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Emergent Behaviors of the Infinite Set of Lohe Hermitian Sphere Oscillators

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Abstract
We study the emergent behaviors of an infinite number of Lohe Hermitian sphere oscillators on the unit Hermitian sphere. For this, we propose an infinite analogue of the Lohe hermitian sphere model, and present sufficient frameworks leading to collective behaviors in terms of system parameters and initial data. Under some network topology, we show that practical synchronization emerges for a heterogeneous ensemble, whereas exponential synchronization can appear for a homogeneous ensemble. Furthermore we have also derived analogous results for the infinite swarm-sphere model. For the sender network topology in which coupling capacities depend only on the sender index number, we show that there are only two possible asymptotic states, namely complete phase synchrony or bi-cluster configuration for a homogeneous ensemble in a positive coupling regime.

Keywords Asymptotic behavior · Infinite particle system · Lohe Hermitian sphere model

Mathematics Subject Classification 34D05 · 34G20 · 70F45

1 Introduction

Collective behaviors of a complex system have received a significant attention due to its wide range of applications in engineering and biological fields [9, 19, 33, 34, 37–39]. They include several group behaviors such as aggregation of bacteria [37], flocking of birds [14], swarming of fish [38] and synchronization of fireflies and neurons [33] etc. Among them, our interest lies in synchronization of weakly coupled limit-cycle oscillators. In 1975, Japanese physicist Yoshiki Kuramoto introduced a first-order particle model [28] following the work of Arthur Winfree [41] to study a simple phase-transition like phenomenon describing synchronization

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among a finite number of phase oscillators. The Kuramoto model is also a nice model to describe the synchronization of oscillators with constant period and it has been studied in various researchers [1, 5, 7, 16, 18, 20, 35]. We refer to [1, 2, 4, 15, 17, 21, 33–35, 39, 42] for a brief survey and introduction to collective dynamics. In this paper, we are interested in the collective behaviors of oscillators on the unit Hermitian sphere embedded in $\mathbb{C}^d$:

$$\mathbb{C}^d = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{d \text{ times}}, \quad [N] := \{1, \ldots, N\}, \quad \mathbb{N} := \{1, 2, \ldots\}.$$

To fix the idea, we begin with the Kuramoto model. Let $\theta_i = \theta_i(t)$ be a real-valued phase of the i-th oscillator. Then, the dynamics of θ_i is governed by the following Cauchy problem:

$$\begin{cases} \dot{\theta}_i = \nu_i + \dfrac{\kappa}{N} \displaystyle\sum_{i \in [N]} \sin(\theta_j - \theta_i), \quad t > 0, \\ \theta_i(0) = \theta_i^{\text{in}}, \quad i \in [N], \end{cases}$$

where $\mathcal{V} = \{\nu_i\}_{i \in [N]}$ and κ are the collection of natural frequencies in $\mathbb{R}$ and nonnegative coupling strength, respectively. Then the dynamics of the complex-valued function $z_i = e^{\mathrm{i}\theta_i}$ satisfies

$$\dot{z}_i = i\nu_i z_i + \frac{K}{2N} \sum_{j \in [N]} \left(z_j - \langle z_j, z_i \rangle z_i \right), \quad i \in [N],$$

where $\langle z_j, z_i \rangle = \bar{z}_j z_i$. This form can be generalized to the swarm sphere model on the unit Euclidean sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

Let $x_i = x_i(t) \in \mathbb{R}^d$ be a position of the i-th swarm sphere oscillator. Then, the dynamics of x_i is governed by the Cauchy problem to the swarm sphere (in short SS) model [29, 30, 32]:

$$\begin{cases} \dot{x}_i = \Omega_i x_i + \dfrac{\kappa}{N} \displaystyle\sum_{j \in [N]} \Big(\langle x_i, x_i \rangle x_j - \langle x_i, x_j \rangle x_i \Big), \quad t > 0, \\ x_i(0) = x_i^{\text{in}}, \quad i \in [N], \end{cases} \tag{1.1}$$

where $\mathbf{\Omega} := \{\Omega_i\}_{i \in [N]}$ and κ are the collections of $d \times d$ skew symmetric matrices, nonnegative coupling strength, respectively, and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$. Then, it is easy to check that the modulus of x_i is conserved along the swarm sphere dynamics (1.1). Hence the unit sphere $\mathbb{S}^{d-1}$ is a positively invariant set. Recently, Ha and Park introduced a particle model for aggregation on $\mathbb{C}^d$ using a finite-dimensional reduction from the Lohe tensor model [24]. More precisely, let $z_i = z_i(t)$ be a state of the i-th Lohe hermitian sphere oscillator on $\mathbb{C}^d$. Then it is governed by the following Cauchy problem to the Lohe hermitian sphere (in short LHS) model:

$$\begin{cases} \dot{z}_i = \Omega_i z_i + \lambda_0 \displaystyle\sum_{j \in [N]} \Big(\langle z_i, z_i \rangle z_j - \langle z_j, z_i \rangle z_i \Big) \\ \qquad\qquad + \lambda_1 \displaystyle\sum_{j \in [N]} \Big(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle \Big) z_i, \quad t > 0, \\ z_i(0) = z_i^{\text{in}}, \quad i \in [N], \end{cases} \tag{1.2}$$

where λ_0 and λ_1 are nonnegative real numbers such that $\lambda_0 + \lambda_1 = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^d$:

$$z = (z^1, \ldots, z^d) \in \mathbb{C}^d, \quad w = (w^1, \ldots, w^d) \in \mathbb{C}^d, \quad \langle z, w \rangle := \sum_{i=1}^d \overline{z^i} w^i, \quad \|z\| := \sqrt{\langle z, z \rangle}.$$

Then we can see that the modulus of z_i is conserved along the LHS dynamics (1.2), and the complex unit sphere(Hermitian sphere) $\mathbb{HS}^{d-1}$ is a positively invariant set. That is why we call this model as the LHS model. The Cauchy problems (1.1) and (1.2) have been extensively studied in a series of works [8, 11–13, 23–27, 29–31, 36, 43].

In this paper, we are interested in the following two questions:

- (Q1): What is the infinite counterpart of the Lohe hermitian sphere model for an infinite ensemble $\{z_i\}_{i\in\mathbb{N}}$?
- (Q2): Once the infinite counterpart is proposed, under what conditions on system parameters and initial data, does the proposed model exhibit collective behaviors?

To describe the mean-field dynamics of an infinite number of Kuramoto oscillators, the Kuramoto-Sakaguchi model was often studied via the corresponding kinetic model for $N \gg 1$. More precisely, to describe the behavior of individual particles, the kinetic model describes the entire configuration by approximating the overall averaged dynamics by a probability density function. Recently, dynamical systems with infinite number of equations have been used in the study of collective dynamics in [22, 40]. In particular, authors' recent work [22] highlights the distinction between the behavior of infinitely many particles and the behavior of particles approximated by a kinetic model. One of motivations to deal with an infinite particle system lies in the construction of measure-valued solutions to the corresponding mean-field kinetic equations with unbounded spatial support. More precisely, in the previous works on kinetic models for collective dynamics, we considered initial data which are compactly supported in phase space. To construct a measure-valued solution with a compact support, particle-in-cell method is often used. In this procedure, since the spatial support is bounded, there are only finite number of cells for a given finite mesh size. Hence, a particle system with a finite system size can be used in the construction of approximate solutions in the form of an empirical measure. However, when the spatial support is bounded, we must have an infinite number of cells for any finite mesh size. Therefore, we are forced to deal with particle model with an infinite system size. This is why we need to study infinite particle systems.

Throughout the paper, we provide answers for the above posed questions (Q1) and (Q2). More specifically, our main results are two-fold. First, we propose an infinite counterpart of the Cauchy problem to the LHS model (1.2) with an infinite network matrix (κ_{ij}):

$$
\begin{cases}
\dot{z}_i = \Omega_i z_i + \lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle z_i, z_i \rangle z_j - \langle z_j, z_i \rangle z_i \right) \\
\qquad\qquad + \lambda_1 \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle \right) z_i, \quad t > 0, \quad i \in \mathbb{N}, \\
z_i(0) = z_i^{\mathrm{in}},
\end{cases}
\tag{1.3}
$$

where the coupling matrix $\boldsymbol{\kappa} = (\kappa_{ij})_{i,j\in\mathbb{N}}$ and a sequence of anti-Hermitian matrices $\boldsymbol{\Omega} = \{\Omega_i\}_{i\in\mathbb{N}}$ satisfy the following conditions: for $i, j \in \mathbb{N}$,

$$
\begin{aligned}
&\Omega_i^\dagger = -\Omega_i, \quad \mathcal{D}(\boldsymbol{\Omega}) := \sup_{i,j\in\mathbb{N}} \|\Omega_i - \Omega_j\|_{\mathrm{op}} < \infty, \quad \|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} := \sup_{i\in\mathbb{N}} \|\Omega_i\|_{\mathrm{op}} < \infty, \\
&\kappa_{ij} > 0, \quad 0 < \|\boldsymbol{\kappa}\|_{-\infty,1} = \inf_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \kappa_{ij} \le \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \kappa_{ij} := \|\boldsymbol{\kappa}\|_{\infty,1} < \infty.
\end{aligned}
\tag{1.4}
$$

Here $\Omega_i^\dagger$ is the Hermitian conjugate of Ω_i, and $\|\Omega_i\|_{\mathrm{op}}$ is the operator norm:

$$\|\Omega_i\|_{\mathrm{op}} := \sup_{x\neq 0} \frac{\|\Omega_i x\|}{\|x\|}.$$

Since $\|\boldsymbol{\kappa}\|_{\infty,1} < \infty$, the infinite sum of right-hand side of (1.3) is well-defined. The global well-posedness of the Cauchy problem (1.3) is presented in Appendix A.
Second, we study the emergent behaviors of (1.3) for the cases in which system parameters and initial condition are given as follows.

$$\begin{cases} \text{Case A: } \boldsymbol{\Omega} : \text{anti-symmetric real matrix}, \quad z_i^{\mathrm{in}} \in \mathbb{R}^d \\ \text{Case B: } \boldsymbol{\Omega} : \text{anti-Hermitian}, \quad z_i^{\mathrm{in}} \in \mathbb{C}^d, \quad \kappa_{ij} \text{ satisfies } \begin{cases} \text{Case B.1: } \kappa_{ij} > 0, \\ \text{Case B.2: } \kappa_{ij} = \kappa_j > 0. \end{cases} \end{cases}$$

Specially, for Case A, we obtain an infinite counterpart of the Cauchy problem to the SS model (1.1) with an infinite network matrix (κ_{ij}):

$$\begin{cases} \dot{x}_i = \Omega_i x_i + \lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle x_i, x_i \rangle x_j - \langle x_j, x_i \rangle x_i \right), \quad t > 0, \\ x_i(0) = x_i^{\mathrm{in}}, \quad i \in \mathbb{N}, \end{cases} \tag{1.5}$$

where the coupling matrix $\boldsymbol{\kappa} = \left(\kappa_{ij}\right)_{i,j\in\mathbb{N}}$ and a sequence of anti-symmetric matrices $\boldsymbol{\Omega} = \{\Omega_i\}_{i\in\mathbb{N}}$ which satisfies conditions (1.4) inherited from the original infinite LHS model (1.3).

For Case A and Case B.1, we derive "practical synchronization" estimate for heterogeneous ensemble:

$$\lim_{t\to\infty} \sup_{i,j\in\mathbb{N}} |z_i(t) - z_j(t)| \le \mathcal{O}(1) \frac{\mathcal{D}(\boldsymbol{\Omega})}{\|\boldsymbol{\kappa}\|_{-\infty,1}}.$$

For Case B.2, we consider the Cauchy problem for a homogeneous ensemble with $\Omega_j = O$:

$$\begin{cases} \dot{z}_j = \lambda_0 \left(\langle z_j, z_j \rangle z_c - \langle z_c, z_j \rangle z_j \right) + \lambda_1 \left(\langle z_j, z_c \rangle - \langle z_c, z_j \rangle \right) z_j, \quad t \ge 0, \\ z_j(0) = z_j^{\mathrm{in}}, \quad \left\| z_j^{\mathrm{in}} \right\| = 1, \quad z_c = \sum_{l\in\mathbb{N}} \kappa_l z_l. \end{cases} \tag{1.6}$$

In Sect. 5, we investigate the roles of each term in the right-hand side of (1.6). More precisely, for $\lambda_1 = 0$, if initial data satisfy

$$\sup_{i,j\in\mathbb{N}} \left| 1 - \left\langle z_i^{\mathrm{in}}, z_j^{\mathrm{in}} \right\rangle \right| < 1 - \delta,$$

we have an exponential synchronization (see Theorem 5.1):

$$\left| 1 - \langle z_i(t), z_j(t) \rangle \right| \le \left| 1 - \left\langle z_i^{\mathrm{in}}, z_j^{\mathrm{in}} \right\rangle \right| \cdot \exp\left(-\delta R_0 t \right).$$

For the whole system (1.6), we show that possible asymptotic states are either one-point cluster or bi-polar state (see Corollary 5.2).

The rest of this paper is organized as follows. In Sect. 2, we study basic properties of the infinite LHS model and discuss its relation to other aggregation models. In Sect. 3, we study emergent dynamics of the infinite SS model as a special case of model (1.3) in which initial data and Ω_i is anti-symmetric real matrix. In Sect. 4, we study the emergent dynamics of the model (1.3) for a homogeneous ensemble with the same Ω_i. In Sect. 5, we present a synchronization estimate for a special case in which the interaction capacity depends only on the sender node, which is different from the presentation in Sect. 3. Finally, Sect. 6 is devoted to a brief summary of our main results.

2 Preliminaries

In this section, we briefly review basic properties such as the conservation of ℓ^2-norm and a global existence of the infinite LHS model and discuss relations with other existing aggregation models such as the Kuramoto model and the Schrödinger-Lohe model.

2.1 The Infinite LHS Model

In this subsection, we briefly study several properties of the Cauchy problem (1.3)–(1.4). First, we show that the unit Hermitian sphere is positively invariant.

Lemma 2.1 *Let $\mathcal{Z} = \{z_i\}_{i\in\mathbb{N}}$ be a global solution to* (1.3)–(1.4)*. Then the modulus of z_i is a conservative quantity:*

$$\|z_i(t)\| = \left\|z_i^{in}(t)\right\|, \quad t \ge 0, \quad i \in \mathbb{N}.$$

Proof We take an inner product $(1.3)_1$ with z_i to find

$$\begin{aligned}
\frac{d}{dt}\|z_i\|^2 &= \langle \dot{z}_i, z_i\rangle + \langle z_i, \dot{z}_i\rangle \\
&= \Big\langle \Omega_i z_i, z_i\Big\rangle + \Big\langle z_i, \Omega_i z_i\Big\rangle \\
&+ \lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij}\Big\langle \langle z_i, z_i\rangle z_j - \big\langle z_j, z_i\big\rangle z_i, z_i\Big\rangle + \lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij}\Big\langle z_i, \langle z_i, z_i\rangle z_j - \big\langle z_j, z_i\big\rangle z_i\Big\rangle \\
&+ \lambda_1 \sum_{j\in\mathbb{N}} \kappa_{ij}\Big\langle \Big(\langle z_i, z_j\rangle - \langle z_j, z_i\rangle\Big) z_i, z_i\Big\rangle + \lambda_1 \sum_{j\in\mathbb{N}} \kappa_{ij}\Big\langle z_i, \Big(\langle z_i, z_j\rangle - \langle z_j, z_i\rangle\Big) z_i\Big\rangle \\
&=: \sum_{i=1}^{6} \mathcal{I}_{1i}.
\end{aligned} \tag{2.1}$$

Below, we estimate the terms $\mathcal{I}_{1i}$ one by one.

• Step A (Estimate of $\mathcal{I}_{11} + \mathcal{I}_{12}$): We use (1.3) and the skew-Hermitian property $\Omega^\dagger = -\Omega$ to get

$$\begin{aligned}
\mathcal{I}_{11} + \mathcal{I}_{12} &= \Big\langle \Omega_i z_i, z_i\Big\rangle + \Big\langle z_i, \Omega_i z_i\Big\rangle = \Big\langle z_i, \Omega_i^\dagger z_i\Big\rangle + \Big\langle z_i, \Omega_i z_i\Big\rangle \\
&= -\Big\langle z_i, \Omega_i z_i\Big\rangle + \Big\langle z_i, \Omega_i z_i\Big\rangle = 0.
\end{aligned}$$

• Step B (Estimate of $\mathcal{I}_{13} + \mathcal{I}_{14}$): We use the sesqui-linearity of $\langle\cdot,\cdot\rangle$ with $\overline{\langle z_i, z_j\rangle} = \langle z_j, z_i\rangle$ to obtain

$$\begin{aligned}
&\Big\langle \langle z_i, z_i\rangle z_j - \big\langle z_j, z_i\big\rangle z_i, z_i\Big\rangle = \langle z_i, z_i\rangle \langle z_j, z_i\rangle - \big\langle z_i, z_j\big\rangle \langle z_i, z_i\rangle, \\
&\Big\langle z_i, \langle z_i, z_i\rangle z_j - \big\langle z_j, z_i\big\rangle z_i\Big\rangle = \langle z_i, z_i\rangle \langle z_i, z_j\rangle - \big\langle z_j, z_i\big\rangle \langle z_i, z_i\rangle.
\end{aligned}$$

These imply

$$\mathcal{I}_{13} + \mathcal{I}_{14} = 0.$$

• Step C (Estimate of $\mathcal{I}_{15} + \mathcal{I}_{16}$): Similar to Step B, we have

$$\Big\langle \Big(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle \Big) z_i, z_i \Big\rangle = \Big(\langle z_j, z_i \rangle - \langle z_i, z_j \rangle \Big) \langle z_i, z_i \rangle ,$$
$$\Big\langle z_i, \Big(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle \Big) z_i \Big\rangle = \Big(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle \Big) \langle z_i, z_i \rangle .$$

Thus, we have

$$\mathcal{I}_{15} + \mathcal{I}_{16} = 0.$$

Finally in (2.1), we combine all the estimates in Step A–Step C to get the desired conservation law:

$$\frac{d}{dt} \|z_i\|^2 = 0, \quad t > 0.$$

□

Remark 2.1 Thanks to the result of Lemma 2.1, we can assume

$$\|z_i\| = 1, \quad t \geq 0, \quad i \in \mathbb{N}$$

without loss of generality.

Lemma 2.2 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to (1.3)–(1.4). Then we have the following estimates:*

$$\text{(i)} \left\| \frac{dz_i}{dt} \right\| \leq \|\mathbf{\Omega}\|_{\infty,op} + 2 \|\boldsymbol{\kappa}\|_{\infty,1} (\lambda_0 + \lambda_1) .$$
$$\text{(ii)} \left\| \frac{d}{dt} \big(z_i - z_j \big) \right\| \leq 2 \|\mathbf{\Omega}\|_{\infty,op} + 4 \|\boldsymbol{\kappa}\|_{\infty,1} (\lambda_0 + \lambda_1) .$$
$$\text{(iii)} \frac{d}{dt} \left\| z_i - z_j \right\|^2 \leq 8 \|\mathbf{\Omega}\|_{\infty,op} + 16 \|\boldsymbol{\kappa}\|_{\infty,1} (\lambda_0 + \lambda_1) .$$
$$\text{(iv)} \left| \frac{d}{dt} \left\| z_i - z_j \right\| \right| \leq 2 \|\mathbf{\Omega}\|_{\infty,op} + 4 \|\boldsymbol{\kappa}\|_{\infty,1} (\lambda_0 + \lambda_1) .$$

Proof (i) and (ii): It follows from (1.3) and the triangle inequality that

$$\begin{aligned}
\left\| \frac{dz_i}{dt} \right\| &\leq \|\mathbf{\Omega}\|_{\infty,\mathrm{op}} + \sum_{j \in \mathbb{N}} \kappa_{ij} \Big(\lambda_0 \left\| z_j - \langle z_j, z_i \rangle z_i \right\| + \lambda_1 \left\| \langle z_i, z_j \rangle - \langle z_j, z_i \rangle \right\| \Big) \\
&\leq \|\mathbf{\Omega}\|_{\infty,\mathrm{op}} + \sum_{j \in \mathbb{N}} \kappa_{ij} \Big(\lambda_0 \left\| z_j \right\| + \lambda_0 \left| \langle z_j, z_i \rangle \right| \|z_i\| + 2\lambda_1 \Big) \\
&\leq \|\mathbf{\Omega}\|_{\infty,\mathrm{op}} + 2 \|\boldsymbol{\kappa}\|_{\infty,1} (\lambda_0 + \lambda_1) .
\end{aligned}$$

The second relation follows from the first relation directly:

$$\left\| \frac{d}{dt} \big(z_i - z_j \big) \right\| \leq \left\| \frac{dz_i}{dt} \right\| + \left\| \frac{dz_j}{dt} \right\| .$$

(iii) and (iv): Note that

$$\begin{aligned}\frac{d}{dt}\left\|z_i - z_j\right\|^2 &\le 2\left|\left\langle z_i - z_j, \frac{d}{dt}\left(z_i - z_j\right)\right\rangle\right| \le 2\left\|z_i - z_j\right\| \cdot \left\|\frac{d}{dt}\left(z_i - z_j\right)\right\| \\ &\le 2\cdot 2\cdot\left(2\left\|\boldsymbol{\Omega}\right\|_{\infty,\mathrm{op}} + 4\left\|\boldsymbol{\kappa}\right\|_{\infty,1}(\lambda_0+\lambda_1)\right) \\ &\le 8\left\|\boldsymbol{\Omega}\right\|_{\infty,\mathrm{op}} + 16\left\|\boldsymbol{\kappa}\right\|_{\infty,1}(\lambda_0+\lambda_1) \quad \text{and}\end{aligned}$$

$$\begin{aligned}\left|\frac{d}{dt}\left\|z_i - z_j\right\|\right| &= \left|\frac{1}{2\sqrt{\left\|z_i - z_j\right\|^2}}\frac{d}{dt}\left\|z_i - z_j\right\|^2\right| \le 2\cdot\frac{1}{2\left\|z_i - z_j\right\|}\left|\left\langle\frac{d}{dt}\left(z_i - z_j\right), z_i - z_j\right\rangle\right| \\ &\le \left\|\frac{d}{dt}\left(z_i - z_j\right)\right\| \le 2\left\|\boldsymbol{\Omega}\right\|_{\infty,\mathrm{op}} + 4\left\|\boldsymbol{\kappa}\right\|_{\infty,1}(\lambda_0+\lambda_1).\end{aligned}$$

□

Note that if we set $\lambda_1 = 0$ in (1.3), it becomes the infinite complex swarm sphere model [30]. Hence we can expect that (1.3) can be reduced to the infinite swarm sphere model as a special case, if initial data and natural frequencies are real. This can be seen in the following lemma.

Lemma 2.3 *Let $\mathcal{Z}$ be a global solution to* (1.3)–(1.4) *satisfying the following two conditions:*

(i) *Initial data are purely real:*

$$\mathfrak{Im}(z_i^{in}) = 0, \quad i \in \mathbb{N},$$

where $\mathfrak{Im}(z)$ is the imaginary part of z.

(ii) $\boldsymbol{\Omega} = \{\Omega_i\}_{i\in\mathbb{N}}$ *is a sequences of $d \times d$ anti-symmetric matrices:*

$$\Omega_i \in \mathbb{R}^{d\times d}, \quad \Omega_i = -\Omega_i^T, \quad i \in \mathbb{N}.$$

Then, we have

$$\mathfrak{Im}(z_i(t)) = 0, \quad i \in \mathbb{N}, \quad t > 0.$$

Proof Since every calculation in the proof of Theorem A.1 can be applied for (1.5), we can show the real counterpart of Theorem A.1 with solution curve $\mathcal{X}$ defined on the real Banach space:

$$(\ell^{\infty,2}, \|\cdot\|_{\infty,2}) := \left\{\mathcal{Y} = \{y_i\}_{i\in\mathbb{N}} : y_i \in \mathbb{R}^d, \quad \|\mathcal{Y}\|_{\infty,2} := \sup_{i\in\mathbb{N}}\|y_i\| < \infty\right\}.$$

Then the real solution $\mathcal{X}$ can be considered as a unique solution of model (1.3) on the unit Hermitian sphere $\mathbb{HS}^{d-1}$. □

Remark 2.2 Let $\mathcal{Z}$ be a real-valued solution to (1.3). Then the second term involving with λ_1 is zero:

$$\sum_{j\in\mathbb{N}}\kappa_{ij}\Big(\langle z_i, z_j\rangle - \langle z_j, z_i\rangle\Big)z_i = 0.$$

Hence (1.3) has the same form as in the infinite swarm sphere model.

Next, we consider a finite truncation of (1.3). For a fixed positive integer N, we assume that

$$\kappa_{ij} = 0, \quad z_i^{\mathrm{in}} = 0, \quad i \ge N+1.$$

Then, the Cauchy problem (1.3) becomes

$$\begin{cases} \dot{z}_i = \Omega_i z_i + \lambda_0 \sum_{j\in[N]} \kappa_{ij}\Big(\langle z_i, z_i\rangle z_j - \big\langle z_j, z_i\big\rangle z_i\Big) \\ \qquad + \lambda_1 \sum_{j\in[N]} \kappa_{ij}\Big(\langle z_i, z_j\rangle - \langle z_j, z_i\rangle\Big) z_i, \quad t>0, \quad \forall\, i \in \mathbb{N}, \\ z_i(0) = z_i^{\text{in}}. \end{cases} \tag{2.2}$$

Lemma 2.4 *Suppose that initial data satisfy*

$$z_i(0) = \begin{cases} z_i^{in}, & 1 \le i \le N, \\ 0, & i \ge N+1, \end{cases}$$

and let $\mathcal{Z}$ be a solution to (2.2). Then, we have

$$z_i(t) = 0, \quad t \ge 0, \quad i \ge N+1.$$

Proof Since the proof is straightforward from Lemma 2.1, we omit its detailed proof. □

Consider a finite-dimensional analogue of (1.3) in which all the coupling strengths are uniform over nonzero nodes and collectiions $\{\Omega_i\}$ are homogeneous:

$$\kappa_{ij} = \begin{cases} \frac{1}{N}, & i, j \le N, \\ 0 & \max(i,j) > N, \end{cases} \qquad \Omega_i = \Omega, \quad i \in \mathbb{N}, \quad z_i = 0, \quad i \ge N+1.$$

In this case, the system (1.3) can be rewritten as

$$\begin{cases} \dot{z}_j = \Omega z_j + \lambda_0\Big(z_c\langle z_j, z_j\rangle - z_j\langle z_c, z_j\rangle\Big) + \lambda_1\Big(\langle z_j, z_c\rangle - \langle z_c, z_j\rangle\Big) z_j, \quad t > 0, \\ z_j(0) = z_j^{\text{in}}, \quad j \in [N]. \end{cases} \tag{2.3}$$

Now, we also consider the homogeneous analogue of (1.3):

$$\begin{cases} \dot{w}_j = \lambda_0\Big(w_c\langle w_j, w_j\rangle - w_j\langle w_c. w_j\rangle\Big) + \lambda_1\Big(\langle w_j, w_c\rangle - \langle w_c, w_j\rangle\Big) w_j, \quad t > 0, \\ w_j(0) = z_j^{\text{in}}, \quad j \in [N], \end{cases} \tag{2.4}$$

where z_c and w_c are averages of $\{z_1, z_2, \dots, z_N\}$ and $\{w_1, w_2, \dots, w_N\}$ respectively:

$$z_c := \frac{1}{N}\sum_{i\in[N]} z_i, \quad w_c := \frac{1}{N}\sum_{i\in[N]} w_i.$$

In the following proposition, we study a relation between (2.3) and (2.4).

Proposition 2.1 (Solution splitting property) [24] *Let $\mathcal{Z} = \{z_j\}$ and $\mathcal{W} = \{w_j\}$ be solutions to* (2.3) *and* (2.4) *with the same initial data $\{z_j^{in}\}$, respectively. Then, one has*

$$z_j = e^{\Omega t} w_j, \quad j \in [N].$$

Proof We first note that

$$\big(e^{\Omega t}\big)^\dagger = (e^{\Omega t})^{-1}.$$

Then $e^{\Omega t}$ is unitary, and we introduce the variable w_j such that

$$z_j = e^{\Omega t} w_j \quad \text{for all } j \in [N]. \tag{2.5}$$

We substitute (2.5) into (2.3) to get

$$e^{\Omega t}\dot{w}_j + \Omega e^{\Omega t} w_j = \Omega e^{\Omega t} w_j + \lambda_0(\langle e^{\Omega t} w^j, e^{\Omega t} w_j\rangle e^{\Omega t} w_c - \langle e^{\Omega t} w_c, e^{\Omega t} w_j\rangle e^{\Omega t} w_j) \\ + \lambda_1(\langle e^{\Omega t} w_j, e^{\Omega t} w_c\rangle - \langle e^{\Omega t} w_c, e^{\Omega t} w_c\rangle) e^{\Omega t} w_j.$$

After simplification, one has

$$\dot{w}_j = \lambda_0(\langle w_j, w_j\rangle w_c - \langle w_c, w_j\rangle w_j) + \lambda_1(\langle w_j, w_c\rangle - \langle w_c, w_j\rangle) w_j.$$

Thus, we obtain the desired result. □

2.2 Reduction to Known Aggregation Models

In this subsection, we discuss three reductions from (1.3) to other related aggregation models.

2.2.1 The Swarm Sphere Model

Consider the finite-dimensional Lohe Hermitian sphere model:

$$\dot{z}_j = \Omega_j z_j + \lambda_0(z_c\langle z_j, z_j\rangle - z_j\langle z_c . z_j\rangle) + \lambda_1(\langle z_j, z_c\rangle - \langle z_c z_j\rangle) z_j. \tag{2.6}$$

It follows from Lemma 2.3 that once initial data lies on the unit Euclidean sphere $\mathbb{S}^{d-1}$, then we have

$$z_i \in \mathbb{R}^d, \quad i \in [N].$$

In this case, the second coupling term in the right-hand side of (2.6) becomes zero:

$$\langle z_j, z_c\rangle - \langle z_c, z_j\rangle = 0.$$

Hence, for a real-valued function z_j, the system (1.3) reduces to the swarm sphere model [30]:

$$\dot{x}_j = \Omega_j x_j + \lambda_0\Big(x_c - x_j\langle x_c, x_j\rangle\Big).$$

2.2.2 The Kuramoto Model

Now, we return to the complex Lohe sphere model (1.3) with $d = 1$, and explain how (1.3) can be related to the Kuramoto model. For this, we set

$$\Omega_j = 0, \quad z_j = r_j e^{\mathrm{i}\theta_j}, \quad j \in [N] \quad \text{and} \quad z_c := r_c e^{\mathrm{i}\phi}. \tag{2.7}$$

We substitute the ansatz (2.7) into (2.6) to see

$$\dot{r}_j e^{\mathrm{i}\theta_j} + \mathrm{i} r_j e^{\mathrm{i}\theta_j}\dot{\theta}_j = \kappa_0 r_j^2 r_c(e^{\mathrm{i}\phi} - e^{\mathrm{i}(2\theta_j-\phi)}) + \kappa_1 r_j^2 r_c(e^{\mathrm{i}\phi} - e^{\mathrm{i}(2\theta_j-\phi)}) \\ = 2(\lambda_0+\lambda_1) r_j^2 r_c \mathrm{i}\sin(\phi-\theta_j) e^{\mathrm{i}\theta_j}.$$

This yields

$$\dot{r}_j + \mathrm{i} r_j\dot{\theta}_j = 2(\kappa_0+\kappa_1)\mathrm{i} r_j^2 r_c \sin(\phi-\theta_j). \tag{2.8}$$

We compare the real and imaginary parts of the above relation (2.8) to get

$$\dot{r}_j = 0 \quad \text{and} \quad \dot{\theta}_j = 2(\lambda_0+\lambda_1) r_j r_c \sin(\phi-\theta_j).$$

These yield

$$r_j(t) = r_j^{\text{in}}, \qquad \dot{\theta}_j = \frac{2(\lambda_0+\lambda_1)}{N}\sum_{k\in[N]} r_j^{\text{in}} r_k^{\text{in}} \sin(\theta_k-\theta_j).$$

Now, we set

$$r_j^{\text{in}} = 1, \quad \kappa := 2(\lambda_0+\lambda_1)$$

to get the Kuramoto model for identical oscillators:

$$\dot{\theta}_i = \frac{\kappa}{N}\sum_{j\in[N]} \sin(\theta_j-\theta_i).$$

2.2.3 The Schrödinger-Lohe Model

In this part, we follow the presentation from [23]. Let $\{\psi_j\}$ be the collection of N complex-valued wave functions in $\mathcal{C}(\mathbb{R}_+; L^2(\mathbb{T}^d))$ whose dynamics is governed by the coupled system of nonlinear Schrödinger equations:

$$\mathrm{i}\partial_t\psi_j = H\psi_j + \frac{\mathrm{i}\kappa}{N}\sum_{k\in[N]}\Big(\psi_k\langle\psi_j,\psi_j\rangle - \langle\psi_k,\psi_j\rangle\psi_j\Big), \qquad (t,x)\in\mathbb{R}_+\times\mathbb{T}^d, \tag{2.9}$$

where $H = -\frac{1}{2}\Delta_x + V$ is one-body Hamiltonian.

Let $\{\phi_k\}$ and $\{E_k\}$ be a countable orthonormal basis consisting of eigenfunctions and their corresponding eigenvalues respectively:

$$H\phi_k = E_k\phi_k, \quad k\in\mathbb{N}.$$

Then the standing wave solution $\Phi_k(t,x) := e^{-\mathrm{i}E_k t}\phi_k(x)$ satisfies the linear Schrödinger equation:

$$\mathrm{i}\partial_t\Phi_k = H\Phi_k, \quad k\in\mathbb{N}.$$

Now we set ψ_j to be the linear combination of $\{\Phi_k\}_{k\in\mathbb{N}}$:

$$\psi_j(t,x) = \sum_{k\in\mathbb{N}} z_j^k(t)\Phi_k(t,x), \quad j\in[N]. \tag{2.10}$$

Suppose that ψ_j satisfies the Schrödinger-Lohe model (2.9) with $\|\psi_j\|_2 = 1$:

$$\mathrm{i}\partial_t\psi_j = H\psi_j + \frac{\mathrm{i}\kappa}{N}\sum_{k\in[N]}(\psi_k - \langle\psi_k,\psi_j\rangle\psi_j). \tag{2.11}$$

We use (2.10) to rewrite the left-hand side of (2.11) to see

$$\mathrm{i}\partial_t\psi_j = \sum_{k\in\mathbb{N}}\Big(z_j^k\mathrm{i}\partial_t\Phi_k + \dot{z}_j^k\mathrm{i}\Phi_k\Big) = \sum_{k\in\mathbb{N}}\Big(z_j^k H\Phi_k + \dot{z}_j^k\mathrm{i}\Phi_k\Big) = H\psi_j + \mathrm{i}\sum_{k\in\mathbb{N}}\dot{z}_j^k\Phi_k. \tag{2.12}$$

Next, we equate (2.11) and (2.12) to get

$$\begin{aligned} H\psi_j + \mathrm{i}\sum_{k\in\mathbb{N}}\dot{z}_j^k\Phi_k &= H\psi_j + \frac{\mathrm{i}\kappa}{N}\sum_{l\in[N]}(\psi_l - \langle\psi_l,\psi_j\rangle\psi_j) \\ &= H\psi_j + \frac{\mathrm{i}\kappa}{N}\sum_{l\in[N]}\sum_{k\in\mathbb{N}}(z_l^k - \langle\psi_l,\psi_j\rangle z_j^k)\Phi_k. \end{aligned}$$

This yields

$$\sum_{k\in\mathbb{N}} \dot{z}_j^k \Phi_k = \frac{\kappa}{N} \sum_{l\in[N]} \sum_{k\in\mathbb{N}} (z_l^k - \langle \psi_l, \psi_j\rangle z_l^k)\Phi_k.$$

Since $\{\Phi_k\}$ is an orthonormal basis, one has

$$\frac{dz_j^k}{dt} = \frac{\kappa}{N} \sum_{l\in[N]} (z_l^k - \langle \psi_l, \psi_j\rangle z_l^k), \quad l \in [N], \quad k \in \mathbb{N}. \tag{2.13}$$

For each $j \in [N]$, we set an infinite complex vector z_j in $(\ell^\infty \cap \ell^2)(\mathbb{Z}_+)$ as follows:

$$z_j = (z_j^1, z_j^2, \dots).$$

Now, we use the definition of $\langle \cdot, \cdot\rangle$, (2.10) and (2.13) to get

$$\langle \psi_l, \psi_j\rangle = \sum_{k,m\in\mathbb{N}} \Big\langle z_l^k \Phi_k, z_j^m \Phi_m \Big\rangle = \sum_{k,m\in\mathbb{N}} \overline{z_l^k} z_j^m \Big\langle \Phi_k, \Phi_m \Big\rangle = \sum_{k\in\mathbb{N}} \overline{z_l^k} z_j^k = \langle z_l, z_j\rangle. \tag{2.14}$$

Finally, we combine (2.13) and (2.14) to derive the complex Lohe sphere model on $(\ell^2 \cap \ell^\infty)(\mathbb{Z}_+)$:

$$\dot{z}_j = \frac{\kappa}{N} \sum_{l\in[N]} (z_l - \langle z_l, z_j\rangle z_j), \quad j \in [N].$$

In the following three sections, we study emergent dynamics of the model (1.3) under the following cases:

$$\begin{cases} \text{Case A:} \quad \Omega_i^T = -\Omega_i, \quad \forall\, i \in \mathbb{N}, \quad z_i^{\text{in}} \in \mathbb{R}^d, \\ \text{Case B.1:} \ \Omega_i^\dagger = -\Omega_i, \quad \forall\, i \in \mathbb{N}, \quad z_i^{\text{in}} \in \mathbb{C}^d, \quad \kappa_{ij} > 0, \\ \text{Case B.2:} \ \Omega_i^\dagger = -\Omega_i, \quad \forall\, i \in \mathbb{N}, \quad z_i^{\text{in}} \in \mathbb{C}^d, \quad \kappa_{ij} = \kappa_j > 0. \end{cases}$$

3 The Infinite Swarm Sphere Model

In this section, we provide a sufficient framework on the emergent dynamics of an infinite set of LHS particles on the unit Euclidean sphere $\mathbb{S}^{d-1}$, and present a practical synchronization.

3.1 Preparatory Lemmas

In this subsection, we study an infinite analogue of the swarm sphere model on the Euclidean unit sphere $\mathbb{S}^{d-1}$. Let $\mathbf{\Omega} = \{\Omega_i\}_{i\in\mathbb{N}}$ be a sequence of $d \times d$ anti-symmetric real matrices:

$$\Omega_i^T = -\Omega_i, \quad i \in \mathbb{N},$$

and we consider the LHS model (1.3) defined on the following real Banach space:

$$(\ell^{\infty,2}, \|\cdot\|_{\infty,2}) := \left\{ \mathcal{Y} = \{y_i\}_{i\in\mathbb{N}} : y_i \in \mathbb{R}^d, \quad \|\mathcal{Y}\|_{\infty,2} := \sup_{i\in\mathbb{N}} \|y_i\| < \infty \right\}.$$

Thanks to Corollary 2.2 and Lemma 2.1, we can see that $\mathbb{S}^{d-1}$ is positively invariant along (1.3). If we set

$$z_i = x_i \in \mathbb{R}^d, \quad \forall\, i \in \mathbb{N},$$

then the system (1.3) is reduced to the infinite analogue of the swarm sphere model:

$$\begin{cases} \dot{x}_i = \Omega_i x_i + \sum_{j\in\mathbb{N}} \kappa_{ij}\left(x_j - \langle x_j, x_i\rangle x_i\right), \quad t>0, \\ x_i(0) = x_i^{\text{in}} \in \mathbb{R}^d, \quad \forall i \in \mathbb{N}, \quad \mathcal{X}^{\text{in}} := \left\{x_i^{\text{in}}\right\}_{i\in\mathbb{N}} \in \ell^{\infty,2}, \quad \left\|x_i^{\text{in}}\right\| = 1, \end{cases} \tag{3.1}$$

with structural conditions:

$$\boldsymbol{\kappa} = \left(\kappa_{ij}\right)_{ij\in\mathbb{N}} \in \ell_+^{\infty,1}, \quad \mathcal{D}(\boldsymbol{\Omega}) < \infty, \quad \|\boldsymbol{\Omega}\|_{\infty,\text{op}} < \infty. \tag{3.2}$$

We call this model as the infinite swarm sphere model(ISS). A global well-posedness of the ISS model can be reduced from the well-posedness of LHS model as a special case.

Lemma 3.1 *Let $\mathcal{X}(t) = \{x_i(t)\}_{i\in\mathbb{N}}$ be a global solution to (3.1)–(3.2). Then one has the following estimates.*

(i) $\left\|\frac{dx_i}{dt}\right\| \le \|\boldsymbol{\Omega}\|_{\infty,op} + 2\|\boldsymbol{\kappa}\|_{\infty,1}, \quad \left\|\frac{d}{dt}\left(x_i - x_j\right)\right\| \le 2\|\boldsymbol{\Omega}\|_{\infty,op} + 4\|\boldsymbol{\kappa}\|_{\infty,1}.$

(ii) $\frac{d}{dt}\left\|x_i - x_j\right\|^2 \le 8\|\boldsymbol{\Omega}\|_{\infty,op} + 16\|\boldsymbol{\kappa}\|_{\infty,1}, \quad \left|\frac{d}{dt}\left\|x_i - x_j\right\|\right| \le 2\|\boldsymbol{\Omega}\|_{\infty,op} + 4\|\boldsymbol{\kappa}\|_{\infty,1}.$

Proof Since the proof is similar to Lemma 2.2. we omit their proofs. □

Recall that the finite swarm sphere model and the finite LHS model exhibit the synchronous behaviors on high-dimensional manifolds [24], and the following diameter functional

$$\mathcal{D}(\mathcal{X}(t)) := \sup_{i,j\in\mathbb{N}} \left\|x_i(t) - x_j(t)\right\| \tag{3.3}$$

plays a key role in the analysis of the emergent dynamics for (3.1). Let $\boldsymbol{\kappa} = (\kappa_{mn})_{m,n\in\mathbb{N}}$ be a given coupling matrix. Then, we denote the i-th row $\{\kappa_{in}\}_{n\in\mathbb{N}}$ by $\boldsymbol{\kappa}_i$, i.e.,

$$\{\kappa_{in}\} := \{\kappa_{i1}, \kappa_{i2}, \ldots\}.$$

Next we briefly discuss a sufficient framework $(\mathcal{F}_A)$ for the emergent dynamics of the ISS model:

- $(\mathcal{F}_A 1)$: There exists a positive constant $\delta \in (0,1)$ such that

$$\mathcal{D}\left(\mathcal{X}^{\text{in}}\right) < \sqrt{2-2\delta} \quad \text{or} \quad \inf_{i,j\in\mathbb{N}} \left\langle x_i^{\text{in}}, x_j^{\text{in}}\right\rangle > \delta.$$

- $(\mathcal{F}_A 2)$: For a given coupling matrix $\boldsymbol{\kappa} = (\kappa_{mn})_{m,n\in\mathbb{N}}$, denote the i-th row $\{\kappa_{in}\}_{n\in\mathbb{N}}$ by $\boldsymbol{\kappa}_i$. Then there exists $r_\kappa \in (0, 1/6)$ such that

$$\left\|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j\right\|_1 \le r_\kappa\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right), \quad \forall\, i,j \in \mathbb{N}. \tag{3.4}$$

- $(\mathcal{F}_A 3)$: Positive constants δ and r_κ satisfy

$$\delta > 3r_\kappa.$$

- $(\mathcal{F}_A 4)$: The natural frequency $\boldsymbol{\Omega}$ satisfies

$$\mathcal{D}(\boldsymbol{\Omega}) < \|\boldsymbol{\kappa}\|_{-\infty,1}(\delta - 3r_\kappa)\,\mathcal{D}(\mathcal{X}^{\text{in}}).$$

Remark 3.1 Before we move on to technical lemmas, we briefly comment on the above conditions on initial data and system parameters one by one.

1. Note that
$$\|x_i - x_j\|^2 \le 2(\|x_i\|^2 + \|x_j\|^2) = 4, \quad \text{i.e.,} \quad \mathcal{D}(\mathcal{X}) \le 2.$$
Therefore, the condition on initial state diameter in $(\mathcal{F}_A 1)$ is a certainly restriction on initial data.

2. If we choose all rows of infinite coupling matrix to be close in ℓ^1-norm, then the condition (3.4) can be achieved.

3. The condition in $(\mathcal{F}_A 4)$ denotes that either the size of natural frequency set is sufficiently small or the coupling strengths are large enough.

4. It follows that $(\mathcal{F}_A 1)$ - $(\mathcal{F}_A 4)$ gives
$$\mathcal{D}_* := \frac{\mathcal{D}(\mathbf{\Omega})}{\|\boldsymbol{\kappa}\|_{-\infty,1}(\delta - 3r_\kappa)} < \mathcal{D}(\mathcal{X}^{\text{in}}) < \sqrt{2-2\delta}.$$

Now, under the above framework $(\mathcal{F}_A)$, we derive a differential inequality for $\|x_i - x_j\|$ and $\mathcal{D}(\mathcal{X})$ in (3.3).

Lemma 3.2 *Suppose the framework $(\mathcal{F}_A 1) - (\mathcal{F}_A 4)$ holds, and let $\mathcal{X} = \{x_i\}_{i\in\mathbb{N}}$ be a global solution to (3.1). Then for $i, j \in \mathbb{N}$, the relative distance $\|x_i - x_j\|$ near $t = 0$ satisfies*
$$\frac{d}{dt}\bigg|_{t=0+} \|x_i - x_j\| \le \mathcal{D}(\mathbf{\Omega}) + \frac{1}{2}\left(\|\boldsymbol{\kappa}_i\|_1 + \|\boldsymbol{\kappa}_j\|_1\right)\Big(-\delta \left\|x_i^{in} - x_j^{in}\right\| + 3r_\kappa \mathcal{D}(\mathcal{X}^{in})\Big).$$

Proof We write $\mathcal{X}^{\text{in}} = \{x_i^{\text{in}}\}_{i\in\mathbb{N}}$ by $\mathcal{X} = \{x_i\}_{i\in\mathbb{N}}$ only in this proof. We use (3.1) to get
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle x_i - x_j, x_i - x_j\rangle &= \langle x_i - x_j, \Omega_i x_i - \Omega_j x_j\rangle \\ &\quad + \sum_{l\in\mathbb{N}} \Big\langle x_i - x_j, \kappa_{il}(x_l - \langle x_l, x_i\rangle x_i) - \kappa_{jl}\big(x_l - \langle x_l, x_j\rangle x_j\big)\Big\rangle \\ &=: \mathcal{I}_{21} + \mathcal{I}_{22}.\end{aligned}$$

Below, we estimate $\mathcal{I}_{21}$ and $\mathcal{I}_{22}$ separately.

- Step A (Bound of $\mathcal{I}_{21}$): For $\mathcal{I}_{21}$, we again use the skew-symmetry of Ω_i to obtain
$$\begin{aligned}\mathcal{I}_{21} &= \langle x_i - x_j, \Omega_i x_i - \Omega_i x_j\rangle + \langle x_i - x_j, \Omega_i x_j - \Omega_j x_j\rangle \\ &= \langle x_i - x_j, \Omega_i(x_i - x_j)\rangle + \langle x_i - x_j, (\Omega_i - \Omega_j)x_j\rangle \\ &= 0 + \langle x_i - x_j, (\Omega_i - \Omega_j)x_j\rangle \le \mathcal{D}(\mathbf{\Omega})\|x_i - x_j\|.\end{aligned}$$

- Step B (Bound of $\mathcal{I}_{22}$): We divide $\mathcal{I}_{22}$ into two terms. Define $\mathcal{I}_{221}$ and $\mathcal{I}_{222}$ by
$$\begin{aligned}\mathcal{I}_{22} &= \sum_{l\in\mathbb{N}} \Big\langle x_i - x_j, \kappa_{il}(x_l - \langle x_l, x_i\rangle x_i) - \kappa_{jl}\big(x_l - \langle x_l, x_j\rangle x_j\big)\Big\rangle \\ &= \sum_{l\in\mathbb{N}} \langle x_i - x_j, \kappa_{il}(x_l - \langle x_l, x_i\rangle x_i)\rangle - \sum_{l\in\mathbb{N}} \Big\langle x_i - x_j, \kappa_{jl}\big(x_l - \langle x_l, x_j\rangle x_j\big)\Big\rangle \\ &= -\sum_{l\in\mathbb{N}} \langle x_j, \kappa_{il}(x_l - \langle x_l, x_i\rangle x_i)\rangle - \sum_{l\in\mathbb{N}} \Big\langle x_i, \kappa_{jl}\big(x_l - \langle x_l, x_j\rangle x_j\big)\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{l\in\mathbb{N}} \left[\kappa_{il} \left\langle x_j, x_l - \langle x_l, x_i\rangle x_i\right\rangle + \kappa_{jl} \left\langle x_i, x_l - \left\langle x_l, x_j\right\rangle x_j\right\rangle\right] \\
&= -\frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} + \kappa_{jl}\right) \left[\left\langle x_i, x_l - \left\langle x_l, x_j\right\rangle x_j\right\rangle + \left\langle x_j, x_l - \langle x_l, x_i\rangle x_i\right\rangle\right] \\
&\quad - \frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} - \kappa_{jl}\right) \left[\left\langle x_j, x_l - \langle x_l, x_i\rangle x_i\right\rangle - \left\langle x_i, x_l - \left\langle x_l, x_j\right\rangle x_j\right\rangle\right] \\
&=: \mathcal{I}_{221} + \mathcal{I}_{222}.
\end{aligned}
$$

◇ Step B.1 (Bound of $\mathcal{I}_{221}$): The term $\mathcal{I}_{221}$ can be estimated as

$$
\begin{aligned}
\mathcal{I}_{221} &= -\frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} + \kappa_{jl}\right) \left[\left\langle x_i, x_l - \left\langle x_l, x_j\right\rangle x_j\right\rangle + \left\langle x_j, x_l - \langle x_l, x_i\rangle x_i\right\rangle \right] \\
&= -\frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} + \kappa_{jl}\right) \left[\langle x_i, x_l\rangle - \left\langle x_l, x_j\right\rangle \left\langle x_i, x_j\right\rangle + \left\langle x_j, x_l\right\rangle - \langle x_l, x_i\rangle \left\langle x_j, x_i\right\rangle\right] \\
&= -\frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} + \kappa_{jl}\right) \left(\langle x_l, x_i\rangle + \left\langle x_l, x_j\right\rangle\right) \left(1 - \left\langle x_i, x_j\right\rangle\right) \\
&= -\sum_{l\in\mathbb{N}} \left(\kappa_{il} + \kappa_{jl}\right) \left(1 - \left\langle x_i, x_j\right\rangle\right) \\
&\quad + \frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} + \kappa_{jl}\right) \left(2 - \langle x_l, x_i\rangle - \left\langle x_l, x_j\right\rangle\right) \left(1 - \left\langle x_i, x_j\right\rangle\right) \\
&\le -\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right) \frac{\left\|x_i - x_j\right\|^2}{2} + \frac{1}{2}\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right) \mathcal{D}\left(\mathcal{X}\right)^2 \cdot \frac{\left\|x_i - x_j\right\|^2}{2}.
\end{aligned}
$$

◇ Step B.2 (Bound of $\mathcal{I}_{222}$): For the summand of $\mathcal{I}_{222}$, one has

$$
\begin{aligned}
&\left|\left\langle x_j, x_l - \langle x_l, x_i\rangle x_i\right\rangle - \left\langle x_i, x_l - \left\langle x_l, x_j\right\rangle x_j\right\rangle\right| \\
&\quad = \left|\left\langle x_i - x_j, x_l - \left\langle x_l, x_j\right\rangle x_j\right\rangle - \left\langle x_j - x_i, x_l - \langle x_l, x_i\rangle x_i\right\rangle\right| \\
&\quad \le \left|\left\langle x_i - x_j, x_l - x_j\right\rangle + \left\langle x_i - x_j, x_l - x_i\right\rangle\right| \\
&\qquad + \left|\left(1 - \left\langle x_l, x_j\right\rangle\right)\left\langle x_i - x_j, x_j\right\rangle - \left(1 - \langle x_l, x_i\rangle\right)\left\langle x_i - x_j, x_j\right\rangle\right| \\
&\quad \le 2\mathcal{D}(\mathcal{X}) \left\|x_i - x_j\right\| + \left|\left\langle x_l, x_i - x_j\right\rangle \left\langle x_i - x_j, x_j\right\rangle\right| \\
&\quad \le 2\mathcal{D}(\mathcal{X}) \left\|x_i - x_j\right\| + \frac{1}{2}\left\|x_i - x_j\right\|^3.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\mathcal{I}_{222} &= -\frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} - \kappa_{jl}\right) \left[\left\langle x_i, x_l - \left\langle x_l, x_j\right\rangle x_j\right\rangle - \left\langle x_j, x_l - \langle x_l, x_i\rangle x_i\right\rangle \right] \\
&\le \sum_{l\in\mathbb{N}} \left|\kappa_{il} - \kappa_{jl}\right| \left[\mathcal{D}(\mathcal{X}) \left\|x_i - x_j\right\| + \frac{1}{4}\left\|x_i - x_j\right\|^3\right] \\
&\le r_\kappa \left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right) \left[\mathcal{D}(\mathcal{X}) \left\|x_i - x_j\right\| + \frac{1}{4}\left\|x_i - x_j\right\|^3\right].
\end{aligned}
$$

• Step C: We combine estimate for $\mathcal{I}_{21}$, $\mathcal{I}_{221}$ and $\mathcal{I}_{222}$ in Step A and Step B to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|x_i-x_j\right\|^2 &\le \mathcal{D}(\boldsymbol{\Omega})\left\|x_i-x_j\right\| - \left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\frac{\left\|x_i-x_j\right\|^2}{2}\\ &\quad+\frac{1}{2}\left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\mathcal{D}(\mathcal{X})^2\cdot\frac{\left\|x_i-x_j\right\|^2}{2}\\ &\quad+r_\kappa\left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\left[\mathcal{D}(\mathcal{X})\left\|x_i-x_j\right\|+\frac{1}{4}\left\|x_i-x_j\right\|^3\right].\end{aligned}$$

With $(\mathcal{F}_A 1)$, this implies

$$\begin{aligned}&\frac{d}{dt}\left\|x_i-x_j\right\|\\ &\le \mathcal{D}(\boldsymbol{\Omega}) - \left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\frac{\left\|x_i-x_j\right\|}{2}\\ &\quad+\frac{1}{2}\left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\mathcal{D}(\mathcal{X})^2\cdot\frac{\left\|x_i-x_j\right\|}{2}+r_\kappa\left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\left[\mathcal{D}(\mathcal{X})+\frac{1}{4}\left\|x_i-x_j\right\|^2\right]\\ &\le \mathcal{D}(\boldsymbol{\Omega}) - \left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\frac{\left\|x_i-x_j\right\|}{2}+\frac{1-\delta}{2}\left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\left\|x_i-x_j\right\|\\ &\quad+\frac{3}{2}r_\kappa\left(\|\boldsymbol{\kappa}_i\|_1+\left\|\boldsymbol{\kappa}_j\right\|_1\right)\mathcal{D}(\mathcal{X}).\end{aligned}$$

□

Thanks to Lemma 3.2, we can study the local behavior of $\mathcal{D}(\mathcal{X}(t))$ in the following lemma.

Lemma 3.3 *Suppose that we can replace $\mathcal{X}^{in}$ in framework $(\mathcal{F}_A 1)$ - $(\mathcal{F}_A 4)$ with $\mathcal{X}(t_0)$ for $t_0 \ge 0$, and let $\mathcal{X}=\{x_i\}_{i\in\mathbb{N}}$ be a global solution to (3.1). Then there exists a positive constant t_δ such that*

$$\mathcal{D}_* = \frac{\mathcal{D}(\boldsymbol{\Omega})}{\|\boldsymbol{\kappa}\|_{-\infty,1}(\delta-3r_\kappa)} \le \mathcal{D}(\mathcal{X}(t)) \le \sqrt{2-2\delta}, \qquad \forall\, t\in[t_0, t_0+t_\delta]. \tag{3.5}$$

Proof We use Lemma 3.1 to get

$$\left\|x_i(t)-x_j(t)\right\| \le \left\|x_i(t_0)-x_j(t_0)\right\| + 2L_1(t-t_0), \quad \forall i,j\in\mathbb{N},$$

for

$$L_1 := \|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} + 2\|\boldsymbol{\kappa}\|_{\infty,1}.$$

This yields the Lipschitz continuity of the following functions near t_0:

$$t\mapsto \left\|x_i(t)-x_j(t)\right\| \quad \text{and} \quad t\mapsto \mathcal{D}(\mathcal{X}(t)).$$

Then we define t_δ by

$$t_\delta := \frac{1}{2L_1}\min\left\{\mathcal{D}(\mathcal{X}^{\mathrm{in}}) - \frac{\mathcal{D}(\boldsymbol{\Omega})}{\|\boldsymbol{\kappa}\|_{-\infty,1}(\delta-3r_\kappa)}, \sqrt{2-2\delta}-\mathcal{D}(\mathcal{X}^{\mathrm{in}})\right\} \tag{3.6}$$

so that the relation (3.5) holds (Fig. 1). □

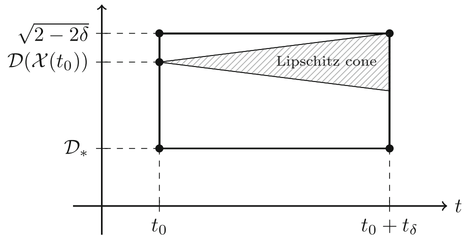

Fig. 1 Schematic diagram for t_δ

3.2 Emergence of the Quasi-Steady State

In this subsection, we consider the following setting:

$$\Omega_i \equiv 0, \quad i \in \mathbb{N} \quad \text{and} \quad \|\boldsymbol{\kappa}\|_{-\infty,1} = 0.$$

Under the above setting, we study the emergence of a "quasi-steady state", which is a non-constant state with a fixed diameter over time. Similar to authors' recent work [22] for the infinite Kuramoto model, we can observe a distinguished phenomenon compared to finite-dimensional particle models. Furthermore, it justifies that the condition

$$\|\boldsymbol{\kappa}\|_{-\infty,1} > 0$$

in Sect. 1 is necessary to guarantee exponential synchronization for a homogeneous ensemble. By the continuity of $t \mapsto \mathcal{D}(\mathcal{X}(t))$, we can see that the set

$$\mathcal{S} := \left\{ t \in [0, \infty) : \mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}^{\text{in}}) \right\} \tag{3.7}$$

is relatively closed subset in $\mathbb{R}_+$. Since the set $\mathcal{S}$ contains 0, it is nonempty. Furthermore, in the following lemma, we show that $\mathcal{S}$ is in fact relatively open.

Lemma 3.4 *Suppose that the framework* $(\mathcal{F}_A 1)$ - $(\mathcal{F}_A 4)$ *holds, and* $\mathcal{D}(\boldsymbol{\Omega}) = 0$, *and let* $\mathcal{X} = \{x_i\}_{i \in \mathbb{N}}$ *be a global solution to* (3.1). *Then there exists two positive constant* t_1 *such that*

$$\mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}^{in}), \quad t \in [0, t_1).$$

Proof By Lemma 3.2, if we can replace $\mathcal{X}^{\text{in}}$ in framework $(\mathcal{F}_A 1)$ - $(\mathcal{F}_A 4)$ with $\mathcal{X}(t_0)$ for $t_0 \ge 0$, we have

$$\left.\frac{d}{dt}\right|_{t=t_0} \|x_i(t) - x_j(t)\| \le \frac{1}{2} \left(\|\boldsymbol{\kappa}_i\|_1 + \|\boldsymbol{\kappa}_j\|_1 \right) \left(-\delta \|x_i(t_0) - x_j(t_0)\| + 3 r_\kappa \mathcal{D}(\mathcal{X}(t_0)) \right). \tag{3.8}$$

By $(\mathcal{F}_A 1)$, there exists $\varepsilon_1 > 0$ such that

$$\mathcal{D}(\mathcal{X}^{\text{in}}) \le \sqrt{2 - 2\delta} - \varepsilon_1.$$

Hence by Lemma 3.1,

$$\mathcal{D}(\mathcal{X}(t)) < \sqrt{2 - 2\delta}, \quad 0 \le t < \frac{\varepsilon_1}{4 \|\boldsymbol{\kappa}\|_{\infty,1}},$$

and (3.8) holds for $t \in \left[0, \frac{\varepsilon_1}{4\|\boldsymbol{\kappa}\|_{\infty,1}}\right)$. Furthermore, we can see that

$$\begin{aligned} & \frac{d}{dt}\bigg|_{t=t_0} \left\|x_i(t) - x_j(t)\right\| < 0 \\ \iff \quad & -\delta \left\|x_i(t_0) - x_j(t_0)\right\| + 3r_\kappa \mathcal{D}(\mathcal{X}(t_0)) < 0 \\ \iff \quad & \left\|x_i(t_0) - x_j(t_0)\right\| > \frac{3r_\kappa}{\delta}\mathcal{D}(\mathcal{X}(t_0)). \end{aligned} \tag{3.9}$$

By using two constants t_1 and ε defined by

$$t_1 := \left(4\|\boldsymbol{\kappa}\|_{\infty,1}\left(2 + \frac{3r_\kappa}{\delta}\right)\right)^{-1}\left(1 - \frac{3r_\kappa}{\delta}\right)\mathcal{D}(\mathcal{X}^{\text{in}}),$$

$$\varepsilon := 4\|\boldsymbol{\kappa}\|_{\infty,1} t_1 = \left(2 + \frac{3r_\kappa}{\delta}\right)^{-1}\left(1 - \frac{3r_\kappa}{\delta}\right)\mathcal{D}(\mathcal{X}^{\text{in}}),$$

we can show that for each $(i, j) \in \mathbb{N} \times \mathbb{N}$,

$$\left\|x_i(t) - x_j(t)\right\| \le \mathcal{D}(\mathcal{X}^{\text{in}}), \quad t \in [0, t_1).$$

- Case A: Let $(i, j) \in \mathbb{N} \times \mathbb{N}$ be the pair of indexes such that

$$\mathcal{D}(\mathcal{X}^{\text{in}}) - \varepsilon \le \left\|x_i^{\text{in}} - x_j^{\text{in}}\right\|,$$

By Lemma 2.2, we have

$$\mathcal{D}(\mathcal{X}^{\text{in}}) - 4\|\boldsymbol{\kappa}\|_{\infty,1} t \le \mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}^{\text{in}}) + 4\|\boldsymbol{\kappa}\|_{\infty,1} t. \tag{3.10}$$

and

$$\mathcal{D}(\mathcal{X}^{\text{in}}) - \varepsilon - 4\|\boldsymbol{\kappa}\|_{\infty,1} t \le \left\|x_i(t) - x_j(t)\right\| \le \mathcal{D}(\mathcal{X}^{\text{in}}) + 4\|\boldsymbol{\kappa}\|_{\infty,1} t, \tag{3.11}$$

From (3.10), we also have

$$\frac{3r_\kappa}{\delta}\mathcal{D}(\mathcal{X}(t)) \le \frac{3r_\kappa}{\delta}\mathcal{D}(\mathcal{X}^{\text{in}}) + \frac{12r_\kappa}{\delta}\|\boldsymbol{\kappa}\|_{\infty,1} t. \tag{3.12}$$

On the other hand, we can observe that the following relation

$$\begin{aligned} & \frac{3r_\kappa}{\delta}\mathcal{D}(\mathcal{X}^{\text{in}}) + \frac{12r_\kappa}{\delta}\|\boldsymbol{\kappa}\|_{\infty,1} t \le \mathcal{D}(\mathcal{X}^{\text{in}}) - \varepsilon - 4\|\boldsymbol{\kappa}\|_{\infty,1} t \\ \iff \quad & 4\|\boldsymbol{\kappa}\|_{\infty,1}\left(1 + \frac{3r_\kappa}{\delta}\right) t \le \left(1 - \frac{3r_\kappa}{\delta}\right)\mathcal{D}(\mathcal{X}^{\text{in}}) - \varepsilon \end{aligned}$$

holds for $t \le t_1$. Hence we can combine (3.11) and (3.12) to conclude that

$$\frac{3r_\kappa}{\delta}\mathcal{D}(\mathcal{X}(t)) \le \left\|x_i(t) - x_j(t)\right\|, \quad t \in [0, t_1).$$

This and (3.9) imply

$$\frac{d}{dt}\left\|x_i(t) - x_j(t)\right\| < 0, \quad t \in [0, t_1).$$

- Case B: Let $(i, j) \in \mathbb{N} \times \mathbb{N}$ be the pair of indexes such that

$$\mathcal{D}(\mathcal{X}^{\text{in}}) - \varepsilon > \left\|x_i^{\text{in}} - x_j^{\text{in}}\right\|.$$

In this case, Lemma 2.2 implies that

$$\left\|x_i(t)-x_j(t)\right\| \le \left\|x_i^{\text{in}}-x_j^{\text{in}}\right\| - \varepsilon + 4\,\|\boldsymbol{\kappa}\|_{\infty,1}\,t \le \left\|x_i^{\text{in}}-x_j^{\text{in}}\right\|, \qquad t\in[0,t_1).$$

□

Remark 3.2 Thanks to the result of Lemma 3.4, the set $\mathcal{S}$ in (3.7) is open, and we can prove that the diameter $\mathcal{D}(\mathcal{X}(t))$ is globally non-increasing.

Proposition 3.1 *Suppose that the framework* $(\mathcal{F}_A1)$ - $(\mathcal{F}_A4)$ *holds, and* $\mathcal{D}(\boldsymbol{\Omega})=0$*, and let* $\mathcal{X}=\{x_i\}_{i\in\mathbb{N}}$ *be a global solution to* (3.1)*. Then we have*

$$\mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}^{in}), \qquad t\in[0,t_1), \quad \forall\, t\in[0,\infty).$$

Proof Thanks to Lemma 3.5 and continuity of the map $t\to\mathcal{D}(\mathcal{X}(t))$, the set $\mathcal{S}$ in (3.7) is a nonempty relatively open and closed subset of $\mathbb{R}_+$. Hence, we have

$$\mathcal{S}=[0,\infty).$$

□

Finally, we are ready to show the existence of quasi-steady state. More precisely, for some well-prepared initial data, we have a non-constant state with a fixed diameter.

Proposition 3.2 *Suppose that the framework* $(\mathcal{F}_A1)$ - $(\mathcal{F}_A4)$ *holds, and* $\mathcal{D}(\boldsymbol{\Omega})=0$*, and let* $\mathcal{X}=\{x_i\}_{i\in\mathbb{N}}$ *be a global solution to* (3.1)*. If there exists two non-overlapping increasing sequence* $\{i_n\}_{n\in\mathbb{N}}$ *and* $\{j_n\}_{n\in\mathbb{N}}$ *of* $\mathbb{N}$ *such that*

$$\lim_{n\to\infty}\left\|\boldsymbol{\kappa}_{i_n}\right\|_1=0, \quad \lim_{n\to\infty}\left\|\boldsymbol{\kappa}_{j_n}\right\|_1=0, \quad \lim_{n\to\infty}\left\|x_{i_n}^{in}-x_{j_n}^{in}\right\|=\mathcal{D}(\mathcal{X}^{in}).$$

Then

$$\mathcal{D}(\mathcal{X}(t))=\mathcal{D}(\mathcal{X}^{in}), \qquad t\ge 0.$$

Proof For each $i, j\in\mathbb{N}$, we use

$$\begin{aligned}\left|\frac{d}{dt}\left\|x_i(t)-x_j(t)\right\|\right| &= \left|\frac{d}{dt}\sqrt{\left\|x_i(t)-x_j(t)\right\|^2}\right| \\ &= \left|\frac{2\left\langle x_i(t)-x_j(t), \dot{x}_i(t)-\dot{x}_j(t)\right\rangle}{2\sqrt{\left\|x_i(t)-x_j(t)\right\|^2}}\right| \le \left\|\dot{x}_i(t)-\dot{x}_j(t)\right\|\end{aligned}$$

and

$$\left\|\dot{x}_i(t)-\dot{x}_j(t)\right\| \le \|\dot{x}_i(t)\| + \left\|\dot{x}_j(t)\right\| \le 2\,\|\boldsymbol{\kappa}_i\|_1 + 2\left\|\boldsymbol{\kappa}_j\right\|_1$$

to conclude that

$$\left|\frac{d}{dt}\left\|x_i(t)-x_j(t)\right\|\right| \le 2\,\|\boldsymbol{\kappa}_i\|_1 + 2\left\|\boldsymbol{\kappa}_j\right\|_1.$$

Hence, for each $t>0$, we have

$$\left\|x_{i_n}(t)-x_{j_n}(t)\right\| \ge \left\|x_{i_n}^{\text{in}}-x_{j_n}^{\text{in}}\right\| - \left(2\left\|\boldsymbol{\kappa}_{i_n}\right\|_1 + 2\left\|\boldsymbol{\kappa}_{j_n}\right\|_1\right)t.$$

Now, we take the limit $n\to\infty$ to get

$$\mathcal{D}(\mathcal{X}(t)) \ge \lim_{n\to\infty}\left\|x_{i_n}(t)-x_{j_n}(t)\right\| \ge \mathcal{D}(\mathcal{X}^{\text{in}}).$$

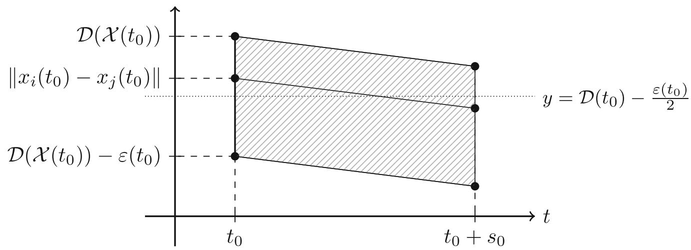

Fig. 2 Upper bound of $\|x_i - x_j\|$ near $\mathcal{D}(\mathcal{X}(t_0))$

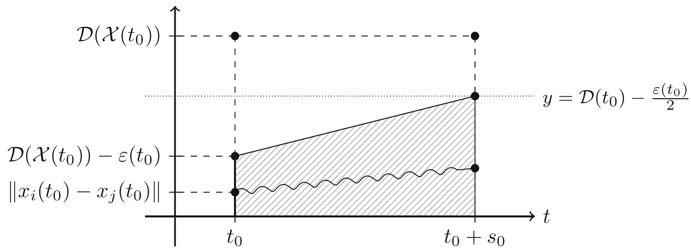

Fig. 3 Upper bound of $\|x_i - x_j\|$ far from $\mathcal{D}(\mathcal{X}(t_0))$

By Proposition 3.1, we have

$$\mathcal{D}(\mathcal{X}(t)) = \mathcal{D}(\mathcal{X}^{\text{in}}), \quad t \geq 0.$$

□

3.3 Local Behavior of the Relative Distances

In this subsection, we study the local behavior of the relative distances in the time interval near $t = t_0$ appearing in the previous subsection (see Lemma 3.3.). To set up the stage, we first introduce an auxiliary function $\varepsilon(t)$ to be used in the sequel (see (3.16) for motivation):

$$\varepsilon(t) := \frac{1}{2\delta}\left((\delta - 3r_\kappa)\,\mathcal{D}(\mathcal{X}(t)) - \frac{\mathcal{D}\,(\boldsymbol{\Omega})}{\|\boldsymbol{\kappa}\|_{-\infty,1}}\right) > 0. \tag{3.13}$$

Then for each $t \in [t_0, t_0 + t_\delta]$, the positivity of $\varepsilon(t_1)$ is guaranteed by (3.5). With this $\varepsilon(t_0)$, we can also set s_0 as follows:

$$s_0 := \min\left\{t_\delta,\ \frac{\varepsilon(t_0)}{4L_1},\ \frac{1}{2\delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}},\ \frac{\delta\varepsilon(t_0)}{2(\delta - 3r_\kappa)L_1}\right\}. \tag{3.14}$$

In what follows, we find that a local upper bound of $\|x_i(t) - x_j(t)\|$ depends on whether the $\|x_i(t) - x_j(t)\|$ is close or far from $\mathcal{D}(\mathcal{X}(t))$ for t in the time interval $[t_0, t_0 + s_0]$ (see Figs. 2 and 3).

More precisely, we claim the following two assertions in Proposition 3.3:

- $*$ If the distance $\|x_i(t) - x_j(t)\|$ is close to $\mathcal{D}(\mathcal{X}(t))$ for $t \in [t_0, t_0 + s_0]$, it is in decreasing mode for $t \in [t_0, t_0 + s_0]$.

* If the distance $\|x_i(t) - x_j(t)\|$ is far away from $\mathcal{D}(\mathcal{X}(t))$ for $t \in [t_0, t_0 + s_0]$, it lies in some Lipschitz cone for $t \in [t_0, t_0 + s_0]$.

Lemma 3.5 *Suppose that we can replace $\mathcal{X}^{in}$ in framework $(\mathcal{F}_A 1)$ - $(\mathcal{F}_A 4)$ with $\mathcal{X}(t_0)$ for $t_0 \geq 0$, and let $\mathcal{X} = \{x_i\}_{i \in \mathbb{N}}$ be a global solution to (3.1). Then, the following assertion holds. If there exists a time interval $[t_1, t_2] \subset [t_0, t_0 + s_0]$ such that*

$$\mathcal{D}(\mathcal{X}(t)) - \varepsilon(t) < \left\|x_i(t) - x_j(t)\right\|, \quad t \in [t_1, t_2], \tag{3.15}$$

then we have

$$\left\|x_i(t) - x_j(t)\right\| \leq \left\|x_i(t_1) - x_j(t_1)\right\| - \delta \left\|\boldsymbol{\kappa}\right\|_{-\infty,1} \int_{t_1}^{t} \varepsilon(s) ds, \quad t \in [t_1, t_2].$$

Proof By Lemma 3.2, (3.15) and (3.13), for $t \in [t_1, t_2]$, we have

$$\begin{aligned}
&\frac{d}{dt}\left\|x_i(t) - x_j(t)\right\| \\
&\qquad \leq \mathcal{D}(\boldsymbol{\Omega}) + \frac{1}{2}\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left(-\delta\left\|x_i(t) - x_j(t)\right\| + 3r_\kappa \mathcal{D}(\mathcal{X}(t))\right) \\
&\qquad \leq \mathcal{D}(\boldsymbol{\Omega}) + \frac{1}{2}\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left(-(\delta - 3r_\kappa)\mathcal{D}(\mathcal{X}(t)) + \delta\varepsilon(t)\right) \\
&\qquad \leq \frac{1}{2}\left(\mathcal{D}(\boldsymbol{\Omega}) - \frac{1}{2}\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)(\delta - 3r_\kappa)\mathcal{D}(\mathcal{X}(t))\right) \\
&\qquad \leq \frac{1}{2}\left(\mathcal{D}(\boldsymbol{\Omega}) - \|\boldsymbol{\kappa}\|_{-\infty,1}(\delta - 3r_\kappa)\mathcal{D}(\mathcal{X}(t))\right) \\
&\quad = -\delta\|\boldsymbol{\kappa}\|_{-\infty,1}\varepsilon(t).
\end{aligned} \tag{3.16}$$

This implies

$$\left\|x_i(t) - x_j(t)\right\| \leq \left\|x_i(t_1) - x_j(t_1)\right\| - \delta \left\|\boldsymbol{\kappa}\right\|_{-\infty,1} \int_{t_1}^{t} \varepsilon(s) ds, \quad t_1 \leq t \leq t_2.$$

□

In the next lemma, we show that the diameter is nonincreasing in the time interval $[t_0, t_0 + s_0]$.

Lemma 3.6 *Suppose that we can replace $\mathcal{X}^{in}$ in framework $(\mathcal{F}_A 1)$ - $(\mathcal{F}_A 4)$ with $\mathcal{X}(t_0)$ for $t_0 \geq 0$, and let $\mathcal{X} = \{x_i\}_{i \in \mathbb{N}}$ be a global solution to (3.1). Then there exist positive constants $\varepsilon(t_0)$, t_δ and L_1 such that $\mathcal{D}(\mathcal{X}(t))$ is nonincreasing for $t \in [t_0, t_0 + s_0]$.*

Proof Let $\mathcal{C}_{ij}(t)$ be the condition depending on $\mathcal{X}(t) = \{x_i(t)\}_{i \in \mathbb{N}}$:

$$\mathcal{C}_{ij}(t) \text{ holds} \quad \Longleftrightarrow \quad \mathcal{D}(\mathcal{X}(t)) - \varepsilon(t) < \left\|x_i(t) - x_j(t)\right\|.$$

For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, there are three cases:

$$\begin{cases}
\text{Case A: } \mathcal{C}_{ij}(t) \text{ holds for all } t \in [t_0, t_0 + s_0]. \\
\text{Case B: } \mathcal{C}_{ij}(t) \text{ holds for } t = t_0, \text{ but there exists a } t_1 \in [t_0, t_0 + s_0] \text{ such that} \\
\qquad\qquad \mathcal{C}_{ij}(t_1) \text{ not holds.} \\
\text{Case C: } \mathcal{C}_{ij}(t_0) \text{ does not hold.}
\end{cases} \tag{3.17}$$

In what follows, we show that $\left\|x_i(t) - x_j(t)\right\|$ is decreasing for Case A. On the other hands, we show that $\left\|x_i(t) - x_j(t)\right\|$ cannot exceed $\mathcal{D}(\mathcal{X}(t_0)) - \frac{\varepsilon(t_0)}{2}$ for Case B and Case C. Finally, by combining Case A - Case C, we conclude that $\mathcal{D}(\mathcal{X}(t))$ is nonincreasing for $t \in [t_0, t_0+s_0]$.

• Case A: Note that by (3.15), we have

$$\frac{d}{dt}\left\|x_i(t)-x_j(t)\right\| \leq -\delta \left\|\boldsymbol{\kappa}\right\|_{-\infty,1} \varepsilon(t) < 0, \quad t \in [t_0, t_0+s_0]. \tag{3.18}$$

This implies that $\left\|x_i(t)-x_j(t)\right\|$ is decreasing for $[t_0, t_0+s_0]$.

• Case B: We define the first entrance time $t_{i,j}$ such that

$$t_{i,j} := \inf\left\{t \in [t_0, t_0+s_0] \,:\, \mathcal{D}(\mathcal{X}(t)) - \varepsilon(t) \geq \left\|x_i(t)-x_j(t)\right\|\right\}. \tag{3.19}$$

Then we use Lemma 3.1 and (3.19) to get

$$\begin{aligned}\left\|x_i(t)-x_j(t)\right\| &\leq \left\|x_i(t_{ij})-x_j(t_{ij})\right\| + 2L_1(t-t_{ij})\\ &\leq \mathcal{D}(\mathcal{X}(t_{ij})) - \varepsilon(t_{ij}) + 2L_1(t-t_{ij})\\ &\leq \frac{1}{2\delta}\left(\delta+3r_\kappa\right)\mathcal{D}(\mathcal{X}(t_{ij})) + \frac{1}{2\delta}\frac{\mathcal{D}\left(\boldsymbol{\Omega}\right)}{\left\|\boldsymbol{\kappa}\right\|_{-\infty,1}} + 2L_1(t-t_{ij}).\end{aligned} \tag{3.20}$$

Next, we claim that the right-hand side of (3.20) is smaller than $\mathcal{D}(\mathcal{X}(t_0))$. This can be seen as follows:

$$\begin{aligned}&\mathcal{D}(\mathcal{X}(t_0)) - \frac{1}{2\delta}\left(\delta+3r_\kappa\right)\mathcal{D}(\mathcal{X}(t_{ij})) - \frac{1}{2\delta}\frac{\mathcal{D}\left(\boldsymbol{\Omega}\right)}{\left\|\boldsymbol{\kappa}\right\|_{-\infty,1}} - 2L_1(t-t_{ij})\\ &= \frac{\delta+3r_\kappa}{2\delta}\left(\mathcal{D}(\mathcal{X}(t_0)) - \mathcal{D}(\mathcal{X}(t_{ij}))\right) + \frac{\delta-3r_\kappa}{2\delta}\mathcal{D}(\mathcal{X}(t_0)) - \frac{1}{2\delta}\frac{\mathcal{D}\left(\boldsymbol{\Omega}\right)}{\left\|\boldsymbol{\kappa}\right\|_{-\infty,1}} - 2L_1(t-t_{ij})\\ &\geq \frac{\delta+3r_\kappa}{2\delta}\cdot 2L_1\left(t_0-t_{ij}\right) + \varepsilon(t_0) + 2L_1(t_{ij}-t) \geq \varepsilon(t_0) - 2L_1 s_0 \geq \frac{\varepsilon(t_0)}{2},\end{aligned}$$

where we used Lemma 3.1 in the first inequality.

• Case C: For (i, j) such that

$$\mathcal{D}(\mathcal{X}(t_0)) - \varepsilon(t_0) \geq \left\|x_i(t_0)-x_j(t_0)\right\|,$$

we use Lemma 3.1, the above inequality and (3.14) to estimate

$$\left\|x_i(t)-x_j(t)\right\| \leq \mathcal{D}(\mathcal{X}(t_0)) - \varepsilon(t_0) + 2L_1(t-t_0) \leq \mathcal{D}(\mathcal{X}(t_0)) - \frac{\varepsilon(t_0)}{2}.$$

Now we combine Case A - Case C to derive the local non-increasing property of the diameter. To see this, let $t \in (t_0, t_0+s_0]$ and $\mathcal{P}$ be the set of pair (i, j) satisfying $\mathcal{C}_{ij}(t)$ for $t \in [t_0, t_0+s_0]$. Then we use (3.18) to see

$$\begin{aligned}\sup_{(i,j)\in\mathcal{P}}\left\|x_i(t)-x_j(t)\right\| &\leq \sup_{(i,j)\in\mathcal{P}}\left[\left\|x_i(t_0)-x_j(t_0)\right\| + \int_{t_0}^{t}\frac{d}{ds}\left\|x_i(s)-x_j(t_0)\right\|ds\right]\\ &\leq \sup_{(i,j)\in\mathcal{P}}\left[\left\|x_i(t_0)-x_j(t_0)\right\| + \int_{t_0}^{t} -\delta\left\|\boldsymbol{\kappa}\right\|_{-\infty,1}\varepsilon(s)ds\right]\\ &\leq \sup_{(i,j)\in\mathcal{P}}\left\|x_i(t_0)-x_j(t_0)\right\| \leq \mathcal{D}(t_0).\end{aligned} \tag{3.21}$$

On the other hand, for $(i, j) \in \mathcal{P}^c$, Case B and Case C imply

$$\sup_{(i,j)\in\mathcal{P}^c}\left\|x_i(t)-x_j(t)\right\| \leq \mathcal{D}(\mathcal{X}(t_0)) - \frac{\varepsilon_0}{2}. \tag{3.22}$$

By (3.21) and (3.22), we have

$$t_0 \le t \le t_0 + s_0 \quad \Longrightarrow \quad \mathcal{D}(\mathcal{X}(t_0)) \ge \mathcal{D}(\mathcal{X}(t)).$$

Furthermore, we perform similar procedure as above to see

$$t_0 \le t \le s \le t_0 + s_0 \quad \Longrightarrow \quad \mathcal{D}(\mathcal{X}(t)) \ge \mathcal{D}(\mathcal{X}(s)).$$

□

Now we are ready to quantify a decrement of $\mathcal{D}(\mathcal{X}(t))$ in the following proposition.

Proposition 3.3 *Suppose that we can replace $\mathcal{X}^{in}$ in framework $(\mathcal{F}_A 1)$ - $(\mathcal{F}_A 4)$ with $\mathcal{X}(t_0)$ for $t_0 \ge 0$, and let $\mathcal{X} = \{x_i\}_{i\in\mathbb{N}}$ be a global solution to (3.1). Then we have*

$$\mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}(t_0)) - \frac{\delta}{2}\,\|\boldsymbol{\kappa}\|_{-\infty,1}\,(t-t_0)\,\varepsilon(t_0), \quad t \in [t_0, t_0+s_0].$$

Proof Below, we use the same classification in (3.17) in Lemma 3.6.

For the pairs in Case B or Case C, we have

$$\left\|x_i(t) - x_j(t)\right\| \le \mathcal{D}(\mathcal{X}(t_0)) - \frac{\varepsilon(t_0)}{2}, \quad t \in [t_0, t_0+s_0]. \tag{3.23}$$

On the other hand, for the pairs in Case A, we have

$$\left\|x_i(t) - x_j(t)\right\| \le \mathcal{D}(\mathcal{X}(t_0)) - \delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}\int_{t_0}^{t}\varepsilon(s)ds, \quad t \in [t_0, t_0+s_0]. \tag{3.24}$$

By the definition of $\varepsilon(t)$ in (3.13), $\varepsilon(t)$ is non-increasing for $t \in [t_0, t_0+s_0]$, since $\varepsilon(t)$ is a linear function of $\mathcal{D}(\mathcal{X}(t))$ and $\mathcal{D}(\mathcal{X}(t))$ is non-increasing. Hence we have

$$\delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}\int_{t_0}^{t}\varepsilon(s)ds \le \delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}\,(t-t_0)\varepsilon(t_0) \le \delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}\,\varepsilon(t_0)s_0 \le \frac{\varepsilon(t_0)}{2}, \tag{3.25}$$

where we used the definition of s_0 in the last inequality [see (3.14)].

We combine (3.23)–(3.25) to have

$$\mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}(t_0)) - \delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}\int_{t_0}^{t}\varepsilon(s)ds, \quad t \in [t_0, t_0+s_0]. \tag{3.26}$$

Meanwhile, by nonincreasing property of $\varepsilon(t)$, we have

$$\int_{t_0}^{t}\varepsilon(s)ds \ge (t-t_0)\varepsilon(t). \tag{3.27}$$

Now, we use the defining relation of $\varepsilon(t)$ in (3.13) and the Lipschitz constant of $\mathcal{D}(\mathcal{X}(t))$ is $2L_1$ to find that

$$\begin{aligned}&\text{Lipschitz constant of } \varepsilon(t)\\ &\quad= \frac{\delta - 3r_\kappa}{2\delta}\cdot(\text{Lipschitz constant of } \mathcal{D}(\mathcal{X}(t))) = \frac{\delta - 3r_\kappa}{2\delta}\cdot 2L_1 = \frac{(\delta - 3r_\kappa)L_1}{\delta}.\end{aligned}$$

This implies

$$\varepsilon(t) \ge \varepsilon(t_0) - \frac{\delta - 3r_\kappa}{2\delta}\cdot 2L_1(t-t_0) \ge \varepsilon(t_0) - \frac{\delta - 3r_\kappa}{\delta}\cdot L_1\cdot s_0 \ge \frac{1}{2}\varepsilon(t_0). \tag{3.28}$$

Finally, we combine (3.26), (3.27) and (3.28) to get the desired estimate:

$$\mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}(t_0)) - \delta \|\boldsymbol{\kappa}\|_{-\infty,1} \int_{t_0}^{t} \varepsilon(s)ds \le \mathcal{D}(\mathcal{X}(t_0)) - \frac{\delta \|\boldsymbol{\kappa}\|_{-\infty,1}}{2}\varepsilon(t_0)(t-t_0).$$

□

3.4 Practical Synchronization

Now, we are ready to show "practical synchronization" of our model (3.1). Our result means that each oscillator x_i can be confined within a small region of $\mathbb{S}^{d-1}$ by increasing the coupling strength $\|\boldsymbol{\kappa}\|_{-\infty,1}$ in this subsection.

Theorem 3.1 *Suppose that the framework* $(\mathcal{F}_A1)$-$(\mathcal{F}_A4)$ *holds, and let* $\mathcal{X} = \{x_i\}_{i\in\mathbb{N}}$ *be a global solution to* (3.1)*. Then* $\mathcal{D}(\mathcal{X})$ *satisfies*

$$\limsup_{t\to\infty} \mathcal{D}(\mathcal{X}(t)) \le \frac{\mathcal{D}(\boldsymbol{\Omega})}{(\delta - 3r_\kappa)\|\boldsymbol{\kappa}\|_{-\infty,1}}.$$

Proof Note that our framework $(\mathcal{F}_A4)$ admits the existence of $\varepsilon_1 \ll 1$ such that

$$(\delta - 3r_\kappa)\mathcal{D}(\mathcal{X}^{\text{in}}) - \frac{\mathcal{D}(\boldsymbol{\Omega})}{\|\boldsymbol{\kappa}\|_{-\infty,1}} > \varepsilon_1.$$

For such $\varepsilon_1 > 0$, we define

$$\mathcal{T}_{\varepsilon_1} := \left\{ \tau \in [0,\infty) : \ (\delta - 3r_\kappa)\mathcal{D}(\mathcal{X}(t)) - \frac{\mathcal{D}(\boldsymbol{\Omega})}{\|\boldsymbol{\kappa}\|_{-\infty,1}} > \varepsilon_1, \quad \forall\, t \in [0,\tau) \right\},$$

and

$$\tilde{s}(\varepsilon_1) := \min\left\{ t_\delta, \ \frac{\varepsilon_1}{4L_1}, \ \frac{1}{2\delta\|\boldsymbol{\kappa}\|_{-\infty,1}}, \ \frac{\delta\varepsilon_1}{2(\delta-3r_\kappa)L_1} \right\}.$$

Here, the definition of $\tilde{s}$ is motivated by that of s_0 in (3.14). Then we have

$$0 \in \mathcal{T}_{\varepsilon_1} \quad \text{and} \quad \tilde{s}(\varepsilon(t_0)) = s_0.$$

Now, we claim that

$$\inf \mathcal{T}_{\varepsilon_1}^c < \infty. \tag{3.29}$$

Proof of (3.29)*:* By Lemma 3.6, we have

$$\begin{aligned} &\{t_0, t_0 + \tilde{s}(\varepsilon_1)\} \subset \mathcal{T}_{\varepsilon_1} \\ &\implies \mathcal{D}(\mathcal{X}(t_0 + \tilde{s}(\varepsilon_1))) \le \mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}(\mathcal{X}(t_0)), \quad t \in [t_0, t_0 + \tilde{s}(\varepsilon_1)] \\ &\implies \big[t_0, t_0 + \tilde{s}(\varepsilon_1)\big] \subset \mathcal{T}_{\varepsilon_1}. \end{aligned}$$

If we have $\{t_0 + n \cdot \tilde{s}(\varepsilon_1)\}_{n\in\mathbb{N}} \subset \mathcal{T}_{\varepsilon_1}$, then $\mathcal{T}_{\varepsilon_1} = [t_0, \infty)$ and

$$\mathcal{D}(\mathcal{X}(t_0 + (n+1)\cdot\tilde{s}(\varepsilon_1))) \le \mathcal{D}(\mathcal{X}(t_0 + n\cdot\tilde{s}(\varepsilon_1))) - \frac{1}{4}\|\boldsymbol{\kappa}\|_{-\infty,1}\tilde{s}(\varepsilon_1)\cdot\varepsilon_1, \quad n \ge 1.$$

This yields that

$$\mathcal{D}(\mathcal{X}(t_0 + n\cdot\tilde{s}(\varepsilon_1))) \le \mathcal{D}(\mathcal{X}(t_0)) - \left[\frac{1}{4}\|\boldsymbol{\kappa}\|_{-\infty,1}\tilde{s}(\varepsilon_1)\cdot\varepsilon_1\right]\cdot n \ \to\ -\infty \quad \text{as} \quad n \to \infty.$$

This contradicts to $\mathcal{D}(\mathcal{X}(t)) > 0$. Thus, we have (3.29). Now, we set

$$t_\infty := \inf \mathcal{T}_{\varepsilon_1}^c < \infty.$$

Note that t_∞ is the first departure time of the set $\mathcal{T}_{\varepsilon_1}$, and it should satisfy

$$\mathcal{D}(\mathcal{X}(t_\infty)) = \frac{\mathcal{D}(\mathbf{\Omega})}{(\delta - 3r_\kappa)\|\boldsymbol{\kappa}\|_{-\infty,1}} + \frac{\varepsilon_1}{\delta - 3r_\kappa}.$$

If there exists $t_1 \in (t_\infty, \infty)$ such that $t_1 \in \mathcal{T}_{\varepsilon_1}$, then by Lemma 3.3, the diameter function $\mathcal{D}(\mathcal{X}(t))$ decreases in the time interval $[t_\infty, t_\infty + \tilde{s}(\varepsilon_1)]$. Hence the intermediate value theorem provides the existence of $t_{\infty,1}$ such that

$$\mathcal{D}(\mathcal{X}(t_\infty)) = \mathcal{D}(\mathcal{X}(t_{\infty,1})), \quad t_{\infty,1} \in [t_\infty + \tilde{s}(\varepsilon_1),\ t_1].$$

We can continue this process to construct the sequence $\left\{t_{\infty,k}\right\}_{k\in\mathbb{N}}$ such that

$$\mathcal{D}(\mathcal{X}(t_\infty)) = \mathcal{D}(\mathcal{X}(t_{\infty,k})), \quad t_{\infty,k+1} \in [t_{\infty,k} + \tilde{s}(\varepsilon_1),\ t_1], \quad k \in \mathbb{N}.$$

This contradicts to the finiteness of t_1. Therefore such t_1 does not exist and

$$\limsup_{t\to\infty} \mathcal{D}(\mathcal{X}(t)) \le \frac{\mathcal{D}(\mathbf{\Omega})}{(\delta - 3r_\kappa)\|\boldsymbol{\kappa}\|_{-\infty,1}} + \frac{\varepsilon_1}{\delta - 3r_\kappa}$$

for arbitrarily small ε_1. Finally, we take $\varepsilon_1 \to 0$ to find the desired result. □

Remark 3.3 Note that our practical synchronization result can cover the case

$$\mathcal{D}(\mathcal{X}^{\text{in}}) \le \mathcal{D}_*.$$

If $\mathcal{X}(t)$ satisfies

$$\mathcal{D}(\mathcal{X}(t)) \le \mathcal{D}_* = \frac{\mathcal{O}(1)}{\|\boldsymbol{\kappa}\|_{-\infty,1}}, \quad t \ge 0,$$

then the oscillators $\{x_i\}_{i\in\mathbb{N}}$ are already confined in a small arc with diameter $\mathcal{D}_*$.

On the other hand, if there exists some $t_0 > 0$ such that

$$\mathcal{D}(\mathcal{X}(t_0)) > \mathcal{D}_*,$$

then by the Lipschitz continuity of $\mathcal{D}(\mathcal{X}(t))$, we can assume the existence of t_* such that

$$t_* := \inf\left\{t > 0\ \ \mathcal{D}(\mathcal{X}(t_0)) < \mathcal{D}(\mathcal{X}(t)) < \sqrt{2-2\delta}\right\}.$$

Then we have $\mathcal{D}(\mathcal{X}(t_*)) = \mathcal{D}(\mathcal{X}(t_0))$ and our Theorem 3.1 can control $\mathcal{X}(t)$ for $t > t_*$.

As a corollary, we have exponential synchronization for a homogeneous ISS ensemble.

Corollary 3.1 *Suppose that the framework* $(\mathcal{F}_A1)$ *-* $(\mathcal{F}_A4)$ *holds, and* $\mathcal{D}(\mathbf{\Omega}) = 0$*, and let* $\mathcal{X} = \{x_i\}_{i\in\mathbb{N}}$ *be a global solution to* (3.1)*. Then, one has asymptotic zero convergence:*

$$\lim_{t\to\infty} \mathcal{D}(\mathcal{X}(t)) = 0.$$

Proof We define two function $\varepsilon, \mathfrak{s} : \mathbb{R}_{\ge 0} \to \mathbb{R}$ by

$$\varepsilon(t) := \frac{1}{2\delta}(\delta - 3r_\kappa)\mathcal{D}(\mathcal{X}(t))$$

$$\mathfrak{s}(t) := \min\left\{\mathcal{D}(\mathcal{X}(t)),\ \sqrt{2-2\delta}-\mathcal{D}(\mathcal{X}(t)),\ \frac{\varepsilon(t)}{4L_1},\ \frac{1}{2\delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}},\ \frac{\delta\varepsilon(t)}{2(\delta-3r_\kappa)L_1}\right\}$$
$$= \min\left\{\frac{2\delta}{\delta-3r_\kappa}\varepsilon(t),\ \sqrt{2-2\delta}-\frac{\delta-3r_\kappa}{2\delta}\varepsilon(t),\ \frac{\varepsilon(t)}{4L_1},\ \frac{1}{2\delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}},\ \frac{\delta\varepsilon(t)}{2(\delta-3r_\kappa)L_1}\right\}.$$

For

$$C_1 := \min\left\{\frac{2\delta}{\delta-3r_\kappa},\ \frac{1}{4L_1},\ \frac{\delta}{2(\delta-3r_\kappa)L_1}\right\} \quad \text{and} \quad C_2 := \frac{1}{2\delta\,\|\boldsymbol{\kappa}\|_{-\infty,1}},$$

we have

$$\mathfrak{s}(t) = \min\left\{C_1\varepsilon(t),\ C_2,\ \sqrt{2-2\delta}-\frac{2\delta}{\delta-3r_\kappa}\varepsilon(t)\right\} =: \min\left\{\tilde{s}_1(t),\ \tilde{s}_2(t),\ \tilde{s}_3(t)\right\}.$$

We can see that $\mathfrak{s}(t)$ attains $\tilde{s}_2(t)$ or $\tilde{s}_3(t)$ implies that the diameter is sufficiently large. More precisely, one has

$$\begin{aligned}
&\tilde{s}_1(t) \le \tilde{s}_2(t) \Longleftrightarrow \mathcal{D}(\mathcal{X}(t)) \le \frac{2\delta C_2}{C_1(\delta-3r_\kappa)} =: \mathcal{D}_1,\\
&\tilde{s}_1(t) \le \tilde{s}_3(t) \Longleftrightarrow \mathcal{D}(\mathcal{X}(t)) \le \sqrt{2-2\delta}\cdot\left(\frac{C_1}{2\delta}(\delta-3r_\kappa)+1\right)^{-1} =: \mathcal{D}_2,\\
&\tilde{s}_2(t) \le \tilde{s}_3(t) \Longleftrightarrow \mathcal{D}(\mathcal{X}(t)) \le \sqrt{2-2\delta}-C_2 =: \mathcal{D}_3.
\end{aligned} \tag{3.30}$$

This yields

$$\begin{aligned}
\mathfrak{s}(t) = \tilde{s}_2(t) \quad &\Longrightarrow \quad \mathcal{D}(\mathcal{X}(t)) \ge \mathcal{D}_1,\\
\mathfrak{s}(t) = \tilde{s}_3(t) \quad &\Longrightarrow \quad \mathcal{D}(\mathcal{X}(t)) \ge \max\{\mathcal{D}_2, \mathcal{D}_3\}.
\end{aligned} \tag{3.31}$$

We divide the remaining proof into two steps. First we claim that the configuration $\mathcal{X}$ shrinks into an arc with diameter $\min\{\mathcal{D}_1, \mathcal{D}_2\}$ in finite time if initial diameter is greater than $\mathcal{D}_1$ or $\max\{\mathcal{D}_2, \mathcal{D}_3\}$. Next, we prove exponential decay of the diameter.

• Step A (Decay for large initial diameter): Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence defined by

$$t_{k+1} := t_k + \mathfrak{s}(t_k), \quad k \ge 0.$$

We first claim that assuming

$$\mathfrak{s}(t_k) = \tilde{s}_2(t_k), \quad k \ge 0 \quad \text{or} \quad \mathfrak{s}(t_k) = \tilde{s}_3(t_k), \quad k \ge 0, \tag{3.32}$$

will leads to contradiction.

⋄ Step A.1: Suppose that $(3.32)_1$. By Proposition 3.3, we have

$$\mathcal{D}(\mathcal{X}(t_{k+1})) \le \left(1-\frac{1}{4}\cdot\frac{\delta-3r_\kappa}{2\delta}\right)\mathcal{D}(\mathcal{X}(t_k)), \quad k \ge 0.$$

Hence the sequence $\{\mathcal{D}(\mathcal{X}(t_k))\}_{k\in\mathbb{N}}$ should decay exponentially, which contradicts to $(3.32)_1$.

⋄ Step A.2: Suppose that $(3.32)_2$. In this case, we apply Proposition 3.3 to get

$$\mathcal{D}(\mathcal{X}(t_{k+1})) \le \left(1-\frac{\|\boldsymbol{\kappa}\|_{-\infty,1}(\delta-3r_\kappa)}{4}\cdot\left(\sqrt{2-2\delta}-\mathcal{D}(t_k)\right)\right)\mathcal{D}(\mathcal{X}(t_k)), \quad k \ge 0, \tag{3.33}$$

and

$$\mathcal{D}(\mathcal{X}(t_{k+1})) \le \mathcal{D}(\mathcal{X}(t_k)), \quad k \ge 0. \tag{3.34}$$

Now we combine (3.33) and (3.34) to conclude

$$\mathcal{D}(\mathcal{X}(t_{k+1})) \le \left(1 - \frac{\|\boldsymbol{\kappa}\|_{-\infty,1}(\delta - 3r_\kappa)}{4} \cdot \left(\sqrt{2-2\delta} - \mathcal{D}(t_0)\right)\right)\mathcal{D}(\mathcal{X}(t_k)), \quad k \ge 0,$$

which gives a similar contradiction to Step A.1.

• Step B (Decay for small initial diameter): Suppose that for some $t_0 \ge 0$, the state diameter satisfies

$$\mathcal{D}(\mathcal{X}(t_0)) \le \min\{\mathcal{D}_1, \mathcal{D}_2\},$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined in (3.30). Then we combine Lemma 3.3 together with (3.13) to derive the existence of some positive constants $C_3(\boldsymbol{\kappa}), C_4(\boldsymbol{\kappa})$ such that

$$\mathcal{D}(\mathcal{X}(t)) \le (1 - C_3(t - t_0))\,\mathcal{D}(\mathcal{X}(t_0)), \quad t_0 \le t \le t_0 + C_4\mathcal{D}(\mathcal{X}(t_0)).$$

Define a sequence $\{t_k\}_{k\in\mathbb{N}}$ by the following recursive relation:

$$t_{k+1} := t_k + C_3\mathcal{D}(t_k).$$

Then we have

$$\begin{aligned} &\mathcal{D}(\mathcal{X}(t_{k+1})) \le (1 - C_3C_4\mathcal{D}(\mathcal{X}(t_k)))\,\mathcal{D}(\mathcal{X}(t_k)) \\ \Longrightarrow \quad &\frac{1}{\mathcal{D}(\mathcal{X}(t_{k+1}))} \ge \frac{1}{\mathcal{D}(\mathcal{X}(t_k))} + \frac{C_3C_4}{1 - C_3C_4\mathcal{D}(\mathcal{X}(t_k))} \ge \frac{1}{\mathcal{D}(\mathcal{X}(t_k))} + C_3C_4. \end{aligned}$$

By induction on k, we have

$$\mathcal{D}(\mathcal{X}(t_k)) \le \frac{1}{\frac{1}{\mathcal{D}(\mathcal{X}^{\mathrm{in}})} + k \cdot C_3C_4}, \quad t_k \lesssim \frac{1}{C_3}\log k.$$

This yields the exponential decay of $\mathcal{D}(\mathcal{X}(t))$. □

4 The Infinite LHS Model A

In this section, we study the emergent behaviors of the infinite Lohe Hermitian sphere model. The overall structure of this section is parallel to those given in Sect. 3, but the difference comes from extra perturbative terms included in the infinite LHS model. Hence, we propose a different framework $(\mathcal{F}_B)$ compared to $(\mathcal{F}_A)$ to control bad terms.

4.1 Preparatory Lemmas

We introduce a new Banach space:

$$(\ell_{\mathbb{C}}^{\infty,2}, \|\cdot\|_{\infty,2}) := \left\{\mathcal{Y} = \{y_i\}_{i\in\mathbb{N}} : y_i \in \mathbb{C}^d, \quad \|\mathcal{Y}\|_{\infty,2} := \sup_{i\in\mathbb{N}} \|y_i\| < \infty\right\}.$$

For each $i \in \mathbb{N}$, let $z_i(t) \in \mathbb{C}^d$ be the position of the i-th particle at time t.

Suppose that $\mathcal{Z}(t) = \{z_i(t)\}_{i\in\mathbb{N}}$ belongs to $\ell^{\infty,2}_{\mathbb{C}}$. Then the dynamics of $\mathcal{Z} := \{z_i\}_{i\in\mathbb{N}}$ is given by the LHS model:

$$\begin{cases} \dot{z}_i = \Omega_i z_i + \lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij}\Big(\langle z_i, z_i\rangle z_j - \langle z_j, z_i\rangle z_i \Big) \\ \qquad\qquad + \lambda_1 \sum_{j\in\mathbb{N}} \kappa_{ij}\Big(\langle z_i, z_j\rangle - \langle z_j, z_i\rangle \Big) z_i, \quad t > 0, \quad \forall\, i \in \mathbb{N}, \\ z_i(0) = z_i^{\mathrm{in}}. \end{cases} \tag{4.1}$$

For a homogeneous ensemble, we may set $\Omega_i \equiv 0$. Next, we state the second sufficient framework $(\mathcal{F}_B)$ compared with the sufficient framework $(\mathcal{F}_A)$ for the ISS model:

- $(\mathcal{F}_B 0)$: Nonnegative constants λ_0 and λ_1 are assumed to be proportional to each other:
$$\lambda_1 = r_1 \lambda_0 \quad \text{for} \quad 0 \le r_1 < 1.$$
- $(\mathcal{F}_B 1)$: There exists a $\delta \in (0, 1)$ such that
$$\mathcal{D}\left(\mathcal{Z}^{\mathrm{in}}\right) < \frac{1-\delta}{2}.$$
- $(\mathcal{F}_B 2)$: Then there exists $r_\kappa \in (0, 1)$ such that
$$\left\|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j\right\|_1 \le r_\kappa \left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right), \quad i, j \in \mathbb{N}, \quad 0 < \|\boldsymbol{\kappa}\|_{-\infty,1}, \quad 4\,(r_\kappa + r_1) < \delta. \tag{4.2}$$
- $(\mathcal{F}_B 3)$: The natural frequencies satisfy
$$\mathcal{D}(\boldsymbol{\Omega}) < \lambda_0 \|\boldsymbol{\kappa}\|_{-\infty,1} \left(\delta - 4\,(r_\kappa + r_1)\right) \mathcal{D}(\mathcal{Z}^{\mathrm{in}}).$$

Note that the framework $(\mathcal{F}_B)$ seems to be very restricted compared to the framework $(\mathcal{F}_A)$ for the ISS model. After we prove Lemma 4.1, we will identify which term in the LHS model prevents synchronization and explain how to deal with these "bad" terms.

Lemma 4.1 *Suppose the framework $(\mathcal{F}_B 0)$ - $(\mathcal{F}_B 3)$ holds, and let $\mathcal{Z} = \{z_i\}_{i\in\mathbb{N}}$ be a global solution to (4.1). Then $\left\|z_i^{in} - z_j^{in}\right\|$ and $\mathcal{D}\left(\mathcal{Z}^{in}\right)$ satisfy*

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \left\|z_i - z_j\right\| &\le \mathcal{D}\left(\boldsymbol{\Omega}\right) - \frac{1}{2}\lambda_0 \left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left(1 - 2\mathcal{D}\left(\mathcal{Z}^{in}\right)\right)\left\|z_i^{in} - z_j^{in}\right\| \\ &\quad + 2\lambda_0 \left\|\kappa_i - \kappa_j\right\|_1 \mathcal{D}(\mathcal{Z}^{in}) + 2\lambda_1 \left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\mathcal{D}(\mathcal{Z}^{in}). \end{aligned}$$

Proof We write $\mathcal{Z}^{\mathrm{in}} = \left\{z_i^{\mathrm{in}}\right\}_{i\in\mathbb{N}}$ by $\mathcal{Z} = \{z_i\}_{i\in\mathbb{N}}$ only in this proof. We use (4.1) to get

$$\begin{aligned} &\frac{d}{dt}\left\langle z_i - z_j, z_i - z_j\right\rangle \\ &\quad = \left\langle z_i - z_j, \Omega_i z_i - \Omega_j z_j\right\rangle + \lambda_0 \sum_{l\in\mathbb{N}} \left\langle z_i - z_j, \kappa_{il}\left(z_l - \langle z_l, z_i\rangle z_i\right) - \kappa_{jl}\left(z_l - \left\langle z_l, z_j\right\rangle z_j\right)\right\rangle \\ &\qquad + \lambda_1 \sum_{l\in\mathbb{N}} \left\langle z_i - z_j, \kappa_{il}\left(\langle z_i, z_l\rangle - \langle z_l, z_i\rangle\right) z_i - \kappa_{jl}\left(\left\langle z_j, z_l\right\rangle - \left\langle z_l, z_j\right\rangle\right) z_j\right\rangle + \text{c.c} \\ &\quad =: \mathcal{I}_{31} + \lambda_0 \mathcal{I}_{32} + \lambda_1 \mathcal{I}_{33} + \text{c.c}. \end{aligned}$$

Here c.c denotes the complex conjugates of the preceding terms.

Next, we deal with $\mathcal{I}_{31}, \mathcal{I}_{32}$ and $\mathcal{I}_{33}$ with their complex conjugates, respectively.

• Step A (Bound for $\mathcal{I}_{31}$ + c.c): By direct calculation,

$$\begin{aligned}\mathcal{I}_{31} + \text{c.c} &= \langle z_i - z_j, \Omega_i z_i - \Omega_i z_j \rangle + \langle z_i - z_j, \Omega_i z_j - \Omega_j z_j \rangle + \text{c.c} \\ &= \langle z_i - z_j, \Omega_i (z_i - z_j) \rangle + \langle z_i - z_j, (\Omega_i - \Omega_j) z_j \rangle + \text{c.c} \\ &= \langle z_i - z_j, \Omega_i (z_i - z_j) \rangle + \langle \Omega_i (z_i - z_j), z_i - z_j \rangle \\ &\quad + \langle z_i - z_j, (\Omega_i - \Omega_j) z_j \rangle + \langle (\Omega_i - \Omega_j) z_j, z_i - z_j \rangle \\ &= 0 + 2\mathfrak{Re} \langle z_i - z_j, (\Omega_i - \Omega_j) z_j \rangle \\ &\leq 2\mathcal{D}(\mathbf{\Omega}) \|z_i - z_j\|,\end{aligned}$$

where $\mathfrak{Re}(z)$ denotes the real part of the complex number z.

• Step B (Bound for $\mathcal{I}_{32}$ + c.c): We divide $\mathcal{I}_{32}$ into two terms by

$$\begin{aligned}\mathcal{I}_{32} + \text{c.c} &= \sum_{l \in \mathbb{N}} \langle z_i - z_j, \kappa_{il} (z_l - \langle z_l, z_i \rangle z_i) - \kappa_{jl} (z_l - \langle z_l, z_j \rangle z_j) \rangle + \text{c.c} \\ &= \sum_{l \in \mathbb{N}} \left[\kappa_{il} \langle z_i - z_j, z_l - \langle z_l, z_i \rangle z_i \rangle - \kappa_{jl} \langle z_i - z_j, z_l - \langle z_l, z_j \rangle z_j \rangle \right] + \text{c.c} \\ &= \frac{1}{2} \sum_{l \in \mathbb{N}} (\kappa_{il} + \kappa_{jl}) \left[\langle z_i - z_j, z_l - \langle z_l, z_i \rangle z_i \rangle - \langle z_i - z_j, z_l - \langle z_l, z_j \rangle z_j \rangle \right] \\ &\quad + \frac{1}{2} \sum_{l \in \mathbb{N}} (\kappa_{il} - \kappa_{jl}) \left[\langle z_i - z_j, z_l - \langle z_l, z_i \rangle z_i \rangle + \langle z_i - z_j, z_l - \langle z_l, z_j \rangle z_j \rangle \right] + \text{c.c} \\ &=: \mathcal{I}_{321} + \mathcal{I}_{322} + \text{c.c.}\end{aligned}$$

Below, we estimate $\mathcal{I}_{321}$ + c.c and $\mathcal{I}_{322}$ + c.c separately.

◇ Step B.1 (Bound of $\mathcal{I}_{321}$ + c.c): We rewrite $\mathcal{I}_{321}$ as

$$\mathcal{I}_{321} = \frac{1}{2} \sum_{l \in \mathbb{N}} (\kappa_{il} + \kappa_{jl}) \left[\langle z_i - z_j, z_l - \langle z_l, z_i \rangle z_i \rangle - \langle z_i - z_j, z_l - \langle z_l, z_j \rangle z_j \rangle + \text{c.c} \right]. \tag{4.3}$$

Then, we can reform the summand in $\mathcal{I}_{321}$ as

$$\begin{aligned}&\langle z_i - z_j, z_l - \langle z_l, z_i \rangle z_i \rangle - \langle z_i - z_j, z_l - \langle z_l, z_j \rangle z_j \rangle + \text{c.c} \\ &\quad = \langle z_i - z_j, \langle z_l, z_j \rangle z_j - \langle z_l, z_i \rangle z_i \rangle + \text{c.c} \\ &\quad = \langle z_i - z_j, \langle z_l, z_j \rangle (z_j - z_i) \rangle + \langle z_i - z_j, \langle z_l, z_j \rangle z_i - \langle z_l, z_i \rangle z_i \rangle + \text{c.c} \\ &\quad = -\langle z_l, z_j \rangle \|z_i - z_j\|^2 + \langle z_l, z_j - z_i \rangle \langle z_i - z_j, z_i \rangle + \text{c.c} \\ &\quad = -\|z_i - z_j\|^2 + \langle z_j - z_l, z_j \rangle \|z_i - z_j\|^2 + \langle z_l - z_i, z_j - z_i \rangle \langle z_i - z_j, z_i \rangle \\ &\qquad + \langle z_i, z_j - z_i \rangle \langle z_i - z_j, z_i \rangle + \text{c.c} \\ &\quad \leq -2 \|z_i - z_j\|^2 + 2\mathcal{D}(\mathcal{Z}) \|z_i - z_j\|^2 + 2\mathcal{D}(\mathcal{Z}) \|z_i - z_j\|^2 + 0.\end{aligned}$$

This gives

$$\mathcal{I}_{321} \leq -\left(\|\boldsymbol{\kappa}_i\|_1 + \|\boldsymbol{\kappa}_j\|_1 \right) (1 - 2\mathcal{D}(\mathcal{Z})) \|z_i - z_j\|^2. \tag{4.4}$$

◇ Step B.2 (Bound of $\mathcal{I}_{322}$ + c.c): Recall $\mathcal{I}_{322}$ + c.c is

$$\mathcal{I}_{322} = \frac{1}{2}\sum_{l\in\mathbb{N}} \left(\kappa_{il} - \kappa_{jl}\right)\left[\left\langle z_i - z_j, z_l - \langle z_l, z_i\rangle z_i\right\rangle + \left\langle z_i - z_j, z_l - \left\langle z_l, z_j\right\rangle z_j\right\rangle\right] + \text{c.c.}$$

Then, we use the inequality

$$\begin{aligned}
&\left|\left\langle z_i - z_j, z_l - \langle z_l, z_i\rangle z_i\right\rangle\right| \\
&\quad \le \left|\left\langle z_i - z_j, z_l - z_i\right\rangle\right| + \left|(1 - \langle z_l, z_i\rangle)\left\langle z_i - z_j, z_i\right\rangle\right| \\
&\quad = \left|\left\langle z_i - z_j, z_l - z_i\right\rangle\right| + \left|\langle z_l - z_i, z_i\rangle\left\langle z_i - z_j, z_i\right\rangle\right| \\
&\quad \le 2\left\|z_i - z_j\right\|\mathcal{D}(\mathcal{Z})
\end{aligned}$$

to estimate

$$\mathcal{I}_{322} \le \frac{1}{2}\sum_{l\in\mathbb{N}}\left|\kappa_{il} - \kappa_{jl}\right|\cdot 8\left\|z_i - z_j\right\|\mathcal{D}(\mathcal{Z}) = 4\left\|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j\right\|_1\left\|z_i - z_j\right\|\mathcal{D}(\mathcal{Z}). \tag{4.5}$$

Now we combine (4.4) and (4.5) to obtain

$$\mathcal{I}_{32} \le -\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)(1 - 2\mathcal{D}(\mathcal{Z}))\left\|z_i - z_j\right\|^2 + 4\left\|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j\right\|_1\left\|z_i - z_j\right\|\mathcal{D}(\mathcal{Z}).$$

• Step C (Bound of $\mathcal{I}_{33}$ + c.c): Note that the $\mathcal{I}_{33}$ + c.c term is given by

$$\mathcal{I}_{33} + \text{c.c} = \sum_{l\in\mathbb{N}}\left\langle z_i - z_j, \kappa_{il}\left(\langle z_i, z_l\rangle - \langle z_l, z_i\rangle\right)z_i - \kappa_{jl}\left(\left\langle z_j, z_l\right\rangle - \left\langle z_l, z_j\right\rangle\right)z_j\right\rangle + \text{c.c.}$$

Then, we use

$$\begin{aligned}
&\sum_{l\in\mathbb{N}}\left\langle z_i - z_j, \kappa_{il}\left(\langle z_i, z_l\rangle - \langle z_l, z_i\rangle\right)z_i\right\rangle \\
&\quad \le \sum_{l\in\mathbb{N}}\left\langle z_i - z_j, \kappa_{il}\left(\langle z_i, z_l - z_i\rangle + \langle z_i - z_l, z_i\rangle\right)z_i\right\rangle \\
&\quad \le \sum_{l\in\mathbb{N}}\left\|z_i - z_j\right\|\kappa_{il}\left(\|z_l - z_i\| + \|z_i - z_l\|\right) \\
&\quad \le 2\left\|z_i - z_j\right\|\|\boldsymbol{\kappa}_i\|_1\mathcal{D}(\mathcal{Z})
\end{aligned}$$

to get

$$\mathcal{I}_{33} + \text{c.c} \le 4\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left\|z_i - z_j\right\|\mathcal{D}(\mathcal{Z}).$$

• Step D (Bound of $\frac{d}{dt}\left\|z_i - z_j\right\|$): We combine all the estimates in Step A to Step C to find

$$\begin{aligned}
\frac{d}{dt}\left\langle z_i - z_j, z_i - z_j\right\rangle \le{}& 2\mathcal{D}(\boldsymbol{\Omega})\left\|z_i - z_j\right\| - \lambda_0\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)(1 - 2\mathcal{D}(\mathcal{Z}))\left\|z_i - z_j\right\|^2 \\
&+ 4\lambda_0\left\|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j\right\|_1\left\|z_i - z_j\right\|\mathcal{D}(\mathcal{Z}) \\
&+ 4\lambda_1\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left\|z_i - z_j\right\|\mathcal{D}(\mathcal{Z}).
\end{aligned}$$

This yields

$$\begin{aligned}
\frac{d}{dt}\left\|z_i - z_j\right\| \le{}& \mathcal{D}(\boldsymbol{\Omega}) - \frac{1}{2}\lambda_0\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)(1 - 2\mathcal{D}(\mathcal{Z}))\left\|z_i - z_j\right\| \\
&+ 2\lambda_0\left\|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j\right\|_1\mathcal{D}(\mathcal{Z}) + 2\lambda_1\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\mathcal{D}(\mathcal{Z}).
\end{aligned}$$

□

Remark 4.1 (i) Among $\mathcal{I}_{31}$, $\mathcal{I}_{321}$, $\mathcal{I}_{322}$ and $\mathcal{I}_{33}$, only $\mathcal{I}_{321}$ contains the term making $\frac{d}{dt}\left\|z_i - z_j\right\|$ be decreasing. Furthermore, we impose $(\mathcal{F}_B 2)$ to control $\mathcal{I}_{32}$ term.

(ii) For a finite ensemble, the authors in [23] derived the Kuramoto model with frustration which contains $\mathcal{I}_{33}$ only:

$$\begin{cases} \dot{\theta}_i = \dfrac{2\kappa_1}{N}\displaystyle\sum_{j=1}^{N} R_{ij}^{\text{in}} \sin\left(\theta_j - \theta_i + \alpha_{ji}\right), \quad t > 0, \\ \theta_i(0) = 0, \quad i \in [N]. \end{cases}$$

We cannot expect the term $\mathcal{I}_{33}$ can contribute to decrement of diameter. Hence we just supressed the effect of $\mathcal{I}_{33}$ term with $(\mathcal{F}_B 0)$ and $(\mathcal{F}_B 2)$.

Following the arguments in the proof of Lemma 4.1, we show that the diameter is decreasing for our configuration $\mathcal{Z}$ near $t = t_0$. In the sequel, we briefly sketch the proof of nonincreasing property of diameter. Let L_1 be a constant appearing in Lemma 2.2, which has the form

$$L_1 := \|\boldsymbol{\Omega}\|_{\infty,\text{op}} + 2\,\|\boldsymbol{\kappa}\|_{\infty,1}\,(\lambda_0 + \lambda_1).$$

We define t_δ, $\varepsilon(t)$, and s_0 motivated by (3.6), (3.13) and (3.14), respectively:

$$\begin{aligned} t_\delta &:= \frac{1}{2L_1}\min\left\{\mathcal{D}(\mathcal{X}^{\text{in}}) - \frac{\mathcal{D}(\boldsymbol{\Omega})}{\lambda_0\,\|\boldsymbol{\kappa}\|_{-\infty,1}\,(\delta - 4(r_\kappa + r_1))},\ \frac{1-\delta}{2} - \mathcal{D}(\mathcal{Z}^{\text{in}})\right\}, \\ \varepsilon(t) &:= \frac{1}{2}\cdot\frac{1}{\delta}\left((\delta - 4\,(r_\kappa + r_1))\,\mathcal{D}(\mathcal{Z}(t)) - \frac{\mathcal{D}(\boldsymbol{\Omega})}{\lambda_0\,\|\boldsymbol{\kappa}\|_{-\infty,1}}\right), \\ s_0 &:= \min\left\{t_\delta,\ \frac{\varepsilon(t_0)}{4L_1},\ \frac{\delta\varepsilon(t_0)}{2(\delta - 4\,(r_\kappa + r_1))L_1},\ \frac{1}{2\delta\lambda_0\,\|\boldsymbol{\kappa}\|_{-\infty,1}}\right\}. \end{aligned}$$

Lemma 4.2 *Suppose that we can replace $\mathcal{Z}^{in}$ in the framework $(\mathcal{F}_B 0)$ - $(\mathcal{F}_B 3)$ with $\mathcal{X}(t_0)$ for $t_0 \geq 0$, and let $\mathcal{Z} = \{z_i\}_{i\in\mathbb{N}}$ be a global solution to (4.1). Then we have*

$$\mathcal{D}(\mathcal{Z}(t)) \leq \mathcal{D}(\mathcal{Z}(t_0)) - \frac{\delta}{2}\lambda_0\,\|\boldsymbol{\kappa}\|_{-\infty,1}\,(t - t_0)\,\varepsilon(t_0), \quad t \in [t_0, t_0 + s_0].$$

Proof We estimate $\left\|z_i(t) - z_j(t)\right\|$ for two groups of oscillators.

- Case A: We choose (i, j) such that

$$\left\|z_i(t) - z_j(t)\right\| \geq \mathcal{D}(\mathcal{Z}(t)) - \varepsilon(t), \quad t \in [t_0, t_0 + s_0].$$

Then for such index pair (i, j), we have

$$\begin{aligned} &\frac{d}{dt}\left\|z_i(t) - z_j(t)\right\| \\ &\quad\leq \mathcal{D}(\boldsymbol{\Omega}) + \frac{1}{2}\lambda_0\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left(-\delta\left\|z_i(t) - z_j(t)\right\| + 4\,(r_\kappa + r_1)\,\mathcal{D}(\mathcal{Z}(t))\right) \\ &\quad\leq \mathcal{D}(\boldsymbol{\Omega}) + \frac{1}{2}\lambda_0\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left(-\,(\delta - 4\,(r_\kappa + r_1))\,\mathcal{D}(\mathcal{Z}(t_0)) + \delta\varepsilon(t)\right) \\ &\quad= \frac{1}{2}\left(\mathcal{D}(\boldsymbol{\Omega}) - \frac{1}{2}\lambda_0\left(\|\boldsymbol{\kappa}_i\|_1 + \left\|\boldsymbol{\kappa}_j\right\|_1\right)\left((\delta - 4\,(r_\kappa + r_1))\,\mathcal{D}(\mathcal{Z}(t))\right)\right) \end{aligned}$$

$$\leq -\delta\lambda_0 \, \|\boldsymbol{\kappa}\|_{-\infty,1} \, \varepsilon(t) < 0.$$

Hence, we have

$$\left\| z_i(t) - z_j(t) \right\| \leq \left\| z_i(t_0) - z_j(t_0) \right\| - \int_{t_0}^{t} \delta\lambda_0 \, \|\boldsymbol{\kappa}\|_{-\infty,1} \, \varepsilon(s) ds, \quad t \in [t_0, t_0 + t_\delta].$$

• Case B: Again we choose an index pair (i, j) such that

$$\left\| z_i(t_1) - z_j(t_1) \right\| \leq \mathcal{D}(\mathcal{Z}(t_1)) - \varepsilon(t_1) \ \text{ for some } \ t_1 \in [t_0, t_0 + s_0].$$

For such (i, j), we have

$$\left\| z_i(t) - z_j(t) \right\| \leq \mathcal{D}(\mathcal{Z}(t_0)) - \frac{\varepsilon(t_0)}{2}, \quad t \in [t_0, t_0 + s_0].$$

Finally, for $t \in [t_0, t_0 + s_0]$, we combine Case A and Case B to obtain

$$\begin{aligned} \left\| z_i(t) - z_j(t) \right\| &\leq \left\| z_i(t_0) - z_j(t_0) \right\| - \min \left(\frac{\varepsilon(t_0)}{2}, \delta\lambda_0 \, \|\boldsymbol{\kappa}\|_{-\infty,1} \int_{t_0}^{t} \varepsilon(s) ds \right) \\ &\leq \left\| z_i(t_0) - z_j(t_0) \right\| - \frac{\delta\lambda_0 \, \|\boldsymbol{\kappa}\|_{-\infty,1}}{2} \varepsilon(t_0) \, (t - t_0) \, . \end{aligned}$$

□

Now we are ready to provide our second main result in the next subsection.

4.2 Practical Synchronization

In previous subsection, we have studied several basic lemmas to be used in the following practical synchronization estimates.

Theorem 4.1 *Suppose that the framework* $(\mathcal{F}_B 0)$ - $(\mathcal{F}_B 3)$ *holds for* $t_0 \geq 0$, *and let* $\mathcal{Z} = \{z_i\}_{i \in \mathbb{N}}$ *be a global solution to* (4.1). *Then* $\mathcal{D}(\mathcal{Z}(t))$ *satisfies the following practical synchronization estimate:*

$$\limsup_{t \to \infty} \mathcal{D}(\mathcal{Z}(t_0)) \leq \frac{\mathcal{D}(\boldsymbol{\Omega})}{\lambda_0 \, \|\boldsymbol{\kappa}\|_{-\infty,1} \, (\delta - 4\,(r_\kappa + r_1))} = \mathcal{O}\left(\frac{1}{\|\boldsymbol{\kappa}\|_{-\infty,1}} \right).$$

Proof The proof is similar to the ISS model case (see the proof of Theorem 3.1). Here we need to define

$$\mathcal{T}_{\varepsilon_1} := \left\{ t \in [0, \infty) : (\delta - 4\,(r_\kappa + r_1))\, \mathcal{D}(\mathcal{Z}(t)) - \frac{\mathcal{D}(\boldsymbol{\Omega})}{\lambda_0 \, \|\boldsymbol{\kappa}\|_{-\infty,1}} \geq \varepsilon_1 \right\},$$

and our framework $(\mathcal{F}_B)$ allows the existence of $\varepsilon_1 \ll 1$ such that $\mathcal{T}_{\varepsilon_1} \ni 0$. By Lemma 4.2, we have

$$\{t_0, t_0 + \tilde{s}(\varepsilon_1)\} \in \mathcal{T}_{\varepsilon_1} \quad \Longrightarrow \quad [t_0, t_0 + \tilde{s}(\varepsilon_1)] \subset \mathcal{T}_{\varepsilon_1},$$

for

$$\tilde{s}(\varepsilon_1) := \min \left\{ t_\delta, \ \frac{\varepsilon_1}{4L_1}, \ \frac{\delta\varepsilon_1}{2(\delta - 4\,(r_\kappa + r_1))L_1}, \ \frac{1}{2\delta\lambda_0 \, \|\boldsymbol{\kappa}\|_{-\infty,1}} \right\}.$$

We can see

$$\limsup_{t\to\infty} \mathcal{D}(\mathcal{Z}(t)) \le \frac{\mathcal{D}(\boldsymbol{\Omega})}{\lambda_0 \|\boldsymbol{\kappa}\|_{-\infty,1}(\delta - 4(r_\kappa + r_1))} + \frac{\varepsilon_1}{\delta - 4(r_\kappa + r_1)}$$

by a similar method in the proof of Theorem 3.1. Finally, we take the limit $\varepsilon_1 \to 0$ to obtain the desired result. □

As on can see in the proof, we can apply the same steps with the proof of Theorem 3.1. Furthermore, the proof of LHS counterpart of Corollary 3.1 is the same. Hence, we can state the result without a detailed proof.

Corollary 4.1 *Suppose the framework* $(\mathcal{F}_B0)$ - $(\mathcal{F}_B3)$ *holds for* $t_0 \ge 0$, *and let* $\mathcal{Z} = \{z_i\}_{i\in\mathbb{N}}$ *be a global solution to* (4.1) *with* $\mathcal{D}(\boldsymbol{\Omega}) = 0$. *Then* $\mathcal{D}(\mathcal{Z}(t))$ *decays to zero exponentially fast.*

Proof Since the proof is almost the same as in the proof of Corollary 3.1, we omit its details. □

5 The Infinite LHS Model B

In this section, we study the emergent behavior of the infinite LHS model on some special network, namely, a "*sender network*" in which interaction capacities depend only on sender nodes.

5.1 Order Parameter and Collision Avoidance

Consider a network topology in which the interaction capacity κ_{ij} froms the j-th node to the i-th node is solely determined by the j-th sending node:

$$\kappa_{ij} = \kappa_j > 0, \quad i, j \in \mathbb{N} \quad \text{and} \quad \sum_{j\in\mathbb{N}} \kappa_j = \|\boldsymbol{\kappa}\|_1 < \infty. \tag{5.1}$$

Then, it is clear to see that this network satisfies the condition (4.2). Hence, the practical synchronization estimate in Theorem 4.1 can be applied for this special case. However, for this special case, we can infer more detailed asymptotic dynamics as can be see in the remaining parts of this section (see Theorem 5.1, Corollary 5.2 and Proposition 5.4). For given state $\{z_i\}$ and a sender network (κ_j), we define a complex order parameter z_c as a weighted sum of z_i:

$$z_c := \sum_{i\in\mathbb{N}} \kappa_i z_i. \tag{5.2}$$

Then, by (5.1), it is well-defined and the square of the modulus of z_c will play the key role in the asymptotic dynamics of the infinite LHS ensemble. For this type of network topology, we can rearrange the homogeneous LHS model as

$$\begin{cases} \dot{z}_i = \lambda_0 (\langle z_i, z_i\rangle z_c - \langle z_c, z_i\rangle z_i) + \lambda_1(\langle z_i, z_c\rangle - \langle z_c, z_i\rangle) z_i, \quad t \ge 0, \\ z_i(0) = z_i^{\text{in}}, \quad \|z_i^{\text{in}}\| = 1, \quad i \in \mathbb{N}. \end{cases} \tag{5.3}$$

For the simplicity of presentation, we set

$$\|\boldsymbol{\kappa}\|_1 = 1, \quad \lambda_0 + \lambda_1 = 1 \tag{5.4}$$

by rescaling time if necessary. We first introduce the basic properties of (5.3).

Lemma 5.1 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to* (5.3). *Then we have*

$$\text{(i)}\ \ \|z_c\| \le 1, \quad \|\dot{z}_i\| \le 2, \quad \|\dot{z}_c\| \le 2, \quad \|\ddot{z}_c\| \le 12.$$

$$\text{(ii)}\ \ \left|\frac{d}{dt}\langle z_i, z_j\rangle\right| \le 4, \quad \left|\frac{d}{dt}\langle z_i, z_c\rangle\right| \le 4, \quad \left|\frac{d^2}{dt^2}\langle z_c, z_c\rangle\right| \le 32.$$

Proof (i) For the first estimate, we use (5.2) and (5.4) to get

$$\|z_c\| = \left\|\sum_{i\in\mathbb{N}} \kappa_i z_i\right\| \le \sum_{i\in\mathbb{N}} \kappa_i \|z_i\| = \sum_{i\in\mathbb{N}} \kappa_i = 1. \tag{5.5}$$

Again, it follows from (5.5) and $(5.3)_1$ that

$$\begin{aligned}
\|\dot{z}_i\| &= \|\lambda_0 \left(\langle z_i, z_i\rangle z_c - \langle z_c, z_i\rangle z_i\right)\| + \lambda_1 \|(\langle z_i, z_c\rangle - \langle z_c, z_i\rangle) z_i\| \\
&\le \lambda_0 \left(\|z_c\| + \|z_c\| \cdot \|z_i\|^2\right) + \lambda_1 \left(\|z_c\|\,\|z_i\|^2 + \|z_c\|\,\|z_i\|^2\right) \\
&= 2(\lambda_0 + \lambda_1)\|z_c\| \le 2\|\boldsymbol{\kappa}\|_1 = 2.
\end{aligned} \tag{5.6}$$

Now, we use (5.5) and (5.6) to find

$$\begin{aligned}
&\|\dot{z}_c\| \le \sum_{j\in\mathbb{N}} \kappa_j \|\dot{z}_j\| \le 2\|\boldsymbol{\kappa}\|_1 = 2, \\
&\|\ddot{z}_c\| \le \sum_{j\in\mathbb{N}} \kappa_j \|\ddot{z}_j\| \le \lambda_0 \sum_{j\in\mathbb{N}} \kappa_j \left\|\left(\dot{z}_c - \langle \dot{z}_c, z_j\rangle z_j - \langle z_c, \dot{z}_j\rangle z_j - \langle z_c, z_j\rangle \dot{z}_j\right)\right\| \\
&\qquad + 2\cdot\lambda_1 \sum_{j\in\mathbb{N}} \kappa_j \left\|\langle \dot{z}_c, z_j\rangle z_j + \langle z_c, \dot{z}_j\rangle z_j + \langle z_c, z_j\rangle \dot{z}_j\right\| \\
&\quad \le \lambda_0 \sum_{j\in\mathbb{N}} \kappa_j \left(4\cdot 2(\lambda_0+\lambda_1)\right) + 2\cdot\lambda_1 \sum_{j\in\mathbb{N}} \kappa_j \left(3\cdot 2(\lambda_0+\lambda_1)\right) \\
&\quad = 12(\lambda_0+\lambda_1)^2 = 12.
\end{aligned}$$

(ii) We use the estimates in (i) to get the following set of estimates:

$$\begin{aligned}
\left|\frac{d}{dt}\langle z_i, z_j\rangle\right| &\le |\langle \dot{z}_i, z_j\rangle| + |\langle z_i, \dot{z}_j\rangle| \le \|\dot{z}_i\|\,\|z_j\| + \|z_i\|\,\|\dot{z}_j\| \le 4, \\
\left|\frac{d}{dt}\langle z_i, z_c\rangle\right| &\le |\langle \dot{z}_i, z_c\rangle| + |\langle z_i, \dot{z}_c\rangle| \le 4(\lambda_0+\lambda_1) = 4, \\
\left|\frac{d^2}{dt^2}\langle z_c, z_c\rangle\right| &\le 2|\langle \ddot{z}_c, z_c\rangle| + 2|\langle \dot{z}_c, \dot{z}_c\rangle| \le 2\cdot 12(\lambda_0+\lambda_1)^2 + 2(2(\lambda_0+\lambda_1))^2 = 32.
\end{aligned}$$

□

In the next lemma, we present the collision avoidance property for a solution to system (5.3).

Lemma 5.2 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to system* (5.3). *Then, for $(i, j) \in \mathbb{N}\times\mathbb{N}$, the following dichotomy holds.*

(i) *If $z_i^{in} \ne z_j^{in}$, then one has*

$$z_i(t) \ne z_j(t), \qquad t > 0.$$

(ii) *If* $z_i^{in} = z_j^{in}$, *then one has*

$$z_i(t) \equiv z_j(t), \qquad t > 0.$$

Proof Suppose that z_i and z_j collides at some positive time t_*. Now we consider a temporal set and its infimum.

$$t_0 := \inf\left\{t > 0 \;:\; z_i(t) = z_j(t)\right\} < \infty. \tag{5.7}$$

By (5.3), we have

$$\frac{d}{dt} z_i(t_0) = \frac{d}{dt} z_j(t_0).$$

Inductively one can see that

$$\left.\frac{d^n}{dt^n}\right|_{t=t_0} z_i(t_0) = \left.\frac{d^n}{dt^n}\right|_{t=t_0} z_j(t_0), \quad n \ge 2.$$

Since $z_i - z_j$ is analytic at $t = t_0$ as the solution of (5.3), there exists $\delta > 0$ such that

$$z_i(t) = z_j(t), \quad t \in (t_0 - \delta, t_0 + \delta)$$

which is contradictory to the choice of t_0 in (5.7).

(ii) Note that the set

$$\mathcal{T} := \left\{t \in [0, \infty) \;\; z_i(t) - z_j(t) = 0\right\}$$

is nonempty closed set. At the collision time t_0 such that

$$z_i(t_0) = z_j(t_0),$$

there exists an open set $(t_0 - \delta, t_0 + \delta)$ containing t_0 by similar argument to (i). Hence $\mathcal{T}$ is an open set and $\mathcal{T} = \mathbb{R}_+$. □

As briefly mentioned before, the roles of mean-field coupling terms

$$\left(\langle z_j, z_j\rangle z_c - \langle z_c, z_j\rangle z_j\right) \quad \text{and} \quad \left(\langle z_j, z_c\rangle - \langle z_c, z_j\rangle\right) z_j \tag{5.8}$$

are somewhat different. In fact, the first term $(5.8)_1$ is mainly responsible for the collective behavior of model (4.1), whereas the second term $(5.8)_2$ can be regarded as a perturbation. More precisely, in order to see the role of each term, we first focus on the collective behaviors of each subsystem

$$\text{Subsystem } A: \quad \begin{cases} \dot{z}_j = \left(\langle z_j, z_j\rangle z_c - \langle z_c, z_j\rangle z_j\right), & t \ge 0, \\ z_j(0) = z_j^{\text{in}}, \quad \left\| z_j^{\text{in}} \right\| = 1, \end{cases}$$

and

$$\text{Subsystem } B: \quad \begin{cases} \dot{z}_j = \left(\langle z_j, z_c\rangle - \langle z_c, z_j\rangle\right) z_j, & t \ge 0, \\ z_j(0) = z_j^{\text{in}}, \quad \left\| z_j^{\text{in}} \right\| = 1. \end{cases}$$

In what follows, the main tool is Barbalat's lemma stated as follows.

Lemma 5.3 (Barbalat [3]) *Let $f : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function satisfying the following two properties:*

$$\exists \lim_{t \to \infty} f(t) \quad \textit{and} \quad f' \textit{ is uniformly continuous.}$$

Then, f' tends to zero, as $t \to \infty$:

$$\lim_{t\to\infty} f'(t) = 0.$$

In the following two subsections, we study the emergent dynamics of Subsystem A and Subsystem B separately.

5.2 Subsystem A

Consider the Cauchy problem to the following Subsystem A:

$$\begin{cases} \dot{z}_i = \left(\langle z_i, z_i \rangle z_c - \langle z_c, z_i \rangle z_i \right), \quad t \geq 0, \\ z_i(0) = z_i^{\text{in}}, \quad \left\| z_i^{\text{in}} \right\| = 1, \quad i \in \mathbb{N}. \end{cases} \tag{5.9}$$

This corresponds to the special case $(\lambda_0, \lambda_1) = (1, 0)$ in (5.3). In next proposition, we show that the two functionals:

$$\|z_c\|^2 \quad \text{and} \quad \sum_{i,j\in\mathbb{N}} \kappa_i \kappa_j \ln \left| 1 - \left\langle z_i, z_j \right\rangle \right|$$

are monotone along the dynamics (5.9).

Proposition 5.1 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to Subsystem A. Then the following assertions hold.*

(i) *The order parameter $\|z_c\|$ is nondecreasing:*

$$\frac{d}{dt} \|z_c\|^2 \geq 0, \quad t > 0.$$

(ii) *If $z_i^{in} \neq z_j^{in}$ for $i \neq j$, the functional $\sum_{i,j\in\mathbb{N}} \kappa_i \kappa_j \ln \left| 1 - \left\langle z_i, z_j \right\rangle \right|$ is nonincreasing.*

Proof (i) It follows from (5.2) and (5.9) that

$$\frac{dz_c}{dt} = \sum_{j\in\mathbb{N}} \kappa_j \dot{z}_j = \sum_{j\in\mathbb{N}} \kappa_j \left(z_c - \left\langle z_c, z_j \right\rangle z_j \right) = z_c - \sum_{j\in\mathbb{N}} \kappa_j \left\langle z_c, z_j \right\rangle z_j.$$

This yields

$$\begin{aligned} \frac{d\, \|z_c\|^2}{dt} &= \left\langle z_c, z_c - \sum_{j\in\mathbb{N}} \kappa_j \left\langle z_c, z_j \right\rangle z_j \right\rangle + \left\langle z_c - \sum_{j\in\mathbb{N}} \kappa_j \left\langle z_c, z_j \right\rangle z_j, z_c \right\rangle \\ &= 2\, \|z_c\|^2 - \sum_{j\in\mathbb{N}} \kappa_j \left\langle z_c, z_j \right\rangle^2 - \sum_{j\in\mathbb{N}} \kappa_j \left\langle z_j, z_c \right\rangle^2 \\ &= 2 \left(\|z_c\|^2 - \sum_{j\in\mathbb{N}} \kappa_j \mathfrak{Re} \left(\left\langle z_c, z_j \right\rangle^2 \right) \right). \end{aligned} \tag{5.10}$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\left| \left\langle z_j, z_c \right\rangle \right|^2 \leq \left\langle z_j, z_j \right\rangle \langle z_c, z_c \rangle, \quad \left| \mathfrak{Re} \left\langle z_c, z_j \right\rangle^2 \right| \leq \left| \left\langle z_c, z_j \right\rangle \right|^2 \leq \langle z_c, z_c \rangle. \tag{5.11}$$

Finally, we combine (5.10) and (5.11) to derive

$$\frac{d\, \|z_c\|^2}{dt} \geq 0.$$

(ii) By Lemma 5.2, the function $\ln\left|1-\langle z_i, z_j\rangle\right|$ is globally well-defined. Again, we use (5.9) to find

$$\begin{aligned}\frac{d}{dt}\left(1-\langle z_i, z_j\rangle\right) &= -\left[\langle z_c - \langle z_c, z_i\rangle z_i, z_j\rangle + \langle z_i, z_c - \langle z_c, z_j\rangle z_j\rangle\right]\\ &= -\left[\langle z_c, z_j\rangle - \langle z_i, z_c\rangle\langle z_i, z_j\rangle + \langle z_i, z_c\rangle - \langle z_c, z_j\rangle\langle z_i, z_j\rangle\right]\\ &= -\left[\langle z_i, z_c\rangle + \langle z_c, z_j\rangle\right]\left[1-\langle z_i, z_j\rangle\right].\end{aligned} \tag{5.12}$$

Now, we use (5.12) to obtain

$$\frac{d}{dt}\left|1-\langle z_i, z_j\rangle\right|^2 = -\left[\langle z_i + z_j, z_c\rangle + \langle z_c, z_i + z_j\rangle\right]\left|1-\langle z_i, z_j\rangle\right|^2.$$

This implies

$$\frac{d}{dt}\ln\left|1-\langle z_i, z_j\rangle\right| = -\frac{1}{2}\left[\langle z_i + z_j, z_c\rangle + \langle z_c, z_i + z_j\rangle\right]. \tag{5.13}$$

Thus, the desired estimates follows from (5.13):

$$\frac{d}{dt}\sum_{i,j\in\mathbb{N}}\kappa_i\kappa_j\ln\left|1-\langle z_i, z_j\rangle\right| = -\frac{1}{2}\sum_{i,j\in\mathbb{N}}\kappa_i\kappa_j\left[\langle z_i + z_j, z_c\rangle + \langle z_c, z_i + z_j\rangle\right] = -2\,\|z_c\|^2 < 0.$$

□

Theorem 5.1 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to (5.3) with*

$$\sup_{i,j\in\mathbb{N}}\left|1-\left\langle z_i^{in}, z_j^{in}\right\rangle\right| < 1-\delta, \quad \textit{for some } \delta\in(0,1). \tag{5.14}$$

Then we have

$$\left|1-\langle z_i(t), z_j(t)\rangle\right| \le \left|1-\left\langle z_i^{in}, z_j^{in}\right\rangle\right|\cdot\exp(-2\delta t), \quad \forall\, t\ge 0.$$

Proof Since the proof is rather lengthy, we leave proof in Appendix B. □

5.3 Subsystem B

Consider the Cauchy problem to the following Subsystem B:

$$\begin{cases}\dot{z}_i = (\langle z_i, z_c\rangle - \langle z_c, z_i\rangle)z_i, & t\ge 0,\\ z_i(0) = z_i^{\mathrm{in}}, \quad \|z_i^{\mathrm{in}}\| = 1.\end{cases} \tag{5.15}$$

In the following proposition, we show that the time-derivative of z_i vanishes asymptotically.

Proposition 5.2 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to (5.15). Then we have*

$$\lim_{t\to\infty}|\dot{z}_i(t)| = 0, \quad \forall i\in\mathbb{N}.$$

Proof We split the proof into two steps.

- Step A: We will use the Babalat lemma to derive the desired estimate. For this, we set

$$f(t) = \langle z_c(t), z_c(t)\rangle = \|z_c(t)\|^2, \quad t\ge 0,$$

and we claim

$$\text{(i) } \exists \lim_{t\to\infty} \langle z_c(t), z_c(t)\rangle. \qquad \text{(ii) } \frac{d}{dt}\langle z_c, z_c\rangle \text{ is uniformly continuous.} \tag{5.16}$$

Below, we check the assertions in (5.16).

(i) First, we show that

$$\frac{d}{dt}\langle z_c, z_c\rangle = -\sum_{i\in\mathbb{N}} \kappa_i \Big(\langle z_c, z_i\rangle - \langle z_i, z_c\rangle \Big)^2 > 0. \tag{5.17}$$

Proof of (5.17): we use

$$\dot{z}_c = \sum_{i\in\mathbb{N}} \kappa_i (\langle z_i, z_c\rangle - \langle z_c, z_i\rangle) z_i$$

to find the desired estimate (5.17):

$$\begin{aligned} \frac{d}{dt}\langle z_c, z_c\rangle &= \sum_{i\in\mathbb{N}} \kappa_i \Big(\langle z_c, z_i\rangle \langle z_i, z_c\rangle - \langle z_i, z_c\rangle \langle z_i, z_c\rangle \Big) \\ &\quad + \sum_{i\in\mathbb{N}} \kappa_i \Big(\langle z_c, z_i\rangle \langle z_i, z_c\rangle - \langle z_c, z_i\rangle \langle z_c, z_i\rangle \Big) \\ &= \sum_{i\in\mathbb{N}} \kappa_i \left(2\, |\langle z_c, z_i\rangle|^2 - \langle z_c, z_i\rangle^2 - \langle z_i, z_c\rangle^2 \right) > 0. \end{aligned} \tag{5.18}$$

On the other hand, we use $\left|\langle z_i, z_j\rangle\right| \le \|z_i\| \left\|z_j\right\| = 1$ to see

$$\langle z_c, z_c\rangle = \left| \sum_{i,j\in\mathbb{N}} \kappa_i\kappa_j \left\langle z_i, z_j\right\rangle \right| \le \sum_{i,j\in\mathbb{N}} \kappa_i\kappa_j = 1 < \infty. \tag{5.19}$$

By (5.18) and (5.19), we have

$$\exists \lim_{t\to\infty} \langle z_c(t), z_c(t)\rangle.$$

(ii) It follows from Lemma 5.1 that

$$\left| \frac{d^2}{dt^2}\langle z_c, z_c\rangle \right| \le 32.$$

This implies the uniform continuity of $\frac{d}{dt}\langle z_c, z_c\rangle$. Then, by the Babalat lemma and (5.17), we have

$$\lim_{t\to\infty} \frac{d}{dt}\langle z_c(t), z_c(t)\rangle = 0, \quad \text{i.e.,} \quad \lim_{t\to\infty} \Big(\langle z_c(t), z_i(t)\rangle - \langle z_i(t), z_c(t)\rangle \Big) = 0, \quad i\in\mathbb{N}. \tag{5.20}$$

• Step B: It follows from $(5.20)_2$ that

$$\lim_{t\to\infty} \dot{z}_i(t) = \lim_{t\to\infty} \left(\langle z_i(t), z_c(t)\rangle - \langle z_c(t), z_i(t)\rangle \right) z_i(t) = 0, \quad i\in\mathbb{N},$$

where we use $\|z_i\| = 1$. □

So far, we have studied collective behaviors of two submodels of (5.3) one by one. In next subsection, we study the collective behavior of the full model for a homogeneous ensemble.

5.4 Asymptotic State-Locking

In this subsection, we consider the Cauchy problem to the full infinite LHS model:

$$\begin{cases} \dot{z}_j = \lambda_0 \left(z_c - \langle z_c, z_j \rangle z_j\right) + \lambda_1 \big(\langle z_j, z_c \rangle - \langle z_c, z_j \rangle\big) z_j, \quad t > 0, \\ z_j(0) = z_j^{\text{in}}, \quad \left\| z_j^{\text{in}} \right\| = 1, \quad j \in \mathbb{N}. \end{cases} \tag{5.21}$$

For a special case with $\lambda_0 = 0$, in the course of proof of Proposition 5.2, we have shown that

$$\lim_{t\to\infty} \frac{d}{dt} \|z_c(t)\|^2 = 0.$$

In the next proposition, we show that the above estimate holds in full generality.

Proposition 5.3 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to the full model* (5.21)*. Then we have*

$$\exists \lim_{t\to\infty} \|z_c(t)\|^2 \quad \textit{and} \quad \lim_{t\to\infty} \frac{d}{dt} \|z_c(t)\|^2 = 0.$$

Proof We basically follow the same strategy employed in the proof of Proposition 5.2.

(i) (Derivation of the first estimate): We split the derivation into two steps.

• Step A: We first claim:

$$\begin{aligned} \frac{d}{dt} \langle z_c, z_c \rangle &= \lambda_0 \sum_{i\in\mathbb{N}} \kappa_i \left(2 \|z_c\|^2 - \langle z_c, z_i \rangle^2 - \langle z_i, z_c \rangle^2\right) \\ &\quad + \lambda_1 \sum_{i\in\mathbb{N}} \kappa_i \left(2 |\langle z_c, z_i \rangle|^2 - \langle z_c, z_i \rangle^2 - \langle z_i, z_c \rangle^2\right). \end{aligned} \tag{5.22}$$

Proof of (5.22): Note that

$$\begin{aligned} \frac{d}{dt} \langle z_i, z_j \rangle &= \langle \dot{z}_i, z_j \rangle + \langle z_i, \dot{z}_j \rangle \\ &= \langle \lambda_0 (z_c - \langle z_c, z_i \rangle z_i) + \lambda_1 (\langle z_i, z_c \rangle - \langle z_c, z_i \rangle) z_i, z_j \rangle \\ &\quad + \left\langle z_i, \lambda_0 \left(z_c - \langle z_c, z_j \rangle z_j\right) + \lambda_1 \left(\langle z_j, z_c \rangle - \langle z_c, z_j \rangle\right) z_j \right\rangle \\ &= \lambda_0 \left(\langle z_c, z_j \rangle - \langle z_i, z_c \rangle \langle z_i, z_j \rangle\right) + \lambda_1 (\langle z_c, z_i \rangle - \langle z_i, z_c \rangle) \langle z_i, z_j \rangle \\ &\quad + \lambda_0 \left(\langle z_i, z_c \rangle - \langle z_c, z_j \rangle \langle z_i, z_j \rangle\right) - \lambda_1 \left(\langle z_c, z_j \rangle - \langle z_j, z_c \rangle\right) \langle z_i, z_j \rangle . \end{aligned} \tag{5.23}$$

We sum up (5.23) over all i and j to get the desired estimate (5.22):

$$
\begin{aligned}
\frac{d}{dt}\langle z_c, z_c\rangle &= \sum_{i,j\in\mathbb{N}} \kappa_i\kappa_j \frac{d}{dt}\langle z_i, z_j\rangle \\
&= \sum_{i,j\in\mathbb{N}} \kappa_i\kappa_j\lambda_0 \left(\langle z_c, z_j\rangle - \langle z_i, z_c\rangle\langle z_i, z_j\rangle\right) + \sum_{i,j\in\mathbb{N}} \kappa_i\kappa_j\lambda_1 \left(\langle z_c, z_i\rangle - \langle z_i, z_c\rangle\right)\langle z_i, z_j\rangle \\
&\quad + \sum_{i,j\in\mathbb{N}} \kappa_i\kappa_j\lambda_0 \left(\langle z_i, z_c\rangle - \langle z_c, z_j\rangle\langle z_i, z_j\rangle\right) - \sum_{i,j\in\mathbb{N}} \kappa_i\kappa_j\lambda_1 \left(\langle z_c, z_j\rangle - \langle z_j, z_c\rangle\right)\langle z_i, z_j\rangle \\
&= \lambda_0\|z_c\|^2 - \lambda_0\sum_{i\in\mathbb{N}}\kappa_i\langle z_i, z_c\rangle^2 + \sum_{i\in\mathbb{N}}\kappa_i\lambda_1|\langle z_c, z_i\rangle|^2 - \sum_{i\in\mathbb{N}}\kappa_i\lambda_1\langle z_i, z_c\rangle^2 \\
&\quad + \lambda_0\|z_c\|^2 - \lambda_0\sum_{i\in\mathbb{N}}\kappa_i\langle z_c, z_i\rangle^2 + \sum_{i\in\mathbb{N}}\kappa_i\lambda_1|\langle z_c, z_i\rangle|^2 - \sum_{i\in\mathbb{N}}\kappa_i\lambda_1\langle z_c, z_i\rangle^2 \\
&= \lambda_0\sum_{i\in\mathbb{N}}\kappa_i\left(2\|z_c\|^2 - \langle z_c, z_i\rangle^2 - \langle z_i, z_c\rangle^2\right) + \lambda_1\sum_{i\in\mathbb{N}}\kappa_i\left(2|\langle z_c, z_i\rangle|^2 - \langle z_c, z_i\rangle^2 - \langle z_i, z_c\rangle^2\right),
\end{aligned}
\tag{5.24}
$$

where we used $\sum_{i\in\mathbb{N}}\kappa_i = 1$.

• Step B: We show that the summand in (5.24) are nonnegative. For this, we use the identity

$$z^2 + \bar{z}^2 = (\Re e(z) + \mathrm{i}\Im m(z))^2 + (\Re e(z) - \mathrm{i}\Im m(z))^2 = 2\Big((\Re e(z))^2 - \Im m(z))^2\Big) \le 2\|z\|^2,$$

the Cauchy-Schwarz inequality and $\|z_i\| = 1$ to find

$$2\|z_c\|^2 = 2\|z_c\|^2\|z_i\|^2 \ge 2|\langle z_c, z_i\rangle|^2 \ge \langle z_c, z_i\rangle^2 + \langle z_i, z_c\rangle^2.$$

This implies the nonnegativity of the right-hand side of (5.24):

$$\frac{d}{dt}\langle z_c, z_c\rangle \ge 0. \tag{5.25}$$

On the other hand, we have

$$\langle z_c, z_c\rangle \le 1. \tag{5.26}$$

Finally, it follows from (5.25) and (5.26) that

$$\exists \lim_{t\to\infty}\langle z_c, z_c\rangle.$$

(ii) (Derivation of the second estimate): We apply the Babalat's lemma with $f(t) = \langle z_c(t), z_c(t)\rangle$. Since we have already shown that

$$\exists \lim_{t\to\infty} f(t),$$

we need to show that f' is uniformly continuous. Thus, it suffices to show that $|f''(t)|$ is uniformly bounded. This is obvious from Lemma 5.1 that

$$\left\|\frac{d^2}{dt^2}\langle z_c, z_c\rangle\right\| \le 32.$$

Finally, we can apply Lemma 5.3 to show

$$\lim_{t\to\infty}\frac{d}{dt}\|z_c\|^2 = 0.$$

□

As a corollary of Proposition 5.3, we have asymptotic behavior of the angle between z_i and z_c.

Corollary 5.1 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to the full model (5.3). Then we have*

$$\lim_{t\to\infty}\left(\|z_c(t)\|^2 - |\langle z_c(t), z_i(t)\rangle|^2\right) = 0 \quad \text{and} \quad \lim_{t\to\infty} \mathfrak{Im}\,\langle z_c(t), z_i(t)\rangle = 0 \quad \text{for } i \in \mathbb{N}.$$

Proof We use $\langle z_c, z_i\rangle = \overline{\langle z_i, z_c\rangle}$ and further rearrange the estimate (5.22) as

$$\begin{aligned}
\frac{d}{dt}\langle z_c, z_c\rangle &= \lambda_0 \sum_{i\in\mathbb{N}} \kappa_i \left(2\,\|z_c\|^2 - \langle z_i, z_c\rangle^2 - \langle z_c, z_i\rangle^2\right) \\
&\quad + \lambda_1 \sum_{i\in\mathbb{N}} \kappa_i \left(2\,|\langle z_c, z_i\rangle|^2 - \langle z_c, z_i\rangle^2 - \langle z_i, z_c\rangle^2\right) \\
&= 2\lambda_0 \sum_{i\in\mathbb{N}} \kappa_i \left(\|z_c\|^2 - \mathfrak{Re}\left(\langle z_i, z_c\rangle^2\right)\right) + 2\lambda_1 \sum_{i\in\mathbb{N}} \kappa_i \left(|\langle z_c, z_i\rangle|^2 - \mathfrak{Re}\left(\langle z_i, z_c\rangle^2\right)\right) \\
&= 2\lambda_0 \sum_{i\in\mathbb{N}} \kappa_i \left(\|z_c\|^2 - \mathfrak{Re}\,(\langle z_i, z_c\rangle)^2 + \mathfrak{Im}\,(\langle z_i, z_c\rangle)^2\right) \\
&\quad + 2\lambda_1 \sum_{i\in\mathbb{N}} \kappa_i \left(|\langle z_c, z_i\rangle|^2 - \mathfrak{Re}\,(\langle z_i, z_c\rangle)^2 + \mathfrak{Im}\,(\langle z_i, z_c\rangle)^2\right) \\
&= 2\lambda_0 \sum_{i\in\mathbb{N}} \kappa_i \left(\|z_c\|^2 - \mathfrak{Re}\,(\langle z_i, z_c\rangle)^2 - \mathfrak{Im}\,(\langle z_i, z_c\rangle)^2 + 2\mathfrak{Im}\,(\langle z_i, z_c\rangle)^2\right) \\
&\quad + 2\lambda_1 \sum_{i\in\mathbb{N}} \kappa_i \left(2 \cdot \mathfrak{Im}\,(\langle z_i, z_c\rangle)^2\right) \\
&= 2\lambda_0 \sum_{i\in\mathbb{N}} \kappa_i \left(\|z_c\|^2 - |\langle z_i, z_c\rangle|^2\right) + 4\,(\lambda_0 + \lambda_1) \sum_{i\in\mathbb{N}} \kappa_i \,|\mathfrak{Im}\,(\langle z_i, z_c\rangle)|^2 .
\end{aligned}$$

This clearly shows that $\frac{d}{dt}\langle z_c, z_c\rangle$ is the sum of nonnegative terms. Finally, by Proposition 5.3, we obtain the desired result. □

So far, we do not show the convergence of our solution $\mathcal{Z}(t)$ as $t \uparrow \infty$, but we can derive an information for how the asymptotic configuration $\mathcal{Z}^\infty$ in unit Hermitian sphere.

Corollary 5.2 *Suppose that for each $i \in \mathbb{N}$, z_i converges to z_i^∞. Then we have*

$$\left\langle z_i^\infty, z_c^\infty\right\rangle \in \{1, -1\}.$$

Proof By the first part of Proposition 5.3 and $\|z_i\| = 1$, one has

$$0 = \lim_{t\to\infty}\left(\|z_c(t)\|^2 - |\langle z_c(t), z_i(t)\rangle|^2\right) = \left\|z_c^\infty\right\|^2 - \left|\left\langle z_i^\infty, z_c^\infty\right\rangle\right|^2 = \left\|z_c^\infty\right\|^2 \|z_i\|^2 - \left|\left\langle z_i^\infty, z_c^\infty\right\rangle\right|^2.$$

Thus, the asymptotic configuration $\left\{z_i^\infty\right\}_{i\in\mathbb{N}}$ satisfies the equality condition of the Cauchy-Schwarz inequality. Hence we have

$$z_i^\infty = a_i z_c^\infty, \quad i \in \mathbb{N},$$

for some $a_i \in \mathbb{C}$ with $|a_i| = 1$. On the other hand, by the second part of Corollary 5.1,

$$\mathfrak{Im}\left\langle z_c^\infty, z_i^\infty\right\rangle = 0.$$

Therefore $a_i \in \{1, -1\}$. □

Remark 5.1 The result of Corollary 5.2 shows that the possible asymptotic configuration is either completely synchronized state or bi-polar state.

In next proposition, we show that z_i becomes stationary asymptotically (see Proposition 5.2 for Subsystem B).

Proposition 5.4 *Let $\mathcal{Z} = \mathcal{Z}(t)$ be a global solution to (5.3). then we have*

$$\lim_{t\to\infty} |\dot{z}_i(t)| = 0, \quad i \in \mathbb{N}.$$

Proof We use (5.21) to see

$$\begin{aligned} \langle \dot{z}_i, \dot{z}_i \rangle &= |\lambda_0|^2 \Big\langle z_c - \langle z_c, z_i \rangle z_i, z_c - \langle z_c, z_i \rangle z_i \Big\rangle \\ &\quad + \lambda_0 \lambda_1 (\langle z_c, z_i \rangle - \langle z_i, z_c \rangle) \Big\langle z_c - \langle z_c, z_i \rangle z_i, z_i \Big\rangle \\ &\quad + \lambda_1 \lambda_0 \overline{(\langle z_c, z_i \rangle - \langle z_i, z_c \rangle)} \Big\langle z_i, z_c - \langle z_c, z_i \rangle z_i \Big\rangle \\ &\quad + |\lambda_1|^2 |\langle z_c, z_i \rangle - \langle z_i, z_c \rangle|^2. \end{aligned} \tag{5.27}$$

Om the other hand, it follows from Corollary 5.2 that

$$\langle z_c, z_i \rangle - \langle z_i, z_c \rangle = \langle z_c, z_i \rangle - \overline{\langle z_c, z_i \rangle} = 2\mathrm{i}\Im m(\langle z_c, z_i \rangle) \to 0, \quad \text{as } t \to \infty. \tag{5.28}$$

By (5.27) and (5.28), one has

$$\lim_{t\to\infty} \langle \dot{z}_i, \dot{z}_i \rangle = \lim_{t\to\infty} |\lambda_0|^2 \Big\langle z_c - \langle z_c, z_i \rangle z_i, z_c - \langle z_c, z_i \rangle z_i \Big\rangle. \tag{5.29}$$

Again, we use Corollary 5.1 to see

$$\begin{aligned} &\langle z_c - \langle z_c, z_i \rangle z_i, z_c - \langle z_c, z_i \rangle z_i \rangle \\ &= \|z_c\|^2 - \langle z_c, z_i \rangle^2 - \langle z_i, z_c \rangle^2 + |\langle z_c, z_i \rangle|^2 \\ &= \|z_c\|^2 - |\langle z_c, z_i \rangle|^2 + 2\,|\langle z_c, z_i \rangle|^2 - \langle z_c, z_i \rangle^2 - \langle z_i, z_c \rangle^2 \to 0, \end{aligned} \tag{5.30}$$

as $t \uparrow \infty$. Finally, we combine (5.29) and (5.30) to get the desired estimate. □

6 Conclusion

In this paper, we have studied the collective behaviors of infinitely many Lohe oscillators on the unit Hermitian sphere in d-dimensional complex Euclidean space. For this, we proposed a new synchronization model governing the dynamics of an infinite set of Lohe Hermitian sphere oscillators and we have also presented several sufficient framework leading to practical and complete synchronization estimates. The proposed model extends author's recent work [22] on the infinite set of Kuramoto oscillators to the infinite set of Lohe Hermitian sphere oscillators in a higher-dimensional setting. In our infinite model with an infinite coupling matrix (κ_{ij}), we cannot find such an average quantity with a similar role as z_c in Sect. 5. That makes our analysis in Sects. 3 and 4 be more delicate. The presented results of this paper can be summarized as follows. First, we presented a sufficient framework for the collective behaviors of the ensemble of infinite oscillators defined on higher-dimensional ambient space with a network topology. Our sufficient framework is given in terms of system parameters and admissible initial data. Second, we have demonstrated how the analysis in [23] can be extended to an infinite ensemble over the sender network. In the previous works, the tool employed to analyze the finite-dimensional swarm sphere model over network topology is the spectral theory of adjacent matrices. However, we use a direct nonlinear functional approach based on the state diameter as a suitable Lyapunov functional.

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Data Availibility We do not analyze or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

Declarations

Conflict of interest The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript.

Appendix A Well-Posedness of the Infinite LHS Model

In this appendix, we present a global well-posedness of the infinite LHS model. For this, we study a local well-posedness using the Cauchy-Lipschitz theorem on a suitable Banach space.

A.1 A Local Well-Posedness

We first recall the Cauchy-Lipschitz theorem. Let E be a Banach space, and $U \subset E$. Let $F : U \to E$ be a local Lipschitz map and let $I = [0, T^*)$ be an interval contained in $\mathbb{R}$ where $T^* \in (0, \infty]$.

Lemma A.1 (Cauchy-Lipschitz) [6, 10] *The Cauchy problem:*

$$\begin{cases} \frac{du}{dt} = F(u(t)), \quad t > 0, \\ u\big|_{t=0+} = u_0. \end{cases}$$

has a unique local solution u in the time interval I.

To apply Lemma A.1 for the infinite LHS model, we need to introduce E and U. We introduce the Banach space $\ell_{\mathbb{C}}^{\infty,2}$:

$$(\ell_{\mathbb{C}}^{\infty,2}, \|\cdot\|_{\infty,2}) := \left\{ \mathcal{Y} = \{y_i\}_{i\in\mathbb{N}} : y_i \in \mathbb{C}^d, \quad \|\mathcal{Y}\|_{\infty,2} := \sup_{i\in\mathbb{N}} \|y_i\| < \infty \right\}.$$

Theorem A.1 (Local existence) *The Cauchy problem* (1.3)–(1.4) *admits a local unique smooth solution $\mathcal{Z} : [0, t_0) \to \ell_{\mathbb{C}}^{\infty,2}$ for some $t_0 > 0$.*

Proof We define $F : \ell_{\mathbb{C}}^{\infty,2} \to \ell_{\mathbb{C}}^{\infty,2}$ as

$$\begin{aligned} &F(z) = \{f_i(z)\}_{i\in\mathbb{N}}, \quad z = \{z_i\}_{i\in\mathbb{N}}, \\ &f_i(z) = \Omega_i z_i + \lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle z_i, z_i\rangle z_j - \langle z_j, z_i\rangle z_i \right) + \lambda_1 \sum_{j\in\mathbb{N}} \kappa_{ij} \Big(\langle z_i, z_j\rangle - \langle z_j, z_i\rangle \Big) z_i. \end{aligned} \tag{A1}$$

We outline the proof strategy in four steps:

- (Step A): Find a local bound of F depending on $\|z\|_{\infty,2}$ for $z \in \ell^{\infty,2}_{\mathbb{C}}$.
- (Step B): Find a local Lipschitz constant of F depending on $\|z\|_{\infty,2}$.
- (Step C): Prove a local existence of integral solution to the infinite LHS model.
- (Step D): Prove $\mathcal{Z}$ is a classical solution of LHS model.

In what follows, we perform the above steps one by one.

◇ Step A (Local boundedness of F): We use (A1) to see

$$\begin{aligned}\|f_i(z)\| &\le \|\Omega_i z_i\| + \lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij} \left\| \langle z_i, z_i\rangle z_j - \langle z_j, z_i\rangle z_i \right\| + \lambda_1 \sum_{j\in\mathbb{N}} \kappa_{ij} \left| \langle z_i, z_j\rangle - \langle z_j, z_i\rangle \right| \|z_i\| \\ &\le \|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} \|\boldsymbol{z}\|_{\infty,2} + 2\lambda_0 \sum_{j\in\mathbb{N}} \kappa_{ij} \|\boldsymbol{z}\|^3_{\infty,2} + 2\lambda_1 \sum_{j\in\mathbb{N}} \kappa_{ij} \|\boldsymbol{z}\|^3_{\infty,2} \\ &\le \|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} \|\boldsymbol{z}\|_{\infty,2} + 2(\lambda_0+\lambda_1) \|\boldsymbol{\kappa}\|_{\infty,1} \|\boldsymbol{z}\|^3_{\infty,2}.\end{aligned}$$

This yields

$$\sup_{i\in\mathbb{N}} \|f_i(z)\| = \|F(\boldsymbol{z})\|_{\infty,2} \le \|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} \|\boldsymbol{z}\|_{\infty,2} + 2(\lambda_0+\lambda_1) \|\boldsymbol{\kappa}\|_{\infty,1} \|\boldsymbol{z}\|^3_{\infty,2}.$$

◇ Step B (Local Lipschitz continuity of F): For $\mathcal{Z}, \tilde{\mathcal{Z}} \in \ell^{\infty,2}_{\mathbb{C}}$, we have

$$\begin{aligned}\left\| F(\mathcal{Z}) - F(\tilde{\mathcal{Z}}) \right\|_\infty &\le \sup_{i\in\mathbb{N}} \|\Omega_i\|_{\mathrm{op}} \|z_i - \tilde{z}_i\| \\ &\quad + \lambda_0 \sup_{i\in\mathbb{N}} \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left\{ \left(\langle z_i, z_i\rangle z_j - \langle z_j, z_i\rangle z_i \right) - \left(\langle \tilde{z}_i, \tilde{z}_i\rangle \tilde{z}_j - \langle \tilde{z}_j, \tilde{z}_i\rangle \tilde{z}_i \right) \right\} \right\| \\ &\quad + \lambda_1 \sup_{i\in\mathbb{N}} \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left\{ \Big(\langle z_i, z_j\rangle - \langle z_j, z_i\rangle \Big) z_i - \Big(\langle \tilde{z}_i, \tilde{z}_j\rangle - \langle \tilde{z}_j, \tilde{z}_i\rangle \Big) \tilde{z}_i \right\} \right\| \\ &=: \mathcal{I}_{41} + \lambda_0 \mathcal{I}_{42} + \lambda_1 \mathcal{I}_{43}.\end{aligned}$$

In the sequel, we show that each term $\mathcal{I}_{41}, \mathcal{I}_{42}$ and $\mathcal{I}_{43}$ can be controlled by $\mathcal{O}(1) \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2}$.

✤ Step B.1 (Estimate of $\mathcal{I}_{41}$): Note that

$$\mathcal{I}_{41} \le \|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2}.$$

✤ Step B.2 (Estimate of $\mathcal{I}_{42}$): By direct calculation, one has

$$\begin{aligned}\mathcal{I}_{42} &= \sup_{i\in\mathbb{N}} \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left\{ \left(\langle z_i, z_i\rangle z_j - \langle z_j, z_i\rangle z_i \right) - \left(\langle \tilde{z}_i, \tilde{z}_i\rangle \tilde{z}_j - \langle \tilde{z}_j, \tilde{z}_i\rangle \tilde{z}_i \right) \right\} \right\| \\ &\le \sup_{i\in\mathbb{N}} \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\|z_i\|^2 z_j - \|\tilde{z}_i\|^2 \tilde{z}_j \right) \right\| + \sup_{i\in\mathbb{N}} \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle z_j, z_i\rangle z_i - \langle \tilde{z}_j, \tilde{z}_i\rangle \tilde{z}_i \right) \right\|\end{aligned}$$

$$=: \mathcal{I}_{421} + \mathcal{I}_{422}.$$

Then for each $i \in \mathbb{N}$, we have

$$\begin{aligned}
&\left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\|z_i\|^2 z_j - \|\tilde{z}_i\|^2 \tilde{z}_j \right) \right\| \\
&\quad \le \sum_{j\in\mathbb{N}} \kappa_{ij} \|z_i\|^2 \left\| z_j - \tilde{z}_j \right\| + \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\|z_i\|^2 - \|\tilde{z}_i\|^2 \right) \left\| \tilde{z}_j \right\| \\
&\quad \le \|\boldsymbol{\kappa}\|_{\infty,1} \|\mathcal{Z}\|_{\infty,2}^2 \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2} + \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\|z_i\| + \|\tilde{z}_i\| \right) \left\| \tilde{z}_j \right\| \|z_i - \tilde{z}_i\| \\
&\quad \le \|\boldsymbol{\kappa}\|_{\infty,1} \left(\|\mathcal{Z}\|_{\infty,2}^2 + \|\mathcal{Z}\|_{\infty,2} \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2} + \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2}^2 \right) \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2},
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle z_j, z_i \rangle z_i - \langle \tilde{z}_j, \tilde{z}_i \rangle \tilde{z}_i \right) \right\| \\
&\quad \le \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \langle z_j, z_i \rangle (z_i - \tilde{z}_i) \right\| + \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle z_j, z_i - \tilde{z}_i \rangle \tilde{z}_i \right) \right\| + \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left(\langle z_j - \tilde{z}_j, \tilde{z}_i \rangle \tilde{z}_i \right) \right\| \\
&\quad \le \sum_{j\in\mathbb{N}} \kappa_{ij} \left\| z_j \right\| \|z_i\| \|z_i - \tilde{z}_i\| + \sum_{j\in\mathbb{N}} \kappa_{ij} \left\| z_j \right\| \|z_i - \tilde{z}_i\| \|\tilde{z}_i\| + \sum_{j\in\mathbb{N}} \kappa_{ij} \left\| z_j - \tilde{z}_j \right\| \|\tilde{z}_i\|^2 \\
&\quad \le \|\boldsymbol{\kappa}\|_{\infty,1} \left(\|\mathcal{Z}\|_{\infty,2}^2 + \|\mathcal{Z}\|_{\infty,2} \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2} + \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2}^2 \right) \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2}.
\end{aligned}$$

These give upper bounds of $\mathcal{I}_{421}$ and $\mathcal{I}_{422}$.

✤ Step B.3 (Estimate of $\mathcal{I}_{43}$): Note that

$$\begin{aligned}
\mathcal{I}_{43} &= \sup_{i\in\mathbb{N}} \left\| \sum_{j\in\mathbb{N}} \kappa_{ij} \left\{ \Big(\langle z_i, z_j \rangle - \langle z_j, z_i \rangle \Big) z_i - \Big(\langle \tilde{z}_i, \tilde{z}_j \rangle - \langle \tilde{z}_j, \tilde{z}_i \rangle \Big) \tilde{z}_i \right\} \right\| \\
&\le \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \kappa_{ij} \left\| \langle z_i, z_j \rangle z_i - \langle \tilde{z}_i, \tilde{z}_j \rangle \tilde{z}_i \right\| + \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \kappa_{ij} \left\| \langle z_j, z_i \rangle z_i - \langle \tilde{z}_j, \tilde{z}_i \rangle \tilde{z}_i \right\| \\
&=: \mathcal{I}_{431} + \mathcal{I}_{432}.
\end{aligned}$$

For each i and j, we use

$$\begin{aligned}
&\left\| \langle z_i, z_j \rangle z_i - \langle \tilde{z}_i, \tilde{z}_j \rangle \tilde{z}_i \right\| \\
&\quad \le \left\| \langle z_i, z_j \rangle z_i - \langle z_i, z_j \rangle \tilde{z}_i \right\| + \left\| \langle z_i, z_j \rangle \tilde{z}_i - \langle z_i, \tilde{z}_j \rangle \tilde{z}_i \right\| + \left\| \langle z_i, \tilde{z}_j \rangle \tilde{z}_i - \langle \tilde{z}_i, \tilde{z}_j \rangle \tilde{z}_i \right\| \\
&\quad \le \left(\|\mathcal{Z}\|_{\infty,2}^2 + \|\mathcal{Z}\|_{\infty,2} \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2} + \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2}^2 \right) \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2}
\end{aligned}$$

to get

$$\mathcal{I}_{431} \le \|\boldsymbol{\kappa}\|_{\infty,1} \left(\|\mathcal{Z}\|_{\infty,2}^2 + \|\mathcal{Z}\|_{\infty,2} \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2} + \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2}^2 \right) \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2}.$$

Similarly we have

$$\mathcal{I}_{432} \le \|\boldsymbol{\kappa}\|_{\infty,1} \left(\|\mathcal{Z}\|_{\infty,2}^2 + \|\mathcal{Z}\|_{\infty,2} \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2} + \left\| \tilde{\mathcal{Z}} \right\|_{\infty,2}^2 \right) \left\| \mathcal{Z} - \tilde{\mathcal{Z}} \right\|_{\infty,2}.$$

Finally, we combine estimates for $\mathcal{I}_{41}, \mathcal{I}_{42}$ and $\mathcal{I}_{43}$ to obtain

$$\begin{aligned} &\left\| F(\mathcal{Z}) - F(\tilde{\mathcal{Z}}) \right\|_\infty \\ &\quad \le \left(\|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} + 2(\lambda_0+\lambda_1)\|\boldsymbol{\kappa}\|_{\infty,1} \left(\|\mathcal{Z}\|_{\infty,2}^2 + \|\mathcal{Z}\|_{\infty,2} \left\|\tilde{\mathcal{Z}}\right\|_{\infty,2} + \left\|\tilde{\mathcal{Z}}\right\|_{\infty,2}^2 \right)\right) \left\|\mathcal{Z} - \tilde{\mathcal{Z}}\right\|_{\infty,2}. \end{aligned} \tag{A2}$$

◇ Step C (Local existence of an integral equation): We integrate the infinite LHS model to see

$$\mathcal{X}(t) = \mathcal{X}^{\mathrm{in}} + \int_0^t F(\mathcal{X}(s))\, ds.$$

Then, the solution to this integral equation is given as a fixed point of the operator:

$$\Phi : \mathcal{C}\left(C_{t_0,r}\right) \to \mathcal{C}\left(C_{t_0,r}\right), \quad (\Phi\mathcal{Z})(t) := \mathcal{Z}^{\mathrm{in}} + \int_0^t F(\mathcal{Z}(s))\, ds \tag{A3}$$

for suitable Banach space $\mathcal{C}\left(C_{t_0,r}\right)$ to be defined below. We set

$$\begin{aligned} &L := 27\left(\|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} + 2(\lambda_0+\lambda_1)\|\boldsymbol{\kappa}\|_{\infty,1}\right), \quad t_0 < \frac{1}{L}, \\ &B_r\left(\mathcal{Z}^{\mathrm{in}}\right) := \left\{ \mathcal{Y} \in \ell_{\mathbb{C}}^{\infty,2} : \left\|\mathcal{Y} - \mathcal{Z}^{\mathrm{in}}\right\|_{\infty,2} \le r \right\}, \quad C_{t_0,r} := [0,t_0] \times B_r\left(\mathcal{Z}^{\mathrm{in}}\right). \end{aligned} \tag{A4}$$

Then, we define a normed space and the associated norm as follows.

$$\mathcal{C}\left(C_{t_0,2}\right) := \left\{ f : [0,t_0] \to B_2\left(\mathcal{Z}^{\mathrm{in}}\right) \mid f \text{ is continuous} \right\}, \quad \|\mathcal{Z}\|_c := \sup_{0\le t\le t_0} \|\mathcal{Z}(t)\|_{\infty,2}.$$

For $\mathcal{X} \in \mathcal{C}\left(C_{t_0,2}\right)$ we have

$$\|\mathcal{Z}(t)\|_{\infty,2} \le \left\|\mathcal{Z}(t) - \mathcal{Z}^{\mathrm{in}}\right\|_{\infty,2} + \left\|\mathcal{Z}^{\mathrm{in}}\right\|_{\infty,2} \le 3.$$

Then, we use (A3) and (A4). to see that the functional Φ defined in (A3) satisfies

$$\begin{aligned} \left\|\Phi\mathcal{Z} - \mathcal{Z}^{\mathrm{in}}\right\|_{\infty,2} &\le \int_0^t \|F(\mathcal{Z}(s))\|_{\infty,2}\, ds \\ &\le \int_0^t \|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} \|\mathcal{Z}(s)\|_{\infty,2} + 2(\lambda_0+\lambda_1)\|\boldsymbol{\kappa}\|_{\infty,1} \|\mathcal{Z}(s)\|_{\infty,2}^3\, ds \\ &\le \int_0^t 3\|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} + 54(\lambda_0+\lambda_1)\|\boldsymbol{\kappa}\|_{\infty,1}\, ds \\ &= \left(3\|\boldsymbol{\Omega}\|_{\infty,\mathrm{op}} + 54(\lambda_0+\lambda_1)\|\boldsymbol{\kappa}\|_{\infty,1}\right) t \\ &< 2, \qquad \text{for } t \le t_0. \end{aligned}$$

We combine (A2) and (A4) gives

$$\|F(\mathcal{Z}_1(t)) - F(\mathcal{Z}_2(t))\|_{\infty,2} \le L\, \|\mathcal{Z}_2(t) - \mathcal{Z}_1(t)\|_{\infty,2}, \quad t \le t_0.$$

Hence we have

$$\begin{aligned} \|\Phi\mathcal{X}_1 - \Phi\mathcal{X}_2\|_\infty &\le \int_0^{t_0} \|F(\mathcal{X}_1(s)) - F(\mathcal{X}_2(s))\|_{\infty,2}\, ds \\ &\le L \int_0^{t_0} \|\mathcal{X}_2(s) - \mathcal{X}_1(s)\|_{\infty,2}\, ds \\ &\le L t_0 \|\mathcal{X}_2 - \mathcal{X}_1\|_c. \end{aligned}$$

Since $t_0 < \frac{1}{L}$, the relation implies that Φ is a contraction mapping. Then, by the Banach fixed point theorem, we obtain the existence of integral solution $\mathcal{X}(t) \in \mathcal{C}\left(C_{t_0,2}\right)$.

• Step D (Local existence of solution): Next, we show that the fixed point $\mathcal{X}(t)$ is differentiable: for each $t, s \in [0, t_1]$,

$$
\begin{aligned}
&\|\mathcal{X}(t) - \mathcal{X}(s) - (t-s)F(\mathcal{X}(s))\|_{\infty,2} = \left\| \int_s^t F(\mathcal{X}(\tau)) - F(\mathcal{X}(s))\, d\tau \right\|_{\infty,2} \\
&\le L \int_s^t \|\mathcal{X}(\tau) - \mathcal{X}(s)\|_{\infty,2}\, d\tau = L \int_s^t \left\| \int_s^\tau F(\mathcal{X}(\sigma))\, d\sigma \right\|_{\infty,2} d\tau \\
&\le L \int_s^t \int_s^\tau \|F(\mathcal{X}(\sigma))\|_{\infty,2}\, d\sigma d\tau \le L^2 \int_s^t \int_s^\tau d\sigma d\tau = \frac{L^2}{2}(t-s)^2 .
\end{aligned}
$$

This gives

$$\lim_{t\to s} \frac{\|\mathcal{X}(t) - \mathcal{X}(s) - (t-s)F(\mathcal{X}(s))\|_{\infty,2}}{|t-s|} = 0.$$

Therefore, the fixed point $\mathcal{X}$ is the desired solution in the time interval $[0, t_0)$:

$$\frac{d}{dt}\mathcal{X}(t) = F(\mathcal{X}(t)).$$

Furthermore by Lemma A.1, this local solution is unique. □

A.2 A Global Well-Posedness

In this part, we provide a global well-posedness by extending the local solution which was constructed in the previous subsection. More precisely, our global well-posedness can be stated as follows.

Theorem A.2 (A global existence) *For any $T \in (0, \infty)$, the Cauchy problem* (1.3)–(1.4) *admits a global unique smooth solution $\mathcal{Z} : [0, T) \to \ell^{\infty,2}_{\mathbb{C}}$.*

Proof By Theorem A.1, we have a local solution $\mathcal{Z} : [0, t_0) \to \ell^{\infty,2}_{\mathbb{C}}$ where t_0 depends on the parameters $\boldsymbol{\kappa}$, $\boldsymbol{\Omega}$, λ_1 and λ_2 in our model. We proceed by induction on $n \ge 1$ to prove the existence of solution $\mathcal{Z}$ in the time interval $[0, nt_0)$. The initial step has already verified in Theorem A.1. For the inductive step, it sufficies to check how the domain can be extended by $[0, 2t_0)$. Since our local solution $\mathcal{Z} : [0, t_0) \to \ell^{\infty,2}_{\mathbb{C}}$ defined as the fixed point of operator

$$\Phi : \mathcal{C}\left(C_{t_0,2}\right) \to \mathcal{C}\left(C_{t_0,2}\right), \quad (\Phi\mathcal{Z})(t) = \mathcal{Z}^{\text{in}} + \int_0^t F(\mathcal{Z}(s))\, ds,$$

where

$$C_{t_0,2} := [0, t_0] \times \left\{ \mathcal{Y} : \left\| \mathcal{Y} - \mathcal{Z}^{\text{in}} \right\|_{\infty,2} \le 2 \right\},$$

$\mathcal{Z}$ cannot blow up at $t = t_0$. Therefore $\mathcal{Z}$ is defined at $[0, t_0]$. By Lemma 2.1, we can consider $\mathcal{Z}(t_0)$ as new initial data, and we can apply Theorem A.1 to extend the local solution to the interval $[t_0, 2t_0]$, since the estimates in Step C of Theorem A.1 depends on the estimate:

$$\left\| \mathcal{Z}^{\text{in}} \right\|_{\infty,2} \le 1.$$

In this way, we have the solution in the time interval $[0, 2t_0]$. □

Appendix B Proof of Theorem 5.1

In this appendix, we provide the lengthy proof of Theorem 5.1 in several steps.

• Step A (A dynamical system for two-point correlation function): We set

$$h_{ij} := \langle z_i, z_j \rangle, \quad R_{ij} := \mathfrak{Re} h_{ij}, \quad I_{ij} := \mathfrak{Im} h_{ij}.$$

We use (5.9) to see

$$\begin{aligned}\frac{dh_{ij}}{dt} &= \langle z_c - \langle z_c, z_i \rangle z_i, z_j \rangle + \langle z_i, z_c - \langle z_c, z_j \rangle z_j \rangle \\ &= \left(\langle z_c, z_j \rangle - \langle z_i, z_c \rangle \langle z_i, z_j \rangle + \langle z_i, z_c \rangle - \langle z_c, z_j \rangle \langle z_i, z_j \rangle \right).\end{aligned} \tag{B1}$$

Then, we take the real and imaginary parts of (B1) to find

$$\begin{aligned}\frac{dR_{ij}}{dt} &= R_{cj} - R_{ic}R_{ij} + R_{ic} - R_{cj}R_{ij} + I_{ic}I_{ij} + I_{cj}I_{ij} \\ &= \left(1 - R_{ij}\right)\left(R_{cj} + R_{ic}\right) + I_{ij}\left(I_{ic} + I_{cj}\right), \\ \frac{dI_{ij}}{dt} &= I_{cj} - I_{ic}R_{ij} - R_{ic}I_{ij} + I_{ic} - I_{cj}R_{ij} - R_{cj}I_{ij} \\ &= \left(1 - R_{ij}\right)\left(I_{cj} + I_{ic}\right) - I_{ij}\left(R_{ic} + R_{cj}\right).\end{aligned} \tag{B2}$$

For notational simplicity, we also use the following handy notation in (B2):

$$R_{ic} := \mathfrak{Re} \langle z_i, z_c \rangle = \sum_{l \in \mathbb{N}} \kappa_l \mathfrak{Re} \langle z_i, z_l \rangle, \quad I_{ic} := \mathfrak{Re} \langle z_i, z_c \rangle = \sum_{l \in \mathbb{N}} \kappa_l \mathfrak{Re} \langle z_i, z_l \rangle.$$

Similarly we can define R_{cj} and I_{cj}. Since we are looking for a sufficient framework in which R_{ij} approaches to one asymptotically, it would be nice to work with $1 - R_{ij}$ instead of R_{ij}. Hence, we set

$$H_{ij} = 1 - R_{ij}, \quad H_{ic} := \sum_{l \in \mathbb{N}} \kappa_l \left(1 - \mathfrak{Re} \langle z_i, z_l \rangle\right) = 1 - R_{ic}.$$

Then the system (B2) can be rewritten as

$$\begin{aligned}\frac{dH_{ij}}{dt} &= -H_{ij}\left(2 - H_{cj} - H_{ic}\right) - I_{ij}\left(I_{ic} + I_{cj}\right), \\ \frac{dI_{ij}}{dt} &= H_{ij}\left(I_{cj} + I_{ic}\right) - I_{ij}\left(2 - H_{ic} - H_{cj}\right).\end{aligned}$$

This is equivalent to

$$\frac{d}{dt}\begin{bmatrix} H_{ij} \\ I_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha_{ij} & -\beta_{ij} \\ \beta_{ij} & -\alpha_{ij} \end{bmatrix}\begin{bmatrix} H_{ij} \\ I_{ij} \end{bmatrix}, \tag{B3}$$

where α_{ij} and β_{ij} in (B3) are given by

$$\alpha_{ij} := 2 - H_{cj} - H_{ic}, \quad \beta_{ij} := I_{ic} + I_{cj}.$$

Now we use (B3) to find the Grönwall-type inequality for $H_{ij}^2 + I_{ij}^2$:

$$\begin{aligned}\frac{d}{dt}\left(H_{ij}^2 + I_{ij}^2\right) &= 2H_{ij}\dot{H}_{ij} + 2I_{ij}\dot{I}_{ij} \\ &= 2H_{ij}\left(-\alpha_{ij}H_{ij} - \beta_{ij}I_{ij}\right) + 2I_{ij}\left(\beta_{ij}H_{ij} - \alpha_{ij}I_{ij}\right) \\ &= -2\alpha_{ij}\left(H_{ij}^2 + I_{ij}^2\right).\end{aligned} \tag{B4}$$

We define

$$\lambda(t) = \sup_{i \neq j} \sqrt{H_{ij}^2(t) + I_{ij}^2(t)}.$$

Then the assumption (5.14) implies

$$\lambda(0) < 1 - \delta.$$

- Step B (Estimation of $\left|1 - \langle z_i(t), z_j(t)\rangle\right|$): By a direct calculation, for each $s < t$,

$$\left|H_{cj}(t) - H_{cj}(s)\right| \le \sum_{l \in \mathbb{N}} \kappa_l \left|H_{jl}(t) - H_{jl}(s)\right| \le \sum_{l \in \mathbb{N}} \kappa_l \left|h_{jl}(t) - h_{jl}(s)\right| \le 8(t - s),$$

since we can see that $h_{ij} = \langle z_i, z_j\rangle$ is Lipschitz:

$$\left|\frac{d}{dt}\langle z_i, z_j\rangle\right| \le \left|\langle \dot{z}_i, z_j\rangle\right| + \left|\langle z_i, \dot{z}_j\rangle\right| \le \|\dot{z}_i\| \left\|z_j\right\| + \|z_i\| \left\|\dot{z}_j\right\| \le 16.$$

Here we obtain local bound

$$\alpha_{ij}(t) \ge \alpha_{ij}(0) - t \cdot 16 \ge \delta, \quad 0 \le t < \frac{\delta}{16} \tag{B5}$$

of α_{ij} from

$$\alpha_{ij}(0) = 2\|\kappa\|_1 - H_{cj} - H_{ic} \ge 2\|\kappa\|_1 - 2\|\kappa\|_1(1 - \delta) = 2\delta,$$
$$\left|\alpha_{ij}(t) - \alpha_{ij}(s)\right| \le \left|H_{cj}(t) - H_{cj}(s)\right| + |H_{ic}(t) - H_{ic}(s)| \le 16(t - s).$$

Then we use (B4) and (B5) to obtain

$$\frac{d}{dt}\left(H_{ij}^2(t) + I_{ij}^2(t)\right) \le -2\delta\left(H_{ij}^2(0) + I_{ij}^2(0)\right), \quad 0 \le t < \frac{\delta}{16} =: t_0.$$

This yields

$$H_{ij}^2(t) + I_{ij}^2(t) \le \left(H_{ij}^2(0) + I_{ij}^2(0)\right) \cdot \exp(-2\delta t), \quad 0 \le t < t_0.$$

By induction on n, we can prove that $H_{ij}^2(t) + I_{ij}^2(t)$ is exponentially decreases for $0 \le t < nt_0$ with exponential decay rate 2δ. For the inductive step, we can consider $\mathcal{Z}(nt_0)$ as new initial data. Then, we have

$$\lambda(nt_0) < 1 - \delta,$$

and we can prove that $H_{ij}^2(t) + I_{ij}^2(t)$ is exponentially decrease with decay rate 2δ for $nt_0 \le t < (n + 1)t_0$ by a similar argument as in the initial step ($n = 1$).

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