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Relativistic Stochastic Mechanics II: Reduced Fokker-Planck Equation in Curved Spacetime

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Abstract
The general covariant Fokker-Planck equations associated with the two different versions of covariant Langevin equation in Part I of this series of work are derived, both lead to the same reduced Fokker-Planck equation for the non-normalized one particle distribution function (1PDF). The relationship between various distribution functions is clarified in this process. Several macroscopic quantities are introduced by use of the 1PDF, and the results indicate an intimate connection with the description in relativistic kinetic theory. The concept of relativistic equilibrium state of the heat reservoir is also clarified, and, under the working assumption that the Brownian particle should approach the same equilibrium distribution as the heat reservoir in the long time limit, a general covariant version of Einstein relation arises.

Keywords Stochastic mechanics · Relativistic · General covariance · Fokker–Planck equation · Einstein relation

1 Introduction

This is Part II of our series of works on relativistic stochastic mechanics. Part I of this series has already been presented in [1]. The major subject of concern in Part I is the construction of manifestly general covariant Langevin equation from the observer's perspective. Two different versions of relativistic Langevin equation (denoted LE_τ and LE_t respectively) were proposed, among which LE_τ takes the proper time τ of the Brownian particle as evolution parameter and LE_t takes the proper time t of some prescribed observer as evolution parameter. It was shown that although LE_τ contains some conceptual issues from the point of view of the

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prescribed observer, there is numerical evidence indicating that LE_τ and LE_t can produce the same distribution in the same space of micro states (SoMS) for the case of $(1+1)$-dimensional Minkowski spacetime, which in turn suggests that we may be able to extract useful probability distributions from LE_τ.

Besides Langevin equation, Fokker-Planck equation (FPE) is another important equation in stochastic mechanics. The route leading from Langevin equation to FPE can be regarded as a bridge from mechanics to statistical physics. The study of FPE was initiated about a hundred years ago [2, 3], and the purpose is to analyze the diffusion phenomena (in the configuration space) of suspended particles in solution. Kolmogoroff [4] gave an explanation of the equation of the same form from the perspective of stochastic processes, therefore the corresponding equation is also called Kolmogoroff equation. Later, Klein [5] and Kramers [6] generalized the equation to the phase space. Chandrasekhar provided a detailed report on the relevant topics [7], and the solution to the Klein-Kramers equation describing a relaxation process was also given. All these works used the transition probability to study the evolution of random variables. With the development of stochastic differential equations, related topics have been extensively studied by use of Ito calculus [8], and some more modern methods about this topic can be found in [9].

In the relativistic regime, there is no Markov process satisfying causality on the spacetime manifold [10, 11]. The only choice is to study FPE on the SoMS — a subspace of the future mass shell bundle. This means that the equation to be considered needs to be of the Klein-Kramers type. However, with the usual abuse of terminology, we still use the name FPE for convenience.

The study of relativistic stochastic process can be traced back to Dudley [10, 12] and Hakim [11, 13], who first discussed the space of states for stochastic processes in a model independent way. The study of concrete relativistic stochastic processes, e.g. the relativistic Ornstein-Uhlenbeck process, was carried out by Debbasch *et al* in [14]. Barbachoux *et al* [15, 16] made some discussions about the corresponding FPE (Kolmogoroff equation). Dunkel *et al* [17–21] also studied similar topics in the special relativistic context, and their model gave an intuitive understanding of the relativistic Brownian motion. Herrmann [22] and Haba [23] extended the studies to general relativistic context, with some emphasis placed on the manifest general covariance.

It is necessary to point out that, in all previous works, the important role played by the observer has not beed sufficiently addressed. In this work, we shall show that properly addressing the role played by the observer is the starting point in understanding different versions of general covariant FPE that arise either directly or indirectly from the Langevin equations LE_τ and/or LE_t proposed in [1]. In particular, the observer plays an important role in the interpretation of various distribution functions that appear in different versions of covariant FPE.

Another important aspect which has not been made sufficiently clear in previous works is the state of the heat reservoir. The description for the non-relativistic Brownian motion of a heavy particle inside a heat reservoir relies on two basic assumptions. First, the heat reservoir should have reached thermodynamic equilibrium, and the only impact of the reservoir on the Brownian particle is provided through thermal fluctuations, of which the fast and slow parts manifest respectively in the Langevin equation in the form of stochastic and damping forces. Second, the stochastic motion of the Brownian particle should be able to mimic a relaxation process, which means that, after sufficiently long time, the probability distribution for the Brownian particle should approach the same equilibrium distribution obeyed by the particles from the heat reservoir. We shall see in Sect. 6 that the concept of equilibrium state for the heat reservoir needs to be re-examined carefully in the relativistic context.

This paper is organized as follows. In Sect. 2, we presents an introduction to the SoMS for the Brownian particle and prepare the notations to be used in the forthcoming sections. The description for the SoMS is placed on the same ground as in relativistic kinetic theory [24–26], with the expectation that the deep connection between these complementary approaches to non-equilibrium statistical physics could be further elucidated. Such a treatment is more appealing than the alternative approaches, e.g. those making use of jet bundles. In Sect. 3, we deduce the covariant FPEs from the Langevin equations with different evolution parameters introduced in [1]. Sect. 4 is devoted to clarifying the relationship between different distribution functions. In this section, we also introduce a new distribution, which is identified to be the one particle distribution function (1PDF) in the sense of relativistic kinetic theory, together with its evolution equation, i.e. the reduced FPE. Sect. 5 introduces some thermodynamic quantities and thermodynamic relations, and the formulation seems to indicate some deep connections between the approaches of stochastic mechanics and relativistic kinetic theory. In Sect. 6, we clarify the meaning of the equilibrium state of the heat reservoir, and, by assuming that the 1PDF should approach the intrinsic equilibrium distribution of the heat reservoir, we deduce a general relativistic version of the Einstein relation. Finally, in Sect. 7, we present a brief summary of the results.

2 The SoMS and Its Geometry

Since this work is a followup to Ref. [1], we use exactly the same notations and conventions as in [1]. In particular, the spacetime manifold $\mathcal{M}$ is taken to be a curved pseudo-Riemannian manifold of dimension $(d+1)$ with a mostly positive signature. The future mass shell bundle Γ_m^+ over $\mathcal{M}$ is defined as

$$\Gamma_m^+ := \{(x,p) \in T\mathcal{M} \mid g_{\mu\nu}(x)p^\mu p^\nu = -m^2 \text{ and } p^\mu Z_\mu(x) < 0\}, \tag{1}$$

in which $Z^\mu(x)$ denotes the proper velocity of some observer field. Later on, we shall omit the word "future" and simply refer to Γ_m^+ as the mass shell bundle. The momentum space of the Brownian particle at the event x is identified as the intersection of the tangent space $T_x\mathcal{M}$ with the mass shell bundle and is referred to simply as the mass shell at x,

$$\left(\Gamma_m^+\right)_x := T_x\mathcal{M} \cap \Gamma_m^+, \tag{2}$$

and the configuration space is labeled by the proper time t of a single prescribed observer, *Alice*, as the level set

$$\mathcal{S}_t := \{x \in \mathcal{M} | t(x) = t = \text{const.}\},$$

where $t(x)$ is an extension of the proper time t over $\mathcal{M}$ as a scalar field. Denoting the proper velocity of Alice also by Z^μ should produce no confusions. The SoMS of the Brownian particle is then given by

$$\Sigma_t := \bigcup_{x\in\mathcal{S}_t} (\Gamma_m^+)_x = \{(x,p) \in \Gamma_m^+ | x \in \mathcal{S}_t\}, \tag{3}$$

which is clearly observer-dependent. The above specification for the SoMS of the Brownian particle naturally falls inline with the tangent bundle formalism of relativistic kinetic theory [24–26]. This will certainly benefit for the communication between the two important branches of non-equilibrium relativistic statistical physics. An immediate benefit is to adopt the Sasaki metric [27] for describing the local geometry of the tangent bundle (and subspaces thereof).

Before proceeding, let us introduce our conventions on indices. We use both concrete and abstract index notations, however with omissions of the abstract indices when there no confusions could arise. Lower-case Greek letters $\alpha, \beta, \mu, \nu, \rho, \ldots$ denote concrete indices and range from 0 to d. Latin capital letters $A, B, \ldots$ and some lower-case Latin letters, such as $i, j, \ldots$, also denote concrete indices. The upper-case Latin indices range from 0 to $2d$ and is associated with tensors on the mass shell bundle, while the lower-case Latin indices $i, j, \ldots$ range from 1 to d. The other lower-case Latin letters $a, b, \ldots$ denote abstract indices. Additionally, we use the calligraphy fonts, like $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{K}$, to denote tensors on the momentum space $(\Gamma^+_m)_x$, and the cursive fonts, like $\mathscr{N}$, $\mathscr{X}$ and $\mathscr{L}$, to denote tensors on the mass shell bundle Γ^+_m.

Since Γ^+_m, $(\Gamma^+_m)_x$ and Σ_t are all subspaces of the tangent bundle $T\mathcal{M}$, it is appropriate to begin by describing the relevant geometric structures on $T\mathcal{M}$. What really matters is the tangent space of the tangent bundle, which can be subdivided into the direct sum of horizontal and vertical subspaces [28, 29],

$$T_{(x,p)}(T\mathcal{M}) = H_{(x,p)} \oplus V_{(x,p)}, \tag{4}$$

where $H_{(x,p)}$ is spanned by

$$e_\mu = \frac{\partial}{\partial x^\mu} - \Gamma^\alpha{}_{\mu\beta} p^\beta \frac{\partial}{\partial p^\alpha}, \tag{5}$$

and $V_{(x,p)}$ is spanned by $\partial/\partial p^\mu$. Here $\Gamma^\alpha{}_{\mu\beta}$ represents the usual Christoffel connection associated with the spacetime metric $g_{\mu\nu}$.

The metric on the tangent bundle $T\mathcal{M}$ is given by the Sasaki metric, which can be written as a direct sum of metrics on the two subspaces,

$$\hat{g}_{ab} := \underbrace{g_{\mu\nu} \mathrm{d}x^\mu{}_a \mathrm{d}x^\nu{}_b}_{\text{the metric of } H_{(x,p)}} + \underbrace{g_{\mu\nu}\, \theta^\mu{}_a \theta^\nu{}_b}_{\text{the metric of } V_{(x,p)}}, \tag{6}$$

where

$$\theta^\mu = \mathrm{d}p^\mu + \Gamma^\mu{}_{\alpha\beta} p^\alpha \mathrm{d}x^\beta. \tag{7}$$

$\{e_\mu, \partial/\partial p^\mu\}$ and $\{\mathrm{d}x^\mu, \theta^\mu\}$ are dual bases on the tangent and cotangent spaces of the tangent bundle respectively. As a hypersurface on the tangent bundle, the mass shell bundle is naturally equipped with an induced metric

$$\hat{h}_{ab} := \hat{g}_{ab} + \hat{N}_a \hat{N}_b, \qquad \hat{N}^a := (m)^{-1} p^\mu \left(\frac{\partial}{\partial p^\mu}\right)^a, \tag{8}$$

where $\hat{N}^a$ is the unit normal vector of the mass shell bundle. The metric of the mass shell bundle can also be written as the direct sum of the metrics of the horizontal subspace and the momentum space,

$$\hat{h}_{ab} = \underbrace{g_{\mu\nu} \mathrm{d}x^\mu{}_a \mathrm{d}x^\nu{}_b}_{\text{the metric of } H_{(x,p)}} + \underbrace{\Delta_{\mu\nu}(p) \theta^\mu{}_a \theta^\nu{}_b}_{\text{the metric of } T_{(x,p)}(\Gamma^+_m)_x}, \tag{9}$$

where

$$\Delta_{\mu\nu}(p) := g_{\mu\nu} + m^{-2} p_\mu p_\nu$$

is the orthogonal projection tensor associated with p^μ. The inverse of the metric $\hat{h}$ reads

$$\hat{h}^{ab} = g^{\mu\nu} e_\mu{}^a e_\nu{}^b + \Delta^{\mu\nu}(p) \left(\frac{\partial}{\partial p^\mu}\right)^a \left(\frac{\partial}{\partial p^\nu}\right)^b. \tag{10}$$

It is obvious that the metric on the momentum space $(\Gamma_m^+)_x$ and its inverse are respectively

$$h_{ab} = \Delta_{\mu\nu}(p)\theta^\mu{}_a\theta^\nu{}_b, \qquad h^{ab} = \Delta^{\mu\nu}(p) \left(\frac{\partial}{\partial p^\mu}\right)^a \left(\frac{\partial}{\partial p^\nu}\right)^b. \tag{11}$$

Please remember that we use $\hat{h}_{ab}$ for the metric on the mass shell bundle and h_{ab} for the metric on the fiber space alone.

Since Σ_t is a hypersurface on the mass shell bundle, there is a normal vector field. This normal vector field is given by

$$\mathscr{Z}^a = Z^\mu e_\mu{}^a. \tag{12}$$

$\mathscr{Z}^a$ is actually an up-lift of the observer field onto the mass shell bundle.

Using the above metrics, it is easy to find the invariant volume elements on $T\mathcal{M}$, Γ_m^+ and $(\Gamma_m^+)_x$, respectively [25],

$$\eta_{T\mathcal{M}} = g\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^d \wedge \mathrm{d}p^0 \wedge ... \wedge \mathrm{d}p^d, \tag{13}$$

$$\eta_{\Gamma_m^+} = \frac{g}{p_0} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^d \wedge \mathrm{d}p^1 \wedge ... \wedge \mathrm{d}p^d, \tag{14}$$

$$\eta_{(\Gamma_m^+)_x} = \frac{\sqrt{g}}{p_0} \mathrm{d}p^1 \wedge ... \wedge \mathrm{d}p^d, \tag{15}$$

where we have introduced $g = |\det(g_{\mu\nu})|$.

As mentioned above, $\{e_\mu, \partial/\partial p^\mu\}$ is the basis of $T_{(x,p)}(T\mathcal{M})$, so an arbitrary tangent vector on $T\mathcal{M}$ can be written as

$$\mathscr{V}^a = V^\mu e_\mu^a + \mathcal{V}^\mu \left(\frac{\partial}{\partial p^\mu}\right)^a. \tag{16}$$

The vectors with vanishing components V^μ can also be treated as tangent vectors on the tangent space, and these will be denoted as

$$\mathcal{V}^a = \mathcal{V}^\mu \left(\frac{\partial}{\partial p^\mu}\right)^a. \tag{17}$$

For tangent vectors on the mass shell $(\Gamma_m^+)_x$, it is convenient to introduce the following vector basis,

$$\left(\frac{\partial}{\partial \breve{p}^i}\right)^a := \left(\frac{\partial}{\partial p^i}\right)^a - \frac{p_i}{p_0}\left(\frac{\partial}{\partial p^0}\right)^a. \tag{18}$$

Notice that, due to the mass shell condition, $(\Gamma_m^+)_x$ has one less dimension than $T_x\mathcal{M}$, and so are their respective tangent spaces. Since $(\Gamma_m^+)_x$ is a hypersurface in $T_x\mathcal{M}$ with normalized normal vector $\hat{N}^a$ given in eq.(8), any tangent vector on $(\Gamma_m^+)_x$ is automatically a tangent vector on $T_x\mathcal{M}$. Therefore, we can also write the tangent vectors on $(\Gamma_m^+)_x$ in terms of the basis $\{(\partial/\partial p^\mu)^a\}$. In other words, any tangent vector $\mathcal{V}^a$ on $(\Gamma_m^+)_x$ acquires two different component representations

$$\mathcal{V}^a = \mathcal{V}^i \left(\frac{\partial}{\partial \breve{p}^i}\right)^a \quad \text{and} \quad \mathcal{V}^a = \mathcal{V}^\mu \left(\frac{\partial}{\partial p^\mu}\right)^a.$$

It is straightforward to check that these two representations are equivalent,

$$\begin{aligned} \mathcal{V}^i \left(\frac{\partial}{\partial \breve{p}^i}\right)^a &= \mathcal{V}^i \left(\frac{\partial}{\partial p^i}\right)^a - \mathcal{V}^i \frac{p_i}{p_0} \left(\frac{\partial}{\partial p^0}\right)^a \\ &= \mathcal{V}^i \left(\frac{\partial}{\partial p^i}\right)^a + \mathcal{V}^0 \left(\frac{\partial}{\partial p^0}\right)^a = \mathcal{V}^\mu \left(\frac{\partial}{\partial p^\mu}\right)^a, \end{aligned} \tag{19}$$

wherein we have used the orthogonal condition $\mathcal{V}^\mu p_\mu = \mathcal{V}^0 p_0 + \mathcal{V}^i p_i = 0$. Similarly, the inverse metric on the momentum space can be expressed in two different bases,

$$h^{ab} = \Delta^{\mu\nu}(p) \left(\frac{\partial}{\partial p^\mu}\right)^a \left(\frac{\partial}{\partial p^\nu}\right)^b = \Delta^{ij}(p) \left(\frac{\partial}{\partial \breve{p}^i}\right)^a \left(\frac{\partial}{\partial \breve{p}^j}\right)^b. \tag{20}$$

In order to describe the different versions of FPE, it is customary to introduce the covariant derivatives on each of the relevant manifolds using the standard conventions with the aid of the metrics introduced above. However, this step can be skiped, because we only need the covariant divergences. For a vector $\mathscr{F}^A = (F^\mu, \mathcal{F}^i)$ on the mass shell bundle, the covariant divergence is simply given by

$$\hat{\nabla}^{(\hat{h})}_A \mathscr{F}^A = \frac{p_0}{g} \frac{\partial}{\partial x^\mu} \left(\frac{g}{p_0} F^\mu\right) + p_0 \frac{\partial}{\partial \breve{p}^i} \left(\frac{1}{p_0} \mathcal{F}^i\right), \tag{21}$$

where $\hat{\nabla}^{(\hat{h})}$ is the covariant derivative on the mass shell bundle. If $F^\mu = 0$, $\mathscr{F}$ reduces into a vector on the momentum space, and the above equation becomes

$$\hat{\nabla}^{(\hat{h})}_A \mathscr{F}^A = \nabla^{(h)}_i \mathcal{F}^i, \tag{22}$$

which is automatically the covariant divergence on the the momentum space, with $\nabla^{(h)}$ being the corresponding covariant derivative.

Finally, let us make some remarks on the notations and conventions. For any vector field $\mathscr{V}^a$ and any scalar field Φ, the map from Φ to $\mathscr{V}^a$ is denoted as $\mathscr{V}^a[\Phi]$. On the contrary, the action of the vector field $\mathscr{V}^a$ on Φ is denoted as $\mathscr{V}(\Phi)$. It is crucial to distinguish these two notations in the following text.

3 Covariant FPEs

In this section, we shall derive the FPE associated with each version of the Langevin equation presented in [1] and try to make sense of the corresponding probability distribution functions (PDFs). In practice, there are different ways to obtain FPE from Langevin equation [30, 31]. To highlight the geometric interpretation, we will adopt the diffusion operator method [32]. A brief review of the method is presented in Appendix A, and the construction below will be made as brief as possible in order to focus on the physical interpretations.

3.1 FPE Associated with LE_τ

The Langevin equation LE_τ is given as follows,

$$\mathrm{d}\tilde{x}^\mu_\tau = \frac{\tilde{p}^\mu_\tau}{m} \mathrm{d}\tau, \tag{23}$$

$$\mathrm{d}\tilde{p}^\mu_\tau = \left[\mathcal{R}^\mu{}_\mathfrak{a} \circ_S \mathrm{d}\tilde{w}^\mathfrak{a}_\tau + \mathcal{F}^\mu_{\mathrm{add}}\mathrm{d}\tau\right] + \mathcal{K}^{\mu\nu}U_\nu \mathrm{d}\tau - \frac{1}{m}\Gamma^\mu{}_{\alpha\beta}\tilde{p}^\alpha_\tau \tilde{p}^\beta_\tau \mathrm{d}\tau, \tag{24}$$

and the meaning of each term is described in detail in [1]. Since the stochastic forces arise from thermal fluctuations from the heat reservoir, it is natural to expect that

$$\mathcal{R}^\mu{}_\mathfrak{a} \to 0, \qquad \mathcal{F}^\mu_{\mathrm{add}} \to 0$$

in the low temperature limit.

Since LE_τ preserves the mass shell condition, not all components of $\tilde{p}^\mu$ could be viewed as independent, and it is appropriate to take only $\tilde{p}^i$ as independent random variables. One can introduce a corresponding probability distribution function (PDF)

$$\Phi_\tau(x^\mu, p^i) := \Pr[\tilde{x}^\mu_\tau = x^\mu, \tilde{p}^i_\tau = p^i] \tag{25}$$

which describes the probability for the Brownian particle to appear at the position x^μ in the spacetime and meanwhile has the momentum p^i at the proper time τ of the Brownian particle itself. This PDF is pathological for two reasons. First, $\Phi_\tau(x^\mu, p^i)$ depends on two time variables τ and x^0, which makes it hard to assign a physical interpretation; Second, $\Phi_\tau(x^\mu, p^i)$ is not a distribution on the SoMS Σ_t, but rather on the full mass shell bundle Γ^+_m. However, there is no technical obstacle which prevents us from constructing the FPE obeyed by $\Phi_\tau(x^\mu, p^i)$.

In order to get the desired FPE, we need to construct the diffusion operator of eq.(24). For Stratonovich type Langevin equation, the diffusion operator can always be written in the form

$$\mathbf{A} = \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2} L_\mathfrak{a} L_\mathfrak{b} + L_0. \tag{26}$$

In the case of eq.(24), we have

$$L_\mathfrak{a} = \mathcal{R}^\mu{}_\mathfrak{a}\frac{\partial}{\partial p^\mu} = \mathcal{R}^i{}_\mathfrak{a}\frac{\partial}{\partial \breve{p}^i}, \tag{27}$$

$$\begin{aligned} L_0 &= \frac{p^\mu}{m}\frac{\partial}{\partial x^\mu} - \frac{1}{m}\Gamma^\mu{}_{\alpha\beta}p^\alpha p^\beta \frac{\partial}{\partial p^\mu} + \left(\mathcal{F}^\mu_{\mathrm{add}} + \mathcal{K}^{\mu\nu}U_\nu\right)\frac{\partial}{\partial p^\mu} \\ &= \frac{1}{m}\mathscr{L} + \left(\mathcal{F}^i_{\mathrm{add}} + \mathcal{K}^{i\nu}U_\nu\right)\frac{\partial}{\partial \breve{p}^i}, \end{aligned} \tag{28}$$

where $\mathscr{L} = p^\mu e_\mu$ is the Liouville vector field [25]. Using the volume element of mass shell bundle eq.(14), the adjoint of coordinate derivative operators can be obtained straightforwardly,

$$\left(\frac{\partial}{\partial x^\mu}\right)^* = -\frac{p_0}{g}\frac{\partial}{\partial x^\mu}\frac{g}{p_0}, \tag{29}$$

$$\left(\frac{\partial}{\partial \breve{p}^i}\right)^* = -\frac{p_0}{g}\frac{\partial}{\partial \breve{p}^i}\frac{g}{p_0} = -\frac{p_0}{\sqrt{g}}\frac{\partial}{\partial \breve{p}^i}\frac{\sqrt{g}}{p_0}. \tag{30}$$

With these operators, the adjoint of $L_\mathfrak{a}$ and L_0 can be obtained, which read

$$L^*_\mathfrak{a} = -\nabla^{(h)}_i \mathcal{R}^i{}_\mathfrak{a}, \qquad L^*_0 = -\frac{1}{m}\mathscr{L} - \nabla^{(h)}_i\left(\mathcal{F}^i_{\mathrm{add}} + \mathcal{K}^{i\nu}U_\nu\right), \tag{31}$$

where we have used $\mathscr{L}^* = -\mathscr{L}$. The FPE can then be written as

$$\partial_\tau \Phi_\tau = \mathbf{A}^*\Phi_\tau$$

$$
\begin{aligned}
&= \left(\frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2} L^*_{\mathfrak{a}} L^*_{\mathfrak{b}} + L^*_0\right)\Phi_\tau \\
&= \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\nabla^{(h)}_i \mathcal{R}^i{}_{\mathfrak{a}} \nabla^{(h)}_j \mathcal{R}^j{}_{\mathfrak{b}} \Phi_\tau - \nabla^{(h)}_i \left(\mathcal{F}^i_{\text{add}}\Phi_\tau + \mathcal{K}^{i\nu} U_\nu \Phi_\tau\right) - \frac{1}{m}\mathscr{L}(\Phi_\tau) \\
&= \nabla^{(h)}_i \left[\frac{1}{2}\mathcal{D}^{ij}\nabla^{(h)}_j \Phi_\tau + \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\left(\mathcal{R}^i{}_{\mathfrak{a}}\nabla^{(h)}_j \mathcal{R}^j{}_{\mathfrak{b}}\right)\Phi_\tau - \mathcal{F}^i_{\text{add}}\Phi_\tau - \mathcal{K}^{i\nu}U_\nu\Phi_\tau\right] \\
&\quad - \frac{1}{m}\mathscr{L}(\Phi_\tau),
\end{aligned} \tag{32}
$$

where we have introduced the diffusion tensor $\mathcal{D}^{\mu\nu} := \mathcal{R}^\mu{}_{\mathfrak{a}}\mathcal{R}^\nu{}_{\mathfrak{a}}$. Defining the vector field

$$
\mathcal{I}^a[\Phi_\tau] := \left[\frac{1}{2}\mathcal{D}^{ij}\nabla^{(h)}_j \Phi_\tau + \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\left(\mathcal{R}^i{}_{\mathfrak{a}}\nabla^{(h)}_j \mathcal{R}^j{}_{\mathfrak{b}}\right)\Phi_\tau - \mathcal{F}^i_{\text{add}}\Phi_\tau - \mathcal{K}^{i\nu}U_\nu\Phi_\tau\right]\left(\frac{\partial}{\partial \breve{p}^i}\right)^a, \tag{33}
$$

the FPE for Φ_τ can be written in more concise form

$$
\partial_\tau \Phi_\tau = \nabla^{(h)}_i \mathcal{I}^i[\Phi_\tau] - \frac{1}{m}\mathscr{L}(\Phi_\tau). \tag{34}
$$

Eq.(34) can be viewed as a continuity equation for the PDF Φ_τ and its associated probability flow $\mathscr{J}[\Phi_\tau]$, which is defined as

$$
\mathscr{J}[\Phi_\tau] := \frac{\Phi_\tau}{m}\mathscr{L} - \mathcal{I}[\Phi_\tau]. \tag{35}
$$

Here, the term proportional to the Liouville vector field corresponds to the contribution from the free motion of the Brownian particle, while $\mathcal{I}[\Phi_\tau]$ represents the contribution from the interaction between the Brownian particle and the heat reservoir.

Please be reminded that we use the term "flow" instead of "current" to refer to the spatial components of the objects which obey the continuity equation. The term "current" is reserved for the full object, including the temporal component. Using the definition (35), eq. (34) can be rewritten in the form

$$
\partial_\tau \Phi_\tau + \hat{\nabla}^{(\hat{h})}_A \mathscr{J}^A[\Phi_\tau] = 0. \tag{36}
$$

Eq.(36) implies that the surface integral

$$
-\int_\Sigma \eta_{\Sigma_t}\mathscr{Z}_A \mathscr{J}^A[\Phi_\tau] \tag{37}
$$

should be the probability that the Brownian particle passes through the subarea Σ in the SoMS Σ_t per unit proper time of the Brownian particle. Although it looks puzzling to understand eq.(36) as a continuity equation because of the presence of two time variables, this equation still plays a key role while making connection to the alternative distribution function to be introduced shortly.

3.2 FPE Associated with LE_t

The second Langevin equation proposed in [1], i.e. LE_t, arises from a reparametrization of LE_τ. The concrete form of LE_t reads

$$
\mathrm{d}\tilde{y}^\mu_t = \frac{\tilde{k}^\mu_t}{m}\gamma^{-1}\mathrm{d}t, \tag{38}
$$

$$\mathrm{d}\tilde{k}_t^\mu = \left[\hat{\mathcal{R}}^\mu{}_{\mathfrak{a}} \circ_S \mathrm{d}\tilde{W}_t^{\mathfrak{a}} + \hat{\mathcal{F}}^\mu_{\mathrm{add}}\mathrm{d}t\right] + \hat{\mathcal{K}}^{\mu\nu}U_\nu \mathrm{d}t - \frac{1}{m}\Gamma^\mu{}_{\alpha\beta}\tilde{k}_t^\alpha \tilde{k}_t^\beta \gamma^{-1}\mathrm{d}t, \tag{39}$$

where t represents the proper time of Alice, the prescribed observer, and

$$\tilde{y}^\mu(t) = \tilde{x}^\mu(\tau(t)), \qquad \tilde{k}^\mu(t) = \tilde{p}^\mu(\tau(t)).$$

$\hat{\mathcal{R}}^\mu{}_{\mathfrak{a}}$, $\hat{\mathcal{K}}^{\mu\nu}$ and $\hat{\mathcal{F}}^\mu_{\mathrm{add}}$ are connected with their respective un-hatted counterparts via

$$\begin{aligned}
&\hat{\mathcal{R}}^\mu{}_{\mathfrak{a}} := \gamma^{-1/2}\mathcal{R}^\mu{}_{\mathfrak{a}},\\
&\hat{\mathcal{K}}^{\mu\nu} := \gamma^{-1}\mathcal{K}^{\mu\nu}, \quad \hat{\mathcal{F}}^\mu_{\mathrm{add}} := \gamma^{-1}\mathcal{F}^\mu_{\mathrm{add}} - \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\mathcal{R}^\mu{}_{\mathfrak{a}}\mathcal{R}^j{}_{\mathfrak{b}}(\gamma^{-1/2}\hat{\nabla}_j\gamma^{-1/2}),
\end{aligned} \tag{40}$$

and

$$\lambda := |\nabla t|, \qquad \gamma(\tilde{x}, \tilde{p}) := -\frac{\lambda Z_\mu \tilde{p}^\mu}{m}. \tag{41}$$

$\gamma(\tilde{x}, \tilde{p})$ plays the role of a local Lorentz factor, i.e. $\mathrm{d}\tau = \gamma^{-1}\mathrm{d}t$, which is also random-valued because of the random motion of the Brownian particle. The fact that the proper time τ of the Brownian particle becomes a random variable from the observer's perspective is the reason why the reparametrization leading from LE_τ to LE_t is unavoidable.

The PDF for the Brownian particle described by eqs.(38)-(39) is

$$\Psi_t(x^\mu, p^i) := \Pr[\tilde{y}_t^\mu = x^\mu, \tilde{k}_t^i = p^i]. \tag{42}$$

Apparently, this PDF is also a two-time distribution, just like $\Phi_\tau(x^\mu, p^i)$ given in eq.(25), which is hard to understand physically. However, the PDF $\Psi_t(x^\mu, p^i)$ actually encodes the physical PDF $f(x^\mu, p^i)$ on Σ_t in the following manner. Recall that Σ_t can be regarded as a hypersurface on the mass shell bundle with normal vector field $\mathscr{Z}^a$. This relationship allows us to introduce an invariant volume form on Σ_t, i.e.

$$(\eta_{\Sigma_t})_{a_1,\ldots,a_{2d}} := \mathscr{Z}^{a_0}(\eta_{\Gamma_m^+})_{a_0,a_1,\ldots,a_{2d}}. \tag{43}$$

Since t is the proper time of Alice, there is no randomness in t, therefore, using the co-area formula [33, 34] of geometric measure theory, we can write

$$\Psi_t(x^\mu, p^i) = \lambda\delta(t(x) - t) f(x^\mu, p^i), \tag{44}$$

in which $f(x^\mu, p^i)$ is the desired physical PDF on Σ_t. Let us stress that the volume elements associated with $\Psi_t(x^\mu, p^i)$ and $f(x^\mu, p^i)$ are, respectively, $\eta_{\Gamma_m^+}$ and η_{Σ_t}.

Following a similar procedure which leads to the FPE (32), we can get the FPE for $\Psi_t(x^\mu, p^i)$, which is associated with LE_t,

$$\begin{aligned}
&\frac{\partial}{\partial t}\Psi_t + \frac{1}{m}\mathscr{L}(\gamma^{-1}\Psi_t)\\
&= \nabla_i^{(h)}\left[\frac{1}{2}\hat{\mathcal{D}}^{ij}\nabla_j^{(h)}\Psi_t + \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\left(\hat{\mathcal{R}}^i_{\mathfrak{a}}\nabla_j^{(h)}\hat{\mathcal{R}}^j_{\mathfrak{b}}\right)\Psi_t - \hat{\mathcal{F}}^i_{\mathrm{add}}\Psi_t - \hat{\mathcal{K}}^{i\nu}U_\nu\Psi_t\right],
\end{aligned} \tag{45}$$

where $\hat{\mathcal{D}}^{\mu\nu} := \gamma^{-1}\mathcal{D}^{\mu\nu}$.

Now since

$$\begin{aligned}
\left[\frac{\partial}{\partial t} + \gamma^{-1}\frac{p^\mu}{m}\frac{\partial}{\partial x^\mu}\right]\delta(t(x) - t) &= \left[-1 + \gamma^{-1}\frac{p^\mu}{m}\frac{\partial t}{\partial x^\mu}\right]\delta'(t(x) - t)\\
&= \left[-1 + \gamma^{-1}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right]\delta'(t(x) - t) = 0,
\end{aligned} \tag{46}$$

substituting eq.(44) into the left hand side of eq.(45) yields

$$\frac{\partial}{\partial t}\Psi_t + \frac{1}{m}\mathscr{L}(\gamma^{-1}\Psi_t) = \delta(t(x)-t)\frac{1}{m}\mathscr{L}(\gamma^{-1}\lambda f). \tag{47}$$

On the other hand, the first three terms in the square bracket on the right hand side of eq.(45) can be rearranged in the form

$$\begin{aligned} &\frac{1}{2}\hat{\mathcal{D}}^{ij}\nabla^{(h)}_j\Psi_t + \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\left(\hat{\mathcal{R}}^i_{\mathfrak{a}}\nabla^{(h)}_j\hat{\mathcal{R}}^j_{\mathfrak{b}}\right)\Psi_t - \hat{\mathcal{F}}^i_{\rm add}\Psi_t \\ &\quad = \frac{1}{2}\mathcal{D}^{ij}\nabla^{(h)}_j(\gamma^{-1}\Psi_t) + \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\left(\mathcal{R}^i_{\mathfrak{a}}\nabla^{(h)}_j\mathcal{R}^j_{\mathfrak{b}}\right)(\gamma^{-1}\Psi_t) - \mathcal{F}^i_{\rm add}(\gamma^{-1}\Psi_t). \end{aligned} \tag{48}$$

Therefore, the substitution of eq.(44) into eq.(45) yields

$$\frac{1}{m}\mathscr{L}(\gamma^{-1}\lambda f) = \nabla^{(h)}_i\mathcal{I}^i[\gamma^{-1}\lambda f], \tag{49}$$

where $\mathcal{I}^i[\gamma^{-1}\lambda f]$ is defined in a similar fashion as in eq.(34).

Notice that the FPEs (34) and (49) have a similar form. By dropping the time derivative term $\partial_\tau\Phi_\tau$ in eq.(34) and replacing Φ_τ with $\gamma^{-1}\lambda f$, eq.(34) can be changed into eq.(49). This is certainly not a coincidence, and we will demonstrate in the next section how eq.(34) is intimately related to eq.(49).

4 Reduced FPE

In Part I of this series of research [1], we used numerical method to investigate whether the random paths generated by LE_τ and LE_t produce the same physical PDF on the SoMS Σ_t. The results in the example case of $(1+1)$-dimensional Minkowski spacetime indicate that nearly identical distributions arise from the two Langevin equations LE_τ and LE_t. In this section, we will provide an analytical proof in generic spacetimes. During this proof, we will introduce a new distribution function, φ, together with its evolution equation, which we call the reduced FPE.

Recall from eq.(37) that the integral $-\int_\Sigma \eta_{\Sigma_t}\mathscr{Z}_A\mathscr{J}^A[\Phi_\tau]$ represents the probability that the Brownian particle passes through the subregion Σ in the SoMS Σ_t per unit proper time of the Brownian particle. From the observer's perspective, the condition "per unit proper time of the Brownian particle" is irrelevant, the actual probability that the Brownian particle passes through the subarea Σ should read

$$\begin{aligned} \Pr[\text{The particle passes through } \Sigma] &= -\int_{\mathbb{R}} \mathrm{d}\tau \int_\Sigma \eta_{\Sigma_t}\mathscr{Z}_A\mathscr{J}^A[\Phi_\tau] \\ &= -\int_\Sigma \eta_{\Sigma_t}\mathscr{Z}_A\mathscr{J}^A[\varphi], \end{aligned} \tag{50}$$

where we have introduced

$$\varphi(x,p) := \int_{\mathbb{R}} \mathrm{d}\tau\, \Phi_\tau(x,p). \tag{51}$$

Since Σ is an arbitrary subregion in the SoMS Σ_t, the integrand $-\mathscr{Z}_A\mathscr{J}^A[\varphi]$ in eq.(50) should be the PDF for the intersection points of the random paths with the SoMS Σ_t, i.e.

$$f = -\mathscr{Z}_A\mathscr{J}^A[\varphi] = -\frac{1}{m}Z_\mu p^\mu\varphi = \gamma\lambda^{-1}\varphi. \tag{52}$$

Now let us consider a scenario in which the random paths of the Brownian particles are infinitely stretched, i.e. extending from $\tau = -\infty$ to $\tau = \infty$. It is natural to introduce the boundary conditions

$$\Phi_{-\infty}(x, p) = \Phi_{+\infty}(x, p) = 0 \tag{53}$$

for the PDF Φ_τ, because otherwise Φ_τ will not be normalizable. Then, by integrating eq.(34) with respect to τ, we can get

$$\frac{1}{m}\mathscr{L}(\varphi) = \nabla_i^{(h)}\mathcal{I}^i[\varphi], \tag{54}$$

where again $\mathcal{I}^i[\varphi]$ is defined in a similar way as in eq.(34).

Since φ differs from the true PDF f by the scalar factor $\gamma\lambda^{-1}$, it cannot be a normalized PDF. Therefore, the equation (54) obeyed by φ will not be referred to as FPE, but rather as *reduced FPE*. Bearing in mind the relationship (52), one can easily recognize that eq.(54) is actually identical to eq.(49). In other words, both the FPEs (34) and (45) give rise to the same reduced FPE. This fact gives an analytical evidence for the correctness of the numerical test presented in [1].

Some remarks are in due here.

(1) Since the reduced FPE (54) is homogeneous in φ and there is no need to normalize φ, there is a freedom to multiply φ with a constant factor which preserves eq.(54). This freedom will be used in the next section while defining the particle number density of the Brownian particle.

(2) There is a common misconception about the role of Φ_τ regarding a particular case, i.e. the stationary distribution $\Phi_\tau(x, p) := \Phi(x, p)$, which is often considered to be identical to the equilibrium distribution for the particles of the heat reservoir, i.e. the Jüttner distribution. Technically it is true that, when Φ_τ is independent of τ, eq. (34) will take the same form as eq.(54). However, this coincidence does not imply that $\varphi(x, p)$ is identical to the stationary distribution $\Phi(x, p)$. There are two primary reasons for this difference: (i) The stationary distribution is a distribution which does not change with the time of some stationary observer, rather than of the Brownian particle; (ii) The identification of $\varphi(x, p)$ with the stationary distribution $\Phi(x, p)$ implies that the reduced FPE can only describe the stationary states, whereas it can actually describe the whole relaxation process, as will be demonstrated in Sect. 5 and Sect. 6.

Following a similar fashion with eq.(35), we can introduce a *current* associated with φ,

$$\mathscr{J}[\varphi] := \frac{\varphi}{m}\mathscr{L} - \mathcal{I}[\varphi].$$

Then the reduced FPE (54) could be rewritten as the current conservation equation

$$\hat{\nabla}_A^{(\hat{h})}\mathscr{J}^A[\varphi] = 0 \tag{55}$$

on the mass shell bundle. Let us stress that $\mathscr{J}[\varphi]$ is now interpreted as a *current*, rather than *flow*, because only a single time variable is present in the above equation which is hidden behind the index A. The conservation of the current $\mathscr{J}^A[\varphi]$ does not correspond to conservation of probability, but rather to conservation of matter. More details on this point will be presented in the next section.

5 Macroscopic Quantities and Interpretation of $\varphi(x, p)$

Let $\mathcal{S}$ be an arbitrary subregion in the configuration space $\mathcal{S}_t$, and $\Sigma := \{(x, p) \in \Gamma_m^+ | x \in \mathcal{S}\}$ is the corresponding subregion in the SoMS. When Alice is not bound together with the coordinate system, the proper time t will be different from the coordinate time x^0, which means that $\mathcal{S}_t$ is not the coordinate hypersurface with fixed x^0, but rather a tilted hypersurface with mixtures between x^0 and x^i. Nevertheless, since the PDF $f(x, p)$ is by definition the probability density on Σ_t, and that $\eta_{\Sigma_t} = \eta_{\mathcal{S}_t} \wedge \eta_{(\Gamma_m^+)_x}$, we can calculate the probability for the Brownian particle to appear in $\mathcal{S}$ at the time t as

$$\begin{aligned} \Pr[\mathcal{S}] = \Pr[\Sigma] &= \int_\Sigma \eta_{\Sigma_t} f \\ &= -\int_\Sigma \eta_{\Sigma_t} \mathscr{Z}_a \mathscr{J}^a[\varphi] = -\int_{\mathcal{S}} \eta_{\mathcal{S}_t} Z_\mu \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \frac{p^\mu}{m} \varphi. \end{aligned} \tag{56}$$

The change from f to φ in the integrand of the last equality reflects the tiltedness of $\mathcal{S}_t$ in the spacetime.

Now consider the case with N non-interacting Brownian particles coexisting in the same heat reservoir. By putting an extra factor N in front of the integrals in eq.(56) and enlarging Σ into Σ_t, we should get N as the final value of the integration. Therefore, by dropping the integral over $\mathcal{S}$, we get the particle number density in the configuration space

$$\bar{n} = -N Z_\mu \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \frac{p^\mu}{m} \varphi. \tag{57}$$

Recall that the particle number density should be defined as

$$\bar{n} = -Z_\mu N^\mu[\varphi],$$

wherein N^μ denotes the particle number current. At present, the particle number current reads

$$N^\mu[\varphi] = \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \frac{p^\mu}{m} N \varphi. \tag{58}$$

It is remarkable that the above form of the particle number current is identical to that given in relativistic kinetic theory (except for the constant factor N), provided that φ is identified with the 1PDF which obeys the relativistic Boltzmann equation. This resemblance reminds us that there may be some deep connections between the approaches of relativistic stochastic mechanics and of relativistic kinetic theory.

Since there is no chemical reactions between the Brownian particles, the particle current must be conserved. This fact can be proved using Stokes' theorem. Let V be an region in the spacetime manifold $\mathcal{M}$, and $\Gamma = \{(x, p) \in \Gamma_m^+ | \; x \in V\}$ is the corresponding region on the mass shell bundle. Let n^μ be the unit normal vector field of ∂V which induces the unit normal vector field $\mathscr{N}$ of $\partial\Gamma$. The Stokes' theorem on the mass shell bundle reads

$$\int_\Gamma \eta_{\Gamma_m^+} \hat{\nabla}_A^{(\hat{h})} \mathscr{J}^A[\varphi] = \int_{\partial\Gamma} \eta_{\partial\Gamma} \mathscr{N}_A \mathscr{J}^A[\varphi]. \tag{59}$$

Using the fact that $\partial\Gamma = \{(x, p) \in \Gamma_m^+ | \; x \in \partial V\}$ and that $\mathscr{N}^a = n^\mu e_\mu{}^a$, we can rewrite the above equation as

$$\int_V \eta_{\mathcal{M}} \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \hat{\nabla}_A^{(\hat{h})} \mathscr{J}^A[\varphi] = \int_{\partial V} \eta_{\partial V} \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \mathscr{N}_A \mathscr{J}^A[\varphi]$$

$$= \frac{1}{N} \int_{\partial V} \eta_{\partial V} n_\mu N^\mu[\varphi] = \frac{1}{N} \int_V \eta_{\mathcal{M}} \nabla_\mu N^\mu[\varphi], \tag{60}$$

where ∇_μ denotes the usual covariant derivative on the spacetime manifold. Due to the arbitrariness of V, we can drop the integration with respect to the measure $\eta_{\mathcal{M}}$ and get

$$\nabla_\mu N^\mu[\varphi] = N \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \hat{\nabla}_A^{(\hat{h})} \mathscr{J}^A[\varphi] = 0, \tag{61}$$

which means that $N^\mu[\varphi]$ is a conservation current on the spacetime.

The energy of a single Brownian particle measured by Alice is defined as

$$E := -p^\mu Z_\mu. \tag{62}$$

Thus the single particle contribution to the average energy flux through the subregion Σ in the SoMS Σ_t should read

$$\bar{E}[\Sigma] := \int_\Sigma \eta_\Sigma \mathscr{Z}_A \mathscr{J}^A[\varphi] p^\mu Z_\mu = \int_{\mathcal{S}} \eta_{\mathcal{S}} Z_\mu Z_\nu \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \frac{p^\nu p^\mu}{m} \varphi. \tag{63}$$

The second integration factor, i.e.

$$T^{\mu\nu}[\varphi] := \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \frac{p^\nu p^\mu}{m} \varphi, \tag{64}$$

is recognized to be the single particle contribution to the energy-momentum tensor, and

$$\rho := Z_\mu Z_\nu T^{\mu\nu}[\varphi] = Z_\mu Z_\nu \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \frac{p^\nu p^\mu}{m} \varphi \tag{65}$$

is naturally the single particle contribution to the energy density.

The single particle contribution to the average energy-momentum vector of the Brownian particle is defined as

$$E^\mu[\varphi] := -Z_\nu T^{\mu\nu}[\varphi].$$

In general, $E^\mu[\varphi]$ is non-conserved because of the joint effects of gravitational work and heat transfer from the heat reservoir. Since

$$\begin{aligned}
&-\int_V \eta_{\mathcal{M}} \nabla_\mu (Z_\nu T^{\mu\nu}[\varphi]) = -\int_{\partial V} \eta_{\partial V} n_\mu Z_\nu T^{\mu\nu}[\varphi] \\
&= -\int_{\partial V} \eta_{\partial V} \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} (Z_\nu p^\nu) n_\mu \frac{p^\mu}{m} \varphi \\
&= \int_{\partial V} \eta_{\partial V} \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \mathscr{N}_A \mathscr{J}^A[\varphi] E = \int_V \eta_{\mathcal{M}} \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \hat{\nabla}_A^{(\hat{h})} (E \mathscr{J}^A[\varphi]),
\end{aligned} \tag{66}$$

we have

$$\begin{aligned}
\nabla_\mu E^\mu[\varphi] &= -\nabla_\mu \left(Z_\nu T^{\mu\nu}[\varphi]\right) = -\int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \hat{\nabla}_A^{(\hat{h})} \left(p^\nu Z_\nu \mathscr{J}^A[\varphi]\right) \\
&= -\int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \mathscr{J}^A[\varphi] \hat{\nabla}_A^{(\hat{h})} \left(p^\nu Z_\nu\right) \\
&= -\int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \left(\frac{\varphi}{m} \mathscr{L}(Z_\nu p^\nu) - Z_\nu \mathcal{I}^\nu[\varphi]\right) \\
&= -T^{\mu\nu}[\varphi] \nabla_\mu Z_\nu + Z_\nu \int_{(\Gamma_m^+)_x} \eta_{(\Gamma_m^+)_x} \mathcal{I}^\nu[\varphi].
\end{aligned} \tag{67}$$

The first term on the right hand side of eq.(67) is the average gravitational power acting on the Brownian particle, i.e.

$$P_{\rm grav}[\varphi] := -T^{\mu\nu}[\varphi]\nabla_\mu Z_\nu = \int_{(\Gamma^+_m)_x} \eta_{(\Gamma^+_m)_x}\mathcal{P}_{\rm grav}(Z)\varphi, \tag{68}$$

where

$$\mathcal{P}_{\rm grav}(Z) = -\frac{p^\mu p^\nu}{m}\nabla_\mu Z_\nu \tag{69}$$

is the the gravitational power along a single trajectory of the particle [35] as measured by Alice. Thus the second term on the right hand side of eq.(67) should be interpreted as the heat transfer rate from the heat reservoir,

$$Q[\varphi] := \int_{(\Gamma^+_m)_x} \eta_{(\Gamma^+_m)_x} Z_\nu \mathcal{I}^\nu[\varphi] = -Z_\nu\nabla_\mu T^{\mu\nu}[\varphi]. \tag{70}$$

In the end, we have

$$\nabla_\mu E^\mu[\varphi] = P_{\rm grav}[\varphi] + Q[\varphi], \tag{71}$$

which is reminiscent to the first law of thermodynamics, but is presented in terms of the divergence of the average energy-momentum vector, the gravitational power and the heat transfer rate. Please note that the last equation is valid for any observer. However, for different observers, the values of $P_{\rm grav}[\varphi]$ and $Q[\varphi]$ can be different.

6 Relativistic Equilibrium State and Einstein Relation

So far, we have not paid a word on the state of the heat reservoir, except for the implicit assumption of an equilibrium state. This does not make any harm to the formal construction of FPE. However, when the solution to the FPE is concerned, an explicit description for the equilibrium state of the reservoir becomes inevitable.

As mentioned in the introduction, there are two basic assumptions in the description for the Brownian motion of a heavy particle in a heat reservoir. In the relativistic context, these assumptions need to be re-examined.

The first problem one encounters is the proper definition for the equilibrium state of the reservoir. It is well known that, in the presence of gravity, a macroscopic system cannot reach the thermodynamic equilibrium in the usual sense, i.e. the one with uniform temperature and chemical potential. The reason lies in that there is a bilateral interaction between thermal and gravitational effects. On the one hand, thermal energy as a form of energy should generate gravity; on the other hand, gravity, as a long range interaction, has nontrivial impact on the relaxation process, leading to the final state with non-uniform temperature and chemical potential.

Meanwhile, the choice of observer also brings about some subtleties in describing the state of the heat reservoir. The importance of the role of observer can be revealed in two different aspects: i) According to the equivalence principle, gravity is locally indistinguishable from acceleration. Therefore, the strength of the gravitational force experienced by the observer and by the macroscopic system being observed could be different, provided the amounts of accelerations are different. ii) There has long been a dispute on the of relativistic transformation rules of thermodynamic parameters, mostly about the transform of temperature, but also include the transform of chemical potential. According to the results of [36], these transformation rules are related to the choice of observer.

Due to the above reasons, we need to answer the following questions in order to clarify the state of the heat reservoir:

Q1. What is a relativistic equilibrium state? Is the equilibrium state observer-dependent?
Q2. What is the equilibrium distribution for particles of the heat reservoir?
Q3. Is this distribution observer-dependent?

Fortunately the answers to these questions can be inferred from the studies on relativistic kinetic theory. To answer Q1, let us infer that equilibrium states could be viewed as the final states of relaxation processes, and a system carrying out a relaxation process should not care about who is observing it. Therefore, the final state of the relaxation process should not be affected by the choice of observer. Given an isolated system, there can only be one *intrinsic equilibrium state*, i.e. the state in detailed balance, which is characterized by several macroscopic features, including the absence of entropy production rate and vanishing collision integral in the Boltzmann equation.

From the point of view of the comoving observer, *Bob*, the equilibrium state has one extra feature, i.e. the absence of transport flows. By definition, the proper velocity of Bob is identical to the proper velocity U^μ of an element of the heat reservoir viewed as a relativistic fluid. The same U^μ also appeared in the damping force term in the Langevin equation. According to [37, 38], the driving forces for the relativistic transports are the generalized gradients for the temperature and chemical potential, which are dependent on the proper velocity of the observer. For the comoving observer Bob, the generalized gradients for the temperature and chemical potential read

$$\mathcal{D}_\nu T_\mathrm{B} = \nabla_\nu T_\mathrm{B} + T_\mathrm{B} U^\rho \nabla_\rho U_\nu = 0, \qquad \mathcal{D}_\nu \mu_\mathrm{B} = \nabla_\nu \mu_\mathrm{B} + \mu_\mathrm{B} U^\rho \nabla_\rho U_\nu = 0, \tag{72}$$

where T_B and μ_B respectively are the temperature and chemical potential of the heat reservoir measured by Bob. One immediately sees that the ordinary gradients $\nabla_\nu T_\mathrm{B}$ and $\nabla_\nu \mu_\mathrm{B}$ are nonzero, unless Bob undergoes geodesic motion, i.e. $U^\rho \nabla_\rho U_\nu = 0$. In the latter case, T_B and μ_B becomes uniform, which is fully consistent with the definition of equilibrium state in the non-relativistic thermodynamics.

The answer to Q2 is also provided by relativistic kinetic theory, and the explicit 1PDF for the heat reservoir is given by the *Jüttner-like distribution* [39]

$$\varphi_\mathrm{HR}(x, p) = \frac{\mathfrak{g}}{\mathrm{e}^{\alpha - B_\mu p^\mu} - \varsigma} = \frac{\mathfrak{g}}{\mathrm{e}^{\alpha - U_\mu p^\mu / T_\mathrm{B}} - \varsigma} = \frac{\mathfrak{g}}{\mathrm{e}^{(\varepsilon_\mathrm{B} - \mu_\mathrm{B})/T_\mathrm{B}} - \varsigma} \tag{73}$$

provided that the background spacetime is stationary, where $\varsigma = 0, \pm 1$, $\mathfrak{g}$ denotes the quantum degeneracy, $\varepsilon_\mathrm{B} = -U_\mu p^\mu$ is the energy of the particle measured by Bob, μ_B is the chemical potential of the heat reservoir, and $\alpha = -\mu_\mathrm{B}/T_\mathrm{B}$ is a constant in spacetime. In order that the distribution (73) indeed describes a state in detailed balance, the vector field $B^\mu = U^\mu / T_\mathrm{B}$ is required to be timelike Killing, i.e.

$$\nabla_{(\mu} B_{\nu)} = 0. \tag{74}$$

The existence of a timelike Killing field implies that the underlying spacetime needs to be stationary.

We *assume that the heat reservoir is consisted of purely classical particles*. In this case, the above 1PDF becomes the *standard Jüttner distribution*

$$\varphi_\mathrm{HR}(x, p) = \mathrm{e}^{-\alpha + U_\mu p^\mu / T_\mathrm{B}} = \mathrm{e}^{(\mu_\mathrm{B} - \varepsilon_\mathrm{B})/T_\mathrm{B}}. \tag{75}$$

The 1PDF $\varphi_{\rm HR}(x, p)$ as presented in the form (75) contains the proper velocity U^μ of Bob and the temperature $T_{\rm B}$ measured by Bob, thus it is explicitly dependent on the choice of observer. This answers Q3. It is remarkable that the distribution (75) has the same form as the non-relativistic Boltzmann distribution. However, due to eq.(72), the above distribution is in fact different from the Boltzmann distribution, because $T_{\rm B}$ and $\mu_{\rm B}$ are now non-uniform.

It is interesting to ask what the distribution (75) would look like from the point of view of Alice. To answer this question, let us remind that all measurements in curved spacetime must be made *on the spot*. Therefore, to consider the distributions of the same particles, Alice and Bob must appear at the same spacetime event, and their proper velocities can differ by at most a *local* Lorentz boost. Let $\gamma_{\rm AB}$ denotes the relative Lorentz factor between Alice and Bob. Then the proper velocity U^μ of Bob can be expressed as

$$U^\mu = \gamma_{\rm AB}\left(Z^\mu + z^\mu\right), \quad \gamma_{\rm AB} = -(U_\mu Z^\mu)^{-1}, \quad z^\mu U_\mu = 0, \tag{76}$$

The energy of the particle observed by Alice reads $\varepsilon_{\rm A} = -Z_\mu p^\mu$. Denoting the temperature and chemical potential of the heat reservoir measured by Alice as $T_{\rm A}$ and $\mu_{\rm A}$ respectively, we get, by inserting the eq.(76) into eq.(75), the following distribution,

$$\begin{aligned}\varphi_{\rm HR}(x, p) &= {\rm e}^{-\alpha+\gamma_{\rm AB}(Z_\mu+z_\mu)p^\mu/T_{\rm B}} = {\rm e}^{[\mu_{\rm B}+\gamma_{\rm AB}(Z_\mu+z_\mu)p^\mu]/T_{\rm B}}\\ &= {\rm e}^{[(\gamma_{\rm AB})^{-1}\mu_{\rm B}+Z_\mu p^\mu+z_\mu p^\mu]/T_{\rm A}} = {\rm e}^{(\mu_{\rm A}-\varepsilon_{\rm A}+z_\mu p^\mu)/T_{\rm A}},\end{aligned} \tag{77}$$

where the temperatures and the chemical potentials measured by both observers are related as [36]

$$T_{\rm A} = (\gamma_{\rm AB})^{-1}T_{\rm B}, \qquad \mu_{\rm A} = (\gamma_{\rm AB})^{-1}\mu_{\rm B}. \tag{78}$$

Now let us proceed with the second basic assumption for the Brownian motion un-altered, hence the the probability distribution for the Brownian particle should approach the same form as the 1PDF for the heat reservoir after sufficient long time, i.e.

$$\varphi(x, p) \to \varphi_{\rm HR}(x, p) = {\rm e}^{-\alpha+U_\mu p^\mu/T_{\rm B}} \quad \text{as} \quad t \to \infty. \tag{79}$$

The above assumption also implies that the long time limit of the heat transfer rate $Q[\varphi]$ should approach zero, because for the Jüttner distribution φ, we always have $\nabla_\mu T^{\mu\nu}[\varphi] = 0$ and thus $Q[\varphi] = Z_\nu \nabla_\mu T^{\mu\nu}[\varphi] = 0$. This result is independent of Z_μ. It is worth noting that the condition $Q[\varphi] = 0$ does not necessarily imply $Z_\mu \mathcal{I}^\mu[\varphi] = 0$. When the latter fails to vanish, it means that the Brownian particle is more likely to absorb heat from in some states and is more likely to transfer heat to the heat reservoir in some other states. Although the heat transfer between different micro states cancels out, this may lead to a deviation from detailed balance in the transition probabilities between different micro states, causing a change in the momentum distribution of the Brownian particle. Therefore, the *detailed* thermal equilibrium between the Brownian particle and the heat reservoir should be given by the stronger condition $Z_\mu \mathcal{I}^\mu[\varphi] = 0$. Due to the arbitrariness in the choice of Z^μ, this condition can be further reduced to $\mathcal{I}[\varphi] = 0$, i.e.

$$\frac{1}{2}\mathcal{D}^{ij}\nabla^{(h)}_j\varphi + \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\left(\mathcal{R}^i{}_{\mathfrak{a}}\nabla^{(h)}_j\mathcal{R}^j{}_{\mathfrak{b}}\right)\varphi - \mathcal{F}^i_{\rm add}\varphi - \mathcal{K}^{i\nu}U_\nu\varphi = 0. \tag{80}$$

Inserting eq.(79) into eq.(80) we have

$$\mathcal{F}^\mu_{\rm add} = \left[\frac{1}{2T_{\rm B}}\mathcal{D}^{\mu\nu} - \mathcal{K}^{\mu\nu}\right]U_\nu + \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\mathcal{R}^\mu{}_{\mathfrak{a}}\nabla^{(h)}_j\mathcal{R}^j{}_{\mathfrak{b}}. \tag{81}$$

In the low temperature limit, both $\mathcal{F}^{\mu}_{\text{add}}$ and $\dfrac{\delta^{\mathfrak{ab}}}{2}\mathcal{R}^{\mu}{}_{\mathfrak{a}}\nabla^{(h)}_{j}\mathcal{R}^{j}{}_{\mathfrak{b}}$ tends to vanish at least as $\mathcal{O}(T_{\text{B}})$. Therefore we get

$$\mathcal{D}^{\mu\nu} = 2T_{\text{B}}\mathcal{K}^{\mu\nu} + \mathcal{O}(T_{\text{B}}^{2}), \tag{82}$$

where the extra $\mathcal{O}(T_{\text{B}}^{2})$ term is dependent on the choice of damping model. When appropriate damping model is taken, e.g. like in [40], this term can be removed completely, yielding

$$\mathcal{D}^{\mu\nu} = 2T_{\text{B}}\mathcal{K}^{\mu\nu}. \tag{83}$$

This relation is the general relativistic analogue of the celebrated Einstein relation.

As a simple intuitive example case, let us consider the isotropic damping model in which the diffusion tensor and tensorial damping coefficients are given as

$$\mathcal{K}^{\mu\nu} = \kappa\Delta^{\mu\nu}(p), \qquad \mathcal{D}^{\mu\nu} = D\Delta^{\mu\nu}(p), \tag{84}$$

where k and D are both scalar functions, i.e. the scalar friction coefficient and diffusion coefficient respectively. Then the Einstein relation will degenerate into

$$D = 2\kappa T_{\text{B}}, \tag{85}$$

which is formally identical to that arises from non-relativistic linear response theory, except that T_{B} now could have a nonvanishing ordinary gradient because of eq.(72). This result suggests that linear response theory should still hold in the relativistic context, at least locally.

When the Einstein relation (83) holds precisely, we have

$$\mathcal{F}^{\mu}_{\text{add}} = \frac{\delta^{\mathfrak{ab}}}{2}\mathcal{R}^{\mu}_{\mathfrak{a}}\nabla^{(h)}_{j}\mathcal{R}^{j}_{\mathfrak{b}}. \tag{86}$$

This result has already been adopted in [1] while constructing the general covariant Langevin equations. Inserting eq.(86) into the definition of $\mathcal{I}[\varphi]$ yields

$$\mathcal{I}[\varphi] = \left[\frac{1}{2}\mathcal{D}^{ij}\nabla^{(h)}_{j}\varphi - \mathcal{K}^{i\nu}U_{\nu}\varphi\right]\frac{\partial}{\partial \breve{p}^{i}}. \tag{87}$$

This formula provides a physical image of how the Brownian particle reaches equilibrium after long time of relaxation. The damping force causes a heat transfer from the Brownian particle to the heat reservoir, while the stochastic force causes a heat transfer form the heat reservoir to the Brownian particle. After long time of relaxation, the damping and stochastic forces balances each other in the statistical sense.

As a final remark, let us mention that, due to the transformation rule (78), the Einstein relation rewritten in terms of the temperature measured by Alice should read

$$\mathcal{D}^{\mu\nu} = 2\gamma_{\text{AB}}T_{\text{A}}\mathcal{K}^{\mu\nu}.$$

7 Conclusion

The major results of the present work can be summarized as follows.

1) The general covariant FPEs associated with both versions of the general relativistic Langevin equation proposed in Ref. [1] are presented, both give rise to the same reduced FPE obeyed by the 1PDF $\varphi(x, p)$ for the Brownian particle. The relationship between different distribution functions is clarified.

2) Several important macroscopic quantities and the quantitative relationships between them are obtained with the aid of the 1PDF which obeys the reduced FPE. These quantities and relationships reveal a close connection between the approaches of stochastic mechanics and relativistic kinetic theory.

3) The meaning of the relativistic equilibrium state of the heat reservoir is properly addressed, and, by assuming that the long time relaxation result for the 1PDF of the Brownian particle should be identical to the 1PDF of the heat reservoir, we derive a general covariant version of the Einstein relation.

These results resolve several common confusions which exist in the literature. Moreover, we hope to use these results as the starting point for further exploring some important subjects in relativistic macroscopic systems, e.g. the origin of irreversibility in relativistic systems, the area law of near horizon entropies, etc. More on these topics will come about later.

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Data Availability This research has no associated data.

Declarations

Competing Interest The authors declare no competing interest.

Appendex A: Diffusion Operator Approach to the FPE

In order to derive the FPE from a stochastic differential equation (SDE), we need to use Ito's lemma to calculate the differential of an arbitrary scalar function, and perform integration by parts twice. When the SDE is defined on a manifold, this procedure can be very complicated.

There is a simpler approach, i.e. the diffusion operator approach [32], for obtaining the FPE on a manifold. Here we give a brief review of this alternative method.

The Ito type SDE on Riemannian manifold or Pseudo-Riemannian manifold (M, g) can be written as

$$\mathrm{d}\tilde{X}_t^\mu = F^\mu \mathrm{d}t + C^\mu{}_{\mathfrak{a}} \circ_I \mathrm{d}\tilde{w}_t^{\mathfrak{a}}. \tag{88}$$

Let h be an arbitrary scalar field on M, then the time differential of $\tilde{h}_t := h(\tilde{X}_t)$ can be derived by Ito's lemma:

$$\mathrm{d}\tilde{h}_t = \left[\frac{\partial h}{\partial x^\mu} F^\mu + \frac{\delta^{\mathfrak{ab}}}{2} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} C^\mu{}_{\mathfrak{a}} C^\nu{}_{\mathfrak{b}}\right] \mathrm{d}t + \frac{\partial h}{\partial x^\mu} C^\mu{}_{\mathfrak{a}} \circ_I \mathrm{d}\tilde{w}_t^{\mathfrak{a}}. \tag{89}$$

Therefore, the expectation of $\mathrm{d}\tilde{h}_t$ is

$$\langle \mathrm{d}\tilde{h}_t \rangle = \left\langle \frac{\partial h}{\partial x^\mu} F^\mu + \frac{\delta^{\mathfrak{ab}}}{2} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} C^\mu{}_{\mathfrak{a}} C^\nu{}_{\mathfrak{b}} \right\rangle \mathrm{d}t. \tag{90}$$

This means $\langle \tilde{h}_t \rangle$ is differentiable with respect to time in spite of the fact that $\tilde{h}_t$ isn't differentiable. Defining the diffusion operator as

$$\mathbf{A} = F^\mu \frac{\partial}{\partial x^\mu} + \frac{\delta^{\mathfrak{ab}}}{2} C^\mu{}_{\mathfrak{a}} C^\nu{}_{\mathfrak{b}} \frac{\partial^2}{\partial x^\mu \partial x^\nu}, \tag{91}$$

the derivative of $\langle \tilde{h}_t \rangle$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tilde{h}_t \rangle = \langle \mathbf{A}\tilde{h}_t \rangle. \tag{92}$$

Let $\Phi_t(x) := \Pr[\tilde{X}_t = x]$ be a PDF associated with the invariant volume element $\sqrt{g}\mathrm{d}^n x$ of M, above equation actually means

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M h\Phi_t\sqrt{g}\mathrm{d}^n x = \int_M h\partial_t\Phi_t\sqrt{g}\mathrm{d}^n x = \int_M \Phi_t\mathbf{A}h\sqrt{g}\mathrm{d}^n x = \int_M (\mathbf{A}^*\Phi_t)h\sqrt{g}\mathrm{d}^n x, \tag{93}$$

where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$. Since h is arbitrary, the above equation implies

$$\partial_t\Phi_t = \mathbf{A}^*\Phi_t, \tag{94}$$

which is the FPE associated with the SDE (88).

There are four rules for computing the adjoint operator:

1. $(\mathbf{A}+\mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$.
2. $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$.
3. $\left(\frac{\partial}{\partial x^\mu}\right)^* = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\sqrt{g}$, where the right hand side needs to be understood as a right associative operator.
4. $(F^\mu)^* = F^\mu$.

Using these rules, the adjoint of the diffusion operator (91) is evaluated to be

$$\mathbf{A}^* = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\sqrt{g}F^\mu + \frac{\delta^{\mathfrak{ab}}}{2}\frac{1}{\sqrt{g}}\frac{\partial^2}{\partial x^\mu\partial x^\nu}\sqrt{g}C^\mu{}_{\mathfrak{a}}C^\nu{}_{\mathfrak{b}}. \tag{95}$$

Since the Stratonovich type SDE

$$\mathrm{d}\tilde{X}^\mu = F^\mu\mathrm{d}t + C^\mu{}_{\mathfrak{a}} \circ_S \mathrm{d}\tilde{w}^{\mathfrak{a}} \tag{96}$$

is equivalent to the Ito type SDE

$$\mathrm{d}\tilde{X}^\mu = \left(F^\mu + \frac{\delta^{\mathfrak{ab}}}{2}C^\nu{}_{\mathfrak{a}}\frac{\partial}{\partial x^\nu}C^\mu{}_{\mathfrak{b}}\right)\mathrm{d}t + C^\mu{}_{\mathfrak{a}} \circ_I \mathrm{d}\tilde{w}^{\mathfrak{a}}, \tag{97}$$

the corresponding diffusion operation reads

$$\begin{aligned}\mathbf{A} &= \left(F^\mu + \frac{\delta^{\mathfrak{ab}}}{2}C^\nu{}_{\mathfrak{a}}\frac{\partial}{\partial x^\nu}C^\mu{}_{\mathfrak{b}}\right)\frac{\partial}{\partial x^\mu} + \frac{\delta^{\mathfrak{ab}}}{2}C^\nu{}_{\mathfrak{a}}C^\mu{}_{\mathfrak{b}}\frac{\partial^2}{\partial x^\mu\partial x^\nu}\\ &= F^\mu\frac{\partial}{\partial x^\mu} + \frac{\delta^{\mathfrak{ab}}}{2}C^\nu{}_{\mathfrak{a}}\frac{\partial}{\partial x^\nu}C^\mu{}_{\mathfrak{b}}\frac{\partial}{\partial x^\mu}.\end{aligned} \tag{98}$$

Introducing the vector fields

$$L_0 = F^\mu\frac{\partial}{\partial x^\mu} \qquad L_{\mathfrak{a}} = C^\mu{}_{\mathfrak{a}}\frac{\partial}{\partial x^\mu}, \tag{99}$$

the diffusion operation can be written as simpler form

$$\mathbf{A} = \frac{\delta^{\mathfrak{ab}}}{2}L_{\mathfrak{a}}L_{\mathfrak{b}} + L_0. \tag{100}$$

It is easy to see that L_0 provides the drift term of FPE and $L_{\mathfrak{a}}$ provides the diffusion term. Notice that the adjoint of the coordinate derivative operator looks like the covariant divergence operator when acting on a vector field. Therefore, the action of the adjoint of **A** on the PDF becomes

$$\begin{aligned}\mathbf{A}^*\Phi_t &= \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}L^*_{\mathfrak{a}}L^*_{\mathfrak{b}}\Phi_t + L^*_0\Phi_t \\ &= \frac{\delta^{\mathfrak{a}\mathfrak{b}}}{2}\nabla_\mu(C^\mu{}_{\mathfrak{a}}(\nabla_\nu C^\nu{}_{\mathfrak{b}}\Phi_t)) - \nabla_\mu(F^\mu\Phi_t).\end{aligned} \tag{101}$$

Inserting this result into eq.(94) gives rise to the Fokker-Planck equation associated with the Stratonovich type SDE (96).

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