A Central Limit Theorem for Diffusion in Sparse Random Graphs

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Abstract
We consider bootstrap percolation and diffusion in sparse random graphs with fixed degrees, constructed by configuration model. Every vertex has two states: it is either active or inactive. We assume that to each vertex is assigned a nonnegative (integer) threshold. The diffusion process is initiated by a subset of vertices with threshold zero which consists of initially activated vertices, whereas every other vertex is inactive. Subsequently, in each round, if an inactive vertex with threshold θ has at least θ of its neighbours activated, then it also becomes active and remains so forever. This is repeated until no more vertices become activated. The main result of this paper provides a central limit theorem for the final size of activated vertices. Namely, under suitable assumptions on the degree and threshold distributions, we show that the final size of activated vertices has asymptotically Gaussian fluctuations.

1 Introduction

Threshold models of contagion have been used to describe many complex phenomena in diverse areas including epidemiology [7, 33, 35, 36, 41], neuronal activity [3, 39], viral marketing [24, 25, 27] and spread of defaults in economic networks [4, 8].

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Consider a connected graph $G = (V, E)$ with n vertices $V = [n] := \{1, 2, \ldots, n\}$. Given two vertices $i, j \in [n]$, we write $i \sim j$ if $\{i, j\} \in E$. Every vertex has two states: it is either active or inactive (sometimes also referred to as infected or uninfected). We assume that to each vertex is assigned a nonnegative (integer) threshold. Let θ_i be the threshold of vertex i. The diffusion is then initiated by the (deterministic) subset of vertices with threshold zero which consists of initially activated vertices $\mathcal{A}_0 := \{i \in V : \theta_i = 0\}$. Subsequently, in each round, if an inactive vertex with threshold θ has at least θ of its neighbours activated, then it also becomes active and remains so forever. Namely, at time $t \in \mathbb{N}$, the (deterministic) set of active vertices is given by

$$\mathcal{A}_t = \{i \in V : \sum_{j:j\sim i} \mathbf{1}\{j \in \mathcal{A}_{t-1}\} \geq \theta_i\}, \tag{1}$$

where $\mathbf{1}\{\mathcal{E}\}$ denotes the indicator of an event $\mathcal{E}$, i.e., this is 1 if $\mathcal{E}$ holds and 0 otherwise.

We study above threshold driven contagion model in sparse random graphs with fixed degrees, constructed by configuration model. The interest in this random graph model stems from its ability to mimic some of the empirical properties of real networks, while allowing for tractability (see e.g., [16, 32, 40]). We describe this random graph model next.

1.1 Configuration Model

For each integer $n \in \mathbb{N}$, we consider a system of n vertices $[n] := \{1, 2, \ldots, n\}$ endowed with a sequence of initial non-negative integer thresholds $(\theta_{n,i})_{i=1}^n$. We are also given a sequence $\mathbf{d}_n = (d_{n,i})_{i=1}^n$ of nonnegative integers $d_{n,1}, \ldots, d_{n,n}$ such that $\sum_{i=1}^n d_{n,i}$ is even. By means of the configuration model we define a random multigraph with given degree sequence, denoted by $\mathcal{G}(n, \mathbf{d}_n)$. It starts with n vertices and $d_{n,i}$ half-edges corresponding to the i-th vertex. Then at each step two half-edges are selected uniformly at random, and a full edge is completed by joining them. The multigraph is constructed when there is no more half edges left. Although self-loops and multiple edges may occur, these become rare (under our regularity conditions below) as $n \to \infty$. It is easy to see conditional on the multigraph being simple graph, we obtain a uniformly distributed random graph with these given degree sequences; see e.g. [40].

Let $u_n(d, \theta)$ be the numer of vertices with degree d and threshold θ:

$$u_n(d, \theta) := \#\{i \in [n] : d_{n,i} = d, \ \theta_{n,i} = \theta\}.$$

We assume the following regularity conditions on the degree sequence and thresholds.

Condition 1.1 *For each $n \in \mathbb{N}$, $\mathbf{d}_n = (d_{n,i})_{i=1}^n$ and $(\theta_{n,i})_{i=1}^n$ are sequence of non-negative integers such that $\sum_{i=1}^n d_{n,i}$ is even and, for some probability distribution $p : \mathbb{N}^2 \to [0, 1]$ independent of n, with $u_n(d, \theta)$ as defined above:*

(C$_1$) *as $n \to \infty$,*

$$\frac{u_n(d, \theta)}{n} := \frac{\#\{i \in [n] : d_{n,i} = d, \ \theta_{n,i} = \theta\}}{n} \longrightarrow p(d, \theta)$$

for every non-negative integers d and θ. We further assume $\sum_{\theta=0}^\infty p(0, \theta) < 1$.

(C$_2$) *for every $A > 1$, we have*

$$\sum_{d=0}^\infty \sum_{\theta=0}^\infty u_n(d, \theta)A^d = \sum_{i=1}^n A^{d_{n,i}} = O(n).$$

Remark 1.2 Let (D_n, Θ_n) be random variables with joint distribution $\mathbb{P}(D_n = d, \Theta_n = \theta) = u_n(d, \theta)/n$, which is the distribution of dergee and threshold of a random vertex in $\mathcal{G}(n, \mathbf{d}_n)$. Moreover, let (D, Θ) be random variables (over nonnegative integers) with joint distribution $\mathbb{P}(D = d, \Theta = \theta) = p(d, \theta)$. Let

$$\lambda := \mathbb{E}[D] = \sum_{d=0}^{\infty} \sum_{\theta=0}^{\infty} d\, p(d, \theta).$$

Then Condition 1.1 can be rewritten as $(D_n, \Theta_n) \xrightarrow{d} (D, \Theta)$ as $n \to \infty$ and $\mathbb{E}[A^{D_n}] = O(1)$ for each $A > 1$, which in particular implies the uniform integrability of D_n, so (as $n \to \infty$)

$$\frac{\sum_{i=1}^{n} d_{n,i}}{n} = \mathbb{E}[D_n] \longrightarrow \mathbb{E}[D] = \lambda \in (0, \infty).$$

Similarly, all higher moments converge.

1.2 Diffusion Process in $\mathcal{G}(n, \mathbf{d}_n)$

The aim of this section is to write the activation process described above as a diffusion process. We start with the graph $\mathcal{G}(n, \mathbf{d}_n)$, and then remove (here the removal of a vertex is the same as activation described above) the initially activated vertex with threshold zero, say $\mathcal{A}_0$. Now the degree of each vertices in the graph induced by $[n] \setminus \mathcal{A}_0$ is less than or equal to the degree of that vertex in $\mathcal{G}(n, \mathbf{d}_n)$. We denote the degree of vertex i at time t in the evolving graph by $d_i^B(t)$ (here the time t is non-negative integer valued, later we will introduce a continuous time). In this process, we remove the vertex i at time t if $d_i^B(t) \leq d_i - \theta_i$ (if there are more than one such i, then we we select one arbitrarily). Note that at the end of this procedure all the vertices that are removed will be the set of active vertices, and the rest will remain inactive.

Let us describe the process by assigning a type A (active) or B (inactive) to each of these vertices. To begin with (at time $t = 0$), the initially activated vertices (with threshold zero) are said to be of type A, and all vertices not in the initially activated vertices are of type B. At time $t > 0$, a vertex is said to be of type A if $d_i^B(t) \leq d_i - \theta_i$, otherwise, we call it of type B. In particular at the beginning of the diffusion process, all vertices in initially activated vertices are of type A, and all vertices not in the initially activated vertices are by definition of type B. As we proceed with the algorithm, $d_i^B(t)$ might decrease and a type B vertices may become of type A. In terms of edges, a half-edge is of type A or B when its endpoint is. As long as there is any half-edge of type A, we choose one such half-edge uniformly at random and remove the edge it belongs to. Note that it may result in changing the other endpoint from B to A (by decreasing $d_i^B(t)$) and thus create new half-edges of type A. When there are no half-edges of type A left, we stop the process and the final set of activated vertices is the set of vertices of type A (which are all isolated).

The next step is to turn this process into a balls and bins problem. In this step we simultaneously run the activation process described above and construct the configuration model. We call a type A (or B) vertex as type A (or B) bins and similarly consider the half-edges as balls of type A (or B) if its end point is type A (or B). At each step, we remove first one random ball from the set of balls in A-bins and then another ball without restriction. We stop when there are no non-empty A-bins. Therefore we alternately remove an A-ball and a random ball. We may just as well say that we first remove a random A-ball. We then remove

balls in pairs, first a ball without restriction, and then a random A-ball, and stop with the ball, leaving no A-ball to remove.

The next step is to run the above deletion process in continuous time. Each ball has an exponentially distributed random lifetime with mean 1, independent of the other balls. In other words balls die and are removed at rate 1, independent of each other. Also when a ball dies we remove a random A ball (in terms of configuration model these two half-edges form an edge). We stop when we should remove an A ball but there is no such a ball. Let $H_A(t)$ and $H_B(t)$ denote the number of A balls and B balls respectively, at time t. Note that these parameters depend on n, but we will omit the subscript as it is clear from the context. We also let $A_n(t)$ and $B_n(t)$ be the number of A bins and B bins at time t.

Let τ_n be the stopping time of the above diffusion process. Note that there are no A balls left at τ_n. However we pretend that we delete a (nonexistent) A ball at τ_n-th step and denote $H_A(\tau) = -1$. Therefore the stopping time τ_n may be characterized by $H_A(\tau) = -1$, and $H_A(t) \geq 0$ for all $0 \leq t < \tau_n$. Denoting by $\mathcal{A}_n^*$ the final set of activated vertices, we observe that $|\mathcal{A}_n^*| = n - B_n(\tau_n)$. Next we consider the balls only. The total number of balls at time t is $H_A(t) + H_B(t)$. In the evolution of this process each ball dies with rate 1 and another ball is removed upon its death. Therefore, $H_A(t) + H_B(t)$ is a death process with rate 2.

1.3 Main Theorems

The main result of this paper provides a central limit theorem for the final size of activated vertices. Namely, when the degree and threshold sequences satisfy Condition 1.1, we show that the final size of activated vertices $|\mathcal{A}_n^*|$ has asymptotically Gaussian fluctuations.

Let

$$b(d, z, \ell) := \mathbb{P}(\mathrm{Bin}(d, z) = \ell) = \binom{d}{\ell} z^\ell (1 - z)^{d-\ell},$$

$$\beta(d, z, \ell) := \mathbb{P}(\mathrm{Bin}(d, z) \geq \ell) = \sum_{r=\ell}^{d} \binom{d}{r} z^r (1 - z)^{d-r},$$

and $\mathrm{Bin}(d, z)$ denotes the binomial distribution with parameters d and z.

For the following theorems, we will use the notation

$$\widehat{a}_n(t) := 1 - \frac{1}{n} \sum_{d=0}^{\infty} \sum_{\theta=1}^{d} u_n(d, \theta) \beta(d, e^{-t}, d - \theta + 1),$$

where $u_n(d, \theta)$ is the number of vertices with degree d and threshold θ, and

$$\widehat{a}(t) := 1 - \sum_{d=0}^{\infty} \sum_{\theta=1}^{d} p(d, \theta) \beta(d, e^{-t}, d - \theta + 1).$$

Theorem 1.3 *Assume that Condition 1.1 holds. Let $\tau_n' \leq \tau_n$ be a stopping time such that $\tau_n' \xrightarrow{P} t_0$ for some $t_0 \geq 0$. Then in $\mathcal{D}[0, \infty)$, as $n \to \infty$,*

$$n^{-1/2} \big(A_n(t \wedge \tau_n') - n\widehat{a}_n(t \wedge \tau_n') \big) \xrightarrow{d} Z_A(t \wedge t_0),$$

for all $t \leq t_0$, *where* Z_A *is a continuous Gaussian process on* $[0, t_0]$ *with mean* 0 *and variance*

$$\sigma_A^2(t) := \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta+1)t}$$

$$\times \left(\Delta_{d,\theta,d-\theta+1}(t) + \sum_{\ell=d-\theta+2}^{\infty} (d-\theta+1) \binom{\ell-1}{d-\theta+1} \right.$$

$$\left. \int_0^t (e^{-s} - e^{-t})^{\ell-(d-\theta+2)} e^{-s} \Delta_{d,\theta,\ell}(s) \, ds \right),$$

where

$$\Delta_{d,\theta,\ell}(t) = p(d,\theta) \left(1 - e^{2\ell t} \beta(d, e^{-t}, \ell) + 2\ell \int_0^t e^{2\ell s} \beta(d, e^{-s}, \ell) \, ds \right).$$

The proof of above theorem is provided in Sect. 3.1.

Let us define

$$h_B(z) := \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} p(d,\theta) \sum_{\ell=d-\theta+1}^{d} \ell b(d, z, \ell) \text{ and } h_A(z) := \lambda z^2 - h_B(z),$$

which also gives us the plausible candidate for the limit of our stopping time

$$\widehat{z} := \sup\{z \in [0, 1] : h_A(z) = 0\}.$$

Note that the solution $\widehat{z}$ always exists since $h_A(0) = 0$. Also, $h_A(z)$ is continuous in z. We will further assume that if $\widehat{z} \neq 0$, then it is not a local minimum of $h_A(z)$, i.e., $\alpha := h'_A(\widehat{z}) > 0$.

Lemma 1.4 *Let* τ_n *be the stopping time of the diffusion process such that* $H_A(\tau_n) = -1$ *for the first time. Then* $\tau_n \xrightarrow{p} -\ln \widehat{z}$ *as* $n \to \infty$, *where* $\xrightarrow{p}$ *denotes the convergence in probability.*

The proof of lemma is provided in Sect. 3.3.

We are now interested in the final number of activated vertices (bins) which is given by $\lim_{t\to\infty} A_n(t \wedge \tau_n)$. Note that from Theorem 1.3 with the stopping time τ_n (as in Lemma 1.4), we have for $t \geq -\ln \widehat{z}$,

$$n^{-1/2} (A_n(t \wedge \tau_n) - n\widehat{a}_n(t \wedge \tau_n)) \to Z_A(t \wedge -\ln \widehat{z}).$$

Therefore for $t \geq \ln \widehat{z}$,

$$n^{-1/2} (A_n(t \wedge \tau_n) - n\widehat{a}_n(t \wedge \tau_n)) \xrightarrow{d} Z_A(-\ln \widehat{z}).$$

The following theorem provides a central limit theorem for the final size of activated vertices. Let us denote by

$$h_A^n(z) := \frac{\sum_{i=1}^n d_{n,i}}{n} z^2 - \frac{1}{n} \sum_{d=1}^{\infty} \sum_{\theta=1}^{\infty} \sum_{\ell=d-\theta+1}^{d} \ell u_n(d, \theta) b(d, z, \ell),$$

and let $\widehat{z}_n$ be the largest $z \in [0, 1]$ such that $h_A^n(z) = 0$.

Theorem 1.5 *Assume Condition* 1.1. *Let* $\widehat{\tau} = -\ln \widehat{z}$ *and* $\widehat{\tau}_n = -\ln \widehat{z}_n$. *We have:*

- *If* $\widehat{z} = 0$ *then asymptotically almost all nodes become active and* $|\mathcal{A}_n^*| = n - o_p(n)$.

- *If $\widehat{z} \neq 0$ and $\widehat{z}$ is a stable solution, i.e. $\alpha := h'_A(\widehat{z}) > 0$, then*

$$n^{-1/2}\left(|\mathcal{A}_n^*| - n\widehat{a}_n(\widehat{\tau}_n)\right) \overset{d}{\longrightarrow} Z_{Act},$$

where $Z_{Act} = \frac{\widehat{a}'(\widehat{\tau})}{\alpha} Z_{HA}(\widehat{\tau}) + Z_A(\widehat{\tau})$, where $(Z_{HA}(\widehat{\tau}), Z_A(\widehat{\tau}))$ jointly follow a Gaussian distribution that is described in Proposition 3.3.

The proof of above theorem is given in Sect. 3.4, where we first derive a joint functional central limit theorem for the processes $(A_n(t), H_A(t))$, from which we derive the theorem.

1.4 Relation to Bootstrap Percolation and *k*-Core

We now discuss our results with respect to related literature.

The diffusion model we consider in this paper can be seen as a generalization of bootstrap percolation and k-core in any graph $G = (V, E)$.

In the case of equal thresholds ($\theta_i = \theta$ over all vertices), our model is equivalent to bootstrap percolation (with deterministic initial activation). This process was introduced by Chalupa, Leath and Reich [13] in 1979 as a simplified model of some magnetic disordered systems. A short survey regarding applications of bootstrap percolation processes can be found in [1]. Recently, bootstrap percolation has been studied on varieties of random graphs models, see e.g., [22] for random graph $G_{n,p}$, [5, 6, 18] for inhomogeneous random graphs and [2, 9, 26] for the configuration model; see also [12, 15, 28, 29, 34].

The k-core of a graph G is the largest induced subgraph of G with minimum vertex degree at least k. The k-core of an arbitrary finite graph can be found by removing vertices of degree less than k, in an arbitrary order, until no such vertices exist. By setting the threshold of vertex i as $\theta_i = (d_i - k + 1)_+ = \max\{d_i - k + 1, 0\}$, we find that $B_n(\tau_n)$ will be the size of k-core in the random graph $G(n, \mathbf{d}_n)$.

The questions concerning the existence, size and structure of the k-core in random graphs, have attracted a lot of attention over the last few decades, see e.g., [30, 37] for random graph $G_{n,p}$, [10, 38] for inhomogeneous random graphs and [14, 17, 20, 21, 31] for the configuration model.

In particular, more closely related to our paper, [21] analyze the asymptotic normality of the k-core for sparse random graph $G_{n,p}$ and for configuration model. We continue on the same line as [21] and generalize partly their results by allowing different threshold levels to each of vertices. Our proof technique is also inspired by [21]. In particular, we look at the spread of activation (or infection) and constructing the configuration model simultaneously. Then we express the number of inactive vertices at a particular time point in terms of a martingale. After that we appeal to a martingale limit theorem from [19] to derive the limiting distribution.

Notation We let $\mathbb{N}$ be the set of nonnegative integers. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathbb{P})$. If $c \in \mathbb{R}$ is a constant, we write $X_n \overset{p}{\longrightarrow} c$ to denote that X_n converges in probability to c. That is, for any $\epsilon > 0$, we have $\mathbb{P}(|X_n - c| > \epsilon) \to 0$ as $n \to \infty$. We write $X_n = o_p(a_n)$, if $|X_n|/a_n$ converges to 0 in probability. We use $\overset{d}{\longrightarrow}$ for convergence in distribution. If $\mathcal{E}_n$ is a measurable subset of Ω, for any $n \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{U}}$ occurs with high probability (**w.h.p.**) if $\mathbb{P}(\mathcal{E}_n) = 1 - o(1)$, as $n \to \infty$. Also, we denote by $\mathsf{Bin}(k, p)$ a binomial distribution corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success p. We will suppress the dependence of parameters on the size of the network n, if it is clear from the context. We use the notation $\mathbf{1}\{\mathcal{E}\}$ for the indicator

of an event $\mathcal{E}$ which is 1 if $\mathcal{E}$ holds and 0 otherwise. We let $\mathcal{D}[0, \infty)$ be the standard space of right-continuous functions with left limits on $[0, \infty)$ equipped with the Skorohod topology (see e.g. [19, 23])

2 Preliminaries

In this section we provide some preliminary lemmas that will be used in our proofs.

2.1 Some Death Process Lemmas

Consider a pure death process with rate 1. This process starts with some number of balls whose lifetimes are i.i.d. rate 1 exponentials.

Lemma 2.1 (Death Process Lemma) *Let $N_n(t)$ be the number of balls alive at time t in a rate* 1 *death process with $N_n(0) = n$. Then*

$$\sup_{t \geq 0} |N_n(t)/n - e^{-t}| \xrightarrow{p} 0 \text{ as } n \to \infty.$$

Proof $1 - N_n(t)/n$ is the empirical distribution function of the n lifetimes, which are i.i.d. random variables with the distribution function $1 - e^{-t}$. Therefore the result follows using Glivenko-Cantelli theorem (see e.g. [23, Proposition 4.24]). □

Lemma 2.2 (Number of Balls Centrality Lemma) *The number of balls $H_A(t) + H_B(t)$ follow a pure death process, and*

$$\sup_{0 \leq t \leq \tau_n} |H_A(t) + H_B(t) - n\lambda e^{-2t}| = o_p(n).$$

Proof In the evolution of this process each ball dies with rate 1 and another ball is removed upon its death. Therefore $A(t) + B(t)$ is a death process with rate 2. Therefore the lemma follows using Lemma 2.1. □

2.2 Martingale Limit Theorems

We recall some martingale theory that are going to be useful in proving Theorem 1.3. Let X be a martingale defined on $[0, \infty)$, we denote its quadratic variation of X by $[X, X]_t$, and the bilinear extension of quadratic variation to two martingales X and Y by $[X, Y]_t$. If X and Y be two martingales with path-wise finite variation, then

$$[X, Y]_t := \sum_{0 < s \leq t} \Delta X(s) \Delta Y(s), \tag{2}$$

where $\Delta X(s) := X(s) - X(s-)$ is the jump of X at s. Similarly, $\Delta Y(s) := Y(s) - Y(s-)$. In our context there will only be countable number of jumps for the martingales under consideration, and the sum in Eq. (2) will be finite. We will assume $[X, Y]_0 = 0$. For vector-valued martingales $X = (X_i)_{i=1}^m$ and $Y = (Y_i)_{i=1}^n$, we define the square bracket $[X, Y]$ to be the matrix $([X_i, Y_j])_{i,j}$. A real-valued martingale $X(s)$ on $[0, t]$ is an L^2 if and only if $\mathbb{E}[X, X]_t < \infty$ and $\mathbb{E}|X(0)|^2 < \infty$, and then $\mathbb{E}|X(t)|^2 = \mathbb{E}[X, X]_t + \mathbb{E}|X(0)|^2$. We will use the following martingale limit theorem from [19], see also [21, Proposition 4.1].

Proposition 2.3 *For each $n \geq 1$, let $M_n(t) = (M_{ni}(t))_{i=0}^q$ be a q-dimensional martingale on $[0, \infty)$ and $M_n(0) = 0$. Also $\Sigma(t)$ be a continuous positive semi-definite function such that for every fixed $t \geq 0$*

$$[M_n, M_n]_t \xrightarrow{p} \Sigma(t) \ as \ n \to \infty,$$
$$\sup_n \mathbb{E}[M_{ni}, M_{ni}]_t < \infty, \ i = 1, \ldots, q.$$

Then $M_n \xrightarrow{d} M$, in $\mathcal{D}[0, \infty)$ where M is continuous q-dimensional Gaussian martingale with $\mathbb{E}M(t) = 0$ and covariances $\mathrm{Cov}(M(t)) = \Sigma(t)$.

In the next section, we will apply Proposition 2.3 to stopped processes.

3 Proofs

In this section we present the proofs of Theorem 1.3, Lemma 1.4 and Theorem 3.4.

Remark 3.1 In the proof of Theorem 1.3, we always consider the processes up to time $\tau_n' \leq \tau_n$. Although, sometimes it will be possible to extend the process (for example by removing non-existent balls), that will not be relevant for us, and thus we will always stop the process at τ_n'.

3.1 Proof of Theorem 1.3

We denote the number of (alive) balls at time t by $W_n(t)$. Clearly $W_n(t) = H_A(t) + H_B(t)$. Let $m_n := \frac{1}{2} \sum_{i=1}^n d_{n,i}$ denotes the total number of edges in $\mathcal{G}(n, \mathbf{d}_n)$. In our construction $W_n(0) = 2m_n - 1$, and W_n decreases by 2 each time a ball dies. The death happens with rate 1, and therefore $W_n(t) + \int_0^t 2W_n(s)\, ds$ is a martingale on $[0, \tau_n']$. In differential form

$$dW_n(t) = -2W_n(t)\, dt + d\mathcal{M}(t), \tag{3}$$

where $\mathcal{M}$ is a martingale. Now by Ito's lemma and (3),

$$d(e^{2t} W_n(t)) = e^{2t}\, dW_n(t) + 2e^{2t} W_n(t)\, dt = e^{2t}\, d\mathcal{M}(t),$$

which implies that $\widehat{W}_n(t) := e^{2t} W_n(t)$ is another martingale. Note that distinct balls die at distinct time with probability one, also all jumps in $W_n(t)$ equals -2. The quadratic variation of $\widehat{W}_n(t)$ is given by

$$
\begin{aligned}
[\widehat{W}_n, \widehat{W}_n]_t &= \sum_{0 < s \leq t} |\Delta \widehat{W}_n(s)|^2 = \sum_{0 < s \leq t} \left(e^{2s} |\Delta W_n(s)|\right)^2 \\
&= \sum_{0 < s \leq t} e^{4s} (W_n(s-) - W_n(s))(W_n(s-) - W_n(s)) \\
&= \sum_{0 < s \leq t} 2e^{4s} (W_n(s-) - W_n(s)) = \int_0^t 2e^{4s}\, d(-W_n(s)) \\
&= -2e^{4t} W_n(t) + 2W_n(0) + \int_0^t 8e^{4s} W_n(s)\, ds. \tag{4}
\end{aligned}
$$

Using the fact that $W_n(t)$ is a decreasing function in t and $W_n(0) = 2m_n - 1$ we get

$$[\widehat{W}_n, \widehat{W}_n]_t \le 2e^{4t} \int_0^t d(-W_n(s)) \le 4m_n e^{4t}.$$

Let the stopped martingale on $[0, \infty)$ be $W_n^*(t) := \frac{1}{n}\widehat{W}(t \wedge \tau_n')$. Then W_n^* is a martingale on $[0, \infty)$ and, for every $T < \infty$, the quadratic variation of this martingale is

$$[W_n^*, W_n^*]_T = \frac{1}{n^2}[W_n, W_n]_{T \wedge \tau_n'} \le \frac{4e^{4T}m_n}{n^2} \longrightarrow 0, \tag{5}$$

as n goes to infinity. Also $0 \le W_n^*(t) \le 2m_n/n = O(1)$. Therefore by Proposition 2.3 on $\mathcal{D}[0, \infty)$, $W_n^*(t) - W_n^*(0) \xrightarrow{p} 0$ uniformly on $[0, T]$, and as $n \to \infty$,

$$n^{-1} \sup_{0 \le t \le T \wedge \tau_n'} |W_n(t) - W_n(0)e^{-2t}| \xrightarrow{p} 0. \tag{6}$$

Now using (4), (5) and (6) we get for every $t \in [0, T \wedge \tau_n']$,

$$\begin{aligned}[\widehat{W}_n, \widehat{W}_n]_t &= -2e^{4t}W_n(0)e^{-2t} + 2W_n(0) + \int_0^t 8e^{4s}W_n(0)e^{-2s}\, ds + o_p(n) \\ &= 2W_n(0)(e^{2t} - 1) + o_p(n) = 2(2m_n - 1)(e^{2t} - 1) + o_p(n) \\ &= 2\lambda n(e^{2t} - 1) + o_p(n).\end{aligned} \tag{7}$$

Note that in the last display, (6) ensures that we can use the approximation $W_n(t) = W_n(0)e^{-2t} + o_p(n)$ on $[0, T \wedge \tau_n']$. Let $W_i(t)$ be the number of balls for vertex (bin) i and denote by (for $d, \theta, \ell \in \mathbb{N}$)

$$\mathcal{B}_{d,\theta,\ell}(t) = \{i \in [n] : d_i = d, \theta_i = \theta, W_i(t) = \ell\},$$

the set of vertices (bins) with degree d, threshold θ and ℓ balls at time t. Let $B_{d,\theta,\ell}(t) = |\mathcal{B}_{d,\theta,\ell}(t)|$. Therefore, the total number of (inactive) B bins at time t is given by

$$B_n(t) = \sum_{d=0}^{\infty}\sum_{\theta=1}^{\infty}\sum_{\ell=d-\theta+1}^{d} B_{d,\theta,\ell}(t) = \sum_{d=0}^{\infty}\sum_{\theta=1}^{\infty} \widetilde{B}_{d,\theta,d-\theta+1}(t),$$

where $\widetilde{B}_{d,\theta,\ell} = \sum_{r=\ell}^{d} B_{d,\theta,r}(t)$ denotes the number of (inactive) B bins at time t with initial degree d and threshold θ with at least ℓ balls. In the rest of the proof we will derive a central limit theorem for the quantity $B_n(t)$, which is the number of inactive vertices at time t. This will immediately give us our desired result since $A_n(t) + B_n(t) = n$.

Note that $\widetilde{B}_{d,\theta,\ell}$ decreases by one only when a ball dies in an uninfected (inactive) bin with initially d balls and threshold θ that has exactly ℓ balls, and there are precisely $\ell B_{d,\theta,\ell}$ many such balls, therefore

$$d\widetilde{B}_{d,\theta,\ell}(t) = -\ell B_{d,\theta,\ell}(t)\, dt + d\mathcal{M}(t), \tag{8}$$

where $\mathcal{M}$ is a martingale.

Let us now define the following quantity

$$\widehat{B}_{d,\theta,\ell}(t) := e^{\ell t}\widetilde{B}_{d,\theta,\ell}(t), \tag{9}$$

which gives

$$d\widehat{B}_{d,\theta,\ell}(t) = \ell e^{\ell t}\widetilde{B}_{d,\theta,\ell}(t)\, dt + e^{\ell t}\, d\widetilde{B}_{d,\theta,\ell}(t).$$

Plugging in (8) we get

$$
\begin{aligned}
d\widehat{B}_{d,\theta,\ell}(t) &= \ell e^{\ell t}\widetilde{B}_{d,\theta,\ell}(t)\,dt - \ell e^{\ell t}B_{d,\theta,\ell}(t)\,dt + e^{\ell t}\,d\mathcal{M}(t)\\
&= \ell e^{\ell t}\widetilde{B}_{d,\theta,\ell+1}(t)\,dt + e^{\ell t}\,d\mathcal{M}(t)\\
&= \ell e^{-t}\widehat{B}_{d,\theta,\ell+1}(t)\,dt + d\mathcal{M}'(t),
\end{aligned}
$$

where $d\mathcal{M}'(t) = e^{\ell t}\,d\mathcal{M}(t)$. Since this is yet another martingale differential, we can define the following martingale for every fixed $d \geq \ell \geq 0$

$$
M_{d,\theta,\ell}(t) := \widehat{B}_{d,\theta,\ell}(t) - \ell \int_0^t e^{-s}\widehat{B}_{d,\theta,\ell+1}(s)\,ds. \tag{10}
$$

The quadratic variation will be same as that of $\widehat{B}_{d,\theta,\ell}(t)$, i.e.,

$$
\begin{aligned}
[M_{d,\theta,\ell}, M_{d,\theta,\ell}]_t &= \sum_{0<s\leq t}|\Delta M_{d,\theta,\ell}(s)|^2 = \sum_{0<s\leq t}|\Delta\widehat{B}_{d,\theta,\ell}(s)|^2\\
&= \sum_{0<s\leq t} e^{2\ell s}|\Delta\widetilde{B}_{d,\theta,\ell}(s)|^2\\
&= \int_0^t e^{2\ell s}\,d(-\widetilde{B}_{d,\theta,\ell}(s)). \tag{11}
\end{aligned}
$$

Let us now define the centered version of $M_{d,\theta,\ell}$ as follows

$$
\widetilde{M}_{d,\theta,\ell}(t) := n^{-1/2}(M_{d,\theta,\ell}(t) - M_{d,\theta,\ell}(0)). \tag{12}
$$

This is of course the centered martingale where for $\ell \leq d$,

$$
M_{d,\theta,\ell}(0) = \widetilde{B}_{d,\theta,\ell}(0) = \sum_{r=\ell}^{\infty}\sum_{i=1}^{n}\mathbf{1}\{i \in \mathcal{B}_{d,\theta,r}(0)\} = \sum_{i=1}^{n}\mathbf{1}\{i \in \mathcal{B}_{d,\theta,d}(0)\}.
$$

The quadratic variation of $\widetilde{M}_{d,\theta,\ell}(t)$ can be calculated using integration by parts as follows

$$
\begin{aligned}
[\widetilde{M}_{d,\theta,\ell}, \widetilde{M}_{d,\theta,\ell}]_t &= \frac{1}{n}[M_{d,\theta,\ell}, M_{d,\theta,\ell}]_t\\
&= \frac{1}{n}\left(\widetilde{B}_{d,\theta,\ell}(0) - e^{2\ell t}\widetilde{B}_{d,\theta,\ell}(t) + 2\ell\int_0^t e^{2\ell s}\widetilde{B}_{d,\theta,\ell}(s)\,ds\right).
\end{aligned}
$$

Recall from (9) that $\widehat{B}_{d,\theta,\ell}(t) := e^{\ell t}\widetilde{B}_{d,\theta,\ell}(t)$, where $\widetilde{B}_{d,\theta,\ell}(t)$ is the number of (uninfected) B bins with at least ℓ balls at time t with initial number of balls (degree) d and threshold θ. Also, recall that balls die independently with rate 1. Let us denote

$$
\mathcal{U}_n(d,\theta) = \{i \in [n] : d_{n,i} = d, \theta_{n,i} = \theta\},
$$

so that $u_n(d,\theta) = |\mathcal{U}_n(d,\theta)|$. This gives also $u_n(d,\theta) = \sum_\ell B_{d,\theta,\ell}(0)$.

Recall that $B_{d,\theta,\ell}(t) = |\mathcal{B}_{d,\theta,\ell}(t)|$ where $\mathcal{B}_{d,\theta,\ell}(t) = \{i \in [n] : d_i = d, \theta_i = \theta, W_i(t) = \ell\}$. Since each ball dies with rate one independent of each other and survival probability of a ball after time t is equal to e^{-t}. A bin from $\mathcal{U}_n(d,\theta)$ has at least ℓ balls at time t with probability

$$
\beta(d, e^{-t}, \ell) = \sum_{r=\ell}^{d}\binom{d}{r}(e^{-t})^r(1 - e^{-t})^{d-r}.
$$

Hence we get

$$\mathbb{E}[\widehat{B}_{d,\theta,\ell}(t)] = e^{\ell t}\mathbb{E}[\widetilde{B}_{d,\theta,\ell}(t)] = e^{\ell t}\sum_{v\in\mathcal{U}_n(d,\theta)}\mathbb{P}(\text{out of } d \text{ at least } \ell \text{ balls survive}),$$

which gives

$$\widehat{b}_{d,\theta,\ell}(t) = \mathbb{E}[\widehat{B}_{d,\theta,\ell}(t)] = e^{\ell t}u_n(d,\theta)\beta(d,e^{-t},\ell).$$

Now note that (for $\theta \geq 1$)

$$\widetilde{B}_{d,\theta,\ell}(0) = u_n(d,\theta) \quad\text{and}\quad \mathbb{E}\big[\widetilde{B}_{d,\theta,\ell}(t)\big] = u_n(d,\theta)\beta(d,e^{-t},\ell).$$

Hence, by using Condition 1.1 and Glivenko-Cantelli's lemma (since each bins are independent), we get

$$\sup_{0\leq t\leq\tau_n'}\left|\frac{1}{n}\widetilde{B}_{d,\theta,\ell}(t) - p(d,\theta)\beta(d,e^{-t},\ell)\right| \xrightarrow{p} 0, \tag{13}$$

as $n \to \infty$. Therefore, the quadratic variation of $\widetilde{M}_{d,\theta,\ell}(t)$ satisfies (for $\theta \geq 1$ and $\ell \leq d$) (see also Eq. 31)

$$[\widetilde{M}_{d,\theta,\ell},\widetilde{M}_{d,\theta,\ell}]_t := p(d,\theta)\left(1 - e^{2\ell t}\beta(d,e^{-t},\ell) + 2\ell\int_0^t e^{2\ell s}\beta(d,e^{-s},\ell)\,ds\right) + o_p(1)$$

$$= \Delta_{d,\theta,\ell}(t) + o_p(1).$$

We will also use the following crude estimate. Using (11) we get that

$$[\widetilde{M}_{d,\theta,\ell},\widetilde{M}_{d,\theta,\ell}]_t = \frac{1}{n}[M_{d,\theta,\ell},M_{d,\theta,\ell}]_t \leq \frac{1}{n}e^{2\ell t}\widetilde{B}_{d,\theta,\ell}(0) = \frac{u_n(d,\theta)}{n}e^{2\ell t}. \tag{14}$$

Therefore we can apply Proposition 2.3 to the stopped process at τ_n'

$$\widetilde{M}_{d,\theta,\ell}(t\wedge\tau_n') \xrightarrow{d} Z_{d,\theta,\ell}(t\wedge t_0) \text{ in } D[0,\infty), \tag{15}$$

where $Z_{d,\theta,\ell}$ is a Gaussian process with $\mathbb{E}Z_{d,\theta,\ell}(t) = 0$ and covariance $\Delta_{d,\theta,\ell}(t)$.

Also note that $\widetilde{B}_{d,\theta,\ell}(t)$ and $\widetilde{B}_{d',\theta',\ell'}(t)$ can not change together almost surely for $(d,\theta,\ell) \neq (d',\theta',\ell')$, therefore $[\widetilde{M}_{d,\theta,\ell},\widetilde{M}_{d',\theta',\ell'}] = 0$. Thus for a finite set S, $\big(\widetilde{M}_{d,\theta,\ell}\big)_{(d,\theta,\ell)\in S}$ converges to $(Z_{d,\theta,\ell})_{(d,\theta,\ell)\in S}$ in distribution, and $(Z_{d,\theta,\ell})$ are independent.

Let us now express $\widehat{B}_{d,\theta,\ell}$ in terms of $M_{d,\theta,\ell}$ so that we can apply the limit theorems for $\widetilde{M}_{d,\theta,\ell}$'s to get limit theorems for $\widehat{B}_{d,\theta,\ell}$.

Using (10) one can write

$$\widehat{B}_{d,\theta,\ell}(t) = M_{d,\theta,\ell}(t) + \sum_{r=\ell+1}^{d}\ell\binom{r-1}{\ell}\int_0^t (e^{-s}-e^{-t})^{r-\ell-1}e^{-s}M_{d,\theta,r}(s)\,ds. \tag{16}$$

To see how to get (16) from (10), note that

$$\widehat{B}_{d,\theta,\ell}(t) := M_{d,\theta,\ell}(t) + \ell\int_0^t e^{-s}\widehat{B}_{d,\theta,\ell+1}(s)\,ds$$

$$= M_{d,\theta,\ell}(t) + \ell\int_0^t e^{-s}M_{d,\theta,\ell+1}(s)\,ds$$

$$+ (\ell+1)\ell\int_{0<s_1<s<t} e^{-s}e^{-s_1}\widehat{B}_{d,\theta,\ell+2}(s_1)\,ds_1\,ds.$$

Proceeding like this by repeatedly using (10), we can obtain (16). Since $M_{d,\theta,\ell}$ is a martingale $\mathbb{E}[M_{d,\theta,\ell}(t)] = M_{d,\theta,\ell}(0)$, and using (16) we get $\mathbb{E}[\widehat{B}_{d,\theta,\ell}(t)] = \widehat{b}_{d,\theta,\ell}(t)$, where we define for $t \geq 0$,

$$\widehat{b}_{d,\theta,\ell}(t) = M_{d,\theta,\ell}(0) + \sum_{r=\ell+1}^{d} \ell \binom{r-1}{\ell} \int_0^t (e^{-s} - e^{-t})^{r-\ell-1} e^{-s} M_{d,\theta,\ell}(0)\, ds. \quad (17)$$

Using (16), (17) and (12), it turns out that the centered version of $\widehat{B}_{d,\theta,\ell}(t)$ satisfies

$$\frac{1}{\sqrt{n}} \left(\widehat{B}_{d,\theta,\ell}(t) - \widehat{b}_{d,\theta,\ell}(t)\right)$$

$$= \frac{1}{\sqrt{n}} \left(M_{d,\theta,\ell}(t) - M_{d,\theta,\ell}(0)\right.$$

$$\left. + \sum_{r=\ell+1}^{\infty} \ell \binom{r-1}{\ell} \int_0^t (e^{-s} - e^{-t})^{r-\ell-1} e^{-s} \left(M_{d,\theta,\ell}(s) - M_{d,\theta,\ell}(0)\right) ds \right)$$

$$= \widetilde{M}_{d,\theta,\ell}(t) + \sum_{r=\ell+1}^{\infty} \ell \binom{r-1}{\ell} \int_0^t (e^{-s} - e^{-t})^{r-\ell-1} e^{-s} \widetilde{M}_{d,\theta,\ell}(s)\, ds. \quad (18)$$

Now using (14) we get (for $\ell \leq d$ and for any $T \geq 0$):

$$\mathbb{E}\left[\widetilde{M}_{d,\theta,\ell}, \widetilde{M}_{d,\theta,\ell}\right]_T \leq e^{2\ell T} \frac{u_n(d,\theta)}{n} \leq \frac{1}{n} e^{2\ell T} \sum_{d \geq \ell} \sum_{\theta} u_n(d,\theta).$$

Then using Condition 1.1 and the simple fact that for $A > 1$,

$$A^\ell \sum_{d \geq \ell} \sum_{\theta} u_n(d,\theta) \leq \sum_{d \geq \ell} \sum_{\theta} u_n(d,\theta) A^d \leq \sum_{d} \sum_{\theta} u_n(d,\theta) A^d,$$

we get for any $T \geq 0$, by setting $A = e^{4T}$, there is a constant C_A such that

$$\mathbb{E}\left[\widetilde{M}_{d,\theta,\ell}, \widetilde{M}_{d,\theta,\ell}\right]_T \leq e^{2\ell T} C_A A^{-\ell} = C_A e^{-2\ell T}.$$

Now using Doob's L^2-inequality we get

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \widetilde{M}_{d,\theta,\ell}^2(t)\right) \leq 4\mathbb{E}\left[\widetilde{M}_{d,\theta,\ell}, \widetilde{M}_{d,\theta,\ell}\right]_T \leq 4C_A e^{-2\ell T}.$$

Therefore using Cauchy–Schwarz inequality,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |\widetilde{M}_{d,\theta,\ell}(t)|\right) \leq C' e^{-\ell T}. \quad (19)$$

Therefore by (15) and Fatou's lemma we get

$$\mathbb{E}\left(\sup_{0 \leq t \leq t_0} \left|Z_{d,\theta,\ell}(t)\right|\right) \leq C' e^{-\ell T}. \quad (20)$$

Let us define

$$R_{d,\theta,\ell}(t) := \sum_{r=\ell+1}^{\infty} \ell \binom{r-1}{\ell} \int_0^t (e^{-s} - e^{-t})^{r-\ell-1} e^{-s} \widetilde{M}_{d,\theta,r}(s)\, ds.$$

Then we have (since $(e^{-s} - e^{-t})^{r-\ell-1} \leq (1 - e^{-t})^{r-\ell-1}$ for $s \in [0, t]$)

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |R_{d,\theta,\ell}(t)|\right] \leq \sum_{r=\ell+1}^{\infty} \ell \binom{r-1}{\ell} \int_0^T (1-e^{-T})^{r-\ell-1} e^{-s} \mathbb{E}\left[\sup_{0 \leq t \leq T} |\widetilde{M}_{d,\theta,r}(t)|\right] ds.$$

Therefore (19) yields

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |R_{d,\theta,\ell}(t)|\right) \leq C' \sum_{r=\ell+1}^{\infty} \ell e^{-rT} \binom{r-1}{\ell}(1 - e^{-T})^{r-\ell}. \tag{21}$$

For fixed T, using the sum formula for negative binomial distribution with parameters $\ell + 1$ and probability of success $1 - e^{-T}(1 - e^{-T})$,

$$\sum_{r=\ell+1}^{\infty} \binom{r-1}{\ell} e^{-T(r-\ell-1)} (1 - e^{-T})^{r-\ell-1} = \left(1 - e^{-T}(1 - e^{-T})\right)^{-(\ell+1)},$$

which implies that (from (21))

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |R_{d,\theta,\ell}(t)|\right) \leq C'' \ell \left(\frac{e^{-T}}{1 - e^{-T}(1 - e^{-T})}\right)^{\ell+1}.$$

Since $1 - e^{-T}(1 - e^{-T}) > e^{-T}$ (from $(1 - e^{-T})^2 > 0$ for $T > 0$), RHS of (21) converges to zero uniformly in n, as $\ell \to \infty$. Now using (16), (17), (21) and applying [11, Theorem 4.2] we get

$$\frac{1}{\sqrt{n}}\left(\widehat{B}_{d,\theta,\ell}(t \wedge \tau_n') - \widehat{b}_{d,\theta,\ell}(t \wedge \tau_n')\right) \xrightarrow{d} \widetilde{Z}_{d,\theta,\ell}(t \wedge t_0),$$

in $\mathcal{D}[0, \infty)$ for each ℓ, where

$$\widetilde{Z}_{d,\theta,\ell}(t) := Z_{d,\theta,\ell}(t) + \sum_{r=\ell+1}^{\infty} \ell \binom{r-1}{\ell} \int_0^t (e^{-s} - e^{-t})^{r-\ell-1} e^{-s} Z_{d,\theta,r}(s) \, ds. \tag{22}$$

Again using (20) we obtain that the sum in (22) almost surely uniformly converges for $t \leq t_0$, and this gives $\widetilde{Z}_{d,\theta,\ell}$ is continuous for each (d, θ, ℓ). Using (9), we can write

$$\frac{e^{-\ell t}}{\sqrt{n}}\left(\widehat{B}_{d,\theta,\ell}(t) - \widehat{b}_{d,\theta,\ell}(t)\right) := \frac{1}{\sqrt{n}}\left(\widetilde{B}_{d,\theta,\ell}(t) - e^{-\ell t}\widehat{b}_{d,\theta,\ell}(t)\right).$$

Now recall that we are interested in the case $\ell = d - \theta + 1$. More particularly, we will derive a central limit theorem of the following quantity

$$
\frac{1}{\sqrt{n}} \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} \left(\widetilde{B}_{d,\theta,d-\theta+1}(t) - e^{-(d-\theta+1)t} \widehat{b}_{d,\theta,d-\theta+1}(t) \right)
$$

$$
= \frac{1}{\sqrt{n}} \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta+1)t} \left(\widehat{B}_{d,\theta,d-\theta+1}(t) - \widehat{b}_{d,\theta,d-\theta+1}(t) \right)
$$

$$
= \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta+1)t} \widetilde{M}_{d,\theta,d-\theta+1}(t)
$$

$$
+ \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta+1)t} \sum_{r=d-\theta+2}^{\infty} (d-\theta+1) \binom{r-1}{d-\theta+1}
$$

$$
\int_{0}^{t} (e^{-s} - e^{-t})^{r-(d-\theta+1)-1} e^{-s} \widetilde{M}_{d,\theta,r}(s) \, ds. \tag{23}
$$

Note that from (15) we know the limiting distribution of $\widetilde{M}_{d,\theta,d-\theta}(t)$, and therefore, it will be sufficient to show that the contribution from the tail of (23) is negligible. To be precise, let us define two terms (we ignore the contribution from $e^{-(d-\theta)t}$, which will always be bounded above by 1)

$$
\widetilde{R}_\ell(t) := \sum_{d=\ell}^{\infty} \sum_{\theta=1}^{d} \widetilde{M}_{d,\theta,d-\theta+1}(t),
$$

and

$$
\widehat{R}_\ell(t) := \sum_{d=\ell}^{\infty} \sum_{\theta=1}^{d} \sum_{r=d-\theta+2}^{d} (d-\theta+1) \binom{r-1}{d-\theta+1} \int_{0}^{t} (e^{-s} - e^{-t})^{r-(d-\theta+1)-1} e^{-s} \widetilde{M}_{d,\theta,r}(s) \, ds.
$$

In the following lemma we show that $\mathbb{E}\left[\sup_{0 \le t \le T} \left|\widetilde{R}_\ell(t)\right|\right]$, and $\mathbb{E}\left[\sup_{0 \le t \le T} \left|\widehat{R}_\ell(t)\right|\right]$ converges to zero as $\ell \to \infty$ uniformly in n.

Lemma 3.2 *We have* $\mathbb{E}\left[\sup_{0 \le t \le T} \left|\widetilde{R}_\ell(t)\right|\right] \to 0$ *and* $\mathbb{E}\left[\sup_{0 \le t \le T} \left|\widehat{R}_\ell(t)\right|\right] \to 0$, *as* $\ell \to \infty$, *uniformly in* n.

Proof From Condition 1.1 and the fact that for all $A > 1$, $A^d u_n(d, \theta) \le \sum_{d \ge \ell} u_n(d, \theta) A^d$ we find that for any $A > 1$ there is a constant C_A such that

$$
\mathbb{E}\left[\widetilde{M}_{d,\theta,\ell}, \widetilde{M}_{d,\theta,\ell}\right]_T \le e^{2\ell T} \frac{u_n(d, \theta)}{n}.
$$

Using Doob's L^2−inequality we get

$$
\mathbb{E}\left[\sup_{0 \le t \le T} \widetilde{M}_{d,\theta,\ell}^2(t)\right] \le 4\mathbb{E}\left[\widetilde{M}_{d,\theta,\ell}, \widetilde{M}_{d,\theta,\ell}\right]_T \le 4e^{2rT} \frac{u_n(d, \theta)}{n}. \tag{24}
$$

Therefore,

$$
\mathbb{E}\left[\sup_{0 \le t \le T} \left|\widetilde{R}_\ell(t)\right|\right] \le \sum_{d=\ell}^{\infty} \sum_{\theta=1}^{d} \mathbb{E}\left(\sup_{0 \le t \le T} \left|\widetilde{M}_{d,\theta,d-\theta}(t)\right|\right).
$$

Using Cauchy–Schwarz inequality, and (24)

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{R}_{N,n}(t)\right|\right] \leq 2\sum_{d=\ell}^{\infty}\sum_{\theta=1}^{d}e^{(d-\theta)T}\sqrt{\frac{u_n(d,\theta)}{n}} \leq \frac{2}{\sqrt{n}}\sum_{d\geq\ell}e^{dT}\sum_{\theta=1}^{d}\sqrt{u_n(d,\theta)}.$$

Using Condition 1.1, for all $A > 1$, $A^d u_n(d,\theta) = O(n)$. Therefore for any $A > 1$ there is a constant C_A such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{R}_{\ell}(t)\right|\right] \leq C_A\sum_{d\geq\ell}de^{dT}A^{-d/2}.$$

In particular, choosing $A = e^{4T}$, we see that $\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{R}_{\ell}(t)\right|\right]$ converges to zero as $\ell \to \infty$ uniformly in n.

Now let us define

$$\widehat{R}_{\ell}(t) := \sum_{d=\ell}^{\infty}\sum_{\theta=1}^{d}\sum_{r=d-\theta+1}^{d}(d-\theta)\binom{r-1}{d-\theta}\int_0^t(e^{-s}-e^{-t})^{r-(d-\theta)-1}e^{-s}\widetilde{M}_{d,\theta,r}(s)\,ds.$$

Observe that the index r is at most d, and therefore using Cauchy–Schwarz inequality, and (24)

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{M}_{d,\theta,r}(t)\right|\right] \leq 2e^{dT}\sqrt{\frac{u_n(d,\theta)}{n}}. \tag{25}$$

We can now compute the following tail bound

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\widehat{R}_{\ell}(t)\right|\right) \leq \sum_{d=\ell}^{\infty}\sum_{\theta=1}^{d}\sum_{r=d-\theta+1}^{d}(d-\theta)\binom{r-1}{d-\theta}(1-e^{-t})^{r-(d-\theta)}2e^{sT}\sqrt{\frac{u_n(d,\theta)}{n}}$$

Again we have $u_n(d,\theta) \leq C_A A^{-d}n$ for some constant $C_A > 0$. Plugging in $A = e^{8T}$ we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widehat{R}_{\ell}(t)\right|\right] \leq C_A\sum_{d=\ell}^{\infty}e^{-3dT}\sum_{\theta=1}^{d}\sum_{r=d-\theta+1}^{\infty}(d-\theta)\binom{r-1}{d-\theta}(1-e^{-T})^{r-(d-\theta)}. \tag{26}$$

We can write

$$\sum_{r=d-\theta+1}^{\infty}(d-\theta)\binom{r-1}{d-\theta}(1-e^{-t})^{r-(d-\theta)} = \sum_{r=0}^{\infty}r\binom{r+d-\theta-1}{r}(1-e^{-T})^r$$

$$= (d-\theta)e^{(d-\theta+1)T}(1-e^{-T}), \tag{27}$$

where in the second equality we used the expectation of negative binomial distribution. Now plugging this in (26) we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widehat{R}_{\ell}(t)\right|\right] \leq C_A\sum_{d\geq\ell}d^2e^{-2dT},$$

which again goes to zero uniformly in n as $\ell \to \infty$. This completes the proof of Lemma 3.2.

□

By using Lemma 3.2, (23) and [11, Theorem 4.2] we get

$$
\frac{1}{\sqrt{n}} \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} \left(\widetilde{B}_{d,\theta,d-\theta+1}(t \wedge \tau'_n) - e^{-(d-\theta+1)(t \wedge \tau'_n)} \widehat{b}_{d,\theta,d-\theta+1}(t \wedge \tau'_n) \right)
$$

$$
\xrightarrow{d} \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta+1)(t \wedge \tau'_n)} \widetilde{Z}_{d,\theta,d-\theta+1}(t \wedge t_0),
\tag{28}
$$

as $n \to \infty$, that is

$$
\frac{1}{\sqrt{n}} \left(B_n(t \wedge \tau'_n) - \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} u_n(d,\theta) \beta(d, e^{-t}, d-\theta+1) \right)
$$

$$
\xrightarrow{d} \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta+1)(t \wedge \tau'_n)} \widetilde{Z}_{d,\theta,d-\theta+1}(t \wedge t_0) =: Z_A(t \wedge t_0).
\tag{29}
$$

We are now done except for showing that $Z_A(t)$ is continuous. Using (18) and (25) we get

$$
\mathbb{E} \left[\sup_{0 \le t \le T} \frac{1}{\sqrt{n}} \left| \widehat{B}_{d,\theta,\ell}(t) - \widehat{b}_{d,\theta,\ell}(t) \right| \right]
$$

$$
\le 2 e^{dT} \sqrt{\frac{u_n(d,\theta)}{n}} + 2 \sum_{r=\ell+1}^{\infty} \ell \binom{r-1}{\ell} (1 - e^{-t})^{r-\ell} e^{dT} \sqrt{\frac{u_n(d,\theta)}{n}}.
$$

Again using $u_n(d,\theta) \le C_A A^{-d} n$ and writing the second term in terms of negative binomial distribution we get

$$
\mathbb{E} \left(\sup_{0 \le t \le T} \frac{1}{\sqrt{n}} \left| \widehat{B}_{d,\theta,\ell}(t) - \widehat{b}_{d,\theta,\ell}(t) \right| \right) \le C_A e^{dT} A^{-d/2} + C_A e^{dT} A^{-d/2} e^{\ell T + T} (1 - e^{-T}) \ell.
$$

Since $\ell \le d$, by choosing $A = e^{4T+2}$, we get

$$
\mathbb{E} \left[\sup_{0 \le t \le T} \frac{1}{\sqrt{n}} \left| \widehat{B}_{d,\theta,\ell}(t) - \widehat{b}_{d,\theta,\ell}(t) \right| \right] \le C_A d e^{-d}.
\tag{30}
$$

Using (30) and Fatou's lemma we get

$$
\mathbb{E} \left[\sup_{0 \le t \le t_0} \left| \widetilde{Z}_{d,\theta,d-\theta}(t) \right| \right] \le C_A d e^{-d}.
$$

This in turn implies that $\sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta)t} \widetilde{Z}_{d,\theta,d-\theta}(t)$, almost surely converges uniformly in $t \le t_0$ and therefore $\sum_{d=1}^{\infty} \sum_{\theta=1}^{d} e^{-(d-\theta)t} \widetilde{Z}_{d,\theta,d-\theta}(t)$ is continuous almost surely. This completes the proof of Theorem 1.3.

3.2 The Stopping Time

Note that we can write

$$
H_B(t) = \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} \sum_{\ell=d-\theta+1}^{d} \ell B_{d,\theta,\ell}(t) = \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} \sum_{\ell=d-\theta+1}^{d} \ell \sum_{v \in [n]} \mathbf{1}\{v \in \mathcal{B}_{d,\theta,\ell}(t)\}.
$$

It is precisely the number of balls that were not initially infected (active), and remain non-infected at time t. First let us consider the term $\sum_{\ell=d-\theta+1}^{d} \ell \sum_{v \in [n]} \mathbf{1}\{v \in \mathcal{B}_{d,\theta,\ell}(t)\}$. Consider those $u_n(d, \theta)$ bins with degree d and threshold $\theta \geq 1$ (not initially infected). For $k = 1, 2 \ldots, u_n(d, \theta)$, let T_k be the time ℓ-th ball is removed from the k-the such bin. Then

$$|\{k : T_k \leq t\}| = \sum_{r=0}^{d-\ell} \sum_{v \in [n]} \mathbf{1}\{v \in \mathcal{B}_{d,\theta,r}(t)\}.$$

For the k-th bin we get

$$\mathbb{P}(T_k \leq t) = \sum_{r=0}^{d-\ell} \binom{d}{r} \left(e^{-t}\right)^r \left(1 - e^{-t}\right)^{d-r} := \sum_{r=0}^{d-\ell} b(d, e^{-t}, r),$$

out of the $u_n(d, \theta)$ bins. Since all bins are independent of each other, using Glivenko-Cantelli lemma we get that

$$\sup_{t \geq 0} \left| \frac{1}{n} \sum_{r=0}^{d-\ell} \sum_{v \in [n]} \mathbf{1}\{v \in \mathcal{B}_{d,\theta,r}(t)\} - \frac{u_n(d, \theta)}{n} \sum_{r=0}^{d-\ell} b(d, e^{-t}, r) \right| \longrightarrow 0,$$

in probability as $n \to \infty$. By Condition 1.1, $u_n(d, \theta) = p(d, \theta)n + o(n)$, therefore we get

$$\sup_{t \geq 0} \left| \frac{1}{n} \sum_{r=0}^{d-\ell} \sum_{v \in [n]} \mathbf{1}\{v \in \mathcal{B}_{d,\theta,r}(t)\} - p(d, \theta) \sum_{r=0}^{d-\ell} b(d, e^{-t}, r) \right| \longrightarrow 0, \tag{31}$$

in probability as $n \to \infty$. Since (31) is true for all $0 \leq \ell \leq d$, taking difference of two consecutive terms we get for each $0 \leq r \leq d$,

$$\sup_{t \geq 0} \left| \frac{1}{n} \sum_{v \in [n]} \mathbf{1}\{v \in \mathcal{B}_{d,\theta,r}(t)\} - p(d, \theta)b(d, e^{-t}, r) \right| \longrightarrow 0,$$

and therefore taking a sum over r, and using Condition 1.1,

$$\sup_{t \geq 0} \left| \frac{1}{n} \widetilde{B}_{d,\theta,\ell}(t) - p(d, \theta)\beta(d, e^{-t}, \ell) \right| \xrightarrow{p} 0.$$

Also combining Condition 1.1 and (31) we get

$$\sup_{t \geq 0} \left| \frac{H_B(t)}{n} - \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} \sum_{\ell=d-\theta+1}^{d} \ell p(d, \theta)b(d, e^{-t}, \ell) \right| \xrightarrow{p} 0, \tag{32}$$

in probability as $n \to \infty$. Recall that the stopping time τ was defined as $H_A(\tau) = -1$. Also note that from (32) and (6) we get that

$$\sup_{t \geq 0} \left| \frac{H_A(t)}{n} - \lambda e^{-2t} + \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} \sum_{\ell=d-\theta+1}^{d} \ell p(d, \theta)b(d, e^{-t}, \ell) \right| \xrightarrow{p} 0, \tag{33}$$

in probability as $n \to \infty$. We can then write $h_B(z) = \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} \sum_{\ell=d-\theta+1}^{d} \ell p(d, \theta)$ $b(d, z, \ell)$ and $h_A(z) = \lambda z^2 - h_B(z)$, as the limit of $\frac{H_B(t)}{n}$ and $\frac{H_A(t)}{n}$, by setting $z = e^{-t}$. Also (33), gives us the plausible candidate for the limit of our stopping time $\widehat{z} := \sup\{z \in [0, 1] : h_A(z) = 0\}$. We further assume that if $\widehat{z}$ is a non-zero then it is not a local minimum of $h_A(z)$.

3.3 Proof of Lemma 1.4

To show this let us take a constant $t_1 > 0$ such that $t_1 < -\ln\widehat{z}$. This means $\widehat{z} < 1$, and hence $\lambda > h_B(1)$. Therefore using the fact that $\lambda z^2 - h_B(z)$ is continuous we get $h_B(z) < \lambda z^2$ for $z \in (\widehat{z}, 1]$. This again gives $h_B(e^{-t}) < \lambda e^{-2t}$ for all $t \leq t_1$. Since $[0, t_1]$ is compact again using the continuity of $h_B(z)$ we get $h_B(e^{-t}) - \lambda e^{-2t} < -c$ for some $c > 0$. Therefore on the set $\tau < t_1$ we will have $h_B(e^{-\tau}) - \lambda e^{-2\tau} < -c$. On the other hand, since $H_A(\tau) = -1$, as $n \to \infty$ in (33) $H_A(\tau)/n \to 0$, which gives a contradiction, and thus $\mathbb{P}(\tau \leq t_1) \to 0$ as $n \to \infty$. Now let us choose $t_2 < \tau$ where $t_2 \in (-\ln\widehat{z}, -\ln(\widehat{z} - \varepsilon))$. By our assumption $\widehat{z}$ is not a local minimum of $h_A(z) = \lambda z^2 - h_B(z)$, therefore there is an $\varepsilon > 0$ such that $\lambda e^{-2t_2} - h_B(e^{-t_2}) = -c$ for some $c > 0$. Now by definition if $\tau > t_2$, then $H_B(t_2) \geq 0$. Plugging these in (33) we get $\frac{H_B(t_2)}{n} - \lambda e^{-2t_2} + h_B(t_2) \geq c$. This gives $\mathbb{P}(\tau \geq t_2) \to 0$ as $n \to \infty$. Since t_1 and t_2 can be arbitrary close to $-\ln\widehat{z}$, the proof of the claim is complete.

3.4 Proof of Theorem 1.5

We first derive a joint functional central limit theorem for the processes $(A_n(t), H_A(t))$, from which we prove the theorem.

Proposition 3.3 *Let $\tau'_n \leq \tau_n$ be a stopping time such that $\tau'_n \xrightarrow{p} t_0$ for some $t_0 \geq 0$. Then in $\mathcal{D}[0, \infty)$, as $n \to \infty$,*

$$n^{-1/2}\left(A_n(t \wedge \tau'_n) - n\widehat{a}_n(t), H_A(t \wedge \tau'_n) - nh_A^n(e^{-t})\right) \xrightarrow{d} (Z_A(t \wedge t_0), Z_{HA}(t \wedge t_0)), \quad (34)$$

with Z_A as in Theorem 1.3, and $Z_{HA}(t)$ is a Gaussian process that is described in the proof of this proposition. (The covariance could be calculated from (29) and (47).)

For the sake of readability, we postpone the proof of the proposition to the end of this section. We continue with the proof of Theorem 1.5. Part (i) follows form [2]. Consider now the case when $\widehat{z} \neq 0$ and $\widehat{z}$ is a stable solution, i.e. $\alpha := h'_A(\widehat{z}) > 0$. Using similar arguments as in the proof of [21, Lemma 2.3], one can show that $\widehat{a}_n$ converges to $\widehat{a}$ and h_A^n converges to h_A, together with their derivatives, i.e., $\widehat{a}'_n$ converges to $\widehat{a}'$ and $h_A^{n\,\prime}$ converges to h'_A, uniformly on $[0, 1]$.

Hence, since for small enough $\delta > 0$, $h_A(\widehat{z} + \delta) > 0$, and $h_A(\widehat{z} - \delta) < 0$, h_A^n has a zero at $\widehat{z}_n$ in $(\widehat{z} - \delta, \widehat{z} + \delta)$ for sufficiently large n. Further, since $h_A^n \to h_A$ uniformly, we have $h_A^n > 0$ in the interval $[\widehat{z} + \delta, 1]$. Since $\delta > 0$ is arbitrary, we have $\widehat{z}_n \to \widehat{z}$ as $n \to \infty$. Let us write $\widehat{\tau}_n = -\ln\widehat{z}_n$, and $\widehat{\tau} = -\ln\widehat{z}$, therefore $\widehat{\tau}_n \to \widehat{\tau}$ as $n \to \infty$.

We now use the Skorohod coupling theorem so that we can consider all the random variables in (34) to be defined on the same probability space and the limit holds almost surely. Now we apply Proposition 3.3 with τ_n, and Lemma 1.4 gives us that $\tau_n \to \widehat{\tau} = -\ln\widehat{z}$. Also, since the limits are continuous almost surely, we can say (34) holds uniformly on $[0, \widehat{\tau} + 1]$. Therefore taking $t = \tau_n$ (this is less than $\widehat{\tau} + 1$ for large n almost surely), we get

$$\begin{aligned}
H_A(\tau_n) &= nh_A^n(e^{-\tau_n}) + n^{1/2}Z_{HA}(\widehat{\tau} \wedge \tau_n) + o(n^{1/2}) \\
&= nh_A^n(e^{-\tau_n}) + n^{1/2}Z_{HA}(\widehat{\tau}) + o(n^{1/2}),
\end{aligned} \quad (35)$$

where using continuity of Z_{HA}, we absorb the error term in $o(n^{1/2})$. Since $H_A(\tau_n) = -1$, therefore

$$h_A^n(e^{-\tau_n}) = n^{-1/2}Z_{HA}(\widehat{\tau}) + o(n^{-1/2}). \quad (36)$$

Using the Mean-Value theorem, we can write for some $\xi_n \in [\widehat{z}_n, z_n]$ or $[z_n, \widehat{z}_n]$,

$$h_A^n(e^{-\tau_n}) = h_A^n(e^{-\tau_n}) - h_A^n(e^{-\widehat{\tau}_n}) = h_A^n(z_n) - h_A^n(\widehat{z}_n) = h_A^{n\prime}(\xi_n)(z_n - \widehat{z}_n). \qquad (37)$$

We have $z_n \to \widehat{z}$ and $\widehat{z}_n \to \widehat{z}$, and therefore $\xi_n \to \widehat{z}$. Now, from $h_A^{n\prime} \to h_A{}'$ uniformly, we get $h_A^{n\prime}(\xi_n) \to h_A{}'(\widehat{z}) = \alpha$. Therefore (36), and (37) we get

$$z_n - \widehat{z}_n = n^{-1/2} \frac{1}{\alpha} \left(Z_{HA}(\widehat{\tau}) + o(1) \right). \qquad (38)$$

Now, we use the Mean value theorem on A_n for some $\xi_n \to \widehat{z}$, and (38) to obtain

$$\begin{aligned}
n^{-1/2} A_n(\tau_n) &= n^{1/2} \widehat{a}_n(\tau_n) + Z_A(\widehat{\tau}) + o(1) \\
&= n^{1/2} \widehat{a}_n(\widehat{\tau}) + n^{1/2} \widehat{a}_n'(e^{-\xi_n})(z_n - \widehat{z}_n) + Z_A(\widehat{\tau}) + o(1) \\
&= n^{1/2} \widehat{a}_n(\widehat{\tau}) + \frac{\widehat{a}'(\widehat{\tau})}{\alpha} Z_{HA}(\widehat{\tau}) + Z_A(\widehat{\tau}) + o(1).
\end{aligned} \qquad (39)$$

In the third equality we have used the fact that $\widehat{a}_n' \to \widehat{a}'$ uniformly in t. The proof is thus complete since $|\mathcal{A}_n^*| = A_n(\tau_n)$. We end this section by presenting the proof of Proposition 3.3.

Proof of Proposition 3.3 First note that we can write the number of (uninfected) B balls at time t as

$$H_B(t) = \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} \sum_{\ell=d-\theta+1}^{d} \ell B_{d,\theta,\ell}(t) = \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} (d-\theta+1) \widetilde{B}_{d,\theta,d-\theta+1} + \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} \sum_{\ell=d-\theta+2}^{d} \widetilde{B}_{d,\theta,\ell}.$$

This gives

$$H_A(t) = W_n(t) - H_B(t) = W_n(t) - \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} (d-\theta+1) \widetilde{B}_{d,\theta,d-\theta+1} - \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} \sum_{\ell=d-\theta+2}^{d} \widetilde{B}_{d,\theta,\ell}(t). \qquad (40)$$

We wish to prove a joint central limit theorem for $(A_n(t), H_A(t))$. Let us first outline the procedure. We have already expressed $\widehat{B}_{d,\theta,\ell}(t)$ (which is the centered and scaled version of $\widetilde{B}_{d,\theta,\ell}(t)$) in terms of $\widetilde{M}_{d,\theta,\ell}(t)$'s in (16), and therefore $H_A(t)$ is also implicitly a linear combination of $W_n(t)$, and $\widetilde{M}_{d,\theta,\ell}(t)$'s (since $\widetilde{B}_{d,\theta,\ell}(t)$'s are also linear combinations of $\widetilde{M}_{d,\theta,\ell}(t)$'s). Therefore once we can prove that the joint distribution $(W_n, \widetilde{M}_{d,\theta,\ell})$ is Gaussian, we can derive the joint distribution of $(A_n(t), H_A(t))$ (since both $A_n(t), H_A(t)$ are linear combination of independent random variables which are jointly Gaussian). To do that, let us define $\widetilde{\widetilde{W}}_n(t) = n^{-1/2} \left(\widehat{W}_n(t) - \widehat{W}_n(0) \right)$, and note that using (7) we get for every $t \in [0, T \wedge \tau_n]$,

$$[\widetilde{\widetilde{W}}_n, \widetilde{\widetilde{W}}_n]_t = 2\lambda(e^{2t} - 1) + o_p(1). \qquad (41)$$

Therefore using Proposition 2.3 on the stopped process

$$\widetilde{\widetilde{W}}_n(t \wedge \tau_n') \xrightarrow{d} \widehat{Z}_W(t \wedge t_0),$$

as $n \to \infty$, where $\widehat{Z}_W$ is a continuous Gaussian process with mean zero, and variance $2\lambda(e^{2t} - 1)$. Now since $\widehat{W}_n(t) := e^{2t} W_n(t)$, we get that

$$n^{-1/2} \left(W_n(t \wedge \tau_n') - e^{-2t} W_n(0) \right) \xrightarrow{d} Z_W(t \wedge t_0), \qquad (42)$$

as $n \to \infty$, where Z_W is a continuous Gaussian process with mean zero, and variance $2\lambda(e^{-2t} - e^{-4t})$. Moreover, for $t \in [0, T \wedge \tau_n]$, $\ell \geq d - \theta + 1$, and $\theta \geq 1$

$$\left[\widetilde{\widehat{W}}_n, \widetilde{M}_{d,\theta,\ell}\right]_t = \frac{1}{n} \sum_{0 < s \leq t} \Delta \widehat{W}_n(s) \Delta M_{d,\theta,\ell}(s) = \frac{1}{n} \sum_{0 < s \leq t} \Delta \widehat{W}_n(s) \Delta \widehat{B}_{d,\theta,\ell}(s)$$
$$= \frac{1}{n} \sum_{0 < s \leq t} e^{(2+\ell)s} \Delta W_n(s) \Delta \widetilde{B}_{d,\theta,\ell}(s). \tag{43}$$

Now when $\widetilde{B}_{d,\theta,\ell}$ jumps by 1 for some $\ell \geq d - \theta + 1$, the W_n jumps by -2, therefore using (13), (43) yields

$$\left[\widetilde{\widehat{W}}_n, \widetilde{M}_{d,\theta,\ell}\right]_t = \frac{1}{n} \int_0^t 2e^{(2+\ell)s} \, d(-\widetilde{B}_{d,\theta,\ell}(s)) = \int_0^t 2e^{(2+\ell)s} \, d(-\beta(d, e^{-s}, \ell)) + o_p(1). \tag{44}$$

Therefore using Proposition 2.3 we get the joint convergence of $(\widetilde{M}_{d,\theta,\ell}(t \wedge \tau_n'), \widetilde{\widehat{W}}_n(t \wedge \tau_n'))$ for $\ell \geq d - \theta + 1$. Thus combining (42), the joint convergence of $(\widetilde{M}_{d,\theta,\ell}(t \wedge \tau_n'), \widetilde{\widehat{W}}_n(t \wedge \tau_n'))$ for $\ell \geq d - \theta + 1$, and (28) we get

$$n^{-1/2} \left(A_n(t \wedge \tau_n') - n\widehat{a}_n(t), H_A(t \wedge \tau_n') - nh_A^n(e^{-t})\right) \xrightarrow{d} (Z_A(t \wedge \tau_n'), Z_{HA}(t \wedge \tau_n')), \tag{45}$$

as $n \to \infty$, where $\widehat{a}_n(t) = 1 - \frac{1}{n} \sum_{d=1}^{\infty} \sum_{\theta=1}^{d} u_n(d, \theta) \beta(d, e^{-t}, d - \theta + 1)$,

$$h_A^n(e^{-t}) = e^{-2t} \frac{2m_n}{n} - \frac{1}{n} \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} (d - \theta + 1) u_n(d, \theta) \beta(d, e^{-t}, d - \theta + 1)$$
$$- \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} \sum_{\ell=d-\theta+2}^{d} u_n(d, \theta) \beta(d, e^{-t}, \ell), \tag{46}$$

and then

$$Z_{HA}(t) := Z_W(t) - \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} (d - \theta + 1) \widetilde{Z}_{d,\theta,d-\theta+1} - \sum_{d=0}^{\infty} \sum_{\theta=1}^{\infty} \sum_{\ell=d-\theta+2}^{d} \widetilde{Z}_{d,\theta,\ell}(t). \tag{47}$$

We can ignore the contribution of the tail in the centered and scaled version of (40) using an almost identical argument as Lemma 3.2. Similarly, we can also ensure that the limit $Z_{HA}(t)$ is continuous. This completes the proof of Proposition 3.3. $\qquad\square$

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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