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A Study of Local Solutions in Linear Bilevel Programming[1]

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Abstract. In this paper, a linear bilevel programming problem (LBP) is considered. Local optimality in LBP is studied via two related problems (P) and P(M). Problem (P) is a one-level model obtained by replacing the innermost problem of LBP by its KKT conditions. Problem P(M) is a penalization of the complementarity constraints of (P) with a penalty parameter M. Characterizations of a (strict) local solution of LBP are derived. In particular, the concept of equilibrium point of P(M) is used to characterize the local optima of (P) and LBP.

Key Words. Bilevel linear programming, local optimization, exact penalty methods, equilibrium constraints.

1. Introduction

In this work, we consider the following linear bilevel program:

$$
\begin{array}{llll}
\text{(LBP)} & \max_{x,y} & f_1(x,y)=c_1^T x + c_2^T y, & \\
 & \text{s.t.} & x \geq 0, \quad y \text{ solves} & \\
 & & \max_y & f_2(x,y)=b^T y, \\
 & & \text{s.t.} & A_1 x + A_2 y \leq a, \\
 & & & y \geq 0,
\end{array}
$$

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where

$$c_1, x \in \mathbb{R}^{n_1}, \quad c_2, b, y \in \mathbb{R}^{n_2}, \quad a \in \mathbb{R}^m, \quad A_1 \in \mathbb{R}^{m \times n_1}, \quad A_2 \in \mathbb{R}^{m \times n_2}.$$

This problem has been studied extensively in the literature; see e.g. Refs. 1–6. The LBP with linear constraints in the first level has also been considered; see e.g. Ref. 7. We refer to Ref. 8 for a bibliographical survey and to Refs. 9–11 for more recent results on bilevel and multilevel programming.

Actually, this problem can be reformulated as a mathematical program with equilibrium constraints (MPEC), since the second-level problem can be replaced by a linear complementarity problem (Ref. 12). The two formulations are equivalent while considering global solutions, but the equivalence does not hold for local solutions. We are going to show that a local optimum of the MPEC formulation may not yield a local optimum of LBP.

Problem LBP belongs to the class of strongly NP-hard problems (Ref. 7). The main difficulties are due to its nonconvexity, which may result in an exponential number of local optima (Ref. 13). On the other hand, the design of algorithms has been made difficult due to the lack of computationally attractive theoretical results for the problem.

Our main aim is to derive necessary and sufficient conditions for local optimality in problem LBP. In particular, we state a characterization of a local optimum of LBP based on the notion of equilibrium point introduced in Ref. 6. This characterization is useful from a numerical point of view and may be used to devise local algorithms. Finding local solutions of nonconvex optimization problems is a meaningful deed itself. In addition, local procedures can be used within global algorithms. For problem LBP, this strategy is applied in Ref. 4, 5 for example.

The paper is organized as follows. Section 2 examines the auxiliary problems that will be used in the development. The approach follows that one presented in Ref. 6. In Section 3, we carry out the local analysis of LBP. We derive characterizations for its local and strict local solutions. The most computationally useful characterizations are based on the notion of equilibrium point, which is further explored in Section 4.

2. Preliminaries

In this section, we consider two problems related to LBP. The first auxiliary problem (P) is the MPEC obtained by replacing the inner linear problem of LBP by its KKT conditions. The second problem, P(M), comes from a penalization of the complementarity constraints in (P) with a parameter $M \geq 0$. Thus, we have the following models:

$$
\begin{aligned}
\text{(P)} \quad & \max \quad c_1^T x + c_2^T y, \\
& \text{s.t.} \quad A_1 x + A_2 y + w = a, \\
& \qquad x \geq 0, \quad y \geq 0, \quad w \geq 0, \\
& \qquad A_2^T u - v = b, \\
& \qquad u \geq 0, \quad v \geq 0, \\
& \qquad u^T w + v^T y = 0,
\end{aligned}
$$

$$
\begin{aligned}
\text{(P}(M)) \quad & \max \quad c_1^T x + c_2^T y - M(u^T w + v^T y), \\
& \text{s.t.} \quad A_1 x + A_2 y + w = a, \\
& \qquad x \geq 0, \quad y \geq 0, \quad w \geq 0, \\
& \qquad A_2^T u - v = b, \\
& \qquad u \geq 0, \quad v \geq 0,
\end{aligned}
$$

where $w \in \mathbb{R}^m$ is the primal slack vector and where $u \in \mathbb{R}^m, v \in \mathbb{R}^{n_2}$ are the dual vectors.

The formulations (P) and P(M) are often used as approaches to identify the global optima of LBP; see e.g. Refs. 3, 4, 6, 14, 15. Such approaches are based on the global equivalence among these problems. Indeed, we show in Ref. 6 that there exists a finite M for which problems (P) and P(M) have the same (empty or nonempty) global solution set, which also yields the global solution set of LBP. It is worth remarking that, besides P(M), other exact penalizations for (P) have been considered in the literature (Refs. 16–18).

Here, we use problems (P) and P(M) in the context of local optimality. We show that there is $\bar{M} \geq 0$ such that (P) and P(M) have the same local solution set for every $M \geq \bar{M}$. Mainly, we derive necessary and sufficient conditions for a local solution of (P) to yield a local optimum of LBP. Furthermore, we present a computationally attractive characterization of the local optima of (P) and LBP by using the notion of equilibrium point of the penalized problem P(M). We show that the penalty parameter can be considered implicitly to get an equilibrium point.

For the sake of convenience, we will study local optimality within neighborhoods given by the infinity norm. Let us recall that the infinity norm of $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ is

$$\|\nu\|_\infty = \max\{|\nu_i| : 1 \leq i \leq p\}$$

and that

$$B_\varepsilon(\bar{\nu}) = \{\nu \in \mathbb{R}^p : \|\nu - \bar{\nu}\|_\infty \leq \varepsilon\}$$

is an ε-neighborhood of $\bar{\nu} \in \mathbb{R}^p$. Observe that

$$B_\varepsilon(\nu, \omega) = B_\varepsilon(\nu) \times B_\varepsilon(\omega), \quad \text{for all } (\nu, \omega) \in \mathbb{R}^p \times \mathbb{R}^q.$$

We adopt the notation introduced in Ref. 6. We consider the block matrices

$$A = [A_1, A_2, I_m] \in \mathbb{R}^{m \times n}, \quad B = \left[0, -I_{n_2}, A_2^T\right] \in \mathbb{R}^{n_2 \times n},$$
$$c^T = \left(c_1^T, c_2^T, 0\right) \in \mathbb{R}^n, \quad z^T = (x^T, y^T, w^T) \in \mathbb{R}^n, \quad s^T = (0, v^T, u^T) \in \mathbb{R}^n,$$

where $n = n_1 + n_2 + m$, I_k is the $k \times k$-identity matrix, and 0 is a null matrix of an appropriate dimension for each case. We define the polyhedra

$$Z = \left\{z \in \mathbb{R}^n_+ : Az = a\right\}, \quad S = \left\{s \in \mathbb{R}^n_+ : Bs = b\right\}.$$

Thus, the auxiliary problems are rewritten as

$$\begin{array}{ll} (\mathrm{P}) & \max\ F(z,s) = c^T z, \\ & \text{s.t.}\quad z \in Z, \quad s \in S, \quad s^T z = 0; \\ (\mathrm{P}(M)) & \max\ F_M(z,s) = c^T z - M s^T z, \\ & \text{s.t.}\quad z \in Z, \quad s \in S. \end{array}$$

To conclude this preliminary section, let us recall the following well-known characterization: x is a vertex of a polyhedral set

$$X = \left\{x \in \mathbb{R}^n_+ : Qx = q\right\},$$

where Q has full row rank if and only if Q can be decomposed into $[D, N]$ such that

$$x_D = D^{-1} q \geq 0, \quad x_N = 0,$$

where D is a nonsingular matrix and x_D, x_N are the components of x related to D, N respectively; see Ref. 19 for example. We denote the vertex set of the polyhedron X by X_v.

3. Local Optimality

For our development, it will be useful to consider the point-to-set functions

$$S(z) = \left\{s \in S : z^T s = 0\right\}, \quad Z(s) = \left\{z \in Z : s^T z = 0\right\},$$

which map a point $z \in Z \subset \mathbb{R}^n_+$ [resp. $s \in S \subset \mathbb{R}^n_+$] into a polyhedron $S(z) \subset S$ [resp. $Z(s) \subset Z$]. These polyhedra have the following property.

Lemma 3.1. For each $z \in Z, S(z)$ is a face of S with vertex set $S_v(z) = S(z) \cap S_v$. For each $s \in S, Z(s)$ is a face of Z with vertex set $Z_v(s) = Z(s) \cap Z_v$.

Proof. Let $z \in Z \subseteq \mathbb{R}^n_+$. Since $S \subseteq \mathbb{R}^n_+$, we have that $S(z) = \emptyset$ or $S(z)$ is the solution set of the linear program $\min\{z^T s : s \in S\}$. So, it is a face of S. Moreover, the vertices of $S(z)$ are all vertices of S lying in $S(z)$. Similarly, we prove the second part. □

The functions $S(\cdot)$ and $Z(\cdot)$ are related to our problems as follows. The feasible region of problem (P) is the graph of $S(\cdot)$,

$$\{(z, s) : z \in Z, s \in S(z)\}.$$

The feasible set of LBP is the domain of $S(\cdot)$,

$$\{z \in Z : S(z) \neq \emptyset\},$$

or equivalently, the image of $Z(\cdot)$,

$$\left\{z \in Z : s^T z = 0, \text{ for some } s \in S\right\}.$$

Note that $s \in S(z)$ if and only if $z \in Z(s)$. Some other relations with the feasible set of LBP are given below.

Lemma 3.2. For $z \in Z$, the following assertions are equivalent:

(i) z is feasible to LBP;
(ii) $S_v(z) = S(z) \cap S_v \neq \emptyset$;
(iii) $z \in Z(s)$, for some $s \in S_v \subset S$.

Proof. Let $z \in Z$. Then, z is feasible to LBP if and only if $S(z) \neq \emptyset$. Since the polyhedron $S(z)$ is included in $\mathbb{R}^n_+$, it has no lines. This implies that $S(z) \neq \emptyset$ if and only if $S_v(z) \neq \emptyset$. By Lemma 3.1,

$$S_v(z) = S(z) \cap S_v.$$

In addition,

$$s \in S_v(z) = S(z) \cap S_v,$$

if and only if $z \in Z(s)$ for $s \in S_v$. Therefore, we get the desired equivalences. □

As a consequence of the lemma above, we obtain a known characterization of the feasible set of LBP in terms of some faces of the polyhedral set Z (Ref. 20). Actually, we have the following result.

Corollary 3.1. The feasible set of LBP is the union of faces of Z. Each of these faces is given by $Z(s)$ for some $s \in S_v$.

Next, we derive some preliminary local properties.

Lemma 3.3. For each $\bar{z} \in Z$, there is $\varepsilon = \varepsilon(\bar{z}) > 0$ such that $S(z) \subseteq S(\bar{z})$ and $S_v(z) \subseteq S_v(\bar{z})$ for every $z \in Z \cap B_\varepsilon(\bar{z})$. For each $\bar{s} \in S$, there is $\varepsilon = \varepsilon(\bar{s}) > 0$ such that $Z(s) \subseteq Z(\bar{s})$ and $Z_v(s) \subseteq Z_v(\bar{s})$ for every $s \in S \cap B_\varepsilon(\bar{s})$.

Proof. By the symmetry of $Z(\cdot)$ and $S(\cdot)$, it is enough to prove the first part of the lemma. Let $\bar{z} \in Z$. If $\bar{z} = 0$, then

$$S(z) \subseteq S = S(\bar{z}), \quad \text{for every } z \in Z.$$

Now, suppose that $\bar{z} \neq 0$. Let us partition the index set $J = \{1, 2, \dots, n\}$ into the subsets

$$J_0 = \{j \in J : \bar{z}_j = 0\}, \quad J_1 = J \backslash J_0.$$

First, we prove that there is $\varepsilon > 0$ such that, if $z \in Z \cap B_\varepsilon(\bar{z})$, then $z_j > 0$ for every $j \in J_1$. Since $\bar{z} \neq 0$, it must be $J_1 \neq \emptyset$. Let us take $j \in J_1$. Consider the closed set

$$C_j = \{z \in \mathbb{R}^n : z_j = 0\}.$$

Therefore, $\bar{z}$ belongs to the open set $\mathbb{R}^n \backslash C_j$. So, there is $\varepsilon_j > 0$ such that $B_{\varepsilon_j}(\bar{z}) \cap C_j = \emptyset$. Define

$$\varepsilon = \min\{\varepsilon_j : j \in J_1\} > 0.$$

As $B_\varepsilon(\bar{z}) \cap C_j = \emptyset$, if $z \in Z \cap B_\varepsilon(\bar{z}) \neq \emptyset$, then $z \in Z \backslash C_j$, that is, $z_j > 0$. Now, we prove the inclusion $S(z) \subseteq S(\bar{z})$ for $z \in Z \cap B_\varepsilon(\bar{z})$. We consider two cases. If $S(z) = \emptyset$, it holds trivially that $S(z) \subseteq S(\bar{z})$ [it may be $S(\bar{z}) = \emptyset$]. Otherwise, let $s \in S(z)$. Then, it must be $s_j = 0$ for every $j \in J_1$. Consequently, we have that

$$\bar{z}^T s = \sum_{j \in J_1} \bar{z}_j s_j + \sum_{j \in J_0} \bar{z}_j s_j = 0.$$

Hence, it results that $s \in S(\bar{z})$, implying that $S(z) \subseteq S(\bar{z})$. In any case, there exists $\varepsilon > 0$ such that $S(z) \subseteq S(\bar{z})$ for every $z \in Z \cap B_\varepsilon(\bar{z})$. In particular, the inclusion of the extreme points holds due to $S_v(z) = S(z) \cap S_v$ for any $z \in Z$. □

Corollary 3.2. For each $(\bar{z}, \bar{s}) \in Z \times S$, there is $\varepsilon = \varepsilon(\bar{z}, \bar{s}) > 0$ such that $S(z) \subseteq S(\bar{z})$, $S_v(z) \subseteq S_v(\bar{z})$, $Z(s) \subseteq Z(\bar{s})$, $Z_v(s) \subseteq Z_v(\bar{s})$ for every $(z, s) \in (Z \times S) \cap B_\varepsilon(\bar{z}, \bar{s})$.

Corollary 3.3. Let $(\bar{z}, \bar{s}) \in Z \times S$. Then, there is $\varepsilon > 0$ such that $s \in S(\bar{z})$ and $z \in Z(\bar{s})$ if $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$ and is feasible to problem (P).

Now, we can characterize a local solution of LBP in terms of a local solution of (P).

Theorem 3.1. A point $\bar{z}$ is a local solution of problem LBP if and only if $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z}, s)$ is a local solution of problem (P) for every vertex $s \in S_v(\bar{z})$.

Proof. First, assume that $\bar{z} \in Z$ is a local solution of LBP. Then, there is $\varepsilon > 0$ such that $c^T z \leq c^T \bar{z}$ for every $z \in Z \cap B_\varepsilon(\bar{z})$ with $S(z) \neq \emptyset$. By Lemma 3.2, we have that $S_v(\bar{z}) \neq \emptyset$. Let $\bar{s} \in S_v(\bar{z})$. So, $(\bar{z}, \bar{s})$ is feasible to (P). Let $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$ feasible to (P). Then, $z \in Z \cap B_\varepsilon(\bar{z})$ and $S(z) \neq \emptyset$ yielding that $c^T z \leq c^T \bar{z}$. Therefore, $(\bar{z}, \bar{s})$ is a local solution of (P). Since $\bar{s}$ is an arbitrary element in $S_v(\bar{z})$, we conclude that $(\bar{z}, s)$ is a local solution of (P) for every $s \in S_v(\bar{z})$.

Conversely, let $\bar{z} \in Z$ be such that $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z}, s)$ is a local solution of (P) for every $s \in S_v(\bar{z})$. By contradiction, suppose that $\bar{z}$ is not a local solution of LBP. Let $\bar{\varepsilon} = \varepsilon(\bar{z}) > 0$ be given according to Lemma 3.3. Then, there is $\hat{z} \in B_{\bar{\varepsilon}}(\bar{z})$, feasible to LBP, such that $c^T \hat{z} > c^T \bar{z}$ and $S_v(\hat{z}) \subseteq S_v(\bar{z})$. By Lemma 3.2, $S_v(\hat{z}) \neq \emptyset$. Let $\hat{s} \in S_v(\hat{z}) \subseteq S_v(\bar{z})$. Let us define

$$z(\alpha) = \alpha \hat{z} + (1 - \alpha) \bar{z}.$$

Since $\hat{z}$ and $\bar{z}$ belong to the convex set $Z(\hat{s})$, then $z(\alpha) \in Z(\hat{s})$ for every $\alpha \in [0, 1]$. Hence, $(z(\alpha), \hat{s})$ is feasible to (P) for every $\alpha \in [0, 1]$. Since $z(\alpha) \to \bar{z}$ as $\alpha \to 0$, for each $\varepsilon > 0$, there is $\alpha \in (0, 1]$ such that $(z(\alpha), \hat{s}) \in B_\varepsilon(\bar{z}, \hat{s})$ is feasible to (P) and satisfies

$$c^T z(\alpha) = c^T \bar{z} + \alpha (c^T \hat{z} - c^T \bar{z}) > c^T \bar{z}.$$

Therefore, there is $\hat{s} \in S_v(\bar{z})$ such that $(\bar{z}, \hat{s})$ is not a local solution of (P), which contradicts the hypothesis. Hence, $\bar{z}$ must be a local solution of LBP. □

Remark 3.1. A point $\bar{z}$ may not be a local solution of problem LBP if $(\bar{z}, \bar{s})$ is a local solution of (P) for only some $\bar{s} \in S_v(\bar{z})$. This situation is

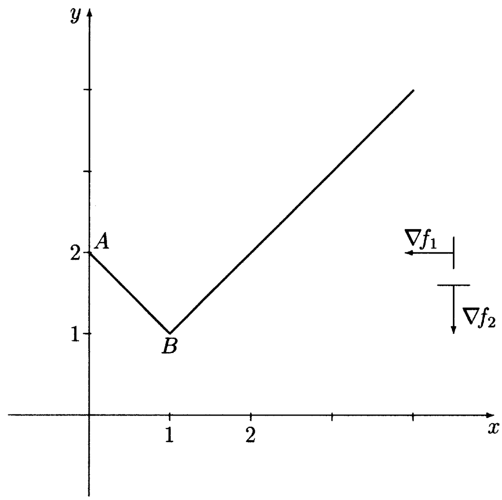

Fig. 1. Illustration of Example E1.

illustrated by the following example:

$$\begin{aligned}
\text{(E1)}\quad \max\quad & f_1(x,y) = -x,\\
\text{s.t.}\quad & x \geq 0, \quad y \text{ solves}\\
\max\quad & f_2(x,y) = -y,\\
\text{s.t.}\quad & x + y \geq 2,\\
& x - y \leq 0,\\
& y \geq 0.
\end{aligned}$$

The feasible set of Example E1 is shown in bold in Figure 1. Point A is the global solution. Let

$$\bar{z} = (\bar{x}, \bar{y}, \bar{w}_1, \bar{w}_2)^T = (1, 1, 0, 0)^T \in Z,$$

corresponding to point B, and let

$$\bar{s} = (0, \bar{v}, \bar{u}_1, \bar{u}_2)^T = (0, 0, 0, 1)^T \in S_v(\bar{z}).$$

Although $(\bar{z}, \bar{s})$ is a local solution of (P), $\bar{z}$ is not a local solution of LBP. On the other hand, for

$$\hat{s} = (0, 0, 1, 0)^T \in S_v(\bar{z}),$$

point $(\bar{z}, \hat{s})$ is not a local solution of (P).

Remark 3.2. The equivalence given in Theorem 3.1 does not hold in the case of strict local optimality. Moreover, it is possible to have a strict local solution $\bar{z}$ of LBP such that $(\bar{z}, s)$ is not a strict local solution of (P) for every $s \in S_v(\bar{z})$. Actually, this situation occurs whenever $S(\bar{z})$ is not a singleton. For instance, let us consider Example E1, where we change the first-level objective function to

$$f_1(x, y) = -y.$$

Now,

$$\bar{z} = (1, 1, 0, 0)^T \in Z,$$

corresponding to point B in Figure 1, is a strict local solution of LBP. Actually, it is the global solution. We have that

$$S_v(\bar{z}) = \{\bar{s} = (0, 0, 0, 1)^T, \hat{s} = (0, 0, 1, 0)^T\}.$$

Since

$$s(\alpha) = \alpha\bar{s} + (1-\alpha)\hat{s} \in S(\bar{z}), \qquad \text{for every} \quad \alpha \in [0, 1],$$

it follows that $(\bar{z}, \bar{s})$ and $(\bar{z}, \hat{s})$ are not strict local solutions of (P). In Corollary 3.5, we are going to establish a characterization for strict local optimality.

Theorem 3.1 suggests searching local solutions of problem LBP among local solutions of (P). These points can be characterized by the penalized problem, bringing about a better computational insight. The characterizations will be stated using the notion of equilibrium point of the penalized problem.

Definition 3.1. A point $(\bar{z}, \bar{s}) \in Z \times S$ is an equilibrium point of the penalized problem P(M) if there is $\bar{M} \geq 0$ such that, for each $M \geq \bar{M}$, it holds that

$$\max\{F_M(\bar{z}, s) : s \in S\} = F_M(\bar{z}, \bar{s}) = \max\{F_M(z, \bar{s}) : z \in Z\}. \tag{1}$$

If the equilibrium point $(\bar{z}, \bar{s})$ additionally satisfies

$$\{\bar{z}\} = \arg\max\{F_M(z, \bar{s}) : z \in Z\}, \tag{2}$$

it is called a primal strict equilibrium point. If the equilibrium point $(\bar{z}, \bar{s})$ verifies also

$$\{\bar{s}\} = \arg\max\{F_M(\bar{z}, s) : s \in S\}, \tag{3}$$

it is said to be a dual strict equilibrium point. A primal and dual strict equilibrium point is simply called a strict equilibrium point.

Let us observe that the definition of an equilibrium point of P(M) was introduced in Ref. 21. It was also considered in Ref. 6 for the penalized problem related to a linear bilevel problem with constraints in the first level. The (nonstrict) equilibrium required in Definition 3.1 extends the equilibrium considered by the mountain climbing algorithm given for a bilinear program (Ref. 22). Indeed, in our case, condition (1) must be fulfilled for every $M \geq \bar{M}$, which means that the equilibrium has to be held by a family of parametric bilinear problems P(M).

In order to characterize local solutions of (P) and LBP, we establish an important property of an equilibrium point.

Lemma 3.4. If $(\bar{z}, \bar{s})$ is an equilibrium point of the penalized problem P(M), then

$$\min\left\{\bar{z}^T s : s \in S\right\} = \min\left\{\bar{s}^T z : z \in Z\right\} = \bar{s}^T \bar{z} = 0. \tag{4}$$

Proof. Let $(\bar{z}, \bar{s})$ be an equilibrium point. Then,

$$\min\{\bar{z}^T s : s \in S\} = \bar{z}^T \bar{s}$$

comes from the first equality given in (1). Now, by contradiction, suppose that

$$\bar{s}^T \bar{z} > \bar{s}^T \hat{z} = \min\{\bar{s}^T z : z \in Z\}.$$

Take

$$M > \max\{\bar{M}, c^T(\bar{z} - \hat{z})/\bar{s}^T(\bar{z} - \hat{z})\}.$$

Thus,

$$c^T \hat{z} - M\bar{s}^T \hat{z} > c^T \bar{z} - M\bar{s}^T \bar{z},$$

which contradicts the second equality in (1). Therefore,

$$\bar{s}^T \bar{z} = \min\{\bar{s}^T z : z \in Z\}.$$

Finally, we show that $\bar{s}^T \bar{z} = 0$. Let

$$\hat{s} \in \arg\min\{\bar{z}^T s : s \in S_v\}.$$

Then,

$$\hat{s} = ((D^{-1}b)^T, 0)^T,$$

where D is a basis of B. Consider the optimal reduced cost

$$\hat{z}^T = \bar{z}^T - \bar{z}_D^T D^{-1} B \geq 0.$$

We have that

$$A\hat{z} = A\bar{z} = a, \quad \text{since } AB^T = 0.$$

Hence, $\hat{z} \in Z$.

Using the first two equalities given in (4) and the definitions of $\hat{z}$ and $\hat{s}$, it follows that

$$\begin{aligned} 0 &\leq \bar{z}^T \bar{s} \leq \hat{z}^T \bar{s} \\ &= \bar{z}^T \bar{s} - \bar{z}_D^T D^{-1} B\bar{s} \\ &= \bar{z}^T \bar{s} - \bar{z}_D^T D^{-1} b \\ &= \bar{z}^T \bar{s} - \bar{z}^T \hat{s} \\ &= 0. \end{aligned}$$

Then, $\bar{s}^T \bar{z} = 0$. □

In particular, for a dual strict equilibrium point, we get the following stronger result.

Corollary 3.4. A point $(\bar{z}, \bar{s})$ is a dual strict equilibrium point of the penalized problem P(M) if and only if it is an equilibrium point and $S(\bar{z}) = S_v(\bar{z}) = \{\bar{s}\}$.

Lemma 3.4 says that an equilibrium point is feasible to (P). With this property, we now show the equivalence among a local solution of (P), a local solution of P(M) for large values for M and an equilibrium point.

Theorem 3.2. The following assertions are equivalent:

(i) $(\bar{z}, \bar{s})$ is a local solution of problem (P);
(ii) $(\bar{z}, \bar{s})$ is a local solution of the penalized problem P(M) for every $M \geq \bar{M}$, for some $\bar{M} \geq 0$;
(iii) $(\bar{z}, \bar{s})$ is an equilibrium point of the penalized problem P(M).

Proof.

(i) $\Rightarrow$ (ii). Assume that $(\bar{z}, \bar{s})$ is a local solution of (P). So, there is $\varepsilon > 0$ such that

$$c^T z \leq c^T \bar{z}, \quad \text{for every } (z, s) \in B_\varepsilon(\bar{z}, \bar{s}),$$

feasible to (P). Define

$$\mathcal{Z} = Z \cap B_\varepsilon(\bar{z}) \quad \text{and} \quad \mathcal{S} = S \cap B_\varepsilon(\bar{s}).$$

Regarding that $\mathcal{Z} \times \mathcal{S}$ is an ε-neighborhood of $(\bar{z}, \bar{s})$ in $Z \times S$, we are going to show that there is $\bar{M} \geq 0$ such that

$$\max_{(z,s) \in \mathcal{Z} \times \mathcal{S}} F_M(z, s) = \max_{(z,s) \in \mathcal{Z}_v \times \mathcal{S}_v} F_M(z, s) = F_M(\bar{z}, \bar{s}), \quad \forall M \geq \bar{M}. \tag{5}$$

For each $M \in \mathbb{R}$, the first equality comes from the fact that $F_M(\cdot, \cdot)$ is a bilinear function and $\mathcal{Z}$ and $\mathcal{S}$ are compact polyhedra (Proposition IX.1 in Ref. 23). To show the second equality, define

$$C = \{(z, s) \in \mathcal{Z}_v \times \mathcal{S}_v : s^T z > 0\}$$

and

$$M_0 = \sup\{(c^T z - c^T \bar{z})/s^T z : (z, s) \in C\}.$$

Note that $M_0 = -\infty$, if $C = \emptyset$; otherwise, $M_0 \in \mathbb{R}$. Thus, let us take $\bar{M} \in \mathbb{R}$ such that $\bar{M} > \max\{0, M_0\}$. Consider an arbitrary $(z, s) \in \mathcal{Z}_v \times \mathcal{S}_v$. If $(z, s) \in C$, the definition of M_0 assures that

$$F_M(z, s) = c^T z - M s^T z < c^T \bar{z} = F_M(\bar{z}, \bar{s}), \quad \forall M \geq \bar{M} > M_0. \tag{6}$$

Otherwise, (z, s) is feasible to (P), which yields that

$$F_M(z, s) = c^T z \leq c^T \bar{z} = F_M(\bar{z}, \bar{s}), \quad \forall M \in \mathbb{R}. \tag{7}$$

By (6)–(7), the first equality in (5) and the fact that $(\bar{z}, \bar{s}) \in \mathcal{Z} \times \mathcal{S}$, we get the second equality in (5) for every $M \geq \bar{M}$. Then, $(\bar{z}, \bar{s})$ is a local solution of P(M) for every $M \geq \bar{M}$.

(ii) $\Rightarrow$ (iii). Assume that there is $\bar{M} \geq 0$ such that $(\bar{z}, \bar{s}) \in Z \times S$ is a local solution of P(M) for every $M \geq \bar{M}$. Consider an arbitrary $M \geq \bar{M}$. Then, the directional derivative of $F_M(\cdot, \cdot)$ at $(\bar{z}, \bar{s})$ in any feasible direction is nonpositive. Since $Z \times S$ is convex, it means that

$$(c - M\bar{s})^T (z - \bar{z}) - M\bar{z}^T (s - \bar{s}) \leq 0, \quad \forall (z, s) \in Z \times S. \tag{8}$$

This expression with either $z=\bar{z}$ or $s=\bar{s}$, respectively, yields

$$-M\bar{z}^T s \leq -M\bar{z}^T \bar{s}, \quad \forall s \in S, \tag{9}$$
$$c^T z - M\bar{s}^T z \leq c^T \bar{z} - M\bar{s}^T \bar{z}, \quad \forall z \in Z. \tag{10}$$

Therefore, the equilibrium equation (1) holds.

(iii) ⇒ (i). Assume that $(\bar{z}, \bar{s})$ is an equilibrium point. Then, there is $\bar{M} \geq 0$ such that

$$F_{\bar{M}}(z, \bar{s}) \leq F_{\bar{M}}(\bar{z}, \bar{s}), \quad \text{for every } z \in Z.$$

In addition, by Lemma 3.4, $(\bar{z}, \bar{s})$ is feasible to (P). Let $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to (P), with $\varepsilon > 0$ according to Corollary 3.3. Then,

$$\bar{s}^T z = \bar{s}^T \bar{z} = 0.$$

The previous expressions imply that

$$c^T z = F_{\bar{M}}(z, \bar{s}) \leq F_{\bar{M}}(\bar{z}, \bar{s}) = c^T \bar{z}. \tag{11}$$

Therefore, $(\bar{z}, \bar{s})$ is a local solution of (P). □

Let us note that the characterization (i)–(iii) stated above is stronger than Theorem 9 in Ref. 6, which establishes a similar equivalence for the LBP with linear constraints in the first level. Indeed, such a theorem uses the feasibility of $(\bar{z}, \bar{s})$ as a premise. Now, we have shown in Lemma 3.4 that the complementarity condition is a consequence of the equilibrium. This fact assures that finding an equilibrium point is enough to have a local solution of (P). Besides, we can observe that P(M) is a local exact penalization for (P).

We establish similar equivalences for strict local solutions as follows.

Theorem 3.3. The following assertions are equivalent:

(i) $(\bar{z}, \bar{s})$ is a strict local solution of problem (P);

(ii) $(\bar{z}, \bar{s})$ is a strict local solution of the penalized problem P(M) for every $M \geq \bar{M}$, for some $\bar{M} \geq 0$;

(iii) $(\bar{z}, \bar{s})$ is a strict equilibrium of the penalized problem P(M).

Proof. Let us follow the proof of Theorem 3.2 and only point out the necessary modifications.

(i) $\Rightarrow$ (ii). Assume that $(\bar{z}, \bar{s})$ is a strict local solution of (P) within the neighborhood $B_\varepsilon(\bar{z}, \bar{s})$. Let $(z, s) \in B_\varepsilon(\bar{z}, \bar{s}) \backslash (\bar{z}, \bar{s})$. Thus, we get a strict inequality in (7) for (z, s) feasible to (P). Moreover, by considering (5)–(6), we conclude that $(\bar{z}, \bar{s})$ is a strict local solution of P(M) for every $M \geq \bar{M}$.

(ii) $\Rightarrow$ (iii). Assume that there is $\bar{M} \geq 0$ such that $(\bar{z}, \bar{s}) \in Z \times S$ is a strict local solution of P(M) for every $M \geq \bar{M}$. Then, a strict inequality holds in (8) for $(z, s) \neq (\bar{z}, \bar{s})$. Hence, we get strict inequalities in (9) and (10) for $s \neq \bar{s}$ and $z \neq \bar{z}$, respectively. Thus, conditions (2)–(3) are attained, which imply that $(\bar{z}, \bar{s})$ is a strict equilibrium point.

(iii) $\Rightarrow$ (i). Assume that $(\bar{z}, \bar{s})$ is a strict equilibrium point. Let $\varepsilon > 0$ verifying Corollary 3.3 and let $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to (P), with $(z, s) \neq (\bar{z}, \bar{s})$. It follows that $s \in S(\bar{z})$. Since Corollary 3.4 ensures that $S(\bar{z}) = \{\bar{s}\}$, it must be $s = \bar{s}$ and $z \neq \bar{z}$. Thus, condition (2) leads to a strict inequality in (11), which implies that $(\bar{z}, \bar{s})$ is a strict local solution of (P). □

A characterization of local solutions of LBP in terms of equilibrium points follows.

Theorem 3.4. A point $\bar{z}$ is a (strict) local solution of LBP if and only if $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z}, s)$ is a (primal strict) equilibrium point of the penalized problem P(M) for every vertex $s \in S_v(\bar{z})$.

Proof. The equivalence in the nonstrict case is a direct consequence of Theorems 3.1 and 3.2.

Now, assume that $\bar{z}$ is a strict local solution of LBP within a neighborhood $B_\varepsilon(\bar{z})$. It remains to be shown that condition (2) holds for every $s \in S_v(\bar{z}) \neq \emptyset$. Let $\bar{s}$ be arbitrary in $S_v(\bar{z})$. Then, $(\bar{z}, \bar{s})$ is an equilibrium point. Given $\bar{M} \geq 0$ according to Definition 3.1, let us consider $\hat{z} \in Z$ such that

$$F_M(\hat{z}, \bar{s}) = \max\{F_M(z, \bar{s}) : z \in Z\}, \quad \text{for every } M \geq \bar{M}.$$

We want to show that $\hat{z} = \bar{z}$. Since $F_M(\bar{z}, \bar{s}) = F_M(\hat{z}, \bar{s})$ for every $M \geq \bar{M}$ and $\bar{s}^T \bar{z} = 0$, it must be

$$\bar{s}^T \hat{z} = 0 \quad \text{and} \quad c^T \hat{z} = c^T \bar{z}.$$

Then, $\hat{z} \in Z(\bar{s})$. By the convexity of $Z(\bar{s})$, there is $\alpha \in (0, 1)$ such that

$$z(\alpha) \in Z(\bar{s}) \cap B_\varepsilon(\bar{z}) \quad \text{and} \quad c^T z(\alpha) = c^T \bar{z},$$

where

$$z(\alpha) = \alpha \hat{z} + (1 - \alpha) \bar{z}.$$

As $\bar{z}$ is a strict local solution and $z(\alpha)$ is feasible to LBP, it must be $z(\alpha) = \bar{z}$ which implies that $\hat{z} = \bar{z}$. Thus, condition (2) holds. Therefore, $(\bar{z}, \bar{s})$ is a primal strict equilibrium point. Moreover, since $\bar{s}$ is arbitrary in $S_v(\bar{z})$, we conclude that $(\bar{z}, s)$ is a primal strict equilibrium point for every $s \in S_v(\bar{z})$.

Conversely, let $\bar{z} \in Z$ such that $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z}, s)$ is a primal strict equilibrium point for every $s \in S_v(\bar{z})$. By Lemma 3.3, there is $\varepsilon = \varepsilon(\bar{z}) > 0$ such that $S_v(z) \subseteq S_v(\bar{z})$ for every $z \in Z \cap B_\varepsilon(\bar{z})$. Consider $z \in B_\varepsilon(\bar{z}) \setminus \{\bar{z}\}$, feasible to LBP. By Lemma 3.2, we conclude that there is $\bar{s} \in S_v(z) \subseteq S_v(\bar{z})$. Thus, $(\bar{z}, \bar{s})$ is a primal strict equilibrium point. Using condition (2), it follows that

$$c^T z = F_{\bar{M}}(z, \bar{s}) < F_{\bar{M}}(\bar{z}, \bar{s}) = c^T \bar{z}.$$

Therefore, $\bar{z}$ is a strict local solution of LBP. □

In Ref. 24, the authors reformulate a disjoint bilinear program (BILD) as a linear maxmin problem (LMM) and show that a local solution of LMM yields a local solution of BILD. They note also that the number of local optima of LMM may be less than the one of BILD. These results resemble the statement of Theorem 3.4. In fact, an LMM is a special case of an LBP, P(M) is a parametric disjoint bilinear problem, and $(\bar{z}, \bar{s})$ is an equilibrium point if and only if there is $\bar{M} \geq 0$ such that $(\bar{z}, \bar{s})$ is a local solution of P(M) for every $M \geq \bar{M}$. It is also worth noting that Theorem 3.4 goes further than Proposition 6 in Ref. 24, in the sense that it presents a characterization.

The next corollary expresses the equivalence corresponding to Theorem 3.1 for strict optimality. Note that, in this case, the condition of $S_v(\bar{z})$ being a singleton is necessary to characterize a strict local solution $\bar{z}$ of LBP.

Corollary 3.5. The following assertions are equivalent:

(i) $(\bar{z}, \bar{s})$ is a strict local solution of problem (P);
(ii) $(\bar{z}, \bar{s})$ is a strict equilibrium point of the penalized problem P(M);
(iii) $\bar{z}$ is a strict local solution of problem LBP and $S_v(\bar{z}) = \{\bar{s}\}$.

Proof. By Corollary 3.4, we conclude that $(\bar{z}, \bar{s})$ is a strict equilibrium point if and only if $(\bar{z}, \bar{s})$ is a primal strict equilibrium point and $S_v(\bar{z}) = \{\bar{s}\}$. Thus, by Theorems 3.3 and 3.4, the result follows. □

4. Equilibrium Points

In this section, we explore the characterization of local solutions of LBP in terms of equilibrium points as given by Theorem 3.4. We aim to find results which are more attractive from a computational point of view. We start giving a characterization for an equilibrium point which does not depend on the parameter M.

Theorem 4.1. A point $(\bar{z}, \bar{s})$ is an equilibrium point of the penalized problem P(M) if and only if $\bar{s} \in S$ and $\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\}$.

Proof. Let $(\bar{z}, \bar{s})$ be an equilibrium point. Then, $\bar{s} \in S$ and $\bar{z} \in Z$. By Lemma 3.4, it must be $\bar{z} \in Z(\bar{s})$. Moreover, we have that

$$\begin{aligned} c^T \bar{z} &\leq \max\{c^T z : z \in Z(\bar{s})\} \\ &\leq \max\{c^T z - \bar{M} \bar{s}^T z : z \in Z\} \\ &= c^T \bar{z}, \end{aligned}$$

where $\bar{M} \geq 0$ is given by Definition 3.1. Hence,

$$\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\}.$$

Conversely, assume that $\bar{s} \in S$ and

$$\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\}.$$

Then, $(\bar{z}, \bar{s})$ is feasible to (P). By Lemma 3.3, there is $\varepsilon = \varepsilon(\bar{s})$ such that $Z(s) \subseteq Z(\bar{s})$ for every $s \in S \cap B_\varepsilon(\bar{s})$. Let $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to (P). Then, $s \in S \cap B_\varepsilon(\bar{s})$ and $z \in Z(s)$. It follows that $z \in Z(\bar{s})$. As

$$\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\},$$

it must be

$$F(z, s) = c^T z \leq c^T \bar{z} = F(\bar{z}, \bar{s}).$$

Hence, $(\bar{z}, \bar{s})$ is a local solution of (P) and by Theorem 3.2 an equilibrium point. □

It is known that the feasible set of LBP is nonconvex in general. However, by Corollary 3.1, it can be decomposed into a finite number of polyhedra, each one defined by $Z(s)$ for some $s \in S_v$. According to the theorem above, an equilibrium point $(\bar{z}, \bar{s}) \in Z_v \times S_v$ yields the best feasible vertex $\bar{z}$ in the face $Z(\bar{s})$ of Z.

Theorem 4.1 establishes also how to find an equilibrium point if LBP is not infeasible nor unbounded, which means that $Z \times S \neq \emptyset$ and

$$\sup\{c^T z : z \in Z(s)\} < +\infty, \quad \text{for any } s \in S.$$

Indeed, by the theorem, it is a matter of finding $\bar{s} \in S$ such that $Z(\bar{s}) \neq \emptyset$. Such a point can be chosen as a solution of the linear problem $\min\{(z^0)^T s : s \in S\}$, where $z^0 \in Z$ is arbitrary; see Proposition 1 in Ref. 6.

In the rest of this section, we derive sufficient conditions for local optimality in LBP to be satisfied by an equilibrium point.

Corollary 3.5 states that a strict equilibrium point $(\bar{z}, \bar{s})$ yields a local optimum $\bar{z}$ of LBP. Actually, the strict equilibrium condition can be relaxed by the dual strict equilibrium condition. In this case, Theorem 3.4 assures that $\bar{z}$ is a local solution of LBP, provided that $S_v(\bar{z}) = \{\bar{s}\}$ by Corollary 3.4. This situation can be recognized by searching the adjacent vertices to $\bar{s}$, as shown in the next corollary.

Corollary 4.1. Let $(\bar{z}, \bar{s})$ be an equilibrium point of the penalized problem P(M). If $\bar{s} \in S_v$ and $\bar{z}^T s > 0$ for every $s \in S_v$ adjacent to $\bar{s}$, then $\bar{z}$ is a local solution of problem LBP.

Proof. Let $(\bar{z}, \bar{s})$ be an equilibrium point with $\bar{s} \in S_v$. By Lemma 3.4, it follows that $\bar{s} \in S_v(\bar{z})$. Thus, $\bar{z}$ is feasible to LBP. In addition, Lemma 3.1 assures that $S(\bar{z})$ is a face of S. By hypothesis, $s \notin S(\bar{z})$ for every vertex $s \in S_v$ adjacent to $\bar{s}$. Hence, $S_v(\bar{z}) = \{\bar{s}\}$. By Theorem 3.4, $(\bar{z}, \bar{s})$ is a local solution of LBP. □

Another sufficient condition for local optimality in LBP is given below. It is weaker but more difficult to be verified computationally.

Theorem 4.2. Let $(\bar{z}, \bar{s})$ be an equilibrium point of the penalized problem P(M). If there is $\varepsilon > 0$ such that

$$\bar{s} \in \bigcap_{z} \{S(z) : z \in Z \cap B_\varepsilon(\bar{z}),\, S(z) \neq \emptyset\}, \tag{12}$$

then $\bar{z}$ is a local solution of problem LBP. Particularly, if $\bar{s} \in S_v$ then $S(z)$ can be replaced by $S_v(z)$ in (12).

Proof. Let $(\bar{z}, \bar{s})$ be an equilibrium point. By Lemma 3.4, it follows that $\bar{s} \in S(\bar{z})$. Then, by Definition 3.1,

$$\max\left\{c^T z - \bar{M}\bar{s}^T z : z \in Z\right\} = c^T \bar{z}, \quad \text{for some } \bar{M} \geq 0.$$

We recall that the feasible set of LBP is $\{z \in Z : S(z) \neq \emptyset\}$. Let us consider $\varepsilon > 0$ satisfying (12) and let us restrict LBP to $B_\varepsilon(\bar{z})$. It follows that

$$\begin{aligned}
&\max \left\{c^T z : z \in Z \cap B_\varepsilon(\bar{z}),\, S(z) \neq \emptyset\right\} \\
&= \max\{c^T z - \bar{M}\bar{s}^T z : z \in Z \cap B_\varepsilon(\bar{z}),\, S(z) \neq \emptyset\} \\
&\leq \max\{c^T z - \bar{M}\bar{s}^T z : z \in Z\} \\
&= c^T \bar{z}. \qquad (13)
\end{aligned}$$

Since $\bar{z} \in Z \cap B_\varepsilon(\bar{z})$ and $S(\bar{z}) \neq \emptyset$, equality holds in (13). Hence, $\bar{z}$ is a local solution of LBP. Besides, as the polyhedron $S(z)$ has no lines, $S_v(z) \neq \emptyset$ if $S(z) \neq \emptyset$. Then, we can replace $S(z)$ by $S_v(z)$ in (12) when $\bar{s} \in S_v$. □

Remark 4.1. Condition (12) is weaker than that one which is used in Corollary 4.1. In fact, that condition implies that $S_v(\bar{z}) = \{\bar{s}\}$. So, for $\varepsilon = \varepsilon(\bar{z})$ given by Lemma 3.3, condition (12) holds trivially. On the other hand, it may be the case that condition (12) does not imply $S_v(\bar{z}) = \{\bar{s}\}$. Indeed, let us consider the example below,

$$\begin{array}{lll}
\text{(E2)} & \max & f_1(x, y) = -x, \\
 & \text{s.t.} & x \geq 0, \quad y \text{ solves} \\
 & \max & f_2(x, y) = -y, \\
 & \text{s.t.} & x + y \geq 2, \\
 & & x - y \leq 0, \\
 & & 2x + y \geq 2, \\
 & & y \geq 0.
\end{array}$$

Its feasible set is shown in bold in Figure 2. Let

$$\bar{z} = (\bar{x}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{w}_3)^T = (0, 2, 0, 2, 0)^T \in Z,$$

related to point A, and let

$$\bar{s} = (0, \bar{v}, \bar{u}_1, \bar{u}_2, \bar{u}_3)^T = (0, 0, 1, 0, 0)^T \in S.$$

We have that $(\bar{z}, \bar{s})$ is an equilibrium point for $\bar{M} = 0$. In addition, let $\varepsilon \in (0, 1)$ and $z = (x, y, w_1, w_2, w_3)^T \in B_\varepsilon(\bar{z})$, feasible to LBP. We can see that z lies on the segment (A, B). Thus, $w_1 = 0$ and so $\bar{s}^T z = 0$. Hence, $(\bar{z}, \bar{s})$ satisfies (12). However, the hypothesis of Corollary 4.1 is not verified, because there is $\hat{s} \in S_v(\bar{z})$, adjacent to $\bar{s}$, namely $\hat{s} = (0, 0, 0, 0, 1)^T$.

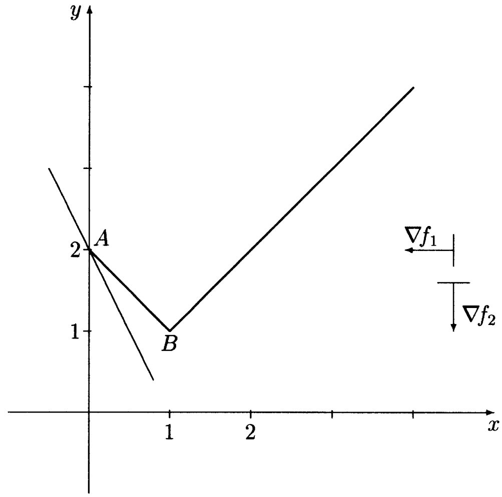

Fig. 2. Illustration of Example E2.

Remark 4.2. Condition (12) is not necessary for an equilibrium point to be a local solution of LBP. Indeed, let us consider Example E2 with the first-level objective function replaced by $f_1(x, y) = -y$. Then, the global solution, which is attained at point B in Figure 2, does not verify condition (12).

Now, we consider an assumption of nondegeneracy in order to get a more interesting characterization. We are going to use the following notation. Let $(\bar{z}, \bar{s}) \in Z_v \times S_v$ be given, where $\bar{z}$ is a nondegenerate vertex defined by the basis E of $A = [E, N]$. We denote by $\mathcal{N}(\bar{z})$ and $\mathcal{B}(\bar{s})$ the index sets of the nonbasic variables at $\bar{z}$ and the basic variables at $\bar{s}$, respectively. For $i \in \mathcal{N}(\bar{z})$, we have that

$$d_i^T = \left(d_{iE}^T, d_{iN}^T\right) = \left(-(E^{-1}N_i)^T, e_i^T\right)$$

is an extreme direction of Z, where N_i and e_i are respectively the columns of the matrices N and I_{n-m} corresponding to the variable z_i.

Theorem 4.3. Let $(\bar{z}, \bar{s}) \in Z_v \times S_v$ be an equilibrium point of the penalized problem P(M), where $\bar{z}$ is a nondegenerate vertex. Let

$$\mathcal{N}^+(\bar{z}) = \{i \in \mathcal{N}(\bar{z}) : c^T d_i > 0\}.$$

Then, $\mathcal{N}^+(\bar{z}) \subseteq \mathcal{B}(\bar{s})$. In addition, $\bar{z}$ is not a local solution of LBP if and only if there are $i \in \mathcal{N}^+(\bar{z})$ and $\hat{s} \in S_v(\bar{z})$ with $\hat{s}_i = 0$.

Proof. Let $(\bar{z}, \bar{s}) \in Z_v \times S_v$ be an equilibrium point such that $\bar{z}$ is nondegenerate. First, we prove that

$$s^T d_i = s_i, \quad \forall i \in \mathcal{N}(\bar{z}), \ \forall s \in S_v(\bar{z}). \tag{14}$$

In fact, let $i \in \mathcal{N}(\bar{z})$ and $s \in S_v(\bar{z})$. Since $s^T \bar{z} = 0$ and $\bar{z}_E > 0$, it must be $s_E = 0$. Hence,

$$s^T d_i = s_i - s_E^T E^{-1} N_i = s_i.$$

To prove the desired inclusion, consider $i \in \mathcal{N}^+(\bar{z})$. By (14), it follows that $\bar{s}^T d_i = \bar{s}_i \geq 0$ because $\bar{s} \in S_v(\bar{z})$. Suppose that $\bar{s}^T d_i = 0$. As $\bar{z}$ is nondegenerate, there is $\alpha > 0$ such that $z = \bar{z} + \alpha d_i \in Z$. Then, $\bar{s}^T z = 0$ and $c^T z - c^T \bar{z} = \alpha c^T d_i > 0$. Thus, it results that $\bar{z} \notin \arg\max\{c^T z : z \in Z(\bar{s})\}$, which contradicts Theorem 4.1. Hence, $\bar{s}_i = \bar{s}^T d_i > 0$ and so $i \in \mathcal{B}(\bar{s})$. Therefore, $\mathcal{N}^+(\bar{z}) \subseteq \mathcal{B}(\bar{s})$.

Now, we prove the claimed equivalence. Assume that $\bar{z}$ is not a local solution of LBP. Since $S_v(\bar{z}) \neq \emptyset$, by Theorem 3.4 there is $\hat{s} \in S_v(\bar{z})$ such that $(\bar{z}, \hat{s})$ is not an equilibrium point. By Theorem 4.1,

$$\bar{z} \notin \arg\max\{c^T z : z \in Z(\hat{s})\}.$$

Since $\bar{z} \in Z_v(\hat{s})$ and is nondegenerate, there must exist $i \in \mathcal{N}^+(\bar{z})$ and $\alpha > 0$ such that $z = \bar{z} + \alpha d_i \in Z(\hat{s})$. Therefore, $\hat{s}^T d_i = 0$ implying, by (14), that $\hat{s}_i = 0$. Conversely, let $i \in \mathcal{N}^+(\bar{z})$ and $\hat{s} \in S_v(\bar{z})$ with $\hat{s}_i = 0$. By (14), $\hat{s}^T d_i = \hat{s}_i = 0$. Let us consider an arbitrary $\varepsilon > 0$. Since $\bar{z}$ is nondegenerate, there is $\alpha > 0$ such that $z = \bar{z} + \alpha d_i \in Z \cap B_\varepsilon(\bar{z})$. In addition, z is feasible to LBP because $\hat{s}^T z = 0$. Moreover,

$$c^T z - c^T \bar{z} = \alpha c^T d_i > 0.$$

Thus, $\bar{z}$ is not a local solution of LBP. □

Finally, let us comment on the case where an equilibrium point $(\bar{z}, \bar{s})$ is such that $\bar{z}$ is a nondegenerate vertex which is not a local solution of LBP. From this point $(\bar{z}, \bar{s})$, we can find an improved equilibrium point $(\hat{z}, \hat{s})$ with $c^T \hat{z} > c^T \bar{z}$ or conclude that LBP is unbounded. Actually, we obtain

$$\hat{s} \in S_v(\bar{z}) \text{ and } \hat{z} \in \arg\max\{c^T z : z \in Z(\hat{s})\},$$

according to Theorems 4.3 and 4.1. Note that

$$\arg\max\{c^T z : z \in Z(\hat{s})\} = \emptyset$$

only if LBP is unbounded.

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