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Controllability of Some Nonlinear Systems in Hilbert Spaces

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Abstract. In this paper, several abstract results concerning the controllability of semilinear evolution systems are obtained. First, approximate controllability conditions for semilinear systems are obtained by means of a fixed-point theorem of the Rothe type; in this case, the compactness of the linear operator is assumed. Next, the exact controllability of semilinear systems with nonlinearities having small Lipschitz constants is derived by means of the Banach fixed-point theorem; in this case, the compactness of the operators is not assumed. In both cases, it is proven that the controllability of the linear system implies the controllability of the associated semilinear system. Finally, these abstract results are applied to the controllability of the semilinear wave and heat equations.

Key Words. Infinite-dimensional systems, linear systems, controllability, fixed-point theorems.

1. Introduction

The problems of controllability of infinite-dimensional nonlinear systems have been studied by many authors; see Refs. 1–9 and the references therein. The approximate controllability of nonlinear systems when the semigroup $S(t), t > 0$, generated by A is compact has been studied also by many authors. The results of Zhou (Ref. 1) and Naito (Ref. 2) give sufficient conditions on B with finite-dimensional range or necessary and sufficient conditions based on more strict assumptions on B. Li and

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Yong (Ref. 3) studied the same problem assuming the approximate controllability of the associated linear system under arbitrary perturbation in $L_\infty(I, L(X))$. Bian (Ref. 4) investigated the approximate controllability for a class of semilinear systems. For abstract nonlinear systems, Carmichael and Quinn (Ref. 5) used the Banach fixed-point theorem to obtain a local exact controllability in the case of nonlinearities with small Lipschitz constants. Zhang (Ref. 6) studied the local exact controllability of semilinear evolution systems. Naito (Ref. 2) and Seidman (Ref. 7) used the Schauder fixed-point theorem to prove the invariance of the reachable set under nonlinear perturbations. Other related abstract results were given by Lasiecka and Triggiani (Ref. 8). Balachandran and Sakthivel (Ref. 9) studied the controllability of semilinear integrodifferential systems in Banach spaces by using the Schaefer fixed-point theorem.

Consider an abstract semilinear equation,

$$y = y_0 + Lv + L_1 F(y, v), \tag{1}$$

and define the following sets:

$$R(F) = \{y \in Y : \text{there exists } v \in Y \text{ such that } y = y_0 + Lv + L_1 F(y, v)\},$$

$$QR(F) = \{Qy : y \in R(F)\}, \quad QR(0) = \{Qz : z \in R(0)\}.$$

Here Y, X, V are Hilbert spaces, $L \in L(V, Y)$, $L_1 \in L(Y, Y)$, $Q \in L(Y, X)$, $F : Y \times V \to Y \times V$ is nonlinear operator, $y_0 \in Y$, $v \in V$. The set $QR(F)$ is the set of points Qy, where y is a solution of (1), attainable from the point y_0. The set $QR(0)$ is the set of points Qz, where z is a solution of

$$z = y_0 + Lv, \tag{2}$$

reachable from y_0. One can see that, for each $h \in X$, $\alpha > 0$, the control

$$v^\alpha = (QL)^*(\alpha I + QL(QL)^*)^{-1}(h - Qy_0) \tag{3}$$

transfers the equation (2) from y_0 to

$$Qz^\alpha = h - \alpha(\alpha I + QL(QL)^*)^{-1}(h - Qy_0),$$

where

$$z^\alpha = y_0 + Lv^\alpha.$$

It is known that $\overline{QR(0)} = X$ if and only if

$$\alpha(\alpha + QL(QL)^*)^{-1} \to 0$$

in the strong operator topology as $\alpha \to 0^+$; see Ref. 10. Thus, the control (3) transfers the system (2) from $y_0 \in Y$ to a small neighborhood of an arbitrary point $h \in X$ if and only if $\overline{QR(0)} = X$.

The same idea is used now to investigate the controllability of the semilinear system (1). To do so, for each $\alpha \geq 0$ and $h \in X$, consider a nonlinear operator T^α from $Y \times V$ to $Y \times V$ defined by

$$T^\alpha(y, v) = (z, w), \tag{4}$$

where

$$z = y_0 + Lw + L_1 F(y, v),$$

$$w = (QL)^*(\alpha I + QL(QL)^*)^{-1}(h - Qy_0 - QL_1 F(y, v)).$$

One can see that, if the operator T^α has a fixed point (y_*^α, v_*^α), then the control v_*^α steers the control system (1) from y_0 to

$$Qy_*^\alpha = h - \alpha(\alpha I + QL(QL)^*)^{-1}(h - Qy_0 - QL_1 F(y_*^\alpha, v_*^\alpha)),$$

if $\alpha > 0$, it steers the control system (1) from y_0 to $Qy_*^\alpha = h$, if $\alpha = 0$. We prove that this point is close to h provided that $\alpha(\alpha + QL(QL)^*)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$. Therefore, to prove the controllability of (1) for each $\alpha \geq 0$ (we consider the cases $\alpha > 0$ and $\alpha = 0$ separately) and $h \in X$, we have to look for a solution of the following equations:

$$y^\alpha = y_0 + Lv^\alpha + L_1 F(y^\alpha, v^\alpha), \tag{5a}$$

$$v^\alpha = (QL)^*(\alpha I + QL(QL)^*)^{-1}(h - Qy_0 - QL_1 F(y^\alpha, v^\alpha)). \tag{5b}$$

It is clear that the fixed points of the nonlinear operator T^α are the solutions of the nonlinear control system (5) and vice versa.

The purpose of this paper is to show the exact and approximate controllability of semilinear systems in Hilbert spaces under simple and fundamental assumptions on the system operators; in particular, to show that the corresponding linear system is appropriately controllable. This is consistent with the classical finite-dimensional theory. Note that the approximate controllability of (1) is derived under the compactness assumptions of the linear operators involved. We prove that the approximate controllability of the linear system (2) implies the approximate controllability of the semilinear system (1). On the other hand, it is known that, if the operator L is compact, then $\mathrm{Im}\, QL \neq X$; that is, the linear system (2) is not exactly controllable. This is why the analogue of the above result is not true for exact controllability and we study the exact controllability of (1) for noncompact operators L and L_1.

In Section 2, several abstract results concerning the controllability of semilinear system (1) are obtained. First, conditions for the approximate controllability of (1) are derived by means of a fixed-point theorem the Rothe type; in this case, the compactness of the linear operators is assumed. Next, conditions for the exact controllability of (1) with nonlinearities having small Lipschitz constants are derived using the Banach fixed-point theorem; in this, the compactness is not assumed. In both cases, it is proven that controllability of (2) implies the controllability of (1). Finally, these abstract results are applied to the controllability of the semilinear integrodifferential equations. These equations serve as an abstract formulation of the partial integrodifferential equations arising in various applications such as viscoelasticity, heat equations, and many other physical phenomena (see Sections 3 and 4).

2. Controllability of Semilinear Systems

Let us impose the following assumptions:

(A1) $F: Y \times V \to Y$ is continuous and there exists $C > 0$ such that $\|F(y, v)\| \leq C$ for all $(y, v) \in Y \times V$.

(A1)$'$ $F: Y \times V \to Y$ is continuous and there exists $C > 0$ such that $\|F(y, v)\| \leq C(1 + \|(y, v)\|)$ for all $(y, v) \in Y \times V$.

(A2) $L: V \to Y$ and $L_1: Y \to Y$ are compact.

(A3) There exists $l > 0$ such that

$$\|F(y_1, v_1) - F(y_2, v_2)\| \leq l(\|y_1 - y_2\| + \|v_1 - v_2\|).$$

(A4) $\alpha(\alpha I + QLL^* Q^*)^{-1}$ converges to zero in the strong operator topology as $\alpha \to 0^+$.

Remark 2.1. Condition (A4) holds if and only if $\overline{\mathrm{Im}(QL)} = \overline{QR(0)} = X$; see Ref. 10.

Definition 2.1. The system (1) is approximately [exactly] controllable if

$$\overline{QR(F)} = X \qquad [QR(F) = X].$$

Theorem 2.1. Assume that (A1), (A2) hold. Then, the approximate controllability of the linear system (2) implies the approximate controllability of the semilinear system (1).

Proof.

Step 1. We show that the operator T^α has a fixed point in $Y \times V$ for all $\alpha > 0$. By the compactness and continuity of the operators involved, we see that T^α is a compact continuous operator. Since $QL(QL)^* \geq 0$, $\alpha I + QL(QL)^*$ has an inverse bounded by $1/\alpha$. On the other hand, there exists $R(\alpha) > 0$ such that, for $\|(y, v)\| = R(\alpha)$, we have

$$\begin{aligned}\|w\| &\leq (1/\alpha)\|QL\|(\|h\| + \|Q\|\|y_0\| + \|Q\|\|L_1F(y, v)\|)\\ &= (1/\alpha)\|Q\|\|L\|\,[R(\alpha)/\|(y, v)\|]\\ &\quad \times (\|h\| + \|Q\|\|y_0\| + \|Q\|\|L_1\|\|F(y, v)\|)\\ &< R(\alpha)/2,\end{aligned}$$

$$\begin{aligned}\|z\| &\leq \|y_0\| + \|L\|\|w\| + \|L_1\|\|F(y, v)\|\\ &\leq [R(\alpha)/\|(y, v)\|]\,(\|y_0\| + \|L\|\|w\| + \|L_1\|\|F(y, v)\|)\\ &< R(\alpha)/2.\end{aligned}$$

Hence, there exists $R(\alpha) > 0$ such that, for $\|(y, v)\| = R(\alpha)$,

$$T^\alpha(y, v) = (z, w) \in B = \{(z, w) \in Y \times V : \|(z, w)\| \leq R(\alpha)\}.$$

Thus, T^α maps the sphere

$$\partial B = \{(y, v) \in Y \times V : \|(y, v)\| = R(\alpha)\}$$

into the ball B. By a fixed-point theorem of the Rothe type (see Ref. 12), for all $\alpha > 0$ T^α has a fixed point in the ball B.

Step 2. Assume $\overline{QR(0)} = X$. By Step 1, the operator (4) has a fixed point (y_*^α, v_*^α). So, (y_*^α, v_*^α) satisfies (5); moreover, it follows that, for all $h \in X$,

$$Qy_*^\alpha - h = -\alpha(\alpha I + QL(QL)^*)^{-1}(h - Qy_0 - QL_1F(y_*^\alpha, v_*^\alpha)). \tag{6}$$

By Assumptions (A1) and (A2), the operator F is bounded and L_1 is compact. So, there exists a subsequence, still denoted by $\{F(y_*^\alpha, v_*^\alpha)\}$, which weakly converges to say $z \in Y$ and $L_1F(y_*^\alpha, v_*^\alpha) \to L_1z$ strongly in Y as $\alpha \to 0^+$. Then, by (6), the inequality

$$\left\|\alpha(\alpha I + QL(QL)^*)^{-1}\right\| \leq 1$$

proved in Step 1, and Remark 2.1, we obtain

$$\begin{aligned}\|Qy_*^\alpha - h\| &\le \left\|\alpha(\alpha I + QL(QL)^*)^{-1}(h - Qy_0 - QL_1 z)\right\| \\ &\quad + \left\|\alpha(\alpha I + QL(QL)^*)^{-1}(QL_1(z - F(y_*^\alpha, v_*^\alpha)))\right\| \\ &\le \left\|\alpha(\alpha I + QL(QL)^*)^{-1}(h - Qy_0 - QL_1 z)\right\| \\ &\quad + \|QL_1(z - F(y_*^\alpha, v_*^\alpha))\| \to 0,\end{aligned}$$

as $\alpha \to 0^+$. Thus, $\overline{QR(F)} = X$. The theorem is proved. □

Theorem 2.2. Let Assumptions (A1)′, (A3) hold. If $\Gamma = QL(QL)^* \ge \gamma I$ and

$$\left(\left\|\Gamma^{-1}\right\| \|L\|^2 \|Q\|^2 + \left\|\Gamma^{-1}\right\| \|L\| \|Q\|^2 + 1\right) l \|L_1\| < 1, \tag{7}$$

then the semilinear system (1) is exactly controllable.

Proof.

Step 1. We show that T^0 has a fixed point, where T^0 is the operator (4) corresponding to $\alpha = 0$. The proof is based on the Banach fixed-point theorem. By the definition of the involved operators, T^0 maps $Y \times V$ into itself.

Now, it is shown that T^0 is a contraction mapping. Let (y_1, v_1) and (y_2, v_2) be arbitrary elements from $Y \times V$. Then,

$$\begin{aligned}&\|T^0(y_1, v_1) - T^0(y_2, v_2)\| \\ &= \|w_1 - w_2\| + \|z_1 - z_2\| \\ &\le \|QL\| \left\|\Gamma^{-1}\right\| \|QL_1\| \|F(y_1, v_1) - F(y_2, v_2)\| \\ &\quad + \|L\| \|w_1 - w_2\| + \|L_1\| \|F(y_1, v_1) - F(y_2, v_2)\| \\ &\le \left(\left\|\Gamma^{-1}\right\| \|L\| \|Q\|^2 + 1\right) \|L_1\| \|F(y_1, v_1) - F(y_2, v_2)\| \\ &\quad + \left\|\Gamma^{-1}\right\| \|L\|^2 \|Q\|^2 \|L_1\| \|F(y_1, v_1) - F(y_2, v_2)\| \\ &\le \left(\left\|\Gamma^{-1}\right\| \|L\|^2 \|Q\|^2 + \left\|\Gamma^{-1}\right\| \|L\| \|Q\|^2 + 1\right) \\ &\quad \times l \|L_1\| \|(y_1, v_1) - (y_2, v_2)\|.\end{aligned}$$

Consequently, if (7) is satisfied, then the mapping T^0 is a contraction mapping and, by the Banach fixed-point theorem, it has a unique fixed point.

Step 2. If (y^0, v^0) is a fixed point of the operator T^0, then the equality (6) holds for $\alpha = 0$; that is, for all $h \in X$, there exists v^0 such that

$$y^0 = y_0 + Lv^0 + L_1 F(y^0, v^0) \quad \text{and} \quad Qy^0 = h.$$

Thus,

$$QR(F) = X.$$

The theorem is proved. □

Obviously, the condition (7) is fulfilled if the Lipschitz constant l is sufficiently small.

3. Integrodifferential Equations

Consider the following integrodifferential system

$$x'(t) = Ax(t) + Bu(t) + f\left(t, x(t), \int_0^t g(t, s, x(s))ds\right), \tag{8a}$$

$$x(0) = x_0, \quad t \in I = [0, T], \tag{8b}$$

with state space X and control space U. Here, both X and U are Hilbert spaces, A is the infinitesimal generator of a strongly continuous semigroup $S(t), t \geq 0$, on X, B is a bounded linear operator from U to X,

$$\Delta = \{(t, s) : 0 \leq s \leq t \leq T\},$$

and $g : \Delta \times X \to X$, $f : I \times X \times X \to X$ are continuous bounded functions.

Theorem 3.1. Suppose that $S(t)$, $t > 0$, is compact. Then, the system (8) is approximately controllable on $[0, T]$ if the corresponding linear system is approximately controllable on $[0, T]$.

Proof. Let

$$Y = L_2(0, T; X), \quad V = L_2(0, T; U), \quad y_0 = S(\cdot)x_0 \in Y.$$

Define the linear operators Q, L, L_1 and the nonlinear operator F by

$$Qy = y(T),$$
$$L(v)(t) = \int_0^t S(t-s)Bv(s)ds,$$
$$L_1(y)(t) = \int_0^t S(t-s)y(s)ds,$$
$$L_1F(y)(t) = \int_0^t S(t-s)f\left(s, y(s), \int_0^s g(s, r, y(r))dr\right)ds,$$

for $y \in Y, v \in V$. By Remark 2.1, the associated linear system is approximately controllable if and only if

$$\alpha(\alpha I + QLL^*Q^*)^{-1} \to 0, \quad \text{as } \alpha \to 0^+,$$

in the strong operator topology. Then, it is easy to see that all the conditions of Theorem 2.1 are satisfied and that (8) is approximately controllable. This completes the proof. □

Theorem 3.2. Let the following assumptions hold:

(i) $g : \Delta \times X \to X$, $f : I \times X \times X \to X$ are continuous and there exists $C > 0$ such that

$$\|g(t,x)\| \le C(1+\|x\|), \quad \|f(t,x,y)\| \le C(1+\|x\|+\|y\|).$$

(ii) There exists $l > 0$ such that

$$\|g(t,x_1) - g(t,x_2)\| \le l\|x_1 - x_2\|,$$
$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le l(\|x_1 - x_2\| + \|y_1 - y_2\|).$$

If the linear system associated to (8) is exactly controllable on $[0,T]$ and if the inequality (7) is satisfied, then the system (8) is exactly controllable.

Proof. The proof follows from Theorem 2.2 □

4. Applications

Example 4.1. Consider a partial differential system of the form

$$x_t(t,\theta) = x_{\theta\theta}(t,\theta) + b(\theta)u(t) + f\left(t, x(t,\theta), \int_0^t g(t,s,x(s,\theta))\,ds\right), \tag{9a}$$

$$x(t,0) = x(t,\pi) = 0, \quad t > 0, \tag{9b}$$

$$x(0) = x_0, \quad 0 < \theta < \pi, \quad 0 \le t \le T, \tag{9c}$$

where

$$u \in L_2[0,T], \quad X = L_2[0,\pi], \quad b \in X,$$

and where

$$f : R \times R \to R, g : R \times R \times R \to R$$

are continuous and uniformly bounded. Let $B \in L(R, X)$ be defined as

$$(Bu)(\theta) = b(\theta)u,$$

where

$$0 \le \theta \le \pi, \quad u \in R, \quad b(\theta) \in L_2[0, \pi],$$

and let $A : X \to X$ be operator defined by $Az = z''$ with domain

$$D(A) = \{z \in X | z, z' \text{ are absolutely continuous, } z'' \in X, z(0) = z(\pi) = 0\}.$$

Then,

$$Az = \sum_{n=1}^{\infty} \left(-n^2\right)(z, e_n)e_n, \quad z \in D(A),$$

where

$$e_n(\theta) = \sqrt{2/\pi}\sin(n\theta), \quad 0 \le x \le \pi, \quad n = 1, 2, \dots .$$

It is known that A generates a compact semigroup $S(t), t > 0$, in X and is given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z, e_n)e_n, \quad z \in X.$$

Therefore, the associated linear system is not exactly controllable but approximately controllable (Ref. 11) provided that

$$\int_0^{\pi} b(\theta)e_n(\theta)d\theta \ne 0, \qquad \text{for } n = 1, 2, 3, \dots .$$

Under the above conditions imposed on f and b, the system (9) will be approximately controllable on $[0, T]$ by Theorem 3.1.

Example 4.2. Consider the controlled wave equation with a control $u(t, \cdot) \in L_2[0, 1]$,

$$y_{tt}(t, \theta) = y_{\theta\theta}(t, \theta) + \chi_{\Omega}(\theta)u(t, \theta) + f\left(t, y(t, \theta), \int_0^t g(t, s, y(s, \theta))\, ds\right), \quad \text{(10a)}$$

$$y(t, 0) = y(t, 1) = 0, \quad t > 0, \quad \text{(10b)}$$

$$y(0, \theta) = \alpha(\theta), y_t(0, \theta) = \beta(\theta), \quad 0 \le \theta \le 1, \quad 0 \le t \le T, \quad \text{(10c)}$$

where Ω is a proper open subset of $(0,1), \alpha, \beta \in L_2[0,1]$, and where $f: R \times R \to R, g: R \times R \times R \to R$ are continuous, Lipschitz continuous in their state variables with Lipschitz constant l and satisfy a linear growth condition. Proceeding in a similar way to that in (Ref. 11), the system (10) can be written in the following abstract form in $X = D\left(A_0^{1/2}\right) \oplus L_2[0,1]$,

$$(d/dt)\begin{bmatrix} y \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}\begin{bmatrix} y \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} u + \begin{bmatrix} 0 \\ f\left(t, y(t), \int_0^t g(t,s,y(s))ds\right) \end{bmatrix},$$
$$\begin{bmatrix} y(0) \\ y_t(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

where

$$A_o h = -h_{\theta\theta}, \quad B_1 u = \chi_\Omega(\theta) u(t,\theta),$$

with domain

$$D(A_0) = \{h \in L_2(0,1) : h, h_\theta \text{ are absolutely continuous, } h_{\theta\theta} \in L_2[0,1] \text{ and } h(0) = 0 = h(1)\}.$$

It is known that A is the infinitesimal generator of a contraction group $S(t)$ on X. Therefore, $S(t)$ is not compact. On the other hand, it is known that the linear system corresponding to (10) is exactly controllable; see Ref. 11. Thus, by Theorem 3.2, the semilinear system (10) is exactly controllable on $[0,T]$ if the inequality (7) is satisfied.

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