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New Generalized Convexity Notion for Set-Valued Maps and Application to Vector Optimization[1]

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Abstract. In this paper, we introduce a new generalized convexity notion for set-valued maps, called ic-cone-convexlikeness, and use it as the main tool to derive an alternative theorem and necessary conditions for efficient, weakly efficient, and Benson properly efficient solutions of the problem of minimizing a set-valued map subject to set-valued constraints. Our results are valid for a class of optimization problems broader than that of the problems considered in Refs. 1–6 and generalize the corresponding results of these references.

Key Words. Convexlikeness, set-valued maps, alternative theorems, efficiency, Benson proper efficiency.

1. Introduction

Let Q be an arbitrary set; let D and E be convex cones of locally convex spaces Y and Z, respectively. Let F (resp. G) be a set-valued map associating to any point $x \in Q$ a nonempty set $F(x)$ [resp. $G(x)$] of Y [resp. Z]. In this paper, we are interested in weakly efficient solutions, efficient solutions, and Benson properly efficient solutions of the following vector optimization problem:

$$\text{(P)} \quad \min \quad F(x) \tag{1a}$$

$$\text{s.t.} \quad x \in V := \{x' \in Q : G(x') \cap (-E) \notin \emptyset\}, \tag{1b}$$

where $\emptyset$ is the empty set. Necessary conditions for efficiency, weak efficiency, proper efficiency and other results related to optimization theory such as minimax theorems, alternative theorems, etcetera have been developed in several papers under some generalized convexity assumptions:

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cone-convexlikeness (Refs. 7–9), cone-subconvexlikeness (Refs. 1–2, 8–10), generalized cone-subconvexlikeness (Refs. 3–4, 11), and near cone-subconvexlikeness (Refs. 5, 12). Among them, the near cone-subconvexlikeness is the most general notion and can be applied successfully to derive necessary conditions for weakly efficient solutions and Benson properly efficient solutions of (P) in terms of Lagrange multipliers (Ref. 5) and saddle points (Ref. 12). The definition of near cone-subconvexlikeness does not require any topological property of the cones D and E. However, the nonemptiness of the interior of these cones must be satisfied when proving optimization results (see Refs. 1–5). Observe that this requirement does not hold in many optimization problems. For example, int $E=\emptyset$ if E is the positive cone of the space L_p or ℓ_p, with $p\geq 1$. Another example is the case when E is the Cartesian product $E'\times E''$ of a trivial cone $E'=\{0\}$ and a cone E'' having a nonempty interior. Such a case was considered in Ref. 6 where necessary conditions were established for weak efficiency under a near cone-semiconvexlikeness assumption.

In this paper, we introduce a new notion of generalized convexity for set-valued maps, called ic-cone-convexlikeness, and prove that this notion can be used as the main tool to obtain an alternative result and necessary conditions for weakly efficient solutions, efficient solutions, and Benson properly efficient solutions for a class of vector optimization problems which is broader than that of the problems considered in Refs. 1–6.

In this paper, it is assumed that X is an arbitrary set and that Y and Z are locally convex spaces with topological duals Y^* and Z^*, respectively. The origin of Y is denoted by 0_Y. When no confusion can arise, we write 0 instead of 0_Y. We use the symbols U_Y and U_Z to denote convex neighborhoods of $0\in Y$ and $0\in Z$, respectively. A set $A\subset Y$ is a cone if $\lambda A\subset A, \forall\lambda>0$. A cone A is pointed if $a\in A\cap(-A)\Rightarrow a=0$. For a cone $A\subset Y$, we set

$$A^+=\{y^*\in Y^*:\langle y^*,a\rangle\geq 0, \forall a\in A\},$$
$$A^{+i}=\{y^*\in Y^*:\langle y^*,a\rangle> 0, \forall a\in A\backslash\{0\}\},$$

where $\langle .,.\rangle$ denotes the canonical bilinear form between Y and Y^*. For a set $A\subset Y$, we write

$$\operatorname{cone} A=\{\lambda a:\lambda>0, a\in A\}. \tag{2}$$

If D is a cone, then it is easy to see that

$$\operatorname{cone}(A+D)=\operatorname{cone} A+D. \tag{3}$$

The closure and interior of a set A are denoted by cl A and int A. A convex subset $\tilde{A}$ of a cone A is a base of A if $0\notin \operatorname{cl}\tilde{A}$ and $A\backslash\{0\}=\operatorname{cone}\tilde{A}$.

Theorem 1.1. See Ref. 13. Let D and K be cones in Y such that $D \cap K = \{0\}$. If D has a compact base and K is closed, then there exists a pointed convex cone D' such that $D \backslash \{0\} \subset \operatorname{int} D'$ and $D' \cap K = \{0\}$.

The following lemma is known in convex analysis.

Lemma 1.1. Let $A \subset Y$ be convex. Then, $\operatorname{cl} A$ is convex. In addition, if $\operatorname{int} A \neq \emptyset$, then $\operatorname{int} A$ is convex, $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$, and $\operatorname{int} A = \operatorname{int} \operatorname{cl} A$.

Lemma 1.2. See Refs. 14–15. Let $A \subset Y$ be an arbitrary subset and let $D \subset Y$ be a convex cone with nonempty interior. Then,

$$\operatorname{cl}(A + D) = \operatorname{cl}(A + \operatorname{int} D),$$
$$\operatorname{int} \operatorname{cl}(A + D) = A + \operatorname{int} D.$$

Remark 1.1. Under the same conditions of Lemma 1.2, we have

$$\operatorname{int}(A + D) = A + \operatorname{int} D.$$

Indeed, since $A + \operatorname{int} D$ is an open set, we get by Lemma 1.2

$$A + \operatorname{int} D \subset \operatorname{int}(A + D) \subset \operatorname{int} \operatorname{cl}(A + D) = A + \operatorname{int} D.$$

This proves the desired equality.

2. ic-Cone-Convexlike Set-Valued Maps

In this section, we introduce the notion of ic-cone-convexlikeness for set-valued maps and we establish an alternative theorem which generalizes the corresponding results of Refs. 1–6. We begin by studying the generalized convexity for sets. Let Y be a locally convex space and let $A \subset Y$ be a nonempty subset.

Definition 2.1. The set A is called int-convex (shortly, i-convex) if $\operatorname{int} A$ is convex and if $A \subset \operatorname{cl} \operatorname{int} A$.

Remark 2.1. From the very definition of the i-convexity of A, it is clear that $\operatorname{int} A \neq \emptyset$, since otherwise the second requirement in Definition 2.1 cannot be satisfied for a nonempty set A.

We now give some characterizations of i-convex sets.

Proposition 2.1. The set A is i-convex if and only if $\operatorname{int} A$ is convex and

$$\operatorname{cl} A = \operatorname{cl}\ \operatorname{int} A. \tag{4}$$

Proof. Obviously, we have

$$A \subset \operatorname{cl}\ \operatorname{int} A \Leftrightarrow \operatorname{cl} A \subset \operatorname{cl}\ \operatorname{int} A \subset \operatorname{cl} A \Leftrightarrow \operatorname{cl} A = \operatorname{cl}\ \operatorname{int} A. \qquad \square$$

Proposition 2.2. The following statements are equivalent:

(a) $\operatorname{cl} A$ in convex and

$$\operatorname{int}\ \operatorname{cl} A = \operatorname{int} A \neq \emptyset. \tag{5}$$

(b) $\operatorname{int} A \neq \emptyset$ and

$$\alpha \operatorname{int} A + (1-\alpha) A \subset \operatorname{int} A, \quad \forall \alpha \in (0, 1).$$

(c) The set A is i-convex.

Before proving this proposition, let us establish the following lemma.

Lemma 2.1. Let A and B be nonempty sets of Y and let $\operatorname{int} A \neq \emptyset$. If $\operatorname{int} A + B$ is convex and if

$$A + B \subset \operatorname{cl}(\operatorname{int} A + B), \tag{6}$$

then

$$\operatorname{int} A + B = \operatorname{int}(A + B) = \operatorname{int}\ \operatorname{cl}(A + B), \tag{7}$$

$$\operatorname{cone}\ \operatorname{int}(A + B) = \operatorname{int}\ \operatorname{cone}(A + B). \tag{8}$$

Proof. We prove first (7). Obviously,

$$\operatorname{int} A + B \subset A + B \subset \operatorname{cl}(A + B).$$

Observing that $\operatorname{int} A + B$ is an open set, we derive that

$$\operatorname{int} A + B \subset \operatorname{int}(A + B) \subset \operatorname{int}\ \operatorname{cl}(A + B).$$

So, all we have to prove is that

$$\operatorname{int} A + B = \operatorname{int}\ \operatorname{cl}(A + B).$$

Indeed, assume to the contrary that there exists a point

$$y \in \text{int cl}(A+B), \tag{9}$$

which does not belong to $\text{int}\, A+B$. Then, by a separation theorem, there exists $y^* \in Y^* \backslash \{0\}$ such that

$$\langle y^*, y\rangle < \langle y^*, y'\rangle, \quad \forall y' \in \text{int}\, A+B.$$

Combining this result with (6) yields

$$\langle y^*, y\rangle \leq \langle y^*, y'\rangle, \quad \forall y' \in \text{cl}(A+B).$$

By (9), there exists a neighborhood U_Y such that

$$y+U_Y \subset \text{cl}(A+B).$$

Thus,

$$\langle y^*, y\rangle \leq \langle y^*, y'\rangle, \quad \forall y' \in y+U_Y. \tag{10}$$

This holds only if $y^*=0$, which is impossible. Thus, (7) is established.

Turning to the proof of (8), we observe first from (7) that $\text{int}\,(A+B)$ is convex and nonempty. So, cone $\text{int}(A+B)$ is also convex and nonempty. Since

$$\text{int}(A+B) \subset \text{int cone}(A+B),$$

we have

$$\text{cone int}(A+B) \subset \text{cone}[\text{int cone}(A+B)] \subset \text{int}[\text{cone}(A+B)].$$

Assume to the contrary that there exists a point

$$y \in \text{int}[\text{cone}(A+B)] \tag{11}$$

such that

$$y \notin \text{cone int}(A+B).$$

By a separation theorem, there exists $y^* \in Y^* \backslash \{0\}$ such that

$$\langle y^*, y\rangle \leq 0 \leq \langle y^*, y'\rangle, \quad \forall y' \in \text{int}(A+B).$$

On the other hand, from (6) and (7), we have

$$A+B \subset \text{cl int}(A+B).$$

Thus,

$$\langle y^*, y\rangle \le 0 \le \langle y^*, y'\rangle, \quad \forall y' \in (A+B),$$

or equivalently,

$$\langle y^*, y\rangle \le 0 \le \langle y^*, y'\rangle, \quad \forall y' \in \operatorname{cone}(A+B). \tag{12}$$

By (11), there exists U_Y such that

$$y + U_Y \subset \operatorname{cone}(A+B).$$

Together with (12), the last inclusion proves that (10) holds. As we have seen above, this implies that $y^* = 0$, which is impossible. □

Proof of Proposition 2.2.
(a) ⇒ (c). Let $M = \operatorname{cl} A$. Then, by (5), $\operatorname{int} M \neq \emptyset$. In view of Lemma 1.1, $\operatorname{int} M$ is convex and

$$M = \operatorname{cl} M = \operatorname{cl}\ \operatorname{int} M.$$

Since $\operatorname{int} M = \operatorname{int} A$ [see (5)], we derive that $\operatorname{int} A$ is convex and

$$A \subset \operatorname{cl} A = M = \operatorname{cl}\ \operatorname{int} M = \operatorname{cl}\ \operatorname{int} A.$$

This shows that A is i-convex.

(c) ⇒ (a). Since $\operatorname{int} A$ is (nonempty and) convex, so is $\operatorname{cl}(\operatorname{int} A)$ (see Lemma 1.1). Therefore, by (4), $\operatorname{cl} A$ is convex. To complete the proof, it remains to observe that (5) can be obtained from (7) by setting $B = \{0\}$.

(b) ⇒ (c). Since $\operatorname{int} A \subset A$, the statement (b) yields

$$\alpha \operatorname{int} A + (1-\alpha)\operatorname{int} A \subset \operatorname{int} A, \quad \forall \alpha \in (0, 1),$$

proving the convexity of $\operatorname{int} A$. It remains to show that $A \subset \operatorname{cl}\ \operatorname{int} A$; i.e.,

$$a \in \operatorname{cl}\ \operatorname{int} A, \quad \text{for any } a \in A.$$

Take a point $a_0 \in \operatorname{int} A$. For any neighborhood V_Y of $0 \in Y$, let us choose $\alpha \in (0, 1)$ such that $\alpha(a_0 - a) \in V_Y$. Since $a + \alpha(a_0 - a) \in \operatorname{int} A$ [see statement (b)], we conclude that $(a + V_Y) \cap \operatorname{int} A \neq \emptyset$. Since this is true for any V_Y, we get $a \in \operatorname{cl}\ \operatorname{int} A$, as required.

(c) ⇒ (b). Assume to the contrary that, for $\alpha \in (0, 1)$, $a_0 \in \operatorname{int} A$, and $a \in A$, we have

$$\tilde{a} := \alpha a_0 + (1-\alpha) a \notin \operatorname{int} A.$$

By a separation theorem, there exists $y^* \in Y^*\backslash\{0\}$ such that $\langle y^*, \tilde{a}\rangle < \langle y^*, a'\rangle$ for all $a' \in \operatorname{int} A$. Since $a_0 \in \operatorname{int} A$ and $a \in \operatorname{cl}\ \operatorname{int} A$, this implies that

$$\langle y^*, \tilde{a}\rangle < \langle y^*, a_0\rangle \text{ and } \langle y^*, \tilde{a}\rangle \leq \langle y^*, a\rangle.$$

Multiplying the first of these inequalities by α and the second by $1-\alpha$, and summing up the obtained inequalities we have

$$\langle y^*, \tilde{a}\rangle < \langle y^*, \tilde{a}\rangle,$$

which is impossible. □

Definition 2.2. The set A is called

(i) nearly convex if there exists $\alpha \in (0, 1)$ such that $\alpha A + (1-\alpha)A \subset A$;
(ii) closely convex if $\operatorname{cl} A$ is convex.

The near convexity was introduced in Ref. 16. For the close convexity, see Refs. 14–15. Let us mention some properties of near convexity.

Lemma 2.2. See Ref. 16. Convexity ⇒ near convexity ⇒ close convexity.

Lemma 2.3. See Refs. 17, 6. Let A be a nearly convex set with nonempty interior. Then,

(i) $\operatorname{int} A$ is convex;
(ii) $\alpha \operatorname{int} A + (1-\alpha)\operatorname{cl} A \subset \operatorname{int} A, \quad \forall \alpha \in (0, 1)$.

The following result is clear from Proposition 2.2 and Lemmas 2.2 and 2.3.

Proposition 2.3. Let $\operatorname{int} A \neq \emptyset$. Consider the following statements:

(a) A is convex.
(b) A is nearly convex.
(c) A is closely convex and $\operatorname{int}\ \operatorname{cl} A = \operatorname{int} A$.
(d) $\alpha \operatorname{int} A + (1-\alpha)A \subset \operatorname{int} A, \quad \forall \alpha \in (0, 1)$.
(e) A is i-convex.

Then, (a) ⇒ (b) ⇒ (c) ⇔ (d) ⇔ (e).

Definition 2.3. The set A is called intcone-convex (shortly, ic-convex) if cone A is i-convex.

Let $D \subset Y$ be a nonempty convex cone.

Definition 2.4. The set A is called ic-D-convex if $A+D$ is ic-convex.

The following proposition is clear.

Proposition 2.4.

(i) The set A is ic-convex if and only if int cone A is convex and

$$A \subset \text{cl int cone}\, A.$$

(ii) The set A is ic-D-convex if and only if int cone$(A+D)$ is convex and

$$A+D \subset \text{cl int cone}(A+D).$$

Proposition 2.5.

$$\text{i-convexity} \Rightarrow \text{ic-convexity} \Rightarrow \text{ic-}D\text{-convexity}. \tag{13}$$

Proof. We start by the first implication in (13). From the very definition of i-convexity, it follows that int$A$ is a nonempty convex set. Hence, cone int$A$ is also a nonempty convex set. Using (8) with $B=\{0\}$, we see that int cone A is nonempty and convex. In addition, from the inclusion $A \subset \text{cl int}\, A$ in Definition 2.1, we obtain

$$A \subset \text{cl int cone}\, A. \tag{14}$$

By Proposition 2.4, A is ic-convex.

To prove the second implication in (13), we note from (14) that

$$\text{cone}\, A+D \subset \text{cl int cone}\, A+D \subset \text{cl}[\text{int cone}A+D]. \tag{15}$$

Observe also from the definition of ic-convexity that int cone A is nonempty and convex. Therefore, int cone $A+D$ is nonempty and convex. Applying (7) with cone A instead of A and with D instead of B, we get

$$\text{int cone}\, A+D = \text{int}\,(\text{cone}\, A+D).$$

This and (15) prove that int (cone $A+D$) is a nonempty convex set and that

$$\text{cone}\, A+D \subset \text{cl int}\,(\text{cone}A+D).$$

To complete our proof, it remains to observe by (3) that, in the above formulation, the set cone $A+D$ can be replaced by cone $(A+D)$. □

Remark 2.2. Examples 2.1 and 2.2 below prove that

i-convexity $\not\Leftarrow$ ic-convexity $\not\Leftarrow$ ic-D-convexity.

Example 2.1. Let Y be the real line R and let $A=\{0,1\}\subset R$. Then, A is ic-convex but it is not i-convex.

Example 2.2. Let Y be the plane R^2 and let $D=\{(\xi,\eta)\in R^2:\eta\geq 0\}$. Then, the following set A is ic-D-convex, but is not ic-convex:

$$A=\left\{(\xi,\eta)\in R^2:\eta\geq 0\right\}\bigcup\left\{(0,\eta)\in R^2:\eta\leq 0\right\}.$$

Applying Proposition 2.3, we obtain the following result.

Proposition 2.6. Let int cone$(A+D)\neq\emptyset$. Consider the following statements:

(a) cone$(A+D)$ is convex.
(b) cone$(A+D)$ is nearly convex.
(c) cone$(A+D)$ is closely convex and

$$\text{int cl cone}(A+D)=\text{int cone}(A+D). \tag{16}$$

(d) α int cone$(A+D)+(1-\alpha)\,$cone$(A+D)\subset$ int cone$(A+D)$,

$\forall\alpha\in(0,1)$.

(e) A is ic-D-convex.

Then, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e).

Corollary 2.1. If cone$(A+D)$ is closely convex and if int $D\neq\emptyset$, then (16) holds and A is ic-D-convex.

Proof. By Proposition 2.6, all we have to prove is equality (16). By virtue of (3), equality (16) means that

$$\text{int cl}[\text{cone}A+D]=\text{int}[\text{cone}A+D].$$

But the last equality is a direct consequence of Lemma 1.2 and Remark 1.1, with cone$A$ instead of A. Thus, (16) holds, as desired. □

Remark 2.3. If $\operatorname{int} D \neq \emptyset$, then statement (c) of Proposition 2.6 holds (see Corollary 2.1). But there exist sets A for which condition (16) and this statement hold even though $\operatorname{int} D = \emptyset$.

Example 2.3. Let

$$A = \{(0,0), (0,1), (1,0)\} \subset R^2 \text{ and } D = \{(0,\eta) \in R^2 : \eta \geq 0\}.$$

Then, cl cone$(A + D)$ coincides with the first orthant R^2_+ of R^2 [i.e., cone$(A + D)$ is closely convex] and (16) holds.

We now establish an alternative result for ic-D-convex sets.

Theorem 2.1. Let the set A be ic-D-convex. Then,

$$\text{either} \quad \text{(a)} \quad 0 \in \operatorname{int} \operatorname{cone}(A + D), \tag{17}$$

$$\text{or} \quad \text{(b)} \quad \exists y^* \in D^+ \backslash \{0\} \text{ s.t. } \inf_{a \in A} \langle y^*, a \rangle \geq 0, \tag{18}$$

but never both.

Proof. We first prove that [not (a)] $\Rightarrow$ (b). Indeed, if $0 \notin \operatorname{int} \operatorname{cone}(A + D)$, then by a separation theorem there exists $y^* \in Y^* \backslash \{0\}$ such that

$$\langle y^*, y \rangle > 0, \quad \forall y \in \operatorname{int} \operatorname{cone}(A + D),$$

which implies that

$$\langle y^*, y \rangle \geq 0, \quad \text{for all } y \in \operatorname{cl} \operatorname{int} \operatorname{cone}(A + D).$$

Since

$$A + D \subset \operatorname{cone}(A + D) \subset \operatorname{cl} \operatorname{int} \operatorname{cone}(A + D),$$

we have

$$\langle y^*, y \rangle \geq 0, \quad \text{for all } y \in (A + D).$$

This result and the positive homogeneity of D prove that $y^* \in D^+$. Thus, statement (b) holds.

To complete our proof, it remains to prove that statements (a) and (b) of Theorem 2.1 cannot be satisfied simultaneously. Indeed, if (a) holds, then for some U_Y we get

$$U_Y \subset \operatorname{cone}(A + D).$$

If (b) holds, then from (b) and the fact that $\langle y^*, d\rangle \geq 0$ for all $d \in D$, we have

$$\langle y^*, y\rangle \geq 0, \quad \text{for all } y \in \text{cone}(A+D).$$

Thus, if (a) and (b) are satisfied simultaneously, then

$$\langle y^*, y\rangle \geq 0, \quad \text{for all } y \in U_Y.$$

This is true only if $y^* = 0$, a contradiction to the assumption that $y^* \neq 0$ in statement (b). □

Remark 2.4. The definition of ic-D-convexity of A requires the validity of two conditions: int cone$(A + D)$ is convex and cone$(A + D) \subset$ cl int cone$(A+D)$. Theorem 2.1 is no longer true if at least one of these conditions is deleted from the definition of ic-D-convexity.

Example 2.4. Let

$$D = \{0\} \subset Y = R^2$$

and let A be as in Example 2.2. Then, int cone$(A+D)$ is nonempty and convex, but

$$\text{cone}(A+D) \not\subset \text{cl int cone}(A+D)$$

and both statements (a) and (b) of Theorem 2.1 fail to hold.

Example 2.5. Let

$$D = \{0\} \subset Y = R^2 \text{ and } A = R^2_+ \cup \left(-R^2_+\right).$$

Then,

$$\text{cone}(A+D) = \text{cl int cone}(A+D),$$

but int cone$(A + D)$ is not convex and both statements (a) and (b) of Theorem 2.1 fail to hold.

Now, for a set-valued map $F : X \to Y$, denote by dom F and im F the domain and the image of F,

$$\text{dom}\, F = \{x \in X : F(x) \neq \emptyset\},$$

$$\text{im}\, F = F(X) = \bigcup_{x \in X} F(x).$$

Definition 2.5. The map F is called i-convexlike [resp. ic-convexlike, ic-D-convexlike] if $\operatorname{im} F$ is i-convex [resp. ic-convex, ic-D-convex].

By Proposition 2.5, we see that

$$\text{i-convexlikeness} \Rightarrow \text{ic-convexlikeness} \Rightarrow \text{ic-}D\text{-convexlikeness};$$

by Remark 2.2, the converse of each of these implications is not true.

The following result is a direct consequence of Proposition 2.6.

Proposition 2.7. Let $\operatorname{int} \operatorname{cone}(\operatorname{im} F + D) \neq \emptyset$. Consider the following statements:

(a) $\operatorname{cone}(\operatorname{im} F + D)$ is convex.
(b) $\operatorname{cone}(\operatorname{im} F + D)$ is nearly convex.
(c) $\operatorname{cl} \operatorname{cone}(\operatorname{im} F + D)$ is convex and

$$\operatorname{int} \operatorname{cl} \operatorname{cone}(\operatorname{im} F + D) = \operatorname{int} \operatorname{cone}(\operatorname{im} F + D). \tag{19}$$

(d) $\alpha \operatorname{int} \operatorname{cone}(\operatorname{im} F + D) + (1-\alpha) \operatorname{cone}(\operatorname{im} F + D) \subset \operatorname{int} \operatorname{cone}(\operatorname{im} F + D)$, $\forall \alpha \in (0, 1)$.
(e) F is ic-D-convexlike.

Then, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e).

Remark 2.5. In Refs. 5 and 12, a map F is called nearly D-subconvexlike if $\operatorname{cl} \operatorname{cone}(\operatorname{im} F + D)$ is convex. Proposition 2.7 proves that the class of ic-D-convexlike maps consists of nearly D-subconvexlike maps F satisfying the additional requirement (19). Observe that (19) holds if $\operatorname{int} D \neq \emptyset$ (see Corollary 2.1), but there exists maps F for which (19) holds even though $\operatorname{int} D = \emptyset$ (see Remark 2.3).

Now, we can formulate an alternative result for ic-D-convexlike maps which is obtained from Theorem 2.1 with $A = \operatorname{im} F$.

Theorem 2.2. Let F be ic-D-convexlike. Then,

either (a) $0 \in \operatorname{int} \operatorname{cone}(\operatorname{im} F + D)$,

or (b) $\exists y^* \in D^+ \setminus \{0\}$ s.t. $\inf_{y \in \operatorname{im} F} \langle y^*, y \rangle \geq 0$,

but never both.

Corollary 2.2. Let cl cone$(\operatorname{im} F + D)$ be convex and let int $D \neq \emptyset$. Then,

either (a) $\operatorname{im} F \cap -\operatorname{int} D \neq \emptyset$,

or (b) $\exists y^* \in D^+ \backslash \{0\}$ s.t. $\inf_{y \in \operatorname{im} F} \langle y^*, y \rangle \geq 0$,

but never both.

Proof. By Remark 2.5, F is ic-D-convexlike. On the other hand, setting $A = \operatorname{im} F$ and making use of Remark 1.1 and equality (3), we see that

$$\begin{aligned} A \cap -\operatorname{int} D \neq \emptyset \quad &\Leftrightarrow \quad 0 \in \operatorname{cone} A + \operatorname{int} D \\ &\Leftrightarrow \quad 0 \in \operatorname{int}(\operatorname{cone} A + D) \\ &\Leftrightarrow \quad 0 \in \operatorname{int}\ \operatorname{cone}(A + D). \end{aligned}$$

Therefore, Corollary 2.2 is a direct consequence of Theorem 2.2. □

Remark 2.6. Corollary 2.2 is exactly Theorem 4.1 of Ref. 5, which generalizes alternative results obtained earlier in Lemma 2.2 of Ref. 1, Lemma 3.3 of Ref. 2, Theorem 3.1 of Ref. 3, Theorem 3.1 of Ref. 4. Thus, among the alternative results, we just mention that our Theorem 2.2 is the most general.

Now, let $G: X \to Z$ be a set-valued map and let E be a convex cone of Z. Assume that F and G have the same domain, denoted by Q,

$$\operatorname{dom} F = \operatorname{dom} G = Q.$$

Let $K = D \times E$ and let $(F \times G)(.) = F(.) \times G(.)$.

Corollary 2.3. Let int $D \neq \emptyset$ and let $F \times G$ be ic-K-convexlike. If the following system is inconsistent (i.e., it has no solution):

$$x \in Q, \quad -F(x) \cap \operatorname{int} D \neq \emptyset, \quad -G(x) \cap E \neq \emptyset, \tag{20}$$

then there exists $(y^*, z^*) \in D^+ \times E^+ \backslash \{(0,0)\}$ such that $\langle y^*, y \rangle + \langle z^*, z \rangle \geq 0$ for all $(y, z) \in \operatorname{im}(F \times G)$. The converse statement is true if $y^* \neq 0$.

Proof. Assume that system (20) is inconsistent. Then,

$$(0,0) \notin \operatorname{int}\ \operatorname{cone}[\operatorname{im}(F \times G) + D \times E].$$

Indeed, otherwise there exist U_Y and U_Z such that

$$U_Y \times U_Z \subset \operatorname{cone}[\operatorname{im}(F \times G) + D \times E].$$

Since $\operatorname{int} D \neq \emptyset$, we can find $d' \in \operatorname{int} D$ such that $-d' \in U_Y \backslash \{0\}$. Thus, there exist $\lambda > 0$, $x \in Q$, $y \in F(x)$, $z \in G(x)$, $d \in D$, and $e \in E$ such that

$$-d' = \lambda(y + d) \quad \text{and} \quad 0 = \lambda(z + e).$$

Equivalently,

$$-y = (1/\lambda)d' + d \in \operatorname{int} D + D \subset \operatorname{int} D \quad \text{and} \quad -z = e \in E.$$

This proves that x is a solution of system (20), which is impossible. Now, applying Theorem 2.2 with $F \times G$ instead of F and with K instead of D, we obtain the desired conclusion of the first part of Corollary 2.3. The second part of Corollary 2.3 is established in Ref. 6. □

Remark 2.7. Corollary 2.3 generalizes Theorem 1 of Ref. 6. Indeed, this theorem claims that the same conclusions of Corollary 2.3 can be formulated under the following assumptions:

(i) $\operatorname{int}[\operatorname{im}(F \times G) + D \times E] \neq \emptyset$,
(ii) $F \times G$ is nearly K-semiconvexlike.

We do not recall the definition of near cone-semiconvexlikeness of Ref. 6, but we observe from Proposition 5 of Ref. 6 that (ii) holds if and only if $\operatorname{im}(F \times G) + D' \times E$ is nearly convex where $D' = \operatorname{int} D$. On the other hand, it is easily seen that (i) implies that $\operatorname{int}[\operatorname{im}(F \times G) + D' \times E] \neq \emptyset$. Therefore, by Proposition 2.3, the set $\operatorname{im}(F \times G) + D' \times E$ is i-convex. By Proposition 2.5, $F \times G$ is ic-K'-convexlike, where $K' := D' \times E$. To obtain Theorem 1 of Ref. 6, it remains to apply Corollary 2.3 with D' instead of D.

Examples can be given to illustrate that our Corollary 2.3 is more general than Theorem 1 of Ref. 6.

Example 2.6. Let

$$X = \{-1, 0, +1\} \subset R, \quad Y = Z = R, \quad D = R_+, \quad E = \{0\}.$$

For $x \in X$, let us set $F(x) = \{0\}$ and $G(x) = \{x\}$. It is easily seen that $F \times G$ is ic-K-convexlike, while the above assumptions of Theorem 1 of Ref. 6 do not hold.

3. Optimization with ic-Cone-Convexlike Set-Valued Maps

In this section, we investigate necessary conditions for solutions of the problem (P) (see Section 1) under some ic-cone-convexlikeness assumption. Let $F{:}X \to Y$ and $G{:}X \to Z$ be set-valued maps with the same domain Q.

Let $D \subset Y$ and $E \subset Z$ be convex cones. From now on, we assume that $D \neq Y$ and $D \neq \{0\}$. Consider the vector optimization problem (P) formulated in Section 1. We will be interested in weakly efficient solutions, efficient solutions, and Benson properly efficient solutions of (P). In this section, we assume that $x_0 \in V$ [see (1)], $y_0 \in F(x_0)$, and $z_0 \in G(x_0) \cap -E$.

Definition 3.1. A point (x_0, y_0) is a weakly efficient solution of (P) if

$$[F(V) - y_0] \cap -\operatorname{int} D = \emptyset.$$

Definition 3.2. A point (x_0, y_0) is an efficient solution of (P) if

$$[F(V) - y_0] \cap -[D \setminus \{0\}\} = \emptyset.$$

Definition 3.3. A point (x_0, y_0) is a Benson properly efficient solution of (P) if

$$[\operatorname{cl}\ \operatorname{cone}(F(V) - y_0 + D)] \cap -[D \setminus \{0\}] = \emptyset.$$

For each $\beta \in [0, 1)$, let us consider a set-valued map $H_\beta : X \to Y \times Z$ whose domain is the set Q,

$$H_\beta(x) = (F(x) - y_0) \times (G(x) - \beta z_0), \quad x \in Q. \tag{21}$$

Let $K = D \times E$. Throughout this section, we make the following assumption.

Assumption (A). There exists $\beta \in [0, 1)$ such that H_β is ic-K-convexlike.

Observe that, in Assumption (A), no topological property is imposed on D and E. So, this assumption can be used not only in studying the weak efficiency in problem (P) with $\operatorname{int} E = \emptyset$, but also in discussing the efficiency and proper efficiency in (P) without requiring that $\operatorname{int} D \neq \emptyset$ and $\operatorname{int}\ E \neq \emptyset$.

Definition 3.4. We say that condition (CQ) holds if

$$\operatorname{cl}\ \operatorname{cone}[\operatorname{im} G + E] = Z. \tag{22}$$

Observe that, for any $\beta \geq 0$,

$$\operatorname{im}(G - \beta z_0) + E \subset \operatorname{im} G + \beta E + E \subset \operatorname{im} G + E.$$

Thus, (22) holds if

$$\operatorname{cl}\ \operatorname{cone}[\operatorname{im}(G - \beta z_0) + E] = Z, \quad \text{for some } \beta \geq 0.$$

Let us give some sufficient conditions for Assumption (A) and condition (CQ) to hold. We begin by the conditions used in Refs. 1–5.

Proposition 3.1. Let $\operatorname{int} D \neq \emptyset$. Let $(F - y_0) \times G$ be nearly K-subconvexlike in the sense of Ref. 5; i.e., let cl cone$[\operatorname{im}((F - y_0) \times G) + K]$ be convex. Let the generalized Slater condition $(\operatorname{im} G) \cap -\operatorname{int} E \neq \emptyset$ be satisfied. Then, Assumption (A) and condition (CQ) hold, with $\beta = 0$.

Proof. Since $\operatorname{int} K = \operatorname{int} D \times \operatorname{int} E \neq \emptyset$, we derive from Remark 2.5 that $(F - y_0) \times G$ is ic-K-convexlike. In other words, Assumption (A) is satisfied with $\beta = 0$. We omit the easy proof of the fact that the generalized Slater condition implies (22). □

Remark 3.1. If we are interested in weak efficiency then, by replacing D by int D if necessary, we may assume that D is an open convex cone. This assumption is used in the following proposition.

Proposition 3.2. Let D be an open convex cone. Let $\beta \in [0, 1)$ be such that $[\operatorname{im} H_\beta + K]$ is nearly convex and, for some $y \in Y$, $(y, 0_Z) \in \operatorname{int}[\operatorname{im} H_\beta + K]$. Then Assumption (A) and condition (CQ) hold.

Proof. From the hypotheses of Proposition 3.2 and Remark 2.7, it is clear that Assumption (A) is satisfied. To prove (22), let us take U_Y and U_Z such that

$$(y, 0_Z) + U_Y \times U_Z \subset \operatorname{im} H_\beta + K.$$

Therefore, $\forall u \in U_Z$, $\exists x \in Q$ such that $(y, u) \in (F(x) - y_0) \times (G(x) - \beta z_0) + D \times E$, which implies that

$$\forall u \in U_Z, \quad \exists x \in Q \text{ s.t. } u \in G(x) - \beta z_0 + E.$$

In other words,

$$U_Z \subset \operatorname{im}(G - \beta z_0) + E;$$

hence, (22) holds, as desired. □

Let $\mathcal{L}_+(E, D)$ be the set of (single-valued) linear continuous maps T from Z into Y such that $T(E) \subset D$. For $x \in Q$ and $L \in \mathcal{L}_+(E, D)$, let

$$L(x, T) = F(x) + T[G(x)] := \bigcup_{(y,z) \in F(x) \times G(x)} [y + T(z)].$$

Theorem 3.1. Let $\operatorname{int} D \neq \emptyset$. Let Assumption (A) be satisfied. Let (x_0, y_0) be a weakly efficient solution of Problem (P). Then:

(i) There exists $(y_0^*, z_0^*) \in D^+ \times E^+ \setminus \{(0,0)\}$ such that

$$\langle y_0^*, y\rangle + \langle z_0^*, z\rangle \geq \langle y_0^*, y_0\rangle, \quad \forall (y,z) \in \operatorname{im}(F \times G), \tag{23}$$

$$\langle z_0^*, z_0'\rangle = 0, \quad \forall z_0' \in G(x_0) \cap (-\operatorname{cl} E). \tag{24}$$

(ii) Under condition (CQ), there exists $(y_0^*, z_0^*) \in D^+ \times E^+$ such that $y_0^* \neq 0$ and conditions (23)–(24) are fulfilled.

(iii) Under condition (CQ), there exists $T_0 \in \mathcal{L}_+(E, D)$ such that $(x_0, y_0 + T_0(z_0))$ is a weakly efficient solution of the following problem:

$$\text{(P)}' \quad \min \quad L(x, T_0), \qquad \text{s.t.} \quad x \in Q.$$

In addition,

$$T_0(z_0') = 0, \quad \forall z_0' \in G(x_0) \cap (-\operatorname{cl} E). \tag{25}$$

Proof. Let

$$S_\beta = \operatorname{im} H_\beta + K \subset Y \times Z.$$

We claim that

$$(0,0) \notin \operatorname{int}\ \operatorname{cone} S_\beta. \tag{26}$$

Indeed, otherwise we find convex neighborhoods U_Y and U_Z such that

$$U_Y \times U_Z \subset \operatorname{cone} S_\beta. \tag{27}$$

Since D is a convex cone with nonempty interior, there exists $d' \in \operatorname{int} D$ such that $-d' \in U_Y \setminus \{0\}$. Using (27) and noting that $(-d', 0) \in U_Y \times U_Z$, we can find $\lambda > 0, x \in Q, (y,z) \in F(x) \times G(x)$, and $(d, e) \in D \times E$ such that

$$-d' = \lambda(y - y_0 + d), \tag{28}$$

$$0 = \lambda(z - \beta z_0 + e). \tag{29}$$

Since $\lambda > 0$, (29) yields

$$z = \beta z_0 - e \in -E.$$

Thus, $z \in G(x) \cap (-E)$ i.e., $x \in V$. On the other hand, we see from (28) that

$$y - y_0 = -(1/\lambda)d' - d \in -\text{int}\, D - D \subset -\text{int}\, D,$$

a contradiction to Definition 3.1. Thus, we have shown that (26) holds. Applying Theorem 2.2, we find $(y_0^*, z_0^*) \in D^+ \times E^+ \backslash \{(0, 0)\}$ such that

$$\langle y_0^*, y\rangle + \langle z_0^*, z\rangle \geq \langle y_0^*, y_0\rangle + \langle z_0^*, \beta z_0\rangle, \quad \forall (y, z) \in \text{im}(F \times G). \tag{30}$$

Setting $y = y_0$ and $z = z_0$ in (30), we have

$$(1-\beta)\langle z_0^*, z_0\rangle \geq 0,$$

which implies that

$$\langle z_0^*, z_0\rangle \geq 0.$$

On the other hand,

$$\langle z_0^*, z_0\rangle \leq 0,$$

since $z_0 \in -E$ and $z_0^* \in E^+$. Thus,

$$\langle z_0^*, z_0\rangle = 0$$

and (30) reduces to (23), as desired.

Observe that $y_0^* \neq 0$ if (22) holds. Indeed, otherwise we see that

$$\begin{aligned} (23) \quad &\Rightarrow \quad \langle z_0^*, z\rangle \geq 0, \quad \forall z \in \text{im}\, G, \\ &\Rightarrow \quad \langle z_0^*, z'\rangle \geq 0, \quad \forall z' \in \text{im}\, G + E, \end{aligned} \tag{31}$$

since $\langle z^*, e\rangle \geq 0$ for all $e \in E$. By the continuity property, we have from (31)

$$\langle z_0^*, z'\rangle \geq 0, \quad \forall z' \in \text{cl cone}[\text{im}\, G + E] = Z.$$

Thus, $z_0^* = 0$. This is impossible, since $(y_0^*, z_0^*) \neq (0, 0)$.

Observe now that condition (24) is derived easily from (23). Indeed, for $y = y_0$ and $z = z_0' \in G(x_0) \cap -\text{cl}\, E$, condition (23) yields

$$\langle z_0^*, z_0'\rangle \geq 0.$$

But

$$\langle z_0^*, z_0'\rangle \leq 0,$$

since $z_0' \in -\text{cl}\, E$. Therefore, we get (24).

To complete the proof, let us observe that the map T_0 required in the third part of Theorem 3.1 can be constructed by a standard approach. Indeed, let (y_0^*, z_0^*) be as in the second part of Theorem 3.1. Since $y_0^* \in D^+ \setminus \{0\}$, we can find $d_0 \in \operatorname{int} D \subset D$ such that $\langle y_0^*, d_0 \rangle = 1$. Let $T_0(\cdot) = \langle z_0^*, \cdot \rangle d_0$. Then, obviously $T_0 \in \mathcal{L}_+(E, D), T_0$ satisfies (25) [see (24)], and $\langle y_0^*, T_0(\cdot)\rangle = \langle z_0^*, \cdot \rangle$. From (23) and from $T_0(z_0) = 0$, we obtain

$$\langle y_0^*, y + T_0(z) - [y_0 + T_0(z_0)]\rangle \geq 0, \quad \forall (y, z) \in \operatorname{im}(F \times G).$$

Since $y_0^* \in D^+ \setminus \{0\}$, this shows that

$$y + T_0(z) - [y_0 + T_0(z_0)] \notin -\operatorname{int} D, \quad \forall (y, z) \in \operatorname{im}(F \times G);$$

i.e., $(x_0, y_0 + T_0(z_0))$ is a weakly efficient solution of (P)′. □

Remark 3.2. Let $\hat{D}$ be a convex cone such that $\emptyset \neq \hat{D}\setminus\{0\} \subset \operatorname{int} D$. In the proof of the third part Theorem 3.1, if we take $d_0 \in \hat{D}\setminus\{0\}$, then the map T_0 constructed in this proof will be an element of $\mathcal{L}_+(E, \hat{D})$.

Theorem 3.2. Let D be pointed. Let Assumption (A) be satisfied. If (x_0, y_0) is an efficient point of (P), then there exists $(y_0^*, z_0^*) \in D^+ \times E^+ \setminus \{(0, 0)\}$ satisfying (23) and (24). In addition, $y_0^* \neq 0$ if condition (CQ) holds.

Proof. We claim that (26) holds. Indeed, otherwise we find U_Y and U_Z such that (27) is satisfied. Let us take $d' \in D\setminus\{0\}$ such that $-d' \in U_Y \setminus \{0\}$. Then, as in the proof of Theorem 3.1, there exist $\lambda > 0, x \in Q$, $(y, z) \in F(x) \times G(x)$, and $(d, e) \in D \times E$ such that (28) and (29) are satisfied. The argument used in the proof of Theorem 3.1 shows that $x \in V$ and

$$y - y_0 = -(1/\lambda)d' - d \in -D - D \subset -D.$$

Observe that $y \neq y_0$, since otherwise we have $d = -(1/\lambda)d'$. Since $d' \neq 0$, we derive that $D \cap (-D) \neq \{0\}$, a contradiction to the pointedness of D. Therefore, $y_0 - y \in D \setminus \{0\}$, which is impossible by the efficiency property of (x_0, y_0). To complete the proof, it remains to apply Theorem 2.2 and use the same argument as in the proof of Theorem 3.1. □

Theorem 3.3. Let $0 \in D$ and let D have a compact base. Let Assumption (A) be satisfied. Let condition (CQ) hold. Let (x_0, y_0) be a Benson properly efficient solution of (P). Then:

(i) There exists $\left(y_0^*, z_0^*\right) \in D^{+i} \times E^+$ satisfying (23) and (24).

(ii) There exists $T_0 \in \mathcal{L}_+(E, D)$ such that $(x_0, y_0 + T_0(z_0))$ is a Benson properly efficient solution of problem (P)$'$ and, in addition, (25) is satisfied.

Proof. Let

$$C = \text{cone}(F(V) - y_0 + D).$$

By Definition 3.3,

$$(-D) \cap \text{cl}\, C = \{0\}.$$

Using Theorem 1.1, we can find a pointed convex cone D_1 such that

$$(-D_1) \cap \text{cl}\, C = \{0\}, \tag{32}$$

$$D \setminus \{0\} \subset \text{int}\, D_1. \tag{33}$$

Since $0 \notin \text{int}\, D_1 \subset D_1$, we derive from (32) that

$$[F_1(V) - y_0] \cap -(\text{int}\, D_1) = \emptyset,$$

where

$$F_1(x) = F(x) + D.$$

Thus, (x_0, y_0) is a weakly efficient solution of a vector optimization problem $(\text{P})_1$ which differs from problem (P) in that F is replaced by F_1 and D is replaced by D_1. On the other hand, setting

$$H_{1\beta}(x) = (F_1(x) - y_0) \times (G(x) - \beta z_0),$$

we see easily from the ic-K-convexlikeness of H_β that $H_{1\beta}$ is ic-K_1-convexlike, where

$$K_1 = D_1 \times E.$$

Indeed, clearly,

$$\begin{aligned} \text{im}\, H_{1\beta} + K_1 &= \text{im}\, H_{1\beta} + \{0_Y\} \times E + D_1 \times \{0_Z\} \\ &= \text{im}\, H_\beta + D \times E + D_1 \times \{0_Z\}, \end{aligned}$$

which together with (3) implies that

$$A_1 = A + B,$$

where

$$A_1 = \text{cone}(\text{im}\, H_{1\beta} + K_1), \quad A = \text{cone}(\text{im}\, H_\beta + K), \quad B = D_1 \times \{0_Z\}.$$

Since H_β is ic-K-convexlike, we see that $\text{int}\, A$ is nonempty and convex and that $A \subset \text{cl int}\, A$. Therefore, $\text{int}\, A + B$ is nonempty and convex and

$$A + B \subset \text{cl int}\, A + B \subset \text{cl int}\, A + \text{cl}\, B \subset \text{cl}(\text{int}\, A + B).$$

Making use of (7), we obtain

$$\text{int}\, A + B = \text{int}(A + B) = \text{int}\, A_1.$$

Thus, $\text{int}\, A_1$ is convex and $A_1 \subset \text{cl int}\, A_1$. This proves the ic-$K_1$-convexlikeness of $H_{1\beta}$, as desired.

Now, applying Theorem 3.1 to problem $(\text{P})_1$ we find $(y_0^*, z_0^*) \in D_1^+ \times E^+ \setminus \{(0,0)\}$ such that (24) holds and

$$\langle y_0^*, y\rangle + \langle z_0^*, z\rangle \geq \langle y_0^*, y_0\rangle, \quad \forall (y,z) \in \text{im}(F_1 \times G), \tag{34}$$

where $y_0^* \neq 0$. Since $y_0^* \in D_1^+ \setminus \{0\}$ and since (33) holds, we get $y_0^* \in D^{+i}$. To complete the proof of the first part of Theorem 3.3, it remains to observe that (34) $\Rightarrow$ (23). To prove the second part, we use again Theorem 3.1. Indeed, applying this theorem to problem $(\text{P})_1$ and using Remark 3.2, we see that there exists $T_0 \in \mathcal{L}_+(E, D)$ such that

$$[y + T_0(z)] - [y_0 + T_0(z_0)] \notin -\text{int}\, D_1, \quad \forall (y,z) \in \text{im}(F_1 \times G), \tag{35}$$

and in addition (25) is satisfied. From (35), it is clear that

$$\text{cl cone}\Bigg\{ \bigcup_{(y,z)\in \text{im}\,(F\times G)} [y + T_0(z)] - [y_0 + T_0(z_0)] + D \Bigg\} \cap (-\text{int}\, D_1) = \emptyset.$$

Together with (33), this proves that

$$\text{cl cone}\Bigg\{ \bigcup_{(y,z)\in \text{im}\,(F\times G)} [y + T_0(z)] - [y_0 + T_0(z_0)] + D \Bigg\} \cap (-D \setminus \{0\}) = \emptyset,$$

that is, $(x_0, y_0 + T_0(z_0))$ is a Benson properly efficient solution of $(\text{P})'$. □

Remark 3.3. In Refs. 2–5, problem (P) is considered under the assumptions of Proposition 3.1. By this reason, we can claim that our Theorem 3.1 generalizes Theorem 5.1 of Ref. 2, Theorem 4.2 of Ref. 4, and Theorem 5.1 of Ref. 5. Our Theorem 3.3 generalizes Theorem 5.1 of Ref. 2 and Theorem 5.1 of Ref. 3. In the just mentioned References 2–5, it is assumed that int $E \neq \emptyset$. When the interior of E is empty, necessary conditions for weak efficiency are given in Theorem 3 of Ref. 6. Observe that the just mentioned theorem is also a special case of our Theorem 3.1. This is because, on the one hand, in studying weak efficiency the cone D can be replaced by int D (see Remark 3.1). On the other hand, the assumptions of Theorem 3 of Ref. 6 assure the validity of Proposition 3.2 (see Remark 2.7); hence, our Theorem 3.1 can be applied to derive this theorem of Ref. 6.

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