JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 123, No. 2, pp. 391–407, November 2004 (© 2004)

Characterizing Nonemptiness and Compactness of the Solution Set of a Convex Vector Optimization Problem with Cone Constraints and Applications[1]

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Communicated by J. P. Crouzeix

Abstract. In this paper, we characterize the nonemptiness and compactness of the set of weakly efficient solutions of a convex vector optimization problem with cone constraints in terms of the level-boundedness of the component functions of the objective on the perturbed sets of the original constraint set. This characterization is then applied to carry out the asymptotic analysis of a class of penalization methods. More specifically, under the assumption of nonemptiness and compactness of the weakly efficient solution set, we prove the existence of a path of weakly efficient solutions to the penalty problem and its convergence to a weakly efficient solution of the original problem. Furthermore, for any efficient point of the original problem, there exists a path of efficient solutions to the penalty problem whose function values (with respect to the objective function of the original problem) converge to this efficient point.

Key Words. Optimization problem with cone constraints, weakly efficient solutions, efficient solutions, penalization methods.

1. Introduction and Preliminaries

Let R^l be the objective space ordered by the nonnegative orthorant R^l_+; namely, for any $z_1, z_2 \in R^l, z_1 \leq_{R^l_+} z_2$ if and only if $z_2 - z_1 \in R^l_+$.

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Denote by $\text{int}R^l_+$ the interior of R^l_+. Let Y be a normed space ordered by a nonempty, closed and convex cone K; i.e., for any $y_1, y_2 \in Y$, $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$. Let Y^* be the dual space of Y endowed with norm topology. Define

$$K^* = \{\mu \in Y^* : \mu(v) \geq 0, \forall v \in K\},$$
$$K^0 = \{\mu \in K^* : \|\mu\| \leq 1\}.$$

First, we give the definition of a convex function in an ordered vector space.

Definition 1.1. Let $(U, \preceq)$ be an ordered vector space. Let $X \subset R^n$ be a convex set and let $h : X \to U$ be a vector-valued function. h is said to be convex on X if

$$h(\alpha x_1 + (1-\alpha)x_2) \preceq \alpha h(x_1) + (1-\alpha)h(x_2), \quad \forall x_1, x_2 \in X, \quad \alpha \in [0, 1].$$

In this paper, we consider the following vector optimization problem:

$$\text{(P)} \quad \min \quad f(x), \quad \text{s.t.} \quad x \in X, \quad g(x) \in -K,$$

where $X \subset R^n$ is nonempty closed convex, $f : X \to R^l$ is convex with each component f_i being lower semicontinuous (lsc) on X, $g : X \to Y$ is convex with $\mu(g)$ being lsc for each $\mu \in K^*$.

Denote by X_0 the feasible set of (P), i.e.,

$$X_0 = \{x \in X : g(x) \in -K\}.$$

Throughout the paper, we assume that $X_0 \neq \emptyset$. Now, we give some definitions of solution concepts associated with vector optimization; see e.g. Ref. 1 for details.

Definition 1.2. A point $x^* \in X_0$ is called a weakly efficient solution of (P) if $f(x) - f(x^*) \notin -\text{int}R^l_+, \forall x \in X_0$; it is called an efficient solution if $f(x) - f(x^*) \notin -R^l_+ \setminus \{0\}$.

The set of all the weakly efficient solutions of (P) is denoted by WE; the set of all the efficient solutions is denoted by E. The objective value of a (weakly) efficient solution is called a (weakly) efficient point.

The following concept of level-boundedness on a set will be used frequently in the sequel.

Definition 1.3. Let $X_1 \subset R^n$ be nonempty and let $u: X \to R^1$ be a real-valued function. u is said to be level bounded on X_1 if, for any sequence $\{x_k\} \subset X_1$ satisfying $\|x_k\| \to +\infty$, we have $u(x_k) \to +\infty$.

Now, we make some remarks about the model problem (P).

(i) If $X = R^n$ and $K = Y$, then (P) is an unconstrained convex vector program. Further, assume that $l = 1$; then, (P) becomes an unconstrained convex scalar optimization problem.

(ii) If $Y = R^m$ and $K = R^m_+$, then (P) is a convex vector optimization problem with a finite number of inequality constraints. Further, assume that $l = 1$; then, (P) becomes a convex optimization problem with several inequality constraints.

(iii) Let $Y = S^m$ and $K = S^m_+$, where S^m and S^m_+ are the spaces of symmetric $m \times m$ matrices and set of positive-semidefinite symmetric $m \times m$ matrices, respectively. Y is equipped with the norm

$$\|A\| = \sqrt{\text{trace}(A^2)}$$

for any $A \in S^m$. Then, (P) is a convex vector semidefinite program, which may arise in multiobjective feedback control problems. Further, suppose that $l = 1$. Then, (P) becomes a convex semidefinite program.

(iv) Let Y be the space of real-valued continuous functions defined on the compact infinite index set T. Y is endowed with the usual norm,

$$\|h\| = \max\{|h(t)| : t \in T\}.$$

Let

$$K = \{h \in Y : h(t) \geq 0, \forall t \in T\}.$$

For each $x \in X$, let

$$g(x) = g(x, t) \in Y.$$

Then, (P) becomes a convex vector semi-infinite program. Furthermore, if $l = 1$, then (P) reduces to a convex scalar semi-infinite program.

The property of nonemptiness and compactness of the optimal set of a constrained optimization problem is very important for the study of penalty type methods (see Refs. 2–5).

This property guarantees the existence of a path of optimal solutions to the penalty problem and its convergence to an optimal solution of the original constrained problem. Characterization of the nonemptiness and compactness of the optimal set of an unconstrained scalar optimization problem was expressed in the form of level-boundedness of the objective function; see e.g. Refs. 6–7. This level-boundedness property of the objective function is in turn equivalent to the coercivity property of the objective function. For a convex vector optimization problem (P), without the cone constraint $g(x) \in -K$, Deng (Ref. 8) gave several characterizations of the nonemptiness and compactness of the weakly efficient solution set. In Ref. 9, for a convex vector optimization problem (P), with explicit inequality constraint $g(x) \in -R^m_+$, the nonemptiness and compactness of its weakly efficient solution set was characterized by the level-boundedness of each composite function $\max\{f_i, g_1(x), \ldots, g_m(x)\}, i = 1, \ldots, l,$ on the set X. This property was applied subsequently to carry out the asymptotic analysis of a class of nonlinear penalty methods.

In this paper, we characterize the nonemptiness and compactness of the weakly efficient solution set of the more general problem (P). This characterization is expressed as the level-boundedness of each component $f_i, i = 1, \ldots, l,$ on the perturbed sets of the feasible set of (P). Finally, we apply this characterization to carry out the asymptotic analysis of a class of penalty methods for (P). In particular, when this result specializes to convex scalar semidefinite programming or convex scalar semi-infinite programming, we obtain the convergence analysis of a class of penalty methods for convex semidefinite programming or convex semi-infinite programming.

2. Characterization of Nonemptiness and Compactness of a Weakly Efficient Solution Set

In this section, we present a new characterization of the nonemptiness and compactness of the set of the weakly efficient solutions of (P).

Consider the following vector optimization problem:

$$
\begin{aligned}
(\mathrm{P}') \quad \min \quad & f(x),\\
\text{s.t.} \quad & x \in X,\\
& g_1(x) =: \sup_{\mu \in K^0} \mu(g(x)) \leq 0.
\end{aligned}
$$

It is routine to verify that $g_1(x)$ is a real-valued lsc convex function because each $\mu(g(x))$ is lsc convex and K^0 is bounded. Thus, (P′) is a convex vector optimization problem with only one explicit inequality constraint.

The following proposition establishes an equivalence between the two vector optimization problems (P) and (P′). The proof is routine by applying the well-known separation for convex sets; see e.g. Ref. 10.

Proposition 2.1. The two problems (P) and (P′) have the same sets of (weakly) efficient solutions (points).

The next result follows directly from Theorem 3.1 of Ref. 9 and Proposition 2.1.

Proposition 2.2. Consider problem (P). Then, the set WE of weakly efficient solutions of (P) is nonempty and compact if and only if each

$$\theta_i(x) = \max\{f_i(x), g_1(x)\}, \quad i = 1, \dots, l, \tag{1}$$

is level-bounded on X, where g_1 is as in problem (P′).

Consider also the following scalar optimization problems:

$$\begin{aligned} (\mathrm{P}_i) \quad \min \quad & f_i(x), \\ \text{s.t.} \quad & x \in X, \\ & g_1(x) \le 0, \end{aligned}$$

where $i = 1, \dots, l$ and g_1 is as in problem (P′).

Denote by S_i the set of optimal solutions of (P_i). The next proposition follows from Lemma 3.1 of Ref. 9.

Proposition 2.3. Let $i \in \{1, \dots, l\}$ and let $\theta_i(x)$ be given by (1). Then, S_i is nonempty and compact if and only if θ_i is level-bounded on X.

The following proposition gives a characterization of the nonemptiness and compactness of S_i in terms of the level-boundedness of f_i on any positive perturbation of the constraint set.

Proposition 2.4. Let $i \in \{1, \dots, l\}$. Then, S_i is nonempty and compact if and only if, for any $v \in K$, $f_i(x)$ is level-bounded on the set $X_v = \{x \in X : g(x) \in v - K\}$.

Proof. Sufficiency. Let $x \in X_0$. Then, $g(x) \in -K$. It follows that

$$g(x) \in v - K, \quad \text{for any } v \in K.$$

Therefore, $X_0 \subset X_v$. As f_i is level-bounded on X_v, we see that f_i is level-bounded on X_0. Combined with the fact that $X_0 \neq \emptyset$, this yields that S_i is nonempty and compact.

Necessity. Suppose to the contrary that there exists $v_0 \in K$ such that f_i is not level-bounded on X_{v0}. Without loss of generality, we assume that there exist $M \in R^1$ and $\{x_k\} \subset X_{v0}$ satisfying $\|x_k\| \to +\infty$ such that

$$f_i(x_k) \leq M. \tag{2}$$

By $\{x_k\} \subset X_{v0,}$ we deduce that

$$g(x_k) \in v_0 - K.$$

Thus,

$$\mu(g(x_k)) \leq \mu(v_0), \quad \forall \mu \in K^0.$$

As a result,

$$\begin{aligned} g_1(x_k) &= \sup_{\lambda \in K^0} \lambda(g(x_k)) \\ &\leq \sup_{\lambda \in K^0} \lambda(v_0) = \beta. \end{aligned} \tag{3}$$

The combination of (2) and (3) yields

$$\begin{aligned} \theta_i(x_k) &= \max\{f_i(x_k), g_1(x_k)\} \\ &\leq \max\{M, \beta\}, \end{aligned}$$

contradicting the conclusion of Proposition 2.3. □

Remark 2.1. Note that $X_0 \subset X_v, \forall v \in K$. Thus, if there exists $v_0 \in K$ such that f_i is level-bounded on X_{v0}, then f_i is level-bounded on X_0. Hence, S_i is nonempty compact. Together with Proposition 2.3, this implies that S_i is nonempty compact if and only if there exists $v_0 \in K$ such that f_i is level-bounded on X_{v0}.

Now, we present the following characterization of the nonemptiness and compactness of WE, which follows immediately from Propositions 2.2 to 2.4.

Theorem 2.1. Consider problem (P). Assume that $X_0 \neq \emptyset$. Then, WE is nonempty and compact if and only if, for each $i \in \{1, \ldots, l\}$ and any $v \in K$, f_i is level-bounded on X_v.

3. Applications to Penalty Type Methods

In this section, we apply the characterizations given in Theorem 2.1 to carry out the asymptotic analysis of a class of penalty-type methods for (P). Specifically, we show that, under the assumption of nonemptiness and compactness of the weakly efficient solution set of (P), the existence of a path of weakly efficient solutions to the penalty problem and its convergence to a weakly efficient solution of the original problem. Furthermore, for any efficient point of (P), there exists a path of efficient solutions to the penalty problem whose function values (with respect to the objective function f) converges to this efficient point.

Let

$$e=(1,\ldots,1)\in R^l;$$

all its components are $1'$s. Consider the following class of penalty functions:

$$p_\gamma(x,r)=f(x)+rd^\gamma_{-K}(g(x))e, \quad x\in X, r>0,$$

where $\gamma>0$ is a constant and $d_z(z)$ denotes the distance from the element z to the set Z, i.e.,

$$d_Z(z)=\inf\left\{d(z',z):z'\in Z\right\}.$$

The corresponding penalty problems are given by

$$(\mathrm{PFP}^\gamma_r) \quad \min\{p_\gamma(x,r):x\in X\}.$$

Remark 3.1.

(i) Suppose that $l=1, Y=R^m$, and $K=R^{m_1}_+\times\{0_{m-m_1}\}$, where $m_1\le m$ and 0_{m-m_1} is the origin of the space R^{m-m_1}. Then:

(a) If Y is endowed with the standard norm

$$\|y\|=\left[\sum_{j=1}^{m} y_i^2\right]^{1/2}, \quad \forall y\in R^m,$$

and $\gamma=2$, then the penalty function

$$p_2(x,r)=f(x)+r\left[\sum_{j=1}^{m_1}(g_j^+(x))^2+\sum_{j=m_1+1}^{m} g_j^2(x)\right],$$

where

$$g_j^+(x) = \max\{g_j(x), 0\},$$

is the classical l_2 penalty function.

(b) Let Y be normed by

$$\|y\| = \sum_{i=1}^{m} |y_i|, \quad \forall y \in R^m.$$

If $\gamma = 1$, then

$$p_1(x,r) = f(x) + r\left[\sum_{j=1}^{m_1} g_j^+(x) + \sum_{j=m_1+1}^{m} |g_j(x)|\right]$$

is the classical l_1 penalty function. If $0 < \gamma < 1$, then

$$p_\gamma(x,r) = f(x) + r\left[\sum_{j=1}^{m_1} g_j^+(x) + \sum_{j=m_1+1}^{m} |g_j(x)|\right]^\gamma,$$

which is the lower-order penalty function used in Ref. 11.

(c) Let Y be endowed with the norm

$$\|y\| = \max_{1\le j\le m}\{|y_j|\}, \quad \forall y \in R^m.$$

Then,

$$p_\gamma(x,r) = f(x) + r\left[\max\left\{g_1^+(x), \ldots, g_{m_1}^+(x), |g_{m_1+1}(x)|, \ldots, |g_m(x)|\right\}\right]^\gamma,$$

which is the class of penalty function considered in Ref. 12. In particular, when $\gamma = 1$, it becomes a classical penalty function; see e.g. Ref. 13.

(ii) If $l > 1$, $Y = R^m$ and $K = R_+^{m_1} \times \{0_{m-m_1}\}$ and Y is endowed with the norm

$$\| y \| = \sum_{j=1}^{m} |y_j|, \quad \forall y \in R^m,$$

and $\gamma = 1$, then

$$p_1(x,r) = f(x) + r\left[\sum_{j=1}^{m_1} g_j^+(x) + \sum_{j=m_1+1}^{m} |g_j(x)|\right] e$$

is the l_1 penalty function for the multiobjective programming considered in Refs. 14–15. When $0 < \gamma < 1$, $P_\gamma(x,r)$ can be seen as a lower-order penalty function for multiobjective optimization.

(iii) Let $Y = S^m$ and $K = S_+^m$, where S^m and S_+^m are as in item (iii) of Section 1. It can be checked that

$$\begin{aligned} d_{-K}(g(x)) &= \left[\text{trace}\left(\frac{g(x)+|g(x)|}{2}\right)^2\right]^{1/2} \\ &= 1/2\sqrt{\text{trace}(g(x)+|g(x)|)^2}, \end{aligned}$$

where $|A|$ is defined as the unique positive-semidefinite square root matrix of A^2 for any $A \in S^m$. As a result,

$$p_\gamma(x,r) = f(x) + r/2\left[\text{trace}(A+|A|)^2\right]^{\gamma/2} e,$$

which can be seen as a class of penalty functions for the vector semidefinite program (P). When $l = 1$ and $\gamma = 2$, it is essentially the quadratic penalty function studied in Ref. 16. When $l = 1$ and $0 < \gamma \le 1$, it is the lower-order penalty function considered in Ref. 17.

(iv) Let $l = 1$. Let Y and K be as in item (iv) of Section 1. Then, it can be verified that

$$d_{-K}(g(x)) = \max_{t \in T} g^+(x,t).$$

Thus, the penalty function becomes

$$p_\gamma(x,r) = f(x) + r\left[\max_{t \in T} g^+(x,t)\right]^\gamma.$$

A special case where $\gamma = 1$ was considered in Ref. 18.

We have the following result concerning the existence of weakly efficient solutions to the penalty problem (PFP_r^γ) when r is large and any selection of these solutions as $r \to +\infty$ is bounded. Moreover, any limit point of these solutions as $r \to +\infty$ is a weakly efficient solution of (P). Furthermore, for any efficient point of (P), there exists a path of efficient solutions to the penalty problem whose function values (with respect to the objective function f) converges to this efficient point.

Denote by WE_r and E_r the sets of the weakly efficient solutions and efficient solutions of (PFP_r^γ), respectively.

First, we have the following result which does not need any convexity assumption.

Theorem 3.1. Suppose that $\mathrm{int}K \neq \emptyset$ and let $k^0 \in \mathrm{int}K$. Consider (P) without convexity assumption and (PFP_r^γ). Assume that $\exists \bar{r} > 0$ and $m_0 \in R^1$ such that

$$f(x) + \bar{r} d_{-K}^\gamma(g(x))e - m_0 e \in R_+^l, \quad \forall x \in X. \tag{4}$$

Further assume that, for each $i \in \{1, \ldots, l\}$, f_i is level-bounded on the set $X_{k^0} = \{x \in X : g(x) \in k^0 - K^*\}$. Then:

(i) The weakly efficient solution set WE of (P) is nonempty and compact; the efficient solution set E of (P) is nonempty and bounded.
(ii) There exists $\bar{r}' > 0$ such that WE_r is nonempty and compact; E_r is nonempty and bounded whenever $r \geq \bar{r}'$.
(iii) Let $\bar{r}' < r_k \to +\infty$. Then, for each selection $x_k^* \in \mathrm{WE}_{r_k}$, we have that $\{x_k^*\}$ is bounded and every limit point of $\{x_k^*\}$ belongs to WE.
(iv) Let $x^* \in \mathrm{E}$. Then, there exists $x_r^* \in \mathrm{E}_r$ such that $f(x^*) = \lim_{r \to +\infty} f(x_r^*)$.

Proof.

(i) Since each f_i is level-bounded on $X_{k^0} \supset X_0$, we see that E is nonempty and WE is bounded. As a result, WE is nonempty and compact as $\mathrm{E} \subset \mathrm{WE}$ and WE is bounded and closed. By virtue of the boundedness of WE and the fact that $\mathrm{E} \subset \mathrm{WE}$, we know that E is bounded. So E is nonempty and bounded.

(ii) Let $x_0 \in X_0$. First, we show that there exists $\bar{r}'' > 0$ such that

$$B(x_0, r) = \{x \in X : f(x) + r d_{-K}^\gamma(g(x))e \leq_{R_+^l} f(x_0)\} \subset X_{k^0}, \quad r \geq \bar{r}''. \tag{5}$$

Otherwise, there exist $0 < r_k \uparrow +\infty$ and $x_k \in X$ such that

$$f(x_k) + r_k d_{-K}^\gamma(g(x_k))e \leq_{R_+^l} f(x_0) \tag{6}$$

and

$$x_k \notin X_{k^0}. \tag{7}$$

From (6), we have

$$f(x_k)+\bar{r}d^{\gamma}_{-K}(g(x_k))e+(r_k-\bar{r})d^{\gamma}_{-K}(g(x_k))e \leq_{R^l_+} f(x_0).$$

Together with (4), this gives us

$$(r_k-\bar{r})d^{\gamma}_{-K}(g(x_k)) \leq f_i(x_0)-m_0, \quad i=1,\ldots,l.$$

Consequently,

$$\lim_{k\to+\infty} d_{-K}(g(x_k))=0.$$

Thus, there exists $v_k \in -K$ such that

$$\lim_{k\to+\infty} \| g(x_k)-v_k \|=0. \tag{8}$$

On the other hand, from (7), we have

$$g(x_k) \notin k^0-K. \tag{9}$$

By (8) and the fact that $k^0 \in \mathrm{int}K$, we deduce that

$$k^0-(g(x_k)-v_k) \in K.$$

When k is sufficiently large. This implies that

$$g(x_k)-v_k \in k^0-K.$$

It follows that $g(x_k) \in k^0-K$, since $v_k \in -K$, contradicting (9). Hence, (5) holds. As f_i is level-bounded on X_{k^0}, so $f_i+rd^{\sigma}_{-K}(g)$ is level-bounded on $B(x_0,r)$ when $r \geq \bar{r}''$. Consequently, both E_r and WE_r are nonempty whenever $r \geq \bar{r}''$.

Now, we show that there exists $\bar{r}' \geq \bar{r}''$ such that WE_r is bounded whenever $r \geq \bar{r}'$. Suppose to the contrary that there exists $0<r_k \uparrow +\infty$ such that WE_{r_k} is unbounded. Then, we can choose $x_k \in \mathrm{WE}_{r_k}$ such that $\| x_k \| \to +\infty$. Let $x_0 \in X_0$. By $x_k \in \mathrm{WE}_{r_k}$, we have

$$f(x_0)-f(x_k)-r_k d^{\gamma}_{-K}(g(x_k))e \notin -R^l_+\backslash\{0\}, \quad \forall k. \tag{10}$$

That is,

$$f(x_0)-f(x_k)-\bar{r}d^{\gamma}_{-K}(g(x_k))e-(r_k-\bar{r})d^{\gamma}_{-K}(g(x_k))e \notin -R^l_+\backslash\{0\}, \quad \forall k.$$

Combined with (4), this implies

$$f(x_0)-m_0e-(r_k-\bar{r})d^{\gamma}_{-K}(g(x_k))e \notin -R^l_+\backslash\{0\}, \quad \forall k.$$

It follows that

$$\max_{1\le i\le l} f_i(x_0) - m_0 \ge (r_k - \bar{r}) d^{\gamma}_{-K}(g(x_k)), \quad \forall k.$$

Consequently, we can show as above that $x_k \in X_{k^0}$ when k is large enough. Moreover, from (10), we have

$$f(x_0) - f(x_k) \notin -R^l_+ \backslash \{0\}, \quad \forall k,$$

contradicting the fact that each f_i is level-bounded on X_{k^0} and that $\| x_k \| \to +\infty$ as $k \to +\infty$. As $\mathrm{E}_r \subset \mathrm{WE}_r$, so we know that E_r is bounded whenever $r \ge \bar{r}'$.

(iii) Suppose to the contrary that there exist $0 < r_k \uparrow +\infty$ and $x_k^* \in \mathrm{WE}_{r_k}$ such that $\{x_k^*\}$ is unbounded. Without loss of generality, assume that $\| x_k^* \| \to +\infty$. Then, as shown in the proof of (ii), a contradiction will arise. So, $\{x_k^*\}$ should be bounded. Let x^* be a limit point of $\{x_k^*\}$. Without loss of generality, assume that $x_k^* \to x^*$ as $k \to +\infty$. Then, from $x_k^* \in \mathrm{WE}_{r_k}$, we deduce that, for any $x_0 \in X_0$, there holds that

$$f(x_0) - f\left(x_k^*\right) - r_k d^{\gamma}_{-K}\left(g\left(x_k^*\right)\right) e \notin -R^l_+ \backslash \{0\},$$

implying

$$f(x_0) - f\left(x_k^*\right) \notin -\mathrm{int} R^l_+.$$

Thus, without loss of generality, we can assume that there exists $i^* \in \{1, \ldots, l\}$ such that

$$f_{i^*}(x_0) \ge f_{i^*}(x_k^*), \quad \forall k.$$

Taking the lower limit as $k \to +\infty$, we get

$$f_{i^*}(x_0) \ge \liminf_{k\to+\infty} f_{i^*}(x_k^*) \ge f_{i^*}(x_k^*).$$

Therefore,

$$f(x_0) - f(x^*) \notin -\mathrm{int} R^l_+.$$

Hence, $x^* \in \mathrm{WE}$ by the arbitrariness of $x_0 \in X_0$.

(iv) Let $x^* \in \mathrm{E}$. Consider the set

$$B(x^*, r) = \{x \in X : f(x) + r d^{\gamma}_{-K}(g(x)) e \le_{R^l_+} f(x^*)\}.$$

Clearly, $B(x^*, r) \ne \emptyset$. We can show also as in the proof of (i) that $B(x^*, r) \subset X_{k^0}$ when r is sufficiently large. Thus, each $f_i + r d^{\gamma}_{-K}(g), i =$

$1,\ldots,l$, is level-bounded on $B(x^*,r)$. So, there exists $x_r^* \in E_r \cap B(x^*,r) \subset$ WE_r when r is large enough. By (iii), $\{x_r^*\}$ is bounded. Suppose that $\bar{x}^*$ is a limit point of $\{x_r^*\}$. From $x_r^* \in B(x^*,r)$, we see that

$$f\left(x_r^*\right) + r d_{-K}^{\gamma}\left(g\left(x_r^*\right)\right) e \leq_{R_+^l} f(x^*).$$

It follows that

$$f\left(x_r^*\right) \leq_{R_+^l} f(x_*).$$

That is,

$$f_i\left(x_r^*\right) \leq f_i(x^*), \quad i=1,\ldots,l. \tag{11}$$

Passing to the lower limit as $r \to +\infty$, we obtain

$$f_i(\bar{x}^*) \leq \liminf_{r\to+\infty} f_i\left(x_r^*\right) \leq f_i(x^*), \quad i=1,\ldots,l. \tag{12}$$

Since $x^* \in \mathrm{E}$, we must have

$$f(\bar{x}^*) = f(x^*). \tag{13}$$

Taking the upper limit in (11) as $r \to +\infty$, we have

$$\limsup_{r\to+\infty} f_i\left(x_r^*\right) \leq f_i(x^*), \quad i=1,\ldots,l.$$

Combined with (12) and (13), this yields

$$\lim_{r\to+\infty} f\left(x_r^*\right) = f(x^*).$$

□

Applying Theorem 3.1 to the convex case [i.e., (P) is a convex program], we obtain the next theorem.

Theorem 3.2. Consider the convex program (P) and the penalty problem (PFP_r^{γ}). Assume that $\mathrm{int}K \neq \emptyset$ and $k^0 \in \mathrm{int}K$. Suppose that the weakly efficient solution set WE of (P) is nonempty and compact. Further, assume that there exist $\bar{r} > 0$ and $m_0 \in R^1$ such that (4) holds. Then:

- (i) The efficient solution set E of (P) is nonempty and bounded.
- (ii) There exists $\bar{r}' > 0$ such that WE_r is nonempty and compact; E_r is nonempty and compact when $r \geq \bar{r}'$.
- (iii) Let $\bar{r}' < r_k \to +\infty$; then, for each selection $x_k^* \in \mathrm{WE}_{r_k}$, we have that $\{x_k^*\}$ is bounded and every limit point of $\{x_k^*\}$ belongs to WE.
- (iv) Let $x^* \in \mathrm{E}$; then, there exists $x_r^* \in \mathrm{E}_r$ such that $f(x^*) = \lim_{r\to+\infty} f(x_r^*)$.

Proof. As WE is nonempty and compact, by Theorem 2.1, each f_i is level-bounded on the set

$$X_{k^0} = \{x \in X : g(x) \in k^0 - K^*\}.$$

The conclusion follows immediately from Theorem 3.1. □

Remark 3.2. If we specify the results of Theorem 3.2 to either case (iii) with $l = 1$ or case (iv) in Remark 3.1, we obtain the asymptotic analysis of a class of penalty methods for convex scalar semidefinite programming or convex semi-infinite programming. Specifically, we obtain the following result.

Assume that the solution set of (P) is nonempty and compact and that there exists $\bar{r} > 0$ and $m_0 \in R^1$ such that

$$f(x) + \bar{r} d^{\gamma}_{-K}(g(x)) \geq m_0, \quad \forall x \in X.$$

Then:

(a) There exists $\bar{r}' > 0$ such that the solution set of the penalty problem, considered in (iii) or (iv) of Remark 3.1, is nonempty and compact.

(b) Let $0 < r_k \uparrow +\infty$; suppose that x_k^* is an optimal solution to the penalty problem with penalty parameter r_k; then, $\{x_k^*\}$ is bounded and each of its limit points is an optimal solution of (P).

In what follows, we assume that Y is a Hilbert space, $\text{int}K \neq \emptyset$, and (P) is a convex program. Applying Lemma 2.1 (v) in Ref. 19, we see that $d_{-K}(g(x))$ is a convex function on X. As a result, $d^{\gamma}_{-K}(g(X))$ is convex on X when $\gamma \geq 1$. Hence, the penalty problem (PFP_r^{γ}) is convex if $\gamma \geq 1$. The next result shows that the nonemptiness and compactness of WE together with the well-known Slater constraint qualification: $\exists x_0 \in X$ such that $g(x_0) \in -\text{int}K$ implies condition (4). We mention here that the Slater constraint qualification was assumed in Refs. 2 and 5 to prove results similar to those of Theorem 3.2 for a class of penalty methods for convex scalar optimization problems with finitely many inequality constraints.

Proposition 3.1. Let $\gamma \geq 1$. Assume that Y is a Hilbert space, int $K \neq \emptyset$. Consider the convex program (P) and its convex penalty problem (PFP_r^{γ}). Suppose that WE is nonempty compact and that $\exists x_0 \in X$ such that $g(x_0) \in -\text{int}K$. Then, (4) holds.

Proof. Suppose to the contrary that there exist $i^* \in \{1, \dots, l\}, 0 < r_k \to +\infty, x_k \in X$ such that

$$f_{i^*}(x_k) + r_k d^{\gamma}_{-K}(g(x_k)) \to -\infty. \tag{14}$$

It follows from (14) that there exists $v_k \in -K$ such that

$$f_{i^*}(x_k) + r_k \|g(x_k) - v_k\|^{\gamma} \to -\infty. \tag{15}$$

Since WE is nonempty and compact, by Theorem 2.1, the following convex scalar optimization problem has a solution $\bar{x}$:

$$\begin{aligned} \min \quad & f_{i^*}(x), \\ \text{s.t.} \quad & x \in X, \\ & g(x) \in -K. \end{aligned}$$

Together with the Slater constraint qualification, this fact implies that there exists $\mu \in K^*$ such that

$$f_{i^*}(\bar{x}) \le f_{i^*}(x) + \mu(g(x)), \quad \forall x \in X. \tag{16}$$

As a direct consequence of (16), we have

$$f_{i^*}(\bar{x}) \le f_{i^*}(x_k) + \mu(v_k) + \|\mu\| \|g(x_k) - v_k\|. \tag{17}$$

As $v_k \in -K$ and $\mu \in K^*$, we have $\mu(v_k) \le 0$. Combined with (17), this yields

$$f_{i^*}(\bar{x}) \le f_{i^*}(x_k) + \|\mu\| \|g(x_k) - v_k\|. \tag{18}$$

The combination of (15) and (18) gives

$$f_{i^*}(\bar{x}) - \|\mu\| \|g(x_k) - v_k\| + r_k \|g(x_k) - v_k\|^{\gamma} \to -\infty. \tag{19}$$

It is obvious from (19) that

$$\|g(x_k) - v_k\| \ge 1, \quad \text{when } k \text{ is sufficiently large.}$$

It follows that

$$\|g(x_k) - v_k\|^{\gamma} \ge \|g(x_k) - v_k\|,$$

when k is large enough. Together with (19), this fact implies

$$f_{i^*}(\bar{x}) + (r_k - \|\mu\|) \|g(x_k) - v_k\| \to -\infty,$$

which is impossible. Hence, (4) must hold. □

The following corollary follows immediately from Theorem 3.2 and Proposition 3.1.

Corollary 3.1. Let $\gamma \geq 1$. Assume that Y is a Hilbert space, $\mathrm{int}K \neq \emptyset$. Consider the convex program (P) and its convex penalty problem (PFP_r^γ). Suppose that WE is nonempty, compact and that $\exists x_0 \in X$ such that $g(x_0) \in -\,\mathrm{int}K$. Then:

(i) The efficient solution set E of (P) is nonempty and bounded.
(ii) There exists $\bar{r}' > 0$ such that WE_r is nonempty and compact; E_r is nonempty and compact when $r \geq \bar{r}'$.
(iii) Let $\bar{r}' < r_k \to +\infty$; then, for each selection $x_k^* \in \mathrm{WE}_{r_k}$, we have that $\{x_k^*\}$ is bounded and every limit point of $\{x_k^*\}$ belongs to WE.
(iv) Let $x^* \in \mathrm{E}$; then, there exists $x_r^* \in \mathrm{E}_r$ such that $f(x^*) = \lim_{r\to+\infty} f(x_r^*)$.

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