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Vector Variational Inequalities and the $(S)_+$ Condition[1]

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Abstract. Let $\mathcal{Z}$ and X be Hausdorff real topological vector spaces and let $\mathcal{L}_b(X, \mathcal{Z})$ be the space of continuous linear mappings from X into $\mathcal{Z}$ equipped with the topology of bounded convergence. In this paper, we define the $(S)_+$ condition for operators from a nonempty subset of X into $\mathcal{L}_b(X, \mathcal{Z})$ and derive some existence results for vector variational inequalities with operators of the class $(S)_+$. Some applications to vector complementarity problems are given.

Key Words. Vector variational inequalities, variational inequalities, vector complementarity problems, $(S)_+$ condition, topology of bounded convergence, Ky Fan lemma.

1. Introduction

Let $\mathcal{Z}$ denote a Hausdorff real topological vector space; we fix a closed convex cone $C \subset \mathcal{Z}$ such that $C \neq \mathcal{Z}$ and $\text{Int}\, C \neq \emptyset$, where $\text{Int}\, C$ is the interior of C in $\mathcal{Z}$. In this paper, we are interested only in real topological vector spaces (tvs). For any given tvs X, let $\mathcal{L}(X, \mathcal{Z})$ be the set of all continuous linear mappings from X into $\mathcal{Z}$. If $\mathcal{Z}$ is the set $\mathbb{R}$ of real numbers, then $\mathcal{L}(X, \mathcal{Z}) = X^*$ is the topological dual space of X. For any $f \in \mathcal{L}(X, \mathcal{Z})$ and $x \in X$, the value of f at x will be denoted by $\langle f, x \rangle$.

Let K be a nonempty subset of X and let $T : K \to \mathcal{L}(X, \mathcal{Z})$ be an operator. The vector variational inequality $\text{VVI}(T, K, C)$, associated with

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T, K, C is to find $\hat{x} \in K$ such that

$$\langle T\hat{x}, x - \hat{x} \rangle \in (-\mathrm{Int}C)^c \quad \text{for all } x \in K,$$

where $(-\mathrm{Int}\, C)^c$ is the complement of $-\mathrm{Int}\, C$ in $\mathcal{Z}$. The vector variational inequality was introduced first by Giannessi (Ref. 1) in finite-dimensional Euclidean spaces in 1980; later, it was extended by Chen and Cheng (Ref. 2) and was studied extensively by Chen and Yang (Ref. 3). Since then, the vector variational inequality was further generalized and studied by considering moving cones instead of a fixed cone (Refs. 4–6), multi-valued mappings instead of single-valued mappings (Refs. 7–10), or both (Ref. 11–14).

When $\mathcal{Z} = \mathbb{R}$ and C is the set of nonnegative real numbers, VVI(T, K, C) becomes the usual variational inequality VI(T, K) associated with T and K, that is, to find $\hat{x} \in K$ such that

$$\langle T\hat{x}, x - \hat{x} \rangle \geq 0, \quad \text{for all } x \in K.$$

This variational inequality has been studied extensively in both finite and infinite dimensional spaces; see e.g. Refs. 15–22.

It is well known that there is a very close connection between optimization problems and variational inequalities. It turns out that the vector variational inequality provides also a very good and useful tool in dealing with vector optimization problems; see e.g. Refs. 2–4, 23–24, and references therein. Certainly, it is worth to pay attention to the research of vector variational inequalities. On the other hand, it is worth observing that most of the results on vector variational inequalities require the monotonicity or algebraic pseudomonotonicity (in the sense of Karamardian) assumption on the operator under consideration; see e.g. Refs. 2, 3, 6, 13, 23, and references therein. There are very few results in the literature without these assumptions. A result due to Guo and Yao (Ref. 25, Theorem 2.1) in reflexive Banach spaces is an effort in this direction; this result is stated below as Theorem 1.1.

The motivation of this paper is to derive more existence results for vector variational inqualities in a Hausdorff tvs without the above monotonicity assumption. The choice of the Hausdorff tvs setting is for generality and we introduce the vectorial $(S)_+$ condition for operators under consideration. In turn, this requires the introduction of the contents of limit superior and limit inferior of nets, which are not so straightforward and obvious in a Hausdorff topological vector space. More precisely, the aim of this paper is to prove an existence result for VVI(T,K,C) which is a vector version of the Guo and Yao result.

Theorem 1.1. Let K be a nonempty weakly compact convex subset of a real reflexive Banach space B and let $T: K \to B^*$ be an operator. Suppose that:

(i) T is of class $(S)_+$;
(ii) T is continuous on finite-dimensional subspaces;
(iii) if $x_n \to x$, then $\{Tx_n\}$ has a weakly convergent subsequence with limit T_x.

Then, $\mathrm{VI}(T, K)$ has a solution.

Recall that an operator $T : K \to B^*$ is said to be of class $(S)_+$ if, for $\{x_n\}_{n=1}^{\infty} \subset K$ converging weakly to x and $\limsup_{n\to\infty} \langle Tx_n, x_n - x\rangle \leq 0$, then $x_n \to x$; see Ref. 26. The operator T is called continuous on finite-dimensional subspaces if T is continuous from the norm topology of $K \cap M$ to the weak* topology of B^* for every finite-dimensional subspace M of B with $K \cap M \neq \emptyset$.

We observe that assumption (iii) of Theorem 1.1 implies that the operator T is demicontinuous; i.e., $x_n \to x$ (in norm) implies that Tx_n converges to T_x weakly. Indeed, suppose that $x_n \to x$, but Tx_n does not converge to Tx weakly. Since B is reflexive, so is B^*; hence, the weak* topology and the weak topology coincide. Then, there is a subsequence Tx_{n_k} of Tx_n such that no subsequence of Tx_{n_k} converges to Tx weakly; see e.g. Ref. 27, Proposition 21.23 (i). Now, applying (iii) of Theorem 1.1 with x_n being replaced by x_{n_k}, we see that there are no subsequences of Tx_{n_k} converging to Tx, and this is a contradiction. Also, it is clear that the demicontinuity of T implies that T is continuous on finite-dimensional subspaces. Therefore, Theorem 1.1 can be restated as follows.

Theorem 1.2. Let K be a nonempty weakly compact convex subset of a real reflexive Banach space B and let $T: K \to B^*$ be an operator. If T is of class $(S)_+$ and demicontinuous, then the $\mathrm{VI}(T, K)$ has a solution.

We remark that an interesting example in Ref. 28, p. 360 shows that the result of Theorem 1.1 may not be true without the assumption that T is of class $(S)_+$. In other word, it is not enough to assure the existence of solutions to $\mathrm{VI}(T, K)$ by simply assuming the demicontinuity of T when K is weakly compact.

The rest of the paper is organized as follows. In Section 2, we define first the limit superior and limit inferior of any net in $\mathcal{Z}$. Then, we state the vectorial $(S)_+$ condition. In Section 3, we introduce a topology on X so that the topology becomes the weak topology on X when $\mathcal{Z}=\mathbb{R}$. This topology is called the $\mathcal{L}$-topology on X and the resulting space is denoted by $X_{\mathcal{L}}$. By use of this topology, we generalize the notion of weak compactness to the notion of $\mathcal{L}$-compactness.

In Section 4, we recall the definition of topology of bounded convergence on $\mathcal{L}(X,\mathcal{Z})$, denote the space by $\mathcal{L}_b(X,\mathcal{Z})$, and prove our main existence result (Theorem 4.1) for VVI(T,K,C) with K a nonempty convex, $\mathcal{L}$-compact subset of a Hausdorff tvs X and an operator $T:K\to\mathcal{L}_b(X,\mathcal{Z})$ satisfying conditions corresponding to those given in the Guo and Yao theorem. To prove the main result, we derive also an existence result (Theorem 4.3) for VVI(T,K,C) with K a nonempty compact convex subset of X and T a continuous operator from K into $\mathcal{L}_b(X,\mathcal{Z})$.

In Section 5, we use the main existence result obtained in Section 4 to prove some existence results for VVI(T,K,C) with K non $\mathcal{L}$-compact. In Section 6, we state vector complementarity problems and derive some existence results for the problems by using the results derived in Section 5.

We use the following notation. For any subset A of a topological space X, let A^c denote the complement of A in X. When X is a tvs, let co(A) denote the convex hull of A.

2. Vectorial (S)$_+$ Condition

To state the vector version of the $(S)_+$ condition, we have to define the limit superior and limit inferior of a net in $\mathcal{Z}$. The definition was given first in Ref. 29 by use of the vectorial superior and vectorial inferior introduced in Ref. 30.

For a subset E of $\mathcal{Z}$, let $\bar{E}$ denote the closure of E in $\mathcal{Z}$. The superior of E with respect to C is defined by

$$\operatorname{Sup}(E,C)=\{x\in\bar{E}:(x+\operatorname{Int}C)\cap E=\emptyset\};$$

the inferior of E with respect to C is defined by

$$\operatorname{Inf}(E,C)=\{x\in\bar{E}:(x-\operatorname{Int}C)\cap E=\emptyset\}.$$

Since C is fixed, we simply write

$$\operatorname{Sup}(E,C)=\operatorname{Sup}E,\quad \operatorname{Inf}(E,C)=\operatorname{Inf}E.$$

By a simple verification, one proves that

$$\operatorname{Sup}E=\operatorname{Sup}\bar{E},\quad \operatorname{Inf}E=\operatorname{Inf}\bar{E}$$

are closed subsets of $\mathcal{Z}$. For a proof, see Ref. 31, Proposition 2.1 and Proposition 2.3. When $\mathcal{Z}=\mathbb{R}$ and C is the set of nonnegative real numbers, $\operatorname{Sup} E$ is either empty of a singleton for any nonempty subset E of $\mathbb{R}$. If we use $\sup E$ to denote the usual supremum of $E\subset\mathbb{R}$, then $\operatorname{Sup} E=\{\sup E\}$ when $\operatorname{Sup} E$ is a singleton and $\sup E=\infty$ if and only if $\operatorname{Sup} E=\emptyset$. Similarly, if we use $\inf E$ to denote the usual infimum of $E\subset\mathbb{R}$, then $\operatorname{Inf} E=\{\operatorname{Inf} E\}$ when $\operatorname{Inf} E$ is a singleton and $\inf E=-\infty$ if and only if $\operatorname{Inf} E=\emptyset$.

Now, we define the limit superior and limit inferior, with respect to C, of a net $\{z_\alpha\}_{\alpha\in I}$ in $\mathcal{Z}$. For any $\alpha\in I$, let $S_\alpha=\{z_\beta:\beta\succeq\alpha\}$ and let

$$\operatorname{Limsup} z_\alpha=\operatorname{Inf}\bigcup_{\alpha\in I}\operatorname{Sup} S_\alpha,\quad \operatorname{Liminf} z_\alpha=\operatorname{Sup}\bigcup_{\alpha\in I}\operatorname{Inf} S_\alpha.$$

In the case where $\mathcal{Z}=\mathbb{R}$ and C is the set of nonnegative real numbers, Limsup z_α and Liminf z_α are both either empty or singletons. If Limsup z_α is a singleton, then

$$\operatorname{Limsup} z_\alpha=\{\limsup z_\alpha\},$$

where $\limsup z_\alpha$ is the usual limit superior of $\{z_\alpha\}$ defined in $\mathbb{R}$ and

$$\limsup z_\alpha=\infty,\quad \text{if and only if } \operatorname{Limsup} z_\alpha=\emptyset.$$

Similarly,

$$\operatorname{Liminf} z_\alpha=\{\liminf z_\alpha\},$$

where $\liminf z_\alpha$ is the usual limit inferior of $\{z_\alpha\}$ and

$$\liminf z_\alpha=-\infty,\quad \text{if and only if } \operatorname{Liminf} z_\alpha=\emptyset.$$

It was proved in Ref. 31, Corollary 3.6 that $\operatorname{Sup} E$ and $\operatorname{Inf} E$ are nonempty if E is a nonempty compact subset of $\mathcal{Z}$. Thus, one proves easily that, if a net $\{z_\alpha\}_{\alpha\in I}$ lies in a compact subset of $\mathcal{Z}$, then $\operatorname{Limsup} z_\alpha$ and $\operatorname{Liminf} z_\alpha$ are nonempty.

Now, we are ready to state the vectorial $(S)_+$ condition. Let X be at tvs. A net $\{x_\alpha\}$ in X is called $\mathcal{L}$-convergent to $x\in X$, denoted by $x_\alpha\xrightarrow{\mathcal{L}}x$, if $f(x_\alpha)\to f(x)$ in $\mathcal{Z}$ for all $f\in\mathcal{L}(X,\mathcal{Z})$. We note that the notion of $\mathcal{L}$-convergence coincides with that of weak convergence when $\mathcal{Z}=\mathbb{R}$. When a net $\{x_\alpha\}$ in X converges to $x\in X$ in the original topology of X, we simply write $x_\alpha\to x$.

Let K be a nonempty subset of X. An operator $T : K \to \mathcal{L}(X, \mathcal{Z})$ is called of class $(S)_+$ if it satisfies the $(S)_+$ condition: For any net $\{x_\alpha\} \subset K$,

$$x_\alpha \xrightarrow{\mathcal{L}} x \text{ and } \operatorname{Limsup} \langle T x_\alpha, x_\alpha - x \rangle \subset (\operatorname{Int} C)^c \Rightarrow x_\alpha \to x.$$

When $\mathcal{Z} = \mathbb{R}$, $\mathcal{L}(X, \mathcal{Z}) = X^*$, the topological dual of X. If C is the set of nonnegative numbers, then an operator $T : K \to X^*$ is said to be of class $(S)_+$ if, for any net $\{x_\alpha\} \subset K$,

$$x_\alpha \longrightarrow x \text{ weakly and } \limsup \langle T x_\alpha, x_\alpha - x \rangle \leq 0 \Rightarrow x_\alpha \longrightarrow x.$$

In the proof of our main existence result for vector variational inequalities, we need the following theorem.

Theorem 2.1. Let $\{z_\alpha\}_{\alpha \in I}$ be a net in $\mathcal{Z}$ convergent to z and let $S_\alpha = \{z_\beta : \beta \succeq \alpha\}$.

(i) If there is an α_0 such that, for every $\alpha \succeq \alpha_0$, there exists $\beta \succeq \alpha$ with $\operatorname{Inf} S_\beta \neq \emptyset$, then $z \in \operatorname{Liminf} z_\alpha$.

(ii) If there is an α_0 such that, for every $\alpha \succeq \alpha_0$, there exists $\beta \succeq \alpha$ with Sup $S_\beta \neq \emptyset$, then $z \in \operatorname{Limsup} z_\alpha$.

Proof. We prove (i). The assertion (ii) follows by a similar argument. We have to show that

$$z \in \overline{\bigcup_{\alpha \in I} \operatorname{Inf} S_\alpha} \text{ and } (z + \operatorname{Int} C) \cap \bigcup_{\alpha \in I} \operatorname{Inf} S_\alpha = \bigcup_{\alpha \in I} \{(z + \operatorname{Int} C) \cap \operatorname{Inf} S_\alpha\} = \emptyset.$$

First, we prove that

$$(z + \operatorname{Int} C) \cap \operatorname{Inf} S_\alpha = \emptyset, \quad \text{for every } \alpha.$$

Suppose on the contrary that there is an α and there is a $v \in \operatorname{Int} C$ such $z + v \in \operatorname{Inf} S_\alpha$. By definition, we have

$$(z + v - \operatorname{Int} C) \cap S_\alpha = \emptyset.$$

Since $z + v - \operatorname{Int} C$ is an open neighborhood of z, there is an $\alpha_0 \succeq \alpha$ such that

$$\beta \succeq \alpha_0 \Longrightarrow z_\beta \in z + v - \operatorname{Int} C.$$

This leads to the contradiction that

$$S_{\alpha_0} \subset (z + v - \operatorname{Int} C) \cap S_\alpha.$$

Next, we prove that every open neighborhood U of z intersects $\bigcup_{\alpha\in I} \operatorname{Inf} S_\alpha$, or equivalently, U intersects some $\operatorname{Inf} S_\alpha$. This implies that $z \in \overline{\bigcup_{\alpha\in I} \operatorname{Inf} S_\alpha}$ and completes the proof.

Since there is an open neighborhood V of z such that $\bar{V} \subset U$ (Ref. 32, Theorem 1.11, p. 10), there exists an $\alpha_V \succeq \alpha_0$ such that $z_\beta \in V$ whenever $\beta \succeq \alpha_V$. This implies that $\operatorname{Inf} S_\beta \subset \overline{S}_\beta \subset \overline{V} \subset U$ whenever $\beta \succeq \alpha_V$. By assumption, there is a $\beta \succeq \alpha_V$ such that $\operatorname{Inf} S_\beta \neq \emptyset$. Thus,

$$U \cap \operatorname{Inf} S_\beta = \operatorname{Inf} S_\beta \neq \emptyset. \qquad \square$$

Corollary 2.1. Let $\{z_\alpha\}_{\alpha\in I}$ be a net in $\mathcal{Z}$ convergent to z. If $\mathcal{Z}$ is locally compact, then

$$z \in (\operatorname{Liminf}\, z_\alpha) \cap (\operatorname{Limsup}\, z_\alpha).$$

Proof. For every α, let S_α be given above. Choose an open neighborhood U of z with $\bar{U}$ compact. There is an α_0 such that $z_\alpha \in U$ whenever $\alpha \succeq \alpha_0$. Thus, $S_\alpha \subset U$ for $\alpha \succeq \alpha_0$ and $\operatorname{Inf} S_\alpha$, $\operatorname{Sup} S_\alpha$ are nonempty. $\square$

3. $\mathcal{L}$-Topology

The main work of this section is to introduce a topology on a tvs X so that it generalizes the concept of the weak topology on X. We call it the $\mathcal{L}$-topology on X. With this topology, the notion of weak compactness is generalized to the notion of $\mathcal{L}$-compactness (cf. Theorem 1.1).

The $\mathcal{L}$-topology on X is the topology having the sets $f^{-1}(U)$ as subbasis, where U is open in $\mathcal{Z}$ and $f \in \mathcal{L}(X, \mathcal{Z})$. Let $X_{\mathcal{L}}$ denote the space X equipped with the $\mathcal{L}$-topology. To proceed, we need the following terminology. Let E be a subset of X.

(a) E is called $\mathcal{L}$-closed (respectively, $\mathcal{L}$-open) if E is closed (respectively, open) in $X_{\mathcal{L}}$.
(b) The closure of E in $X_{\mathcal{L}}$ is denoted by $\bar{E}^{\mathcal{L}}$.
(c) E is called $\mathcal{L}$-compact if it is compact in $X_{\mathcal{L}}$.

Remark 3.1. Let X be a tvs. The following assertions are immediate consequences of the definition.

(i) Every $\mathcal{L}$-open ($\mathcal{L}$-closed) subset of X is open (closed) in X.

(ii) A net $\{x_\alpha\}$ converges to x in $X_{\mathcal{L}}$ if and only if $x_\alpha \xrightarrow{\mathcal{L}} x$. From this, one proves easily that $X_{\mathcal{L}}$ is a tvs.

In the rest of this section, we derive some properties that we need in the sequel. To employ the technique of Guo and Yao used in the proof of Theorem 1.1, we recall that a tvs is locally convex if its zero vector has a local base whose members are convex. We need also the following definition.

Let X be a tvs and let $\mathcal{F}_X$ denote the family of finite-dimensional subspaces of X. A subset K of X is called finitely compact if $K \cap Y$ is compact for every $Y \in \mathcal{F}_X$.

In the proof of Theorem 1.1, Guo and Yao have used the following facts on locally convex Hausdorff tvs X.

(a) X^* seperates points in X; i.e., for any nonzero $x \in X$, there is an $f \in X^*$ such that $f(x) \neq 0$ (Ref. 32, Corollary of Theorem 3.4, p. 59). Consequently, the weak topology on X is Hausdorff.
(b) Every weakly compact subset of X is finitely compact.

In general, $X_{\mathcal{L}}$ is not necessarily Hausdorff. However, from Ref. 32, p. 61, we obtain the following theorem.

Theorem 3.1. If X is a locally convex Hausdorff tvs, then $\mathcal{L}(X, \mathcal{Z})$ separates points in X; that is, for any given two distinct points x and x' of X, there is an $f \in \mathcal{L}(X, \mathcal{Z})$ such that $f(x) \neq f(x')$. Therefore, $X_{\mathcal{L}}$ is Hausdorff.

For the proof of Theorem 3.1, we need an immediate consequence of Ref. 33, Theorem 8.4.8 , and its proof. For later use, we state it as the following proposition.

Proposition 3.1. If X is a locally convex Hausdorff tvs and if Y is a finite-dimensional subspace of X, then there is a continuous linear map $\Pi_Y : X \longrightarrow Y$ such that $\Pi_Y(y) = y$ for all $y \in Y$.

Proof of Theorem 3.1. It suffices to show that, for any nonzero $x_0 \in X$, there is an $f \in \mathcal{L}(X, \mathcal{Z})$ such that $f(x_0) \neq 0$. Let Y be the subspace of X generated by x_0 and let Π_Y be given in Proposition 3.1. Note that any nonzero vector z_0 in $\mathcal{Z}$ induces a continuous linear map $\varphi : Y \longrightarrow \mathcal{Z}$ defined by

$$\varphi(\lambda x_0) = \lambda z_0, \quad \text{for } \lambda \in \mathbb{R}.$$

The composition $f = \varphi \circ \Pi_Y$ lies in $\mathcal{L}(X, \mathcal{Z})$ with $f(x_0) = z_0$. □

Corollary 3.1. If X is a finite-dimensional Hausdorff tvs, then $X = X_{\mathcal{L}}$.

Corollary 3.2. If X is a locally convex Hausdorff tvs, then every $\mathcal{L}$-compact subset of X is finitely compact.

Proof. Let K be any $\mathcal{L}$-compact subset of X and let $Y \in \mathcal{F}_X$ be arbitrary. Note that Y is closed in both X and $X_{\mathcal{L}}$. The compactness of K in $X_{\mathcal{L}}$ implies that of $K \cap Y$ in $Y_{\mathcal{L}} = Y$. □

4. Existence Results

In this section, we prove a vector version of Theorem 1.1. To this end, we need a topology on $\mathcal{L}(X, \mathcal{Z})$, where X is a tvs.

Recall that a subset E of X is called bounded if, for any 0-neighborhood U in X, there is a $\lambda > 0$ such that $E \subset \lambda U$. Let $\mathcal{B}_X$ denote the family of all bounded subsets of X and let $\mathcal{N}_{\mathcal{Z}}$ be a neighborhood base of 0 in $\mathcal{Z}$. For $S \in \mathcal{B}_X$ and for $V \in \mathcal{N}_{\mathcal{Z}}$, let

$$[S, V] = \{f \in \mathcal{L}(X, \mathcal{Z}) : f(S) \subset V\}.$$

The family $\{[S, V] : S \in \mathcal{B}_X \text{ and } V \in \mathcal{N}_{\mathcal{Z}}\}$ is a 0-neighborhood base in $\mathcal{L}(X, \mathcal{Z})$ for a unique translation-invariant topology, called the topology of bounded convergence. Let $\mathcal{L}_b(X, \mathcal{Z})$ denote the space $\mathcal{L}(X, \mathcal{Z})$ equipped with the topology of bounded convergence. Note that $\mathcal{L}_b(X, \mathcal{Z})$ is a tvs; see Ref. 34, p. 79. For a full discussion on the topology of bounded convergence, see e.g. Refs. 33 and 34.

Remark 4.1. If X and $\mathcal{Z}$ are normed spaces, the norm

$$u \longmapsto \|u\| = \sup\{|u(x)| : |x| \leq 1\}$$

generates the topology of bounded convergence on $\mathcal{L}(X, \mathcal{Z})$; see e.g. Ref. 34, p. 81.

Theorem 4.1. Let X be a locally convex Hausdorff tvs, let K be a nonempty convex and $\mathcal{L}$-compact subset of X, and let $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ be an operator. Assume that:

(i) T is continuous and of class $(S)_+$;
(ii) for any convergent net $\{x_\alpha\}$ in K and for any $x \in K$, there is an α_0 such that, for every $\alpha \succeq \alpha_0$, there exists $\beta \succeq \alpha$ with $\mathrm{Sup}\,\{\langle T x_{\beta'}, x - x_{\beta'}\rangle : \beta' \succeq \beta\} \neq \emptyset$.

Then, $\mathrm{VVI}(T, K, C)$ has a solution.

The following is an immediate consequence of Theorem 4.1 and Corollary 2.1.

Corollary 4.1. Let X be a locally convex Hausdorff tvs, let K be a nonempty convex and $\mathcal{L}$-compact subset of X, and let $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ be continuous and of class $(S)_+$. If $\mathcal{Z}$ is locally compact, then $\mathrm{VVI}(T, K, C)$ has a solution.

When $\mathcal{Z} = \mathbb{R}$, then $\mathcal{L}(X, \mathcal{Z}) = X^*$ and the topology of bounded convergence of X^* is called the strong topology (denoted by σ) of X^*. The following results are consequences of Corollary 4.1.

Corollary 4.2. Let X be a locally convex Hausdorff tvs let K be a nonempty convex and weakly compact subset of X; and let $T : K \longrightarrow (X^*, \sigma)$ be continuous and of class $(S)_+$. Then, $\mathrm{VI}(T, K)$ has a solution.

For the proof of Theorem 4.1, we need an existence result for $\mathrm{VVI}(T, K, C)$, where K is compact, stated as Theorem 4.3, whose proof is based on the Ky Fan lemma (Ref. 35). Let E be a nonempty subset of a tvs X and let 2^X denote the family of all nonempty subsets of X. A set-valued function $\Phi : E \longrightarrow 2^X$ is said to be a KKM mapping if, for any nonempty finite set $A \subset E$,

$$\mathrm{co}(A) \subset \bigcup_{x \in A} \Phi(x).$$

Theorem 4.2. See Ky Fan, Lemma 1, Ref. 35. Let K be a nonempty convex subset of a Hausdorff tvs X and let $\Phi : K \longrightarrow 2^X$ be a KKM mapping. If $\Phi(x)$ is closed in X for every $x \in K$, and if there is a point $x_0 \in K$ such that $\Phi(x_0)$ is compact, then $\bigcap_{x \in K} \Phi(x) \neq \emptyset$.

Theorem 4.3. Let X be a Hausdorff tvs and let K be a nonempty compact and convex subset of X. If $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ is a continuous operator, then $\mathrm{VVI}(T, K, C)$ has a solution.

Proof. Let $\Phi : K \longrightarrow 2^X$ be defined by

$$\Phi(x) = \{y \in K : \langle Ty, x - y \rangle \in (-\mathrm{Int}\, C)^c\}.$$

Clearly, $\Phi(x)$ contains x and is nonempty. We complete the proof by use of Theorem 4.2. We have to show that every $\Phi(x)$ is closed in X and that Φ is a KKM mapping.

First, we prove that Φ is a KKM mapping. Let $\{x_1,\ldots,x_n\}$ be any finite subset of K and let $\lambda_j \geq 0, 1 \leq j \leq n$, be such that $\sum_{j=1}^{n} \lambda_j = 1$. Write

$$x = \sum_{j=1}^{n} \lambda_j x_j.$$

Suppose that $x \notin \Phi(x_j)$ for all j with $1 \leq j \leq n$, i.e.,

$$\langle Tx, x_j - x \rangle \in -\mathrm{Int}\, C, \quad \text{for all } j.$$

By the convexity of $-\mathrm{Int}\, C$, we have the following:

$$0 = \langle Tx, x - x \rangle = \sum_{j=1}^{n} \lambda_j \langle Tx, x_j - x \rangle \in -\mathrm{Int}\, C,$$

which is a contradiction, since $C \neq \mathcal{Z}$. Therefore, Φ is a KKM mapping.

Note that every $\Phi(x)$ is closed if the function $\varphi : K \to \mathcal{Z}$, defined by

$$\varphi(y) = \langle Ty, x - y \rangle,$$

is continuous. This is equivalent to showing that, if $\{y_\lambda\}$ is a net in K converging to $y \in K$, then the net $\{\varphi(y_\lambda)\}$ converges to $\varphi(y)$.

Note that

$$\varphi(y_\lambda) - \varphi(y) = \langle Ty_\lambda - Ty, x - y_\lambda \rangle + \langle Ty, y - y_\lambda \rangle.$$

Since Ty is a continuous linear map on X and since $y - y_\lambda \longrightarrow 0$, we have

$$\langle Ty, y - y_\lambda \rangle \longrightarrow 0.$$

It remains to show that

$$\langle Ty_\lambda - Ty, x - y_\lambda \rangle \longrightarrow 0.$$

Let V be any 0-neighborhood in $\mathcal{Z}$. There is a 0-neighborhood U in $\mathcal{Z}$ such that $U = -U$ and $U + U \subset V$ (Ref. 32, Theorem 1.10). Note that $K + (-K)$ is bounded in X (Ref. 34, Chapter I, Theorem 5.1). Consider the open set $[K + (-K), U + U]$ in $\mathcal{L}_b(X, \mathcal{Z})$. By the continuity of T, there is a λ_V such that

$$\lambda \succeq \lambda_V \Longrightarrow Ty_\lambda - Ty \in [K + (-K), U + U].$$

This implies that

$$\langle Ty_\lambda - Ty, x - y_\lambda \rangle \in U + U \subset V, \quad \text{for } \lambda \succeq \lambda_V.$$

The proof is complete. □

Proof of Theorem 4.1. Without loss of generality, we may assume that $0 \in K$ and hence $K \cap Y \neq \emptyset$ for any $Y \in \mathcal{F}_X$. For any subspace Y of X, the inclusion map $P_Y : Y \longrightarrow X$ induces a linear map $P_Y^* : \mathcal{L}_b(X, \mathcal{Z}) \longrightarrow \mathcal{L}_b(Y, \mathcal{Z})$, defined by

$$P_Y^*(f) = f \circ P_Y.$$

For any $Y \in \mathcal{F}_X$, the map

$$T_Y = P_Y^* T P_Y : K \cap Y \to \mathcal{L}_b(Y, \mathcal{Z})$$

is continuous. It follows from Corollary 3.2 and Theorem 4.3 that there is an $x_Y \in Y \cap K$ such that

$$\langle Tx_Y, x - x_Y \rangle = \langle T_Y x_Y, x - x_Y \rangle \in (-\text{Int}\, C)^c, \quad \text{for all } x \in K \cap Y.$$

For any $Y \in \mathcal{F}_X$, consider the following subset of K:

$$K_Y = \{x_V \in K : Y \subset V \text{ and } V \in \mathcal{F}_X\}.$$

For $V, V' \in \mathcal{F}_X$, let $Y \in \mathcal{F}_X$ be such that $V \cup V' \subset Y$. Then,

$$\bar{K}_Y^{\mathcal{L}} \subset \bar{K}_V^{\mathcal{L}} \cap \bar{K}_{V'}^{\mathcal{L}}.$$

This proves that the family $\{\bar{K}_Y^{\mathcal{L}} : Y \in \mathcal{F}_X\}$ has the finite intersection property. Therefore, $\bigcap_{Y \in \mathcal{F}_X} \bar{K}_Y^{\mathcal{L}}$ is nonempty since K is $\mathcal{L}$-compact.

Let

$$\hat{x} \in \bigcap_{Y \in \mathcal{F}_X} \bar{K}_Y^{\mathcal{L}}.$$

Now, we complete the proof by showing that

$$\langle T\hat{x}, x - \hat{x} \rangle \in (-\text{Int}\, C)^c, \quad \text{for any } x \in K.$$

Let Y be the subspace of X generated by x and $\hat{x}$. Since $\hat{x} \in \bar{K}_Y^{\mathcal{L}}$, there is a net $\{x_\alpha\}$ in K_Y such that $x_\alpha \longrightarrow \hat{x}$ in $X_{\mathcal{L}}$, or equivalently, $x_\alpha \xrightarrow{\mathcal{L}} \hat{x}$; see Remark 3.1 (i). Note that $x_\alpha = x_{V_\alpha} \in V_\alpha \cap K$ for every α, where $Y \subset V_\alpha \in \mathcal{F}_X$.

Since

$$\langle Tx_\alpha, x_\alpha - \hat{x}\rangle \in (\operatorname{Int} C)^c, \quad \text{for every } \alpha,$$

we have

$$\operatorname{Limsup} \langle Tx_\alpha, x_\alpha - \hat{x}\rangle \subset (\operatorname{Int} C)^c.$$

As T is of class $(S)_+, x_\alpha \longrightarrow \hat{x}$ in X. Then, $\{Tx_\alpha\}$ converges to $T\hat{x}$ in $\mathcal{L}_b(X, \mathcal{Z})$ since T is continuous. We claim that

$$\lim_\alpha \langle Tx_\alpha, \hat{x} - x_\alpha\rangle = 0. \tag{1}$$

This implies that

$$\begin{aligned}\langle T\hat{x}, x - \hat{x}\rangle &= \lim_\alpha \langle Tx_\alpha, x - \hat{x}\rangle \\ &= \lim_\alpha (\langle Tx_\alpha, \hat{x} - x_\alpha\rangle + \langle Tx_\alpha, x - \hat{x}\rangle) \\ &= \lim_\alpha \langle Tx_\alpha, x - x_\alpha\rangle.\end{aligned}$$

Since

$$\langle Tx_\alpha, x - x_\alpha\rangle \in (-\operatorname{Int} C)^c, \quad \text{for all } \alpha,$$

by Theorem 2.1 and (ii), we have that

$$\langle T\hat{x}, x - \hat{x}\rangle = \lim_\alpha \langle Tx_\alpha, x - x_\alpha\rangle \in \operatorname{Limsup} \langle Tx_\alpha, x - x_\alpha\rangle \subset (-\operatorname{Int} C)^c.$$

It remains to prove equation (1). Since $x_\alpha \longrightarrow \hat{x}$ in X, there is a compact subset E of K containing $\hat{x}$ and all x_α. Thus,

$$\hat{x} - x_\alpha \in E - E = E_0.$$

Note that E_0 is compact in X and is bounded in X. Let W be any 0-neighborhood in $\mathcal{Z}$. There is a 0-neighborhood W_0 in $\mathcal{Z}$ such that $W_0 = -W_0$ and $W_0 + W_0 \subset W$. Consider the 0-neighborhood $[E_0, W_0]$ in $\mathcal{L}_b(X, \mathcal{Z})$. There is an α_W such that

$$\begin{aligned}\alpha \succeq \beta_W &\Longrightarrow Tx_\alpha - T\hat{x} \in [E_0, W_0] \text{ and } \langle T\hat{x}, \hat{x} - x_\alpha\rangle \in W_0 \\ &\Longrightarrow (Tx_\alpha - T\hat{x})(E_0) \subset W_0 \text{ and } \langle T\hat{x}, \hat{x} - x_\alpha\rangle \in W_0.\end{aligned}$$

Now, for $\alpha \succeq \alpha_W$, we have

$$\langle Tx_\alpha, \hat{x} - x_\alpha\rangle = \langle Tx_\alpha - T\hat{x}, \hat{x} - x_\alpha\rangle + \langle T\hat{x}, \hat{x} - x_\alpha\rangle \in W_0 + W_0 \subset W.$$

The proof of (1) is complete. □

5. Non $\mathcal{L}$-Compact Case

In this section, we derive some existence results for vector variational inequalities where the underlying domains may not be $\mathcal{L}$-compact. The first result of this section is the following.

Theorem 5.1. Let X be a locally convex Hausdorff tvs, let K be a nonempty convex subset of X, and let $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ be an operator. Assume that the following conditions hold:

(i) T is continuous and of class $(S)_+$.
(ii) For any convergent net $\{x_\alpha\}$ in K and for any $x \in K$, there is an α_0 such that, for every $\alpha \succeq \alpha_0$, there exists $\beta \succeq \alpha$ with $\mathrm{Sup}\,\{\langle Tx_{\beta'}, x - x_{\beta'}\rangle : \beta' \succeq \beta\} \neq \emptyset$.
(iii) There exist a nonempty $\mathcal{L}$-compact set $K_0 \subset K$ and a nonempty convex and $\mathcal{L}$-compact set $K_1 \subset K$ such that, if $x \in K \cap K_0^c$, then $\langle Tx, y - x\rangle \in -\mathrm{Int}\, C$ for some $y \in K_1$.

Then, VVI(T, K, C) has a solution.

Proof. Let Ω be the family of all nonempty finite subsets of K. For any $E \in \Omega$, let

$$\mathcal{S}_E = \{x \in K_0 : \langle Tx, y - x\rangle \in (-\mathrm{Int}\, C)^c, \quad \text{for all } y \in \mathrm{co}(K_1 \cup E)\}.$$

Since K_1 is convex, $\mathcal{L}$-compact, and since E is finite, the set $\mathrm{co}(K_1 \cup E)$ is $\mathcal{L}$-compact; see Ref. 33, Section 5.4.4, p. 73. By Theorem 4.1, there exists $x_E \in \mathrm{co}(K_1 \cup E)$ such that

$$\langle Tx_E, x - x_E\rangle \in (-\mathrm{Int}\, C)^c, \quad \text{for all } x \in \mathrm{co}(K_1 \cup E). \tag{2}$$

If $x_E \in K_0^c$, then $x_E \in K \cap K_0^c$; hence, from the assumption, there exists $y \in K_1$ such that

$$\langle Tx_E, y - x_E\rangle \in -\mathrm{Int}\, C,$$

which is a contradiction to (2). Therefore, $x_E \in K_0$; consequently, $\mathcal{S}_E \neq \emptyset$ for any $E \in \Omega$.

Clearly, the family $\{\mathcal{S}_E : E \in \Omega\}$ has the finite intersection property. Since K_0 is $\mathcal{L}$-compact, we deduce that

$$\bigcap_{E \in \Omega} \bar{\mathcal{S}}_E^{\mathcal{L}} \neq \emptyset. \tag{3}$$

By employing the same argument as in the proof of Theorem 4.1, one can show that any element in the intersection given in (3) is a solution of VVI(T, K, C). The proof is now complete. □

Next, we derive some necessary and sufficient conditions for the existence of solutions to the vector variational inequality. Let X be a Hausdorff tvs. A subset K of X is called inductive if there is a directed index set $\mathcal{A}$ such that

$$K = \bigcup_{\alpha \in \mathcal{A}} K_\alpha,$$

where each K_α is $\mathcal{L}$-compact in X and the sets K_α are increasing in the sense that $K_\alpha \subset K_\beta$ whenever $\alpha \preceq \beta$.

Theorem 5.2. Let X be a Hausdorff tvs let, $K = \bigcup_{\alpha \in \mathcal{A}} K_\alpha$ be a nonempty $\mathcal{L}$-closed and inductive subset of X, and let $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ be an operator. If the solution set $\mathcal{S}$ of VVI(T, K, C) is nonempty, then the following condition holds:

(i) There is a net $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ in an $\mathcal{L}$-compact subset of K and there is an $\alpha_0 \in \mathcal{A}$ such that x_α lies in the solution set $\mathcal{S}(\alpha)$ of VVI(T, K_α, C) for every $\alpha \succeq \alpha_0$.

Conversely, $\mathcal{S} \neq \emptyset$ if condition (i) holds together with the following conditions:

(ii) T is continuous and of class $(S)_+$.
(iii) For any convergent net $\{x_\alpha\}$ in K and for any $x \in K$, there is an α_0 such that, for every $\alpha \succeq \alpha_0$, there exists $\beta \succeq \alpha$ with $\mathrm{Sup}\,\{\langle T x_{\beta'}, x - x_{\beta'} \rangle : \beta' \succeq \beta\} \neq \emptyset$.

Proof. It is clear that (i) holds if $\mathcal{S} \neq \emptyset$. Conversely, assume that the conditions (i), (ii), and (iii) hold. Let $\{x_\alpha\}$ be a net in an $\mathcal{L}$-compact subset of K with $x_\alpha \in \mathcal{S}(\alpha)$ for each $\alpha \succeq \alpha_0$. Without loss of generality, we may assume that $x_\alpha \xrightarrow{\mathcal{L}} \hat{x}$. Since K is $\mathcal{L}$-closed, $\hat{x} \in K$. For each $x \in K$, let $\beta \in \mathcal{A}$ be such that $\{\hat{x}, x\} \subset K_\beta$. Then, $x \in K_\alpha$ for all $\alpha \succeq \beta$. Since $\langle T x_\alpha, x_\alpha - \hat{x} \rangle \in (\mathrm{Int}\, C)^c$ for all $\alpha \succeq \alpha_0$ and since $\alpha \succeq \beta$, we have

$$\mathrm{Limsup}\, \langle T x_\alpha, x_\alpha - \hat{x} \rangle \subset (\mathrm{Int}\, C)^c;$$

hence, $x_\alpha \longrightarrow \hat{x}$ in X because T is of class $(S)_+$. Now, by employing the same argument as in proof of Theorem 4.1, we conclude that

$$\langle T\hat{x}, x-\hat{x}\rangle \in (-\operatorname{Int} C)^c.$$

Hence, $\hat{x} \in \mathcal{S}$ and $\mathcal{S} \neq \emptyset$. □

6. Vector Complementarity Problems

Let X be a tvs, let K be a closed convex cone in X, and let $T: X \longrightarrow \mathcal{L}(X, \mathcal{Z})$ be an operator. The vector complementarity problem, denoted by VCP(T, K, C) is to find $\hat{x} \in K$ such that

$$\langle T\hat{x}, \hat{x}\rangle \notin \operatorname{Int} C \text{ and } \langle T\hat{x}, x\rangle \notin -\operatorname{Int} C, \quad \text{for all } x \in K.$$

The VCP(T, K, C) was introduced by Chen and Yang in 1990 (Ref. 3). When $\mathcal{Z} = \mathbb{R}$ and C is the set of all nonnegative real numbers, the VCP(T, K, C) coincides with the generalized complementarity problem (CP), that is, to find $\hat{x} \in K$ such that

$$\langle T\hat{x}, \hat{x}\rangle = 0 \quad \text{and} \quad T\hat{x} \in K^*,$$

where

$$K^* = \{\ell \in X^* : \langle \ell, x\rangle \geq 0, \text{ for all } x \in K\}$$

is the dual cone of K. Problem (CP) was introduced by Karamardian (Ref. 36) and has been investigated extensively in the literature; see for example Refs. 21, Refs. 37–40, and references therein.

In this section, we obtain some existence results for the problem VCP(T, K, C) by employing the existence results derived in Section 5. First, we have the following lemma giving the equivalence relationship between VVI(T, K, C) and VCP(T, K, C). The proof can be found in Ref. 6, Lemma 4.1.

Lemma 6.1. Let K be a closed convex cone in a tvs X and let $T: K \to \mathcal{L}(X, \mathcal{Z})$ be an operator.

(i) If $\hat{x}$ is a solution to VVI(T, K, C), then $\hat{x}$ is a solution to VCP(T, K, C).
(ii) If $\hat{x}$ is a solution to VCP(T, K, C) and if $\langle T\hat{x}, \hat{x}\rangle \in -C$, then $\hat{x}$ is a solution to VVI(T, K, C).

Combining Theorem 5.1 and Lemma 6.1, we have the following existence result for VCP(T, K, C).

Theorem 6.1. Let K be a closed convex cone of a locally convex Hausdorff tvs X and let $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ be continuous and of class $(S)_+$. Assume that the following conditions hold:

(i) For any convergent net $\{x_\alpha\}$ in K and for any $x \in K$, there is an α_0 such that, for every $\alpha \succeq \alpha_0$, there exists $\beta \succeq \alpha$ with Sup $\{\langle T x_{\beta'}, x - x_{\beta'} \rangle : \beta' \succeq \beta\} \neq \emptyset$.

(ii) There exist a nonempty $\mathcal{L}$-compact set $K_0 \subset K$ and a nonempty convex and $\mathcal{L}$-compact set $K_1 \subset K$ such that, if $x \in K \cap K_0^c$, then $\langle T x, y - x \rangle \in -\mathrm{Int}\, C$ for some $y \in K_1$.

Then, VCP(T, K, C) has a solution.

Finally, we have the following result, which is a consequence of Theorem 5.2 and Lemma 6.1.

Theorem 6.2. Let X be a Hausdorff tvs and let K be an $\mathcal{L}$-closed convex cone in X which is also inductive, given by $K = \bigcup_{\alpha \in \mathcal{A}} K_\alpha$. Let $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ be continuous and of class $(S)_+$. Assume that the following conditions hold:

(i) For any convergent net $\{x_\alpha\}$ in K and for any $x \in K$, there is an α_0 such that, for every $\alpha \succeq \alpha_0$, there exists $\beta \succeq \alpha$ with $\mathrm{Sup}\, \{\langle T x_{\beta'}, x - x_{\beta'} \rangle : \beta' \succeq \beta\} \neq \emptyset$.

(ii) There exists a net $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ in an $\mathcal{L}$-compact subset of K and there is an α_1 such that x_α lies in the solution set $\mathcal{S}(\alpha)$ of VVI(T, K_α, C) for every $\alpha \succeq \alpha_1$.

Then, VCP(T, K, C) has a solution.

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