Journal of Statistical Physics (2023) 190:27
https://doi.org/10.1007/s10955-022-03034-x

Self-similar Profiles for Homoenergetic Solutions of the Boltzmann Equation for Non-cutoff Maxwell Molecules

Bernhard Kepka[1]

Received: 27 April 2022 / Accepted: 2 November 2022 / Published online: 3 December 2022

Abstract

We consider a modified Boltzmann equation which contains, together with the collision operator, an additional drift term which is characterized by a matrix A. Furthermore, we consider a Maxwell gas, where the collision kernel has an angular singularity. Such an equation is used in the study of homoenergetic solutions to the Boltzmann equation. Under smallness assumptions on the drift term, we prove that the longtime asymptotics is given by self-similar solutions. We work in the framework of measure-valued solutions with finite moments of order $p > 2$ and show existence, uniqueness and stability of these self-similar solutions for sufficiently small A. Furthermore, we prove that they have finite moments of arbitrary order if A is small enough. In addition, the singular collision operator allows to prove smoothness of these self-similar solutions. Finally, we study the asymptotics of particular homoenergetic solutions. This extends previous results from the cutoff case to non-cutoff Maxwell gases.

Keywords Boltzmann equation · Homoenergetic solutions · Long-range interactions · Self-similar solutions · Maxwell molecules · Non-equilibrium

Mathematics Subject Classification 35Q20 · 82C40 · 35C06

1 Introduction

The inhomogeneous Boltzmann equation is given by

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \tag{1}$$

where $f = f(t, x, v) : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ is the one-particle distribution of a dilute gas in whole space. In this paper, we restrict ourselves to the physically most relevant

Communicated by Eric A. Carlen.

✉ Bernhard Kepka
kepka@iam.uni-bonn.de

1 Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

case of three dimensions, although our study can be extended to dimensions $d \geq 3$ without any additional difficulties.

On the right-hand side we have Boltzmann's collision kernel

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|, n \cdot \sigma)(f'_* f' - f_* f) d\sigma dv_*,$$

where $n = (v - v_*)/|v - v_*|$ and $f'_* = f(v'_*)$, $f' = f(v')$, $f_* = f(v_*)$, with the pre-collisional velocities (v, v_*) resp. post-collisional velocities (v', v'_*). One parameterization of the post-collisional velocities is given by the σ-representation, i.e. for $\sigma \in S^2$

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

For an introduction into the physical and mathematical theory of the Boltzmann equation (1) see for instance [7, 33].

The collision kernel is given by $B(|v - v_*|, n \cdot \sigma)$ and it can be obtained from an analysis of the binary collisions of the gas molecules. For instance, power-law potentials $1/r^{q-1}$ with $q > 2$ lead to (see e.g. [7, Section II.5])

$$B(|v - v_*|, n \cdot \sigma) = |v - v_*|^{\gamma} \, b(n \cdot \sigma), \quad \gamma = (q - 5)/(q - 1), \tag{2}$$

where $b : [-1, 1) \to [0, \infty)$ has a non-integrable singularity of the form

$$\sin\theta \, b(\cos\theta) \sim \theta^{-1-2/(q-1)}, \quad \text{as } \theta \to 0, \tag{3}$$

where $\cos\theta = n \cdot \sigma$, with θ being the deviation angle. It is customary to classify the collision kernels according to their homogeneity γ with respect to $|v - v_*|$. There are three cases: *hard potentials* ($\gamma > 0$), *Maxwell molecules* ($\gamma = 0$) and *soft potentials* ($\gamma < 0$). In this paper, we consider the case of Maxwell molecules, hence B does not depend on $|v - v_*|$, cf. (2). This corresponds to $q = 5$ for power-law interactions.

Collision kernels with an angular singularity of the form (3) are called non-cutoff kernels. When $\gamma = 0$, one refers to non-cutoff or true Maxwell molecules. This singularity reflects the fact that for power-law interactions the average number of *grazing collisions*, i.e. collisions with $v \approx v'$, diverges. In kinetic theory the Boltzmann equation has often been studied assuming that the collision kernel B is integrable in the angular variable (Grad's cutoff assumption), since the mathematical analysis is usually simpler.

In this paper, we analyze a particular class of solutions to (1) namely the so-called *homoenergetic solutions*, which have been studied in particular in [5, 20] in the case of cutoff Maxwell molecules. We show that the results obtain in their papers extend to non-cutoff Maxwell molecules.

1.1 Homoenergetic Solutions and Existing Results

Our study concerns solutions to (1) of the form

$$f(t, x, v) = g(t, v - L(t)x), \quad w = v - L(t)x, \tag{4}$$

for $L(t) \in \mathbb{R}^{3\times 3}$ and a function $g = g(t, w) : [0, \infty) \times \mathbb{R}^3 \to [0, \infty)$ to be determined. One can check that solutions to (1) of the form (4) for large classes of functions g exist if and only if g and L satisfy

$$\partial_t g - L(t)w \cdot \nabla_w g = Q(g, g), \quad \frac{d}{dt}L(t) + L(t)^2 = 0. \tag{5}$$

The second equation allows the reduction to the variable w. In particular, the collision operator acts on g through the variable w. The second equation can be solved explicitly $L(t) = L(0)(I + tL(0))^{-1}$. Note that the inverse matrix might not be defined for all times, although this situation will not be considered here.

Solutions to (5) are called *homoenergetic solutions* and were introduced by Truesdell [30] and Galkin [15]. They studied their properties via moment equations in the case of Maxwell molecules. As is known since the work by Truesdell and Muncaster [31], it is possible to write a closed systems of ordinary differential equations for the moments up to any arbitrary order for such interactions. This allows to derive properties about the solution to (5). In particular, this approach has been applied in [15–17, 30]. More recently, this method has also been used in [18] (and references therein) in order to obtain information on homoenergetic solutions to the Boltzmann equation, as well as other kinetic models like BGK. The case of mixtures of gases has been studied there as well. The well-posedness of (5) for a large class of initial data, was proved by Cercignani [8]. Furthermore, the shear flow of a granular material for Maxwell molecules was studied in [9, 10].

A systematic analysis of the longtime behavior of solutions to (5) for kernels with arbitrary homogeneities has been undertaken in [5, 19–21]. In [19] they discussed the case of dominant collision term, see also [22]. Furthermore, they proved the existence of a class of self-similar solutions in the case of cutoff Maxwell molecules in [20]. The uniqueness and stability of these self-similar solutions have been proved in [5] and the regularity has been obtained in [13]. Homoenergetic solutions for the two-dimensional Boltzmann equation with hard sphere interactions, as well as for a class of Fokker-Planck equations have been studied in [24].

It is worth mentioning that homoenergetic solutions to (1) can be interpreted in a wider framework introduced in [11, 12]. There the authors studied a formulation of the molecular dynamics of many interacting particle systems with symmetries. In particular, if the particles of the system of molecules of a gas interact by means of binary collisions one obtains the functional form (4) for the particle distribution.

In this paper, we extensively use the Fourier transform method, which was introduced by Bobylev [2, 3] to study the homogeneous Boltzmann equation for Maxwell gases. This method has also been applied in [5] for homoenergetic solutions with cutoff Maxwell molecules.

The main contribution of this paper is to adapt the techniques in [5, 20] and well established methods for the non-cutoff Boltzmann equation to extend the results to the case of non-cutoff Maxwell molecules. The main difficulty is the singular behavior of the collision kernel (3).

1.2 Overview and Main Results

Notation. We denote by $\mathscr{P}(\mathbb{R}^3)$ the set of Borel probability measures on $\mathbb{R}^3$ and by $\mathscr{P}_p(\mathbb{R}^3) \subset \mathscr{P}(\mathbb{R}^3)$ the set of those which have finite moments of order p, i.e. $\mu \in \mathscr{P}_p$ if

$$\|\mu\|_p = \int_{\mathbb{R}^3} |v|^p \mu(dv) < \infty.$$

The action of $\mu \in \mathscr{P}$ on a test function ψ via integration is abbreviated by $\langle \psi, \mu \rangle$. The Fourier transform or characteristic function of a probability measure $\mu \in \mathscr{P}$ is defined by

$$\varphi(k) = \mathscr{F}[\mu](k) = \int_{\mathbb{R}^3} e^{-ik\cdot x} d\mu(x).$$

We denote by $\mathscr{F}_p$ the set of all characteristic functions of probability measures $\mu \in \mathscr{P}_p$. Furthermore, we write $\psi \in C^k$ for k-times continuously differentiable functions and $\psi \in C_b^k$ if the standard norm $\|\psi\|_{C^k}$ is finite.

We also use the notation $\langle k\rangle := \sqrt{1+|k|^2}$ and denote the space of functions $h : \mathbb{R}^3 \to \mathbb{R}$ such that $\langle k\rangle^m h(k) \in L^2(\mathbb{R}^3)$ by $L^2_m(\mathbb{R}^3)$. For matrices $A \in \mathbb{R}^{3\times 3}$ we use the matrix norm $\|A\| = \sum_{ij} |A_{ij}|$. Finally, $\mathbb{I}_B$ is the indicator function for some set B.

Assumption on the kernel. We consider non-cutoff Maxwell molecules, i.e. the collision kernel has the form $B = b(n \cdot \sigma) = b(\cos\theta)$. The function $b : [-1, 1) \to [0, \infty)$ is measurable, locally bounded and has the angular singularity

$$\sin\theta b(\cos\theta)\theta^{1+2s} \to K_b > 0, \quad \text{as } \theta \to 0 \tag{6}$$

for some $s \in (0, 1)$ and $K_b > 0$. This implies

$$\Lambda = \int_0^\pi \sin\theta\, b(\cos\theta)\, \theta^2 d\theta < \infty. \tag{7}$$

In particular, this covers inverse power-law interactions with $q = 5$, cf. (2) and (3).

Main result. In our study we consider the following modified Boltzmann equation, which is a variant of equation (5),

$$\partial_t f = div(Av\, f) + Q(f, f), \quad f(0, \cdot) = f_0(\cdot). \tag{8}$$

In contrast to the previous equation, $A \in \mathbb{R}^{3\times 3}$ is a time-independent matrix. However, the study of solutions to (5) can be reduced to this situation using a change of variables and perturbation arguments, see Sect. 4. We work with weak solutions with finite energy.

Definition 1.1 A family of probability measures $(f_t)_{t\geq 0} \subset \mathscr{P}_p$ with $p \geq 2$ is a weak solution to (8) if for all $\psi \in C_b^2$ and all $0 \leq t < \infty$ it holds

$$\begin{aligned} \langle \psi, f_t\rangle = \langle \psi, f_0\rangle &- \int_0^t \langle Av \cdot \nabla\psi, f_r\rangle\, dr \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \int_{S^2} b(n\cdot\sigma) \left\{\psi'_* + \psi' - \psi_* - \psi\right\} d\sigma\, f_r(dv) f_r(dv_*) dr. \end{aligned} \tag{9}$$

Here, we also assume that the integrands in the time integrals are measurable with respect to the time variable.

Above we abbreviated $\psi'_* = \psi(v'_*)$, etc. This formulation is motivated by testing (8) with ψ and applying the usual pre-postcollisional change of variables $(v, v_*) \leftrightarrow (v', v'_*)$ as well as $v \leftrightarrow v_*$. See also e.g. [20, 23] concerning the above definition. For brevity we will sometimes denote the term involving the collision operator $\langle \psi, Q(f_r, f_r)\rangle$. Note that this is well-defined due to the moment assumption $f_t \in \mathscr{P}_p$, $p \geq 2$, in conjunction with the estimate (see e.g. [23, 32])

$$\left|\int_{S^2} b(n\cdot\sigma)\left\{\psi'_* + \psi' - \psi_* - \psi\right\} d\sigma\right| \leq 2\pi\Lambda \left(\max_{|\xi|\leq\sqrt{|v|^2+|v_*|^2}} |D^2\psi(\xi)|\right) |v - v_*|^2. \tag{10}$$

Using this and an approximation one can also use test functions $\psi \in C^2$, which satisfy the condition $|D^2\psi(v)| \leq C(1 + |v|^{p-2})$, in the weak formulation.

Let us mention that one can always consider, without loss of generality, the case of vanishing momentum/mean $\int_{\mathbb{R}^3} v f_0(dv) = 0$. To get a solution F with initial mean $U \in \mathbb{R}^3$

from f_t, one defines $F(t, v) = f_t(v - e^{tA}U)$ interpreted as a push-forward. However, as we will see, solutions with initial condition different from a Dirac measure are smooth for positive times due to the regularizing effect of the angular singularity.

Let us also define the following Fourier-based metric on probability measures.

Definition 1.2 For two probability measures $\mu,\ \nu \in \mathscr{P}_p$ with finite moments of order $p \geq 2$ we define a distance using the Fourier transforms $\varphi = \mathscr{F}[\mu],\ \psi = \mathscr{F}[\nu]$ via

$$d_2(\mu, \nu) := \sup_k \frac{|\varphi(k) - \psi(k)|}{|k|^2}.$$

Note that $d_2(\mu, \nu) < \infty$ is finite if μ, ν have equal first moments. We sometimes write $d_2(\varphi, \psi)$.

Theorem 1.3 *Consider the equation (8). Let* $2 < p \leq 4$. *There is a constant* $\varepsilon_0 = \varepsilon_0(p, b) > 0$ *such that if* $\|A\| \leq \varepsilon_0$, *the following holds.*

(i) There is $\bar{\beta} = \bar{\beta}(A)$ *and* $f_{st} \in \mathscr{P}_p$ *so that (8) has a self-similar solution*

$$f(v,t) = e^{-3\bar{\beta}t} f_{st}\left(\frac{v - e^{-tA}U}{e^{\bar{\beta}t}}\right), \quad U \in \mathbb{R}^3,$$

where f_{st} *has moments*

$$\int_{\mathbb{R}^3} v f_{st}(dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f_{st}(dv) = K\bar{N}_{ij}.$$

Here, $K \geq 0$ *and* $\bar{N} = \bar{N}(A) \in \mathbb{R}^{3\times 3}$ *is a uniquely given positive definite, symmetric matrix with* $\|\bar{N}\| = 1$. *For* $K = 0$, *we have* $f_{st} = \delta_0$, *a Dirac measure in zero.*

Furthermore, when $K > 0$ *the self-similar solutions are smooth*

$$f(t,\cdot) \in L^1(\mathbb{R}^3) \cap \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^3).$$

(ii) Let $(f_t)_t \subset \mathscr{P}_p$ *be a weak solution to (8) with initial condition* $f_0 \in \mathscr{P}_p$ *and*

$$U = \int_{\mathbb{R}^3} v f_0(dv).$$

Then there is $\alpha = \alpha(f_0) \in \mathbb{R}$, $C = C(f_0, p) > 0$, $\theta = \theta(\varepsilon_0) > 0$ *such that the rescaled function*

$$\tilde{f}(t, v) := e^{3\bar{\beta}t} f\left(e^{\bar{\beta}t} v + e^{-At}U, t\right)$$

satisfies

$$d_2\left(\tilde{f}(t,\cdot), f_{st}(t,\cdot)\right) \leq Ce^{-\theta t},$$

where f_{st} *is given in (i) with second moments* $\alpha^2\bar{N}$, $K = \alpha^2$. *In particular, the self-similar solution in (i) is unique for given* $K \geq 0$.

(iii) In addition, for all $M \in \mathbb{N}$, $M \geq 3$ *there is* $\varepsilon_M \leq \varepsilon_0$ *such that the self-similar solution from (i) has finite moments of order* M *if* $\|A\| \leq \varepsilon_M$.

Remark 1.4 Note that f_{st} in (i) solves

$$div((A+\beta)v\, f_{st}) + Q(f_{st}, f_{st}) = 0, \quad \beta = \bar{\beta}(A). \tag{11}$$

Furthermore, $\bar{N}$ is a stationary solution to the second order moment equations ($\bar{b} \in \mathbb{R}$ depending only on the collision kernel, see Lemma 3.2)

$$2\bar{\beta}\bar{N} - A\bar{N} - (A\bar{N})^\top - 2\bar{b}\left(\bar{N} - \frac{\mathrm{tr}\,(\bar{N})}{3} I\right) = 0.$$

As we will see, $\bar{\beta} = \bar{\beta}(A)$ is chosen such that $2\bar{\beta} \in \mathbb{R}$ is the simple eigenvalue with largest real part. The corresponding eigenvector is given by $\bar{N} = \bar{N}(A)$.

The uniqueness result in (ii) can now be formulated in a more precise way: within the class of probability measures $\mathscr{P}_p$, $p > 2$, there is a unique solution $f_{st} \in \mathscr{P}_p$ to the stationary equation (11) with $\beta = \bar{\beta}(A)$ having moments

$$\int_{\mathbb{R}^3} v f_{st}(dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f_{st}(dv) = \bar{N}_{ij}.$$

Since $f_{st}(K^{-1/2}v)K^{-3/2}$ solves (11) and has second moments $K\bar{N}_{ij}$, $K > 0$, it is the respective self-similar profile in (i). For $K = 0$ this is a Dirac in zero.

Remark 1.5 The above theorem is similar to the results in [5, 20], where *cutoff* Maxwell molecules have been considered. A comparison with Theorem 1.3, which covers the *non-cutoff* case, shows that all results hold true under the same assumptions. Here, the smoothness statement in (i) is a consequence of the regularizing effect of the non-cutoff collision kernel, in contrast to the cutoff case [13], where this has been obtained in a perturbative framework close to a Maxwellian.

Remark 1.6 Regarding part (iii) in Theorem 1.3 it might be that for small but fixed $A \neq 0$ the self-similar solutions do not have finite moments of arbitrary order, but that they have power-law tails. For shear flow this is suggested by numerical experiments, see [18].

Let us also mention that the smallness of $\|A\|$ is crucial for our perturbation arguments. The precise behavior of solutions to (8) for large values of A remains open (see also Remark 4.2).

The paper is organized in the following way. In Sect. 2 we discuss the well-posedness theory of equation (8) and in Sect. 3 the proof of Theorem 1.3. Finally, in Sect. 4 we study the self-similar asymptotics of homoenergetic solution in the case of simple and planar shear.

2 Well-Posedness of the Modified Boltzmann Equation

The following result summarizes the well-posedness theory of equation (8), needed in our study. The assumption $p > 2$ can be relaxed, however we only need this case in the sequel.

Proposition 2.1 *Under our general assumptions, the following statements hold.*

(i) For all $f_0 \in \mathscr{P}_p$, $p > 2$, there is a weak measure-valued solution $(f_t)_t \subset \mathscr{P}_p$ to (8). In addition, every weak solution has the property $t \mapsto \langle \psi, f_t \rangle \in C^1([0,\infty); \mathbb{R})$ for all test functions $\psi \in C^2$ with $\left\| D^2\psi \right\|_\infty < \infty$.

(ii) *For two weak solutions $(f_t)_t$, $(g_t)_t \subset \mathscr{P}_p$ to (8), $p > 2$, such that f_0, g_0 have equal first moments, it holds*

$$d_2(f_t, g_t) \leq e^{2\|A\|t} d_2(f_0, g_0). \tag{12}$$

In particular, solutions are unique.

(iii) *If the initial datum $f_0 \in \mathscr{P}_p$, $p > 2$, is not a Dirac measure, the solution is smooth, i.e. for $t > 0$*

$$f(t, \cdot) \in L^1(\mathbb{R}^3) \cap \bigcap_{k \in \mathbb{N}} H^k(\mathbb{R}^3).$$

Remark 2.2 The setting of measure-valued solutions was also used in [20] for homoenergetic solutions. Measure-valued solutions to the homogeneous Boltzmann equation ($A = 0$ in (8)) were considered in e.g. [23, 27] for both hard and soft potentials with homogeneity $\gamma \geq -2$. In [27] solutions with infinite energies are studied as well, see also [6, 26] for the case of Maxwell molecules.

The metric in Definition 1.2 is also termed Toscani metric and appeared first in [14] for the study of convergence to equilibrium of the homogeneous Boltzmann equation with true Maxwell molecules. Furthermore, it was used to prove uniqueness of respective solutions in [29], by showing that solutions are contractive w.r.t. d_2. Inequality (12) is the extension of this Lipschitzianity to homoenergetic solutions.

A key ingredient in the proof of Theorem 1.3 is the following comparison principle between solutions to (8). A similar result was used in [5, Section 5].

Proposition 2.3 *Consider two weak solutions $(f_t)_t$, $(g_t)_t \subset \mathscr{P}_p$, $p > 2$, to (8) with zero momentum. Let φ, $\psi \in C([0, \infty); \mathscr{F}_p)$ be the corresponding Fourier transforms. Suppose that*

$$|\varphi_0(k) - \psi_0(k)| \leq C_1 |k|^p + C_2 |k|^2, \quad \forall k \in \mathbb{R}^3.$$

Then, we have for all $t \geq 0$ and $k \in \mathbb{R}^3$

$$|\varphi_t(k) - \psi_t(k)| \leq C_1 e^{-(\lambda(p) - p\|A\|)t} |k|^p + C_2 e^{2\|A\|t} |k|^2.$$

Here, $\lambda(p) > 0$ is defined in Lemma 2.5 and depends only on the collision kernel.

In the proof of both propositions we use an approximation by the cutoff problem. To this end, let us introduce an arbitrary cutoff sequence $b_n : [-1, 1) \to [0, \infty)$, $b_n \not\equiv 0$, with $b_n \nearrow b$, $\|b_n\|_\infty < \infty$, e.g. $b_n := \min(b, n)$ and denote the corresponding collision operators by Q_n. Furthermore, let $\Lambda_n \leq \Lambda$ be the corresponding constant as defined in (7) with b_n replacing b.

Let us mention that (12) follows from Proposition 2.3 for $C_1 = 0$. However, in the proof we rely on the uniqueness of solutions due to our approximation procedure.

Proof of Proposition 2.1. (i). The proof follows well-known methods for the homogeneous Boltzmann equation (i.e. $A = 0$). We only give the essential arguments.

First of all, for all $f_0 \in \mathscr{P}_p$, $p > 2$ one can prove the existence of a unique weak solution $(f_t^n)_t \in C([0, \infty); \mathscr{P}_p)$ of the corresponding cutoff equation with collision kernel b_n using e.g. semigroup theory [20, Section 4.1].

To get a solution to the non-cutoff equation on $[0, T]$ we use a weak compactness argument, see e.g. [23]. One can obtain the a priori bound $\|f_t^n\|_p \leq Ce^{C(p,A)T} \|f_0\|_p$ via a Gronwall argument, which yields tightness of the sequence $(f_t^n)_n$ for all $t \in [0, T]$. Furthermore, the a priori bound and the weak formulation (9) imply the following continuity property independent of $n \in \mathbb{N}$: for any test function $\psi \in C^2$ with $\left\|D^2\psi\right\|_{C^2} < \infty$ and $0 \leq s < t \leq T$

$$\left|\langle\psi, f_t^n\rangle - \langle\psi, f_s^n\rangle\right| \leq (t-s)C\left(T, \left\|D^2\psi\right\|_\infty, A, \Lambda\right)\|f_0\|_2 .$$

Hence, we conclude that there is a weakly converging subsequence $f_t^{n_k} \rightharpoonup f_t$ for all $t \in [0, T]$. We pass to the limit in the weak formulation as in [23, Section 4].

Finally, the stated regularity property $t \mapsto \langle\psi, f_t\rangle \in C^1$ follows from the weak formulation. □

We give a proof of part (ii) of Proposition 2.1 and Proposition 2.3 in the next subsection using the Fourier transform method. Part (iii) of Proposition 2.1 is proved in Sect. 2.2.

2.1 The Modified Boltzmann Equation in Fourier Space

We reformulate the problem (8) via the Fourier transform. Consider a weak solution $(f_t)_t \subset \mathscr{P}_p$, $p > 2$ and its Fourier transform $\varphi_t(k) = \mathscr{F}[f_t](k)$. For a fixed $k \in \mathbb{R}^3$, we use $\psi(v) = e^{-ik\cdot v}$ as a test function in the weak formulation of (8) yielding

$$\partial_t\varphi_t(k) + A^\top k \cdot \nabla\varphi_t(k) = \hat{Q}(\varphi_t, \varphi_t)(k). \tag{13}$$

Note that part (i) in Proposition 2.1 implies that $t \mapsto \varphi_t(k) \in C^1$ for any $k \in \mathbb{R}^3$. The last term in (13) corresponds to the collision operator, which has the form (Bobylev's formula [2, 3])

$$\hat{Q}(\varphi, \varphi)(k) = \int_{S^2} b(\hat{k}\cdot\sigma)\left\{\varphi(k_+)\varphi(k_-) - \varphi(k)\varphi(0)\right\} d\sigma,$$

where $k_\pm = (k \pm |k|\sigma)/2$, $\hat{k} = k/|k|$. Let us write $\hat{Q}_n$ for the Fourier representation of the collision operator corresponding to a cutoff sequence $0 \leq b_n \nearrow b$. We will often consider a decomposition of it in a gain and loss term

$$\hat{Q}_n^+(\varphi, \varphi)(k) = \int_{S^2} b_n(\hat{k}\cdot\sigma)\varphi(k_+)\varphi(k_-)d\sigma, \quad \hat{Q}_n^-(\varphi, \varphi)(k) = S_n\varphi(k).$$

In the last equation, we used $\varphi(0) = 1$ for characteristic functions and the constant

$$S_n := \int_{S^2} b_n(e\cdot\sigma)d\sigma, \quad e \in S^2. \tag{14}$$

Observe that the integral does not depend on $e \in S^2$ by rotational invariance. This integral measures the average number of collisions and, since b is singular, we have $S_n \nearrow +\infty$ as $n \to \infty$.

Finally, let us recall the following property of characteristic functions.

Lemma 2.4 *Consider $\mu \in \mathscr{P}_p$, $p > 0$, then its characteristic function satisfies $\varphi \in C_b^{\lfloor p\rfloor, p-\lfloor p\rfloor}$ if $p \notin \mathbb{N}$ and $\varphi \in C_b^p$ if $p \in \mathbb{N}$. Furthermore, $\|\varphi\|_C \leq 1$ and $\overline{\varphi(k)} = \varphi(-k)$.*

2.1.1 Linearization and Lipschitz Property of the Gain Term

For the Fourier transform of the cutoff operator $\hat{Q}_n$ we introduce the linearization of $\hat{Q}_n^+$ defined by

$$\mathscr{L}_n(\varphi)(k) = \int_{S^2} b_n(\hat{k}\cdot\sigma)(\varphi(k_+) + \varphi(k_-))d\sigma, \tag{15}$$

where $\varphi \in C_b$, say. The following lemma can be proved as in [5, Theorem 5.8].

Lemma 2.5 *Let us define*

$$\begin{aligned} w_p(s) &:= 1 - \left(\frac{1+s}{2}\right)^{p/2} - \left(\frac{1-s}{2}\right)^{p/2}, \\ \lambda_n(p) &:= \int_{S^2} b_n(e\cdot\sigma)\, w_p(e\cdot\sigma) d\sigma, \quad \lambda(p) := \int_{S^2} b(e\cdot\sigma)\, w_p(e\cdot\sigma) d\sigma. \end{aligned} \tag{16}$$

Then, $\lambda(p)$ is well-defined for $p \geq 2$ and $\lambda_n(p) \to \lambda(p)$. Furthermore, $\lambda(p)$ is strictly increasing w.r.t. $p \geq 2$. In particular, we have $\lambda(p) > \lambda(2) = 0$ for $p > 2$.

Remark 2.6 We remark that $|k|^p$, $p > 0$, can be interpreted as an eigenfunction of the operator $(\mathscr{L}_n - S_n I)$ w.r.t. the eigenvalue $-\lambda_n(p)$, since we have

$$(\mathscr{L}_n - S_n I)|k|^p = -\lambda_n(p)|k|^p.$$

The following result is an adaptation of [5, Lemma 3.1], where we made the dependence on the constant S_n explicit. Such an estimate was termed $\mathscr{L}$-Lipschitz in [4, Definition 3.1].

Lemma 2.7 *Consider two characteristic functions $\varphi, \psi \in \mathscr{F}_p$, $p \geq 2$, and a cutoff sequence $b_n \nearrow b$. Then, we have with $\varphi = \mathscr{F}[f]$, $\psi = \mathscr{F}[g]$*

$$|\hat{Q}_n^+(\varphi,\varphi) - \hat{Q}_n^+(\psi,\psi)|(k) \leq \mathscr{L}_n(|\varphi - \psi|)(k) \leq S_n d_2(f,g)|k|^2. \tag{17}$$

Proof The first inequality follows from

$$|\varphi(k_+)\varphi(k_-) - \psi(k_+)\psi(k_-)| \leq |\varphi(k_+) - \psi(k_+)| + |\varphi(k_-) - \psi(k_-)|$$

The second one is a consequence of a straightforward estimation and $|k_+|^2 + |k_-|^2 = |k|^2$. □

2.1.2 Uniqueness of Weak Solutions

We turn to the proof of part (ii) of Proposition 2.1. The argument is similar to the ones for the homogeneous Boltzmann equation in [29]. We only give the essential steps.

Proof of Proposition 2.1. (ii). Let $(f_t)_t$, $(g_t)_t$ be two weak solutions and $\varphi_t(k) = \mathscr{F}[f_t](k)$, $\psi_t(k) = \mathscr{F}[g_t](k)$ be the corresponding Fourier transforms. Assuming $d_2(f_0, g_0) < \infty$, it follows that the first moments are equal initially and hence for all times. As a consequence $d_2(f_t, g_t) < \infty$ for all $t \geq 0$. Using a priori bounds of the moments of order $p \geq 2$ we get for $t \in [0, T]$, $T > 0$ arbitrary but fixed,

$$R_n(t,k) := \frac{1}{|k|^2}|(\hat{Q} - \hat{Q}_n)(\varphi_t,\varphi_t) - (\hat{Q} - \hat{Q}_n)(\psi_t,\psi_t)|(k) \leq C(T) r_n.$$

Here $r_n = \Lambda - \Lambda_n \to 0$ as $n \to \infty$.

Let us abbreviate $E_t = e^{tA^\top}$. A calculation shows that for $k \neq 0$

$$\begin{aligned} \frac{d}{dt}\left[\frac{e^{S_n t}(\varphi_t - \psi_t)(E_t k)}{|E_t k|^2}\right] =& \frac{2\,\langle A E_t k, E_t k\rangle}{|E_t k|^2} \frac{e^{S_n t}(\varphi_t - \psi_t)(E_t k)}{|E_t k|^2} \\ &+ \frac{e^{S_n t}}{|E_t k|^2}\left[\hat{Q}_n^+(\varphi_t,\varphi_t)(E_t k) - \hat{Q}_n^+(\psi_t,\psi_t)(E_t k)\right] + e^{S_n t} R_n. \end{aligned}$$

Here, we used a splitting of $\hat{Q}_n$ into gain and loss part. We estimate this term by term, in particular using (17) in Lemma 2.7 for the gain term. Abbreviating $h_t(k) := (\varphi - \psi)(t, e^{tA}k)/|e^{tA}k|^2$ and applying Gronwall's lemma yields

$$e^{S_n t} \|h_t\|_\infty \leq \|h_0\|_\infty e^{[2\|A\|+S_n]t} + C(T) r_n \int_0^t e^{S_n r} e^{[2\|A\|+S_n](t-r)} dr. \tag{18}$$

We divide by $e^{S_n t}$ and let $n \to \infty$. This concludes the proof since $\|h_t\|_\infty = d_2(f_t, g_t)$. □

2.1.3 Comparison Principle in Fourier Space

For the proof of Proposition 2.3 we consider the linearization of the cutoff equation given by

$$\partial_t \varphi + A^\top k \cdot \nabla \varphi = (\mathscr{L}_n - S_n I)(\varphi)(k), \quad \varphi(0, \cdot) = \varphi_0(\cdot). \tag{19}$$

Recall that $\mathscr{L}_n$, S_n are defined in (15) and (14), respectively. As in [5], one can see that the operator $\mathscr{L}_n : C_p \to C_p$ is bounded, where

$$C_p(\mathbb{R}^3) := \left\{ \varphi \in C(\mathbb{R}^3) \ : \ \|\varphi\|_{C_p} := \sup_k |\varphi(k)|/(1 + |k|^p) < \infty \right\}$$

for $p \geq 2$. Hence, the equation (19) defines a semigroup $\mathcal{P}_t^n : C_p \to C_p$.

In the non-cutoff case, the linear semigroup $\mathcal{P}_t^n$ is in general not well-defined for arbitrary functions u_0 as $n \to \infty$. However, the term $(\mathscr{L}_n - S_n)u$ still makes sense for $n \to \infty$ when u satisfies $u(0) = 0$ and $u \in C_b^2$. Let us hence define $u_{n,p} \in C_p$ via

$$u_{n,p}(k, t) := |k|^p \exp(-(\lambda_n(p) - p\,\|A\|)t),$$

where $\lambda_n(p)$ is given in (16).

Proof of Proposition 2.3 We approximate φ, ψ by solutions φ^n, $\psi^n \in C([0, \infty); \mathscr{F}_p)$ to equation (13) with cutoff kernel $0 \leq b_n$ and initial datum φ_0 resp. ψ_0. Let us define $U(k) := C_1|k|^p + C_2|k|^2$.

We can write in mild form

$$\begin{aligned} \varphi_t^n(k) - \psi_t^n(k) &= \varphi_0(k) - \psi_0(k) \\ &+ \int_0^t e^{-(t-r)(S_n + A^\top k \cdot \nabla)} \left[\hat{Q}_n^+(\varphi_r^n, \varphi_r^n) - \hat{Q}_n^+(\psi_r^n, \psi_r^n) \right](k) dr. \end{aligned}$$

Here, we used the semigroup notation $e^{-tA^\top k \cdot \nabla} \varphi(k) = \varphi(e^{-tA^\top} k)$. Set $v_t^n(k) := \varphi_t^n(k) - \psi_t^n(k)$ and estimate using the $\mathscr{L}$-Lipschitz property in Lemma 2.7 to get

$$|v_t^n(k)| \leq |v_0(k)| + \int_0^t e^{-(t-r)(S_n + A^\top k \cdot \nabla)} \mathscr{L}_n(|v_r^n|)(k) dr.$$

A comparison principle for the linear equation implies $|v_t^n(k)| \leq \mathcal{P}_t^n[|v_0|](k)$. Since $\mathscr{L}_n$ is positivity preserving, one can conclude that $\mathcal{P}_t^n$ is monotonicity preserving. Hence, $\mathcal{P}_t^n[|v_0|](k) \leq \mathcal{P}_t^n[U](k)$ due to our assumption $|v_0| \leq U$.

Now, we estimate $\mathcal{P}_t^n[U]$. It is straightforward to prove

$$u_{n,p}(k, t) \geq e^{-t(S_n + A^\top k \cdot \nabla)} |k|^p + \int_0^t e^{-(t-r)(S_n + A^\top k \cdot \nabla)} \mathscr{L}_n(u_{n,p}(\cdot, r)) dr.$$

A comparison principle for the linear equation yields

$$\mathcal{P}_t^n[|\cdot|^p](k) \leq u_{n,p}(t,k)$$

and we infer

$$\mathcal{P}_t^n\big[U\big](k) \leq C_1 u_{n,p}(t,k) + C_2 u_{n,2}(t,k).$$

We combining all estimates to get

$$|\varphi_t^n(k) - \psi_t^n(k)| \leq C_1 u_{n,p}(t,k) + C_2 u_{n,2}(t,k).$$

Since weak convergence implies pointwise convergence of the characteristic function, we can pass to the limit in the preceding inequality. Recall also $\lambda_n(p) \to \lambda(p)$ from Lemma 2.5. □

2.2 Regularity of Weak Solutions

We finally prove the regularity result in Proposition 2.1 (iii). We sketch the arguments following [28], which covers the homogeneous Boltzmann equation, i.e. $A = 0$.

Proof of Proposition 2.1. (iii). Let $(\psi_t)_t$ be the Fourier transform of a weak solution $(f_t)_t \subset \mathscr{P}_2$.

Step 1. Let us first state a coercivity estimate analogous to the one in [28, Lemma 1.4]. As in the original work, the non-cutoff assumption (6) is essential as well as the assumption that f_0 differs from a Dirac. There is $T_0 > 0$ and a constant $C > 0$, both depending on $f_0 \in \mathscr{P}_2$, such that for all $h \in L_2^2(\mathbb{R}^3)$ and all $t \in [0, T_0]$

$$\begin{aligned} &t \int_{\mathbb{R}^3} \langle \xi \rangle^{2s} |h(\xi)|^2 d\xi \\ &\leq C \left\{ \int_{\mathbb{R}^3} \int_{S^2} b(\hat{\xi} \cdot \sigma)(1 - |\psi(t, \xi_-)|) d\sigma |h(\xi)|^2 \, d\xi + \int_{\mathbb{R}^3} |h(\xi)|^2 \, d\xi \right\}. \end{aligned} \tag{20}$$

The constant $s \in (0, 1)$ is given in (6).

The proof of this estimate in [28] still works in our case, since in most arguments only the continuity of ψ and $\partial_t \psi$ is used. Only in the case when f_0 is supported on a straight line, the equation (13) is used. However, the same arguments can be applied to the function $\psi(t, e^{-A^\top t}\xi)$ along the characteristics of the drift term. Since $t \leq T_0$ is chosen sufficiently small, $e^{-A^\top t}$ is close to the identity and the original line of reasoning works.

Step 2. As in [28, Proof of Thm. 1.3] we prove smoothness of the solutions for $0 < t \leq T_0/2$. To this end, we test equation (13) with $M_\delta^2 \overline{\psi}$, where

$$M_\delta(t, \xi) := \langle \xi \rangle^{Nt^2 - 4} \langle \delta \xi \rangle^{-NT_0^2 - 4}, \quad N \in \mathbb{N}.$$

Here, N is chosen large enough such that $M_\delta \psi \in L_2^2$ for $t \leq T_0/2$. We use straightforward estimates for the drift term and bounds from the original proof in [28], which rely on (20), to get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |M_\delta(t, \xi) \psi(t, \xi)|^2 \, d\xi &\leq C(T_0, A) \int_{\mathbb{R}^3} |M_\delta(t, \xi) \, \psi(t, \xi)|^2 d\xi \\ &\quad + t \int_{\mathbb{R}^3} \left[4N \log \langle \xi \rangle - C_2 \langle \xi \rangle^{2s} \right] |M_\delta(t, \xi) \, \psi(t, \xi)|^2 d\xi. \end{aligned}$$

Since $\langle\xi\rangle^{2s} / \log\langle\xi\rangle \to \infty$ as $|\xi| \to \infty$, the last term on the right can be absorbed in the first term. Using Gronwall's lemma and letting $\delta \to 0$ one obtains

$$\int_{\mathbb{R}^3} |\langle\xi\rangle^{Nt^2-4}\psi(t,\xi)|^2\,d\xi \le C\int_{\mathbb{R}^3} |\langle\xi\rangle^{-4}\psi_0(\xi)|^2\,d\xi.$$

Since this holds for all $N \in \mathbb{N}$, we have $f(t,\cdot) \in \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^3)$ for $0 < t \le T_0/2$.

Step 3. Here, we extend the smoothness to times $t \ge T_0/2$. By the smoothness we infer that f_{t_0} has finite entropy for $t_0 \in (0, T_0/2)$, i.e.

$$H(f_{t_0}) := \int_{\mathbb{R}^3} f(t_0,v)\log f(t_0,v)\,dv < \infty.$$

An a priori estimation yields for some arbitrary but fixed time $T' > t_0$

$$H(f_t) \le H(f_{t_0}) + C(T', A), \quad t \in [t_0, T'].$$

To make this rigorous, we use a construction of weak solutions in L^1_2 with finite entropy initiating from f_{t_0}. Here, L^1_2 is the weighted L^1-space with weight $(1+|v|^2)$. Let us mention that this was done in [8] in the case of homoenergetic solutions for cutoff kernels. Using weak L^1-compactness arguments, following from the Dunford-Pettis theorem, yields solutions for the non-cutoff problem. See e.g. [32, Section 4] for such a construction in the case of the homogeneous Boltzmann equation. These solutions are unique by Proposition 2.1.

As was noticed in [28], using the result [1, Lemma 3], the estimate (20) holds now without the condition of small times. Thus, as above we get $f(t,\cdot) \in \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^3)$ for $t \ge t_0$. □

3 Self-similar Solutions and Self-similar Asymptotics

In this section, we give the proof of Theorem 1.3. Let us briefly summarize the strategy, which is partly guided by [5, 20]. We first study the linear equations satisfied by the second moments of a solution. Here, we use perturbation arguments to gain information of the eigenvalues and eigenvectors. Then, the existence of self-similar solutions follows from a fixed point argument.

The convergence to the self-similar solution in Theorem 1.3 (ii), is a consequence of the comparison principle in Proposition 2.3 and a longtime analysis of the second moments.

Finally, Theorem 1.3 (iii), is a result of successive application of the Povzner estimate.

3.1 Existence of Self-similar Solutions

Let us recall the following version of the Povzner estimate due to Mischler and Wennberg [25, Section 2]. As was noticed e.g. in [32, Appendix], their calculation also works in the non-cutoff case.

Lemma 3.1 *Let $\varphi(v) = |v|^{2+\delta}$ for $\delta > 0$. Then we have the following decomposition*

$$\int_{S^2} b(n\cdot\sigma)\left\{\varphi'_* + \varphi' - \varphi_* - \varphi\right\}d\sigma = G(v,v_*) - H(v,v_*)$$

with G, H satisfying

$$G(v,v_*) \le C\Lambda(|v||v_*|)^{1+\delta/2}, \quad H(v,v_*) \ge c\Lambda(|v|^{2+\delta} + |v_*|^{2+\delta})\left(1 - \mathbb{I}_{\{|v|/2<|v_*|<2|v|\}}\right). \tag{21}$$

Hence, for any $f \in \mathscr{P}_p$, *with* $2 < p \leq 4$, $p = 2 + \delta$ *we have for some* $C', \ c' > 0$

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} (G(v, v_*) - H(v, v_*)) f(dv) f(dv_*) \leq C' \Lambda \, \|f\|_2^2 - c' \Lambda \, \|f\|_p \,. \tag{22}$$

Proof The definition and estimates for G, H can be found in [25, Section 2], see also [32, Appendix]. To derive (22) note that $\delta = p - 2 \leq 2$, thus $1 + \delta/2 \leq 2$. We conclude by applying (21) and

$$(|v|^{2+\delta} + |v_*|^{2+\delta}) \mathbb{I}_{\{|v|/2 < |v_*| < 2|v|\}} \leq 8(|v||v_*|)^{1+\delta/2}.$$

□

The following result follows by choosing $\varphi_{jk}(v) = v_j v_k$ in the weak formulation (9), recalling that $t \mapsto \langle \varphi_{jk}, f_t \rangle$ is continuously differentiable, see [20, Prop. 4.10] or [5, Section 6].

Lemma 3.2 *The second moments* $M_{jk}(t) := \langle v_j v_k, f_t \rangle$ *of a solution to (8) satisfy the equations*

$$\frac{dM_t}{dt} = -AM_t - (AM_t)^\top - 2\bar{b} \left(M_t - \frac{tr\,(M_t)}{3} I \right) =: \mathcal{A}(\bar{b}, A) M_t \tag{23}$$

with the constant

$$\bar{b} = \frac{3\pi}{4} \int_0^\pi b(\cos\theta) \sin^3\theta d\theta. \tag{24}$$

Here, the linear operator $\mathcal{A}(\bar{b}, A) : \mathbb{R}^{3\times3}_{sym} \to \mathbb{R}^{3\times3}_{sym}$ acts on symmetric 3×3 matrices. As noticed in Remark 1.4, a self-similar solution f_{st} is a steady state of the equation (8) with A replaced by $A + \bar{\beta} I$. Hence, as in the cutoff case [20, Lemma 4.16], we study the linear map $\mathcal{A}(\bar{b}, A + \beta I) = \mathcal{A}(\bar{b}, A) - 2\beta I$.

Lemma 3.3 *Consider the linear operator* $\mathcal{A}(\bar{b}, A)$ *from Lemma 3.2. There is a sufficiently small constant* $\varepsilon_0 = \varepsilon_0(b) > 0$ *such that for all* $A \in \mathbb{R}^3$ *with* $\|A\| \leq \varepsilon_0$ *the following holds.*

- *(i) The eigenvalue* $2\bar{\beta} > 0$, $\bar{\beta} = \bar{\beta}(\bar{b}, A)$, *with largest real part is unique and simple. One can uniquely choose a corresponding eigenvector* $\bar{N} = \bar{N}(\bar{b}, A) \in \mathbb{R}^{3\times3}_{sym}$ *with* $\|N\| = 1$ *which is positive definite.*
- *(ii) The nonzero eigenvalues of* $\mathcal{A}(\bar{b}, A) - 2\bar{\beta} I$ *have real part less than* $-\nu$, *for some* $\nu > 0$.
- *(iii) In addition, there is* $c_0 > 0$ *such that* $|\bar{\beta}(\bar{b}, A)| \leq c_0 \varepsilon_0$.

Proof This is a perturbation argument noting that $\mathcal{A}(\bar{b}, A) : \mathbb{R}^{3\times3}_{sym} \to \mathbb{R}^{3\times3}_{sym}$ depends smoothly on A. For $A = 0$ there are the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2\bar{b}$ with a one-dimensional subspace of eigenvectors given by $M = KI$, $K \in \mathbb{R}$, respectively, a five-dimensional subspace of eigenvectors defined by $\{\mathrm{tr}\,(M) = 0\}$. The statement now follows by continuity results for eigenvalues when $\|A\|$ is small. We choose $2\bar{\beta}(\bar{b}, A)$ to be the eigenvalue close to $\lambda_1 = 0$ and let $\bar{N}(\bar{b}, A) \in \mathbb{R}^{3\times3}_{sym}$ be the corresponding normalized eigenvector close to I.

□

In the fixed point argument compactness is a consequence of the following estimate.

Lemma 3.4 *Consider a weak solution* $(f_t)_t \in C([0, \infty); \mathscr{P}_p)$ *to (8),* $2 < p \leq 4$, *with matrix* A *replaced by* $A + \bar{\beta} I$. *Assume that* $\|A\| \leq \varepsilon_0$ *with* $\varepsilon_0 > 0$ *from Lemma 3.3 and that the initial condition has zero mean as well as second moment* $K\bar{N}$. *Then, we have for all* $t \geq 0$

$$\int_{\mathbb{R}^3} v f_t(dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f_t(dv) = K\bar{N}_{ij}. \tag{25}$$

Furthermore, by decreasing $\varepsilon_0 = \varepsilon_0(b, p) > 0$*, if necessary, there is* $C_* = C_*(K)$ *such that for all* $t \geq 0$

$$\|f_0\|_p \leq C_* \implies \|f_t\|_p \leq C_*.$$

Proof As was mentioned already in the introduction, the first moment remains zero for all times. Since $\bar{N}$ is a stationary solution to the equation (23), we obtain (25). For the final statement, we use the Povzner estimate from Lemma 3.1

$$\begin{aligned}\frac{d}{dt}\|f_t\|_p &= \frac{d}{dt}\left\langle |v|^p, f_t\right\rangle \leq p\left\|A + \bar{\beta}I\right\|\|f_t\|_p + C'\Lambda\|f_r\|_2^2 - c'\Lambda\|f_t\|_p \\ &\leq \left[p\varepsilon_0(1 + c_0) - c'\Lambda\right]\|f_t\|_p + C'\Lambda K^2.\end{aligned}$$

For ε_0 sufficiently small we have $\delta := c'\Lambda - p\varepsilon_0(1 + c_0) > 0$ and hence from a Gronwall type argument, in conjunction with $\|f_0\|_p \leq C_*$,

$$\|f_t\|_p \leq C_* e^{-\delta t} + \frac{C'\Lambda K^2}{\delta} = C_* + \left(1 - e^{-\delta t}\right)\left(\frac{C'\Lambda K^2}{\delta} - C_*\right)$$

We conclude by choosing $C_* = C_*(K)$ sufficiently large. □

For convenience let us recall the following fact concerning the topology induced by the metric d_2, see e.g. [29, Lemma 1, Lemma 2].

Lemma 3.5 *Define* $D_e \subset \mathscr{P}_2$ *by*

$$D_e = \left\{ f \in \mathscr{P}_2 \,:\, \int v f(dv) = 0, \quad \int |v|^2 f(dv) = e \right\}, \quad e \geq 0.$$

Consider $f^n, f \in \mathscr{P}_2$ *for* $n \in \mathbb{N}$*. Then, the following statements are equivalent:*

(i) $f_n, f \in D_e$ *and* $f^n \rightharpoonup f$ *weakly, i.e.* $\langle \psi, f^n \rangle \to \langle \psi, f \rangle$ *as* $n \to \infty$ *for all* $\psi \in C_b$*;*
(ii) $d_2(f_n, f) \to 0$ *as* $n \to \infty$*.*

Proof of Theorem 1.3. (i) We use similar arguments as in [20, Section 4.3]. Let us define the set $\mathscr{U} \subset \mathscr{P}_p$, $2 < p \leq 4$, consisting of measures $f \in \mathscr{P}_p$ with

$$\int_{\mathbb{R}^3} v f(dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f(dv) = K\bar{N}_{ij}, \quad \|f\|_p \leq C_*.$$

Here, $\bar{N}$ is given in Lemma 3.3 and we assume that $\|A\| \leq \varepsilon_0$ as in Lemmas 3.3, 3.4. Note that $\mathscr{U}$ is a convex, compact subset of the space $\mathscr{M}_f(\mathbb{R}^3)$ of signed Radon measures on $\mathbb{R}^3$ with finite total variation, equipped with the weak-$*$ topology. With this topology $\mathscr{M}_f(\mathbb{R}^3)$ is a locally convex space. Note that weak convergence within $\mathscr{U}$ implies convergence w.r.t. the metric d_2 by Lemma 3.5.

Let us define the nonlinear semigroup $\mathscr{S}_t : \mathscr{P}_p \to \mathscr{P}_p$ mapping any f_0 to f_t, where $(f_t)_t$ is the unique solution to the equation (8) with matrix $A + \bar{\beta}I$ replacing A and initial condition f_0. By Lemma 3.4 we have $\mathscr{S}_t : \mathscr{U} \to \mathscr{U}$. Furthermore, $f \mapsto \mathscr{S}_t f$ is continuous on $\mathscr{U}$ for each $t \geq 0$, as follows from (12). We can now apply Schauder's fixed point theorem to $\mathscr{S}_{1/n} : \mathscr{U} \to \mathscr{U}$ yielding a fixed point f_{st}^n. By compactness of $\mathscr{U}$ we have for a subsequence $f_{st}^{n_k} \to f_{st}$ as $k \to \infty$. As a consequence of the semigroup property, it holds $\mathscr{S}_{m/n_k} f_{st}^{n_k} = f_{st}^{n_k}$ for any $k, m \in \mathbb{N}$.

Now, let $t \geq 0$ be arbitrary. We can find a sequence of integers $m_k \in \mathbb{N}$ with $m_k/n_k \to t$ as $k \to \infty$ and write

$$f_{st} = \lim_{k\to\infty} f_{st}^{n_k} = \lim_{k\to\infty} \mathscr{S}_{m_k/n_k} f_{st}^{n_k} = \mathscr{S}_t f_{st}.$$

To verify the last equality, we use (12) and estimate

$$d_2\left(\mathscr{S}_{m_k/n_k} f_{st}^{n_k}, \mathscr{S}_t f_{st}\right) \le d_2\left(\mathscr{S}_{m_k/n_k} f_{st}^{n_k}, \mathscr{S}_{m_k/n_k} f_{st}\right) + d_2\left(\mathscr{S}_{m_k/n_k} f_{st}, \mathscr{S}_t f_{st}\right)$$
$$\le e^{2(t+1)\|A+\bar{\beta}I\|} d_2\left(f_{st}^{n_k}, f_{st}\right) + d_2\left(\mathscr{S}_{m_k/n_k} f_{st}, \mathscr{S}_t f_{st}\right).$$

The first term goes to zero, since $f_{st}^{n_k} \to f_{st}$ in $\mathscr{U}$. By an approximation we obtain from Proposition 2.1 (i) that $t \mapsto \langle \psi, f_t \rangle$ is continuous for any $\psi \in C_b$. Since the second moments are $K\bar{N}$, we conclude with Lemma 3.5 that the last term goes to zero.

This yields a self-similar solution with zero momentum. To obtain mean $U \in \mathbb{R}^3$ we use the change of variables $v \mapsto v - e^{-tA}U$. For $K > 0$ any self-similar profile is smooth by Proposition 2.1 (iii). Finally, one can see that the Dirac measure $f_{st} = \delta_0$ is a weak solution to (11), yielding a self-similar profile with $K = 0$. This concludes the existence proof. □

3.2 Uniqueness and Stability of Self-similar Solutions

Here, we prove that any solution to (8) converges to a self-similar solution after a change of variables.

Proof of Theorem 1.3 (ii) Let us denote by $\Psi = \mathscr{F}[f_{st}]$ the characteristic function of the profile $f_{st} \in \mathscr{P}_p, 4 \ge p > 2$ with second moments $\bar{N}$. We assume $\|A\| \le \varepsilon_0$, where $\varepsilon_0 > 0$ is chosen sufficiently small, such that part (i) of Theorem 1.3 holds.

For a solution $(f_t)_t \subset \mathscr{P}_p$, $2 < p \le 4$ to (8) we take $(\tilde{f}_t)_t$ as in Theorem 1.3 (ii), which yields a solution to (8) with matrix $A + \bar{\beta}I$ and zero momentum. Let us denote the characteristic functions of $(\tilde{f}_t)_t$ by $(\varphi_t)_t$ and the second moments by $(M_t)_t$.

By Lemma 3.2, $(M_t)_t$ satisfies the equation $M_t' = (\mathcal{A}(\bar{b}, A) - 2\bar{\beta}I)M_t$. Furthermore, by Lemma 3.3 the nonzero eigenvalues of $\mathcal{A}(\bar{b}, A) - 2\bar{\beta}I$ have real part less than $-\nu < 0$. The steady states are given by the span of $\bar{N}$. Thus, there is $C = C(M_0) \ge 0$ and $\alpha = \alpha(M_0) \ge 0$ such that

$$\left\| M_t - \alpha^2 \bar{N} \right\| \le Ce^{-\nu t}. \tag{26}$$

Using a Povzner estimate as in the proof of Lemma 3.4 we get $\sup_{t\ge0} \left\| \tilde{f}_t \right\|_p < \infty$ as long as $\|A\| \le \varepsilon_0$ is sufficiently small. Note that the second moments are uniformly bounded by (26). This yields a uniform estimate of $\|\varphi_t\|_{C^{2,p-2}}$.

Observe that $\Psi(\alpha\cdot)$ is the characteristic function of the steady state $\alpha^{-3} f_{st}(v/\alpha)$ with second moments $\alpha^2\bar{N}$. We estimate the characteristic functions

$$|\varphi_t(k) - \Psi(\alpha k)| \le \left|\varphi_t(k) - 1 + \frac{1}{2}M_t : k\otimes k\right| + \frac{1}{2}\left\|M_t - \alpha^2\bar{N}\right\| |k|^2$$
$$+ \left|1 - \frac{1}{2}\alpha^2\bar{N} : k\otimes k - \Psi(\alpha k)\right|.$$

For the first term we use a Taylor expansion, in conjunction with the fact that $D^2\varphi_t$ is at least $(p-2)$-Hölder continuous with $\left\|D^2\varphi_t\right\|_{C^{p-2}} \le C_*$. We can assume here w.l.o.g $p < 3$. We have

$$\left|\varphi_t(k) - 1 + \frac{1}{2}M_t : k\otimes k\right| \le C_*|k|^p.$$

The last term is treated similarly due to $\Psi \in \mathscr{F}_p$. For the second term we apply (26). This yields

$$|\varphi_t(k) - \Psi(\alpha k)| \leq C|k|^p + Ce^{-\nu t}|k|^2.$$

Now, we apply the comparison principle in Proposition 2.3 starting at time T to obtain

$$|\varphi_{T+t}(k) - \Psi(\alpha k)| \leq Ce^{-(\lambda(p)-p\|A+\bar{\beta}I\|)t}|k|^p + Ce^{-\nu T+2\|A+\bar{\beta}I\|t}|k|^2.$$

Now, we further assume that $\varepsilon_0 > 0$ is small enough to ensure

$$\left\|A + \bar{\beta}I\right\| \leq (1 + c_0)\, \|A\| \leq \min\left(\frac{\lambda(p)}{2p}, \frac{\nu}{4}\right).$$

Thus, we get for $t = T$ and $\theta' = \min(\frac{\lambda(p)}{2}, \frac{\nu}{2})$

$$|\varphi_{2T}(k) - \Psi(\alpha k)| \leq Ce^{-\theta' T}\left(|k|^p + |k|^2\right), \tag{27}$$

where $C = C(\varphi_0, p)$. Now, we apply the following inequality valid for all $\varphi, \psi \in \mathscr{F}_p$

$$d_2(\varphi, \psi) \leq c_p(\gamma + \gamma^{2/p}), \quad \gamma := \sup_k \frac{|\varphi - \psi|(k)}{|k|^2 + |k|^p}. \tag{28}$$

This can be proved by splitting the supremum in $d_2(\varphi, \psi)$ into $|k| \leq R$ and $|k| \geq R$ and minimizing over R. Combining both (27) and (28) yields for some $\theta > 0$

$$d_2(\varphi_t, \Psi(\alpha\cdot)) \leq Ce^{-\theta t}.$$

This concludes the proof. □

3.3 Finiteness of Higher Moments

To prove part (iii) of Theorem 1.3, we need an extension of Lemma 3.4.

Lemma 3.6 *Let $M \in \mathbb{N}$, $M \geq 3$ and $p \geq M$. Consider a solution $(f_t)_t \in C([0, \infty); \mathscr{P}_p)$ to (8) with A replaced by $A + \bar{\beta}I$ satisfying (25). Let $\|A\| \leq \varepsilon_0$ and $\varepsilon_0 > 0$ from Lemma 3.3.*

Then, there is $\varepsilon_M \leq \varepsilon_0$ and $C_* = C_*(K, M)$ such that: if $\|A\| \leq \varepsilon_M$ we have for all $t \geq 0$

$$\|f_0\|_M \leq C_* \implies \|f_t\|_M \leq C_*.$$

Proof This can be proved by induction over M by applying repeatedly Lemma 3.1. The case $M = 3, 4$ is covered by Lemma 3.4 and at each step one has to choose $\varepsilon_M \leq \varepsilon_{M-1}$ and $\|A\| \leq \varepsilon_M$ to absorb the drift term.

□

Proof of Theorem 1.3. (iii). We argue as for (i) of Theorem 1.3. However, now we include the uniform bound $\|f\|_M \leq C_*(M, K)$ in the definition of the sets $\mathscr{U}$. The so constructed stationary solutions coincide with the ones in (i) by uniqueness. □

4 Application to Simple and Planar Shear

In this section, we discuss the longtime behavior of homoenergetic solutions in the case of simple and planar shear. Recall that homoenergetic flows have the form $g(t, x, v) = f(t, v - L(t)x)$ and $f = f(t, v)$ satisfies

$$\partial_t f - L(t)v \cdot \nabla f = Q(f, f) \tag{29}$$

with the matrix $L(t) = (I + tL_0)^{-1}L_0$. Under the assumption $\det(I + tL_0) > 0$ for all $t \geq 0$, one can study the form of $L(t)$ as $t \to \infty$ (see [20, Section 3]). We consider the case of simple shear resp. planar shear ($K \neq 0$)

$$L(t) = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{resp.} \quad L(t) = \frac{1}{t}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{t^2}\right) \quad (t \to \infty). \tag{30}$$

In the first case, (29) preserves mass, since $\operatorname{tr} L = 0$, and our study applies for K sufficiently small. Alternatively, one can assume a largeness condition on the kernel b, see the assumption below.

Let us now turn to planar shear and write $L(t) = A/(1 + t) + \tilde{A}(t)$ with $\operatorname{tr} A = 1$, $\left\|\tilde{A}(t)\right\| \leq \mathcal{O}\left(1/(1 + t)^2\right)$. First, let us introduce the time-change $\log(1 + t) = \tau$ and set $f(t, v) = F(\tau, v)/(t + 1)$ yielding the equation (after multiplying with $(1 + t)^2$)

$$\partial_\tau F - div((A + B(\tau))v \cdot F) + \operatorname{tr} B(\tau)F = Q(F, F) \tag{31}$$

where $B(\tau) = (1 + t)\tilde{A}(t) = \mathcal{O}\left(1/(1 + t)\right) = \mathcal{O}(e^{-\tau})$. The well-posedness theory of (31) does not change compared to (8) and so we omit further details about existence, uniqueness and regularity. We apply our results to (31) yielding a self-similar asymptotics. More precisely, we have the following result (note that we write t instead of τ in the theorem).

Theorem 4.1 *Consider (31) with $A \in \mathbb{R}^{3\times3}$ and $B_t \in C([0, \infty); \mathbb{R}^{3\times3})$ such that $\|B_t\| = \mathcal{O}(e^{-t})$. Let $(F_t)_t \subset \mathscr{P}_p$, $2 < p$, be a weak solution to (31) with $F_0 \in \mathscr{P}_p$ and first moments*

$$\int_{\mathbb{R}^3} vF_0(dv) = U.$$

We define $m_t \in \mathbb{R}$, $E_t \in \mathbb{R}^{3\times3}$ as follows

$$m_t = \int F_t(v)dv = \exp\left(-\int_0^t tr\, B_s\, ds\right), \quad \lim_{t\to\infty} m_t = m_\infty, \quad E_t' = (A + B_t)E_t, \quad E_0 = I.$$

There is a constant $\varepsilon_0 = \varepsilon_0(m_\infty \bar{b}, p) > 0$ such that for $\|A\| \leq \varepsilon_0$, the following holds. Defining

$$\tilde{F}_t := \frac{e^{3\bar{\beta}t}}{m_t} F_t\left(e^{\bar{\beta}t}v + E_tU\right), \quad \tilde{f}_{st}(v) = f_{st}(v\alpha_\infty^{-1})\alpha_\infty^{-3}$$

for a constant $\alpha_\infty = \alpha_\infty(F_0)$ we have for $\lambda > 0$

$$d_2(\tilde{F}_t, \tilde{f}_{st}) \leq Ce^{-\lambda t}.$$

Here, $f_{st} \in \mathscr{P}_p$ is the solution to

$$\div((A + \bar{\beta}I)v \cdot f_{st}) + m_\infty Q(f_{st}, f_{st}) = 0, \quad \int_{\mathbb{R}^3} v_i v_j\, f_{st}(v)\, dv = \bar{N}_{ij},$$

as in Theorem 1.3 with the corresponding objects $\bar{\beta} = \bar{\beta}(m_\infty \bar{b}, A)$*,* $\bar{N} = \bar{N}(m_\infty \bar{b}, A)$.

With this let us now go back to solutions $(f_t)_t$ to equation (29) with $L(t) = A/(1+t) + \tilde{A}(t)$. To apply the previous result, we need $\|A\| \le \varepsilon_0$. This might not be true for A coming from the matrix $L(t)$ above. However, one can instead assume a largeness condition on the kernel b. To see this, let us rescale time $\tau \mapsto \tau M$ yielding

$$\partial_\tau F - \frac{1}{M} div((A + B(\tau))v \cdot F) + \frac{1}{M} \operatorname{tr} B(\tau) F = \frac{1}{M} Q(F, F).$$

In particular, the collision kernel is given by b/M. We can hence consider the following assumption. A similar condition was also used in [20, Section 5.2].
Assumption. Assume that the kernel b is chosen such that

$$\|A/M\| \le \varepsilon_0 (m_\infty \bar{b}/M)$$

is satisfied for some $M > 0$. Recall the definition of $\bar{b}$ in (24).

Under this assumption, we can apply Theorem 4.1 to obtain the asymptotics in terms of $(f_t)_t$ solving (29). For this we undo the above transformations yielding

$$\frac{e^{t/M} e^{3\bar{\beta}t}}{m_t} f\left(e^{t/M} - 1, e^{\bar{\beta}t} v + E_t U\right) \to f_{st}(v\alpha^{-1})\alpha^{-3} \qquad \text{as } t \to \infty. \tag{32}$$

Here, $U \in \mathbb{R}^3$ is the mean of the initial condition $f_0 \in \mathscr{P}_p$ and $\alpha = \alpha(f_0)$ is as in Theorem 4.1. For $B_\tau := e^\tau \tilde{A}(e^\tau - 1)$ we defined

$$m_t = \exp\left(-\frac{1}{M}\int_0^t \operatorname{tr} B_s\, ds\right), \quad E_t' = \frac{1}{M}(A + B_t)E_t, \quad E_0 = I.$$

The convergence in (32) appears with an order $\mathcal{O}(e^{-\lambda\tau}) = \mathcal{O}(t^{-\lambda M})$ w.r.t the metric d_2.

Finally, let us give the main arguments for the proof of Theorem 4.1 following the analysis in Sect. 3.

Proof of Theorem 4.1 *Preparation.* We rescale the solution $G_t(v) = F_t(v + E_t U)/m_t$ so that the mass is one and the momentum is zero. This solves

$$\partial_t G - \div((A + B_t)v \cdot G) = m_t Q(G, G), \quad \int G_t(dv) = 1, \quad \int v G_t(dv) = 0.$$

The assumption $\|B_t\| = \mathcal{O}(e^{-t})$ implies $m_t \to m_\infty > 0$ and $|m_{T+t} - m_T| \le Ce^{-T}$. We introduce the self-similar variables $G_t(v) = f_t(ve^{-\bar{\beta}t})e^{-3\bar{\beta}t}$ and get

$$\partial_t f - \div((A + \bar{\beta}I + B_t)v \cdot f) = m_t Q(f, f). \tag{33}$$

where $\bar{\beta} = \bar{\beta}(A, m_\infty \bar{b})$ is as in Theorem 1.3 or Lemma 3.3 when considering the collision kernel $m_\infty \bar{b}$.

Now, the plan is as follows. First, we study the longtime behavior of the second moments M_t of f_t in Step 1. Then, in Step 2, we want to compare (33) to solutions $g^{(T)}$ to

$$\partial_t g^{(T)} = \div\left((A + \bar{\beta}I)v \cdot g^{(T)}\right) + m_\infty Q\left(g^{(T)}, g^{(T)}\right), \quad g_0^{(T)} = f_T. \tag{34}$$

This equation has the stationary solution f_{st}. In Step 3, we apply Theorem 1.3 to $g^{(T)}$ to obtain $g^{(T)} \to f_{st}(\alpha_T^{-1}\cdot)\alpha_T^{-3}$. Altogether, we conclude $f_t \to f_{st}(\alpha_\infty^{-1}\cdot)\alpha_\infty^{-3}$. Here, α_T, α_∞ are constants, which precise values will be apparent below.

Step 1. Let M_t be the second moments of f_t, which satisfy (see also Lemma 3.2)

$$\frac{dM_t}{dt} = \mathcal{A}(m_t\bar{b}, A + 2\bar{\beta}I)M_t + \mathcal{B}_t M_t,$$

where $\mathcal{A}(m_t\bar{b}, A + \bar{\beta}I)$, $\mathcal{B}_t$ are linear operators $\mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}^{3\times 3}_{sym}$ and $\|\mathcal{B}_t\| \leq Ce^{-t}$. The first operator corresponds to the drift term with matrix $A + \bar{\beta}I$ and the collision operator. The second operator captures the drift term with B_t. Due to the linear dependence of $\mathcal{A}$ w.r.t. $m_\infty\bar{b}$ we can write

$$\frac{dM_t}{dt} = \mathcal{A}(m_\infty\bar{b}, A + \bar{\beta}I)M_t + \mathcal{R}_t M_t.$$

Since $|m_t - m_\infty| \leq Ce^{-t}$ we still have $\|\mathcal{R}_t\| \leq Ce^{-t}$. The results of Lemma 3.3 hold for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A} := \mathcal{A}(m_\infty\bar{b}, A + \bar{\beta}I)$. Using Duhamel's formula one can prove that

$$e^{\mathcal{A}t}M_T \to \alpha_T^2\bar{N}, \quad M_t \to \alpha_\infty^2\bar{N}$$

as $t \to \infty$ for all $T \geq 0$ with a convergence of order $Ce^{-\nu t}$. Furthermore, $|\alpha_\infty^2 - \alpha_T^2| \leq Ce^{-T}$ where the constants $C > 0$ are always independent of T.

Step 2. Now, we compare f with $g^{(T)}$ satisfying (34) via the following estimate for all $t, T \geq 0$

$$d_2\left(f_{t+T}, g_t^{(T)}\right) \leq Ct\, e^{-T+2\|A+\bar{\beta}I\|t}. \tag{35}$$

Here, C is independent of t, T. This inequality can be proved as part (ii) in Proposition 2.1. The difference here is the coefficient m_t in front of the collision operator, as well as the term due to B_t in (33). Both of them lead to a term of order e^{-T}. We get analogously to (18)

$$e^{m_\infty S_n t}d_2(\varphi_{t+T}, \psi_t) \leq \left(r_n + Ce^{-T}\right)\int_0^t e^{m_\infty S_n r}e^{[2\|A+\bar{\beta}I\|+m_\infty S_n](t-r)}dr.$$

Dividing by $e^{m_\infty S_n t}$ and sending $n \to \infty$ yields (35).

Step 3. Now, we apply Theorem 1.3 to the solutions $g^{(T)}$ to (34). For this, let f_{st} be the stationary solution to (34) with second moments $\bar{N}$ and $\Psi = \mathscr{F}[f_{st}]$. We get in Fourier space $\psi_t^{(T)} = \mathscr{F}[g^{(T)}]$, with α_T as in Step 1,

$$d_2\left(\psi_t^{(T)}, \Psi(\alpha_T\,\cdot)\right) \leq Ce^{-\theta t}.$$

The only problem now is that the constant C might depend on the initial condition f_T and thus on T. If we trace back the dependence of this constant in the proof of Theorem 1.3 (ii), then two constants C_1, C_2 contribute. The first one satisfies

$$\left\|e^{\mathcal{A}_{\bar{\beta}}t}M_T - \alpha_T^2\bar{N}\right\| \leq C_1 e^{-\nu t}$$

and depends only on M_T, which is uniformly bounded. The second constant is a uniform bound on the moments of order $4 \geq p > 2$, see Step 2 in the proof of Theorem 1.3 (ii). Looking at the arguments there, we see that it suffices to show $\sup_t \|f_t\|_p < \infty$ in order to obtain $\sup_{t,T}\left\|g_t^{(T)}\right\|_p < \infty$. This can be proved again by an application of the Povzner estimate to the equation (33). The difference here is an additional term due to B_t. Since this is integrable in time one can choose $\varepsilon_0 > 0$ small enough in exactly the same way.

Conclusion. Let us combine all our estimates in Fourier space $\varphi_t = \mathscr{F}[f_t]$, $\psi_t^{(T)} = \mathscr{F}[g_t^{(T)}]$

$$d_2\left(\varphi_{t+T}, \Psi(\alpha_\infty \cdot)\right) \leq d_2\left(\varphi_{t+T}, \psi_t^{(T)}\right) + d_2\left(\psi_t^{(T)}, \Psi(\alpha_T \cdot)\right) + d_2\left(\Psi(\alpha_T \cdot), \Psi(\alpha_\infty \cdot)\right)$$
$$\leq C t\, e^{-T+2\|A+\bar{\beta}I\| t} + Ce^{-\theta t} + Ce^{-T}.$$

The first two estimates follow from Step 2 and Step 3. The last one follows from a Taylor expansion and $|\alpha_\infty^2 - \alpha_T^2| \leq Ce^{-T}$. Let us now choose $t = T$ and ensure $2\left\|A + \bar{\beta}I\right\| \leq 2(1 + c_0)\|A\| \leq 1/2$, by choosing $\|A\|$ sufficiently small. This concludes the proof. □

Remark 4.2 Let us comment on the smallness condition on A, which was used at three different points: (1) in Lemma 3.3 when studying the eigenvalues resp. eigenvectors, (2) in Lemma 3.4 for a uniform bound in time of moments of order $p > 2$ and (3) in the proof of the convergence to the self-similar profile. The first two incidences concerned the existence of self-similar solutions. In the case of simple shear, i.e. A is given by the first matrix in (30), Lemma 3.3 has been extended for large values of K via explicit computations in [20, Section 5.1]. Furthermore, they formulated a condition to extend (2) for such matrices A. However, this condition has not been studied in further detail. Concerning the stability result, different convergence methods would be needed, which take into account the effect of the drift term.

Acknowledgements The author thanks Juan J.L. Velázquez for the suggestion of the problem and helpful discussions.

Funding. Open Access funding enabled and organized by Projekt DEAL. The author has been supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the collaborative research centre *The mathematics of emerging effects* (CRC 1060, Project-ID 211504053) and the Bonn International Graduate School of Mathematics at the Hausdorff Center for Mathematics (EXC 2047/1, Project-ID 390685813).

Data availability. Not applicable.

Declarations

Conflicts of interest/Competing interests. The author has no conflicts of interest to declare that are relevant to the content of this article.

References

1. Alexandre, R., Desvillettes, L., Villani, C., Wennberg, B.: Entropy dissipation and long-range interactions. Arch. Ration. Mech. Anal. **152**(4), 327–355 (2000)
2. Bobylev, A.V.: Fourier transform method in the theory of the Boltzmann equation for Maxwellian molecules. Dokl. Akad. Nauk SSSR **225**, 1041–1044 (1975)
3. Bobylev, A.V.: The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. Sov. Sci. Rev. C **7**, 111–233 (1988)

4. Bobylev, A.V., Cercignani, C., Gamba, I.M.: On the self-similar asymptotics for generalized nonlinear kinetic Maxwell models. Commun. Math. Phys. **291**(3), 599–644 (2009)
5. Bobylev, A.V., Nota, A., Velázquez, J.J.L.: Self-similar asymptotics for a modified Maxwell-Boltzmann equation in systems subject to deformations. Commun. Math. Phys. **380**(1), 409–448 (2020)
6. Cannone, M., Karch, G.: Infinite energy solutions to the homogeneous Boltzmann equation. Commun. Pure Appl. Math. **63**(6), 747–778 (2010)
7. Cercignani, C.: The Boltzmann Equation and Its Applications. Springer, New York (1988)
8. Cercignani, C.: Existence of homoenergetic affine flows for the Boltzmann equation. Arch. Ration. Mech. Anal. **105**(4), 377–387 (1989)
9. Cercignani, C.: Shear flow of a granular material. J. Stat. Phys. **102**(5), 1407–1415 (2001)
10. Cercignani, C.: The Boltzmann equation approach to the shear flow of a granular material. R. Soc. Lond. Philos. Trans. Ser. A **360**(1792), 407–414 (2002). Discrete modelling and simulation of fluid dynamics (Corse, 2001)
11. Dayal, K., James, R.D.: Nonequilibrium molecular dynamics for bulk materials and nanostructures. J. Mech. Phys. Solids **58**(2), 145–163 (2010)
12. Dayal, K., James, R.D.: Design of viscometers corresponding to a universal molecular simulation method. J. Fluid Mech. **691**, 461–486 (2012)
13. Duan, R., Liu, S.: The Boltzmann equation for uniform shear flow. Arch. Ration. Mech. Anal. **242**(3), 1947–2002 (2021)
14. Gabetta, G., Toscani, G., Wennberg, B.: Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation. J. Stat. Phys. **81**(5), 901–934 (1995)
15. Galkin, V.: On a class of solutions of Grad's moment equations. J. Appl. Math. Mech. **22**(3), 532–536 (1958)
16. Galkin, V.: One-dimensional unsteady solution of the equation for the kinetic moments of a monatomic gas. J. Appl. Math. Mech. **28**(1), 226–229 (1964)
17. Galkin, V.S.: Exact solutions of the kinetic-moment equations of a mixture of monatomic gases. Fluid Dyn. **1**(5), 29–34 (1966)
18. Garzó, V., Santos, A.: Kinetic Theory of Gases in Shear Flows. Springer, Amsterdam (2003)
19. James, R.D., Nota, A., Velázquez, J.J.L.: Long-time asymptotics for homoenergetic solutions of the Boltzmann equation: collision-dominated case. J. Nonlinear Sci. **29**(5), 1943–1973 (2019)
20. James, R.D., Nota, A., Velázquez, J.J.L.: Self-similar profiles for homoenergetic solutions of the Boltzmann equation: particle velocity distribution and entropy. Arch. Ration. Mech. Anal. **231**(2), 787–843 (2019)
21. James, R.D., Nota, A., Velázquez, J.J.L.: Long time asymptotics for homoenergetic solutions of the Boltzmann equation. Hyperbolic-dominated case. Nonlinearity **33**(8), 3781–3815 (2020)
22. Kepka, B.: Longtime behavior for homoenergetic solutions in the collision dominated regime for hard potentials (2022). arXiv:2202.09074
23. Lu, X., Mouhot, C.: On measure solutions of the Boltzmann equation, part I: moment production and stability estimates. J. Differ. Equ. **252**(4), 3305–3363 (2012)
24. Matthies, K., Theil, F.: Rescaled objective solutions of Fokker-Planck and Boltzmann equations. SIAM J. Math. Anal. **51**(2), 1321–1348 (2019)
25. Mischler, S., Wennberg, B.: On the spatially homogeneous Boltzmann equation. Ann. l'I.H.P. Anal. Non Linéaire **16**(4), 467–501 (1999)
26. Morimoto, Y.: A remark on Cannone-Karch solutions to the homogeneous Boltzmann equation for Maxwellian molecules. Kinetic Relat. Models **5**(3), 551–561 (2012)
27. Morimoto, Y., Wang, S., Yang, T.: Measure valued solutions to the spatially homogeneous Boltzmann equation without angular cutoff. J. Stat. Phys. **165**(5), 866–906 (2016)
28. Morimoto, Y., Yang, T.: Smoothing effect of the homogeneous Boltzmann equation with measure valued initial datum. Ann. l'I.H.P. Anal. Non Linéaire **32**(2), 429–442 (2015)
29. Toscani, G., Villani, C.: Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. J. Stat. Phys. **94**(3), 619–637 (1999)
30. Truesdell, C.: On the Pressures and the Flux of Energy in a Gas according to Maxwell's Kinetic Theory, II, volume 5. Indiana University Mathematics Department, 2020/11/09/ (1956)
31. Truesdell, C., Muncaster, R.G.: Fundamentals of Maxwell's kinetic theory of a simple monatomic gas. In: volume 83 of Pure and Applied Mathematics. Academic Press, New York (1980)
32. Villani, C.: On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. Arch. Ration. Mech. Anal. **143**(3), 273–307 (1998)

33. Villani, C.: A Review of Mathematical Topics in Collisional Kinetic Theory. In: Volume 1 of Handbook of Mathematical Fluid Dynamics. North-Holland (2002)