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Compactness Property of the Linearized Boltzmann Collision Operator for a Mixture of Monatomic and Polyatomic Species

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Abstract

The linearized Boltzmann collision operator has a central role in many important applications of the Boltzmann equation. Recently some important classical properties of the linearized collision operator for monatomic single species were extended to multicomponent monatomic gases and polyatomic single species. For multicomponent polyatomic gases, the case where the polyatomicity is modelled by a discrete internal energy variable was considered lately. Here we consider the corresponding case for a continuous internal energy variable. Compactness results, stating that the linearized operator can be decomposed into a sum of a positive multiplication operator, the collision frequency, and a compact operator, bringing e.g., self-adjointness, is extended from the classical result for monatomic single species, under reasonable assumptions on the collision kernel. With a probabilistic formulation of the collision operator as a starting point, the compactness property is shown by a decomposition, such that the terms are, or at least are uniform limits of, Hilbert–Schmidt integral operators and therefore are compact operators. Moreover, bounds on—including coercivity of—the collision frequency are obtained for hard sphere like, as well as hard potentials with cutoff like, models, from which Fredholmness of the linearized collision operator follows, as well as its domain.

1 Introduction

The Boltzmann equation is a fundamental equation of kinetic theory of gases, e.g., for computations of the flow around a space shuttle in the upper atmosphere during reentry [1]. Studies of the main properties of the linearized collision operator are of great importance in gaining increased knowledge about related problems, see, e.g., [12] and references therein.

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The linearized collision operator, obtained by considering deviations of an equilibrium, or Maxwellian, distribution, can in a natural way be written as a sum of a positive multiplication operator—the collision frequency—and an integral operator $-K$. Compact properties of the integral operator K (for angular cut-off kernels) are extensively studied for monatomic single species, see, e.g., [12, 14, 18, 21], and more recently for monatomic multi-component mixtures [4, 8]. Extensions to polyatomic single species, where the polyatomicity is modeled by either a discrete, or, a continuous internal energy variable [4, 5] and polyatomic multicomponent mixtures [2], where the polyatomicity is modeled by discrete internal energy variables, have also been conducted. For models, assuming a continuous internal energy variable, see also [7] for the case of molecules undergoing resonant collisions (for which internal energy and kinetic energy, respectively, are conserved under collisions), and [10, 11] for diatomic and polyatomic gases, respectively—with more restrictive—concerning the models considered in [16], allowing some others in return—assumptions on the collision kernels than in [5], but also a more direct approach. The integral operator can be written as the sum of a Hilbert–Schmidt integral operator and an approximately Hilbert–Schmidt integral operator—which is a uniform limit of Hilbert–Schmidt integral operators (cf. Lemma 4 in Sect. 4) [17], and so compactness of the integral operator K can be obtained. In this work, we extend the results of [4, 5] for monatomic multicomponent mixtures and polyatomic single species, where the polyatomicity is modeled by a continuous internal energy variable [9, 16], to the case of multicomponent mixtures of monatomic and/or polyatomic gases, where the polyatomicity is modeled by a continuous internal energy variable [1, 13]. To consider mixtures of monatomic and polyatomic molecules are of highest relevance in, e.g., the upper atmosphere [1].

Following the lines of [2, 4, 5], motivated by an approach by Kogan in [20, Sect. 2.8] for the monatomic single species case, a probabilistic formulation of the collision operator is considered as the starting point. With this approach, it is shown that the integral operator K can be written as a sum of Hilbert–Schmidt integral operators and operators, which are uniform limits of Hilbert–Schmidt integral operators— and so compactness of the integral operator K follows. The operator K is self-adjoint, as well as the collision frequency. Thus the linearized collision operator, as the sum of two self-adjoint operators whereof (at least) one is bounded, is also self-adjoint.

For models corresponding to hard sphere models, as well as hard potentials with cut off models, in the monatomic case, bounds on the collision frequency are obtained. Here we also want to point out reference [15], where the corresponding upper bound in [5] for the single species case was improved. Then the collision frequency is coercive and becomes a Fredholm operator. The set of Fredholm operators is closed under addition with compact operators. Therefore, also the linearized collision operator becomes a Fredholm operator by the compactness of the integral operator K. The Fredholm property is vital in the Chapman-Enskog process, and the Fredholmness of the linearized operator supply with the Fredholm property, taken for granted in [1], for those models. Note that for monatomic species, the linearized operator is not Fredholm for soft potential models, unlike for hard potential models. The domain of collision frequency—and, hence, of the linearized collision operator as well—follows directly by the obtained bounds

For hard sphere like models the linearized collision operator satisfies all the properties of the general linear operator in the abstract half-space problem considered in [3], and, hence, the existence results in [3] apply.

The rest of the paper is organized as follows. In Sect. 2, the model considered is presented. The probabilistic formulation of the collision operators considered and its relations to more classical formulations [1, 13] are accounted for in Sect. 2.1. Some results for the collision

operators in Sect. 2.2 and the linearized collision operator in Sect. 2.3 are reviewed. Section 3 is devoted to the main results of this paper, while the main proofs are addressed in Sects. 4 and 5; a proof of compactness of the integral operators K is presented in Sect. 4, while a proof of the bounds on the collision frequency appears in Sect. 5.

2 Model

This section concerns the model considered. Probabilistic formulations of the collision operators are considered, whose relations to more classical formulations are accounted for. Known properties of the models and corresponding linearized collision operators are also reviewed.

Consider a multicomponent mixture of s species $a_1, \ldots, a_s$, with s_0 monatomic and $s_1 := s - s_0$ polyatomic species, and masses $m_1, \ldots, m_s$, respectively, and introduce the index sets

$$\mathcal{I} = \{1, \ldots, s\}, \mathcal{I}_{mono} = \{\alpha; \ a_\alpha \text{ is monatomic}\} = \{1, \ldots, s_0\}, \text{ and}$$

$$\mathcal{I}_{poly} = \{\alpha; \ a_\alpha \text{ is polyatomic }\} = \{s_0 + 1, \ldots, s\}.$$

Here the polyatomicity is modeled by a continuous internal energy variable $I \in \mathbb{R}_+$ [6].

The distribution functions are of the form $f = (f_1, \ldots, f_s)$, where for $\alpha \in \mathcal{I}$ the component $f_\alpha = f_\alpha(t, \mathbf{x}, \mathbf{Z})$, with

$$\mathbf{Z} = \mathbf{Z}_\alpha := \begin{cases} \boldsymbol{\xi} & \text{for } \alpha \in \mathcal{I}_{mono} \\ (\boldsymbol{\xi}, I) & \text{for } \alpha \in \mathcal{I}_{poly} \end{cases}, \tag{1}$$

$\{t, I\} \subset \mathbb{R}_+$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, and $\boldsymbol{\xi} = (\xi_x, \xi_y, \xi_z) \in \mathbb{R}^3$, is the distribution function for species a_α.

Moreover, consider the real Hilbert space

$$\mathfrak{h} := \left(L^2\left(d\boldsymbol{\xi}\right)\right)^{s_0} \times \left(L^2\left(d\boldsymbol{\xi} dI\right)\right)^{s_1},$$

with inner product

$$(f, g) = \sum_{\alpha=1}^{s_0} \int_{\mathbb{R}^3} f_\alpha g_\alpha \, d\boldsymbol{\xi} + \sum_{\alpha=s_0+1}^{s} \int_{\mathbb{R}^3 \times \mathbb{R}_+} f_\alpha g_\alpha \, d\boldsymbol{\xi} dI, \text{ for } \{f, g\} \subset \mathfrak{h}.$$

The evolution of the distribution functions is (in the absence of external forces) described by the (vector) Boltzmann equation

$$\frac{\partial f}{\partial t} + (\boldsymbol{\xi} \cdot \nabla_{\mathbf{x}}) f = Q(f, f), \tag{2}$$

where the (vector) collision operator $Q = (Q_1, \ldots, Q_s)$ is a quadratic bilinear operator that accounts for the change of velocities and internal energies of particles due to binary collisions (assuming that the gas is rarefied, such that other collisions are negligible). Here the component Q_α, with $\alpha \in \mathcal{I}$, is the collision operator for species a_α.

A collision between two particles of species a_α and a_β, where $\{\alpha, \beta\} \subseteq \mathcal{I}$, respectively, can be represented by two pre-collisional elements, each element consisting of a microscopic velocity and possibly also an internal energy, $\mathbf{Z}$ and $\mathbf{Z}_*$, and two corresponding post-collisional elements, $\mathbf{Z}'$ and $\mathbf{Z}'_*$. The notation for pre- and post-collisional pairs may be interchanged as well. Due to momentum and total energy conservation, the following relations have to be satisfied by the elements

$$m_\alpha \boldsymbol{\xi} + m_\beta \boldsymbol{\xi}_* = m_\alpha \boldsymbol{\xi}' + m_\beta \boldsymbol{\xi}'_*$$

and if $(\alpha, \beta) \in \mathcal{I}_{mono}^2$ (mono/mono-interactions)

$$m_\alpha \left|\boldsymbol{\xi}\right|^2 + m_\beta \left|\boldsymbol{\xi}_*\right|^2 = m_\alpha \left|\boldsymbol{\xi}'\right|^2 + m_\beta \left|\boldsymbol{\xi}'_*\right|^2,$$

if $(\alpha, \beta) \in \mathcal{I}_{mono} \times \mathcal{I}_{poly}$ (mono/poly-interactions)

$$m_\alpha \left|\boldsymbol{\xi}\right|^2 + m_\beta \left|\boldsymbol{\xi}_*\right|^2 + 2I_* = m_\alpha \left|\boldsymbol{\xi}'\right|^2 + m_\beta \left|\boldsymbol{\xi}'_*\right|^2 + 2I'_*,$$

if $(\alpha, \beta) \in \mathcal{I}_{poly} \times \mathcal{I}_{mono}$ (poly/mono-interactions)

$$m_\alpha \left|\boldsymbol{\xi}\right|^2 + m_\beta \left|\boldsymbol{\xi}_*\right|^2 + 2I = m_\alpha \left|\boldsymbol{\xi}'\right|^2 + m_\beta \left|\boldsymbol{\xi}'_*\right|^2 + 2I',$$

while if $(\alpha, \beta) \in \mathcal{I}_{poly}^2$ (poly/poly-interactions)

$$m_\alpha \left|\boldsymbol{\xi}\right|^2 + m_\beta \left|\boldsymbol{\xi}_*\right|^2 + 2I + 2I_* = m_\alpha \left|\boldsymbol{\xi}'\right|^2 + m_\beta \left|\boldsymbol{\xi}'_*\right|^2 + 2I' + 2I'_*.$$

2.1 Collision Operator

The (vector) collision operator $Q = (Q_1, \dots, Q_s)$ has components that can be written in the following form

$$Q_\alpha(f, f) = \sum_{\beta=1}^s Q_{\alpha\beta}(f, f) = \sum_{\beta=1}^s \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} W_{\alpha\beta} \Lambda_{\alpha\beta}(f) \, d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*,$$

where $\Lambda_{\alpha\beta}(f) := \dfrac{f'_\alpha f'_{\beta*}}{\varphi_\alpha(I') \varphi_\beta(I'_*)} - \dfrac{f_\alpha f_{\beta*}}{\varphi_\alpha(I) \varphi_\beta(I_*)}$

and $\mathcal{Z}_\gamma = \begin{cases} \mathbb{R}^3 \text{ if } \gamma \in \mathcal{I}_{mono} \\ \mathbb{R}^3 \times \mathbb{R}_+ \text{ if } \gamma \in \mathcal{I}_{poly} \end{cases}.$

Here and below the abbreviations

$$f_{\alpha*} = f_\alpha (t, \mathbf{x}, \mathbf{Z}_*), \ f'_\alpha = f_\alpha (t, \mathbf{x}, \mathbf{Z}'), \text{ and } f'_{\alpha*} = f_\alpha (t, \mathbf{x}, \mathbf{Z}'_*),$$

where $\mathbf{Z}_*, \mathbf{Z}',$ and $\mathbf{Z}'_*,$ are defined as the natural extension of definition (1), i.e. denoting $\mathbf{Z}_* = \mathbf{Z}_{*\alpha} = \begin{cases} \boldsymbol{\xi}_* & \text{for } \alpha \in \mathcal{I}_{mono} \\ (\boldsymbol{\xi}_*, I_*) & \text{for } \alpha \in \mathcal{I}_{poly} \end{cases}$ etc., are used for $\alpha \in \mathcal{I}$. Moreover, the degeneracies or renormalization weights $\varphi_\alpha = \varphi_\alpha(I)$, $\alpha \in \mathcal{I}$, with $\varphi_\alpha = 1$ for $\alpha \in \mathcal{I}_{mono}$, are positive functions for $I > 0$. A typical choice of the degeneracies is [16]

$$\varphi_\alpha(I) = I^{\delta^{(\alpha)}/2 - 1}, \alpha \in \mathcal{I},$$

where $\delta^{(1)} = \dots = \delta^{(s_0)} = 2$, while $\delta^{(\alpha)}$, with $\delta^{(\alpha)} \geq 2$, denote the number of internal degrees of freedom of the species for $\alpha \in \mathcal{I}_{poly}$. Our main results in Sect. 3 below will be stated and proven for this particular choice of degeneracies.

Note that in the literature it is usual to use a slightly different setting [1, 9, 13], where already renormalized distribution functions are considered, opting to consider a weighted measure—where the renormalization weights appear as weights—with respect to I. However, this is merely due to a different scaling of the distribution functions considered.

The transition probabilities $W_{\alpha\beta}$ are of the form, cf. [4, 5],

$$W_{\alpha\beta} = W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \left| \mathbf{Z}', \mathbf{Z}'_* \right)$$

$$= (m_\alpha + m_\beta)^2 m_\alpha m \varphi_\alpha(I') \varphi_\beta(I'_*) \sigma'_{\alpha\beta} \frac{|\mathbf{g}'|}{|\mathbf{g}|} \widehat{\delta}_1 \widehat{\delta}_3$$

$$= \left(m_\alpha + m_\beta\right)^2 m_\alpha m_\beta \varphi_\alpha\left(I\right) \varphi_\beta\left(I_*\right) \sigma_{\alpha\beta} \frac{|\mathbf{g}|}{|\mathbf{g}'|} \widehat{\delta}_1 \widehat{\delta}_3, \text{ where}$$

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}\left(|\mathbf{g}|, \cos\theta, I, I_*, I', I_*'\right) > 0 \text{ and}$$
$$\sigma_{\alpha\beta}' = \sigma_{\alpha\beta}\left(|\mathbf{g}'|, \cos\theta, I', I_*', I, I_*\right) > 0 \text{ a.e., with}$$

$$\widehat{\delta}_1 = \delta_1\left(\frac{1}{2}\left(m_\alpha |\boldsymbol{\xi}|^2 + m_\beta |\boldsymbol{\xi}_*|^2 - m_\alpha |\boldsymbol{\xi}'|^2 - m_\beta |\boldsymbol{\xi}_*'|^2\right) - \Delta I\right),$$

$$\widehat{\delta}_3 = \delta_3\left(m_\alpha \boldsymbol{\xi} + m_\beta \boldsymbol{\xi}_* - m_\alpha \boldsymbol{\xi}' - m_\beta \boldsymbol{\xi}_*'\right), \cos\theta = \frac{\mathbf{g} \cdot \mathbf{g}'}{|\mathbf{g}| |\mathbf{g}'|},$$

$$\mathbf{g} = \boldsymbol{\xi} - \boldsymbol{\xi}_*, \mathbf{g}' = \boldsymbol{\xi}' - \boldsymbol{\xi}_*', \text{ and } \Delta I = \left(I' - I\right) \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} + \left(I_*' - I_*\right) \mathbf{1}_{\beta \in \mathcal{I}_{poly}}. \quad (3)$$

Here δ_3 and δ_1 denote the Dirac's delta function in $\mathbb{R}^3$ and $\mathbb{R}$, respectively—$\widehat{\delta}_1$ and $\widehat{\delta}_3$ taking the conservation of momentum and total energy into account. Note that for $\alpha \in \mathcal{I}_{mono}$ the scattering cross sections $\sigma_{\alpha\beta}$ are independent of I and I', while correspondingly, for $\beta \in \mathcal{I}_{mono}$ the scattering cross sections $\sigma_{\alpha\beta}$ are independent of I_* and I_*'. Moreover, we have chosen—even if this means not being completely consistent—to not indicate the dependence of species in the notation, if—like for the energy gap ΔI—the dependence is only up to if the species are monatomic or polyatomic.

It is assumed that the scattering cross sections $\sigma_{\alpha\beta}$ for $(\alpha, \beta) \in \mathcal{I}^2$ satisfy the microreversibility conditions

$$\varphi_\alpha\left(I\right) \varphi_\beta\left(I_*\right) |\mathbf{g}|^2 \sigma_{\alpha\beta}\left(|\mathbf{g}|, \cos\theta, I, I_*, I', I_*'\right)$$
$$= \varphi_\alpha\left(I'\right) \varphi_\beta\left(I_*'\right) |\mathbf{g}'|^2 \sigma_{\alpha\beta}\left(|\mathbf{g}'|, \cos\theta, I', I_*', I, I_*\right). \quad (4)$$

Furthermore, to obtain invariance of change of particles in a collision, it is assumed that the scattering cross sections $\sigma_{\alpha\beta}$ for $(\alpha, \beta) \in \mathcal{I}^2$ satisfy the symmetry relations

$$\sigma_{\alpha\beta}\left(|\mathbf{g}|, \cos\theta, I, I_*, I', I_*'\right) = \sigma_{\beta\alpha}\left(|\mathbf{g}|, \cos\theta, I_*, I, I_*', I'\right), \quad (5)$$

while

$$\sigma_{\alpha\alpha} = \sigma_{\alpha\alpha}\left(|\mathbf{g}|, |\cos\theta|, I, I_*, I', I_*'\right) = \sigma_{\alpha\alpha}\left(|\mathbf{g}|, |\cos\theta|, I_*, I, I', I_*'\right)$$
$$= \sigma_{\alpha\alpha}\left(|\mathbf{g}|, |\cos\theta|, I_*, I, I_*', I'\right). \quad (6)$$

The invariance under change of particles in a collision, which follows directly by the definition of the transition probability (3) and the symmetry relations (5), (6) for the collision frequency, and the microreversibility of the collisions (4), implies that the transition probabilities (3) satisfy the relations

$$W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* | \mathbf{Z}', \mathbf{Z}_*') = W_{\beta\alpha}(\mathbf{Z}_*, \mathbf{Z} | \mathbf{Z}_*', \mathbf{Z}')$$
$$W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* | \mathbf{Z}', \mathbf{Z}_*') = W_{\alpha\beta}(\mathbf{Z}', \mathbf{Z}_*' | \mathbf{Z}, \mathbf{Z}_*)$$
$$W_{\alpha\alpha}(\mathbf{Z}, \mathbf{Z}_* | \mathbf{Z}', \mathbf{Z}_*') = W_{\alpha\alpha}(\mathbf{Z}, \mathbf{Z}_* | \mathbf{Z}_*', \mathbf{Z}'). \quad (7)$$

Applying known properties of Dirac's delta function, the transition probabilities may be transformed to

$$W_{\alpha\beta} = W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* | \mathbf{Z}', \mathbf{Z}_*')$$
$$= \left(m_\alpha + m_\beta\right)^2 m_\alpha m_\beta \varphi_\alpha\left(I'\right) \varphi_\beta\left(I_*'\right) \sigma_{\alpha\beta}' \frac{|\mathbf{g}'|}{|\mathbf{g}|} \delta_1 \left(\frac{\mu_{\alpha\beta}}{2}\left(|\mathbf{g}|^2 - |\mathbf{g}'|^2\right) - \Delta I\right)$$
$$\times \delta_3 \left(\left(m_\alpha + m_\beta\right)\left(\mathbf{G}_{\alpha\beta} - \mathbf{G}_{\alpha\beta}'\right)\right)$$

$$= 2\varphi_\alpha \left(I'\right) \varphi_\beta \left(I'_*\right) \sigma'_{\alpha\beta} \frac{|\mathbf{g}'|}{|\mathbf{g}|} \delta_1 \left(|\mathbf{g}|^2 - |\mathbf{g}'|^2 - \frac{2}{\mu_{\alpha\beta}} \Delta I\right) \delta_3 \left(\mathbf{G}_{\alpha\beta} - \mathbf{G}'_{\alpha\beta}\right)$$

$$= \varphi_\alpha \left(I'\right) \varphi_\beta \left(I'_*\right) \sigma'_{\alpha\beta} \frac{1}{|\mathbf{g}|} \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2 > 2\Delta I} \delta_3 \left(\mathbf{G}_{\alpha\beta} - \mathbf{G}'_{\alpha\beta}\right)$$

$$\times \delta_1 \left(\sqrt{|\mathbf{g}|^2 - \frac{2\Delta I}{\mu_{\alpha\beta}}} - |\mathbf{g}'|\right)$$

$$= \varphi_\alpha \left(I\right) \varphi_\beta \left(I_*\right) \sigma_{\alpha\beta} \frac{|\mathbf{g}|}{|\mathbf{g}'|^2} \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2 > 2\Delta I} \delta_1 \left(\sqrt{|\mathbf{g}|^2 - \frac{2\Delta I}{\mu_{\alpha\beta}}} - |\mathbf{g}'|\right)$$

$$\times \delta_3 \left(\mathbf{G}_{\alpha\beta} - \mathbf{G}'_{\alpha\beta}\right),$$

with $\mathbf{G}_{\alpha\beta} = \dfrac{m_\alpha \boldsymbol{\xi} + m_\beta \boldsymbol{\xi}_*}{m_\alpha + m_\beta}, \mathbf{G}'_{\alpha\beta} = \dfrac{m_\alpha \boldsymbol{\xi}' + m_\beta \boldsymbol{\xi}'_*}{m_\alpha + m_\beta}$, and $\mu_{\alpha\beta} = \dfrac{m_\alpha m_\beta}{m_\alpha + m_\beta}$.

Remark 1 Note that

$$\delta_1 \left(\frac{\mu_{\alpha\beta}}{2} \left(|\mathbf{g}|^2 - |\mathbf{g}'|^2\right) - \Delta I\right) = \delta_1 \left(E_{\alpha\beta} - E'_{\alpha\beta}\right),$$

where $E_{\alpha\beta} = \dfrac{\mu_{\alpha\beta}}{2} |\mathbf{g}|^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I_*$ is the total energy in the center of mass frame, and, correspondingly, $E'_{\alpha\beta} = \dfrac{\mu_{\alpha\beta}}{2} |\mathbf{g}'|^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I' + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I'_*$.

Observe that, by a series of change of variables:
$\{\boldsymbol{\xi}', \boldsymbol{\xi}'_*\} \rightarrow \left\{\mathbf{g}' = \boldsymbol{\xi}' - \boldsymbol{\xi}'_*, \mathbf{G}'_{\alpha\beta} = \dfrac{m_\alpha \boldsymbol{\xi}' + m_\beta \boldsymbol{\xi}'_*}{m_\alpha + m_\beta}\right\}$, followed by a change to spherical

coordinates $\left\{\mathbf{g}'\right\} \rightarrow \left\{|\mathbf{g}'|, \boldsymbol{\sigma} = \dfrac{\mathbf{g}'}{|\mathbf{g}'|}\right\}$,

$$d\boldsymbol{\xi}' d\boldsymbol{\xi}'_* = d\mathbf{G}'_{\alpha\beta} d\mathbf{g}' = |\mathbf{g}'|^2 d\mathbf{G}'_{\alpha\beta} d|\mathbf{g}'| d\boldsymbol{\sigma}$$

while, if a_β is polyatomic, $\beta \in \mathcal{I}_{poly}$, by applying the additional change of variables
$\left\{|\mathbf{g}'|, I'_*\right\} \rightarrow \left\{R = \dfrac{\mu_{\alpha\beta}}{2} \dfrac{|\mathbf{g}'|^2}{E_{\alpha\beta}}, E'_{\alpha\beta} = \dfrac{\mu_{\alpha\beta}}{2} |\mathbf{g}'|^2 + I' + I'_*\right\}$,

$$d\boldsymbol{\xi}' d\boldsymbol{\xi}'_* dI'_* = \sqrt{2} \left(\frac{E_{\alpha\beta}}{\mu_{\alpha\beta}}\right)^{3/2} R^{1/2} dR d\boldsymbol{\sigma} d\mathbf{G}'_{\alpha\beta} dE'_{\alpha\beta},$$

and, finally, if additionally $\alpha \in \mathcal{I}_{poly}$, the change $\{I'\} \rightarrow \left\{r = \dfrac{I'}{(1 - R) E_{\alpha\beta}}\right\}$ leads to

$$d\boldsymbol{\xi}' d\boldsymbol{\xi}'_* dI' dI'_* = \frac{\sqrt{2}}{\mu_{\alpha\beta}^{3/2}} E_{\alpha\beta}^{5/2} (1 - R) R^{1/2} dr dR d\boldsymbol{\sigma} d\mathbf{G}'_{\alpha\beta} dE'_{\alpha\beta}. \tag{8}$$

Then for two monatomic species, i.e., $(\alpha, \beta) \in \mathcal{I}_{mono}^2$, (mono/mono-case)

$$Q_{\alpha\beta}(f, f) = \int_{(\mathbb{R}^3)^2 \times \mathbb{R}_+ \times \mathbb{S}^2} W_{\alpha\beta} |\mathbf{g}'|^2 \left(f'_\alpha f'_{\beta*} - f_\alpha f_{\beta*}\right) d\boldsymbol{\xi}_* d\mathbf{G}'_{\alpha\beta} d|\mathbf{g}'| d\boldsymbol{\sigma}$$

$$= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{0\alpha\beta} \left(f'_\alpha f'_{\beta*} - f_\alpha f_{\beta*}\right) d\boldsymbol{\xi}_* d\boldsymbol{\sigma}, \text{ with } B_{0\alpha\beta} = \sigma_{\alpha\beta} |\mathbf{g}|,$$

or, with $\varphi_\alpha(I) = I^{\delta^{(\alpha)}/2-1}$, $\alpha \in \mathcal{I}$, for a monatomic species and a polyatomic species, i.e., either $(\alpha, \beta) \in \mathcal{I}_{mono} \times \mathcal{I}_{poly}$ (mono/poly-case)

$$
Q_{\alpha\beta}(f, f) = \int_{(\mathbb{R}^3)^2 \times \mathbb{R}_+^2 \times [0,1] \times \mathbb{S}^2} W_{\alpha\beta} \sqrt{2} \left(\frac{E_{\alpha\beta}}{\mu_{\alpha\beta}} \right)^{3/2} \left(\frac{f'_\alpha f'_{\beta*}}{\left(I'_* \right)^{\delta^{(\beta)}/2-1}} - \frac{f_\alpha f_{\beta*}}{I_*^{\delta^{(\beta)}/2-1}} \right)
$$
$$
\times R^{1/2} d\boldsymbol{\xi}_* d\mathbf{G}'_{\alpha\beta} dR d\boldsymbol{\sigma} dI_* dE'_{\alpha\beta}
$$
$$
= \int_{\mathbb{R}^3 \times \mathbb{R}_+ \times [0,1] \times \mathbb{S}^2} B_{1\alpha\beta} \left(\frac{f'_\alpha f'_{\beta*}}{\left(I'_* \right)^{\delta^{(\beta)}/2-1}} - \frac{f_\alpha f_{\beta*}}{I_*^{\delta^{(\beta)}/2-1}} \right) (1-R)^{\delta^{(\beta)}/2-1}
$$
$$
\times R^{1/2} I_*^{\delta^{(\beta)}/2-1} d\boldsymbol{\xi}_* dR d\boldsymbol{\sigma} dI_*,
$$
$$
\text{with } B_{1\alpha\beta} = \frac{\sigma_{\alpha\beta} \sqrt{2/\mu_{\alpha\beta}} \, |\mathbf{g}|}{\left(I'_* \right)^{\delta^{(\beta)}/2-1} \sqrt{|\mathbf{g}|^2 - \dfrac{2\Delta I}{\mu_{\alpha\beta}}}} E_{\alpha\beta}^{(\delta^{(\beta)}+1)/2} = \frac{\sigma_{\alpha\beta} |\mathbf{g}| E_{\alpha\beta}}{R^{1/2}(1-R)^{\delta^{(\beta)}/2-1}}
$$

or, $(\alpha, \beta) \in \mathcal{I}_{poly} \times \mathcal{I}_{mono}$ (poly/mono-case)

$$
Q_{\alpha\beta}(f, f) = \int_{(\mathbb{R}^3)^2 \times \mathbb{R}_+ \times [0,1] \times \mathbb{S}^2} W_{\alpha\beta} \sqrt{2} \left(\frac{E_{\alpha\beta}}{\mu_{\alpha\beta}} \right)^{3/2} \left(\frac{f'_\alpha f'_{\beta*}}{\left(I' \right)^{\delta^{(\alpha)}/2-1}} - \frac{f_\alpha f_{\beta*}}{I^{\delta^{(\alpha)}/2-1}} \right)
$$
$$
\times R^{1/2} d\boldsymbol{\xi}_* d\mathbf{G}'_{\alpha\beta} dR d\boldsymbol{\sigma} dE'_{\alpha\beta}
$$
$$
= \int_{\mathbb{R}^3 \times [0,1] \times \mathbb{S}^2} B_{1\beta\alpha} \left(\frac{f'_\alpha f'_{\beta*}}{\left(I' \right)^{\delta^{(\alpha)}/2-1}} - \frac{f_\alpha f_{\beta*}}{I^{\delta^{(\alpha)}/2-1}} \right) (1-R)^{\delta^{(\alpha)}/2-1} R^{1/2}
$$
$$
\times I^{\delta^{(\alpha)}/2-1} d\boldsymbol{\xi}_* dR d\boldsymbol{\sigma}
$$

and, finally, again with $\varphi_\alpha(I) = I^{\delta^{(\alpha)}/2-1}$, $\alpha \in \mathcal{I}$, for two polyatomic species, i.e., with $(\alpha, \beta) \in \mathcal{I}_{poly}^2$ (poly/poly-case)

$$
Q_{\alpha\beta}(f, f) = \int_{(\mathbb{R}^3)^2 \times \mathbb{R}_+^2 \times [0,1]^2 \times \mathbb{S}^2} \left(\frac{f'_\alpha f'_{\beta*}}{\left(I' \right)^{\delta^{(\alpha)}/2-1} \left(I'_* \right)^{\delta^{(\beta)}/2-1}} - \frac{f_\alpha f_{\beta*}}{I^{\delta^{(\alpha)}/2-1} I_*^{\delta^{(\beta)}/2-1}} \right)
$$
$$
\times W_{\alpha\beta} \frac{\sqrt{2}}{\mu_{\alpha\beta}^{3/2}} E_{\alpha\beta}^{5/2} (1-R) R^{1/2} d\boldsymbol{\xi}_* dr dR d\boldsymbol{\sigma} d\mathbf{G}'_{\alpha\beta} dE'_{\alpha\beta} dI_*
$$
$$
= \int_{\mathbb{R}^3 \times \mathbb{R}_+ \times [0,1]^2 \times \mathbb{S}^2} B_{2\alpha\beta} \left(\frac{f'_\alpha f'_{\beta*}}{\left(I' \right)^{\delta^{(\alpha)}/2-1} \left(I'_* \right)^{\delta^{(\beta)}/2-1}} - \frac{f_\alpha f_{\beta*}}{I^{\delta^{(\alpha)}/2-1} I_*^{\delta^{(\beta)}/2-1}} \right)
$$
$$
\times r^{\delta^{(\alpha)}/2-1} (1-r)^{\delta^{(\beta)}/2-1} (1-R)^{(\delta^{(\alpha)}+\delta^{(\beta)})/2-1} R^{1/2}
$$
$$
\times I^{\delta^{(\alpha)}/2-1} I_*^{\delta^{(\beta)}/2-1} d\boldsymbol{\xi}_* dr dR d\boldsymbol{\sigma} dI_*, \text{ with}
$$
$$
B_{2\alpha\beta} = \sigma_{\alpha\beta} \frac{\sqrt{2/\mu_{\alpha\beta}} \, |\mathbf{g}| E_{\alpha\beta}^{(\delta^{(\alpha)}+\delta^{(\beta)}+1)/2}}{\left(I' \right)^{\delta^{(\alpha)}/2-1} \left(I'_* \right)^{\delta^{(\beta)}/2-1} \sqrt{|\mathbf{g}|^2 - \dfrac{2\Delta I}{\mu_{\alpha\beta}}}}
$$
$$
= \frac{\sigma_{\alpha\beta} |\mathbf{g}| E_{\alpha\beta}^2}{r^{\delta^{(\alpha)}/2-1} (1-r)^{\delta^{(\beta)}/2-1} (1-R)^{(\delta^{(\alpha)}+\delta^{(\beta)})/2-2} R^{1/2}}
$$

where

$$
\begin{cases}
\boldsymbol{\xi}' = \mathbf{G}_{\alpha\beta} + \boldsymbol{\sigma} \dfrac{m_\beta}{m_\alpha + m_\beta} \sqrt{|\mathbf{g}|^2 - \dfrac{2\Delta I}{\mu_{\alpha\beta}}} \\[3mm]
\boldsymbol{\xi}'_* = \mathbf{G}_{\alpha\beta} - \boldsymbol{\sigma} \dfrac{m_\alpha}{m_\alpha + m_\beta} \sqrt{|\mathbf{g}|^2 - \dfrac{2\Delta I}{\mu_{\alpha\beta}}}
\end{cases} .
$$

This results in more familiar forms of the Boltzmann collision operators for mixtures of monatomic and/or polyatomic molecules modeled with a continuous energy variable, cf., e.g., [1, 13]. Here and below the internal energy gaps are given by $\Delta I = 0$ in the mono/mono-case, $\Delta I = I'_* - I_*$ in the mono/poly-case, $\Delta I = I' - I$ in the poly/mono-case, while in the poly/poly-case $\Delta I = I' + I'_* - I - I_*$.

2.2 Collision Invariants and Maxwellian Distributions

The following lemma follows directly by the relations (7).

Lemma 1 *The measures*

$$
dA_{\alpha\beta} = W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) d\mathbf{Z} d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*
$$

are invariant under the (ordered) interchange

$$
(\mathbf{Z}, \mathbf{Z}_*) \leftrightarrow (\mathbf{Z}', \mathbf{Z}'_*), \tag{9}
$$

of variables for $(\alpha, \beta) \in \mathcal{I}^2$, while

$$
dA_{\alpha\beta} + dA_{\beta\alpha} \text{ for } (\alpha, \beta) \in \mathcal{I}^2
$$

are invariant under the (ordered) interchange of variables

$$
(\mathbf{Z}, \mathbf{Z}') \leftrightarrow (\mathbf{Z}_*, \mathbf{Z}'_*). \tag{10}
$$

The weak form of the collision operator $Q(f, f)$ reads

$$
\begin{aligned}
(Q(f, f), g) &= \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Lambda_{\alpha\beta}(f) g_\alpha \, dA_{\alpha\beta} \\
&= \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Lambda_{\alpha\beta}(f) g_{\beta*} \, dA_{\alpha\beta} \\
&= -\sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Lambda_{\alpha\beta}(f) g'_\alpha \, dA_{\alpha\beta} \\
&= -\sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Lambda_{\alpha\beta}(f) g'_{\beta*} \, dA_{\alpha\beta}
\end{aligned}
$$

for any function $g = (g_1, \ldots, g_s)$, with $g_\alpha = g_\alpha(\mathbf{Z})$, such that the first integrals are defined for all $(\alpha, \beta) \in \mathcal{I}^2$, while the following equalities are obtained by applying Lemma 1.

Denote for any function $g = (g_1, \ldots, g_s)$, with $g_\alpha = g_\alpha(\mathbf{Z})$,

$$\Delta_{\alpha\beta}(g) := g_\alpha + g_{\beta*} - g'_\alpha - g'_{\beta*}.$$

We have the following proposition.

Proposition 1 *Let $g = (g_1, \ldots, g_s)$, where $g_\alpha = g_\alpha(\mathbf{Z})$, be such that*

$$\int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Lambda_{\alpha\beta}(f) g_\alpha \, dA_{\alpha\beta}, \text{ with } \Lambda_{\alpha\beta}(f) = \frac{f'_\alpha f'_{\beta*}}{\varphi_\alpha(I')\varphi_\beta(I'_*)} - \frac{f_\alpha f_{\beta*}}{\varphi_\alpha(I)\varphi_\beta(I_*)},$$

is defined for any $(\alpha, \beta) \in \mathcal{I}^2$. Then

$$(Q(f,f), g) = \frac{1}{4} \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Lambda_{\alpha\beta}(f) \Delta_{\alpha\beta}(g) \, dA_{\alpha\beta}.$$

Definition 1 A function $g = (g_1, \ldots, g_s)$, where $g_\alpha = g_\alpha(\mathbf{Z})$, is a collision invariant if

$$\Delta_{\alpha\beta}(g) \, W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) = 0 \text{ a.e.}$$

for all $(\alpha, \beta) \in \mathcal{I}^2$.

It is clear that $e_1, \ldots, e_s, m\xi_x, m\xi_y, m\xi_z$, and $m\,|\boldsymbol{\xi}|^2 + 2\mathbb{I}$, denoting here $m = (m_1, \ldots, m_s)$, $\mathbb{I} = \underbrace{(0, \ldots, 0,}_{s_0} \underbrace{I, \ldots, I)}_{s_1 = s - s_0}$, and by $\{e_1, \ldots, e_s\}$ the standard basis of $\mathbb{R}^s$, are collision invariants—corresponding to conservation of mass(es), momentum, and total energy.

In fact, we have the following proposition, cf. [1, 13].

Proposition 2 *Let $m = (m_1, \ldots, m_s)$, $\mathbb{I} = \underbrace{(0, \ldots, 0,}_{s_0} \underbrace{I, \ldots, I)}_{s_1}$, and $\{e_1, \ldots, e_s\}$ be the standard basis of $\mathbb{R}^s$. Then the vector space of collision invariants is generated by*

$$\left\{ e_1, \ldots, e_s, m\xi_x, m\xi_y, m\xi_z, m\,|\boldsymbol{\xi}|^2 + 2\mathbb{I} \right\}.$$

Define

$$\mathcal{W}[f] := \left(Q(f,f), \log\left(\varphi^{-1} f\right) \right),$$

where $\varphi = \varphi(I) = \text{diag}\left(\varphi_1(I), \ldots, \varphi_s(I)\right)$. It follows by Proposition 1 that

$$\mathcal{W}[f] = -\frac{1}{4} \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \left(\frac{\varphi_\alpha(I)\varphi_\beta(I_*) f'_\alpha f'_{\beta*}}{f_\alpha f_{\beta*}\varphi_\alpha(I')\varphi_\beta(I'_*)} - 1 \right)$$

$$\times \log\left(\frac{\varphi_\alpha(I)\varphi_\beta(I_*) f'_\alpha f'_{\beta*}}{f_\alpha f_{\beta*}\varphi_\alpha(I')\varphi_\beta(I'_*)} \right) \frac{f_\alpha f_{\beta*}}{\varphi_\alpha(I)\varphi_\beta(I_*)} \, dA_{\alpha\beta}.$$

Since $(x - 1)\log(x) \geq 0$ for $x > 0$, with equality if and only if $x = 1$,

$$\mathcal{W}[f] \leq 0,$$

with equality if and only if for all $(\alpha, \beta) \in \mathcal{I}^2$

$$\Lambda_{\alpha\beta}(f) W_{\alpha\beta} = 0 \text{ a.e.,} \tag{11}$$

or, equivalently, if and only if

$$Q(f, f) \equiv 0.$$

For any equilibrium, or, Maxwellian, distribution $M = (M_1, \ldots, M_s)$, it follows by identity (11), since $Q(M, M) \equiv 0$, that

$$\left(\log \frac{M_\alpha}{\varphi_\alpha (I)} + \log \frac{M_{\beta*}}{\varphi_\beta (I_*)} - \log \frac{M'_\alpha}{\varphi_\alpha (I')} - \log \frac{M'_{\beta*}}{\varphi_\beta (I'_*)} \right) W_{\alpha\beta} = 0 \text{ a.e.}.$$

Hence, $\log \left(\varphi^{-1} M \right) = \left(\log \dfrac{M_1}{\varphi_1 (I)}, \ldots, \log \dfrac{M_s}{\varphi_s (I)} \right)$ is a collision invariant, and the components of the Maxwellian distributions $M = (M_1, \ldots, M_s)$ are of the form

$$M_\alpha = \begin{cases} \dfrac{n_\alpha m_\alpha^{3/2}}{(2\pi k_B T)^{3/2}} e^{-m_\alpha |\boldsymbol{\xi} - \mathbf{u}|^2/(2k_B T)} & \text{if } \alpha \in \mathcal{I}_{mono} \\[4mm] \dfrac{n_\alpha \varphi_\alpha (I) \, m_\alpha^{3/2} e^{-\left(m_\alpha |\boldsymbol{\xi} - \mathbf{u}|^2 + 2I \right)/(2k_B T)}}{(2\pi)^{3/2} (k_B T)^{3/2} q_\alpha} & \text{if } \alpha \in \mathcal{I}_{poly} \end{cases},$$

where $n_\alpha = \left(M, e_\alpha \right)$, $\mathbf{u} = \dfrac{1}{\rho} (M, m\boldsymbol{\xi})$, and $T = \dfrac{1}{3nk_B} \left(M, m |\boldsymbol{\xi} - \mathbf{u}|^2 \right)$, with mass vector $m = (m_1, \ldots, m_s)$, $n = \sum\limits_{\alpha=1}^{s} n_\alpha$, $\rho = \sum\limits_{\alpha=1}^{s} m_\alpha n_\alpha$, and normalization factors $q_\alpha = \int_0^\infty \varphi_\alpha (I) e^{-I/(k_B T)} \, dI$ for $\alpha \in \mathcal{I}_{poly}$, while k_B denote the Boltzmann constant.

For the typical case $\varphi_\alpha (I) = I^{\delta^{(\alpha)}/2 - 1}, \alpha \in \mathcal{I}$,

$$q_\alpha = (k_B T)^{\delta^{(\alpha)}/2} \Gamma \left(\delta^{(\alpha)}/2 \right),$$

where $\Gamma = \Gamma(n)$ denote the Gamma function $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx$.

Note that, by Eq. (11), any Maxwellian distribution $M = (M_1, \ldots, M_s)$ satisfies the relations

$$\Lambda_{\alpha\beta}(M) W_{\alpha\beta} = 0 \text{ a.e.} \tag{12}$$

for any $(\alpha, \beta) \in \mathcal{I}^2$.

Remark 2 Introducing the $\mathcal{H}$-functional

$$\mathcal{H}[f] = \left(f, \log \left(\varphi^{-1} (I) f \right) \right),$$

an $\mathcal{H}$-theorem can be obtained, cf. [1, 13].

2.3 Linearized Collision Operator

Consider a deviation of a centered and normalized Maxwellian $M = (M_1, \ldots, M_s)$, where

$$M_\alpha = \begin{cases} \dfrac{n_\alpha m_\alpha^{3/2}}{(2\pi)^{3/2}} e^{-m_\alpha |\boldsymbol{\xi}|^2/2} & \text{if } \alpha \in \mathcal{I}_{mono} \\[4mm] \dfrac{n_\alpha \varphi_\alpha (I) \, m_\alpha^{3/2}}{(2\pi)^{3/2} \overline{q}_\alpha} e^{-m_\alpha |\boldsymbol{\xi}|^2/2} e^{-I} & \text{if } \alpha \in \mathcal{I}_{poly} \end{cases}$$

with $\overline{q}_\alpha = \int_0^\infty \varphi_\alpha (I) e^{-I} \, dI$ for $\alpha \in \mathcal{I}_{poly}$, of the form

$$f = M + M^{1/2} h.$$

For $\alpha \in \mathcal{I}$, the typical case $\varphi_\alpha (I) = I^{\delta^{(\alpha)}/2 - 1}$ implies that $\overline{q}_\alpha = \Gamma \left(\delta^{(\alpha)}/2 \right)$.

Insertion in the Boltzmann equation (2) results in the system

$$\frac{\partial h}{\partial t} + (\boldsymbol{\xi} \cdot \nabla_{\mathbf{x}}) \, h + \mathcal{L}h = \Gamma \, (h, h) \,,$$

where the components of the linearized collision operator $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_s)$ are given by

$$\mathcal{L}_\alpha h = -M_\alpha^{-1/2} \left(Q_\alpha(M, M^{1/2}h) + Q_\alpha(M^{1/2}h, M) \right)$$

$$= \sum_{\beta=1}^s \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} \left(\frac{M_{\beta*} M'_\alpha M'_{\beta*}}{\varphi_\alpha (I) \, \varphi_\beta (I_*) \, \varphi_\alpha (I') \, \varphi_\beta (I'_*)} \right)^{1/2} \Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) W_{\alpha\beta} d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*$$

$$= \nu_\alpha h_\alpha - K_\alpha \, (h) \,, \tag{13}$$

with

$$\nu_\alpha = \sum_{\beta=1}^s \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} \frac{M_{\beta*}}{\varphi_\alpha (I) \, \varphi_\beta (I_*)} W_{\alpha\beta} d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*,$$

$$K_\alpha = \sum_{\beta=1}^s \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} \left(\frac{h'_\alpha}{\left(M'_\alpha\right)^{1/2}} + \frac{h'_{\beta*}}{\left(M'_{\beta*}\right)^{1/2}} - \frac{h_{\beta*}}{M_{\beta*}^{1/2}} \right) W_{\alpha\beta}$$

$$\times \left(\frac{M_{\beta*} M'_\alpha M'_{\beta*}}{\varphi_\alpha (I) \, \varphi_\beta (I_*) \, \varphi_\alpha (I') \, \varphi_\beta (I'_*)} \right)^{1/2} d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*, \tag{14}$$

while the components of the quadratic term $\Gamma = (\Gamma_1, \ldots, \Gamma_s)$ are given by

$$\Gamma_\alpha \, (h, h) = M_\alpha^{-1/2} Q_\alpha(M^{1/2}h, M^{1/2}h).$$

for $\alpha \in \mathcal{I}$. The multiplication operator Λ defined by

$$\Lambda(f) = \nu f, \text{ where } \nu = \text{diag} \, (\nu_1, \ldots, \nu_s) \,,$$

is a closed, densely defined, self-adjoint operator on $\mathfrak{h}$. It is Fredholm, as well, if and only if Λ is coercive.

The following lemma follows immediately by Lemma 1.

Lemma 2 *For any* $(\alpha, \beta) \in \mathcal{I}^2$ *the measure*

$$d\widetilde{A}_{\alpha\beta} = \left(\frac{M_\alpha M_{\beta*} M'_\alpha M'_{\beta*}}{\varphi_\alpha (I) \, \varphi_\beta (I_*) \, \varphi_\alpha (I') \, \varphi_\beta (I'_*)} \right)^{1/2} dA_{\alpha\beta}$$

is invariant under the (ordered) interchange (9) *of variables, while*

$$d\widetilde{A}_{\alpha\beta} + d\widetilde{A}_{\beta\alpha} \text{ for } (\alpha, \beta) \in \mathcal{I}^2$$

is invariant under the (ordered) interchange (10) *of variables*

The weak form of the linearized collision operator $\mathcal{L}$ reads

$$(\mathcal{L}h, g) = \sum_{\alpha, \beta=1}^s \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) \frac{g_\alpha}{M_\alpha^{1/2}} \, d\widetilde{A}_{\alpha\beta}$$

$$= \sum_{\alpha, \beta=1}^s \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) \frac{g_{\beta*}}{M_{\beta*}^{1/2}} \, d\widetilde{A}_{\alpha\beta}$$

$$= - \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) \frac{g_\alpha'}{(M_\alpha')^{1/2}} \, d\widetilde{A}_{\alpha\beta}$$

$$= - \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) \frac{g_{\beta*}'}{\left(M_{\beta*}'\right)^{1/2}} \, d\widetilde{A}_{\alpha\beta},$$

for any function $g = (g_1, \ldots, g_s)$, with $g_\alpha = g_\alpha(\mathbf{Z})$, such that the first integrals are defined for all $\{\alpha, \beta\} \subseteq \mathcal{I}$, while the following equalities are obtained by applying Lemma 2. The following lemma follows.

Lemma 3 *Let $g = (g_1, \ldots, g_s)$, where $g_\alpha = g_\alpha(\mathbf{Z})$, be such that*

$$\int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) \frac{g_\alpha}{M_\alpha^{1/2}} \, d\widetilde{A}_{\alpha\beta},$$

is defined for any $(\alpha, \beta) \in \mathcal{I}^2$. Then

$$(\mathcal{L}h, g) = \frac{1}{4} \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) \Delta_{\alpha\beta} \left(\frac{g}{M^{1/2}} \right) \, d\widetilde{A}_{\alpha\beta}.$$

Proposition 3 *The linearized collision operator is symmetric and nonnegative,*

$$(\mathcal{L}h, g) = (h, \mathcal{L}g) \text{ and } (\mathcal{L}h, h) \geq 0,$$

and the kernel of $\mathcal{L}$, $\ker \mathcal{L}$, is generated by

$$\left\{ \mathcal{M}^{1/2} e_1, \ldots, \mathcal{M}^{1/2} e_s, \mathcal{M}^{1/2} m\xi_x, \mathcal{M}^{1/2} m\xi_y, \mathcal{M}^{1/2} m\xi_z, \mathcal{M}^{1/2} \left(m \, |\boldsymbol{\xi}|^2 + 2\mathbb{I} \right) \right\},$$

where $m = (m_1, \ldots, m_s)$, $\mathbb{I} = (\underbrace{0, \ldots, 0}_{s_0}, \underbrace{I, \ldots, I}_{s_1})$, and $\mathcal{M} = \mathrm{diag}\, (M_1, \ldots, M_s)$.

Proof By Lemma 3, it is immediate that $(\mathcal{L}h, g) = (h, \mathcal{L}g)$, and

$$(\mathcal{L}h, h) = \frac{1}{4} \sum_{\alpha,\beta=1}^{s} \int_{\mathcal{Z}_\alpha^2 \times \mathcal{Z}_\beta^2} \left(\Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) \right)^2 \, d\widetilde{A}_{\alpha\beta}.$$

Furthermore, $h \in \ker \mathcal{L}$ if and only if $(\mathcal{L}h, h) = 0$, which will be fulfilled if and only if for all $(\alpha, \beta) \in \mathcal{I}^2$

$$\Delta_{\alpha\beta} \left(\frac{h}{M^{1/2}} \right) W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \, |\mathbf{Z}', \mathbf{Z}_*') = 0 \text{ a.e.,}$$

i.e., if and only if $\mathcal{M}^{-1/2}h$ is a collision invariant. The last part of the lemma follows by Proposition 2. $\square$

Remark 3 Note also that the quadratic term is orthogonal to the kernel of $\mathcal{L}$, i.e., we have that $\Gamma\,(h, h) \in (\ker \mathcal{L})^{\perp_\mathfrak{h}}$.

3 Main Results

This section is devoted to the main results, concerning compact properties in Theorem 1 and bounds of collision frequencies in Theorem 2. Below we restrict to the particular case $\varphi_\alpha(I) = I^{\delta^{(\alpha)}/2 - 1}$ for $\alpha \in \mathcal{I}$.

Assume that for some positive number γ, such that $0 < \gamma < 1$, there is for all $(\alpha, \beta) \in \mathcal{I}^2$ a bound

$$0 \leq \sigma_{\alpha\beta}\left(|\mathbf{g}|, \cos\theta, I, I_*, I', I'_*\right) \leq C \frac{\Psi_{\alpha\beta} + \left(\Psi_{\alpha\beta}\right)^{\gamma/2}}{|\mathbf{g}|^2} \Upsilon_{\alpha\beta},$$

$$\text{where } \Upsilon_{\alpha\beta} = \frac{\varphi_\alpha\left(I'\right)\varphi_\beta\left(I'_*\right)}{\mathcal{E}_{\alpha\beta}^{\delta^{(\alpha)}/2}\left(\mathcal{E}_{\alpha\beta}^*\right)^{\delta^{(\beta)}/2}} \text{ and } \Psi_{\alpha\beta} = |\mathbf{g}|\sqrt{|\mathbf{g}|^2 - \frac{2\Delta I}{\mu_{\alpha\beta}}}, \tag{15}$$

for $\mu_{\alpha\beta}|\mathbf{g}|^2 > 2\Delta I$, on the scattering cross sections, or, equivalently, the bound

$$0 \leq B_{i\alpha\beta}\left(|\mathbf{g}|, \cos\theta, I, I_*, I', I'_*\right) \leq C E_{\alpha\beta}^{1/2}\left(1 + \frac{1}{\Psi_{\alpha\beta}^{1-\gamma/2}}\right) \tag{16}$$

for $\mu_{\alpha\beta}|\mathbf{g}|^2 > 2\Delta I$, on the collision kernels. Here, for $(\alpha, \beta) \in \mathcal{I}^2$, $\mathcal{E}_{\alpha\beta} = 1$ if $\alpha \in \mathcal{I}_{mono}$ and $\mathcal{E}_{\alpha\beta}^* = 1$ if $\beta \in \mathcal{I}_{mono}$, while, otherwise,

$$\mathcal{E}_{\alpha\beta} = \mathcal{E}_{\alpha\beta}^* = \mu_{\alpha\beta}|\mathbf{g}|^2/2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I_*$$

$$= \mu_{\alpha\beta}|\mathbf{g}'|^2/2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I' + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I'_*.$$

Remark 4 In the single polyatomic case, i.e., with $s = s_1 = 1$, assumption (15) (or, (16)), differs from the one in [4], by an extra factor $E^{1/2} = E_{11}^{1/2}$ in the denominator, resulting in the factor $E^{1/2} = E_{11}^{1/2}$ instead of E in assumption (16). This rules out the superhard potential like models considered in [4], while it opens up for some soft potential like models, not covered in [4], in return.

The following result may be obtained.

Theorem 1 *Assume that for all $\{\alpha, \beta\} \subseteq \mathcal{I}$ the scattering cross sections $\sigma_{\alpha\beta}$ satisfy the bound* (15) *for some positive number γ, $0 < \gamma < 1$.*

Then the operator $K = (K_1, \ldots, K_s)$, with the components K_α given by (14) *is a self-adjoint compact operator on $\mathfrak{h} = \left(L^2(d\boldsymbol{\xi})\right)^{s_0} \times \left(L^2(d\boldsymbol{\xi}dI)\right)^{s_1}$.*

The proof of Theorem 1 will be addressed in Sect. 4.

Corollary 1 *The linearized collision operator $\mathcal{L}$, with scattering cross sections satisfying* (15), *is a closed, densely defined, self-adjoint operator on $\mathfrak{h}$.*

Proof By Theorem 1, the linear operator $\mathcal{L} = \Lambda - K$ is closed as the sum of a closed and a bounded operator, and densely defined, since the domains of the linear operators $\mathcal{L}$ and Λ are equal; $D(\mathcal{L}) = D(\Lambda)$. Furthermore, it is a self-adjoint operator, since the set of self-adjoint operators is closed under addition of bounded self-adjoint operators, see Theorem 4.3 of Chapter V in [19]. □

Now consider the scattering cross sections

$$\sigma_{\alpha\beta} = C_{\alpha\beta} \frac{\sqrt{|\mathbf{g}|^2 - \dfrac{2\Delta I}{\mu_{\alpha\beta}}}}{|\mathbf{g}|\, E_{\alpha\beta}^{\eta/2}} \frac{\varphi_\alpha\left(I'\right)\varphi_\beta\left(I'_*\right)}{\mathcal{E}_{\alpha\beta}^{\delta^{(\alpha)}/2}\left(\mathcal{E}_{\alpha\beta}^*\right)^{\delta^{(\beta)}/2}}, \text{ if } \mu_{\alpha\beta}\,|\mathbf{g}|^2 > 2\Delta I, \tag{17}$$

for some positive constants $C_{\alpha\beta} > 0$ for all $(\alpha, \beta) \in \mathcal{I}^2$ and nonnegative number η less than 1, $0 \leq \eta < 1$,—cf. hard sphere models for $\eta = 0$.

In fact, it would be enough with the bounds (for $\mu_{\alpha\beta}\,|\mathbf{g}|^2 > 2\Delta I$)

$$C_- \frac{\sqrt{|\mathbf{g}|^2 - \dfrac{2\Delta I}{\mu_{\alpha\beta}}}}{|\mathbf{g}|\, E_{\alpha\beta}^{\eta/2}} \Upsilon_{\alpha\beta} \leq \sigma_{\alpha\beta} \leq C_+ \frac{\sqrt{|\mathbf{g}|^2 - \dfrac{2\Delta I}{\mu_{\alpha\beta}}}}{|\mathbf{g}|\, E_{\alpha\beta}^{\eta/2}} \Upsilon_{\alpha\beta},$$

$$\text{with } \Upsilon_{\alpha\beta} = \frac{\varphi_\alpha\left(I'\right)\varphi_\beta\left(I'_*\right)}{\mathcal{E}_{\alpha\beta}^{\delta^{(\alpha)}/2}\left(\mathcal{E}_{\alpha\beta}^*\right)^{\delta^{(\beta)}/2}}, \tag{18}$$

for some nonnegative number η less than 1, $0 \leq \eta < 1$, and some positive constants $C_\pm > 0$, on the scattering cross sections—cf. hard potential with cut-off models.

The following bounds restricted to single species were obtained in [5, 15]. In [15] the improved—compared to the one in [5]—upper bound below was shown for single species.

Theorem 2 *The linearized collision operator $\mathcal{L}$, with scattering cross sections (17) (or (18)), can be split into a positive multiplication operator Λ, where $\Lambda(f) = \nu f$, with $\nu = \nu(\mathbf{Z})$, minus a compact operator K on $\mathfrak{h}$, such that there exist positive numbers ν_- and ν_+, where $0 < \nu_- < \nu_+$, for which, for any $\alpha \in \mathcal{I}$*

$$\nu_-\left(1 + |\boldsymbol{\xi}| + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}}\sqrt{I}\right)^{1-\eta} \leq \nu_\alpha \leq \nu_+\left(1 + |\boldsymbol{\xi}| + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}}\sqrt{I}\right)^{1-\eta}. \tag{19}$$

The decomposition follows by decomposition (13), (14) and Theorem 1, while the bounds (19) on the collision frequency will be proven in Sect. 5.

Corollary 2 *The linearized collision operator $\mathcal{L}$, with scattering cross section (17) (or (18)), is a Fredholm operator with domain*

$$D(\mathcal{L}) = \left(L^2\left((1 + |\boldsymbol{\xi}|)^{1-\eta}\, d\boldsymbol{\xi}\right)\right)^{s_0} \times \left(L^2\left(\left(1 + |\boldsymbol{\xi}| + \sqrt{I}\right)^{1-\eta} d\boldsymbol{\xi}\, dI\right)\right)^{s_1}.$$

Proof By Theorem 2 the multiplication operator Λ is coercive, and thus it is a Fredholm operator. Furthermore, the set of Fredholm operators is closed under addition of compact operators, see Theorem 5.26 of Chapter IV in [19] and its proof, so, by Theorem 2, $\mathcal{L}$ is a Fredholm operator. □

We stress that Corollary 2 finally yields the Fredholmness of the linearized operator assumed in [1] for kernels of the form (17) or (18).

For hard sphere like models we obtain the following result.

Corollary 3 *For the linearized collision operator $\mathcal{L}$, with scattering cross section (17 (or (18)) where $\eta = 0$, there exists a positive number λ, $0 < \lambda < 1$, such that*

$$(h, \mathcal{L}h) \geq \lambda\,(h, \nu(\mathbf{Z})h) \geq \lambda\nu_-\,(h, (1 + |\boldsymbol{\xi}|)\,h)$$

for all $h \in D(\mathcal{L}) \cap \operatorname{Im}\mathcal{L}$.

Proof Let $h \in D(\mathcal{L}) \cap (\ker \mathcal{L})^{\perp} = D(\mathcal{L}) \cap \mathrm{Im}\mathcal{L}$. As a Fredholm operator, $\mathcal{L}$ is closed with a closed range, and as a compact operator, K is bounded, and so there are positive constants $\nu_0 > 0$ and $c_K > 0$, such that

$$(h, \mathcal{L}h) \geq \nu_0(h, h) \text{ and } (h, Kh) \leq c_K(h, h).$$

Let $\lambda = \dfrac{\nu_0}{\nu_0 + c_K}$. Then the corollary follows, since

$$\begin{aligned}(h, \mathcal{L}h) &= (1 - \lambda)(h, \mathcal{L}h) + \lambda(h, (\nu(\mathbf{Z}) - K)h) \\ &\geq (1 - \lambda)\nu_0(h, h) + \lambda(h, \nu(\mathbf{Z})h) - \lambda c_K(h, h) \\ &= (\nu_0 - \lambda(\nu_0 + c_K))(h, h) + \lambda(h, \nu(\mathbf{Z})h) = \lambda(h, \nu(\mathbf{Z})h).\end{aligned}$$

$\square$

Remark 5 By Proposition 3 and Corollary 1-3 the linearized operator $\mathcal{L}$ fulfills the properties assumed on the linear operators in [3], and, hence, the results for the abstract linearized half-space problem therein can be applied to hard sphere like models for mixtures of monatomic and polyatomic gases.

4 Compactness

This section concerns the proof of Theorem 1. Note that in the proof the kernels are rewritten in such a way that $\mathbf{Z}_*$—and not $\mathbf{Z}'$ or $\mathbf{Z}'_*$—always will be arguments of the distribution functions. As for single species, either $\mathbf{Z}_*$ is an argument in the loss term (like $\mathbf{Z}$) or in the gain term (unlike $\mathbf{Z}$) of the collision operator. However, in the latter case, unlike for single species, for mixtures one has to differ between two different cases; either $\mathbf{Z}_*$ is associated to the same species as $\mathbf{Z}$, or not. The kernels of the terms from the loss part of the collision operator will be shown to be Hilbert–Schmidt in a quite direct way. The kernels of some of the terms—for which $\mathbf{Z}_*$ is associated to the same species as $\mathbf{Z}$—from the gain parts of the collision operators will be shown to be uniform limits of Hilbert–Schmidt integral operators, i.e., approximately Hilbert–Schmidt in the sense of Lemma 4. Furthermore, it will be shown that the kernels of the remaining terms—for which $\mathbf{Z}_*$ is associated to the opposite species to $\mathbf{Z}$—from the gain parts of the collision operators are Hilbert–Schmidt.

To show the compactness properties—when the terms are not necessarily Hilbert–Schmidt integral operators themselves—the following result will be applied. Denote, for any (nonzero) natural number N,

$$\mathfrak{h}_N := \left\{ (\mathbf{Z}, \mathbf{Z}_*) \in \mathbb{Y} \times \mathbb{Y}_* : |\boldsymbol{\xi} - \boldsymbol{\xi}_*| \geq \frac{1}{N}; \ |\boldsymbol{\xi}| \leq N \right\}, \text{ and}$$

$$b^{(N)} = b^{(N)}(\mathbf{Z}, \mathbf{Z}_*) := b(\mathbf{Z}, \mathbf{Z}_*)\mathbf{1}_{\mathfrak{h}_N}.$$

Here, either $\mathbf{Z} = \boldsymbol{\xi}$ and $\mathbb{Y} = \mathbb{R}^3$, or, $\mathbf{Z} = (\boldsymbol{\xi}, I)$ and $\mathbb{Y} = \mathbb{R}^3 \times \mathbb{R}_+$, and correspondingly, either $\mathbf{Z}_* = \boldsymbol{\xi}_*$ and $\mathbb{Y}_* = \mathbb{R}^3$, or, $\mathbf{Z}_* = (\boldsymbol{\xi}_*, I_*)$ and $\mathbb{Y}_* = \mathbb{R}^3 \times \mathbb{R}_+$. Then we have the following lemma, cf. Glassey [17, Lemma 3.5.1] and Drange [14].

Lemma 4 Assume that $Tf(\mathbf{Z}) = \int_{\mathbb{Y}_*} b(\mathbf{Z}, \mathbf{Z}_*) f(\mathbf{Z}_*) d\mathbf{Z}_*$, with $b(\mathbf{Z}, \mathbf{Z}_*) \geq 0$. Then T is compact on $L^2(d\mathbf{Z})$ if

(i) $\int_{\mathbb{Y}} b(\mathbf{Z}, \mathbf{Z}_*) d\mathbf{Z}$ is bounded in $\mathbf{Z}_*$;
(ii) $b^{(N)} \in L^2(d\mathbf{Z} \, d\mathbf{Z}_*)$ for any (nonzero) natural number N;

(iii) $\sup\limits_{\mathbf{Z}\in\mathbb{Y}} \int_{\mathbb{Y}_*} b(\mathbf{Z}, \mathbf{Z}_*) - b^{(N)}(\mathbf{Z}, \mathbf{Z}_*) \, d\mathbf{Z}_* \to 0$ as $N \to \infty$.

Then T is the uniform limit of Hilbert–Schmidt integral operators [17, Lemma 3.5.1] and we say that the kernel $b(\mathbf{Z}, \mathbf{Z}_*)$ is approximately Hilbert–Schmidt, while T is an approximately Hilbert–Schmidt integral operator. The reader is referred to Glassey [17, Lemma 3.5.1] for a proof of Lemma 4.

Now we turn to the proof of Theorem 1. Note that throughout the proof C will denote a generic positive constant. Moreover, remind that $\varphi_\alpha(I) = I^{\delta^{(\alpha)}/2-1}$ for $\alpha \in \mathcal{I}$ below.

Proof For $\alpha \in \mathcal{I}$ rewrite expression (14) as

$$K_\alpha = (M_\alpha)^{-1/2} \sum_{\beta=1}^{s} \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*)$$

$$\times \left(\frac{h'_\alpha}{\left(M'_\alpha\right)^{1/2}} + \frac{h'_{\beta*}}{\left(M'_{\beta*}\right)^{1/2}} - \frac{h_{\beta*}}{M_{\beta*}^{1/2}} \right) d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*,$$

with

$$w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) = \left(\frac{M_\alpha M_{\beta*} M'_\alpha M'_{\beta*}}{\varphi_\alpha(I)\,\varphi_\beta(I_*)\,\varphi_\alpha(I')\,\varphi_\beta(I'_*)} \right)^{1/2} W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*).$$

Due to relations (7), the relations

$$w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) = w_{\beta\alpha}(\mathbf{Z}_*, \mathbf{Z} \,|\, \mathbf{Z}'_*, \mathbf{Z}')$$
$$w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) = w_{\alpha\beta}(\mathbf{Z}', \mathbf{Z}'_* \,|\, \mathbf{Z}, \mathbf{Z}_*) \qquad (20)$$
$$w_{\alpha\alpha}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) = w_{\alpha\alpha}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}'_*, \mathbf{Z}')$$

are satisfied. By renaming $\{\mathbf{Z}_*\} \leftrightarrows \{\mathbf{Z}'_*\}$,

$$\int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) \frac{h'_{\beta*}}{\left(M'_{\beta*}\right)^{1/2}} \, d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*$$

$$= \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}'_* \,|\, \mathbf{Z}', \mathbf{Z}_*) \frac{h_{\beta*}}{M_{\beta*}^{1/2}} \, d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*.$$

Moreover, by renaming $\{\mathbf{Z}_*\} \leftrightarrows \{\mathbf{Z}'\}$,

$$\int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* \,|\, \mathbf{Z}', \mathbf{Z}'_*) \frac{h'_{\alpha,k}}{\left(M'_\alpha\right)^{1/2}} \, d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*$$

$$= \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}' \,|\, \mathbf{Z}_*, \mathbf{Z}'_*) \frac{h_{\alpha*}}{M_{\alpha*}^{1/2}} \, d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}'_*.$$

It follows that

$$K_\alpha(h) = \sum_{\beta=1}^{s} \int_{\mathcal{Z}_\gamma} k_{\alpha\beta}\left(\boldsymbol{\xi}, \boldsymbol{\xi}_*, I, I_*\right) h_* \, d\mathbf{Z}_*, \text{ where}$$

$$k_{\alpha\beta} h_* = \delta_{\alpha\gamma} k_{\alpha\beta}^{(\alpha)} h_{\alpha*} + \delta_{\beta\gamma} k_{\alpha\beta}^{(\beta)} h_{\beta*}$$

$$= \delta_{\alpha\gamma} k_{\alpha\beta}^{(\alpha)} h_{\alpha*} + \delta_{\beta\gamma} \left(k_{\alpha\beta2}^{(\beta)} - k_{\alpha\beta1}^{(\beta)} \right) h_{\beta*}, \text{ with}$$

$$k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathcal{Z}_\beta^2} \frac{w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}' | \mathbf{Z}_*, \mathbf{Z}_*')}{(M_\alpha M_{\alpha*})^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*',$$

$$k_{\alpha\beta1}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_* | \mathbf{Z}', \mathbf{Z}_*')}{\left(M_\alpha M_{\beta*}\right)^{1/2}} d\mathbf{Z}' d\mathbf{Z}_*', \text{ and}$$

$$k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}_*' | \mathbf{Z}', \mathbf{Z}_*)}{\left(M_\alpha M_{\beta*}\right)^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*'. \tag{21}$$

By applying the second relation of (20) and renaming $\{\mathbf{Z}'\} \leftrightarrows \{\mathbf{Z}_*'\}$,

$$k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathcal{Z}_\beta^2} \frac{w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}' | \mathbf{Z}_*, \mathbf{Z}_*')}{(M_\alpha M_{\alpha*})^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*'$$

$$= \int_{\mathcal{Z}_\beta^2} \frac{w_{\alpha\beta}(\mathbf{Z}_*, \mathbf{Z}' | \mathbf{Z}, \mathbf{Z}_*')}{(M_\alpha M_{\alpha*})^{1/2}} d\mathbf{Z}' d\mathbf{Z}_*' = k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}). \tag{22}$$

Moreover,

$$k_{\alpha\beta}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) = k_{\beta\alpha1}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}) - k_{\beta\alpha2}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}) = k_{\beta\alpha}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}), \tag{23}$$

since, by applying the first relation of (20) and renaming $\{\mathbf{Z}'\} \leftrightarrows \{\mathbf{Z}_*'\}$,

$$k_{\alpha\beta1}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\beta\alpha}(\mathbf{Z}_*, \mathbf{Z} | \mathbf{Z}_*', \mathbf{Z}')}{\left(M_\alpha M_{\beta*}\right)^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*'$$

$$= \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\beta\alpha}(\mathbf{Z}_*, \mathbf{Z} | \mathbf{Z}', \mathbf{Z}_*')}{\left(M_\alpha M_{\beta*}\right)^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*' = k_{\beta\alpha1}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}),$$

while, by renaming $\{\mathbf{Z}'\} \leftrightarrows \{\mathbf{Z}_*'\}$, after applying the first two relations of (20)

$$k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\beta\alpha}(\mathbf{Z}_*', \mathbf{Z} | \mathbf{Z}_*, \mathbf{Z}')}{\left(M_\alpha M_{\beta*}\right)^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*'$$

$$= \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\beta\alpha}(\mathbf{Z}_*, \mathbf{Z}' | \mathbf{Z}_*', \mathbf{Z})}{\left(M_\alpha M_{\beta*}\right)^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*'$$

$$= \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\beta\alpha}(\mathbf{Z}_*, \mathbf{Z}_*' | \mathbf{Z}', \mathbf{Z})}{\left(M_\alpha M_{\beta*}\right)^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}_*' = k_{\beta\alpha2}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}).$$

Now the compactness of the three different types of collision kernel will be considered separately. Note that, if $\alpha = \beta$, by applying the last relation of (20), $k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) = k_{\alpha\beta}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*)$, and we will remain with only two cases—the first two below. Even if $m_\alpha = m_\beta$, the kernels $k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*)$ and $k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*)$ are structurally equal, and we (in principle) remain with (first) two cases (the second one twice).

I. Compactness of $K_{\alpha\beta1} = \int_{\mathcal{Z}_\beta} k_{\alpha\beta1}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \, h_{\beta*} \, d\mathbf{Z}_*$ for any indices $(\alpha, \beta) \in \mathcal{I}^2$.
By a change of variables, cf. Figure 1,

$$\left\{ \boldsymbol{\xi}', \boldsymbol{\xi}_*' \right\} \to \left\{ |\mathbf{g}'| = |\boldsymbol{\xi}' - \boldsymbol{\xi}_*'|, \, \boldsymbol{\sigma} = \frac{\mathbf{g}'}{|\mathbf{g}'|}, \, \mathbf{G}_{\alpha\beta}' = \frac{m_\alpha \boldsymbol{\xi}' + m_\beta \boldsymbol{\xi}_*'}{m_\alpha + m_\beta} \right\},$$
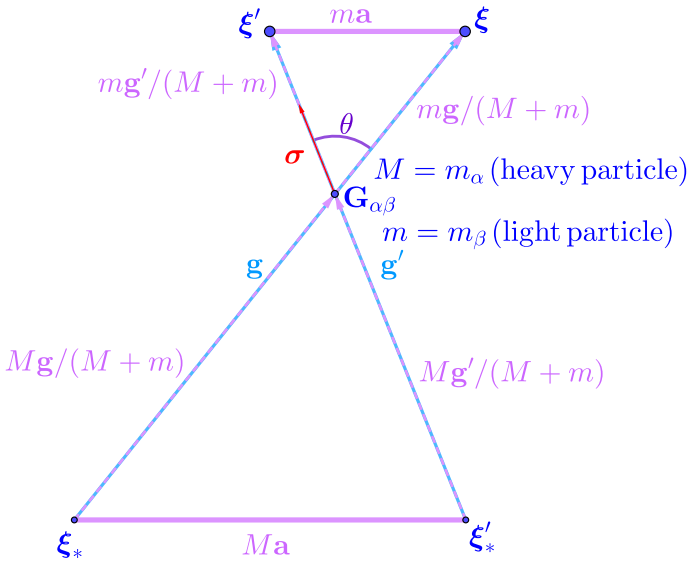
Fig. 1 Classical representation of an inelastic collision between particles of different species

noting identity (8) and using relation (12), expression (21) of $k_{\alpha\beta1}^{(\beta)}$ may be written as

$$k_{\alpha\beta1}^{(\beta)}(\mathbf{Z},\mathbf{Z}_*) = \int_{\mathcal{Z}_\alpha\times\mathcal{Z}_\beta} \frac{\left(M'_\alpha M'_{\beta*}\right)^{1/2} W_{\alpha\beta}}{\left(\varphi_\alpha(I)\,\varphi_\beta(I_*)\,\varphi_\alpha(I')\,\varphi_\beta\left(I'_*\right)\right)^{1/2}}\, d\mathbf{Z}' d\mathbf{Z}'_*$$

$$= \int_{\mathbb{R}^3\times\mathbb{R}^3_+\times\mathbb{S}^2} \frac{\left(M'_\alpha M'_{\beta*}\right)^{1/2} W_{\alpha\beta}\mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}}}{\left(\varphi_\alpha(I)\,\varphi_\beta(I_*)\,\varphi_\alpha(I')\,\varphi_\beta\left(I'_*\right)\right)^{1/2}}\, \left|\mathbf{g}'\right|^2 d\mathbf{G}'_{\alpha\beta} d\left|\mathbf{g}'\right| d\boldsymbol{\sigma}\, dI'dI'_*$$

$$= \left(M_\alpha M_{\beta*}\right)^{1/2}|\mathbf{g}|\int_{\mathbb{S}^2\times\mathbb{R}^2_+} \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I}\mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}}\sigma_{\alpha\beta}\, d\boldsymbol{\sigma}\, dI'dI'_*.$$

Note that

$$\text{if } (\alpha,\beta)\in\mathcal{I}^2_{mono},\, \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I}\mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}} = \mathbf{1}_{I'\leq1}\mathbf{1}_{I'_*\leq1}, \text{ while}$$

$$\mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I}\mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}} = \mathbf{1}_{I'\leq1}\mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I} = \mathbf{1}_{I'\leq1}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}}$$

$$\text{if } (\alpha,\beta)\in\mathcal{I}_{mono}\times\mathcal{I}_{poly},$$

$$\mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I}\mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}} = \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I}\mathbf{1}_{I'_*\leq1} = \mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq1}$$

$$\text{if } (\alpha,\beta)\in\mathcal{I}_{poly}\times\mathcal{I}_{mono}, \text{ and if } (\alpha,\beta)\in\mathcal{I}^2_{poly}$$

$$\mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I}\mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}} = \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2>2\Delta I} = \mathbf{1}_{I'+I'_*\leq\mathcal{E}_{\alpha\beta}}.$$

Since $E_{\alpha\beta}\geq C\Psi_{\alpha\beta}$, it follows, by assumption (15), that

$$\left(k_{\alpha\beta1}^{(\beta)}(\mathbf{Z},\mathbf{Z}_*)\right)^2$$

$$\leq C\left(\int_{\mathbb{S}^2\times\mathbb{R}^2_+} \Upsilon_{\alpha\beta}\left(\Psi_{\alpha\beta}+\Psi_{\alpha\beta}^{\gamma/2}\right)\mathbf{1}_{I'\leq\mathcal{E}_{\alpha\beta}}\mathbf{1}_{I'_*\leq\mathcal{E}^*_{\alpha\beta}}\, d\boldsymbol{\sigma}\, dI'dI'_*\right)^2$$

$$\leq C \frac{M_\alpha M_{\beta*}}{|\mathbf{g}|^2} \left(1 + E_{\alpha\beta}\right)^2 \left(\int_0^{\mathcal{E}_{\alpha\beta}} \frac{(I')^{\delta^{(\alpha)}/2-1}}{\mathcal{E}_{\alpha\beta}^{\delta^{(\alpha)}/2}} \, dI'\right)^2 \left(\int_0^{\mathcal{E}_{\alpha\beta}^*} \frac{(I'_*)^{\delta^{(\beta)}/2-1}}{\left(\mathcal{E}_{\alpha\beta}^*\right)^{\delta^{(\beta)}/2}} \, dI'_*\right)^2$$

$$= C \frac{M_\alpha M_{\beta*}}{\varphi_\alpha (I) \varphi_\beta (I_*)} \frac{\varphi_\alpha (I) \varphi_\beta (I_*)}{|\mathbf{g}|^2} \left(1 + E_{\alpha\beta}\right)^2 .$$

Then, by assumption (15) and

$$m_\alpha \frac{|\boldsymbol{\xi}|^2}{2} + m_\beta \frac{|\boldsymbol{\xi}_*|^2}{2} + \mathbf{1}_{\alpha\in\mathcal{I}_{poly}} I + \mathbf{1}_{\beta\in\mathcal{I}_{poly}} I_* = \frac{m_\alpha + m_\beta}{2} |\mathbf{G}_{\alpha\beta}|^2 + E_{\alpha\beta},$$

where $E_{\alpha\beta} = \mu_{\alpha\beta} |\mathbf{g}|^2 /2 + \mathbf{1}_{\alpha\in\mathcal{I}_{poly}} I + \mathbf{1}_{\beta\in\mathcal{I}_{poly}} I_*$, the bound

$$\left(k_{\alpha\beta 1}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*)\right)^2 \leq C e^{-(m_\alpha+m_\beta)|\mathbf{G}_{\alpha\beta}|^2/2 - E_{\alpha\beta}} \frac{\varphi_\alpha (I) \varphi_\beta (I_*)}{|\mathbf{g}|^2} \left(1 + E_{\alpha\beta}^2\right)$$

$$\leq C e^{-(m_\alpha+m_\beta)|\mathbf{G}_{\alpha\beta}|^2/2 - E_{\alpha\beta}} \frac{\left(1 + |\mathbf{g}|^2\right)^2}{|\mathbf{g}|^2}$$

$$\times \left(1 + \mathbf{1}_{\beta\in\mathcal{I}_{poly}} I_*\right)^2 \varphi_\beta (I_*) \left(1 + \mathbf{1}_{\alpha\in\mathcal{I}_{poly}} I\right)^2 \varphi_\alpha (I) \qquad (24)$$

may be obtained. Hence, by applying the bound (24) and first changing variables of integration $\{\boldsymbol{\xi}, \boldsymbol{\xi}_*\} \to \{\mathbf{g}, \mathbf{G}_{\alpha\beta}\}$, with unitary Jacobian, and then to spherical coordinates,

$$\int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \left(k_{\alpha\beta 1}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*)\right)^2 d\mathbf{Z}' d\mathbf{Z}'_*$$

$$\leq C \int_{(\mathbb{R}^3)^2} e^{-(m_\alpha+m_\beta)|\mathbf{G}_{\alpha\beta}|^2/2} e^{-\mu_{\alpha\beta}|\mathbf{g}|^2/2} \frac{1 + |\mathbf{g}|^4}{|\mathbf{g}|^2} d\mathbf{g} \, d\mathbf{G}_{\alpha\beta}$$

$$\times \int_0^\infty (1 + I)^2 e^{-I} I^{\delta^{(\alpha)}/2-1} dI \int_0^\infty (1 + I_*)^2 e^{-I_*} I_*^{\delta^{(\beta)}/2-1} dI_*$$

$$= C \int_0^\infty e^{-\mu_{\alpha\beta}r^2/4} \left(1 + r^4\right) dr \int_0^\infty R^2 e^{-R^2} dR = C$$
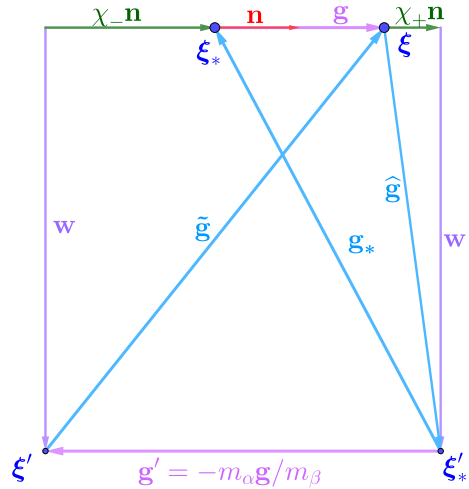
Therefore,

$$K_{\alpha\beta 1} = \int_{\mathcal{Z}_\beta} k_{\alpha\beta 1}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) h_{\beta*} \, d\mathbf{Z}_*$$

are Hilbert–Schmidt integral operators and as such compact on $L^2 (d\mathbf{Z})$, see, e.g., Theorem 7.83 in [22], for $(\alpha, \beta) \in \mathcal{I}^2$.

II. Compactness of $K_{\alpha\beta 3} = \int_{\mathcal{Z}_\alpha} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) h_{\alpha*} \, d\mathbf{Z}_*$ for any indices $(\alpha, \beta) \in \mathcal{I}^2$.
Note that, cf. Figure 2,

$$W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}' | \mathbf{Z}_*, \mathbf{Z}'_*) = (m_\alpha + m_\beta)^2 m_\alpha m_\beta \varphi_\alpha (I) \varphi_\beta (I') \delta_3 (m_\alpha \mathbf{g} + m_\beta \mathbf{g}')$$

$$\times \frac{|\tilde{\mathbf{g}}|}{|\mathbf{g}_*|} \tilde{\sigma}_{\alpha\beta} \delta_1 \left(|\mathbf{g}| m_\alpha \left(\chi_+ - \frac{m_\alpha - m_\beta}{2m_\beta} |\mathbf{g}|\right) - \Delta I_*\right)$$

$$= \frac{|\tilde{\mathbf{g}}| (m_\alpha + m_\beta)^2}{|\mathbf{g}_*| |\mathbf{g}| m_\beta^2} \varphi_\alpha (I) \varphi_\beta (I') \tilde{\sigma}_{\alpha\beta} \delta_3 \left(\frac{m_\alpha}{m_\beta} \mathbf{g} + \mathbf{g}'\right)$$

$$\times \delta_1 \left(\chi_+ - \frac{m_\alpha - m_\beta}{2m_\beta} |\mathbf{g}| - \frac{\Delta I_*}{m_\alpha |\mathbf{g}|}\right),$$

Fig. 2 Typical collision of $K_{\alpha\beta3}$

with $\widetilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta}\left(|\widetilde{\mathbf{g}}|, \dfrac{\widetilde{\mathbf{g}} \cdot \mathbf{g}_*}{|\widetilde{\mathbf{g}}| \, |\mathbf{g}_*|}, I, I', I_*, I'_* \right)$, $\mathbf{g} = \boldsymbol{\xi} - \boldsymbol{\xi}_*$, $\mathbf{g}' = \boldsymbol{\xi}' - \boldsymbol{\xi}'_*$, $\widetilde{\mathbf{g}} = \boldsymbol{\xi} - \boldsymbol{\xi}'$,

$\mathbf{g}_* = \boldsymbol{\xi}_* - \boldsymbol{\xi}'_*$, $\Delta I_* = \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} (I_* - I) + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} (I'_* - I')$, and $\chi_+ = \left(\boldsymbol{\xi}'_* - \boldsymbol{\xi}\right) \cdot \mathbf{n}$, where

$\mathbf{n} = \dfrac{\mathbf{g}}{|\mathbf{g}|}$.

By performing the change of variables $\{\boldsymbol{\xi}', \boldsymbol{\xi}'_*\} \to \{\mathbf{g}' = \boldsymbol{\xi}' - \boldsymbol{\xi}'_*, \ \widehat{\mathbf{g}} = \boldsymbol{\xi}'_* - \boldsymbol{\xi}\}$, with

$$d\boldsymbol{\xi}' d\boldsymbol{\xi}'_* = d\mathbf{g}' d\widehat{\mathbf{g}} = d\mathbf{g}' d\chi_+ d\mathbf{w}, \text{ where } \mathbf{w} = \boldsymbol{\xi}'_* - \boldsymbol{\xi} - \chi_+ \mathbf{n},$$

the expression (21) of $k_{\alpha\beta}^{(\alpha)}$ may be rewritten in the following way

$$k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathcal{Z}_\beta^2} \left(M'_\beta M'_{\beta*}\right)^{1/2} \frac{W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}' \,|\mathbf{Z}_*, \mathbf{Z}'_*) \mathbf{1}_{I' \leq \widetilde{\mathcal{E}}_{\alpha\beta}^*} \mathbf{1}_{I'_* \leq \widetilde{\mathcal{E}}_{\alpha\beta}^*}}{\left(\varphi_\alpha\left(I\right) \varphi_\beta\left(I'\right) \varphi_\alpha\left(I_*\right) \varphi_\beta\left(I'_*\right)\right)^{1/2}} d\mathbf{g}' d\widehat{\mathbf{g}} dI' dI'_*$$

$$= \frac{\varphi_\alpha^{1/2}\left(I\right)}{\varphi_\alpha^{1/2}\left(I_*\right)} \int_{(\mathbb{R}^3)^{\perp \mathbf{n}} \times \mathbb{R}_+^2} \frac{\left(m_\alpha + m_\beta\right)^2}{m_\beta^2} \frac{|\widetilde{\mathbf{g}}| \left(M'_\beta M'_{\beta*}\right)^{1/2}}{|\mathbf{g}_*| \, |\mathbf{g}|} \frac{\varphi_\beta^{1/2}\left(I'\right)}{\varphi_\beta^{1/2}\left(I'_*\right)}$$

$$\times \widetilde{\sigma}_{\alpha\beta} \mathbf{1}_{\mu_{\alpha\beta} |\widetilde{\mathbf{g}}|^2 > 2\Delta I_*} \mathbf{1}_{I' \leq \widetilde{\mathcal{E}}_{\alpha\beta}^*} \mathbf{1}_{I'_* \leq \widetilde{\mathcal{E}}_{\alpha\beta}^*} d\mathbf{w} dI' dI'_*.$$

Here, see Fig. 2,

$$\begin{cases} \boldsymbol{\xi}' = \boldsymbol{\xi}_* + \mathbf{w} - \chi_- \mathbf{n} \\ \boldsymbol{\xi}'_* = \boldsymbol{\xi} + \mathbf{w} - \chi_+ \mathbf{n} \end{cases}, \text{ with } \chi_\pm = \frac{\Delta I_*}{m_\alpha |\mathbf{g}|} \pm \frac{m_\alpha - m_\beta}{2m_\beta} |\mathbf{g}|,$$

implying that

$$\frac{|\boldsymbol{\xi}'|^2}{2} + \frac{|\boldsymbol{\xi}'_*|^2}{2} = \left|\frac{\boldsymbol{\xi} + \boldsymbol{\xi}_*}{2} - \frac{\Delta I_*}{m_\alpha |\mathbf{g}|}\mathbf{n} + \mathbf{w}\right|^2 + \frac{m_\alpha^2}{4m_\beta^2} |\boldsymbol{\xi} - \boldsymbol{\xi}_*|^2$$

$$= \left|\frac{(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\perp n}}{2} + \mathbf{w}\right|^2 + \left(\frac{(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_n}{2} - \frac{\Delta I_*}{m_\alpha |\mathbf{g}|}\right)^2 + \frac{m_\alpha^2}{4m_\beta^2} |\mathbf{g}|^2$$

$$= \left| \frac{(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\perp_n}}{2} + \mathbf{w} \right|^2 + \frac{\left(m_\alpha \left(|\boldsymbol{\xi}_*|^2 - |\boldsymbol{\xi}|^2 \right) + 2\Delta I_* \right)^2}{4m_\alpha^2 |\mathbf{g}|^2} + \frac{m_\alpha^2}{4m_\beta^2} |\mathbf{g}|^2,$$

where

$$(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\mathbf{n}} = (\boldsymbol{\xi} + \boldsymbol{\xi}_*) \cdot \mathbf{n} = \frac{|\boldsymbol{\xi}|^2 - |\boldsymbol{\xi}_*|^2}{|\boldsymbol{\xi} - \boldsymbol{\xi}_*|} \text{ and } (\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\perp_n} = \boldsymbol{\xi} + \boldsymbol{\xi}_* - (\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\mathbf{n}} \mathbf{n}.$$

Denote $\widetilde{\mathcal{E}}_{\alpha\beta} = 1$ if $\alpha \in \mathcal{I}_{mono}$ and $\widetilde{\mathcal{E}}^*_{\alpha\beta} = 1$ if $\beta \in \mathcal{I}_{mono}$, while, otherwise,

$$\widetilde{\mathcal{E}}_{\alpha\beta} = \widetilde{\mathcal{E}}^*_{\alpha\beta} = \mu_{\alpha\beta} |\widetilde{\mathbf{g}}|^2 / 2 + \mathbf{1}_{\alpha\in\mathcal{I}_{poly}} I + \mathbf{1}_{\beta\in\mathcal{I}_{poly}} I'$$
$$= \mu_{\alpha\beta} |\mathbf{g}_*|^2 / 2 + \mathbf{1}_{\alpha\in\mathcal{I}_{poly}} I_* + \mathbf{1}_{\beta\in\mathcal{I}_{poly}} I'_*.$$

Let $0 \leq \varpi \leq 1$, with $\varpi \equiv 0$ if $\alpha \in \mathcal{I}_{mono}$. If $\alpha \in \mathcal{I}_{poly}$, then

$$\frac{(II_*)^{\delta^{(\alpha)}/4 - 1/2}}{\widetilde{\mathcal{E}}_{\alpha\beta}^{\delta^{(\alpha)}/2}} \leq \frac{C}{\widetilde{\mathcal{E}}_{\alpha\beta}^\varpi I^\kappa I_*^{1-\kappa-\varpi}} \leq \frac{C}{|\mathbf{g}|^{2\varpi} I^\kappa I_*^{1-\kappa-\varpi}} \text{ for } 0 \leq \kappa \leq 1 - \varpi. \quad (25)$$

Denoting $\left(\mathbb{R}^3\right)^{\perp \mathbf{n}} = \left\{ \mathbf{w} \in \mathbb{R}^3; \ \mathbf{w} \perp \mathbf{n} \right\}$, we obtain the bound

$$\int_{(\mathbb{R}^3)^{\perp \mathbf{n}}} \left(1 + \frac{\mathbf{1}_{\mu_{\alpha\beta}|\widetilde{\mathbf{g}}|^2 > 2\Delta I_*}}{\widetilde{\Psi}_{\alpha\beta}^{1-\gamma/2}} \right) \exp \left(-\frac{m_\beta}{2} \left| \frac{(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\perp_n}}{2} + \mathbf{w} \right|^2 \right) d\mathbf{w}$$

$$\leq C \int_{(\mathbb{R}^3)^{\perp \mathbf{n}}} \left(1 + |\mathbf{w}|^{\gamma-2} \right) \exp \left(-\frac{m_\beta}{2} \left| \frac{(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\perp_n}}{2} + \mathbf{w} \right|^2 \right) d\mathbf{w}$$

$$\leq C \left(\int_{|\mathbf{w}|\leq 1} 1 + |\mathbf{w}|^{\gamma-2} d\mathbf{w} + 2 \int_{|\mathbf{w}|\geq 1} \exp \left(-\frac{m_\beta}{2} \left| \frac{(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\perp_n}}{2} + \mathbf{w} \right|^2 \right) d\mathbf{w} \right)$$

$$\leq C \left(\int_{|\mathbf{w}|\leq 1} 1 + |\mathbf{w}|^{\gamma-2} d\mathbf{w} + \int_{(\mathbb{R}^3)^{\perp \mathbf{n}}} e^{-|\widetilde{\mathbf{w}}|^2} d\widetilde{\mathbf{w}} \right)$$

$$= C \left(\int_0^1 1 + r^{\gamma-1} dr + \int_0^\infty r e^{-r^2} dr \right) = C, \text{ where } \widetilde{\Psi}_{\alpha\beta} = |\widetilde{\mathbf{g}}| \, |\mathbf{g}_*|. \quad (26)$$

By the bounds (25) and (26), and assumption (15), for any number κ, $0 \leq \kappa \leq 1 - \varpi$,

$$k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \leq \frac{C}{|\mathbf{g}|^{1+2\varpi}} \left(\mathbf{1}_{\alpha\in\mathcal{I}_{mono}} + \frac{\mathbf{1}_{\alpha\in\mathcal{I}_{poly}}}{I^\kappa I_*^{1-\kappa-\varpi}} \right)$$

$$\times \int_{\mathbb{R}_+^2} \exp \left(-m_\beta \frac{\left(m_\alpha \left(|\boldsymbol{\xi}_*|^2 - |\boldsymbol{\xi}|^2 \right) + 2\Delta I_* \right)^2}{8m_\alpha^2 |\mathbf{g}|^2} - \frac{m_\alpha^2}{8m_\beta} |\mathbf{g}|^2 \right)$$

$$\times \int_{(\mathbb{R}^3)^{\perp \mathbf{n}}} \left(1 + \frac{\mathbf{1}_{\mu_{\alpha\beta}|\widetilde{\mathbf{g}}|^2 > 2\Delta I_*}}{\widetilde{\Psi}_{\alpha\beta}^{1-\gamma/2}} \right) \exp \left(-\frac{m_\beta}{2} \left| \frac{(\boldsymbol{\xi} + \boldsymbol{\xi}_*)_{\perp_n}}{2} + \mathbf{w} \right|^2 \right) d\mathbf{w}$$

$$\times e^{-(I'+I_*')/2} \frac{\left(I'I_*'\right)^{\delta^{(\beta)}/2-1}}{\widetilde{\mathcal{E}}_{\alpha\beta}^{\delta^{(\beta)}/2}} dI'dI_*'$$

$$\leq \frac{C}{|\mathbf{g}|^{1+2\varpi}} \left(\mathbf{1}_{\alpha\in\mathcal{I}_{mono}} + \frac{\mathbf{1}_{\alpha\in\mathcal{I}_{poly}}}{I^\kappa I_*^{1-\kappa-\varpi}}\right) \int_{\mathbb{R}_+^2} \frac{e^{-(I'+I_*')/2}}{\left(I'I_*'\right)^{1-\delta^{(\beta)}/4}}$$

$$\times \exp\left(-\frac{m_\beta}{8}\left(|\mathbf{g}| - 2|\boldsymbol{\xi}|\cos\varphi + 2\chi_\alpha\right)^2 - \frac{m_\alpha^2}{8m_\beta}|\mathbf{g}|^2\right) dI'dI_*',$$

$$\text{where } \chi_\alpha = \frac{\Delta I_*}{m_\alpha|\mathbf{g}|} \text{ and } \cos\varphi = \mathbf{n}\cdot\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}. \tag{27}$$

For $0 \leq \kappa \leq 1 - \varpi$, by the bound (27) and the Cauchy-Schwarz inequality,

$$\left(k_{\alpha\beta}^{(\alpha)}(\mathbf{Z},\mathbf{Z}_*)\right)^2$$

$$\leq \frac{C}{|\mathbf{g}|^{2+4\varpi}} \left(\mathbf{1}_{\alpha\in\mathcal{I}_{mono}} + \frac{\mathbf{1}_{\alpha\in\mathcal{I}_{poly}}}{I^{2\kappa} I_*^{2-2\kappa-2\varpi}}\right) \int_{\mathbb{R}_+^2} \frac{e^{-(I'+I_*')/2}}{\left(I'I_*'\right)^{1-\delta^{(\beta)}/4}} dI'dI_*' \int_{\mathbb{R}_+^2} \frac{e^{-(I'+I_*')/2}}{\left(I'I_*'\right)^{1-\delta^{(\beta)}/4}}$$

$$\times \exp\left(-\frac{m_\beta}{4}\left(|\mathbf{g}| - 2|\boldsymbol{\xi}|\cos\varphi + 2\chi_\alpha\right)^2 - \frac{m_\alpha^2}{4m_\beta}|\mathbf{g}|^2\right) dI'dI_*'$$

$$\leq C \left(\int_{\mathbb{R}_+^2} \exp\left(-\frac{m_\beta}{4}\left(|\mathbf{g}| - 2|\boldsymbol{\xi}|\cos\varphi + 2\chi_\alpha\right)^2\right) \frac{e^{-(I'+I_*')/2}}{\left(I'I_*'\right)^{1-\delta^{(\beta)}/4}} dI'dI_*'$$

$$\times \frac{\mathbf{1}_{\alpha\in\mathcal{I}_{poly}}}{|\mathbf{g}|^{4\varpi} I^{2\kappa} I_*^{2-2\kappa-2\varpi}} + \mathbf{1}_{\alpha\in\mathcal{I}_{mono}}\right) \frac{e^{-m_\alpha^2|\mathbf{g}|^2/(4m_\beta)}}{|\mathbf{g}|^2}, \text{ with } \cos\varphi = \mathbf{n}\cdot\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}. \tag{28}$$

If $\alpha \in \mathcal{I}_{poly}$, by the change of variables $I_* \to \Phi = |\mathbf{g}| - 2|\boldsymbol{\xi}|\cos\varphi + 2\chi_\alpha$, noting that

$$dI_* = \frac{m_\alpha}{2}|\mathbf{g}|\,d\Phi,$$

for any positive number $a > 0$, the bound

$$\int_{\mathbb{R}_+^3} \exp\left(-\frac{m_\beta}{a}\left(|\mathbf{g}| - 2|\boldsymbol{\xi}|\cos\varphi + 2\chi_\alpha\right)^2\right) \frac{e^{-(I'+I_*')/2}}{\left(I'I_*'\right)^{1-\delta^{(\beta)}/4}} dI_*dI'dI_*'$$

$$\leq C|\mathbf{g}| \int_{-\infty}^{\infty} e^{-m_\beta\Phi^2/a}\,d\Phi \left(\int_0^\infty \frac{e^{-I'/2}}{(I')^{1-\delta^{(\beta)}/4}} dI'\right)^2 = C|\mathbf{g}| \tag{29}$$

may be obtained. Note also that

$$\int_{\mathbb{R}_+} \left(\frac{1}{|\mathbf{g}|^4}\mathbf{1}_{I\leq 1} + \frac{1}{I^2}\mathbf{1}_{I\geq 1}\right) dI \leq C\left(1 + \frac{1}{|\mathbf{g}|^4}\right) \text{ and } \frac{1}{|\mathbf{g}|} \leq 1 + \frac{1}{|\mathbf{g}|^4}. \tag{30}$$

Then, by the bounds (28) for $\kappa = 1 - \varpi = \begin{cases} 0 \text{ if } I \leq 1 \\ 1 \text{ if } I > 1 \end{cases}$, (29), and (30), one have that, $k_{\alpha\beta}^{(\alpha)}(\mathbf{Z},\mathbf{Z}_*)\mathbf{1}_{\mathfrak{h}_N} \in L^2(d\mathbf{Z}d\mathbf{Z}_*)$ for any (nonzero) natural number N, since

$$\int_{\mathfrak{h}_N} \left(k_{\alpha\beta}^{(\alpha)}(\mathbf{Z},\mathbf{Z}_*)\right)^2 d\mathbf{Z}d\mathbf{Z}_*$$

$$\leq C \int_{|\mathbf{g}| \geq \frac{1}{N}} \left(\frac{1}{|\mathbf{g}|} + \frac{1}{|\mathbf{g}|^5} \right) e^{-m_\alpha^2 |\mathbf{g}|^2/(4m_\beta)} \, d\mathbf{g} \int_{|\boldsymbol{\xi}| \leq N} d\boldsymbol{\xi}$$

$$= C \int_{1/N}^\infty \left(R + \frac{1}{R^3} \right) e^{-m_\alpha^2 R^2/(4m_\beta)} dR \int_0^N r^2 \, dr$$

$$\leq C N^3 \int_0^\infty e^{-m_\alpha^2 R^2/(4m_\beta)} (1 + R^4) N^3 dR = C N^6.$$

Note that, by the bound (27) for $\kappa = 0$ and if $I_* > 1$, $\varpi \equiv 0$,

$$k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*)$$

$$\leq \frac{C}{|\mathbf{g}|} \int_{\mathbb{R}_+^2} \exp\left(-\frac{m_\beta}{8} \left(|\mathbf{g}| - 2 |\boldsymbol{\xi}| \cos\varphi + 2\chi_\alpha \right)^2 - \frac{m_\alpha^2}{8m_\beta} |\mathbf{g}|^2 \right) \frac{e^{-(I'+I_*')/2}}{(I'I_*')^{1-\delta^{(\beta)}/4}} dI' dI_*' \Psi_{*\varpi}^\alpha,$$

with $\Psi_{*\varpi}^\alpha = \left(\dfrac{1}{|\mathbf{g}|^{2\varpi} I_*^{1-\varpi}} \mathbf{1}_{I_* \leq 1} + \dfrac{\mathbf{1}_{\alpha \in \mathcal{I}_{poly}}}{I_*} \mathbf{1}_{I_* \geq 1} \right)$

$$\leq \left(1 + \frac{1}{|\mathbf{g}|^{2\varpi}} \right) \left(\frac{1}{I_*^{1-\varpi}} \mathbf{1}_{I_* \leq 1} + \frac{\mathbf{1}_{\alpha \in \mathcal{I}_{poly}}}{I_*} \mathbf{1}_{I_* \geq 1} \right). \tag{31}$$

The integral of $k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*)$ with respect to $\mathbf{Z}$ over $\mathcal{Z}_\alpha$ is bounded in $\mathbf{Z}_*$. Indeed, due to the symmetry (22) of the kernel $k_{\alpha\beta}^{(\alpha)}$, follows, by the bounds (29) and (31), that

$$\int_{\mathcal{Z}_\alpha} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \, d\mathbf{Z} = \int_{\mathcal{Z}_\alpha} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}) \, d\mathbf{Z}$$

$$\leq \int_0^\infty \int_{\mathbb{R}^3} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}_*, \mathbf{Z}) \left(\mathbf{1}_{I \leq 1} + I \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \right) d\mathbf{g} \, dI$$

$$\leq \int_{\mathbb{R}^3} \frac{C}{|\mathbf{g}|} \left(\left(1 + \frac{1}{|\mathbf{g}|} \right) \int_0^1 \frac{1}{I^{1/2}} dI \left(\int_0^\infty \frac{e^{-I'/2}}{(I')^{1-\delta^{(\beta)}/4}} dI' \right)^2 + |\mathbf{g}| \right)$$

$$\times e^{-m_\alpha^2 |\mathbf{g}|^2/(8m_\beta)} d\mathbf{g}$$

$$\leq C \int_0^\infty \left(1 + R^2 \right) e^{-m_\alpha^2 R^2/(8m_\beta)} dR = C.$$

Aiming to prove that

$$\int_{\mathcal{Z}_\alpha} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \, d\mathbf{Z}_* = \int_{\mathbb{R}^3 \times \mathbb{R}_+} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \left(\mathbf{1}_{I_* \leq 1} + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \mathbf{1}_{I_* \geq 1} \right) dI_* d\boldsymbol{\xi}_*$$

$$\leq C \frac{1 + \log(1 + |\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}, \tag{32}$$

for $|\boldsymbol{\xi}| \neq 0$, split the domain $\mathbb{R}^3 \times \mathbb{R}_+$ into two subdomains

$$\mathcal{D}_+ = \left\{ \mathbb{R}^3 \times \mathbb{R}_+; \ I_* \geq |\boldsymbol{\xi}| \right\} \quad \text{and} \quad \mathcal{D}_- = \left\{ \mathbb{R}^3 \times \mathbb{R}_+; \ I_* \leq |\boldsymbol{\xi}| \right\}.$$

By the bounds (29) and (31) (for $\varpi = 0$), it can be shown that the restriction of bound (32) to the domain $\mathcal{D}_+$ is satisfied for $|\boldsymbol{\xi}| \neq 0$

$$\int_{\mathbb{R}^3} \int_{|\boldsymbol{\xi}|}^\infty k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \left(\mathbf{1}_{I_* \leq 1} + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \mathbf{1}_{I_* \geq 1} \right) dI_* d\boldsymbol{\xi}_*$$

$$\leq \frac{C}{|\boldsymbol{\xi}|} \int_{\mathbb{R}^3} \int_0^\infty k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) I_* \left(\mathbf{1}_{I_* \leq 1} + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \right) dI_* d\boldsymbol{\xi}_*$$

$$\leq \frac{C}{|\boldsymbol{\xi}|} \int_{\mathbb{R}^3} \left(1 + \frac{1}{|\mathbf{g}|} \right) e^{-m_\alpha^2 |\mathbf{g}|^2/(8m_\beta)} d\mathbf{g}$$

$$= \frac{C}{|\boldsymbol{\xi}|} \int_0^\infty \left(R + R^2 \right) e^{-m_\alpha^2 R^2/(8m_\beta)} dR = \frac{C}{|\boldsymbol{\xi}|}$$

$$\leq C \frac{1 + \log(1 + |\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}.$$

On the other hand, by the bound (31) (for $\varpi = 1/2$ if $\alpha \in \mathcal{I}_{poly}$), it can be shown that the restriction of bound (32) to the domain $\mathcal{D}_-$ is satisfied for $|\boldsymbol{\xi}| \neq 0$

$$\int_{\mathbb{R}^3} \int_0^{|\boldsymbol{\xi}|} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \left(\mathbf{1}_{I_* \leq 1} + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \mathbf{1}_{I_* \geq 1} \right) dI_* d\boldsymbol{\xi}_*$$

$$\leq C \int_{\mathbb{R}_+^2} \int_0^{|\boldsymbol{\xi}|} \int_{\mathbb{R}^3} \exp\left(-\frac{m_\beta}{8} (|\mathbf{g}| - 2|\boldsymbol{\xi}| \cos\varphi + \chi_\alpha)^2 - \frac{m_\alpha^2}{8m_\beta} |\mathbf{g}|^2 \right) \frac{1 + |\mathbf{g}|}{|\mathbf{g}|^2} d\mathbf{g}$$

$$\times \left(\frac{1}{I_*^{1/2}} \mathbf{1}_{I_* \leq 1} + \frac{1}{I_*} \mathbf{1}_{I_* \geq 1} \right) \frac{e^{-(I' + I_*')/2}}{(I' I_*')^{1 - \delta^{(\beta)}/4}} dI_* dI' dI_*'$$

$$\leq \frac{C}{|\boldsymbol{\xi}|} \left(\int_0^1 \frac{1}{I_*^{1/2}} dI_* + \mathbf{1}_{|\boldsymbol{\xi}| \geq 1} \int_1^{|\boldsymbol{\xi}|} \frac{1}{I_*} dI_* \right) \left(\int_0^\infty \frac{e^{-I'/2}}{(I')^{1 - \delta^{(\beta)}/4}} dI' \right)^2$$

$$\leq \frac{C}{|\boldsymbol{\xi}|} \left(1 + \mathbf{1}_{|\boldsymbol{\xi}| \geq 1} \log|\boldsymbol{\xi}| \right) \leq C \frac{1 + \log(1 + |\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}.$$

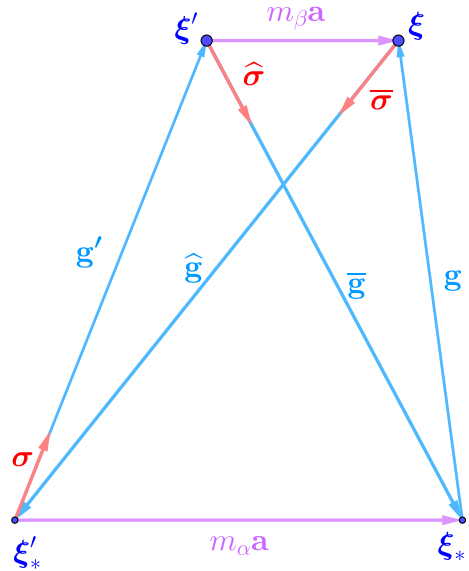Here the second inequality follows for $|\boldsymbol{\xi}| \neq 0$, with $\chi_R^\alpha = \Delta I_*/(m_\alpha R)$, by

$$C \int_{\mathbb{R}^3} \frac{1 + |\mathbf{g}|}{|\mathbf{g}|^2} \exp\left(-\frac{m_\beta}{8} (|\mathbf{g}| - 2|\boldsymbol{\xi}| \cos\varphi + 2\chi_\alpha)^2 - \frac{m_\alpha^2}{8m_\beta} |\mathbf{g}|^2 \right) d\mathbf{g}$$

$$= C \int_0^\infty \int_0^\pi (1 + R) \exp\left(-\frac{m_\beta}{8} (R - 2|\boldsymbol{\xi}| \cos\varphi + 2\chi_R^\alpha)^2 - \frac{m_\alpha^2}{8m_\beta} R^2 \right) \sin\varphi \, d\varphi \, dR$$

$$= \frac{C}{|\boldsymbol{\xi}|} \int_0^\infty \int_{R + 2\chi_R^\alpha - 2|\boldsymbol{\xi}|}^{R + 2\chi_R^\alpha + 2|\boldsymbol{\xi}|} e^{-m_\beta \Phi^2/8} (1 + R) e^{-m_\alpha^2 R^2/(8m_\beta)} d\Phi dR$$

$$\leq \frac{C}{|\boldsymbol{\xi}|} \int_0^\infty e^{-m_\alpha^2 R^2/(8m_\beta)} (1 + R) \, dR \int_{-\infty}^\infty e^{-m_\beta \Phi^2/8} d\Phi = \frac{C}{|\boldsymbol{\xi}|},$$

which is obtained by a change to spherical coordinates followed by the change of variables $\varphi \to \Phi = R - 2|\boldsymbol{\xi}| \cos\varphi + 2\chi_R^\alpha$, with $d\Phi = |\boldsymbol{\xi}| \sin\varphi \, d\varphi$.

Note that $f(x) = \dfrac{1 + \log(1 + x)}{x}$ is a decreasing function for $x > 0$. Therefore, by the bounds (29), (31), and (32),

$$\sup_{\mathbf{Z} \in \mathcal{Z}_\alpha} \int_{\mathcal{Z}_\alpha} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) - k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \mathbf{1}_{\mathfrak{h}_N} \, d\mathbf{Z}_*$$

$$\leq \sup_{\mathbf{Z} \in \mathcal{Z}_\alpha} \int_0^\infty \int_{|\mathbf{g}| \leq \frac{1}{N}} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*) \left(\mathbf{1}_{I_* \leq 1} + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \mathbf{1}_{I_* \geq 1} \right) d\mathbf{g} \, dI_* + \sup_{|\boldsymbol{\xi}| \geq N} \frac{1 + \log(1 + |\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}$$

Fig. 3 Typical collision of $K_{\alpha\beta2}$

$$\leq \sup_{\mathbf{Z}\in\mathcal{Z}_\alpha} \int_0^\infty \int_{|\mathbf{g}|\leq\frac{1}{N}} k_{\alpha\beta}^{(\alpha)}(\mathbf{Z},\mathbf{Z}_*)\left(\mathbf{1}_{I_*\leq 1}+\mathbf{1}_{\alpha\in\mathcal{I}_{poly}}I_*\right)d\mathbf{g}\,dI_* + C\frac{1+\log(1+N)}{N}$$

$$\leq \int_{|\mathbf{g}|\leq\frac{1}{N}}\frac{C}{|\mathbf{g}|}\left(\frac{1}{|\mathbf{g}|}\int_0^1\frac{1}{I_*^{1/2}}dI_*+|\mathbf{g}|\right)d\mathbf{g}+C\frac{1+\log(1+N)}{N}$$

$$\leq C\left(\int_{|\mathbf{g}|\leq\frac{1}{N}}\frac{1}{|\mathbf{g}|^2}d\mathbf{g}+\frac{1+\log(1+N)}{N}\right)$$

$$= C\left(\int_0^{\frac{1}{N}}d\,|\mathbf{g}|+\frac{1+\log(1+N)}{N}\right)\to 0 \text{ as } N\to\infty.$$

Hence, by Lemma 4, the operators

$$K_{\alpha\beta3}=\int_{\mathcal{Z}_\alpha}k_{\alpha\beta}^{(\alpha)}(\mathbf{Z},\mathbf{Z}_*)\,h_*\,d\mathbf{Z}_*$$

are compact on $L^2\,(d\mathbf{Z}_\alpha)$ for $(\alpha,\beta)\in\mathcal{I}^2$.

III. Compactness of $K_{\alpha\beta2}=\int_{\mathcal{Z}_\beta}k_{\alpha\beta}^{(\beta)}(\mathbf{Z},\mathbf{Z}_*)\,h_{\beta*}\,d\mathbf{Z}_*$ for any indices $(\alpha,\beta)\in\mathcal{I}^2$.

Assume that $m_\alpha\neq m_\beta$ and denote

$$\mathbf{g}=\boldsymbol{\xi}-\boldsymbol{\xi}_*,\mathbf{g}'=\boldsymbol{\xi}'-\boldsymbol{\xi}'_*,\widehat{\mathbf{g}}=\boldsymbol{\xi}'_*-\boldsymbol{\xi},\overline{\mathbf{g}}=\boldsymbol{\xi}_*-\boldsymbol{\xi}',\mathbf{g}_{\alpha\beta}=\frac{m_\alpha\boldsymbol{\xi}-m_\beta\boldsymbol{\xi}_*}{m_\alpha-m_\beta},$$

$$\mathbf{g}'_{\alpha\beta}=\frac{m_\alpha\boldsymbol{\xi}'-m_\beta\boldsymbol{\xi}'_*}{m_\alpha-m_\beta},\widehat{\sigma}_{\alpha\beta}=\sigma_{\alpha\beta}\left(|\widehat{\mathbf{g}}|,\frac{\widehat{\mathbf{g}}\cdot\overline{\mathbf{g}}}{|\widehat{\mathbf{g}}|\,|\overline{\mathbf{g}}|},I,I'_*,I',I_*\right),$$

$$\widehat{\Delta}_{\alpha\beta}I_\#=\frac{m_\beta-m_\alpha}{m_\alpha m_\beta}\Delta I_\#,\text{ and }\Delta I_\#=\mathbf{1}_{\alpha\in\mathcal{I}_{poly}}\left(I'-I\right)+\mathbf{1}_{\beta\in\mathcal{I}_{poly}}\left(I_*-I'_*\right).$$

Furthermore, denote $\widehat{\mathcal{E}}_{\alpha\beta} = 1$ if $\alpha \in \mathcal{I}_{mono}$ and $\widehat{\mathcal{E}}^*_{\alpha\beta} = 1$ if $\beta \in \mathcal{I}_{mono}$, while, otherwise,

$$
\begin{aligned}
\widehat{\mathcal{E}}_{\alpha\beta} = \widehat{\mathcal{E}}^*_{\alpha\beta} &= \mu_{\alpha\beta} \, |\widehat{\mathbf{g}}|^2 / 2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I'_* \\
&= \mu_{\alpha\beta} \, |\overline{\mathbf{g}}|^2 / 2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I' + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I_*.
\end{aligned}
$$

Note that, cf. Figure 3,

$$
\begin{aligned}
W_{\alpha\beta} &\left(\mathbf{Z}, \mathbf{Z}'_* \left| \mathbf{Z}', \mathbf{Z}_* \right. \right) \\
&= (m_\alpha + m_\beta)^2 \, m_\alpha m_\beta \varphi_\alpha (I) \, \varphi_\beta (I'_*) \, \widehat{\sigma}_{\alpha\beta} \delta_3 \left((m_\alpha - m_\beta) \left(\mathbf{g}_{\alpha\beta} - \mathbf{g}'_{\alpha\beta} \right) \right) \\
&\quad \times \frac{|\widehat{\mathbf{g}}|}{|\overline{\mathbf{g}}|} \delta_1 \left(\frac{m_\alpha m_\beta}{2 (m_\alpha - m_\beta)} \left(|\mathbf{g}'|^2 - |\mathbf{g}|^2 \right) - \Delta I_\# \right) \\
&= \frac{(m_\alpha + m_\beta)^2}{(m_\alpha - m_\beta)^2} \varphi_\alpha (I) \, \varphi_\beta (I'_*) \, \widehat{\sigma}_{\alpha\beta} \mathbf{1}_{|\mathbf{g}|^2 > 2\widehat{\Delta}_{\alpha\beta} I_\#} \frac{|\widehat{\mathbf{g}}|}{|\mathbf{g}'| \, |\overline{\mathbf{g}}|} \delta_3 \left(\mathbf{g}_{\alpha\beta} - \mathbf{g}'_{\alpha\beta} \right) \\
&\quad \times \delta_1 \left(|\mathbf{g}'| - \sqrt{|\mathbf{g}|^2 - 2\widehat{\Delta}_{\alpha\beta} I_\#} \right),
\end{aligned}
\tag{33}
$$

Then, by a change of variables $\{\boldsymbol{\xi}', \boldsymbol{\xi}'_*\} \to \left\{ |\mathbf{g}'|, \sigma = \dfrac{\mathbf{g}'}{|\mathbf{g}'|}, \mathbf{g}'_{\alpha\beta} = \dfrac{m_\alpha \boldsymbol{\xi}' - m_\beta \boldsymbol{\xi}'_*}{m_\alpha - m_\beta} \right\}$, with $\mathbf{g}' = \boldsymbol{\xi}' - \boldsymbol{\xi}'_*$, where

$$
d\boldsymbol{\xi}' d\boldsymbol{\xi}'_* = d\mathbf{g}' d\mathbf{g}'_{\alpha\beta} = |\mathbf{g}'|^2 \, d \, |\mathbf{g}'| \, d\mathbf{g}'_{\alpha\beta} d\sigma,
$$

and substitution of expression (33) in the expression (21) of $k^{(\beta)}_{\alpha\beta2}$, one obtain

$$
\begin{aligned}
k^{(\beta)}_{\alpha\beta2}(\mathbf{Z}, \mathbf{Z}_*) &= \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta} \frac{w_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}'_* | \mathbf{Z}', \mathbf{Z}_*)}{(M_\alpha M_{\beta*})^{1/2}} \, d\mathbf{Z}' d\mathbf{Z}'_* \\
&= \int_{\mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}^3_+} \left(M'_\alpha M'_{\beta*} \right)^{1/2} \frac{W_{\alpha\beta}(\mathbf{Z}, \mathbf{Z}'_* | \mathbf{Z}', \mathbf{Z}_*) \, |\mathbf{g}'|^2}{\left(\varphi_\alpha (I) \, \varphi_\beta (I'_*) \, \varphi_\alpha (I') \, \varphi_\beta (I_*) \right)^{1/2}} \\
&\quad \times \mathbf{1}_{|\mathbf{g}|^2 > 2\widehat{\Delta}_{\alpha\beta} I_\#} \, \mathbf{1}_{I' \le \widehat{\mathcal{E}}_{\alpha\beta}} \mathbf{1}_{I'_* \le \widehat{\mathcal{E}}^*_{\alpha\beta}} \, d\mathbf{g}'_{\alpha\beta} d\sigma d \, |\mathbf{g}'| \, dI' dI'_* \\
&= \frac{(m_\alpha + m_\beta)^2}{(m_\alpha - m_\beta)^2} \int_{\mathbb{S}^2 \times \mathbb{R}^2_+} \left(M'_\alpha M'_{\beta*} \right)^{1/2} \frac{|\widehat{\mathbf{g}}| \, |\mathbf{g}'|}{|\overline{\mathbf{g}}|} \left(\frac{\varphi_\alpha (I) \, \varphi_\beta (I'_*)}{\varphi_\alpha (I') \, \varphi_\beta (I_*)} \right)^{1/2} \\
&\quad \times \widehat{\sigma}_{\alpha\beta} \mathbf{1}_{|\mathbf{g}|^2 > 2\widehat{\Delta}_{\alpha\beta} I_\#} \, \mathbf{1}_{I' \le \widehat{\mathcal{E}}_{\alpha\beta}} \mathbf{1}_{I'_* \le \widehat{\mathcal{E}}^*_{\alpha\beta}} \, d\sigma dI' dI'_*.
\end{aligned}
\tag{34}
$$

However, also

$$
\begin{aligned}
k^{(\beta)}_{\alpha\beta2}(\mathbf{Z}, \mathbf{Z}_*) &= \int_{\mathbb{S}^2 \times \mathbb{R}^2_+} |\widehat{\mathbf{g}}| \left(M'_\alpha M'_{\beta*} \right)^{1/2} \left(\frac{\varphi_\alpha (I) \, \varphi_\beta (I'_*)}{\varphi_\alpha (I') \, \varphi_\beta (I_*)} \right)^{1/2} \widehat{\sigma}_{\alpha\beta} \\
&\quad \times \mathbf{1}_{\mu_{\alpha\beta} |\widehat{\mathbf{g}}|^2 > 2\Delta I_\#} \mathbf{1}_{I' \le \widehat{\mathcal{E}}_{\alpha\beta}} \mathbf{1}_{I'_* \le \widehat{\mathcal{E}}^*_{\alpha\beta}} \, d\widehat{\sigma} dI' dI'_*, \text{ with } \widehat{\sigma} = \frac{\widehat{\mathbf{g}}}{|\widehat{\mathbf{g}}|},
\end{aligned}
\tag{35}
$$

and, furthermore,

$$
k^{(\beta)}_{\alpha\beta2}(\mathbf{Z}, \mathbf{Z}_*) = \int_{\mathbb{S}^2 \times \mathbb{R}^2_+} |\overline{\mathbf{g}}| \left(M'_\alpha M'_{\beta*} \right)^{1/2} \left(\frac{\varphi_\alpha (I') \, \varphi_\beta (I_*)}{\varphi_\alpha (I) \, \varphi_\beta (I'_*)} \right)^{1/2} \overline{\sigma}_{\alpha\beta}
$$

$$\times \mathbf{1}_{\mu_{\alpha\beta}|\overline{\mathbf{g}}|^2 > -2\Delta I_\#} \mathbf{1}_{I' \leq \widehat{\mathcal{E}}_{\alpha\beta}} \mathbf{1}_{I'_* \leq \widehat{\mathcal{E}}^*_{\alpha\beta}} d\overline{\sigma} \, dI' dI'_*, \text{ with}$$

$$\overline{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta}\left(|\overline{\mathbf{g}}|, \frac{\widehat{\mathbf{g}} \cdot \overline{\mathbf{g}}}{|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}|}, I', I_*, I, I'_*\right) \text{ and } \overline{\sigma} = \frac{\widehat{\mathbf{g}}}{|\widehat{\mathbf{g}}|}. \tag{36}$$

By straightforward calculations, with $\boldsymbol{\sigma} = \mathbf{g}'/|\mathbf{g}'|$,

$$\begin{cases} \boldsymbol{\xi}' = \mathbf{g}'_{\alpha\beta} - \dfrac{m_\beta}{m_\alpha - m_\beta}\mathbf{g}' = \mathbf{g}_{\alpha\beta} - \dfrac{m_\beta \boldsymbol{\sigma}}{m_\alpha - m_\beta}\sqrt{|\mathbf{g}|^2 - 2\widehat{\Delta}_{\alpha\beta} I_\#} \\ \boldsymbol{\xi}'_* = \mathbf{g}'_{\alpha\beta} - \dfrac{m_\alpha}{m_\alpha - m_\beta}\mathbf{g}' = \mathbf{g}_{\alpha\beta} - \dfrac{m_\alpha \boldsymbol{\sigma}}{m_\alpha - m_\beta}\sqrt{|\mathbf{g}|^2 - 2\widehat{\Delta}_{\alpha\beta} I_\#} \end{cases}.$$

It follows, again by straightforward calculations, that

$$m_\alpha \left|\boldsymbol{\xi}'\right|^2 + m_\beta \left|\boldsymbol{\xi}'_*\right|^2 = (m_\alpha + m_\beta)\left(\left|\mathbf{g}'_{\alpha\beta}\right|^2 + \frac{m_\alpha m_\beta}{(m_\alpha - m_\beta)^2}|\mathbf{g}'|^2\right) - \frac{4m_\alpha m_\beta}{m_\alpha - m_\beta}\mathbf{g}'_{\alpha\beta} \cdot \mathbf{g}'$$

$$= \left(\sqrt{m_\alpha} - \sqrt{m_\beta}\right)^2 \left(\left|\mathbf{g}'_{\alpha\beta}\right|^2 + \frac{m_\alpha m_\beta}{(m_\alpha - m_\beta)^2}|\mathbf{g}'|^2\right)$$

$$+ 2\sqrt{m_\alpha m_\beta}\left(\mathbf{g}'_{\alpha\beta} - \frac{\sqrt{m_\alpha m_\beta}}{m_\alpha - m_\beta}\mathbf{g}'\right)^2$$

$$\geq \left(\sqrt{m_\alpha} - \sqrt{m_\beta}\right)^2 \left|\mathbf{g}'_{\alpha\beta}\right|^2 + 4A_2 |\mathbf{g}'|^2$$

$$= \left(\sqrt{m_\alpha} - \sqrt{m_\beta}\right)^2 \left|\mathbf{g}_{\alpha\beta}\right|^2 + 4A_2 \left(|\mathbf{g}|^2 - 2\widehat{\Delta}_{\alpha\beta} I_\#\right), \text{where}$$

$$A_2 = \frac{m_\alpha m_\beta}{4\left(\sqrt{m_\alpha} + \sqrt{m_\beta}\right)^2} > 0. \tag{37}$$

Applying the Cauchy-Schwarz inequality, we obtain that

$$\left(\int_{\mathbb{S}^2 \times \mathbb{R}^2_+} \left(M'_\alpha M'_{\beta*}\right)^{1/4} \frac{\left(\varphi_\alpha(I)\,\varphi_\beta(I'_*)\,\varphi_\alpha(I')\,\varphi_\beta(I_*)\right)^{1/2}}{\widehat{\mathcal{E}}_{\alpha\beta}^{\delta(\alpha)/2}\left(\widehat{\mathcal{E}}^*_{\alpha\beta}\right)^{\delta(\beta)/2}} e^{-(I'+I'_*)/4}d\boldsymbol{\sigma}\,dI'dI'_*\right)^2$$

$$\leq \int_{\mathbb{S}^2 \times \mathbb{R}^2_+} \left(M'_\alpha M'_{\beta*}\right)^{1/2} \frac{\left(\varphi_\alpha(I)\,\varphi_\beta(I'_*)\,\varphi_\alpha(I')\,\varphi_\beta(I_*)\right)^{1/2}}{\widehat{\mathcal{E}}_{\alpha\beta}^{\delta(\alpha)}\left(\widehat{\mathcal{E}}^*_{\alpha\beta}\right)^{\delta(\beta)}} e^{-(I'+I'_*)/4}d\boldsymbol{\sigma}\,dI'dI'_*$$

$$\times \int_{\mathbb{S}^2 \times \mathbb{R}^2_+} e^{-(I'+I'_*)/4}d\boldsymbol{\sigma}\,dI'dI'_*$$

$$\leq 64\pi \int_{\mathbb{S}^2 \times \mathbb{R}^2_+} \frac{\left(M'_\alpha M'_{\beta*}\right)^{1/2}}{\widehat{\mathcal{E}}_{\alpha\beta}^2\left(\widehat{\mathcal{E}}^*_{\alpha\beta}\right)^2} e^{-(I'+I'_*)/4}d\boldsymbol{\sigma}\,dI'dI'_*, \tag{38}$$

since for $(\alpha, \beta) \in \mathcal{I}^2$

$$\widehat{\mathcal{E}}_{\alpha\beta}^{\delta(\alpha)-2}\left(\widehat{\mathcal{E}}^*_{\alpha\beta}\right)^{\delta(\beta)-2} \geq \left(II'\right)^{\delta(\alpha)/2-1}\left(I_*I'_*\right)^{\delta(\beta)/2-1} = \varphi_\alpha(I)\,\varphi_\beta(I'_*)\,\varphi_\alpha(I')\,\varphi_\beta(I_*).$$

Note that $\max\left(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|\right) \geq |\mathbf{g}|$, cf. Figure 3.

H1. If $\min(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|) \geq 1$, then $|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}| \geq \max(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|) \geq |\mathbf{g}|$, and

$$
\frac{|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}| + (|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}|)^{\gamma/2}}{|\widehat{\mathbf{g}}|^2} = \frac{|\overline{\mathbf{g}}|}{|\widehat{\mathbf{g}}|} \left(1 + \frac{1}{(|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}|)^{1-\gamma/2}} \right)
$$

$$
\leq \frac{|\overline{\mathbf{g}}|}{|\widehat{\mathbf{g}}|} \left(1 + \frac{1}{|\mathbf{g}|^{1-\gamma/2}} \right) \leq \frac{|\overline{\mathbf{g}}|}{|\widehat{\mathbf{g}}|} \left(1 + \frac{1}{|\mathbf{g}|} \right).
$$

Observe that $|\mathbf{g}'|^2 \left(M'_\alpha M'_{\beta*} \right)^{1/4} \leq C e^{-(I' + I'_*)/4}$, since $x^2 e^{-ax^2} \leq 1/(ae)$ for any positive number $a > 0$. Hence, by expression (34), assumption (15), and inequalities (37)-(38), one may obtain the following bound

$$
\left(k_{\alpha\beta 2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 \leq C \int_{\mathbb{R}_+^2} \frac{\left(\varphi_\alpha(I') \varphi_\beta(I'_*) \right)^{1/2}}{\widehat{\mathcal{E}}_{\alpha\beta}^2 \left(\widehat{\mathcal{E}}_{\alpha\beta}^* \right)^2} e^{2A_2 \widehat{\Delta}_{\alpha\beta} I_\# - (I' + I'_*)/2} \mathbf{1}_{|\mathbf{g}|^2 > 2\widehat{\Delta}_{\alpha\beta} I_\#} dI' dI'_*
$$

$$
\times \left(1 + \frac{1}{|\mathbf{g}|^2} \right) e^{-(\sqrt{m_\alpha} - \sqrt{m_\beta})^2 |\mathbf{g}_{\alpha\beta}|^2/4} e^{-A_2 |\mathbf{g}|^2} \tag{39}
$$

H2. If $\min(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|) < 1$, then either of the two cases below apply:

(i) $|\widehat{\mathbf{g}}| = \max(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|) \geq |\mathbf{g}|$ and $|\overline{\mathbf{g}}| = \min(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|) < 1$, and, hence,

$$
\frac{|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}| + (|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}|)^{\gamma/2}}{|\widehat{\mathbf{g}}|^2} \leq \frac{1}{|\widehat{\mathbf{g}}|} \left(1 + \frac{1}{|\mathbf{g}|^{1-\gamma/2}} \right) \leq \frac{C}{|\widehat{\mathbf{g}}|} \left(1 + \frac{1}{|\mathbf{g}|} \right),
$$

and then, by expression (35), assumption (15), and inequalities (37)–(38), the bound (39) is again satisfied.

(ii) $|\overline{\mathbf{g}}| = \max(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|) \geq |\mathbf{g}|$ and $|\widehat{\mathbf{g}}| = \min(|\widehat{\mathbf{g}}|, |\overline{\mathbf{g}}|) < 1$, implying, correspondingly, that

$$
\frac{|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}| + (|\widehat{\mathbf{g}}| \, |\overline{\mathbf{g}}|)^{\gamma/2}}{|\overline{\mathbf{g}}|^2} \leq \frac{C}{|\overline{\mathbf{g}}|} \left(1 + \frac{1}{|\mathbf{g}|} \right),
$$

and, hence, by expression (36), assumption (15), and inequalities (37)–(38), the bound (39) is again obtained.

Note that

$$
4A_2 \widehat{\Delta}_{\alpha\beta} I_\# - \left(\mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I' + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I'_* \right)
$$

$$
= 2A_1 \Delta I_\# - \left(\mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I' + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I'_* \right)
$$

$$
= 2A_1 \left(\mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I - \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I_* \right) - \frac{\mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \sqrt{m_\alpha} I' + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} \sqrt{m_\beta} I'_*}{\sqrt{m_\alpha} + \sqrt{m_\beta}}, \tag{40}
$$

where

$$
A_1 = \frac{\sqrt{m_\alpha} - \sqrt{m_\beta}}{2 \left(\sqrt{m_\alpha} + \sqrt{m_\beta} \right)} \quad \text{and} \quad A_2 = \frac{m_\alpha m_\beta}{4 \left(\sqrt{m_\alpha} + \sqrt{m_\beta} \right)^2} > 0.
$$

Moreover, for $\alpha \in \mathcal{I}_{poly}$

$$
\widehat{\mathcal{E}}_{\alpha\beta} \geq I^{1-\zeta_\alpha - \pi_\alpha} \left(I' \right)^{\zeta_\alpha} I \, |\mathbf{g}|^{2\pi_\alpha}, \tag{41}
$$

and for $\beta \in \mathcal{I}_{poly}$

$$
\widehat{\mathcal{E}}_{\alpha\beta}^* \geq I_*^{1-\zeta_\beta - \pi_\beta} \left(I'_* \right)^{\zeta_\beta} |\mathbf{g}|^{2\pi_\beta}, \tag{42}
$$

where $0 \leq \pi_\gamma \leq 1 - \zeta_\gamma$ and $0 \leq \zeta_\gamma < 1/2$ for $\gamma \in \{\alpha, \beta\}$. If the species a_γ is monatomic, $\gamma \in \mathcal{I}_{mono}$, let $1 - \zeta_\gamma = \pi_\gamma = 0$.

Therefore, by the bounds (39), (41), and (42), and expression (40), changing variables of integration $\{\boldsymbol{\xi}, \boldsymbol{\xi}_*\} \to \{\mathbf{g}, \mathbf{g}_{\alpha\beta}\}$, with unitary Jacobian,

$$
\int_{\mathbb{R}^3 \times \Omega} \left(k_{\alpha\beta 2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 d\boldsymbol{\xi} d\boldsymbol{\xi}_*
$$
$$
\leq C \int_\Omega \frac{e^{A_1 \left(\mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I - \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I_* \right) - A_2 |\mathbf{g}|^2}}{I^{2-2\zeta_\alpha - 2\pi_\alpha} I_*^{2-2\zeta_\beta - 2\pi_\beta} |\mathbf{g}|^{4(\pi_\alpha + \pi_\beta)}} \left(1 + \frac{1}{|\mathbf{g}|^2} \right) d\mathbf{g}
$$
$$
\times \int_{\mathbb{R}^3} e^{-(\sqrt{m_\alpha} - \sqrt{m_\beta})^2 |\mathbf{g}_{\alpha\beta}|^2 / 4} d\mathbf{g}_{\alpha\beta}
$$
$$
\times \int_0^\infty (I')^{\delta^{(\alpha)}/4 - 1/2 - 2\zeta_\alpha \mathbf{1}_{\alpha \in \mathcal{I}_{poly}}} e^{-\sqrt{m_\alpha} I'/(2(\sqrt{m_\alpha} + \sqrt{m_\beta}))} dI'
$$
$$
\times \int_0^\infty (I'_*)^{\delta^{(\beta)}/4 - 1/2 - 2\zeta_\beta \mathbf{1}_{\beta \in \mathcal{I}_{poly}}} e^{-\sqrt{m_\beta} I'_*/(2(\sqrt{m_\alpha} + \sqrt{m_\beta}))} dI'_*
$$
$$
\leq C \int_\Omega \frac{e^{A_1 \left(\mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I - \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I_* \right) - A_2 |\mathbf{g}|^2}}{I^{2-2\zeta_\alpha - 2\pi_\alpha} I_*^{2-2\zeta_\beta - 2\pi_\beta} |\mathbf{g}|^{4(\pi_\alpha + \pi_\beta)}} \left(1 + \frac{1}{|\mathbf{g}|^2} \right) d\mathbf{g}, \tag{43}
$$

for $\Omega \subseteq \mathbb{R}^3$, since, by a change to spherical coordinates,

$$
\int_{\mathbb{R}^3} e^{-(\sqrt{m_\alpha} - \sqrt{m_\beta})^2 |\mathbf{g}_{\alpha\beta}|^2 / 2} d\mathbf{g}_{\alpha\beta} = C \int_0^\infty R^2 e^{-R^2} dR = C.
$$

Without loss of generality we can assume that $m_\alpha > m_\beta$, and then $A_1 > 0$. On the other hand, for $\Omega \subseteq \mathbb{R}^3$, by inequality (37), and the bounds (39), (41), and (42), also

$$
\int_{\mathbb{R}^3 \times \Omega} \left(k_{\alpha\beta 2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 d\boldsymbol{\xi} d\boldsymbol{\xi}_*
$$
$$
\leq C \int_{\Omega \times \mathbb{R}_+^2} \frac{e^{-A_2 \left(|\mathbf{g}|^2 - 2\widehat{\Delta}_{\alpha\beta} I_\# \right)} \mathbf{1}_{|\mathbf{g}|^2 > 2\widehat{\Delta}_{\alpha\beta} I_\#}}{I^{2-2\zeta_\alpha - 2\pi_\alpha} I_*^{2-2\zeta_\beta - 2\pi_\beta} |\mathbf{g}|^{4(\pi_\alpha + \pi_\beta)}} \left(1 + \frac{1}{|\mathbf{g}|^2} \right) d\mathbf{g}
$$
$$
\times (I')^{\delta^{(\alpha)}/4 - 1/2 - 2\zeta_\alpha \mathbf{1}_{\alpha \in \mathcal{I}_{poly}}} e^{-I'/2} (I'_*)^{\delta^{(\beta)}/4 - 1/2 - 2\zeta_\beta \mathbf{1}_{\beta \in \mathcal{I}_{poly}}} e^{-I'_*/2} dI' dI'_*
$$
$$
\times \int_{\mathbb{R}^3} e^{-(\sqrt{m_\alpha} - \sqrt{m_\beta})^2 |\mathbf{g}_{\alpha\beta}|^2 / 4} d\mathbf{g}_{\alpha\beta}
$$
$$
\leq C \int_\Omega \frac{1}{I^{2-2\zeta_\alpha - 2\pi_\alpha} I_*^{2-2\zeta_\beta - 2\pi_\beta} |\mathbf{g}|^{4(\pi_\alpha + \pi_\beta)}} \left(1 + \frac{1}{|\mathbf{g}|^2} \right) d\mathbf{g}
$$
$$
\times \int_0^\infty (I')^{\delta^{(\alpha)}/4 - 1/2 - 2\zeta_\alpha \mathbf{1}_{\alpha \in \mathcal{I}_{poly}}} e^{-I'/2} dI'
$$
$$
\times \int_0^\infty (I'_*)^{\delta^{(\beta)}/4 - 1/2 - 2\zeta_\beta \mathbf{1}_{\beta \in \mathcal{I}_{poly}}} e^{-I'_*/2} dI'_*
$$
$$
\leq C \int_\Omega \frac{1}{I^{2-2\zeta_\alpha - 2\pi_\alpha} I_*^{2-2\zeta_\beta - 2\pi_\beta} |\mathbf{g}|^{4(\pi_\alpha + \pi_\beta)}} \left(1 + \frac{1}{|\mathbf{g}|^2} \right) d\mathbf{g}. \tag{44}
$$

By the bound (43), with $\zeta_\gamma = 4\pi_\gamma = 4/9$ if $\gamma \in \mathcal{I}_{poly}$—remind that $1 - \zeta_\gamma = \pi_\gamma = 0$ if $\gamma \in \mathcal{I}_{mono}$—

$$\int_{(\mathbb{R}^3 \times [0,1])^2} \left(k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 d\boldsymbol{\xi} d\boldsymbol{\xi}_* dI_* \, dI$$

$$\leq C \int_{\mathbb{R}^3} \frac{(1 + |\mathbf{g}|^2) e^{-A_2 |\mathbf{g}|^2}}{|\mathbf{g}|^{2 + 4(\pi_\alpha + \pi_\beta)}} d\mathbf{g} \int_0^1 I^{2(\zeta_\alpha + \pi_\alpha - 1)} \, dI \int_0^1 I_*^{2(\zeta_\beta + \pi_\beta - 1)} \, dI_*$$

$$\leq C \int_{\mathbb{R}^3} \frac{1 + R^2}{R^{4(\pi_\alpha + \pi_\beta)}} e^{-A_2 R^2} dR = C.$$

This covers the case $(\alpha, \beta) \in \mathcal{I}_{mono}^2$ completely.

Now assume that $\beta \in \mathcal{I}_{poly}$, and consider the case $I_* \geq I$, assuming that $I_* > 1$—else $(I, I_*) \in [0,1]^2$. Then, by the bound (43), with $\zeta_\alpha = 2\pi_\alpha = \begin{cases} 2/5 \text{ if } I \leq 1 \\ 0 \text{ if } I > 1 \end{cases}$ if $\alpha \in \mathcal{I}_{poly} - \pi_\alpha = 0$ if $\alpha \in \mathcal{I}_{mono}$—and $\zeta_\beta = 10\pi_\beta = 10/21$,

$$\int_1^\infty \int_0^{I_*} \int_{(\mathbb{R}^3)^2} \left(k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 \left(\mathbf{1}_{I \leq 1} \mathbf{1}_{\alpha \in \mathcal{I}_{mono}} + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \right) d\boldsymbol{\xi} d\boldsymbol{\xi}_* dI dI_*$$

$$\leq C \left(1 + \int_0^\infty \frac{\mathbf{1}_{\alpha \in \mathcal{I}_{poly}}}{I^{2 - 2\zeta_\alpha - 2\pi_\alpha}} dI \right) \int_{\mathbb{R}^3} \frac{(1 + |\mathbf{g}|^2) e^{-A_2 |\mathbf{g}|^2}}{|\mathbf{g}|^{2 + 4(\pi_\alpha + \pi_\beta)}} d\mathbf{g}$$

$$\times \int_0^\infty \frac{e^{-A_1(I_* - I)}}{(I_* - I)^{2 - 2\zeta_\beta - 2\pi_\beta}} d(I_* - I)$$

$$\leq C \left(1 + \int_0^1 I^{-4/5} dI + \int_1^\infty I^{-2} dI \right) \int_0^\infty \frac{e^{-A_2 R^2}}{R^{4(\pi_\alpha + \pi_\beta)}} (1 + R^2) dR = C.$$

This covers the case $(\alpha, \beta) \in \mathcal{I}_{mono} \times \mathcal{I}_{poly}$ completely, and the case $(\alpha, \beta) \in \mathcal{I}_{poly}^2$ partly.

On the other hand, if $\alpha \in \mathcal{I}_{poly}$, considering the case $I \geq I_*$, assume that $I > 1$—else $(I, I_*) \in [0,1]^2$. Then, by the bound (44), with $\pi_\alpha = \zeta_\alpha = 0$ and $\zeta_\beta = 2\pi_\beta = 2/5$ if $\beta \in \mathcal{I}_{poly}$,

$$\int_1^\infty \int_0^I \int_{|\mathbf{g}| \leq 1} \left(k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 \left(\mathbf{1}_{I_* \leq 1} \mathbf{1}_{\beta \in \mathcal{I}_{mono}} + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} \right) d\boldsymbol{\xi} d\boldsymbol{\xi}_* dI_* \, dI$$

$$\leq C \int_1^\infty I^{-2} \left(1 + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} \int_0^I I_*^{-4/5} dI_* \right) dI \int_{|\mathbf{g}| \leq 1} |\mathbf{g}|^{-2 - 4/5} d\mathbf{g}$$

$$\leq C \int_1^\infty I^{-2} I^{1/5} dI \int_0^1 R^{-4/5} dR = C \int_1^\infty I^{-9/5} dI = C.$$

If, additionally, assuming that the species a_β is polyatomic, $\beta \in \mathcal{I}_{poly}$, then, again by the bound (44), with $\zeta_\alpha = \zeta_\beta = 0$ and $1 - 2\pi_\alpha = \pi_\beta = 3/5$,

$$\int_1^\infty \int_0^I \int_{|\mathbf{g}| \geq 1} \left(k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 d\boldsymbol{\xi} d\boldsymbol{\xi}_* dI_* \, dI$$

$$\leq C \int_1^\infty I^{-8/5} \int_0^I I_*^{-4/5} dI_* dI \int_{|\mathbf{g}| \geq 1} |\mathbf{g}|^{-16/5} d\mathbf{g}$$

$$= C \int_1^\infty I^{-7/5} dI \int_1^\infty R^{-6/5} dR = C.$$

This completes the case $(\alpha, \beta) \in \mathcal{I}^2_{poly}$.

Assume now instead that, additionally, the species a_β is monatomic, $\beta \in \mathcal{I}_{mono}$. Then, If $A_1 I \leq A_2 |\mathbf{g}|^2$, by the bound (43), with $\zeta_\alpha = \zeta_\beta = 1 - \pi_\beta = \pi_\alpha = 0$,

$$
\int_{|\mathbf{g}| \geq 1} \int_1^{A_2 |\mathbf{g}|^2 / A_1} \left(k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 dI d\boldsymbol{\xi} d\boldsymbol{\xi}_*
$$

$$
\leq C \int_{|\mathbf{g}| \geq 1} |\mathbf{g}|^{-4} \int_1^{A_2 |\mathbf{g}|^2 / A_1} e^{A_1 I - A_2 |\mathbf{g}|^2} dI d\mathbf{g} \leq C \int_1^\infty R^{-2} dR = C.
$$

On the other hand, if $A_1 I \geq A_2 |\mathbf{g}|^2$, noting that

$$
\frac{A_2}{A_1} \left(|\mathbf{g}'|^2 - |\mathbf{g}|^2 \right) + I - I' = \frac{m_\alpha m_\beta}{2 (m_\alpha - m_\beta)} \left(|\mathbf{g}'|^2 - |\mathbf{g}|^2 \right) - \Delta I_\# = 0,
$$

and, hence,

$$
I' = I + \frac{A_2}{A_1} \left(|\mathbf{g}'|^2 - |\mathbf{g}|^2 \right) \geq I - \frac{A_2}{A_1} |\mathbf{g}|^2 \geq 0,
$$

then, by the bounds (39), (37), and (41), with $\zeta_\alpha = 1 - \pi_\alpha = 0$, changing variables of integration $\{\boldsymbol{\xi}, \boldsymbol{\xi}_*\} \to \{\mathbf{g}, \mathbf{g}_{\alpha\beta}\}$, with unitary Jacobian,

$$
\int_{|\mathbf{g}| \geq 1} \int_{A_2 |\mathbf{g}|^2 / A_1}^\infty \left(k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 dI d\boldsymbol{\xi} d\boldsymbol{\xi}_*
$$

$$
\leq C \int_1^\infty |\mathbf{g}|^{-4} \int_{A_2 |\mathbf{g}|^2 / A_1}^\infty e^{-(I - A_2 |\mathbf{g}|^2 / A_1)/4} dI d\mathbf{g}
$$

$$
\times \int_{\mathbb{R}^3} e^{-(\sqrt{m_\alpha} - \sqrt{m_\beta})^2 |\mathbf{g}_{\alpha\beta}|^2 / 4} d\mathbf{g}_{\alpha\beta} \int_0^\infty (I')^{\delta^{(\alpha)}/2 - 1} e^{-I'/4} dI'
$$

$$
\leq C \int_1^\infty R^{-2} dR = C.
$$

Then the case $(\alpha, \beta) \in \mathcal{I}_{poly} \times \mathcal{I}_{mono}$ is covered completely.

Concluding, $\left(k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) \right)^2 \in L^2 (d\mathbf{Z} \, d\mathbf{Z}_*)$, implying that

$$
K_{\alpha\beta2} = \int_{\mathcal{Z}_\beta} k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) h_{\beta*} d\mathbf{Z}_*
$$

are Hilbert–Schmidt integral operators, and as such continuous and compact on $L^2 (d\mathbf{Z})$, see [22, Theorem 7.83], for $(\alpha, \beta) \in \mathcal{I}^2$.

On the other hand, if $m_\alpha = m_\beta$, then

$$
k_{\alpha\beta2}^{(\beta)}(\mathbf{Z}, \mathbf{Z}_*) = \int_{(\mathbb{R}^3)^{\perp \mathbf{n}} \times \mathbb{R}_+^2} 4 \frac{|\widehat{\mathbf{g}}| \left(M'_\alpha M'_{\beta*} \right)^{1/2}}{|\overline{\mathbf{g}}| |\mathbf{g}|} \left(\frac{\varphi_\alpha (I) \varphi_\beta (I')}{\varphi_\alpha (I') \varphi_\beta (I_*)} \right)^{1/2}
$$

$$
\times \widetilde{\sigma}_{\alpha\beta} \mathbf{1}_{m_\alpha |\widehat{\mathbf{g}}|^2 > 4 \Delta I_\#} \mathbf{1}_{I' \leq \widetilde{\mathcal{E}}_{\alpha\beta}} \mathbf{1}_{I'_* \leq \widetilde{\mathcal{E}}^*_{\alpha\beta}} d\mathbf{w} dI' dI'_*,
$$

with $\widehat{\mathbf{g}} = \boldsymbol{\xi} - \boldsymbol{\xi}'_*$ and $\overline{\mathbf{g}} = \boldsymbol{\xi}_* - \boldsymbol{\xi}$. Here, with $\mathbf{g} = \boldsymbol{\xi} - \boldsymbol{\xi}_*$ and $\mathbf{n} = \mathbf{g} / |\mathbf{g}|$,

$$
\begin{cases} \boldsymbol{\xi}' = \boldsymbol{\xi} + \mathbf{w} - \chi_\alpha \mathbf{n} \\ \boldsymbol{\xi}'_* = \boldsymbol{\xi}_* + \mathbf{w} - \chi_\alpha \mathbf{n} \end{cases}, \text{ where } \mathbf{w} \perp \mathbf{g} \text{ and } \chi_\alpha = \frac{\Delta I}{m_\alpha |\mathbf{g}|}.
$$

Then similar arguments to the ones for $k_{\alpha\beta}^{(\alpha)}(\mathbf{Z}, \mathbf{Z}_*)$ (with $m_\alpha = m_\beta$) above, can be applied.

Concluding, the operator

$$K = (K_1, \ldots, K_s) = \sum_{\beta=1}^{s} (K_{1\beta 2} + K_{1\beta 3} - K_{1\beta 1}, \ldots, K_{s\beta 2} + K_{s\beta 3} - K_{s\beta 1}),$$

is a compact self-adjoint operator on $\mathfrak{h}$. Self-adjointness is due to the symmetry relations (22), (23), cf. [23, p. 198]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5 Bounds on the Collision Frequency

This section concerns the proof of Theorem 2. Note that throughout the proof, C will denote a generic positive constants. Moreover, remind that $\varphi_\alpha(I) = I^{\delta^{(\alpha)}/2 - 1}$ for $\alpha \in \mathcal{I}$ below.

Proof Noting identity (8), under assumption (17), the collision frequencies ν_α equal

$$
\begin{aligned}
\nu_\alpha &= \sum_{\beta=1}^{s} \int_{\mathcal{Z}_\alpha \times \mathcal{Z}_\beta^2} \frac{M_{\beta*}}{\varphi_\alpha(I)\,\varphi_\beta(I_*)} W_{\alpha\beta} d\mathbf{Z}_* d\mathbf{Z}' d\mathbf{Z}_*' \\
&= \sum_{\beta=1}^{s} \int_{\mathcal{Z}_\beta \times \mathbb{R}^3 \times (\mathbb{R}_+)^3 \mathbb{S}^2} M_{\beta*} \sigma_{\alpha\beta} |\mathbf{g}|\, \delta_1\left(\sqrt{|\mathbf{g}|^2 - \frac{2\Delta I}{\mu_{\alpha\beta}}} - |\mathbf{g}'|\right) \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2 > 2\Delta I} \\
&\quad \times \mathbf{1}_{I' \leq \mathcal{E}_{\alpha\beta}} \mathbf{1}_{I_*' \leq \mathcal{E}_{\alpha\beta}^*} \delta_3\left(\mathbf{G}_{\alpha\beta} - \mathbf{G}_{\alpha\beta}'\right) d\mathbf{Z}_* d\mathbf{G}_{\alpha\beta}' d\,|\mathbf{g}'|\, d\boldsymbol{\sigma}\, dI' dI_*' \\
&= \sum_{\beta=1}^{s} C_{\alpha\beta} \int_{\mathbb{S}^2} d\boldsymbol{\sigma} \int_{\mathbb{R}^3 \times (\mathbb{R}_+)^3} \varphi_\beta(I_*)\, e^{-I_*} e^{-m_\beta |\boldsymbol{\xi}_*|^2/2} \sigma_{\alpha\beta} \\
&\quad \times \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2 > 2\Delta I} \mathbf{1}_{I' \leq \mathcal{E}_{\alpha\beta}} \mathbf{1}_{I_*' \leq \mathcal{E}_{\alpha\beta}^*} |\mathbf{g}|\, d\boldsymbol{\xi}_* dI_* dI' dI_*' \\
&= \sum_{\beta=1}^{s} C_{\alpha\beta} \int_{\mathbb{R}^3 \times (\mathbb{R}_+)^3} e^{-m_\beta |\boldsymbol{\xi}_*|^2/2} e^{-I_*} \frac{\varphi_\alpha(I')\,\varphi_\beta(I_*)\,\varphi_\beta(I_*')}{\mathcal{E}_{\alpha\beta}^{\delta^{(\alpha)}/2} \left(\mathcal{E}_{\alpha\beta}^*\right)^{\delta^{(\beta)}/2} E_{\alpha\beta}^{\eta/2}} \\
&\quad \times \mathbf{1}_{\mu_{\alpha\beta}|\mathbf{g}|^2 > 2\Delta I} \mathbf{1}_{I' \leq \mathcal{E}_{\alpha\beta}} \mathbf{1}_{I_*' \leq \mathcal{E}_{\alpha\beta}^*} \sqrt{|\mathbf{g}|^2 - 2\widetilde{\Delta}_{\alpha\beta} I}\, d\boldsymbol{\xi}_* dI_* dI' dI_*'
\end{aligned}
$$

for $\alpha \in \mathcal{I}$. Clearly,

$$
\begin{aligned}
\nu_\alpha &\geq C_{\alpha\alpha} \int_{\mathbb{R}^3 \times (\mathbb{R}_+)^3} \frac{\varphi_\alpha(I_*)\,\varphi_\alpha(I')\,\varphi_\alpha(I_*')\, e^{-I_*}}{\mathcal{E}_{\alpha\alpha}^{\delta^{(\alpha)}} E_{\alpha\alpha}^{\eta/2}} \sqrt{|\mathbf{g}|^2 - \frac{4}{m_\alpha}\Delta I} \\
&\quad \times e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2} \mathbf{1}_{I' + I_*' \leq \mathcal{E}_{\alpha\alpha}} d\boldsymbol{\xi}_* dI_* dI' dI_*'.
\end{aligned}
$$

Then, if the species a_α is monatomic, $\alpha \in \mathcal{I}_{mono}$,

$$
\begin{aligned}
\nu_\alpha &\geq C \int_{\mathbb{R}^3 \times (\mathbb{R}_+)^3} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2} |\mathbf{g}|^{1-\eta} d\boldsymbol{\xi}_* \\
&\geq C \int_{\mathbb{R}^3 \times (\mathbb{R}_+)^3} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2} \left(\big||\boldsymbol{\xi}| - |\boldsymbol{\xi}_*|\big|^2\right)^{(1-\eta)/2} d\boldsymbol{\xi}_*.
\end{aligned}
$$

while if a_α is polyatomic, $\alpha \in \mathcal{I}_{poly}$,

$$
\begin{aligned}
\nu_\alpha &\geq C \int_{\mathbb{R}^3 \times \mathbb{R}_+} \int_{I' + I'_* \leq E_{\alpha\alpha}/2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2} \frac{\varphi_\alpha(I_*)\,\varphi_\alpha(I')\,\varphi_\alpha(I'_*)\,e^{-I_*}}{\mathcal{E}_{\alpha\alpha}^{\delta^{(\alpha)}} E_{\alpha\alpha}^{\eta/2}} \\
&\quad \times \sqrt{E_{\alpha\alpha} - (I' + I'_*)}\,d\boldsymbol{\xi}_* dI_* dI' dI'_* \\
&\geq C \int_{\mathbb{R}^3 \times \mathbb{R}_+} \frac{E_{\alpha\alpha}^{1/2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2}}{\mathcal{E}_{\alpha\alpha}^{\delta^{(\alpha)}} E_{\alpha\alpha}^{\eta/2}} \varphi_\alpha(I_*)\, e^{-I_*} \left(\int_0^{E_{\alpha\alpha}/4} (I')^{\delta^{(\alpha)}/2 - 1} dI' \right)^2 d\boldsymbol{\xi}_* dI_* \\
&= C \int_{\mathbb{R}^3 \times \mathbb{R}_+} E_{\alpha\alpha}^{(1-\eta)/2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2} \varphi_\alpha(I_*)\, e^{-I_*}\, d\boldsymbol{\xi}_* dI_* \\
&\geq C \int_{\mathbb{R}^3} \left(|\mathbf{g}|^2 + I \right)^{(1-\eta)/2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2}\, d\boldsymbol{\xi}_* \int_0^\infty I_*^{\delta^{(\alpha)}/2 - 1} e^{-I_*}\, dI_* \\
&\geq C \int_{\mathbb{R}^3} \left(\big||\boldsymbol{\xi}| - |\boldsymbol{\xi}_*|\big|^2 + I \right)^{(1-\eta)/2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2}\, d\boldsymbol{\xi}_*.
\end{aligned}
$$

Now it follows that

$$
\begin{aligned}
\nu_\alpha &\geq C \int_{\mathbb{R}^3} \left(\left(|\boldsymbol{\xi}| - |\boldsymbol{\xi}_*| \right)^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2} \\
&\quad \times \left(\mathbf{1}_{|\boldsymbol{\xi}_*| \leq 1/2} \mathbf{1}_{|\boldsymbol{\xi}| \geq 1} + \mathbf{1}_{|\boldsymbol{\xi}_*| \geq 2} \mathbf{1}_{|\boldsymbol{\xi}| \leq 1} \right) d\boldsymbol{\xi}_* \\
&\geq C \left(\left(|\boldsymbol{\xi}|^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} \mathbf{1}_{|\boldsymbol{\xi}| \geq 1} \int_{|\boldsymbol{\xi}_*| \leq 1/2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2}\, d\boldsymbol{\xi}_* \right. \\
&\quad \left. + \left(1 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} \mathbf{1}_{|\boldsymbol{\xi}| \leq 1} \int_{|\boldsymbol{\xi}_*| \geq 2} e^{-m_\alpha |\boldsymbol{\xi}_*|^2/2}\, d\boldsymbol{\xi}_* \right) \\
&\geq C \left(\left(|\boldsymbol{\xi}|^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} \mathbf{1}_{|\boldsymbol{\xi}| \geq 1} + \left(1 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} \mathbf{1}_{|\boldsymbol{\xi}| \leq 1} \right) \\
&\geq C \left(1 + |\boldsymbol{\xi}|^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} \\
&\geq C \left(1 + |\boldsymbol{\xi}| + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \sqrt{I} \right)^{1-\eta}.
\end{aligned}
$$

Hence, there is a positive constant $\upsilon_- > 0$, such that for all $\alpha \in \mathcal{I}$

$$
\nu_\alpha \geq \upsilon_- \left(1 + |\boldsymbol{\xi}| + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \sqrt{I} \right)^{1-\eta}.
$$

On the other hand,

$$
\begin{aligned}
\nu_\alpha &\leq C \sum_{\beta=1}^s \int_{\mathbb{R}^3 \times (\mathbb{R}_+)^3} e^{-m_\beta |\boldsymbol{\xi}_*|^2/2} e^{-I_*} E_{\alpha\beta}^{(1-\eta)/2} \frac{\varphi_\alpha(I')\,\varphi_\beta(I_*)\,\varphi_\beta(I'_*)}{\mathcal{E}_{\alpha\beta}^{\delta^{(\alpha)}/2} \left(\mathcal{E}_{\alpha\beta}^* \right)^{\delta^{(\beta)}/2}} \\
&\quad \times \mathbf{1}_{I' \leq \mathcal{E}_{\alpha\beta}} \mathbf{1}_{I'_* \leq \mathcal{E}_{\alpha\beta}^*}\, d\boldsymbol{\xi}_* dI_* dI' dI'_* \\
&= C \sum_{\beta=1}^s \int_{\mathbb{R}^3 \times \mathbb{R}_+} e^{-m_\beta |\boldsymbol{\xi}_*|^2/2} e^{-I_*} \int_0^{\mathcal{E}_{\alpha\beta}} \frac{(I')^{\delta^{(\alpha)}/2 - 1}}{\mathcal{E}_{\alpha\beta}^{\delta^{(\alpha)}/2}} dI' \int_0^{\mathcal{E}_{\alpha\beta}^*} \frac{(I'_*)^{\delta^{(\beta)}/2 - 1}}{\left(\mathcal{E}_{\alpha\beta}^* \right)^{\delta^{(\beta)}/2}} dI'_* \\
&\quad \times \varphi_\beta(I_*)\, E_{\alpha\beta}^{(1-\eta)/2}\, dI_* d\boldsymbol{\xi}_*
\end{aligned}
$$

$$= C \sum_{\beta=1}^{s} \int_{\mathbb{R}^3 \times \mathbb{R}_+} e^{-m_\beta |\xi_*|^2/2} e^{-I_*} \varphi_\beta \left(I_* \right) E_{\alpha\beta}^{(1-\eta)/2} dI_* d\xi_*$$

implying, since, clearly, $E_{\alpha\beta} \leq \left(1 + \mu_{\alpha\beta} |\mathbf{g}|^2 /2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right) \left(1 + \mathbf{1}_{\beta \in \mathcal{I}_{poly}} I_* \right)$, that

$$\nu_\alpha \leq C \sum_{\beta=1}^{s} \int_{\mathbb{R}^3} \left(1 + \frac{\mu_{\alpha\beta}}{2} |\mathbf{g}|^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} e^{-m_\beta |\xi_*|^2/2} d\xi_*$$

$$\times \int_0^\infty (1 + I_*)^{(1-\eta)/2} I_*^{\delta^{(\beta)}/2 - 1} e^{-I_*} dI_*$$

$$\leq C \left(1 + |\xi|^2 + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} I \right)^{(1-\eta)/2} \int_{\mathbb{R}^3} \left(1 + |\xi_*|^2 \right)^{(1-\eta)/2} e^{-m_\beta |\xi_*|^2/2} d\xi_*$$

$$\leq C \left(1 + |\xi| + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \sqrt{I} \right)^{1-\eta} \int_0^\infty \left(1 + r^2 \right)^{(1-\eta)/2} r^2 e^{-m_\beta r^2/2} dr$$

$$= C \left(1 + |\xi| + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \sqrt{I} \right)^{1-\eta}.$$

Hence, there is a positive constant $\nu_+ > 0$, such that for all $\alpha \in \mathcal{I}$

$$\nu_\alpha \leq \nu_+ \left(1 + |\xi| + \mathbf{1}_{\alpha \in \mathcal{I}_{poly}} \sqrt{I} \right)^{1-\eta}.$$

$\square$

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

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