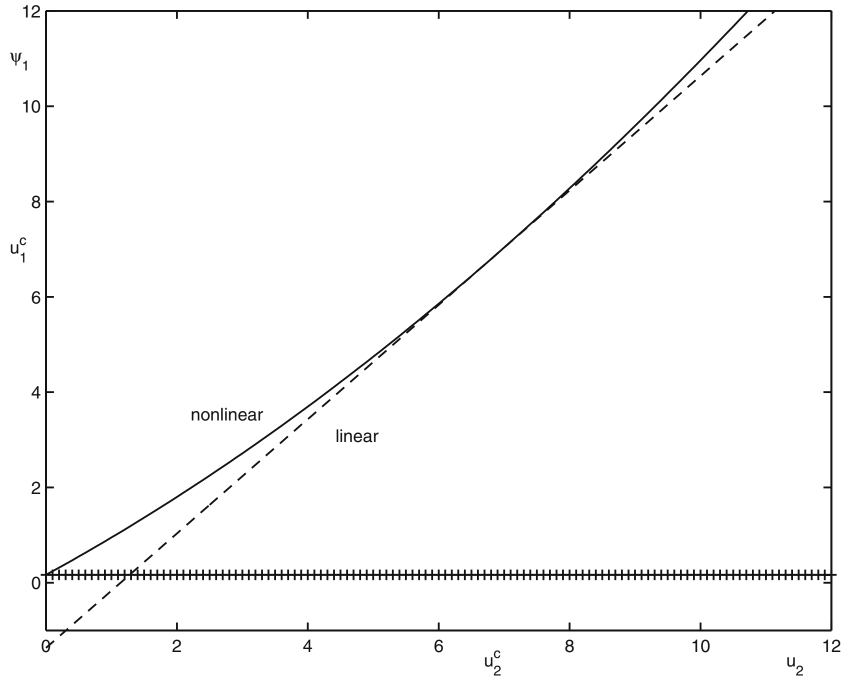

Fig. 2. Knowledge accumulation game: Linear and nonlinear incentive strategies.

5. Concluding Remarks

Incentive strategies could be an interesting device for sustaining cooperation over time of a cooperative agreement, if they happen to be credible. We provided in this paper conditions for testing the credibility of such strategies in the class of linear-state differential games. The two illustrative economic examples allowed us to do two things; first, to show that these conditions are rather simple to use; second, to point out that nonlinear incentive strategies may be required to obtain credibility. An interesting future research project would be to attempt to derive similar conditions for other classes of differential games, with as first candidate the well studied linear-quadratic class.

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