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Higher-Dimensional Nonlinear Thermodynamic Formalism

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Abstract
We introduce a higher-dimensional version of the nonlinear thermodynamic formalism introduced by Buzzi and Leplaideur, in which a potential is replaced by a family of potentials. In particular, we establish a corresponding variational principle and we discuss the existence, characterization, and number of equilibrium measures for this higher-dimensional version.

Keywords Nonlinear thermodynamic formalism · Variational principle · Equilibrium measures

Mathematics Subject Classification Primary 28D20 · 37D35

1 Introduction

Recently, Buzzi and Leplaideur [7] introduced a variation of the thermodynamic formalism, which they called *nonlinear thermodynamic formalism*. Roughly speaking, this amounts to compute the topological pressure replacing Birkhoff sums by images of them under a given function (that may be nonlinear and thus the name). Our main aim is twofold:

(1) to introduce a higher-dimensional version of their notion of topological pressure, replacing a potential by a family of potentials, and to establish a corresponding variational principle;
(2) to discuss the existence, characterization, and number of equilibrium measures, with special attention to the new phenomena that occur in this higher-dimensional version.

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We also give a characterization of the nonlinear pressure as a Carathéodory dimension, which allows us to extend the notion to noncompact sets.

The most basic notion of the mathematical thermodynamic formalism is topological pressure. It was introduced by Ruelle [24] for expansive maps and by Walters [28] in the general case. For a continuous map $T\colon X \to X$ on a compact metric space, the *topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ is defined by

$$P(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp S_n\varphi(x), \tag{1}$$

with the supremum taken over all (n, ε)-separated sets E and where $S_n\varphi = \sum_{k=0}^{n-1} \varphi \circ T^k$. An important relation between the topological pressure and the Kolmogorov–Sinai entropy is given by the variational principle

$$P(\varphi) = \sup_\mu \left(h_\mu(T) + \int_X \varphi\, d\mu \right), \tag{2}$$

with the supremum taken over all T-invariant probability measures μ on X and where $h_\mu(T)$ denotes the entropy with respect to μ. This was established by Ruelle [24] for expansive maps and by Walters [28] in the general case. The theory is now a broad and active independent field of study with many connections to other areas of mathematics. We refer the reader to the books [2, 6, 14, 15, 20, 21, 25, 29] for many developments.

Building on work on the Curie–Weiss mean-field theory in [17], the nonlinear topological pressure was introduced in [7] as a generalization of (1) as follows (more precisely, we give an equivalent formulation using separated sets instead of covers). Given a continuous function $F\colon \mathbb{R} \to \mathbb{R}$, the *nonlinear topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ is given by

$$P_F(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\left[nF\left(\frac{S_n\varphi(x)}{n} \right) \right], \tag{3}$$

with the supremum taken over all (n, ε)-separated sets E. For $F(x) = x$ we recover the classical topological pressure. Buzzi and Leplaideur also established a version of the variational principle in (2). Namely, assuming that the pair (T, Φ) has an abundance of ergodic measures (see Sect. 2.1 for the definition), they proved that

$$P_F(\varphi) = \sup_\mu \left(h_\mu(T) + F\left(\int_X \varphi\, d\mu \right) \right), \tag{4}$$

with the supremum taken over all T-invariant probability measures μ on X. In addition, they characterized the equilibrium measures of this thermodynamic formalism, that is, the invariant probability measures at which the supremum in (4) is attained, and they showed that a new type of phase transition can occur. Namely, one may have more than one equilibrium measure, although we still have a central limit theorem (see also [16, 26]).

As described above, our main aim in the paper is to understand whether and how the results in [7] extend to the higher-dimensional case. This corresponds to replace the functions F and φ in (3), respectively, by a continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$ and by a family $\Phi = \{\varphi_1, \ldots, \varphi_d\}$ of continuous functions $\varphi_i\colon X \to \mathbb{R}$ for $i = 1, \ldots, d$. The *nonlinear topological pressure* of Φ is then defined by

$$P_F(\Phi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\left[nF\left(\frac{S_n\varphi_1(x)}{n}, \ldots, \frac{S_n\varphi_d(x)}{n} \right) \right],$$

with the supremum taken over all (n, ε)-separated sets E. Whenever possible, we follow a similar approach to obtain a variational principle and to discuss the existence, characterization, and number of equilibrium measures.

In particular, assuming that the pair (T, Φ) has an abundance of ergodic measures, we establish the variational principle

$$P_F(\Phi) = \sup_\mu \left(h_\mu(T) + F\left(\int_X \varphi_1 \, d\mu, \dots, \int_X \varphi_d \, d\mu \right) \right), \tag{5}$$

with the supremum taken over all T-invariant probability measures μ on X. As in [7], for a certain class of pairs (T, Φ) we also characterize the equilibrium measures, that is, the invariant probability measures at which the supremum in (5) is attained. Consider the sets

$$L(\Phi) = \left\{ \left(\int_X \varphi_1 \, d\mu, \dots, \int_X \varphi_d \, d\mu \right) : \mu \text{ is } T\text{-invariant} \right\} \subset \mathbb{R}^d$$

and

$$\mathcal{M}(z) = \left\{ \mu \text{ is } T\text{-invariant} : \left(\int_X \varphi_1 \, d\mu, \dots, \int_X \varphi_d \, d\mu \right) = z \right\}.$$

We reduce the problem of finding equilibrium measures to the problem of finding maximizers of the function $E \colon L(\Phi) \to \mathbb{R}$ defined by

$$E(z) = h(z) + F(z),$$

where

$$h(z) = \sup\left\{ h_\mu(T) : \mu \in \mathcal{M}(z) \right\}.$$

We note that the function E first appeared in [17]. It turns out that $h(z)$ coincides with the topological entropy of the map T on the set

$$C_z(\Phi) = \left\{ x \in X : \left(\lim_{n\to\infty} \frac{S_n\varphi_1(x)}{n}, \dots, \lim_{n\to\infty} \frac{S_n\varphi_d(x)}{n} \right) = z \right\}$$

(see (6)). In general $C_z(\Phi)$ need not be compact and so here we need the notion of topological entropy for noncompact sets (see Sect. 2.4 for the definition). In fact, we show that for each $z \in \operatorname{int} L(\Phi)$ maximizing E there exists a unique equilibrium measure ν_z. This is actually a classical equilibrium measure for a certain function ψ_z that depends on the family of functions Φ. In addition, we give conditions for the uniqueness of the equilibrium measures, both for $d = 1$ and for $d > 1$ (see Theorems 10 and 11).

Before proceeding, we highlight the main elements and difficulties of passing to the higher-dimensional case. To the possible extent, our streamlined proof of the variational principle follows arguments in [7] for a single function, considering covers by balls instead of covers by intervals. Our main result (Theorem 7) gives a characterization of equilibrium measures and uses in an essential way the higher-dimensional multifractal analysis developed in [3] (see the following paragraph for further details). It was crucial to make sure that all was prepared so that we could apply this higher-dimensional theory, which allows us to give a description of the equilibrium measures for the nonlinear topological pressure as equilibrium measures of certain functions in $\operatorname{span}\{\varphi_1, \dots, \varphi_d, 1\}$. In addition, in Sect. 5 we describe criteria for the uniqueness of equilibrium measures both for $d = 1$ and $d > 1$, and we give conditions for the coincidence of equilibrium measures for two systems in terms of the notion of cohomology.

As noted above, to a relevant extent we use in the proofs the higher-dimensional multifractal analysis developed in [3]. This gives once more a connection between the thermodynamic formalism and multifractal analysis, which is a principal characteristic of our work. In particular, that other work includes a conditional variational principle, which shows for example that the topological entropy of the level sets of pointwise dimensions, local entropies, and Lyapunov exponents can be approximated simultaneously by the entropy of ergodic measures. More precisely, for a continuous map $T\colon X \to X$ on a compact metric space with upper semicontinuous entropy, it is shown in [3] that if $\Phi = \{\varphi_1, \dots, \varphi_d\}$ is composed of continuous functions such that each element of $\operatorname{span}\{\varphi_1, \dots, \varphi_d, 1\}$ has unique equilibrium measure (for the classical topological pressure), then for each $z \in \operatorname{int} L(\Phi)$ the set $C_z(\Phi)$ is nonempty and has topological entropy

$$h(T|_{C_z(\Phi)}) = \sup\bigl\{h_\mu(T) : \mu \in \mathcal{M}(z)\bigr\} = \inf_{q\in\mathbb{R}^d} P\bigl(\langle q, \Phi - z\rangle\bigr). \tag{6}$$

In addition, there exists an ergodic equilibrium measure $\mu_z \in \mathcal{M}(z)$ with

$$\mu_z(C_z(\Phi)) = 1 \quad \text{and} \quad h_{\mu_z}(T) = h(T|_{C_z(\Phi)}).$$

We note that some phenomena absent in classical multifractal analysis for a single potential may occur in a higher-dimensional multifractal spectrum. For example, the domain of the spectrum may not be convex and its interior may be empty or have more than one connected component.

Finally, we also detail further the motivation for introducing the nonlinear thermodynamical formalism and for our own work with a higher-dimensional version of this formalism. In statistical mechanics, particularly in the study of magnetic systems, the Curie–Weiss–Potts model is generally seen as an extension of the Curie–Weiss model, which can be considered as a mean-field version of the Ising model (see for example [11–13] for detailed discussions). Leplaideur and Watbled traced a parallel between statistical mechanics and ergodic theory for general spin spaces, introducing a generalized Curie–Weiss model in [17] and a generalized Curie–Weiss–Potts model in [18] (the latter model can be seen as a higher-dimensional generalized Curie–Weiss model). When $X = \{-1, 1\}^{\mathbb{N}}$, T is the shift map, φ is a Hölder continuous function and $F\colon \mathbb{R} \to \mathbb{R}$ is given by

$$F(z) = \frac{\beta}{2} z^2, \quad \text{where } \beta \geq 0 \text{ is a physical parameter,}$$

we recover the generalized Curie–Weiss model. Again for the shift map T on X, when $\Phi = \{\varphi_1, \dots, \varphi_d\}$ is a family of Hölder continuous functions and $F\colon \mathbb{R}^d \to \mathbb{R}$ is given by

$$F(z) = \frac{\beta}{2} \|z\|^2, \quad \text{where } \beta \geq 0 \text{ is a physical parameter}$$

and $\|\cdot\|$ is a given norm on $\mathbb{R}^d$, we recover the generalized Curie–Weiss–Potts model. In this sense, while [7] and more recently [8] extend the study of the generalized Curie–Weiss model for any continuous function $F\colon \mathbb{R} \to \mathbb{R}$ and any map T, analogously our work extends the study of the generalized Curie–Weiss–Potts model to include any continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$ and any map T, both under suitable assumptions.

We note that our results and the higher-dimensional results in [8] are mostly overlapping, but there are some differences. Namely, in [8] the authors use tools from convex analysis, for example to characterize the nonlinear equilibrium measures. They also discuss phase transitions in the nonlinear context, study equidistribution of Gibbs ensembles, and obtain a generic result on the uniqueness of nonlinear equilibrium measures. On the other hand, we characterize the nonlinear equilibrium measures using a conditional variational principle coming from higher-dimensional multifractal analysis. For instance, in our characterization we are able to specify the support of the nonlinear equilibrium measure. Moreover, we study in more detail the cohomology relations in the nonlinear context, and we obtain a simple criteria for uniqueness of nonlinear equilibrium measures.

2 Nonlinear Topological Pressure

2.1 Basic Notions

We first recall the notion of nonlinear topological pressure introduced by Buzzi and Leplaideur in [7] as an extension of the classical topological pressure. Let $T\colon X \to X$ be a continuous map on a compact metric space $X = (X, d)$. For each $n \in \mathbb{N}$ we consider the distance

$$d_n(x, y) = \max\bigl\{d(T^k(x), T^k(y)) : k = 0, \dots, n-1\bigr\}.$$

Take $n \in \mathbb{N}$ and $\varepsilon > 0$. A set $C \subset X$ is said to be an (n, ε)-*cover* of X if $\bigcup_{x\in C} B_n(x, \varepsilon) = X$, where

$$B_n(x, \varepsilon) = \bigl\{y \in X : d_n(y, x) < \varepsilon\bigr\}$$

(usually the set $B_n(x, \varepsilon)$ is called a Bowen ball). Given a continuous function $F\colon \mathbb{R} \to \mathbb{R}$, the *nonlinear topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ is defined by

$$P_F(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \inf_C \sum_{x\in C} \exp\left[nF\left(\frac{S_n\varphi(x)}{n}\right)\right],$$

where $S_n\varphi = \sum_{k=0}^{n-1} \varphi \circ T^k$, with the infimum taken over all (n, ε)-covers C.

Let $\mathcal{M}$ be the set of T-invariant probability measures on X. Following [7], we say that the pair (T, φ) has an *abundance of ergodic measures* if for each $\mu \in \mathcal{M}$, $h < h_\mu(T)$ and $\varepsilon > 0$ there exists an ergodic measure $\nu \in \mathcal{M}$ such that

$$h_\nu(T) > h \quad \text{and} \quad \left|\int_X \varphi\, d\nu - \int_X \varphi\, d\mu\right| < \varepsilon.$$

Assuming that (T, φ) has an abundance of ergodic measures, they obtained the variational principle

$$P_F(\varphi) = \sup_{\mu\in\mathcal{M}} \left\{h_\mu(T) + F\left(\int_X \varphi\, d\mu\right)\right\}. \tag{7}$$

They also established (7) when F is a convex function (without assuming that the pair (T, φ) has an abundance of ergodic measures). We say that $\nu \in \mathcal{M}$ is an *equilibrium measure for* (F, φ) with respect to T if

$$P_F(\varphi) = h_\nu(T) + F\left(\int_X \varphi\, d\nu\right).$$

2.2 Higher-Dimensional Version

In this paper we consider a higher-dimensional generalization of the nonlinear topological pressure.

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, a set $E \subset X$ is said to be (n, ε)-*separated* if $d_n(x, y) > \varepsilon$ for every $x, y \in E$ with $x \neq y$. Since X is compact, any (n, ε)-separated set has finite cardinality. Let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function and let $\Phi = \{\varphi_1, \dots, \varphi_d\}$ be a family of continuous functions $\varphi_i\colon X \to \mathbb{R}$ for $i = 1, \dots, d$. The *nonlinear topological pressure* of the family Φ is defined by

$$P_F(\Phi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\left[nF\left(\frac{S_n\varphi_1(x)}{n}, \dots, \frac{S_n\varphi_d(x)}{n}\right)\right] \tag{8}$$

with the supremum taken over all (n, ε)-separated sets E. One can easily verify that the function

$$\varepsilon \mapsto \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\left[nF\left(\frac{S_n\varphi_1(x)}{n}, \dots, \frac{S_n\varphi_d(x)}{n}\right)\right]$$

is nondecreasing and so $P_F(\Phi)$ is well defined. Notice that we only need to consider F on the compact set

$$\big[-\|\varphi_1\|_\infty, \|\varphi_1\|_\infty\big] \times \cdots \times \big[-\|\varphi_d\|_\infty, \|\varphi_d\|_\infty\big] \subset \mathbb{R}^d.$$

We also describe briefly a characterization of the nonlinear topological pressure using (n, ε)-covers. Let

$$\mathcal{W}_n(C) = \sum_{x\in C} \exp\left[nF\left(\frac{S_n\varphi_1(x)}{n}, \dots, \frac{S_n\varphi_d(x)}{n}\right)\right].$$

Following closely arguments in [2], one can show that

$$P_F(\Phi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \inf_C \mathcal{W}_n(C) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \inf_C \mathcal{W}_n(C), \tag{9}$$

with the infimum taken over all (n, ε)-covers C of X.

2.3 Dependence on the Potentials

In this section we discuss briefly how the nonlinear topological pressure depends on the potentials. Given a family of continuous functions $\Phi = \{\varphi_1, \dots, \varphi_d\}$, we define the norm

$$\|\Phi\| = \max_{j\in\{1,\dots,d\}} \|\varphi_j\|_\infty.$$

Recall that F is said to be *Hölder continuous* with constants $C, \alpha > 0$ if

$$|F(x) - F(y)| \le C\|x - y\|_\infty^\alpha \quad \text{for every } x, y \in \mathbb{R}^d.$$

Proposition 1 *Let Φ and Ψ be families of continuous functions and let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Then the following properties hold:*

(1) *the map $\Phi \mapsto P_F(\Phi)$ is continuous;*
(2) *if F is Hölder continuous with constants $C, \alpha > 0$, then*

$$|P_F(\Phi) - P_F(\Phi)| \le C\|\Phi - \Psi\|^\alpha; \tag{10}$$

in particular, if F is Lipschitz, then $\Phi \mapsto P_F(\Phi)$ is also Lipschitz with the same Lipschitz constant.

Proof We first prove property (1). By the uniform continuity of F, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|F(x) - F(y)| < \varepsilon$ whenever $\|x - y\|_\infty < \delta$. Consider a family of continuous functions Ψ such that $\|\Psi - \Phi\| < \delta$. Then

$$\left\| \left(\frac{S_n\psi_1(x)}{n}, \ldots, \frac{S_n\psi_d(x)}{n} \right) - \left(\frac{S_n\varphi_1(x)}{n}, \ldots, \frac{S_n\varphi_d(x)}{n} \right) \right\|_\infty < \delta$$

for any $n \in \mathbb{N}$ and $x \in X$. By the uniform continuity of F, we obtain

$$nF\left(\frac{S_n\psi_1(x)}{n}, \ldots, \frac{S_n\psi_d(x)}{n} \right) < nF\left(\frac{S_n\varphi_1(x)}{n}, \ldots, \frac{S_n\varphi_d(x)}{n} \right) + n\varepsilon$$

for any $n \in \mathbb{N}$ and $x \in X$. It follows from the definition of the topological pressure in (8) that $P_F(\Psi) - P_F(\Phi) < \varepsilon$. One can show in the same manner that $P_F(\Phi) - P_F(\Psi) < \varepsilon$. Therefore,

$$|P_F(\Psi) - P_F(\Phi)| < \varepsilon,$$

which establishes the first property in the proposition.

Now assume that F is Hölder continuous with constants $C, \alpha > 0$. Then for any families of continuous functions Φ and Ψ we have

$$\left| F\left(\frac{S_n\varphi_1(x)}{n}, \ldots, \frac{S_n\varphi_d(x)}{n} \right) - F\left(\frac{S_n\psi_1(x)}{n}, \ldots, \frac{S_n\psi_d(x)}{n} \right) \right| \le C\|\Phi - \Psi\|^\alpha$$

for any $n \in \mathbb{N}$ and $x \in X$. Proceeding as in the proof of property (1), we readily obtain inequality (10). □

2.4 Extension to Noncompact Sets

Based on work of Pesin and Pitskel' in [22], we give a characterization of the nonlinear topological pressure as a Carathéodory dimension. In particular, this allows us to extend the notion to noncompact sets. We expect that this extension plays an important role in an appropriate version of multifractal analysis associated with the nonlinear topological pressure.

We continue to consider a continuous map $T\colon X \to X$ on a compact metric space. Given a finite open cover $\mathcal{U}$ of X, for each $n \in \mathbb{N}$ let $\mathcal{X}_n$ be the set of strings $U = (U_1, \ldots, U_n)$ with $U_i \in \mathcal{U}$ for $i = 1, \ldots, n$. We write $l(U) = n$ and we define

$$X(U) = \{x \in X : T^{k-1} \in U_k \text{ for } k = 1, \ldots, l(U)\}.$$

We say that $\Gamma \subset \bigcup_{n\in\mathbb{N}} \mathcal{X}_n$ *covers* a set $Z \subset X$ if $Z \subset \bigcup_{U\in\Gamma} X(U)$.

Given a family of continuous functions $\Phi = \{\varphi_1, \ldots, \varphi_d\}$, for each $n \in \mathbb{N}$ we define $S_n\Phi = (S_n\varphi_1, \ldots, S_n\varphi_d)$. Moreover, given a function $F\colon \mathbb{R}^d \to \mathbb{R}$, for each $U \in \mathcal{X}_n$ let

$$F_\Phi(U) = \begin{cases} \sup_{X(U)} nF\left(\frac{1}{n}S_n\Phi\right) & \text{if } X(U) \neq \emptyset, \\ -\infty & \text{if } X(U) = \emptyset. \end{cases}$$

Finally, given a set $Z \subset X$ and a number $\alpha \in \mathbb{R}$, we define

$$M_Z(\alpha, \Phi, \mathcal{U}) = \lim_{n\to\infty} \inf_\Gamma \sum_{U\in\Gamma} \exp(-\alpha l(U) + F_\Phi(U)),$$

with the infimum taken over all $\Gamma \subset \bigcup_{k\geq n} \mathcal{X}_k$ covering Z and with the convention that $\exp(-\infty) = 0$. One can easily verify that the map $\alpha \mapsto M_Z(\alpha, \Phi, \mathcal{U})$ goes from $+\infty$ to zero at a unique $\alpha \in \mathbb{R}$ and so one can define

$$P_F(Z, \Phi, \mathcal{U}) = \inf\bigl\{\alpha \in \mathbb{R} : M_Z(\alpha, \Phi, \mathcal{U}) = 0\bigr\}.$$

One can proceed as in the proof of Theorem 2.2.1 in [2] to show that the limit

$$P_F(Z, \Phi) = \lim_{\operatorname{diam} \mathcal{U} \to 0} P_F(Z, \Phi, \mathcal{U})$$

exists. One could also introduce the number $P_F(Z, \Phi)$ using (n, ε)-separated sets or (n, ε)-covers (covers by Bowen balls), in a similar manner to that, for example, in Appendix D in [1].

When $Z = X$ we recover the notion of nonlinear topological pressure for any convex function F.

Theorem 2 *If the function F is convex, then $P_F(\Phi) = P_F(X, \Phi)$.*

Proof The proof is obtained modifying arguments in Sect. 4.2.3 of [2] and so we only give a brief sketch. Given a finite open cover $\mathcal{U}$ of X, we define

$$Z_n(\Phi, \mathcal{U}) = \inf_{\Gamma} \sum_{U \in \Gamma} \exp F_\Phi(U),$$

with the infimum taken over all $\Gamma \subset \mathcal{X}_n$ covering X. Given $\Gamma_1 \subset \mathcal{X}_{n_1}$ and $\Gamma_2 \subset \mathcal{X}_{n_2}$, let

$$\Gamma' = \bigl\{UV : U \in \Gamma_1 \text{ and } V \in \Gamma_2\bigr\}.$$

Note that if Γ_1 and Γ_2 cover X, then Γ' also covers X. Moreover, since F is convex, it follows readily from the identity

$$\frac{S_{m+n}\varphi(x)}{m+n} = \frac{m}{m+n} \cdot \frac{S_m\varphi(x)}{m} + \frac{n}{m+n} \cdot \frac{S_n\varphi(T^m(x))}{n}$$

that

$$F_\Phi(UV) \leq F_\Phi(U) + F_\Phi(V)$$

for each $UV \in \Gamma'$. We have

$$\begin{aligned} Z_{n_1+n_2}(\Phi, \mathcal{U}) &\leq \sum_{UV \in \Gamma'} \exp F_\Phi(UV) \\ &\leq \sum_{U \in \Gamma_1} \exp F_\Phi(U) \sum_{V \in \Gamma_2} \exp F_\Phi(V) \end{aligned}$$

and so

$$Z_{n_1+n_2}(\Phi, \mathcal{U}) \leq Z_{n_1}(\Phi, \mathcal{U}) Z_{n_2}(\Phi, \mathcal{U}).$$

Therefore, one can define

$$Z(\Phi, \mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\Phi, \mathcal{U}).$$

Finally, it follows as in Lemmas 2.2.5 and 2.2.6 in [2] that

$$\lim_{\operatorname{diam} \mathcal{U} \to 0} Z(\Phi, \mathcal{U}) = P_F(X, \Phi)$$

and

$$P_F(\Phi) = \lim_{\operatorname{diam} \mathcal{U} \to 0} Z(\Phi, \mathcal{U}).$$

This yields the desired result. □

Taking $F = \mathrm{id}$ and $\Phi = 0$ we recover the notion of *topological entropy*

$$h(T|_Z) = P_{\mathrm{id}}(Z, 0)$$

of T on the set Z introduced by Pesin and Pitskel' in [22]. It coincides with the notion of topological entropy for noncompact sets introduced earlier by Bowen in [4]. We emphasize that Z need not be compact nor T-invariant. When $Z = X$ we recover the usual notion of topological entropy.

3 Variational Principle

In this section we establish a variational principle for the nonlinear topological pressure.

Let $T: X \to X$ be a continuous map on a compact metric space and let $\Phi = \{\varphi_1, \ldots, \varphi_d\}$ be a family of continuous functions. We say that the pair (T, Φ) has an *abundance of ergodic measures* if for each $\mu \in \mathcal{M}$, $h < h_\mu(T)$ and $\varepsilon > 0$ there exists an ergodic measure $\nu \in \mathcal{M}$ such that $h_\nu(T) > h$ and

$$\left| \int_X \varphi_i \, d\nu - \int_X \varphi_i \, d\mu \right| < \varepsilon \quad \text{for } i = 1, \ldots, d.$$

Moreover, we say that T has *entropy density of ergodic measures* if for every $\mu \in \mathcal{M}$ there exist ergodic measures $\nu_n \in \mathcal{M}$ for $n \in \mathbb{N}$ such that $\nu_n \to \mu$ in the weak* topology and $h_{\nu_n}(T) \to h_\mu(T)$ when $n \to \infty$. Note that if T has entropy density of ergodic measures, then the pair (T, Φ) has an abundance of ergodic measures for any family of continuous functions Φ.

In order to give examples of pairs with an abundance of ergodic measures we first recall a few notions. Given $\delta > 0$, we say that T has *weak specification at scale* δ if there exists $\tau \in \mathbb{N}$ such that for every $(x_1, n_1), \ldots, (x_k, n_k) \in X \times \mathbb{N}$ there are $y \in X$ and times $\tau_1, \ldots, \tau_{k-1} \in \mathbb{N}$ such that $\tau_i \le \tau$ and

$$d_{n_i}(T^{s_{i-1}+\tau_{i-1}}(y), x_i) < \delta \quad \text{for } i = 1, \ldots, k,$$

where $s_i = \sum_{i=1}^{i} n_i + \sum_{i=1}^{i-1} \tau_i$ with $n_0 = \tau_0 = 0$. When one can take $\tau_i = \tau$ for $i = 1, \ldots, k-1$, we say that T has *specification at scale* δ. Finally, we say that T has *weak specification* if it has weak specification at every scale δ and, analogously, we say that T has *specification* if it has specification at every scale δ.

It was shown earlier in [10,Theorem B] and [23,Theorem 2.1] that mixing subshifts of finite type and mixing locally maximal hyperbolic sets have entropy density of ergodic measures. More recently, it was shown in [9] that a continuous map $T: X \to X$ on a compact metric space with the weak specification property such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semicontinuous, has entropy density of ergodic measures. In particular, this implies that the pair (T, Φ) has an abundance of ergodic measures for any family of continuous functions Φ. Some examples of maps with an abundance of ergodic measures include expansive maps with specification or with weak specification, topologically transitive locally maximal hyperbolic sets for diffeomorphisms, and transitive topological Markov chains.

The following theorem establishes a variational principle for the nonlinear topological pressure.

Theorem 3 *Let $T\colon X\to X$ be a continuous map on a compact metric space and let $\Phi=\{\varphi_1,\dots,\varphi_d\}$ be a family of continuous functions. Given a continuous function $F\colon\mathbb{R}^d\to\mathbb{R}$, if the pair (T,Φ) has an abundance of ergodic measures, then*

$$P_F(\Phi)=\sup_{\mu\in\mathcal{M}}\left\{h_\mu(T)+F\left(\int_X\Phi\,d\mu\right)\right\},\tag{11}$$

where $\int_X\Phi\,d\mu=\left(\int_X\varphi_1\,d\mu,\dots,\int_X\varphi_d\,d\mu\right)$.

Proof To the possible extent we follow arguments in [7] for a single function. We divide the proof into two lemmas.

Lemma 1 *Let $T\colon X\to X$ be a continuous map on a compact metric space and let Φ be a family of continuous functions. Then:*

(1)

$$P_F(\Phi)\ge\sup_{\mu\in\mathcal{M}_{\rm erg}}\left\{h_\mu(T)+F\left(\int_X\Phi\,d\mu\right)\right\};$$

(2) if, in addition, the pair (T,Φ) has an abundance of ergodic measures, then

$$P_F(\Phi)\ge\sup_{\mu\in\mathcal{M}}\left\{h_\mu(T)+F\left(\int_X\Phi\,d\mu\right)\right\}.$$

Proof of the lemma Given $r>0$, since X is compact there exist $\delta,\varepsilon>0$ such that

$$|\varphi_i(x)-\varphi_i(y)|<\delta/2\quad\text{whenever } d(x,y)<\varepsilon$$

for $i=1,\dots,d$ and

$$|F(v)-F(w)|<r\quad\text{whenever } \|v-w\|<\delta.$$

For definiteness we shall take the ℓ^∞ norm on $\mathbb{R}^d$. Now let $\mu\in\mathcal{M}$ be an ergodic measure. By Birkhoff's ergodic theorem and the Brin–Katok local entropy formula, together with Egorov's theorem, there exist a set $A\subset X$ of measure $\mu(A)>1-r$ and $N\in\mathbb{N}$ such that

$$\left|\frac{S_n\varphi_i(x)}{n}-\int_X\varphi_i\,d\mu\right|<\delta/2\tag{12}$$

for all $i=1,\dots,d$ and

$$\left|\frac{1}{n}\log\mu(B_n(x,2\varepsilon))+h_\mu(T)\right|<r,\tag{13}$$

for $x\in A$ and $n>N$.

Now let C be an arbitrary (n,ε)-cover and let $D\subset C$ be a minimal (n,ε)-cover of A. For each $x\in D$, the ball $B_n(x,\varepsilon)$ intersects A at some point y (otherwise one could discard the point x in D). Note that

$$d(T^k(x),T^k(y))<\varepsilon\quad\text{for } k=0,\dots,n-1.$$

Hence, it follows from (12) that

$$\left|\frac{S_n\varphi_i(x)}{n}-\int_X \varphi_i\,d\mu\right| \le \frac{1}{n}|S_n\varphi_i(x)-S_n\varphi_i(y)| + \left|\frac{S_n\varphi_i(y)}{n}-\int_X \varphi_i\,d\mu\right| < \delta/2+\delta/2=\delta$$

for $i=1,\dots,d$ and so

$$\left|F\left(\frac{S_n\varphi_1(x)}{n},\dots,\frac{S_n\varphi_d(x)}{n}\right)-F\left(\int_X \Phi\,d\mu\right)\right|<r.$$

Moreover, $B_n(x,\varepsilon)\subset B_n(y,2\varepsilon)$ and so it follows from (13) that

$$1-r<\mu(A)\le |D|\max_{x\in D}\mu(B_n(x,\varepsilon))\le |D|e^{-n(h_\mu(T)-r)},$$

where $|D|$ denotes the cardinality of D. Therefore,

$$\begin{aligned}\mathcal{W}_n(C) &\ge |D|\exp\left[nF\left(\int_X \Phi\,d\mu\right)-r\right]\\ &\ge (1-r)\exp[n(h_\mu(T)-r)]\exp\left[nF\left(\int_X \Phi\,d\mu\right)-r\right]\end{aligned}$$

for any sufficiently large $n\in\mathbb{N}$. It follows from (9) that

$$P_F(\Phi)\ge h_\mu(T)+F\left(\int_X \Phi\,d\mu\right)-2r.$$

Finally, by the arbitrariness of $r>0$ we obtain

$$P_F(\Phi)\ge h_\mu(T)+F\left(\int_X \Phi\,d\mu\right). \tag{14}$$

This yields the first property in the lemma.

Now we consider an arbitrary measure $\nu\in\mathcal{M}$. If (T,Φ) has an abundance of ergodic measures, then for each $h<h_\nu(T)$ and $\varepsilon>0$ there exists an ergodic measure $\mu\in\mathcal{M}$ such that

$$\left|F\left(\int_X \Phi\,d\nu\right)-F\left(\int_X \Phi\,d\mu\right)\right|<\varepsilon \quad\text{and}\quad h_\mu(T)>h$$

(since F is continuous). By (14) we obtain

$$P_F(\Phi)\ge h_\mu(T)+F\left(\int_X \Phi\,d\mu\right)>h+F\left(\int_X \Phi\,d\nu\right)-\varepsilon \tag{15}$$

and it follows from the arbitrariness of h and ε that

$$P_F(\Phi)\ge h_\nu(T)+F\left(\int_X \Phi\,d\nu\right).$$

This yields the second property in the lemma. □

Now we obtain the reverse inequality, without requiring that there are an abundance of ergodic measures.

Lemma 2 *Let $T : X \to X$ be a continuous map on a compact metric space and let Φ be a family of continuous functions. Then*

$$P_F(\Phi) \le \sup_{\mu\in\mathcal{M}} \left\{ h_\mu(T) + F\left(\int_X \Phi\, d\mu \right) \right\}.$$

Proof of the lemma Given $p < P_F(\Phi)$, take $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \inf_C \mathcal{W}_n(C) > p$$

with the infimum taken over all (n, ε)-covers C. Since each maximal (n, ε)-separated set E_n is an (n, ε)-cover, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathcal{W}_n(E_n) > p$$

and given $r > 0$, there exists a diverging subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\mathcal{W}_{n_k}(E_{n_k}) \ge \exp[n_k(p - r)] \quad \text{for } k \in \mathbb{N}. \tag{16}$$

We cover the compact set $\Phi(X)$ by open balls $B(z_i, r_i)$ for $i = 1, \dots, L$ such that $|F(z) - F(z_i)| < r$ for all $z \in B(z_i, r_i)$ and $i = 1, \dots, L$. Now let

$$\Lambda_k^i = \left\{ x \in E_{n_k} : \left(\frac{S_{n_k}\varphi_1(x)}{n_k}, \dots, \frac{S_{n_k}\varphi_d(x)}{n_k} \right) \in B(z_i, r_i) \right\}.$$

Note that

$$\mathcal{W}_{n_k}(E_{n_k}) \le \sum_{i=1}^{L} \mathcal{W}_{n_k}(\Lambda_k^i) \le L\mathcal{W}_{n_k}(\Lambda_k^i) \quad \text{for some } i \in \{1, \dots, L\}$$

and so it follows from (16) that

$$\begin{aligned} \exp[n_k(p - r)] &\le \mathcal{W}_{n_k}(E_{n_k}) \\ &\le L\mathcal{W}_{n_k}(\Lambda_k^i) \le L|\Lambda_k^i| \exp[n_k(F(z_i) + r)]. \end{aligned}$$

This implies that

$$|\Lambda_k^i| \ge \exp[n_k(p - F(z_i) - 3r)] \tag{17}$$

for any sufficiently large k. Proceeding as in the proof of the variational principle in [19], we also consider the measures

$$\mu_k^i = \frac{1}{|\Lambda_k^i|} \sum_{x\in\Lambda_k^i} \delta_x \quad \text{and} \quad \nu_k^i = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mu_k^i T^{-j}.$$

Without loss of generality, one can assume that ν_k^i converges to a T-invariant measure μ^i in the weak* topology satisfying

$$h_{\mu^i}(T) \ge \limsup_{n_k\to\infty} \frac{1}{n_k} \log |\Lambda_k^i|. \tag{18}$$

By the definition of ν_k^i we have

$$\begin{aligned}\int_X \Phi\, d\mu^i &= \lim_{k\to\infty} \int_X \Phi\, d\nu_k^i \\ &= \lim_{k\to\infty} \left(\int_X \frac{S_{n_k}\varphi_1}{n_k}\, d\mu_k^i, \ldots, \int_X \frac{S_{n_k}\varphi_d}{n_k}\, d\mu_k^i \right) \in \overline{B(z_i, r_i)}.\end{aligned}$$

Hence, by (17) and (18) we obtain

$$\begin{aligned}h_{\mu^i}(T) + F\left(\int_X \Phi\, d\mu^i\right) &\geq p - F(z_i) - 3r + F(z_i) - r \\ &= p - 4r.\end{aligned}$$

The desired result follows from the arbitrariness of r and p. □

Lemmas 1 and 2 establish the statement in the theorem. □

For a general continuous map T, we obtain a variational principle for an arbitrary convex function F.

Theorem 4 *Let $T: X \to X$ be a continuous map on a compact metric space and let $\Phi = \{\varphi_1, \ldots, \varphi_d\}$ be a family of continuous functions. If $F: \mathbb{R}^d \to \mathbb{R}$ is a convex continuous function, then identity* (11) *holds.*

Proof It follows from the first property in Lemma 1 that

$$P_F(\Phi) \geq h_\mu(T) + F\left(\int_X \Phi\, d\mu\right)$$

for every ergodic measure $\mu \in \mathcal{M}$. Now let $\nu \in \mathcal{M}$ be an arbitrary measure and consider its ergodic decomposition with respect to T. It is described by a probability measure τ on $\mathcal{M}$ that is concentrated on the subset of ergodic measures $\mathcal{M}_{\text{erg}}$. We recall that for every bounded measurable function $\psi: X \to \mathbb{R}$ we have

$$\int_X \psi\, d\nu = \int_{\mathcal{M}} \left(\int_X \psi\, d\mu\right) d\tau(\mu).$$

For a convex function F one can use Jensen's inequality to obtain

$$\begin{aligned}F\left(\int_X \Phi\, d\nu\right) &= F\left(\int_{\mathcal{M}} \left(\int_X \varphi_1\, d\mu\right) d\tau(\mu), \ldots, \int_{\mathcal{M}} \left(\int_X \varphi_d\, d\mu\right) d\tau(\mu)\right) \\ &\leq \int_{\mathcal{M}} F\left(\int_X \Phi\, d\mu\right) d\tau(\mu).\end{aligned}$$

Moreover, we also have

$$h_\nu(T) = \int_{\mathcal{M}} h_\mu(T)\, d\tau(\mu)$$

(see for example Theorem 9.6.2 in [27]). Hence,

$$h_\nu(T) + F\left(\int_X \Phi\, d\nu\right) \leq \int_{\mathcal{M}} \left[h_\mu(T) + F\left(\int_X \Phi\, d\mu\right)\right] d\tau(\mu) \leq P_F(\Phi). \tag{19}$$

The desired result follows now readily from Lemma 2. □

We also obtain a variational principle over the ergodic measures.

Corollary 5 *Let $T\colon X \to X$ be a continuous map on a compact metric space, let $\Phi = \{\varphi_1, \dots, \varphi_d\}$ be a family of continuous functions, and let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function. If the pair (T, Φ) has an abundance of ergodic measures or F is convex, then*

$$P_F(\Phi) = \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) + F\left(\int_X \Phi \, d\mu \right) \right\}. \tag{20}$$

Proof Since $\mathcal{M}_{\text{erg}} \subset \mathcal{M}$, we have

$$\sup_{\mu \in \mathcal{M}} \left\{ h_\mu(T) + F\left(\int_X \Phi \, d\mu \right) \right\} \ge \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) + F\left(\int_X \Phi \, d\mu \right) \right\}.$$

Now we assume that the pair (T, Φ) has an abundance of ergodic measures and we establish the reverse inequality. It follows from (15) that for each $\nu \in \mathcal{M}$, $h < h_\nu(T)$ and $\varepsilon > 0$, there exists an ergodic measure $\mu \in \mathcal{M}$ such that

$$h_\mu(T) + F\left(\int_X \Phi \, d\mu \right) > h + F\left(\int_X \Phi \, d\nu \right) - \varepsilon.$$

Since h and ε are arbitrary, this readily implies that

$$\sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) + F\left(\int_X \Phi \, d\mu \right) \right\} \ge \sup_{\nu \in \mathcal{M}} \left\{ h_\nu(T) + F\left(\int_X \Phi \, d\nu \right) \right\}.$$

Finally, it follows from Theorem 3 that identity (20) holds.

Now assume that F is convex. It follows from (19) that

$$h_\nu(T) + F\left(\int_X \Phi \, d\nu \right) \le \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) + F\left(\int_X \Phi \, d\mu \right) \right\}$$

for each $\nu \in \mathcal{M}$. Therefore,

$$\sup_{\nu \in \mathcal{M}} \left\{ h_\nu(T) + F\left(\int_X \Phi \, d\nu \right) \right\} \le \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) + F\left(\int_X \Phi \, d\mu \right) \right\}$$

and applying Theorem 4 we also obtain identity (20). □

Remark Without the assumptions of abundance of ergodic measures or convexity of the function F we are only able to show that

$$\sup_{\mu \in \mathcal{M}} \left\{ h_\mu(T) + F\left(\int_X \Phi d\mu \right) \right\} \ge P_F(\Phi) \ge \sup_{\mu \in \mathcal{M}_{erg}} \left\{ h_\mu(T) + F\left(\int_X \Phi d\mu \right) \right\}.$$

In fact, if we drop both assumptions, then the variational principle may fail (see Example 2.5 in [7]).

4 Equilibrium Measures: Existence and Characterization

In this section we consider the problem of characterizing the equilibrium measures of the nonlinear topological pressure.

4.1 Existence of Equilibrium Measures

In view of Theorem 3, we say that $\mu \in \mathcal{M}$ is an *equilibrium measure for* (F, Φ) with respect to T if

$$P_F(\varphi) = h_\mu(T) + F\biggl(\int_X \Phi \, d\mu\biggr).$$

We first formulate a result on the existence of equilibrium measures.

Theorem 6 *Let $T\colon X \to X$ be a continuous map on a compact metric space such that the map $\mu \mapsto h_\mu(T)$ is upper semicontinuous, let $\Phi = \{\varphi_1, \dots, \varphi_d\}$ be a family of continuous functions, and let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function. If the pair (T, Φ) has an abundance of ergodic measures or F is convex, then there exists at least one equilibrium measure for (F, Φ).*

Proof Since the map $\mu \mapsto h_\mu(T)$ is upper semicontinuous, F is continuous and the map $\mu \mapsto \int_X \psi \, d\mu$ is continuous for each continuous function $\psi\colon X \to \mathbb{R}$, we conclude that $\mu \mapsto h_\mu(T) + F(\int_X \Phi \, d\mu)$ is upper semicontinuous. Together with the compactness of $\mathcal{M}$, this guarantees that there exists a measure $\mu_\Phi \in \mathcal{M}$ such that

$$\sup_{\mu \in \mathcal{M}} \biggl\{ h_\mu(T) + F\biggl(\int_X \Phi \, d\mu\biggr) \biggr\} = h_{\mu_\Phi}(T) + F\biggl(\int_X \Phi \, d\mu_\Phi\biggr).$$

Hence, it follows from the variational principles in Theorems 3 and 4 that μ_Φ is an equilibrium measure for (F, Φ). □

In some cases one can pass to the one-dimensional setting of the nonlinear thermodynamic formalism.

Example 1 Consider the function $F\colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$F(z_1, \dots, z_d) = f(\alpha_1 z_1 + \cdots + \alpha_d z_d),$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $\alpha_j \in \mathbb{R}$ for $j = 1, \dots, d$. Then

$$F\biggl(\int_X \Phi \, d\mu\biggr) = F\biggl(\int_X \varphi_1 \, d\mu, \dots, \int_X \varphi_d \, d\mu\biggr) = f\biggl(\int_X \varphi \, d\mu\biggr)$$

for every $\mu \in \mathcal{M}$, where

$$\varphi = \alpha_1 \varphi_1 + \cdots + \alpha_d \varphi_d.$$

Moreover, $P_F(\Phi) = P_f(\varphi)$ and this implies that (F, Φ) and (f, φ) have the same equilibrium measures. In other words, for a function F as above the study of equilibrium measures can be reduced to the case when $d = 1$.

Of course, in general the continuous function F can be much more complicated. For instance, as mentioned in the introduction, the Curie–Weiss–Potts model involves the study of the topological pressure for the function

$$F(z_1, \dots, z_d) = \frac{\beta}{2}(z_1^2 + \cdots + z_d^2)^{1/2},$$

where $\beta \geq 0$ is a physical parameter.

For the following example, we recall the notion of cohomology. We say that a function $\varphi\colon X \to \mathbb{R}$ is *cohomologous* to a function $\psi\colon X \to \mathbb{R}$ (with respect to T) if there exists a measurable bounded function $q\colon X \to \mathbb{R}$ such that

$$\varphi(x) = \psi(x) + q(T(x)) - q(x) \quad \text{for } x \in X.$$

Example 2 *(Reduction of dimension via cohomology)* Let T be a continuous map on a compact metric space and let $\Phi = \{\varphi_1, \ldots, \varphi_d\}$ be a family of continuous functions such that the pair (T, Φ) has an abundance of ergodic measures. Let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function and assume that φ_1 is cohomologous to φ_d. This implies that $\int_X \varphi_1\,d\mu = \int_X \varphi_d\,d\mu$ for every $\mu \in \mathcal{M}$. Therefore,

$$\begin{aligned} F\left(\int_X \Phi\,d\mu\right) &= F\left(\int_X \varphi_1\,d\mu, \int_X \varphi_2\,d\mu, \ldots, \int_X \varphi_1\,d\mu\right) \\ &= G\left(\int_X \varphi_1\,d\mu, \ldots, \int_X \varphi_{d-1}\,d\mu\right) \end{aligned}$$

for every $\mu \in \mathcal{M}$, where

$$G(z_1, \ldots, z_{d-1}) = F(z_1, z_2, \ldots, z_{d-1}, z_1)$$

for each $(z_1, \ldots, z_d) \in \mathbb{R}^d$. The cohomology assumption also implies that

$$\|S_n\varphi_1 - S_n\varphi_d\|_\infty / n \to 0 \quad \text{when } n \to \infty.$$

Together with the continuity of F, this implies that $P_F(\Phi) = P_G(\Psi)$, where $\Psi = \{\varphi_1, \ldots, \varphi_{d-1}\}$. Hence, the pairs (F, Φ) and (G, Ψ) have the same equilibrium measures. More generally, in order to further reduce the dimension of the problem, one could consider additional cohomology relations between any two functions in Φ. For instance, if φ_1 is cohomologous to all functions $\varphi_j \in \Phi$, then the problem reduces to the one-dimensional case.

Example 3 *(Reduction to the classical case via cohomology)* Let T be a continuous map on a compact metric space and let $\Phi = \{\varphi_1, \varphi_2\}$ be a pair of continuous functions such that (T, Φ) has an abundance of ergodic measures. Moreover, assume that φ_1 is cohomologous to φ_2 and consider the function $F\colon \mathbb{R}^2 \to \mathbb{R}$ given by $F(z_1, z_2) = (z_1^3 + z_2^3)^{1/3}$. This implies that $\int_X \varphi_1\,d\mu = \int_X \varphi_2\,d\mu$ for every $\mu \in \mathcal{M}$ and so

$$\begin{aligned} h_\mu(T) + F\left(\int_X \Phi\,d\mu\right) &= h_\mu(T) + F\left(\int_X \varphi_1\,d\mu, \int_X \varphi_1\,d\mu\right) \\ &= h_\mu(T) + \int_X 2^{1/3}\varphi_1\,d\mu \end{aligned}$$

for every $\mu \in \mathcal{M}$. Letting $\psi = 2^{1/3}\varphi_1$, it follows from the definitions that $P_F(\Phi) = P(\psi)$, where P denotes the classical topological pressure. Hence, ν is an equilibrium measure for (F, Φ) if and only if ν is an equilibrium measure for ψ.

Recall that a continuous function $\varphi\colon X \to \mathbb{R}$ is said to have the *Bowen property* if there exist $K > 0$ and $\varepsilon > 0$ such that whenever

$$d(T^k(x), T^k(y)) < \varepsilon \quad \text{for } k = 0, 1, \ldots, n-1$$

we have

$$|S_n\varphi(x) - S_n\varphi(y)| \le K.$$

If $T : X \to X$ is an expansive map with specification and φ_1 (or φ_2) is a continuous function with the Bowen property, then there exists a unique equilibrium measure μ_ψ for ψ (see [5]). Therefore, μ_ψ is also the unique equilibrium measure for (F, Φ).

We observe that this example can be easily generalized to the case when

$$F(z_1, \dots, z_d)^n = H_n(z_1, \dots, z_d),$$

where H_n is a homogeneous polynomial of degree n, assuming additional cohomology relations between some pairs of functions in Φ.

4.2 Characterization of Equilibrium Measures

Now we consider the problem of characterizing the equilibrium measures. Given a pair (T, Φ), we consider the set

$$L(\Phi) = \left\{ \int_X \Phi \, d\mu : \mu \in \mathcal{M} \right\}.$$

Since the map $\mu \mapsto \int_X \psi \, d\mu$ is continuous for each continuous function $\psi : X \to \mathbb{R}$ and $\mathcal{M}$ is compact and connected, the set $L(\Phi)$ is a compact and connected subset of $\mathbb{R}^d$. For each $z \in \mathbb{R}^d$, we also consider the level sets

$$\mathcal{M}(z) = \left\{ \mu \in \mathcal{M} : \int_X \Phi \, d\mu = z \right\}$$

and

$$C_z(\Phi) = \left\{ x \in X : \lim_{n\to\infty} \frac{S_n\Phi(x)}{n} = z \right\}. \tag{21}$$

Following closely [7], we say that the pair (T, Φ) is C^r *regular* (for some $2 \le r \le \omega$, where ω refers to the analytic case) if the following holds:

(1) each function in $\operatorname{span}\{\varphi_1, \dots, \varphi_d, 1\}$ has a unique equilibrium measure for the classical topological pressure and $\operatorname{int} L(\Phi) \ne \emptyset$;
(2) for each $z \in \operatorname{int} L(\Phi)$ the map $q \mapsto P(\langle q, \Phi - z\rangle)$, where P is the classical topological pressure and $\langle \cdot, \cdot \rangle$ is the usual inner product, takes only finite values, is of class C^r, is strictly convex, and its second derivative is a positive definite bilinear form for each $q \in \mathbb{R}^d$;
(3) the entropy map $\mu \mapsto h_\mu(T)$ is upper semicontinuous and bounded.

Examples of C^r regular pairs (T, Φ) include topologically mixing subshifts of finite type, $C^{1+\varepsilon}$ expanding maps, and $C^{1+\varepsilon}$ diffeomorphisms with a locally maximal hyperbolic set, with Φ composed of Hölder continuous functions. Finally, we say that the family of functions $\Phi = \{\varphi_1, \dots, \varphi_d\}$ is *cohomologous* to a constant $c = (c_1, \dots, c_d)$ if φ_i is cohomologous to c_i for $i = 1, \dots, d$. Then $L(\Phi) = \{c\}$ and so $\operatorname{int} L(\Phi) = \emptyset$.

The following theorem is our main result. Given a function $F : \mathbb{R}^d \to \mathbb{R}$, we consider the set

$$K(F, \Phi) = \left\{ \int_X \Phi \, d\mu : \mu \text{ is an equilibrium measure for } (F, \Phi) \right\} \subset L(\Phi).$$

We also consider the function $h\colon L(\Phi) \to \mathbb{R}$ defined by

$$h(z) = \sup\bigl\{h_\mu(T) : \mu \in \mathcal{M}(z)\bigr\}. \tag{22}$$

Theorem 7 *Let $T\colon X \to X$ be a continuous map on a compact metric space and let $\Phi = \{\varphi_1, \dots, \varphi_d\}$ be a family of continuous functions such that the pair (T, Φ) is C^1 regular. For each continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$, the following properties hold:*

(1) *$K(F, \Phi)$ is a nonempty compact set;*
(2) *$K(F, \Phi)$ is the set of maximizers of the function $z \mapsto h(z) + F(z)$;*
(3) *if $K(F, \Phi) \subset \operatorname{int} L(\Phi)$, then the equilibrium measures for (F, Φ) are the elements of $\{\nu_z : z \in K(F, \Phi)\}$, where each $\nu_z \in \mathcal{M}$ satisfies:*
   - *ν_z is ergodic;*
   - *ν_z is the unique invariant measure in $\mathcal{M}(z)$ supported on the level set $C_z(\Phi)$ such that $h_{\nu_z}(T) = h(z)$;*
   - *ν_z is the unique equilibrium measure for a function*

   $$\psi_z = \langle q(z), \Phi - z\rangle \tag{23}$$

   in $\operatorname{span}\{\varphi_1, \dots, \varphi_d, 1\}$, *for some $q(z) \in \mathbb{R}^d$.*

Proof We divide the proof into steps.

Lemma 3 *$K(F, \Phi)$ is a nonempty compact subset of $L(\Phi)$.*

Proof of the lemma Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $K(F, \Phi)$ converging to a point $z \in L(\Phi)$. For each $n \in \mathbb{N}$ there exists an equilibrium measure $\mu_n \in \mathcal{M}$ for (F, Φ) such that $z_n = \int_X \Phi\, d\mu_n$. Passing eventually to a subsequence, we may assume that there exists $\mu \in \mathcal{M}$ such that $\mu_n \to \mu$ when $n \to \infty$ in the weak* topology. Since the map $\mu \mapsto h_\mu(T)$ is upper semicontinuous, we obtain

$$P_F(\Phi) = \limsup_{n\to\infty} \left[h_{\mu_n}(T) + F\biggl(\int_X \Phi\, d\mu_n\biggr)\right] \le h_\mu(T) + F\biggl(\int_X \Phi\, d\mu\biggr),$$

which implies that μ is an equilibrium measure for (F, Φ). Since $z = \int_X \Phi\, d\mu$, we conclude that $z \in K(F, \Phi)$. Hence, $K(F, \Phi)$ is closed. Moreover, since

$$K(F, \Phi) \subset [-\|\varphi_1\|_\infty, \|\varphi_1\|_\infty] \times \cdots \times [-\|\varphi_d\|_\infty, \|\varphi_d\|_\infty],$$

the set $K(F, \Phi)$ is also bounded. By Theorem 6 it is nonempty. □

Lemma 4 *For each $z \in \operatorname{int} L(\Phi)$ there exists an ergodic measure $\nu_z \in \mathcal{M}$ such that $\int_X \Phi\, d\nu_z = z$. In fact, ν_z is the unique equilibrium measure for the function ψ_z given by (23).*

Proof of the lemma For each $z \in L(\Phi)$ we consider the function

$$\Delta_z(q) = P\bigl(\langle q, \Phi - z\rangle - h(T|_{C_z(\Phi)})\bigr),$$

where P is the classical topological pressure and $h(T|_{C_z(\Phi)})$ is the topological entropy of T on the set $C_z(\Phi)$ (see Sect. 2.4 for the definition). By Lemmas 1 and 2 in [3] we have

$$\inf_{q\in\mathbb{R}^d} \Delta_z(q) \ge 0 \quad \text{for } z \in L(\Phi),$$

$$\inf_{q\in\mathbb{R}^d} \Delta_z(q) = 0 \quad \text{for } z \in \operatorname{int} L(\Phi),$$

and there exists at least one point $q(z) \in \mathbb{R}^d$ such that $\Delta_z(q(z)) = 0$. Since the map $q \mapsto \Delta_z(q)$ is of class C^1 and Δ_z has a minimum at $q(z)$, we conclude that $\partial_q \Delta_z(q(z)) = 0$. Now let ν_z be the equilibrium measure of the function ψ_z in (23). One can proceed as in the proof of Theorem 8 in [3] to verify that ν_z is ergodic with

$$\nu_z(C_z(\Phi)) = 1 \quad \text{and} \quad \int_X \Phi\, d\nu_z = z.$$

Moreover, since $\psi_z \in \operatorname{span}\{\varphi_1, \ldots, \varphi_d, 1\}$, it follows from the notion of C^1 regular pair that ν_z is the unique equilibrium measure for ψ_z. □

Lemma 5 *For each $z \in L(\Phi)$ there exists $\mu \in \mathcal{M}(z)$ with $h(z) = h_\mu(T)$. Moreover, when $z \in \operatorname{int} L(\Phi)$ this measure is unique and coincides with ν_z.*

Proof of the lemma Take $z \in L(\Phi)$. By the definition of $L(\Phi)$, there exists $\mu \in \mathcal{M}$ such that $\int_X \Phi\, d\mu = z$, that is, $\mathcal{M}(z) \neq \emptyset$. By the compactness of $\mathcal{M}(z)$ and the upper semicontinuity of the map $\mu \mapsto h_\mu(T)$, there exists $\mu \in \mathcal{M}(z)$ maximizing the metric entropy.

Now take $z \in \operatorname{int} L(\Phi)$. By Lemma 4, there exists a measure $\nu_z \in \mathcal{M}$ such that $\int_X \Phi\, d\nu_z = z$, where ν_z is the unique equilibrium measure for the function ψ_z in (23). Let $\mu \in \mathcal{M}(z)$ be a measure maximizing the metric entropy. Since $\int_X \Phi\, d\mu = \int_X \Phi\, d\nu_z$, it follows readily from (23) that

$$\int_X \psi_z\, d\mu = \int_X \psi_z\, d\nu_z.$$

Therefore,

$$h_\mu(T) + \int_X \psi_z\, d\mu \geq h_{\nu_z}(T) + \int_X \psi_z\, d\nu_z = P(\psi_z),$$

which implies that μ is also an equilibrium measure for ψ_z (for the classical topological pressure). Since ψ_z has a unique equilibrium measure, we conclude that $\mu = \nu_z$. □

Now consider the function $E\colon L(\Phi) \to \mathbb{R}$ defined by $E(z) = h(z) + F(z)$.

Lemma 6 *$z \in K(F, \Phi)$ if and only if z maximizes the function E.*

Proof of the lemma First assume that $z \in L(\Phi)$ maximizes the function E. By Lemma 5, there exists $\mu \in \mathcal{M}(z)$ such that $h(z) = h_\mu(T)$ and so

$$h_\mu(T) + F\left(\int_X \Phi\, d\mu\right) = h(z) + F(z) = \sup_{\mu\in\mathcal{M}} \left\{h_\mu(T) + F\left(\int_X \Phi\, d\mu\right)\right\}.$$

This implies that μ is an equilibrium measure for (F, Φ) and so $z \in K(F, \Phi)$.

Now assume that $z \in K(F, \Phi)$. Then there exists an equilibrium measure μ for (F, Φ) such that $z = \int_X \Phi\, d\mu$ and so

$$\begin{aligned} E(z) &= h(z) + F(z) \\ &\geq h_\mu(T) + F\left(\int_X \Phi\, d\mu\right) \\ &= \sup_{\nu \in \mathcal{M}} \left\{ h_\nu(T) + F\left(\int_X \Phi\, d\nu\right)\right\} \\ &= \sup_{w \in L(\Phi)} \sup_{\nu \in \mathcal{M}(w)} \left\{ h_\nu(T) + F\left(\int_X \Phi\, d\nu\right)\right\} \\ &= \sup_{w \in L(\Phi)} \sup_{\nu \in \mathcal{M}(w)} \{h_\nu(T) + F(w)\} \\ &= \sup_{w \in L(\Phi)} \{h(w) + F(w)\} = \sup_{w \in L(\Phi)} E(w). \end{aligned}$$

This shows that z maximizes E. □

Lemmas 3 and 6 give items (1) and (2) in the theorem. Now we establish item (3). For each $z \in K(F, \Phi)$ there exists an equilibrium measure μ for (F, Φ) such that $\int_X \Phi\, d\mu = z$. When $K(F, \Phi) \subset \operatorname{int} L(\Phi)$, it follows from Lemmas 4 and 5 that μ is the unique measure with $\int_X \Phi\, d\mu = z$ and that $\mu = \nu_z$, where ν_z is ergodic and is the unique equilibrium measure for some function ψ_z. □

It is shown in [8] that the condition $K(F, \Phi) \subset \operatorname{int} L(\Phi)$ in the last property of Theorem 7 holds for a certain class of pairs (T, Φ) that they call C^r Legendre (we refer to that paper for the definition).

Remark In the proof of Lemma 4, for each $z \in \operatorname{int} L(\Phi)$ the point $q(z)$ minimizing $\Delta_z(q)$ might not be unique. Therefore, one may have more than one function ψ_z as in (23). On the other hand, Lemma 5 guarantees that all possible functions ψ_z have the same equilibrium measure ν_z.

5 Number of Equilibrium Measures

In this section we consider the problem of how many equilibrium measures a C^r regular system has.

5.1 Preliminary Results

We start with some auxiliary results about the function h in (22). Note that

$$h(z) = \sup \left\{ h_\mu(T) : \int_X \Phi\, d\mu = z \text{ with } \mu \in \mathcal{M} \right\}.$$

Proposition 8 *For a C^1 regular pair (T, Φ) the function $h\colon L(\Phi) \to \mathbb{R}$ is upper semicontinuous, concave and finite.*

Proof Take $z \in L(\Phi)$ and consider a sequence $(z_n)_{n \in \mathbb{N}}$ in $L(\Phi)$ such that $z_n \to z$ when $n \to \infty$. By Lemma 5, eventually passing to a subsequence one can assume that for each

$n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{M}(z_n)$ such that $h(z_n) = h_{\mu_n}(T)$ and $\mu_n \to \mu$ when $n \to \infty$ for some $\mu \in \mathcal{M}$ in the weak* topology. We also have

$$\int_X \Phi \, d\mu = \lim_{n\to\infty} \int_X \Phi \, d\mu_n = \lim_{n\to\infty} z_n = z$$

and so $\mu \in \mathcal{M}(z)$. Moreover, since $\mu \mapsto h_\mu(T)$ is upper semicontinuous, we obtain

$$\limsup_{n\to\infty} h(z_n) = \limsup_{n\to\infty} h_{\mu_n}(T) \le h_\mu(T) \le h(z)$$

and so h is upper semicontinuous on $L(\Phi)$.

Now we prove the concavity property. Take $z_1, z_2 \in L(\Phi)$ and $\mu_1 \in \mathcal{M}(z_1)$, $\mu_2 \in \mathcal{M}(z_2)$ such that $h(z_1) = h_{\mu_1}(T)$ and $h(z_2) = h_{\mu_2}(T)$. Since the entropy map is affine, for each $t \in [0, 1]$ we have

$$\begin{aligned} h(tz_1 + (1-t)z_2) \ge h_{t\mu_1+(1-t)\mu_2}(T) &= th_{\mu_1}(T) + (1-t)h_{\mu_2}(T) \\ &= th(z_1) + (1-t)h(z_2). \end{aligned}$$

The upper semicontinuity of h on $L(\Phi)$ together with the compactness of $L(\Phi)$ and the fact that $\mathcal{M}(z) \ne \emptyset$ for each $z \in L(\Phi)$, guarantee that h is finite on $L(\Phi)$. □

As pointed out in the recent work [30], in strong contrast to what happens for $d = 1$, the function $z \mapsto h(z)$ need not be continuous on $L(\Phi)$.

Proposition 9 *If the pair (T, Φ) is C^r regular, then the function $h|_{\operatorname{int} L(\Phi)}$ is C^{r-1}. Moreover, if (T, Φ) is C^ω regular, then $h|_{\operatorname{int} L(\Phi)}$ is analytic.*

Proof It follows from Theorem 12 in [3] that if (T, Φ) is C^r regular, then the map $\operatorname{int} L(\Phi) \ni z \mapsto h(T|_{C_z(\Phi)})$ (the topological entropy of T on $C_z(\Phi)$) is of class C^{r-1}, and that if the pair is C^ω regular, then this map is analytic. Since $h(z) = h(T|_{C_z(\Phi)})$ for $z \in \operatorname{int} L(\Phi)$, we obtain the desired statement. □

For $d = 1$, Corollary 1.11 in [7] says that if the pair (T, Φ) is C^ω and F is analytic on $\operatorname{int} L(\Phi)$, then the set $K(F, \Phi)$ is finite. In particular, there exist finitely many equilibrium measures.

5.2 Equilibrium Measures I

For $d = 1$, it was shown in [7] that no point on $\partial L(\Phi)$ maximizes the function $E = h + F$. By Lemma 6, this implies that $K(F, \Phi) \subset \operatorname{int} L(\Phi)$. It is also shown that $h''(z) < 0$ for every $z \in \operatorname{int} L(\Phi)$ and so $h\colon L(\Phi) \to \mathbb{R}$ is a strictly concave function. Note that for $d = 1$ we have $L(\varphi) = [A, B]$, where $A = \inf_{\mu\in\mathcal{M}} \int \varphi \, d\mu$ and $B = \sup_{\mu\in\mathcal{M}} \int \varphi \, d\mu$.

The next result is a criterion for uniqueness of equilibrium measures.

Theorem 10 *Let (T, φ) be a C^r regular pair and let $F\colon \mathbb{R} \to \mathbb{R}$ be a C^r function that is concave on $[A, B]$. Then there exists a unique equilibrium measure for (F, φ). Moreover, the equilibrium measure is ergodic.*

Proof Since F is concave and h is strictly concave, the function $E = h+F$ is strictly concave. This implies that E has at most one maximizer in (A, B). Since there is no maximizer of E on $\partial L(\varphi) = \{A, B\}$ and $K(F, \varphi) \ne \emptyset$, we conclude that there exists a unique point $z^* \in (A, B)$ maximizing E. Hence, it follows from Lemma 6 that $K(F, \varphi) = \{z^*\}$. By Theorem 4.3 in [7] together with Lemma 4, we conclude that there exists a unique equilibrium measure for (F, φ) and that this measure is ergodic. □

The following example illustrates various possibilities.

Example 4 Let $\Sigma = \{-1, 1\}$ and let $T\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be the two-sided shift. We consider the function $\varphi\colon \Sigma \to \mathbb{R}$ defined by $\varphi(\cdots\omega_{-1}\omega_0\omega_1\cdots) = \omega_0$. Then $L(\varphi) = [-1, 1]$ and the entropy function $h\colon L(\varphi) \to \mathbb{R}$ is given by

$$h(z) = -\frac{1-z}{2}\log\left(\frac{1-z}{2}\right) - \frac{1+z}{2}\log\left(\frac{1+z}{2}\right). \tag{24}$$

For the function $F\colon L(\Phi) \to \mathbb{R}$ defined by $F(z) = \alpha/(z^2 - 2)$, where $\alpha \in \mathbb{R}$, we have

$$F''(z) = 2\alpha(3z^2 + 2)/(z^2 - 2)^3.$$

Notice that for $\alpha > 0$ we have $F'' < 0$ on $\operatorname{int} L(\varphi)$. Since $F \equiv 0$ for $\alpha = 0$, the function F is concave on $\operatorname{int} L(\varphi)$ whenever $\alpha \geq 0$. Hence, by Theorem 10 there exists a unique equilibrium measure ν_{z^*} for (F, φ), where $z^* = 0$ (see Fig. 1). For $\alpha < 0$, the number of equilibrium measures may vary and is the number of absolute maximizers of E on $(-1, 1)$. For instance, for $\alpha = -1$ there is one equilibrium measure, while for $\alpha = -2.3$ there are two equilibrium measures (see also Fig. 1).

Theorem 10 also shows that in order to have finitely many equilibrium measures it is not necessary that the pair (T, φ) is C^ω and that the function F is analytic. We give an example in the nonanalytic C^∞ case.

Example 5 Consider the pair (T, φ) in Example 4 and let $F\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$F(z) = \begin{cases} 3\exp(-1/z) & \text{if } z > 0, \\ 0 & \text{if } z \leq 0. \end{cases}$$

One can show that F is C^∞ but not analytic. For $-1 \leq z \leq 0$, we have

$$E = h + F = h + 0 = h.$$

It follows from (24) that E has a local maximum $y_1 = 1$ at $z_1^* = 0$. For $0 < z \leq 1$, one can verify that E has a local maximum $y_2 \approx 1.33$ at $z_2^* \approx 0.75$. Since $y_1 < y_2$, the function E has a unique global maximum at $z_2^* \in (0, 1) \subset \operatorname{int} L(\varphi)$ (see Fig. 2). By Theorem 7, we conclude that $\nu_{z_2^*}$ is the unique equilibrium measure for (F, φ).

5.3 Equilibrium Measures II

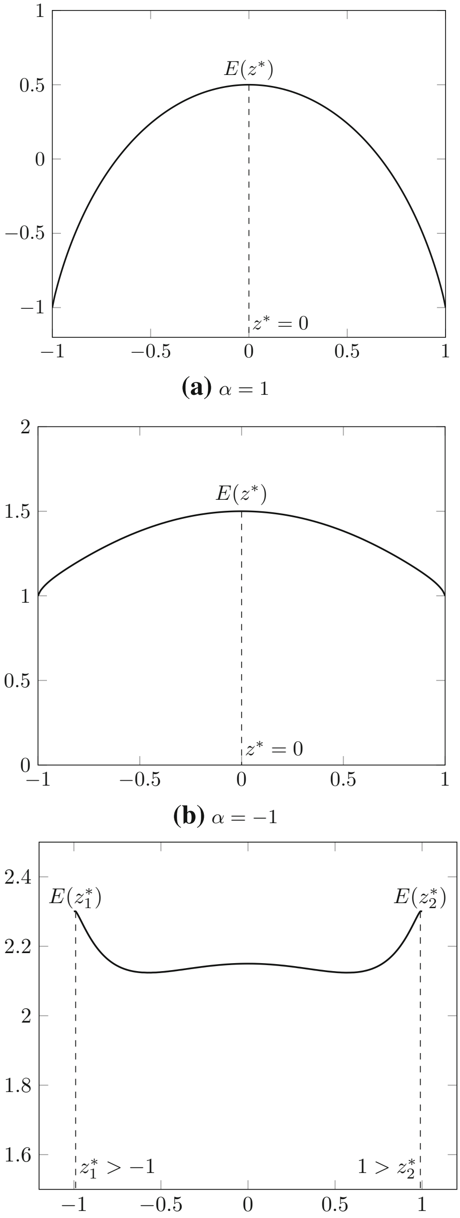
As in the one-dimensional case, for $d > 1$ no point in $\partial L(\Phi)$ can maximize the function E, that is, $K(F, \Phi) \subset \operatorname{int} L(\Phi)$ (see the Claim in the proof of Theorem 4.15 in [8]). This is possible because C^r regular pairs are C^r Legendre (see Proposition 4.10 in [8]).

The following statement is a version of the uniqueness result in Theorem 10 for $d > 1$.

Theorem 11 *Let (T, Φ) be a C^r regular pair and let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a C^r function that is strictly concave on $L(\Phi)$. Then there exists a unique equilibrium measure for (F, Φ). Moreover, the equilibrium measure is ergodic.*

Proof By Proposition 8, the map $z \mapsto h(z)$ is upper semicontinuous on $L(\Phi)$. Since F is C^r on $L(\Phi)$, we conclude that $z \mapsto E(z)$ is upper semicontinuous on $L(\Phi)$. Together with the compactness of $L(\Phi)$, this guarantees the existence of at least one point in $L(\Phi)$ maximizing

Fig. 1 The number of equilibrium measures depends on the parameter α

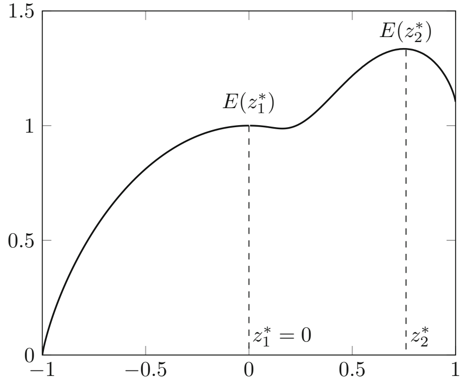


(a) $\alpha = 1$

(b) $\alpha = -1$

(c) $\alpha = -2.3$

Fig. 2 Nonanalytic C^∞ case



the function E. On the other hand, by Propositions 8 and 9 and the strict concavity of F, the function E is strictly concave on $L(\Phi)$ and C^{r-1} on int $L(\Phi)$. The concavity property of E implies that there exists at most one maximizer in $L(\Phi)$. Since there are no maximizers of E in $\partial L(\Phi)$, the unique point z^* maximizing E must be in int $L(\Phi)$. It follows now from Theorem 7 that $K(F, \Phi) = \{z^*\}$, that is, ν_{z^*} is the unique equilibrium measure for (F, Φ). Moreover, by Lemma 4, ν_{z^*} is an ergodic measure. □

In Example 4, we have $h|_{\partial L(\varphi)} \equiv 0$. It turns out that this behavior at the boundary of $L(\varphi)$ is typical for some C^r regular systems, even for $d > 1$. Let $\mathcal{H}_\theta$ be the space of Hölder continuous functions with Hölder exponent $\theta > 0$. The following result is a particular case of Theorem 14 in [3].

Theorem 12 *Let T be a subshift of finite type, a $C^{1+\varepsilon}$ diffeomorphism with a hyperbolic set, or a $C^{1+\varepsilon}$ map with a repeller, that is assumed to be topologically mixing. Then there exists a residual set $\mathcal{O} \subset (\mathcal{H}_\theta)^d$ such that for each $\Phi \in \mathcal{O}$ we have*

$$h|_{\partial L(\Phi)} \equiv 0 \quad \text{and} \quad L(\Phi) = \overline{\operatorname{int} L(\Phi)}. \tag{25}$$

We also note that in Example 4 with $\alpha > 0$, the function F satisfies

$$F|_{\operatorname{int} L(\varphi)} > \max_{z \in \partial L(\varphi)} F(z), \tag{26}$$

where $F_{\operatorname{int} L(\varphi)}$ is the restriction to int $L(\varphi)$. In fact, condition (26) together with the continuity of F implies that F must actually be constant on $\partial L(\Phi)$, as it happens in Example 4. This scenario is a more general situation in which $E = h + F$ attains its maximum on int $L(\Phi)$:

$$\max_{z \in \operatorname{int} L(\Phi)} E(z) > \max_{z \in \partial L(\Phi)} E(z). \tag{27}$$

Note that this condition may depend not only on F, but also on the family of functions Φ.

A similar idea works for typical C^r regular systems in the sense that they belong to the residual set $\mathcal{O}$ in Theorem 12. Let (T, Φ) be a C^r regular pair satisfying (25). In particular, int $L(\Phi) \neq \emptyset$. Now let $F \colon \mathbb{R}^d \to \mathbb{R}$ be a function satisfying (26) with φ replaced by Φ. Since $h \geq 0$, we have

$$\begin{aligned}\max_{z \in \partial L(\Phi)} E(z) &\leq \max_{z \in \partial L(\Phi)} h(z) + \max_{z \in \partial L(\Phi)} F(z) \\ &\leq h|_{\operatorname{int} L(\Phi)} + \max_{z \in \partial L(\Phi)} F(z) \\ &< h|_{\operatorname{int} L(\Phi)} + F|_{\operatorname{int} L(\Phi)} = E_{\operatorname{int} L(\Phi)},\end{aligned}$$

which implies that property (27) holds. Therefore, $K(F, \Phi) \subset \operatorname{int} L(\Phi)$ and so one can apply item (3) of Theorem 7.

It was shown recently in [8] that condition (27) is satisfied for C^r Legendre pairs. This implies that we always have $K(F, \Phi) \subset \operatorname{int} L(\Phi)$ in our setup.

For $d = 1$ and C^r regular systems, the function h in (22) is strictly concave. The next example (which should be compared with the Curie–Weiss–Potts model for 3 colors) illustrates that this may still happen for $d > 1$, but unfortunately we are not able to describe for which C^r regular pairs the function h is strictly concave.

Example 6 Let $T\colon X \to X$ be the two-sided shift with $X = \{1, 2, 3\}^{\mathbb{Z}}$ and let $\varphi_1 = \chi_{C_1}$ and $\varphi_2 = \chi_{C_3}$, where C_i is the set of all sequences

$$(\cdots \omega_{-1}\omega_0\omega_1 \cdots) \in X$$

with $\omega_0 = i$. Since $\int_X \varphi_1 d\mu = \mu(C_1)$ and $\int_X \varphi_2 d\mu = \mu(C_3)$ for each $\mu \in \mathcal{M}$, we have

$$L(\Phi) = \big\{(\mu(C_1), \mu(C_3)) : \mu \in \mathcal{M}\big\}.$$

By Theorem 8 in [3], we obtain

$$\begin{aligned} h(z_1, z_2) &= \max_{\mu\in\mathcal{M}} \big\{h_\mu(T) : (\mu(C_1), \mu(C_2)) = (z_1, z_2)\big\} \\ &= -z_1 \log z_1 - z_2 \log z_2 - z_3 \log z_3. \end{aligned}$$

On the other hand, since $\mu(C_1) + \mu(C_2) + \mu(C_3) = 1$ for each $\mu \in \mathcal{M}$, we have

$$L(\Phi) = \big\{(z_1, z_2) \in [0, 1] \times [0, 1] : z_1 + z_2 \le 1\big\}$$

and

$$h(z_1, z_2) = -z_1 \log z_1 - z_2 \log z_2 - (1 - z_1 - z_2) \log(1 - z_1 - z_2).$$

Note that $\operatorname{int} L(\Phi) \neq \emptyset$ and that $\partial L(\Phi)$ is the set

$$\big((\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \cup \{(z_1, z_2) : z_1 + z_2 = 1\}\big) \cap ([0, 1] \times [0, 1]).$$

For $(z_1, z_2) = (1/2, 0), (0, 1/2), (1/2, 1/2) \in \partial L(\Phi)$ we have $h(z_1, z_2) = 1 > 0$ and so the system is not typical. On the other hand, one can easily verify that the map $(z_1, z_2) \mapsto h(z_1, z_2)$ is still strictly concave on $L(\Phi)$.

Now consider the function $F(z_1, z_2) = \beta(z_1^2 + z_2^2)/2$ with $\beta \in \mathbb{R}$. One can verify that the determinant of the Hessian matrix of $E = h + F$ is given by

$$\det H_E(z_1, z_2) = \beta^2 + \beta \frac{z_1(1 - z_1) + z_2(1 - z_2)}{z_1 z_2(1 - z_1 - z_2)} + \frac{1}{z_1 z_2(1 - z_1 - z_2)}.$$

Since $\det H_E(z_1, z_2) > 0$ for $(z_1, z_2) \in \operatorname{int} L(\Phi)$ and $\beta \ge 0$, every critical point of E is nondegenerate for all $\beta \ge 0$. Hence, for each $\beta \ge 0$, the function E has at most finitely many critical points. In addition, it was shown in [8] that condition (27) always holds. So E attains its maximal value only at critical points. It follows from Theorem 7 that the pair (F, Φ) has finitely many equilibrium measures.

On the other hand, for $\beta < 0$ the function F is strictly concave and one can use Theorem 11 to conclude that (F, Φ) has a unique equilibrium measure.

Remark In Example 6 the parameter β is related to the absolute temperature and the model has physical meaning only when $\beta \ge 0$. However, the general concave case (with $\beta < 0$) might be useful for possible applications in other contexts.

5.4 Coincidence of Equilibrium Measures

The following result gives a sufficient condition so that two systems share equilibrium measures. We say that $\Phi_1 = \{\varphi_{1,1}, \dots, \varphi_{1,d}\}$ is *cohomologous* to $\Phi_2 = \{\varphi_{2,1}, \dots, \varphi_{2,d}\}$ if $\varphi_{1,i}$ is cohomologous to $\varphi_{2,i}$ for $i = 1, \dots, d$. Then

$$\int_X \Phi_1 \, d\mu = \int_X \Phi_2 \, d\mu \quad \text{for each } \mu \in \mathcal{M},$$

which readily implies that $L(\Phi_1) = L(\Phi_2)$.

Proposition 13 *Let (T, Φ_1) and (T, Φ_2) be C^r regular pairs such that Φ_1 is cohomologous to Φ_2 and let $F_1 \colon L(\Phi_1) \to \mathbb{R}$ and $F_2 \colon L(\Phi_2) \to \mathbb{R}$ be continuous functions. If a point $z \in \operatorname{int} L(\Phi_1) \cap \operatorname{int} L(\Phi_2)$ is simultaneously a maximizer for the functions $E_1 = h_1 + F_1$ and $E_2 = h_2 + F_2$, then ν_z is an equilibrium measure for (F_1, Φ_1) and (F_2, Φ_2).*

Proof Since Φ_1 is cohomologous to Φ_2, we have $L := L(\Phi_1) = L(\Phi_2)$. Now take $z \in \operatorname{int} L$ and consider the functions

$$\Delta_1(q) = P(\langle q, \Phi_1 - z\rangle) - h_1(z) \quad \text{and} \quad \Delta_2(q) = P(\langle q, \Phi_2 - z\rangle) - h_2(z),$$

where P denotes the classical topological pressure and where each h_i is the corresponding entropy function (see (22)). By the cohomology assumption, we have

$$\lim_{n\to\infty} \frac{\|S_n\varphi_{1,i} - S_n\varphi_{2,i}\|_\infty}{n} = 0 \quad \text{for } i = 1, \dots, d$$

and so $C_z(\Phi_1) = C_z(\Phi_2)$ for all $z \in \mathbb{R}^d$ (see (21)). In particular, this implies that $h := h_1 = h_2$. Therefore,

$$[\langle q, \Phi_1 - z\rangle - h_1(z)] - [\langle q, \Phi_2 - z\rangle - h_2(z)] = \langle q, \Phi_1 - \Phi_2\rangle$$

for $q \in \mathbb{R}^d$. Again since Φ_1 is cohomologous to Φ_2, we conclude that

$$\Delta_1(q) = \Delta_2(q) \quad \text{for } q \in \mathbb{R}^d. \tag{28}$$

On the other hand, by the proof of Theorem 8 in [3] the function $q \mapsto \Delta_1(q)$ attains its minimum at a point $q_1(z)$ and $\nu_{1,z}$ is the unique equilibrium measure for the function $\langle q_1(z), \Phi_1 - z\rangle - h(z)$. Similarly, $q \mapsto \Delta_2(q)$ attains its minimum at a point $q_2(z)$ and $\nu_{2,z}$ is the unique equilibrium measure for the function $\langle q_2(z), \Phi_2 - z\rangle - h(z)$. By (28), one can take $q_1(z) = q_2(z)$ and so $\nu_z := \nu_{1,z} = \nu_{2,z}$. The desired result follows now from Theorem 7. □

A direct consequence of Proposition 13 is that if Φ_1 is cohomologous to Φ_2 and the functions E_1 and E_2 attain maximal values at the same points, then (F_1, Φ_1) and (F_2, Φ_2) have the same equilibrium measures (in particular, this happens when $F_1 = F_2$). For the converse to hold we need stronger conditions so that the coincidence of two equilibrium measures yields a cohomology relation.

Theorem 14 *Let X be a topologically mixing locally maximal hyperbolic set for a diffeomorphism T and let Φ_1 and Φ_2 be families of Hölder continuous functions. Moreover, let F_1 and F_2 be continuous functions. If (F_1, Φ_1) and (F_2, Φ_2) have the same equilibrium measures, then for each z_1 and z_2 maximizing E_1 and E_2, respectively, there exist $q_1, q_2 \in \mathbb{R}^d$ such that $\langle q_1, \Phi_1 - z_1\rangle$ is cohomologous to $\langle q_2, \Phi_2 - z_2\rangle$.*

Proof By Theorem 7, each equilibrium measure for (F_i, Φ_i) is a measure ν_{z_i} with $z_i \in K(F_i, \Phi_i)$ that is the unique equilibrium measure for

$$\psi_i = \langle q_i(z_i), \Phi_i - z_i \rangle - h_i(z_i),$$

where $q_i(z_i)$ is a minimizer of the function

$$\Delta_i(q) = P(\langle q, \Phi_i - z_i \rangle) - h_i(z_i).$$

Since by hypotheses $\nu_{z_1} = \nu_{z_2}$, the function $\psi_1 - \psi_2$ is cohomologous to $P(\psi_1) - P(\psi_2) \in \mathbb{R}$. But since

$$\Delta_1(q_1(z_1)) = \Delta_2(q_2(z_2)) = 0$$

(see Lemma 2 in [3]), we have $P(\psi_1) = P(\psi_2)$. So there exists a continuous function $S = S(z_1, z_2) : X \to \mathbb{R}$ such that $\psi_1 - \psi_2 = S \circ T - S$, that is,

$$S \circ T - S = \langle q_1(z_1), \Phi_1 - z_1 \rangle - \langle q_2(z_2), \Phi_2 - z_2 \rangle - h_1(z_1) + h_2(z_2).$$

Again since $\nu_{z_1} = \nu_{z_2}$, by Lemma 5 we have

$$h_1(z_1) = h_{\nu_{z_1}}(T) = h_{\nu_{z_2}}(T) = h_2(z_2).$$

Hence, for each z_1 and z_2 maximizing E_1 and E_2, respectively, there exist points $q_1 = q_1(z_1), q_2 = q_2(z_2) \in \mathbb{R}^d$ as in the statement of the theorem. □

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