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On Solutions of the Modified Boltzmann Equation

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Abstract

We study the self-similar solutions and related questions for the modified Boltzmann equation. This equation formally coincides with the classical spatially homogeneous Boltzmann equation in the presence of an artificial force term proportional to a matrix A. The modification is connected with applications to homoenergetic solutions of the spatially inhomogeneous Boltzmann equation. Our study is restricted to the case of Maxwell-type interactions. We investigate existence and uniqueness of self-similar solutions and their role as attractors for large values of time. Similar questions were studied recently under assumption of *sufficient smallness of norm* $\|A\|$ without explicit estimates of that smallness. In this paper we fill this gap and prove, in particular, that all important facts related to self-similar solutions remain valid for moderately small values $\|A\| = O(10^{-1})$ in appropriate dimensionless units.

Keywords Boltzmann equation · Shear flow · Homoenergetic solutions

1 Introduction

There exists a class of particular solutions to the spatially inhomogeneous Boltzmann equation that corresponds to so-called *homoenergetic affine flows*. These solutions were introduced in 1956 independently by Galkin [17] and Truesdell [25]. There are many related references, we just mention two books [18, 26], and contributions of Cercignani to this area [12–14].

Roughly speaking, these are spatially inhomogeneous flows of gas having the linear (with respect to spatial variable $x \in \mathbb{R}^3$) profile of the bulk velocity. This assumption allows to reduce the problem to the modified spatially homogeneous Boltzmann equation, which is considered in the present paper (see also [4]). Our aim is to study solutions of this equation having the self-similar form. Self-similar solutions of nonlinear equations of mathematical physics are always interesting for both physicists and mathematicians. The recent interest to this problem (see also [15]) is partly caused by papers [4, 19–21] related to that kind of flows. In particular, we mention the first proof of existence of self-similar solutions in [19] and more detailed study of these solutions and their asymptotic properties in [4].

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The modified Boltzmann equation from [4] depends on a small parameter related to a bulk velocity of the flow. This small parameter is assumed in these papers to be "as small as we want".

Obviously this assumption is not very convenient for applications. It is important to know more precisely the conditions of validity of mathematically rigorous results. One of the main goals of the present paper is to prove that practically all results of our previous paper [4] remain valid also for moderately small values of the parameter, when the contribution of the perturbation term in the equation does not exceed roughly 10% of the contribution of the main term.

The paper is organized as follows. The connection of the modified Boltzmann equation with homoenergetic affine flows is explained in the beginning of Sect. 2. We confine ourselves in this paper to the case of Maxwell-type interactions. The statement of the problem of constructing the self-similar solution to the modified Boltzmann equation in $\mathbb{R}^d$, $d \geq 2$, is discussed in Sect. 2. The intensity of the extra term in the equation is characterized by the norm $\|A\|$ of a matrix A of order d. In Sect. 3 we perform a detailed study of tensor B of second moments of the solution. This problem was previously briefly considered in [4] with assumption that $\|A\|$ is "sufficiently small". It is shown in Sect. 3 (Lemmas 3.1 and 3.2) that the same results for corresponding eigenvalue problem for matrix B can be proved for moderate values of $\|A\|$, having roughly the order $O(10^{-1})$ in dimensionless units. The self-similar profile is constructed in the Fourier representation in Sect. 4. Here we follow (in slightly more elementary way convenient for physicists) the same way as in [4], but with much weaker assumptions on smallness of $\|A\|$. The convergence of solutions of the Cauchy problem to self-similar solutions is proved in Sect. 5 under the same restrictions on $\|A\|$ as in Sect. 4. The main results of the paper are formulated in Theorems 1 and 2 of Sects. 4 and 5 respectively. The results and some open problems are briefly discusses in Conclusions.

2 Homoenergetic Affine Flows and Modified Boltzmann Equation

We consider the spatially inhomogeneous Boltzmann equation for the distribution function $f(x, v, t)$, where $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, $d \geq 2$, and $t \in \mathbb{R}_+$ denotes respectively the particle position, velocity and time. The equation reads

$$\partial_t f + v \cdot f_x = Q(f, f), \tag{2.1}$$

where $Q(f, f)$ is the collision integral

$$Q(f, f)(x, v, t) = \int_{\mathbb{R}^d \times S^{d-1}} dw dn \, g\left(|u|, \hat{u} \cdot n\right) \times$$
$$\times \left[f(x, v', t) f(x, w', t) - f(x, v, t) f(x, w, t) \right],$$
$$n \in S^{d-1}; \quad u = v - w, \quad \hat{u} = u/|u|,$$
$$v' = \tfrac{1}{2} (v + w + |u|n), \ w' = \tfrac{1}{2} (v + w - |u|n). \tag{2.2}$$

The kernel $g(|u|, \eta)$ is usually considered in mathematical works as a given non-negative function. In the present paper we consider mainly the pseudo-Maxwell molecules with the kernel $g(\eta)$ which does not depend on velocities. Note that the bilinear operator $Q(\cdot, \cdot)$ acts only on v-variable and commutes with translations in v-space. It is easy to see that Eq. (2.1) formally admits a particular class of solutions such that

$$f(x, v, t) = \tilde{f}(\tilde{v}, t), \quad \tilde{v} = v - K(t)x, \tag{2.3}$$

where $K(t)$ is a time-dependent matrix of order d. Indeed, we can substitute (2.3) into Eq. (2.1) and obtain the following equations for $\tilde{f}(\tilde{v}, t)$ and $K(t)$:

$$\partial_t f - K(t) v \cdot f_v = Q(f, f), \quad K_t + K^2 = 0,$$

where tildes are omitted. The general formula for $K(t)$ reads

$$K(t) = (1 + tA)^{-1}A, \quad A = K(0).$$

The solutions (2.3) of the Boltzmann equation (2.1) are called *the homoenergetic affine flows*. They were already briefly discussed in Introduction. Many related details and references can be found in [10, 12–14, 18, 19, 26].

The most famous solution of this kind is the so-called shear flow, which formally corresponds to such matrix $A = \{a_{ij}; \ i, j = 1, ..., d\}$ that $a_{12} = a = const$ and all other elements of A are zeros. In that case we obtain

$$A^2 = 0 \Rightarrow K(t) = K(0) = A.$$

Moreover the Eq. (2.1) for $f(v, t)$ can be written as

$$f_t - \operatorname{div}_v Avf = Q(f, f), \quad v \in \mathbb{R}^d. \tag{2.4}$$

This equation for *arbitrary* constant real matrix $A \in M_{d \times d}(\mathbb{R})$ is called the modified Boltzmann equation [4]. It was shown in [10] (for $d = 2$) and [19] (for $d = 3$) that many interesting cases can be reduced by change of variables to Eq. (2.4) with some constant matrix A.

We further simplify the problem and consider below only the case of Maxwell-type interactions, which correspond to the kernel $g(|u|, \eta) = g(\eta)$ in (2.2). It is also assumed below that

$$g(\eta) \geq 0, \quad \eta \in [-1, 1]; \quad \int_{S^{d-1}} d\omega g(\omega \cdot n) = 1, \quad \omega \in S^{d-1};$$

$$f(v, 0) = f_0(v) \geq 0, \quad \int_{\mathbb{R}^d} dv \, f_0(v)v = 0, \quad \int_{\mathbb{R}^d} dv \, f_0(v) = 1. \tag{2.5}$$

Formally the second term in (2.4) describes the action of the external force

$$F = -Av, \quad A \in M_{d \times d}(\mathbb{R}), \tag{2.6}$$

which looks like the anisotropic *friction* force proportional to components of the particle velocity. Let us consider e.g., the simplest case of Eq. (2.4) with

$$A = aI, \quad a \in \mathbb{R}, \tag{2.7}$$

where I is the unit matrix, a is a constant with any sign. If $a > 0$ this is just a regular friction force $F = -av$. The solution $f(v, t)$ of (2.4) under assumption (2.7) leads to the following behaviour of the second moment (energy):

$$d\mathcal{E}(t)/dt = -2a\mathcal{E}(t) \Rightarrow \mathcal{E}(t) = \mathcal{E}(0)e^{-2at},$$

where

$$\mathcal{E}(t) = \int_{\mathbb{R}^d} dv f(v, t)|v|^2.$$

This equality gives an idea to consider the Eq. (2.4) with $A = aI$ in self-similar variables by substitution

$$f(v, t) = e^{dat} \tilde{f}(\tilde{v}, t), \quad \tilde{v} = ve^{at}. \tag{2.8}$$

Then after simple calculations, we obtain the familiar spatially homogeneous Boltzamnn equation for $\tilde{f}(\tilde{v}, t)$

$$\hat{f}_t = Q(\tilde{f}, \tilde{f}), \quad \tilde{f}|_{t=0} = f_0(\tilde{v}). \tag{2.9}$$

If, in addition,

$$\mathcal{E}(0) = \int_{\mathbb{R}^d} dv f_0(v)|v|^2 = d,$$

then we know (H-theorem for (2.9)) that

$$\tilde{f}(\tilde{v}, t) \rightarrow \tilde{f}_M(\tilde{v}) = (2\pi)^{-d/2} e^{-|\tilde{v}|^2/2}, \quad \text{as } t \rightarrow 0.$$

Hence, coming back to initial variables, we (1) obtain the simplest self-similar solution of (2.4), (2.7), namely

$$f_{s-s}(v, t) = \left(2\pi e^{-2at}\right)^{-d/2} \exp\left(-|v e^{at}|^2/2\right), \tag{2.10}$$

and (2) show that this particular solution is an attractor for various classes of initial data. It is obvious, that this simple example is valid also for $a < 0$ in (2.7) (accelerating forces) and for arbitrary kernel (not necessary the Maxwellian one) in the collision integral.

We note that the spatially homogeneous Boltzmann equation for Maxwell-type interactions, i.e. Eq. (2.4) with $A = 0$, also has self-similar solutions of the form

$$f(v, t) = e^{-dct} F(v e^{-ct}), \quad c = \text{const.}$$

Such solutions were constructed and studied for $d = 3$ in papers [1–3] (see also [6, 11, 23]). The drawback of these solutions is that they have an infinite energy $\mathcal{E}(t)$. Nevertheless they describe the large time asymptotics for various classes of initial data with infinite energy similarly to the above discussed elementary example (2.10). This property is typical for a wide class of Maxwell models [5, 6].

Roughly speaking, our task is to prove that the situation is, to some extent, similar in the case of arbitrary matrix A in (2.4) provided that its norm is not too large. In fact, all proofs were already done in our previous paper [4] with standard formulations of results like as "There exists such $\varepsilon_0 > 0$ that the following property holds under assumption that $\|A\| \leq \varepsilon_0...$ ". This approach allows to avoid some technical work, but it does not show true limits (in terms of $\|A\|$) of the results. The main aim of this paper is to partly clarify this question. Here and below we use the so-called operator norm for matrices [22]. Its properties are discussed in the next section.

Following [4] we pass to the Fourier-representation (see [6, 9] for details) of the Eq. (2.4) and introduce the characteristic function [16] $\varphi(k, t)$

$$\varphi(k, t) = \int_{\mathbb{R}^d} dv\, f(v, t) e^{-ik \cdot v}, \ k \in \mathbb{R}^d. \tag{2.11}$$

Then we obtain

$$\partial_t \varphi + (Ak) \cdot \partial_k \varphi = \mathcal{I}^+(\varphi, \varphi) - \varphi_{|k=0}\varphi, \tag{2.12}$$

where

$$\mathcal{I}^+(\varphi, \varphi)(k) = \int_{S^{d-1}} dn g\left(\hat{k} \cdot n\right) \varphi(k_+)\varphi(k_-),$$

$$k_{\pm} = \frac{1}{2}(k \pm |k|n), \quad \hat{k} = \frac{k}{|k|}. \tag{2.13}$$

The initial condition becomes

$$\varphi(k, 0) = \varphi_0(k) = \int_{\mathbb{R}^d} dv \, f_0(v) e^{-ik \cdot v}, \quad \varphi_0(0) = 1.$$

Note that (2.4) implies the mass conservation. Therefore

$$\varphi(0, t) = \varphi_0(0) = 1, \tag{2.14}$$

and we obtain from (2.12)

$$\partial_t \varphi + \varphi + (Ak) \cdot \partial_k \varphi = \mathcal{I}^+(\varphi, \varphi) = \Gamma(\varphi). \tag{2.15}$$

For brevity we consider in the rest of this section the self-similar solution only. Following [4], we look for such solution in the form

$$\varphi_{s-s}(k, t) = \Psi(ke^{\beta t}), \ \beta \in \mathbb{R}. \tag{2.16}$$

Note that it corresponds to the distribution function (2.10), where $a = -\beta$. The parameter β will be defined below.

Then we pass to self-similar variables in (2.15) by substitution

$$\varphi(k, t) = \tilde{\varphi}(\tilde{k}, t), \quad \tilde{k} = ke^{\beta t}, \tag{2.17}$$

and obtain omitting tildes

$$\partial_t \varphi + \varphi + (A_\beta k) \cdot \partial_k \varphi = \Gamma[\varphi], \quad A_\beta = A + \beta I. \tag{2.18}$$

It is clear that the self-similar solution (2.16) of Eq. (2.15) becomes a stationary solution for Eq. (2.18). The differential form of the stationary solution is obvious from (2.18). Its integral form can be obtained at the formal level from the operator identity

$$\int_0^\infty dt \, e^{-t(1+\hat{D})} = \left(1 + \hat{D}\right)^{-1}, \tag{2.19}$$

where $\hat{D}$ is an abstract operator. We refer to [4] for conditions of equivalence of these integral and differential forms of equation for $\Psi(k)$. The integral equation reads [4]

$$\Psi(k) = \int_0^\infty dt \, E_\beta(t) \Gamma[\Psi(k)], \tag{2.20}$$

where $\Gamma[\Psi(k)] = \mathcal{I}^+(\Psi, \Psi)$ is given in (2.13),

$$E_\beta(t) = \exp\left[-t\left(1 + A_\beta k \cdot \partial_k\right)\right]. \tag{2.21}$$

It is easy to see that the action of the operator $E_\beta(t)$ on any function $\varphi(k)$ is given by formula

$$E_\beta(t)\varphi(k) = e^{-t}\varphi\left[e^{-\beta t}\left(e^{-tA}k\right)\right]. \tag{2.22}$$

The Eq. (2.20) will be solved below with all necessary estimates. We begin in the next section with definition of β and some preliminary estimates.

3 Eigenvalue Problem for Matrices

We can apply the operator $\left(1 + A_\beta k \cdot \partial_k\right)$ to the Eq. (2.20) and obtain the equation for $\Psi(k)$ in differential form (see also (2.18))

$$\left(1 + \beta k \cdot \partial_k + Ak \cdot \partial_k\right)\Psi(k) = \Gamma[\Psi](k). \tag{3.1}$$

It is always assumed below that $\Psi(k)$ is a characteristic function (the Fourier transform of a probability measure in $\mathbb{R}^d$) and have the following asymptotic behaviour for small $|k|$:

$$\Psi(k) = 1 - \frac{1}{2}B : k \otimes k + O\left(|k|^p\right) \tag{3.2}$$

for some $2 < p \leq 4$. The notation $B = \{b_{ij}; i, j = 1, \cdots, d\}$ is used for symmetric positively defined matrix. We also denote for brevity

$$B : k \otimes k = \sum_{i,j=1}^{n} b_{ij} k_i k_j.$$

The formula (3.2) means that the corresponding distribution function, i.e. the inverse Fourier transform of $\Psi(k)$, has finite moments of the order $2 + \varepsilon, \varepsilon > 0$ (see [4] for details). It can be shown that the matrix B and the parameter β satisfy the following equation (see Eq. (2.7) in [4]):

$$\beta B + \theta\left(B - \frac{\mathrm{Tr}B}{d}I\right) + \langle BA \rangle = 0, \tag{3.3}$$

where

$$\theta = \frac{qd}{4(d-1)}, \quad q = \int_{S^{d-1}} dng(\omega \cdot n)[1 - (\omega \cdot n)^2],$$

$$\omega \in S^{d-1}; \quad \mathrm{Tr}B = \sum_{i=1}^{d} b_{ii}, \quad \langle BA \rangle = \tfrac{1}{2}[BA + (BA)^T], \tag{3.4}$$

here the upper index T denotes the transposed matrix. This equation can be easily obtained by substitution of (3.2) into Eq. (3.1). We are interested in solution (β, B) of the eigenvalue problem (3.3) such that the eigenvalue β has the largest (as compared to other eigenvalues) real part. In addition, the real symmetric matrix B must have only positive eigenvalues. The existence of such solution (β, B) was proved in Lemma 7.3 in [4] under assumption that $\|A\| \leq \varepsilon_0$ for sufficiently small $\varepsilon_0 > 0$, where

$$\|A\| = \sup_{|k|=1} |Ak|, \quad k \in \mathbb{R}^d. \tag{3.5}$$

No estimates of ε_0 was given in [4]. Our aim in this paper is to fill this gap and to show that main results of that paper remain valid for moderately small values of ε_0.

To this goal we construct the solution of the problem (3.3) below in explicit form of power series in $\|A\|$. It is convenient to denote in (3.3)

$$\beta = \theta(\lambda - 1), \quad A = -\|A\|\tilde{A}, \quad \varepsilon = \frac{\|A\|}{\theta}. \tag{3.6}$$

Then we obtain a new equation for eigenvalue λ and symmetric matrix B

$$\lambda B = \frac{\mathrm{Tr}B}{d}I + \varepsilon\langle BA \rangle, \quad \|A\| = 1, \tag{3.7}$$

where tildes are omitted.

Note that the dimension of the linear space of symmetric $(d \times d)$-matrices is equal to $d(d + 1)/2$. If $\varepsilon = 0$ we have a very simple problem

$$\lambda B = \hat{P} B = \frac{\text{Tr} B}{d} I,$$

where the operator $\hat{P}$ is the projector, since $\hat{P}^2 = \hat{P}$. Obviously this problem has two eigenvalues: $\lambda = 1$ and $\lambda = 0$. The corresponding "eigenmatrices" are: $B = I$ for $\lambda = 1$ and any linear combination of matrices with zero trace for $\lambda = 0$. It is clear that we need to consider in the problem (3.7) the perturbation of the largest eigenvalue $\lambda = 1$. By using standard procedure we assume that

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n, \quad B = \sum_{n=0}^{\infty} \varepsilon^n B_n, \quad \lambda_0 = 1, \quad B_0 = I. \tag{3.8}$$

Then we obtain for $n \geq 1$

$$\sum_{k=0}^{n} \lambda_k B_{n-k} = \frac{\text{Tr} B_n}{d} I + \langle B_{n-1} A \rangle.$$

Note that $\text{Tr} B_0 = d$. Without any loss of generality we can assume that $\text{Tr} B_0 = 0$ for all $n \geq 1$. Hence, we obtain the following recurrent formulas for $n \geq 1$

$$\lambda_n = d^{-1} \text{Tr} \langle B_{n-1} A \rangle, \quad B_n = \langle B_{n-1} A \rangle - \sum_{k=1}^{n} \lambda_k B_{n-k}, \tag{3.9}$$

where λ_0 and B_0 are given in (3.8).

We shall use below the following well-known properties of the norm (3.5), which are valid also for complex-valued matrices (in that case $k \in \mathbb{C}^d$ in (3.5)):

$$\|cA\| = |c| \, \|A\|, \quad c \in \mathbb{C}; \quad |\text{Tr} A| \leq d \, \|A\|;$$
$$\|A_1 + A_2\| \leq \|A_1\| + \|A_2\|; \quad \|A_1 A_2\| \leq \|A_1\| \, \|A_2\|, \tag{3.10}$$

where A, A_1 and A_2 are arbitrary quadratic matrices with complex elements (all details can be found in [22]). Inequalities (3.10) imply that

$$\|\langle B_{n-1} A \rangle\| \leq \|B_{n-1}\|, \quad |\text{Tr} \langle B_{n-1} A \rangle| \leq d \|B_{n-1}\|, \quad n \geq 1,$$

since $\|A\| = 1$ in (3.7). Therefore it follows from (3.9) that

$$|\lambda_n| \leq \|B_{n-1}\|, \quad \|B_n\| \leq \|B_{n-1}\| + \sum_{k=1}^{n} \|B_{k-1}\| \, \|B_{n-k}\|, \quad n \geq 1,$$

whereas $\lambda_0 = 1$, $\|B_0\| = 1$.

Let us consider a function

$$y(x) = \sum_{n=1}^{\infty} y_n x^n, \quad x \geq 0, \tag{3.11}$$

defined by recurrent formulas

$$y_0 = 1; \quad y_n = y_{n-1} + \sum_{k=1}^{n} y_{k-1} y_{n-k}, \quad n \geq 1. \tag{3.12}$$

Obviously we have the estimates

$$\|B_n\| \le y_n, \quad |\lambda_n| \le y_{n-1}, \quad n \ge 1.$$

Therefore

$$|\lambda - 1| \le \sum_{n=1}^{\infty} |\lambda_n| \varepsilon^n \le \varepsilon[1 + y(\varepsilon)],$$

$$\|B - I\| \le \sum_{n=1}^{\infty} \|B_n\| \varepsilon^n \le y(\varepsilon). \tag{3.13}$$

It is straightforward to get a quadratic equation for $y(x)$ from (3.11). We obtain

$$y(x) = x[1 + y(x)][2 + y(x)], \quad y(0) = 0.$$

Hence,

$$y(x) = \frac{1}{2x}[1 - 3x - \sqrt{(1 - 3x)^2 - 8x^2}]. \tag{3.14}$$

The radius of convergence of series for $y(x)$ is equal to

$$r_0 = 3 - \sqrt{8} = (3 + \sqrt{8})^{-1} > 1/6. \tag{3.15}$$

If $0 < \varepsilon < r_0$ in (3.8) we have estimates for $\|B - I\|$ and $|\lambda - 1|$ given in (3.13). By using more general methods it is possible to prove that the radius of convergence of series (3.8) is greater than or equal to $r_1 = 1/2$ (note that $r_0 \approx 1/6$), as it follows from [22] (Chapter II, §3.5, Theorem 3.9). We need, however, more precise estimates also for λ and B. Note that $y(0) = 0$, $y(1/6) = 1$, $y'(x) > 0$, $y''(x) > 0$ for all $x \in [0, r_0)$. Hence, $0 < y(x) < 6x$ if $0 < x < 1/6$, and we obtain from (3.13)

$$|\lambda(\varepsilon) - 1| < 2\varepsilon, \quad \|B(\varepsilon) - I\| < 6\varepsilon \ \text{ if } \ \varepsilon \in [0, 1/6). \tag{3.16}$$

The result can be formulated in the following way.

Lemma 3.1 *We consider the eigenvalue problem (3.7), (3.4), where B is an unknown symmetric matrix of order $d \ge 2$ and A is a given real matrix of the same order. For any matrix A and any $\varepsilon \ge 0$, satisfying conditions*

$$0 \le \varepsilon \le 1/6, \quad \|A\| \le 1, \tag{3.17}$$

in the notation of equations (3.5), there exists a unique solution (λ, B) of (3.7) such that

(i) $\lambda = \lambda(\varepsilon)$ and $B = B(\varepsilon)$ are represented by Taylor series (3.8) convergent for any $\varepsilon \in [0, r_0)$, where $r_0 > 1/6$ is given in (3.15);
(ii) the matrix $B(\varepsilon)$ is real and positive definite, it is uniquely defined by condition $\operatorname{Tr} B = d$ and satisfies the estimate (3.16);
(iii) the eigenvalue $\lambda(\varepsilon)$ is real and simple, it satisfies (3.16) and also inequality $\lambda > |\lambda'|$, where λ' is any other eigenvalue of the problem (3.7).

Proof The convergence of series (3.8) and the validity of the estimate (3.16) are already proved above. The generalization of all results to the case $\|A\| \le 1$ (instead of $\|A\| = 1$ in (3.7)) is obvious. The fact that $B(\varepsilon)$ is positive definite follows from its estimate (3.16). If there is another solution (λ, B') of (3.7) with the same λ, then it is easy to see that we can choose B' in such a way that $\operatorname{Tr} B' = 0$. Hence,

$$\lambda B' = \varepsilon \langle B' A \rangle \Rightarrow |\lambda| \le \varepsilon.$$

This inequality contradicts to estimate (3.16) and therefore λ is simple. It remains to compare λ with other eigenvalues. Let (λ', B') be any other solution of the problem (3.7), which differs from the solution (λ, B) under consideration. The equation for (λ', B') reads

$$\lambda' B' = \frac{\operatorname{Tr} B'}{d} I + \varepsilon \langle B' A \rangle, \quad \lambda' \neq \lambda.$$

It can be re-written in the form

$$\lambda' B' = \lambda \frac{\operatorname{Tr} B'}{d} B + \varepsilon \langle \bar{B}' A \rangle, \quad \bar{B}' = B' - \frac{\operatorname{Tr} B'}{d} B. \tag{3.18}$$

Taking the trace, we obtain

$$(\lambda' - \lambda) \operatorname{Tr} B' = \varepsilon \operatorname{Tr} \langle \bar{B}' A \rangle.$$

We can also derive from (3.18) the equation for the traceless matrix $\bar{B}'$. It reads

$$\lambda' \bar{B}' + \frac{\operatorname{Tr} B'}{d} (\lambda' - \lambda) B = \varepsilon \langle \bar{B}' A \rangle.$$

Then we finally obtain

$$\lambda' \bar{B}' = \varepsilon \left(\langle \bar{B}' A \rangle - \frac{\operatorname{Tr} \langle \bar{B}' A \rangle}{d} B \right).$$

It remains to use inequalities (3.10) and (3.17) for $\|A\|$. By assumption $\|B'\| \neq 0$ and therefore $\bar{B}' \neq 0$, otherwise $B' = B$ and $\lambda' = \lambda$. Hence, we obtain for $\varepsilon \in [0, 1/6)$

$$|\lambda'| \leq \varepsilon(1 + \|B\|) < 3\varepsilon.$$

Note that this estimate is valid for both complex and real eigenvalues λ', since all properties (3.10) hold for complex-valued matrices [22]. Combining the estimate for $|\lambda'|$ with the first inequality in (3.16) we obtain

$$\lambda - |\lambda'| > 1 - 5\varepsilon > 1/6 \quad \text{if} \quad \varepsilon \in [0, 1/6). \tag{3.19}$$

This completes the proof of Lemma 3.1.

$\square$

The corresponding results for the problem (3.3) are formulated in the next lemma.

Lemma 3.2 *For any real matrix A such that*

$$\|A\| < \frac{\theta}{6}, \quad \theta = \frac{qd}{4(d-1)},$$

$$q = \int dn g(\hat{k} \cdot n)[1 - (\hat{k} \cdot n)^2], \quad \hat{k} \in S^{d-1}, \tag{3.20}$$

the eigenvalue problem (3.3), (3.4) has a solution (β, B) that is connected with the pair $[\lambda(\varepsilon), B(\varepsilon)]$ from Lemma 3.1 by transformation

$$\beta = \theta[\lambda(\varepsilon) - 1], \quad B = B(\varepsilon), \quad \varepsilon = \|A\|/\theta. \tag{3.21}$$

All properties of the pair (β, B) follow from Lemma 3.1. In particular,

$$\operatorname{Tr} B = d, \quad |\beta| < 2\|A\|,$$
$$\beta - \Re\beta' \geq \theta - 5\|A\|, \quad \|B - I\| < 1 \tag{3.22}$$

provided the condition (3.20) is satisfied.

Proof is straightforward, since the eigenvalue problems (3.3) and (3.7) are equivalent. The estimate for $(\beta - \Re\beta')$ follows from Eqs. (3.21) and (3.19). □

This lemma will be directly applied to construction of the self-similar profile in the next section.

4 Construction of the Self-similar Profile

We return to the integral equation (2.20) for the self-similar profile $\Psi(k)$ from (2.16). The parameter β and the tensor (matrix) B are assumed below to be chosen in accordance with Lemma 3.2 from Sect. 3. Moreover we choose the function

$$\Psi_0(k) = \exp\left[-\frac{1}{2}B : k \otimes k\right], \quad k \in \mathbb{R}^d \tag{4.1}$$

as the first approximation for $\Psi(k)$ in the iteration process

$$\Psi_{n+1}(k) = I(\Psi_n) = \int_0^\infty dt\, E_\beta(t)\Gamma[\Psi_n(k)], \quad n = 0, 1, \cdots, \tag{4.2}$$

in the notation of Eqs. (2.20)–(2.22).

The same iteration process is actually considered in [4] in the proof of Theorem 7.1. Therefore we can omit some details in order to avoid repetitions. Our main goal is to prove similar theorem under more definite assumptions (without assuming that the perturbation term in Eq. (2.18) is "as small, as we want").

First we need to estimate the difference between $\Psi_0(k)$ and $\Psi_1(k)$. The elementary inequality yields

$$\left|e^{-x} - 1 + x\right| \leq \frac{1}{2}x^2, \quad x \geq 0. \tag{4.3}$$

Therefore

$$\Psi_0(k) = 1 - \frac{1}{2}B : k \otimes k + \delta_0(k), \quad |\delta_0(k)| \leq \frac{1}{8}\|B\|^2 |k|^4 \tag{4.4}$$

in the notation of Eq. (3.5). Note that

$$\Gamma[\Psi_0(k)] = \int_{S^{d-1}} dn\, g(\hat{k} \cdot n)\Psi_0(k_+)\Psi_0(k_-)$$

$$= \int_{S^{d-1}} dn\, g(\hat{k} \cdot n) \exp\left[-\frac{1}{2}B : (k_+ \otimes k_+ + k_- \otimes k_-)\right],$$

where $k_\pm = (k \pm |k|n)/2$. We substitute $\Gamma[\Psi_0(k)]$ into Eq. (4.2) for $n = 0$ and obtain by using the estimate (4.3) in the integrand:

$$\Psi_1(k) = 1 - \Psi_1^{(1)}(k) + \delta_1(k),$$

where

$$\Psi_1^{(1)}(k) = \frac{1}{2} \int_0^\infty dt\, E_\beta(t) \int_{S^{d-1}} dn\, g(\hat{k} \cdot n) B : (k_+ \otimes k_+ + k_- \otimes k_-),$$

$$|\delta_1(k)| \le C_0 \|B\|_1^2 \int_0^\infty dt\, e^{-t[1+4(\beta - \|A\|)]} |k|^4,$$

as it follows from Eqs. (4.3), (2.22). Here C_0 is an absolute constant. By formal construction (see the transition from Eq. (2.18) to its integral form (2.20)) we have

$$\Psi_1^{(1)}(k) = \frac{1}{2} B : k \otimes k.$$

The detailed proof of this equality under assumption that $\|A\| < \beta + 0.5$ is given in the proof of Theorem 7.1 in [4]. In fact the convergence of the above integral for $\delta_1(k)$ assumes a stronger restriction, namely

$$\|A\| < \beta + \frac{1}{4}. \tag{4.5}$$

We assume below that this condition is satisfied. Then it follows from above considerations that

$$|\Psi_1(k) - \Psi_0(k)| \le \frac{C_0' \|B\|_1^2 |k|^4}{1 + 4(\beta - \|A\|)}, \quad k \in \mathbb{R}^d, \tag{4.6}$$

with some absolute constant C_0'.

Obviously, both $\Psi_0(k)$ and $\Psi_1(k)$ are characteristic functions. Therefore

$$|\Psi_0(k)| \le 1, \quad |\Psi_1(k)| \le 1, \tag{4.7}$$

and we obtain the inequality

$$|\Psi_1(k) - \Psi_0(k)| \le \min[C|k|^4, 2], \quad k \in \mathbb{R}^d \tag{4.8}$$

with appropriate constant C. Then we note that

$$|\Psi_{n+1}(k) - \Psi_n(k)| \le$$
$$\int_0^\infty dt\, E_\beta(t) |\Gamma[\Psi_n(k)] - \Gamma[\Psi_{n-1}(k)]|, \quad n \ge 1, \tag{4.9}$$

in accordance with equations (4.2). On the other hand, for any pair $\varphi(k)$ and $\Psi(k)$ of characteristic functions, the following estimate holds

$$|\Gamma[\Psi(k)] - \Gamma[\varphi(k)]| \le \mathcal{L}[|\Psi - \varphi|](k), \tag{4.10}$$

where

$$\mathcal{L}[\varphi](k) = \int_{S^{d-1}} dn\, g(\hat{k} \cdot n)[\varphi(k_+) + \varphi(k_-)],$$

$$k_\pm = (k \pm |k|n)/2, \quad n \in S^{d-1}, \quad \hat{k} = k/|k|.$$

This estimate follows from Lemma 3.1 of [4]. Note that $|k_\pm|^2 = |k|^2 (1 \pm \hat{k} \cdot n)/2$, therefore

$$\mathcal{L}(|k|^4) = \gamma |k|^4,$$
$$\gamma = \frac{1}{2} \int\limits_{S^{d-1}} dn g(\hat{k} \cdot n)[1 + (\hat{k} \cdot n)^2] = 1 - q/2, \tag{4.11}$$

in the notation of Eq. (3.4).

Coming back to iterations (4.2) we can assume by induction that

$$|\Psi_{j+1}(k) - \Psi_j(k)| \leq C_{j+1}|k|^4 \quad 0 \leq j \leq n-1,$$

for some $n \geq 2$. Then we apply estimates (4.10), (4.11) and obtain

$$|\Psi_{n+1}(k) - \Psi_n(k)| \leq C_n \gamma \int\limits_0^\infty dt E_\beta(t)|k|^4.$$

It follows from Eq. (2.22) that

$$|\Psi_{n+1}(k) - \Psi_n(k)| \leq C_{n+1}|k|^4,$$
$$C_{n+1} \leq C_n \gamma [1 + 4(\beta - \|A\|]^{-1}.$$

Hence, the iterations (4.2) converge if $C_{n+1} < C_n$ or equivalently,

$$\gamma < 1 + 4(\beta - \|A\|). \tag{4.12}$$

We use the estimate (3.22) for β and Eq. (4.11) for γ and obtain the sufficient condition for pointwise convergence of iterations (4.2) in the form

$$\|A\| < q/24,$$
$$q = \int\limits_{S^{d-1}} dn g(\omega \cdot n)[1 - (\omega \cdot n)^2], \quad \omega \in S^d. \tag{4.13}$$

Note that this condition does not depend on dimension d. It is a bit stronger than the condition (3.20) of Lemma 3.2, as expected.

The final result can be formulated in the following way.

Theorem 1 *We consider the integral equation (2.20) and assume that the condition (4.13) for matrix A is fulfilled. It is also assumed that the solution $\Psi(k)$ of this equation has asymptotic behaviour for small $|k|$ in accordance with Eq. (3.2) for some $p \in (2, 4]$.*

(i) Then the parameter β in (2.20) and the symmetric matrix B in (3.2) normalized by condition $\mathrm{Tr} B = d$ are uniquely defined by the solution (β, B) of the eigenvalue problem (3.3), (3.4) constructed in Lemma 3.2.

(ii) For β and B from the item (i) there is a unique characteristic function $\Psi(k)$ that solves Eq. (2.20) and satisfies the asymptotic formula (3.2) with $p = 4$.

Proof is almost done above. The iteration process (4.2) leads to pointwise convergence to characteristic function

$$\Psi(k) = \lim \Psi_n(k), \quad k \in \mathbb{R}^n, \tag{4.14}$$

because all $\Psi_n(k), n \geq 0$, are characteristic functions [16]. In fact the convergence is uniform on any compact set in $\mathbb{R}^d$.

It remains to prove the uniqueness of $\Psi(k)$. It is clear that

$$\left|\Psi(k) - 1 + \frac{1}{2}B : k \otimes k\right| \leq C|k|^4. \tag{4.15}$$

Let us assume that there is another solution $\Psi^{(1)}(k)$ of Eq. (2.20), which also satisfies similar inequality. Then we obtain

$$|\Psi(k) - \Psi^{(1)}(k)| \leq C^{(1)}|k|^4, \quad C^{(1)} = \text{const.}$$

In particular, we can choose

$$C^{(1)} = d(\Psi, \Psi^{(1)}) = \sup \frac{|\Psi(k) - \Psi^{(1)}(k)|}{|k|^4},$$

where $d(\Psi, \Psi^{(1)})$ is the distance between two characteristic functions used in [4].

On the other hand it follows from Eq. (2.20) that

$$|\Psi(k) - \Psi^{(1)}(k)| \leq \int_0^\infty dt\, E_\beta(t)|\Gamma[\Psi(k)] - \Gamma[\Psi^{(1)}(k)]|.$$

This estimate is almost a copy of (4.9). Therefore we repeat the same consideration as before and obtain

$$C^{(1)} \leq C^{(1)}\gamma[1 + 4(\beta - \|A\|)]^{-1}.$$

This contradicts to the assumption (4.13) of the theorem, which guarantees the estimate (4.12). Hence, the solution $\Psi(k)$ is unique in the class of functions satisfying the condition (4.13). This completes the proof.   □

5 Convergence to Self-similar Solution

The final step in our study is an improvement of Theorem 8.1 from [4]. We can show that this theorem with $p = 4$ holds under conditions of Theorem 1 from Sect. 4 , i.e. for all such A that $\|A\| < q/24$, where q is given in (4.13). It is assumed below that this condition is fulfilled.

We briefly consider in this section the initial value problem for characteristic function $\varphi(k, t)$ in self-similar coordinates (2.15). We obtain (see Eq. (2.18))

$$\varphi_t + A_\beta k \cdot \varphi_k + \varphi = \Gamma(\varphi), \quad \varphi|_{t=0} = \varphi_0(k), \quad k \in \mathbb{R}^d, \tag{5.1}$$

where $A_\beta = A + \beta I$, $\beta \in \mathbb{R}$. It is assumed that

$$\left|\varphi_0(k) - \left(1 - \frac{1}{2}G_0 : k \otimes k\right)\right| \leq C_0|k|^4,$$
$$C_0 = \text{const.}, \quad k \in \mathbb{R}^d, \tag{5.2}$$

in the notation analogous to Eq. (3.2). Then it is known from [4] that there exists a unique characteristic function $\varphi(k, t)$ that solves the problem (5.1) and satisfies the inequality

$$\left|\varphi(k, t) - \left(1 - \frac{1}{2}G(t) : k \otimes k\right)\right| \leq C_1|k|^4, \quad C_1 = \text{const.}, \tag{5.3}$$

where $G(t)$ is a time-dependent symmetric $(d \times d)$-matrix that solves the problem

$$\frac{1}{2}G_t + \beta G + \theta \left(G - \frac{1}{d}\mathrm{Tr}GI \right) + \langle GA \rangle = 0,$$
$$G|_{t=0} = G_0, \tag{5.4}$$

in the notation of equations (3.4) with $B = G$. Since $\varphi(k, t)$ is the characteristic function, the matrix $G(t)$ is positive definite for any $t \geq 0$, except for trivial solution $\varphi(k, t) = 1$.

We assume below that the parameter β in Eqs. (5.1), (2.15) coincides with β from Theorem 1. Then the matrix B from Theorem 1 is obviously a stationary solution of Eqs. (5.4) ans so is $G_{st} = c^2 B$ for any $c > 0$. Note that G_{st} is also positive definite matrix (like B). It follows from standard theory of linear ODE that the solution $G(t)$ of (5.4) has the following form

$$G(t) = c^2 B + \sum_{i=1}^{s} r_i e^{\gamma_i t} P_i(t), \quad t \geq 0, \tag{5.5}$$

for some integer $s \in [1, N_d - 1]$, $N_d = d(d + 1)/2$, constant parameters $\gamma_i \in \mathbb{C}$ (distinct nonzero eigenvalues) and coefficients $c^2, r_i \in \mathbb{C}, \quad i = 1, \cdots, s$. Polynomials $P_i(t)$ can appear in the sum (5.5) in case of multiple eigenvalues. It is clear that $G(t) \to c^2 B$, as $t \to \infty$, if $\Re\gamma_i < 0$ for any $i \in [1, s]$. It follows from linear algebra that all $\gamma_1, \cdots, \gamma_s$ can be found from the eigenvalue problem

$$\beta' B' + \theta \left(B' - \frac{\mathrm{Tr}B'}{d}I \right) + \langle B'A \rangle = 0 \tag{5.6}$$

for symmetric matrix B'. Then $\gamma_i = 2(\beta'_i - \beta), \quad i = 1, \cdots, s$, where $\{\beta'_1, \cdots, \beta'_s\}$ are distinct eigenvalues of the problem (5.6) and such that $\beta'_i \neq \beta$. We apply one of estimates (3.22) and obtain

$$\Re\gamma_i < -2(\theta - 5\|A\|) < -q/12, \quad \|A\| < q/24$$

under assumptions of Theorem 1. Hence,

$$G(t) = c^2 B + O[\exp(-qt/12)]. \tag{5.7}$$

The result can be formulated as follows.

Lemma 5.1 *Let the symmetric positive-definite $(d \times d)$-matrix $G(t)$ be a solution of the Cauchy problem (5.4) with β from Lemma 3.2 and $\|A\| < q/24$. Then there exists a constant $c > 0$ such that the equality (5.7) with B from Lemma 3.2 holds for all $t \geq 0$.*

Proof is already done above.   □

Lemma 5.1 is an analogue of Lemma 8.3 from [4]. Note that all parameters are defined in Lemma 5.1 explicitly. Finally we want to prove the convergence of $\varphi(k, t)$ from (5.1) to a stationary solution of that equation, namely, to prove the equality

$$\lim_{t \to \infty} \varphi(k, t) = \Psi(ck), \quad k \in \mathbb{R}^d, \tag{5.8}$$

where the constant $c > 0$ is defined in Lemma 5.1.

Remark 5.1 It is easy to see that the function $\Psi(k)$ constructed in Theorem 1 is even, i.e. $\Psi(-k) = \Psi(k)$ (see also [4]).

To this goal we follow the scheme of [4] and introduce an auxiliary function

$$\tilde{\varphi}(k, t) = e^{-\frac{1}{2}G(t):k\otimes k}, \quad k \in \mathbb{R}^d, \tag{5.9}$$

where $G(t)$ is the same as in Lemma 5.1. We also assume that $\varphi_0(k)$ in (5.1) satisfies the condition (5.2). Then we consider the difference

$$|\varphi(k, t) - \Psi(ck)| \le |\varphi(k, t) - \tilde{\varphi}(k, t)| + |\tilde{\varphi}(k, t) - \Psi(ck)|.$$

It is straightforward to repeat arguments from [4] (the proof of Theorem 8.1) combined with Lemma 5.1 and obtain the following estimate for some $t = T > 0$:

$$|\varphi(k, t) - \Psi(ck)| \le C \left(|k|^4 + |k|^2 e^{-qT/12} \right),$$

provided $\|A\| < q/24$. Then we follow the same proof from [4] and obtain for any $T > 0$ and $L > 0$

$$\begin{aligned} &|\varphi(k, T + L) - \Psi(ck)| \\ &\le C\{|k|^4 \exp[-(q/2 - 4\|A_\beta\|)L] \\ &\quad + |k|^2 \exp[-\left(qT/12 - 2\|A_\beta\|\right)L]\}. \end{aligned}$$

It follows from Lemma 3.2 that $\|A_\beta\| < 3\|A\|$. Taking $L = T/3$, we obtain

$$|\varphi(k, 4T/3) - \Psi(ck)| \le C \left(|k|^2 + |k|^4 \right) e^{-\delta T},$$
$$\delta = \frac{q - 24\|A\|}{12}, \quad k \in \mathbb{R}^d. \tag{5.10}$$

Since $\delta > 0$, this proves the pointwise convergence (5.8) for all $k \in \mathbb{R}^d$. Hence, the following statement is proved.

Theorem 2 *Let $\varphi(k, t)$ be a solution of the problem (5.1), where $\varphi_0(k)$ is a characteristic function (the Fourier transform of the probability distribution in $\mathbb{R}^d$) satisfying (5.2). Let the parameter β in (5.1) and the function $\Psi(k)$ be the same as described in Theorem 1. Let $\|A\| < q/24$ in the notation of Eq. (4.13). Then there exist two constants $c > 0$ and $C > 0$ such that*

$$|\varphi(k, t) - \Psi(ck)| \le C \left(|k|^2 + |k|^4 \right) e^{-\mu t},$$
$$\mu = (q - 24\|A\|)/16, \quad k \in \mathbb{R}^d, \quad t \ge 0. \tag{5.11}$$

Proof is already done above. The estimate (5.11) obviously follows from (5.10). □

This theorem can be considered as certain improvement and clarification of Theorem 8.1 from [4]. We omit the translation of results to the language of distribution functions in the velocity space $\mathbb{R}^d$ because it would be almost a repetition of Section 10 from [4] with new conditions of applicability of the results.

6 Conclusions

We have considered the modified spatially homogeneous Maxwell – Boltzmann equation (2.4). The equation contains an additional force term $\text{div}Avf$, where $v \in \mathbb{R}^d$, A is an arbitrary constant $(d \times d)$-matrix. Applications of this equation are connected with well-known homoenergetic solutions to the spatially inhomogeneous Boltzmann equation studied

by many authors since 1950s. The self-similar solutions and related questions for Eq. (2.4) were recently considered in detail in [4] by using the Fourier transform and some properties of the Boltzmann collision operator in the Fourier representation [5]. Main results of [4] were obtained under assumption of "sufficiently small norm of A" in (2.4) without explicit estimates of this "smallness". Our aim in this paper was to fill this gap and to prove that most of the results related to self-similar solutions remain valid for moderately small matrices A with norm $\|A\| = O(10^{-1})$ in dimensionless units. This is important for applications because it shows boundaries for the approach based on the perturbation theory. The main results of the paper are formulated in Theorems 1 and 2 from Sects. 4 and 5, respectively. These theorems extend the corresponding results of [4] to moderate values of $\| A \|$. The main idea of proofs of new estimates is based on detailed study of the eigenvalue problem (3.3), see Lemma 3.1 from Sect. 3. A by-product result is the proof of existence of the bounded fourth moment of the self-similar profile for moderate values of $\| A \|$. The question of existence of *all* moments for the self-similar profile $F(v)$ remains open even in the class of arbitrarily small norm of A.

Data Availability   Data sharing is not applicable to this article as no new datasets were generated or analysed during the current study.

References

1. Bobylev, A.V., Cercignani, C.: Self-similar solutions of the Boltzmann equation and their applications. J. Stat. Phys. **106**(5–6), 1039–1071 (2002)
2. Bobylev, A.V., Cercignani, C.: Exact eternal solutions of the Boltzmann equation. J. Stat. Phys. **106**(5–6), 1019–1038 (2002)
3. Bobylev, A.V., Cercignani, C.: Self-similar asymptotics for the Boltzmann equation with elastic and inelastic interactions. J. Stat. Phys. **110**, 333–375 (2003)
4. Bobylev, A.V., Nota, A., Velazquez, J.L.: Self-similar asymptotics for a modified Maxwell–Boltzmann equation in systems subject to deformations. Commun. Math. Phys. **380**, 409–448 (2020)
5. Bobylev, A.V., Cercignani, C., Gamba, I.M.: On the self-similar asymptotics for generalized non-linear kinetic Maxwell models. Commun. Math. Phys. **291**, 599–644 (2009)
6. Bobylev, A.V.: Kinetic Equations, vol. 1: Boltzmann Equation, Maxwell Models and Hydrodynamics beyond Navier–Stokes. De Gruyter, Berlin (2020)
7. Bobylev, A.V.: On exact solutions of the Boltzmann equation. Dokl. Acad. Nauk SSSR **225**, 1296–1299 (1975). English translation in Sov. Phys. Dokl. **20**, 822–824 (1976)
8. Bobylev, A.V.: A class of invariant solutions of the Boltzmann equation. Dokl. Acad. Nauk SSSR **231**, 571–574 (1976). English translation in Sov. Phys. Dokl. **21**, 632–635 (1976)
9. Bobylev, A.V.: Fourier transform method in the theory of the Boltzmann equation for Maxwell molecules. Dokl. Acad. Nauk SSSR **225**, 1041–1044 (1975). English translation in Sov. Phys. Dokl. **20**, 820-822 (1976)
10. Bobylev, A.V., Caraffini, G.L., Spiga, G.: On group invariant solutions of the Boltzmann equation. J. Math. Phys. **37**, 2787–2795 (1996)
11. Cannone, M., Carch, G.: Infinite energy solutions to the homogeneous Boltzmann equation. Commun. Pure Appl. Math. **63**(6), 747–778 (2010)
12. Cercignani, C.: Existence of homogeneous affine flows for the Boltzmann equation. Arch. Ration. Mech. Anal. **105**(4), 377–387 (1989)
13. Cercignani, C.: Shear flow of a granular material. J. Stat. Phys. **102**(5), 1407–1415 (2001)
14. Cercignani, C.: The Boltzmann equation approach to the shear flow of a granular material. Philos. Trans. R. Soc. **360**, 437–451 (2002)
15. Duan, R., Liu, Sh.: The Boltzmann equation for uniform shear flow. arXiv:2008.0255 IvI [math. AP] (2020)
16. Feller, W.: An Introduction to Probability Theory and Applications. Wiley, New York (1971)
17. Galkin, V.S.: On a class of solutions of Grade's moment equation. PMM **20**, 445–446 (1956)

18. Garzo, V., Santos, A.: Kinetic Theory of Gases in Shear Flow. Nonlinear Transport. Kluwer Academic Publishers, Dordrech (2003)
19. James, R.D., Nota, A., Velazquez, J.J.L.: Self-similar profiles for homo-energetic solutions of the Boltzmann equation: particle velocity distribution and entropy. Arch. Ration. Mech. Anal. **231**(2), 787–843 (2019)
20. James, R.D., Nota, A., Velazquez, J.J.L.: Long-time asymptotics for homoenergetic solutions of the Boltzmann equation: collision-dominated case. J. Nonlin. Sci. **29**(5), 1943–1973 (2019)
21. James, R.D., Nota, A., Velazquez, J.J.L.: Long-time asymptotics for homoenergetic solutions of the Boltzmann equation: hyperbolic - dominated case. Nonlinearity **33**(8), 3781–3815 (2020)
22. Kato, T.: Perturbation Theory for Linear Operators, Classics in Mathematics. Springer, New York (1976)
23. Morimoto, Y., Yang, T., Zhao, H.: Convergence to the self-similar solutions to the spatially homogeneous Boltzmann equation. J. Eur. Math. Soc. **19**, 2041–2067 (2017)
24. Pomea, Y.: Shock at very large Mach number in simple gases: a physicist approach. Transp. Theory Stat. Phys. **16**, 727–734 (1987)
25. Truesdell, C.: On the pressures and flux of energy in a gas according to Maxwell's kinetic theory I. J. Ration. Mech. Anal. **5**, 55–128 (1956)
26. Truesdell, C., Muncaster, R.G.: Fundamentals of Maxwell's Kinetic Theory for a Simple Monatomic Gas. Academic Press, London (1980)