Journal of Statistical Physics (2021) 183:39
https://doi.org/10.1007/s10955-021-02766-6

Bose–Einstein Condensation for Two Dimensional Bosons in the Gross–Pitaevskii Regime

Cristina Caraci[1] · Serena Cenatiempo[2] · Benjamin Schlein[1]

Received: 17 December 2020 / Accepted: 19 April 2021 / Published online: 20 May 2021

Abstract
We consider systems of N bosons trapped on the two-dimensional unit torus, in the Gross-Pitaevskii regime, where the scattering length of the repulsive interaction is exponentially small in the number of particles. We show that low-energy states exhibit complete Bose–Einstein condensation, with almost optimal bounds on the number of orthogonal excitations.

Keywords Two dimensional Bose gas · Bose–Einstein condensation · Gross-Pitaevskii regime · Interacting bosons

Mathematics Subject Classification 46N50 · 81V70 · 81V73 · 82B10 · 82B27 · 82D03

1 Introduction

We consider $N \in \mathbb{N}$ bosons trapped in the two-dimensional box $\Lambda = [-1/2; 1/2]^2$ with periodic boundary conditions. In the Gross-Pitaevskii regime, particles interact through a repulsive pair potential, with a scattering length exponentially small in N. The Hamilton operator is given by

---

Communicated by Alessandro Giuliani.

C.C. and S.C. gratefully acknowledge the support from the GNFM Gruppo Nazionale per la Fisica Matematica - INDAM through the project "Derivation of effective theories for large quantum systems". B. S. gratefully acknowledges partial support from the NCCR SwissMAP, from the Swiss National Science Foundation through the Grant "Dynamical and energetic properties of Bose–Einstein condensates" and from the European Research Council through the ERC-AdG CLaQS.

✉ Cristina Caraci
cristina.caraci@math.uzh.ch

Serena Cenatiempo
serena.cenatiempo@gssi.it

Benjamin Schlein
benjamin.schlein@math.uzh.ch

[1] University of Zurich, Winterthurerstr. 190, 8057 Zurich, Switzerland

[2] Gran Sasso Science Institute, Viale F. Crispi 7, 67100 L'Aquila, Italy

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} e^{2N} V(e^N(x_i - x_j)) \tag{1}$$

and acts on a dense subspace of $L^2_s(\Lambda^N)$, the Hilbert space consisting of functions in $L^2(\Lambda^N)$ that are invariant with respect to permutations of the N particles. We assume here $V \in L^3(\mathbb{R}^2)$ to be compactly supported and pointwise non-negative (i.e. $V(x) \geq 0$ for almost all $x \in \mathbb{R}^2$).

We denote by $\mathfrak{a}$ the scattering length of the unscaled potential V. We recall that in two dimensions and for a potential V with finite range R_0, the scattering length is defined by

$$\frac{2\pi}{\log(R/\mathfrak{a})} = \inf_{\phi} \int_{B_R} \left[|\nabla \phi|^2 + \frac{1}{2} V |\phi|^2 \right] dx \tag{2}$$

where $R > R_0$, B_R is the disk of radius R centered at the origin and the infimum is taken over functions $\phi \in H^1(B_R)$ with $\phi(x) = 1$ for all x with $|x| = R$. The unique minimizer of the variational problem on the r.h.s. of (2) is non-negative, radially symmetric and satisfies the scattering equation

$$-\Delta \phi^{(R)} + \frac{1}{2} V \phi^{(R)} = 0$$

in the sense of distributions. For $R_0 < |x| \leq R$, we have

$$\phi^{(R)}(x) = \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})} .$$

By scaling, $\phi_N(x) := \phi^{(e^N R)}(e^N x)$ is such that

$$-\Delta \phi_N + \frac{1}{2} e^{2N} V(e^N x) \phi_N = 0$$

We have

$$\phi_N(x) = \frac{\log(|x|/\mathfrak{a}_N)}{\log(R/\mathfrak{a}_N)} \qquad \forall x \in \mathbb{R}^2 : e^{-N} R_0 < |x| \leq R ,$$

for all $x \in \mathbb{R}^2$ with $e^{-N} R_0 < |x| \leq R$. Here $\mathfrak{a}_N = e^{-N}\mathfrak{a}$.

The spectral properties of trapped two dimensional bosons in the Gross-Pitaevskii regime (in the more general case where the bosons are confined by external trapping potentials) have been first studied in [13,14,16]. These results can be translated to the Hamilton operator (1), defined on the torus, with no external potential. They imply that the ground state energy E_N of (1) is such that

$$E_N = 2\pi N\big(1 + O(N^{-1/5})\big) . \tag{3}$$

Moreover, they imply Bose–Einstein condensation in the zero-momentum mode $\varphi_0(x) = 1$ for all $x \in \Lambda$, for any approximate ground state of (1). More precisely, it follows from [13] that, for any sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle = 2\pi, \tag{4}$$

the one-particle reduced density matrix $\gamma_N = \mathrm{tr}_{2,\ldots,N} |\psi_N\rangle\langle\psi_N|$ is such that

$$1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leq C N^{-\bar{\delta}} \tag{5}$$

for a sufficiently small $\bar{\delta} > 0$. The estimate (5) states that, in many-body states satisfying (4) (approximate ground states), almost all particles are described by the one-particle orbital φ_0, with at most $N^{1-\delta} \ll N$ orthogonal excitations.

Similar results have been obtained starting from a three dimensional Bose gas, trapped by a potential which is strongly confining in one direction, so that the system becomes effectively two-dimensional [22]. Finally, let us also mention [5,10], where rigorous results on the time-evolution in the two-dimensional Gross-Pitaevskii regime have been established (in [5], the focus is on the dynamics of a three-dimensional gas, with strong confinement in one direction).

For $V \in L^3(\mathbb{R}^2)$, our main theorem improves (3) and (5) by providing more precise bounds on the ground state energy and on the number of excitations.

Theorem 1 *Let $V \in L^3(\mathbb{R}^2)$ have compact support, be spherically symmetric and pointwise non-negative. Then there exists a constant $C > 0$ such that the ground state energy E_N of (1) satisfies*

$$2\pi N - C \leq E_N \leq 2\pi N + C \log N \,. \tag{6}$$

Furthermore, consider a sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and such that

$$\langle \psi_N, H_N \psi_N \rangle \leq 2\pi N + K \tag{7}$$

for a $K > 0$. Then the reduced density matrix $\gamma_N = \mathrm{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$ associated with ψ_N is such that

$$1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leq \frac{C(1+K)}{N} \tag{8}$$

for all $N \in \mathbb{N}$ large enough.

Remark We expect that the bounds of Theorem 1 can be extended to two-dimensional systems of bosons trapped by an external potential (in three dimensions, similar estimates have been recently established in [7,19]). In this case, the system exhibits condensation in the minimizer of the Gross-Pitaevskii energy functional, as shown in [13,14,16].

It is interesting to compare the Gross-Pitaevskii regime with the thermodynamic limit, where a Bose gas of N particles interacting through a fixed potential with scattering length $\mathfrak{a}$ is confined in a box with area L^2, so that $N, L \to \infty$ with the density $\rho = N/L^2$ kept fixed. Let $b = |\log(\rho\mathfrak{a}^2)|^{-1}$. Then, in the dilute limit $\rho\mathfrak{a}^2 \ll 1$, the ground state energy per particle in the thermodynamic limit is expected to satisfy

$$e_0(\rho) = 4\pi\rho^2 b\Big(1 + b\log b + \big(1/2 + 2\gamma + \log\pi\big)b + o(b)\Big)\,, \tag{9}$$

with γ the Euler's constant. The leading order term on the r.h.s. of (9) has been first derived in [21] and then rigorously established in [15], with an error rate $b^{-1/5}$. The corrections up to order b have been predicted in [1,18,20]. To date, there is no rigorous proof of (9). Some partial result, based on the restriction to quasi-free states, has been recently obtained in [9, Theorem 1].

Extrapolating from (9), in the Gross-Pitaevskii regime we expect $|E_N - 2\pi N| \leq C$. While our estimate (6) captures the correct lower bound, the upper bound is off by a logarithmic correction. Eq. (8), on the other hand, is expected to be optimal (but of course, by (6), we need to choose $K = C\log N$ to be sure that (7) can be satisfied). This bound can be used as starting point to investigate the validity of Bogoliubov theory for two dimensional bosons in the Gross-Pitaevskii regime, following the strategy developed in [3] for the three dimensional case; we plan to proceed in this direction in a separate paper.

The proof of Theorem 1 follows the strategy that has been recently introduced in [4] to prove condensation for three-dimensional bosons in the Gross-Pitaevskii limit. There are, however, additional obstacles in the two-dimensional case, requiring new ideas. To appreciate the difference between the Gross-Pitaevskii regime in two- and three-dimensions, we can compute the energy of the trivial wave function $\psi_N \equiv 1$. The expectation of (1) in this state is of order N^2. It is only through correlations that the energy can approach (6). Also in three dimensions, uncorrelated many-body wave functions have large energy, but in that case the difference with respect to the ground state energy is only of order N ($N\widehat{V}(0)/2$ rather than $4\pi\mathfrak{a}N$). This observation is a sign that correlations in two-dimensions are stronger and play a more important role than in three dimensions (this creates problems in handling error terms that, in the three dimensional setting, were simply estimated in terms of the integral of the potential).

The paper is organized as follows. In Sect. 2 we introduce our setting, based on a description of orthogonal excitations of the condensate on a truncated Fock space. Factoring out the condensate, we introduce an excitation Hamiltonian $\mathcal{L}_N$, unitarily equivalent to H_N. In Sects. 3 and 4 we define two additional unitary maps, modelling the correlation structure characterising low-energy states. The first map is a generalized Bogoliubov transformation, given by the exponential of an anti-symmetric operator B, quadratic in creation and annihilation operators, see Eq. (33). Its action on $\mathcal{L}_N$ leads to a second excitation Hamiltonian $\mathcal{G}_{N,\alpha}$, whose vacuum expectation matches (6), at leading order. Unfortunately, $\mathcal{G}_{N,\alpha}$ is not coercive enough to directly show Bose–Einstein condensation. To overcome this difficulty, we conjugate the main part of $\mathcal{G}_{N,\alpha}$ (later denoted by $\mathcal{G}^{\text{eff}}_{N,\alpha}$) with a second unitary map, given by the exponential of an operator A, cubic in creation and annihilation operators, see Eq. (44). This defines a renormalized excitation Hamiltonian $\mathcal{R}_{N,\alpha}$, where the singular interaction is regularized. In Sect. 5 we combine the bounds on $\mathcal{G}_{N,\alpha}$ and $\mathcal{R}_{N,\alpha}$ with a localization argument proposed in [11] for the number of excitations to conclude the proof of Theorem 1. Section 6 and App. 1 are devoted to the proof of the bounds on $\mathcal{G}_{N,\alpha}$ and on $\mathcal{R}_{N,\alpha}$ stated in Sects. 3 and 4, respectively. Finally, in App. 1, we establish some properties of the solution of the Neumann problem associated with the two-body potential V.

2 The Excitation Hamiltonian

Low-energy states of (1) exhibit condensation in the zero-momentum mode φ_0 defined by $\varphi_0(x) = 1$ for all $x \in \Lambda = [-1/2; 1/2]^2$. Similarly as in [2,4,11], we are going to describe excitations of the condensate on the truncated bosonic Fock space

$$\mathcal{F}_+^{\leq N} = \bigoplus_{k=0}^{N} L^2_\perp(\Lambda)^{\otimes_s k}$$

constructed on the orthogonal complement $L^2_\perp(\Lambda)$ of φ_0 in $L^2(\Lambda)$. To reach this goal, we define a unitary map $U_N : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N}$ by requiring that $U_N\psi_N = \{\alpha_0, \alpha_1, \dots, \alpha_N\}$, with $\alpha_j \in L^2_\perp(\Lambda)^{\otimes_s j}$, if

$$\psi_N = \alpha_0\varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes(N-1)} + \cdots + \alpha_N$$

With the usual creation and annihilation operators, we can write

$$U_N\, \psi_N = \bigoplus_{n=0}^{N} (1 - |\varphi_0\rangle\langle\varphi_0|)^{\otimes n} \frac{a(\varphi_0)^{N-n}}{\sqrt{(N-n)!}}\, \psi_N$$

for all $\psi_N \in L^2_s(\Lambda^N)$. It is then easy to check that $U_N^* : \mathcal{F}_+^{\leq N} \to L^2_s(\Lambda^N)$ is given by

$$U_N^*\, \{\alpha^{(0)}, \ldots, \alpha^{(N)}\} = \sum_{n=0}^{N} \frac{a^*(\varphi_0)^{N-n}}{\sqrt{(N-n)!}}\, \alpha^{(n)}$$

and that $U_N^* U_N = 1$, i.e. U_N is unitary.

With U_N, we can define the excitation Hamiltonian $\mathcal{L}_N := U_N H_N U_N^*$, acting on a dense subspace of $\mathcal{F}_+^{\leq N}$. To compute the operator $\mathcal{L}_N$, we first write the Hamiltonian (1) in momentum space, in terms of creation and annihilation operators a_p^*, a_p, for momenta $p \in \Lambda^* = 2\pi\mathbb{Z}^2$. We find

$$H_N = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/e^N) a_{p+r}^* a_q^* a_p a_{q+r} \tag{10}$$

where

$$\widehat{V}(k) = \int_{\mathbb{R}^2} V(x) e^{-ik\cdot x} dx$$

is the Fourier transform of V, defined for all $k \in \mathbb{R}^2$ (in fact, (1) is the restriction of (10) to the N-particle sector of the Fock space). We can now determine $\mathcal{L}_N$ using the following rules, describing the action of the unitary operator U_N on products of a creation and an annihilation operator (products of the form $a_p^* a_q$ can be thought of as operators mapping $L^2_s(\Lambda^N)$ to itself). For any $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^2\backslash\{0\}$, we find (see [11]):

$$\begin{aligned} U_N\, a_0^* a_0\, U_N^* &= N - \mathcal{N}_+ \\ U_N\, a_p^* a_0\, U_N^* &= a_p^* \sqrt{N - \mathcal{N}_+} \\ U_N\, a_0^* a_p\, U_N^* &= \sqrt{N - \mathcal{N}_+}\, a_p \\ U_N\, a_p^* a_q\, U_N^* &= a_p^* a_q\, . \end{aligned} \tag{11}$$

where $\mathcal{N}_+ = \sum_{p\in\Lambda_+^*} a_p^* a_p$ is the number of particles operator on $\mathcal{F}_+^{\leq N}$. We conclude that

$$\mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} \tag{12}$$

with

$$\begin{aligned}
\mathcal{L}_N^{(0)} &= \frac{1}{2}\widehat{V}(0)(N-1)(N-\mathcal{N}_+) + \frac{1}{2}\widehat{V}(0)\mathcal{N}_+(N-\mathcal{N}_+) \\
\mathcal{L}_N^{(2)} &= \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + N \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\left[b_p^* b_p - \frac{1}{N} a_p^* a_p\right] \\
&\quad + \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\left[b_p^* b_{-p}^* + b_p b_{-p}\right] \\
\mathcal{L}_N^{(3)} &= \sqrt{N} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/e^N)\left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q}\right] \\
\mathcal{L}_N^{(4)} &= \frac{1}{2} \sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -p,-q}} \widehat{V}(r/e^N) a_{p+r}^* a_q^* a_p a_{q+r}\,,
\end{aligned} \tag{13}$$

where we introduced generalized creation and annihilation operators

$$b_p^* = U_N a_p^* U_N^* = a_p^* \sqrt{\frac{N-\mathcal{N}_+}{N}}, \qquad \text{and} \quad b_p = U_N a_p U_N^* = \sqrt{\frac{N-\mathcal{N}_+}{N}}\, a_p$$

for all $p \in \Lambda_+^*$.

On states exhibiting complete Bose–Einstein condensation in the zero-momentum mode φ_0, we have $a_0, a_0^* \simeq \sqrt{N}$ and we can therefore expect that $b_p^* \simeq a_p^*$ and that $b_p \simeq a_p$. From the canonical commutation relations for the standard creation and annihilation operators a_p, a_p^*, we find

$$\begin{aligned}
[b_p, b_q^*] &= \left(1 - \frac{\mathcal{N}_+}{N}\right)\delta_{p,q} - \frac{1}{N} a_q^* a_p \\
[b_p, b_q] &= [b_p^*, b_q^*] = 0\,.
\end{aligned} \tag{14}$$

Furthermore,

$$[b_p, a_q^* a_r] = \delta_{pq} b_r, \qquad [b_p^*, a_q^* a_r] = -\delta_{pr} b_q^*$$

for all $p, q, r \in \Lambda_+^*$; this implies in particular that $[b_p, \mathcal{N}_+] = b_p$, $[b_p^*, \mathcal{N}_+] = -b_p^*$. It is also useful to notice that the operators b_p^*, b_p, like the standard creation and annihilation operators a_p^*, a_p, can be bounded by the square root of the number of particles operators; we find

$$\|b_p \xi\| \leq \|\mathcal{N}_+^{1/2}\xi\|\,, \qquad \|b_p^* \xi\| \leq \|(\mathcal{N}_+ + 1)^{1/2}\xi\|$$

for all $\xi \in \mathcal{F}_+^{\leq N}$. Since $\mathcal{N}_+ \leq N$ on $\mathcal{F}_+^{\leq N}$, the operators b_p^*, b_p are bounded, with $\|b_p\|, \|b_p^*\| \leq (N+1)^{1/2}$.

3 Quadratic Renormalization

From (13) we see that conjugation with U_N extracts, from the original quartic interaction in (10), some large constant and quadratic contributions, collected in $\mathcal{L}_N^{(0)}$ and $\mathcal{L}_N^{(2)}$ respectively. In particular, the expectation of $\mathcal{L}_N$ on the vacuum state Ω is of order N^2, this being an indication of the fact that there are still large contributions to the energy hidden among cubic and quartic terms in $\mathcal{L}_N^{(3)}$ and $\mathcal{L}_N^{(4)}$. Since U_N only removes products of the zero-energy mode φ_0, correlations among particles remain in the excitation vector $U_N \psi_N$. Indeed, correlations play a crucial role in the two dimensional Gross-Pitaevskii regime and carry an energy of order N^2.

To take into account the short scale correlation structure on top of the condensate, we consider the solution f_ℓ of the equation

$$\Big(-\Delta + \frac{1}{2} V(x) \Big) f_\ell(x) = \lambda_\ell \, f_\ell(x) \tag{15}$$

associated with the smallest possible eigenvalue λ_ℓ, on the ball $|x| \leq e^N \ell$, with Neumann boundary conditions and normalized so that $f_\ell(x) = 1$ for $|x| = e^N \ell$. Here and in the following we omit the N-dependence in the notation for f_ℓ and for λ_ℓ. By scaling, we observe that $f_\ell(e^N \cdot)$ satisfies

$$\Big(-\Delta + \frac{e^{2N}}{2} V(e^N x) \Big) f_\ell(e^N x) = e^{2N} \lambda_\ell \, f_\ell(e^N x)$$

on the ball $|x| \leq \ell$. We choose $\ell < 1/2$, so that the ball of radius ℓ is contained in the box $\Lambda = [-1/2; 1/2]^2$. We extend then $f_\ell(e^N .)$ to Λ, by setting $f_{N,\ell}(x) = f_\ell(e^N x)$, if $|x| \leq \ell$ and $f_{N,\ell}(x) = 1$ for $x \in \Lambda$, with $|x| > \ell$. Then, assuming also that $R_0 e^{-N} < \ell$ (later we will choose $\ell = N^{-\alpha}$, so this condition is satisfied, for all N large enough),

$$\Big(-\Delta + \frac{e^{2N}}{2} V(e^N x) \Big) f_{N,\ell}(x) = e^{2N} \lambda_\ell \, f_{N,\ell}(x) \chi_\ell(x)\,, \tag{16}$$

where χ_ℓ is the characteristic function of the ball of radius ℓ. The Fourier coefficients of the function $f_{N,\ell}$ are given by

$$\widehat{f}_{N,\ell}(p) := \int_\Lambda f_\ell(e^N x) e^{-ip\cdot x} dx$$

for all $p \in \Lambda^*$. We introduce also the function $w_\ell(x) = 1 - f_\ell(x)$ for $|x| \leq e^N \ell$ and extend it by setting $w_\ell(x) = 0$ for $|x| > e^N \ell$. Its re-scaled version is defined by $w_{N,\ell} : \Lambda \to \mathbb{R}$ $w_{N,\ell}(x) = w_\ell(e^N x)$ if $|x| \leq \ell$ and $w_{N,\ell} = 0$ if $x \in \Lambda$ with $|x| > \ell$.

The Fourier coefficients of the re-scaled function $w_{N,\ell}$ are given by

$$\widehat{w}_{N,\ell}(p) = \int_\Lambda w_\ell(e^N x) e^{-ip\cdot x} dx = e^{-2N} \widehat{w}_\ell \left(e^{-N} p \right). \tag{17}$$

We find $\widehat{f}_{N,\ell}(p) = \delta_{p,0} - e^{-2N} \widehat{w}_\ell(e^{-N} p)$. From the Neumann problem (16) we obtain

$$-p^2 e^{-2N} \widehat{w}_\ell(e^{-N} p) + \frac{1}{2} \sum_{q \in \Lambda^*} \widehat{V}(e^{-N}(p-q)) \widehat{f}_{N,\ell}(q) = e^{2N} \lambda_\ell \sum_{q \in \Lambda^*} \widehat{\chi}_\ell(p-q) \widehat{f}_{N,\ell}(q). \tag{18}$$

where we used the notation $\widehat{\chi}_\ell$ for the Fourier coefficients of the characteristic function on the ball of radius ℓ. Note that $\widehat{\chi}_\ell(p) = \ell^2 \, \widehat{\chi}(\ell p)$ with $\widehat{\chi}(p)$ the Fourier coefficients of the characteristic function on the ball of radius one.

In the next lemma, we collect some important properties of the solution of (15).

Lemma 1 *Let $V \in L^3(\mathbb{R}^2)$ be non-negative, compactly supported (with range R_0) and spherically symmetric, and denote its scattering length by $\mathfrak{a}$. Fix $0 < \ell < 1/2$, N sufficiently large and let f_ℓ denote the solution of (16). Then*

(i)

$$0 \le f_\ell(x) \le 1 \qquad \forall\, |x| \le e^N \ell\,.$$

(ii) We have

$$\left|\lambda_\ell - \frac{2}{(e^N\ell)^2 \log(e^N\ell/\mathfrak{a})}\right| \le \frac{C}{(e^N\ell)^2 \log^2(e^N\ell/\mathfrak{a})} \tag{19}$$

(iii) There exists a constant $C > 0$ such that

$$\left|\int dx\, V(x) f_\ell(x) - \frac{4\pi}{\log(e^N\ell/\mathfrak{a})}\right| \le \frac{C}{\log^2(e^N\ell/\mathfrak{a})} \tag{20}$$

(iv) There exists a constant $C > 0$ such that

$$\begin{aligned} |w_\ell(x)| &\le \begin{cases} C & \text{if } |x| \le R_0 \\ C\,\frac{\log(e^N\ell/|x|)}{\log(e^N\ell/\mathfrak{a})} & \text{if } R_0 \le |x| \le e^N\ell \end{cases} \\ |\nabla w_\ell(x)| &\le \frac{C}{\log(e^N\ell/\mathfrak{a})}\frac{1}{|x|+1} \qquad \text{for all } |x| \le e^N\ell \end{aligned} \tag{21}$$

(v) Let $w_{N,\ell} = 1 - f_{N,\ell}$ with $f_{\ell,N} = f_\ell(e^N x)$. Then the Fourier coefficients of the function $w_{N,\ell}$ defined in (17) are such that

$$|\widehat{w}_{N,\ell}(p)| \le \frac{C}{p^2 \log(e^N\ell/\mathfrak{a})}. \tag{22}$$

Proof The proof of points (i)–(iv) is deferred in Appendix B. To prove point v) we use the scattering equation (18):

$$\widehat{w}_\ell(e^{-N}p) = \frac{e^{2N}}{2p^2}\sum_{q\in\Lambda^*}\widehat{V}(e^{-N}(p-q))\widehat{f}_{N,\ell}(q) - \frac{e^{4N}}{p^2}\lambda_\ell \sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\widehat{f}_{N,\ell}(q).$$

Using the fact that $e^{2N}\lambda_\ell \le C\ell^{-2}|\ln(e^N\ell/\mathfrak{a})|^{-1}$ and that $0 \le f_\ell \le 1$, we end up with

$$\begin{aligned} |\widehat{w}_\ell(e^{-N}p)| &\le \frac{e^{2N}}{2p^2}\left[\left|(\widehat{V}(e^{-N}\cdot) * \widehat{f}_{N,\ell})(p)\right| + 2e^{2N}\lambda_\ell\left|(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p)\right|\right] \\ &\le \frac{e^{2N}}{2p^2}\left[\int V(x) f_\ell(x) dx + C\ell^{-2}|\log(e^N\ell/\mathfrak{a})|^{-1}\int \chi_\ell(x) f_\ell(e^N x) dx\right] \\ &\le \frac{Ce^{2N}}{p^2 \log(e^N\ell/\mathfrak{a})}. \end{aligned}$$

□

We now define $\check{\eta} : \Lambda \to \mathbb{R}$ through

$$\check{\eta}(x) = -N w_{N,\ell}(x) = -N w_\ell(e^N x)\,. \tag{23}$$

With (21) we find

$$|\check{\eta}(x)| \le \begin{cases} CN & \text{if } |x| \le e^{-N}R_0 \\ C\log(\ell/|x|) & \text{if } e^{-N}R_0 \le |x| \le \ell \end{cases} \tag{24}$$

and in particular, recalling that $e^{-N}R_0 < \ell \le 1/2$,

$$|\check{\eta}(x)| \le C\max(N, \log(\ell/|x|)) \le CN \tag{25}$$

for all $x \in \Lambda$. Using (24) we find

$$\|\eta\|^2 = \|\check{\eta}\|^2 \leq C \int_{|x|\leq \ell} |\log(\ell/|x|)|^2 d^2x \leq C\ell^2 \int_0^1 (\log r)^2 r dr \leq C\ell^2 .$$

In the following we choose $\ell = N^{-\alpha}$, for some $\alpha > 0$ to be fixed later, so that

$$\|\eta\| \leq CN^{-\alpha} . \tag{26}$$

This choice of ℓ will be crucial for our analysis, as commented below. Notice, on the other hand, that the H^1-norms of η diverge, as $N \to \infty$. From (23) and Lemma 1, part iv) we find

$$\begin{aligned} \|\check{\eta}\|_{H_1}^2 &= \int_{|x|\leq \ell} e^{2N} N^2 |(\nabla w_\ell)(e^N x)|^2 d^2x = \int_{|x|\leq e^N \ell} N^2 |\nabla w_\ell(x)|^2 d^2x \\ &\leq C \int_{|x|\leq e^N \ell} \frac{1}{(|x|+1)^2} d^2x \leq CN \end{aligned}$$

for $N \in \mathbb{N}$ large enough. We denote with $\eta : \Lambda^* \to \mathbb{R}$ the Fourier transform of $\check{\eta}$, or equivalently

$$\eta_p = -N\widehat{w}_{N,\ell}(p) = -Ne^{-2N}\widehat{w}_\ell(p/e^N) . \tag{27}$$

With (22) we can bound (since $\ell = N^{-\alpha}$)

$$|\eta_p| \leq \frac{C}{|p|^2} \tag{28}$$

for all $p \in \Lambda_+^* = 2\pi\mathbb{Z}^2\backslash\{0\}$, and for some constant $C > 0$ independent of N, if N is large enough. From (26) we also have

$$\|\eta\|_\infty \leq CN^{-\alpha} . \tag{29}$$

Moreover, (18) implies the relation

$$p^2\eta_p + \frac{N}{2}(\widehat{V}(./e^N) * \widehat{f}_{N,\ell})(p) = Ne^{2N}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p) \tag{30}$$

or equivalently, expressing also the other terms through the coefficients η_p,

$$\begin{aligned} p^2\eta_p + \frac{N}{2}\widehat{V}(p/e^N) + \frac{1}{2}\sum_{q\in\Lambda^*} \widehat{V}((p-q)/e^N)\eta_q \\ = Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p) + e^{2N}\lambda_\ell \sum_{q\in\Lambda^*} \widehat{\chi}_\ell(p-q)\eta_q . \end{aligned} \tag{31}$$

We will mostly use the coefficients η_p with $p \neq 0$. Sometimes, however, it will be useful to have an estimate on η_0 (because Eq. (31) involves η_0). From (27) and Lemma 1, part iv) we find

$$|\eta_0| \leq N \int_{|x|\leq \ell} w_\ell(e^N x) d^2x \leq C \int_{|x|\leq \ell} \log(\ell/|x|) d^2x + CNe^{-N} \leq C\ell^2 . \tag{32}$$

With the coefficients (27) we define the antisymmetric operator

$$B = \frac{1}{2}\sum_{p\in\Lambda_+^*} \left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p}\right) \tag{33}$$

and we consider the unitary operator

$$e^B = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p}\right)\right]. \tag{34}$$

We refer to operators of the form (34) as generalized Bogoliubov transformations. In contrast with the standard Bogoliubov transformations

$$e^{\widetilde{B}} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p a_p^* a_{-p}^* - \bar{\eta}_p a_p a_{-p}\right)\right] \tag{35}$$

defined in terms of the standard creation and annihilation operators, operators of the form (34) leave the truncated Fock space $\mathcal{F}_+^{\leq N}$ invariant. On the other hand, while the action of standard Bogoliubov transformation on creation and annihilation operators is explicitly given by

$$e^{-\widetilde{B}} a_p e^{\widetilde{B}} = \cosh(\eta_p) a_p + \sinh(\eta_p) a_{-p}^*$$

there is no such formula describing the action of generalized Bogoliubov transformations.

Conjugation with (34) leaves the number of particles essentially invariant, as confirmed by the following lemma.

Lemma 2 *Assume B is defined as in (33), with $\eta \in \ell^2(\Lambda^*)$ and $\eta_p = \eta_{-p}$ for all $p \in \Lambda_+^*$. Then, for every $n \in \mathbb{N}$ there exists a constant $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$,*

$$e^{-B}(\mathcal{N}_+ + 1)^n e^B \leq C e^{C\|\eta\|}(\mathcal{N}_+ + 1)^n . \tag{36}$$

as an operator inequality on $\mathcal{F}_+^{\leq N}$.

The proof of (36) can be found in [6, Lemma 3.1] (a similar result has been previously established in [23]).

With the generalized Bogoliubov transformation $e^B : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, we define a new, renormalized, excitation Hamiltonian $\mathcal{G}_{N,\alpha} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by setting

$$\mathcal{G}_{N,\alpha} = e^{-B}\mathcal{L}_N e^B = e^{-B} U_N H_N U_N^* e^B . \tag{37}$$

In the next proposition, we collect important properties $\mathcal{G}_{N,\alpha}$. We will use the notation

$$\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p \quad \text{and} \quad \mathcal{V}_N = \frac{1}{2}\sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r/e^N) a_{p+r}^* a_q^* a_{q+r} a_p \tag{38}$$

for the kinetic and potential energy operators, restricted on $\mathcal{F}_+^{\leq N}$, and $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$. We also introduce a renormalized interaction potential $\omega_N \in L^\infty(\Lambda)$, which is defined as the function with Fourier coefficients $\widehat{\omega}_N$

$$\widehat{\omega}_N(p) := g_N\, \widehat{\chi}(p/N^\alpha), \qquad g_N = 2N^{1-2\alpha} e^{2N}\lambda_\ell \tag{39}$$

for any $p \in \Lambda_+^*$, and

$$\widehat{\omega}_N(0) = g_N \widehat{\chi}(0) = \pi g_N . \tag{40}$$

with $\widehat{\chi}(p)$ the Fourier coefficients of the characteristic function of the ball of radius one. From (19) and $\ell = N^{-\alpha}$ one has $|g_N| \leq C$. Note in particular that the potential $\widehat{\omega}_N(p)$

decays on momenta of order N^α, which are much smaller than e^N. From Lemma 1 parts (i) and (iii) we find

$$\left|\widehat{\omega}_N(0) - N\|Vf_\ell\|_1\right| \le \frac{C}{N}, \qquad \left|\widehat{\omega}_N(0) - 4\pi\left(1+\alpha\,\tfrac{\log N}{N}\right)\right| \le \frac{C}{N}. \tag{41}$$

Proposition 1 *Let $V \in L^3(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{G}_{N,\alpha}$ be defined as in (37) and define*

$$\begin{aligned}\mathcal{G}^{eff}_{N,\alpha} &:= \frac{1}{2}\widehat{\omega}_N(0)(N-1)\left(1-\frac{\mathcal{N}_+}{N}\right) + \left[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\right]\mathcal{N}_+\left(1-\frac{\mathcal{N}_+}{N}\right)\\ &+\frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)(b_pb_{-p}+h.c.) + \sqrt{N}\sum_{\substack{p,q\in\Lambda_+^*:\\ p+q\neq 0}}\widehat{V}(p/e^N)\left[b^*_{p+q}a^*_{-p}a_q + h.c.\right]\\ &+\mathcal{H}_N\,.\end{aligned} \tag{42}$$

Then there exists a constant $C > 0$ such that $\mathcal{E}_\mathcal{G} = \mathcal{G}_{N,\alpha} - \mathcal{G}^{eff}_{N,\alpha}$ is bounded by

$$\begin{aligned}|\langle\xi,\mathcal{E}_\mathcal{G}\,\xi\rangle| &\le C\big(N^{1/2-\alpha} + N^{-1}(\log N)^{1/2}\big)\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\\ &\quad + CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2 + C\|\xi\|^2\end{aligned} \tag{43}$$

for all $\alpha > 1$, $\xi \in \mathcal{F}_+^{\le N}$ and $N \in \mathbb{N}$ large enough.

The proof of Proposition 1 is very similar to the proof of [3, Prop. 4.2]. For completeness, we discuss the changes in Appendix A.

4 Cubic Renormalization

Conjugation with the generalized Bogoliubov transformation (35) renormalizes constant and off-diagonal quadratic terms on the r.h.s. of (42). In order to estimate the number of excitations $\mathcal{N}_+$ through the energy and show Bose–Einstein condensation, we still need to renormalize the diagonal quadratic term (the part proportional to $N\widehat{V}(0)\mathcal{N}_+$, on the first line of (42)) and the cubic term on the second line of (42). To this end, we conjugate $\mathcal{G}^{\text{eff}}_{N,\alpha}$ with an additional unitary operator, given by the exponential of the anti-symmetric operator

$$A := \frac{1}{\sqrt{N}}\sum_{r,v\in\Lambda_+^*}\eta_r\left[b^*_{r+v}a^*_{-r}a_v - \text{h.c.}\right] \tag{44}$$

with η_p defined in (27).

An important observation is that while conjugation with e^A allows to renormalize the large terms in $\mathcal{G}_{N,\alpha}$, it does not substantially change the number of excitations. The following proposition can be proved similarly to [4, Proposition 5.1].

Proposition 2 *Suppose that A is defined as in (44). Then, for any $k \in \mathbb{N}$ there exists a constant $C > 0$ such that the operator inequality*

$$e^{-A}(\mathcal{N}_++1)^k e^A \le C(\mathcal{N}_++1)^k$$

holds true on $\mathcal{F}_+^{\le N}$, for any $\alpha > 0$ (recall the choice $\ell = N^{-\alpha}$ in the definition (27) of the coefficients η_r), and N large enough.

We will also need to control the growth of the expectation of the energy $\mathcal{H}_N$ with respect to the cubic conjugation. This is the content of the following proposition, which is proved in Sect. 6.1.

Proposition 3 *Let A be defined as in (44). Then there exists a constant $C > 0$ such that*

$$e^{-sA}\mathcal{H}_N e^{sA} \leq C\mathcal{H}_N + CN(\mathcal{N}_+ + 1) \tag{45}$$

for all $\alpha \geq 1$, $s \in [0; 1]$ and $N \in \mathbb{N}$ large enough.

We use now the cubic phase e^A to introduce a new excitation Hamiltonian, obtained by conjugating the main part $\mathcal{G}^{\text{eff}}_{N,\alpha}$ of $\mathcal{G}_{N,\alpha}$. We define

$$\mathcal{R}_{N,\alpha} := e^{-A}\,\mathcal{G}^{\text{eff}}_{N,\alpha}\,e^A \tag{46}$$

on a dense subset of $\mathcal{F}^{\leq N}_+$. Conjugation with e^A renormalizes both the contribution proportional to $\mathcal{N}_+$ (in the first line on the r.h.s. of (42)) and the cubic term on the r.h.s. of (42), effectively replacing the singular potential $\widehat{V}(p/e^N)$ by the renormalized potential $\widehat{\omega}_N(p)$ defined in (39). This follows from the following proposition.

Proposition 4 *Let $V \in L^3(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{R}_{N,\alpha}$ be defined in (46) and define*

$$\begin{aligned}\mathcal{R}^{\mathit{eff}}_{N,\alpha} &= \frac{1}{2}(N-1)\,\widehat{\omega}_N(0)(1-\mathcal{N}_+/N) + \frac{1}{2}\widehat{\omega}_N(0)\,\mathcal{N}_+\,(1-\mathcal{N}_+/N)\\ &\quad + \widehat{\omega}_N(0)\sum_{p\in\Lambda^*_+} a^*_p a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \frac{1}{2}\sum_{p\in\Lambda^*_+}\widehat{\omega}_N(p)\big[b^*_p b^*_{-p} + b_p b_{-p}\big]\\ &\quad + \frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda^*_+:\\ r\neq -v}}\widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + h.c.\big] + \mathcal{H}_N\,.\end{aligned} \tag{47}$$

Then for $\ell = N^{-\alpha}$ and $\alpha > 2$ there exists a constant $C > 0$ such that $\mathcal{E}_{\mathcal{R}} = \mathcal{R}_{N,\alpha} - \mathcal{R}^{\mathit{eff}}_{N,\alpha}$ is bounded by

$$\pm\,\mathcal{E}_{\mathcal{R}} \leq C[N^{2-\alpha} + N^{-1/2}(\log N)^{1/2}](\mathcal{H}_N + 1)\,, \tag{48}$$

for $N \in \mathbb{N}$ sufficiently large.

The proof of Proposition 4 will be given in Sect. 6. We will also need more detailed information on $\mathcal{R}^{\text{eff}}_{N,\alpha}$, as contained in the following proposition.

Proposition 5 *Let $\mathcal{R}^{\mathit{eff}}_{N,\alpha}$ be defined in (47). Then, for every $c > 0$ there is a constant $C > 0$ (large enough) such that*

$$\mathcal{R}^{\mathit{eff}}_{N,\alpha} \geq 2\pi N + \frac{\widehat{\omega}_N(0)}{2}\mathcal{N}_+ + \frac{c}{\log N}\mathcal{H}_N - C(\log N)^2\,\frac{\mathcal{N}^2_+}{N} - C \tag{49}$$

for all $\alpha > 2$ and $N \in \mathbb{N}$ large enough.

Moreover, let $f, g : \mathbb{R} \to [0; 1]$ be smooth, with $f^2(x) + g^2(x) = 1$ for all $x \in \mathbb{R}$. For $M \in \mathbb{N}$, let $f_M := f(\mathcal{N}_+/M)$ and $g_M := g(\mathcal{N}_+/M)$. Then there exists $C > 0$ such that

$$\mathcal{R}^{\mathit{eff}}_{N,\alpha} = f_M\,\mathcal{R}^{\mathit{eff}}_{N,\alpha}\,f_M + g_M\,\mathcal{R}^{\mathit{eff}}_{N,\alpha}\,g_M + \Theta_M \tag{50}$$

with

$$\pm\Theta_M \leq \frac{C\log N}{M^2}\big(\|f'\|_\infty^2+\|g'\|_\infty^2\big)\big(\mathcal{H}_N+1\big)$$

for all $\alpha > 2$, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof From (47), using that $|\widehat{\omega}_N(0)| \leq C$ we have

$$\begin{aligned}\mathcal{R}^{\text{eff}}_{N,\alpha} \geq\; & \frac{N}{2}\,\widehat{\omega}_N(0) + \widehat{\omega}_N(0)\,\mathcal{N}_+ + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\big[b_p^*b_{-p}^* + b_pb_{-p}\big] \\ & + \frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda_+^*:\\ r\neq -v}}\widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] + \mathcal{H}_N - C\,\frac{\mathcal{N}_+^2}{N} - C\,.\end{aligned} \tag{51}$$

For the cubic term on the r.h.s. of (51), with

$$\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2} \leq C\log N \tag{52}$$

we can bound

$$\begin{aligned}&\Big|\frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}}\widehat{\omega}_N(r)\langle\xi, b^*_{r+v}a^*_{-r}a_v\xi\rangle\Big| \\ &\quad\leq \frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}}|\widehat{\omega}_N(r)|\|(\mathcal{N}_++1)^{-1/2}b_{r+v}a_{-r}\xi\|\|(\mathcal{N}_++1)^{1/2}a_v\xi\| \\ &\quad\leq \frac{1}{\sqrt{N}}\Big[\sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}}|r|^2\|(\mathcal{N}_++1)^{-1/2}b_{r+v}a_{-r}\xi\|^2\Big]^{1/2} \\ &\qquad\times\Big[\sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}}\frac{|\widehat{\omega}_N(r)|^2}{|r|^2}\|(\mathcal{N}_++1)^{1/2}a_v\xi\|^2\Big]^{1/2} \\ &\quad\leq \frac{C(\log N)^{1/2}}{\sqrt{N}}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)\xi\|\,.\end{aligned} \tag{53}$$

As for the off-diagonal quadratic term on the r.h.s of (51), we combine it with part of the kinetic energy to estimate. For any $0 < \mu < 1$, we have

$$\begin{aligned}&\frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\big[b_p^*b_{-p}^* + b_{-p}b_p\big] + (1-\mu)\sum_{p\in\Lambda_+^*}p^2a_p^*a_p \\ &\quad= (1-\mu)\sum_{p\in\Lambda_+^*}p^2\left[b_p^* + \frac{\widehat{\omega}_N(p)}{2(1-\mu)p^2}b_{-p}\right]\left[b_p + \frac{\widehat{\omega}_N(p)}{2(1-\mu)p^2}b_{-p}^*\right] \\ &\qquad - \frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}b_pb_p^* + (1-\mu)\sum_{p\in\Lambda_+^*}p^2a_p^*\frac{\mathcal{N}_+}{N}a_p\end{aligned} \tag{54}$$

since $a_p^* a_p - b_p^* b_p = a_p^* (\mathcal{N}_+/N) a_p$. With (14), we conclude that

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{\omega}_N(p)\big[b_p^* b_{-p}^* + b_{-p} b_p\big] + (1-\mu)\sum_{p\in\Lambda_+^*} p^2 a_p^* a_p \\ \geq -\frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2} a_p^* a_p - \frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}\,.\end{aligned}$$

With the choice $\mu = C/\log N$ and with (52), we obtain

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{\omega}_N(p)\big[b_p^* b_{-p}^* + b_{-p} b_p\big] + (1-\mu)\sum_{p\in\Lambda_+^*} p^2 a_p^* a_p \\ \geq -\frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2} a_p^* a_p - \frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2} - C\,.\end{aligned} \tag{55}$$

To bound the first terms on the r.h.s. of the last equation, we use the term $\widehat{\omega}_N(0)\mathcal{N}_+$, in (51). To this end, we observe that, with (41),

$$\frac{|\widehat{\omega}_N(p)|^2}{4(1-\mu)p^2} \leq \frac{|\widehat{\omega}_N(0)|^2}{4(1-\mu)p^2} \leq \frac{\widehat{\omega}_N(0)}{4(1-\mu)\pi}\left(1 + C\frac{\log N}{N}\right) \leq \frac{\widehat{\omega}_N(0)}{2}$$

for every $p \in \Lambda_+^*$ (notice that $|p| \geq 2\pi$, for every $p \in \Lambda_+^*$) and for N large enough (recall the choice $\mu = C/\log N$). Inserting (53) and (55) in (51) and using the kinetic energy $\mu\mathcal{K} = C(\log N)^{-1}\mathcal{K}$ (remaining after subtracting the term $(1-\mu)\mathcal{K}$ needed on the l.h.s. of (55)) to bound the r.h.s. of (53), we find

$$\begin{aligned}\mathcal{R}_{N,\alpha}^{\text{eff}} \geq \frac{N}{2}\widehat{\omega}_N(0) - \frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2} + \frac{\widehat{\omega}_N(0)}{2}\mathcal{N}_+ + \frac{c}{\log N}\mathcal{H}_N \\ - C\frac{(\log N)^2}{N}\mathcal{N}_+^2 - C.\end{aligned} \tag{56}$$

Let us now consider the second term on the r.h.s more carefully. Using that, from (39), $\widehat{\omega}_N(p) = g_N\widehat{\chi}(p/N^\alpha)$, we can bound, for any fixed $K > 0$,

$$\frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2} \leq C + \frac{1}{4}\sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}}\frac{|\widehat{\omega}_N(p)|^2}{p^2}\,.$$

With $|\widehat{\omega}_N(p) - \widehat{\omega}_N(0)| \leq C|p|/N^\alpha$, we obtain

$$\frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2} \leq C + \frac{|\widehat{\omega}_N(0)|^2}{4}\sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}}\frac{1}{p^2} \leq C + 4\pi^2\sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}}\frac{1}{p^2}\,. \tag{57}$$

For $q \in \mathbb{R}^2$, let us define $h(q) = 1/p^2$, if q is contained in the square of side length 2π centered at $p \in \Lambda_+^*$ (with an arbitrary choice on the boundary of the squares). We can then estimate, for K large enough,

$$4\pi^2\sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}}\frac{1}{p^2} \leq \int_{K/2<|q|\leq N^\alpha+K} h(q)dq\,.$$

For q in the square centered at $p \in \Lambda_+^*$, we bound

$$\left| h(q) - \frac{1}{q^2} \right| = \frac{|p^2 - q^2|}{p^2 q^2} \le \frac{C}{|q|^3} .$$

Hence

$$4\pi^2 \sum_{\substack{p \in \Lambda_+^*: \\ K < |p| \le N^\alpha}} \frac{1}{p^2} \le \int_{K/2 < |q| < N^\alpha + K} \frac{1}{q^2} dq + C \le 2\pi\alpha \log N + C .$$

Inserting in (57), we conclude that

$$\frac{1}{4} \sum_{p \in \Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2} \le 2\pi\alpha \log N + C .$$

Combining the last bound with (41) (and noticing that the contribution proportional to $\log N$ cancels exactly), from (56) we obtain

$$\mathcal{R}_{N,\alpha}^{\text{eff}} \ge 2\pi N + \frac{\widehat{\omega}_N(0)}{2} \mathcal{N}_+ + \frac{c}{\log N} \mathcal{H}_N - C \frac{(\log N)^2}{N} \mathcal{N}_+^2 - C$$

which proves (49).

Next we prove (50). From (47), with $|\widehat{\omega}_N(0)| \le C$, the bound (53) and since, by (52),

$$\begin{aligned} \left| \sum_{p \in \Lambda_+^*} \widehat{\omega}_N(p) \langle \xi, b_p^* b_{-p}^* \xi \rangle \right| &\le \sum_{p \in \Lambda_+^*} |\widehat{\omega}_N(p)| \|b_p \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \\ &\le \left[\sum_{p \in \Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2} \right]^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\| \\ &\le C(\log N)^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\| \end{aligned}$$

it follows that

$$\mathcal{R}_{N,\alpha}^{\text{eff}} = 2\pi N + \mathcal{H}_N + \theta_{N,\alpha} \tag{58}$$

where for arbitrary $\delta > 0$, there exists a constant $C > 0$ such that

$$\pm \theta_{N,\alpha} \le \delta \mathcal{H}_N + C(\log N)\,(\mathcal{N}_+ + 1) . \tag{59}$$

We now note that for $f : \mathbb{R} \to \mathbb{R}$ smooth and bounded and $\theta_{N,\alpha}$ defined above, there exists a constant $C > 0$ such that

$$\pm [f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), \theta_{N,\alpha}]] \le C \frac{\log N}{M^2} \|f'\|_\infty^2 (\mathcal{H}_N + 1) \tag{60}$$

for all $\alpha > 2$ and $N \in \mathbb{N}$ large enough. The proof of (60) follows analogously to the one for (59), since the bounds leading to (59) remain true if we replace the operators $b_p^\#$, $\# = \{\cdot, *\}$, and $a_p^* a_q$ with $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), b_p^\#]]$ or $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), a_p^* a_q]]$ respectively, provided we multiply the r.h.s. by an additional factor $M^{-2} \|f'\|_\infty^2$, since, for example

$$[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), b_p]] = \big(f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 1)/M)\big)^2 b_p$$

and $\|f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 1)/M)\| \le CM^{-1}\|f'\|_\infty$. With an explicit computation we obtain

$$\mathcal{R}^{\text{eff}}_{N,\alpha} = f_M \mathcal{R}^{\text{eff}}_{N,\alpha} f_M + g_M \mathcal{R}^{\text{eff}}_{N,\alpha} g_M + \frac{1}{2}\Big([f_M, [f_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]] + [g_M, [g_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]]\Big)$$

Writing $\mathcal{R}^{\text{eff}}_{N,\alpha}$ as in (58) and using (60) we get

$$\pm\Big([f_M, [f_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]] + [g_M, [g_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]]\Big) \le \frac{C \log N}{M^2}\big(\|f'\|^2_\infty + \|g'\|^2_\infty\big)\big(\mathcal{H}_N + 1\big)\,.$$

□

5 Proof of Theorem 1

The next proposition combines the results of Propositions 1, 4 and 5. Its proof makes use of localization in the number of particle and is an adaptation of the proof of [4, Proposition 6.1]. The main difference w.r.t. [4] is that here we need to localize on sectors of $\mathcal{F}^{\le N}$ where the number of particles is $o(N)$, in the limit $N \to \infty$.

Proposition 6 *Let $V \in L^3(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{G}_{N,\alpha}$ be the renormalized excitation Hamiltonian defined as in (37). Then, for every $\alpha \ge 5/2$, there exist constants $C, c > 0$ such that*

$$\mathcal{G}_{N,\alpha} - 2\pi N \ge c\,\mathcal{N}_+ - C \tag{61}$$

for all $N \in \mathbb{N}$ sufficiently large.

Proof Let $f, g : \mathbb{R} \to [0; 1]$ be smooth, with $f^2(x) + g^2(x) = 1$ for all $x \in \mathbb{R}$. Moreover, assume that $f(x) = 0$ for $x > 1$ and $f(x) = 1$ for $x < 1/2$. For a small $\varepsilon > 0$, we fix $M = N^{1-\varepsilon}$ and we set $f_M = f(\mathcal{N}_+/M)$, $g_M = g(\mathcal{N}_+/M)$. It follows from Proposition 5 that

$$\begin{aligned}\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \ge\ & f_M\big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big) f_M + g_M\big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big) g_M \\ & - CN^{2\varepsilon-2}(\log N)(\mathcal{H}_N + 1)\end{aligned} \tag{62}$$

Let us consider the first term on the r.h.s. of (62). From Proposition 5, for all $\alpha > 2$ there exist $c, C > 0$ such that

$$\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \ge c\,\mathcal{N}_+ - \frac{C}{N}(\log N)^2\,\mathcal{N}_+^2 - C\,. \tag{63}$$

On the other hand, with (58) and (59) we also find

$$\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \ge c\mathcal{H}_N - C(\log N)\,(\mathcal{N}_+ + 1) \tag{64}$$

for all $\alpha > 2$ and N large enough. Moreover, due to the choice $M = N^{1-\varepsilon}$, we have

$$\frac{(\log N)^2}{N} f_M \mathcal{N}_+^2 f_M \le \frac{(\log N)^2}{N^\varepsilon} f_M^2 \mathcal{N}_+\,.$$

With the last bound, Eq. (63) implies that

$$f_M\Big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\Big) f_M \ge c f_M^2 \mathcal{N}_+ - C \tag{65}$$

for N large enough.

Let us next consider the second term on the r.h.s. of (62). We claim that there exists a constant $c > 0$ such that

$$g_M\Big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\Big)g_M \geq cNg_M^2 \tag{66}$$

for all N sufficiently large. To prove (66) we observe that, since $g(x) = 0$ for all $x \leq 1/2$,

$$g_M\Big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\Big)g_M \geq \left[\inf_{\xi\in\mathcal{F}^{\leq N}_{\geq M/2}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{R}^{\text{eff}}_{N,\alpha}\xi\rangle - 2\pi\right] Ng_M^2$$

where $\mathcal{F}^{\leq N}_{\geq M/2} = \{\xi \in \mathcal{F}^{\leq N}_+ : \xi = \chi(\mathcal{N}_+ \geq M/2)\xi\}$ is the subspace of $\mathcal{F}^{\leq N}_+$ where states with at least $M/2$ excitations are described (recall that $M = N^{1-\varepsilon}$). To prove (66) it is enough to show that there exists $C > 0$ with

$$\inf_{\xi\in\mathcal{F}^{\leq N}_{\geq M/2}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{R}^{\text{eff}}_{N,\alpha}\xi\rangle - 2\pi \geq C \tag{67}$$

for all N large enough. On the other hand, using the definitions of $\mathcal{G}_{N,\alpha}$ in (42), $\mathcal{R}_{N,\alpha}$ and $\mathcal{R}^{\text{eff}}_{N,\alpha}$ in (47), we obtain that the ground state energy E_N of the system is given by

$$E_N = \inf_{\xi\in\mathcal{F}^{\leq N}_+:\|\xi\|=1} \big\langle\xi, e^{-A}\mathcal{G}_{N,\alpha}e^{A}\xi\big\rangle = \inf_{\xi\in\mathcal{F}^{\leq N}_+:\|\xi\|=1} \big\langle\xi, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} + \mathcal{E}_L\big)\xi\big\rangle$$

with $\mathcal{E}_L = \mathcal{E}_\mathcal{R} + e^{-A}\mathcal{E}_\mathcal{G}e^{A}$. The bounds (43) and (48), together with Propositions 2 and 3, imply that for any $\alpha \geq 5/2$ there exists $C > 0$ such that

$$\begin{aligned}\pm\mathcal{E}_L &\leq CN^{-1/2}(\log N)^{1/2}\big[(\mathcal{H}_N + 1) + e^{-A}\big(N^{-1}(\mathcal{H}_N + 1) + (\mathcal{N}_+ + 1)\big)e^{A}\big] + C \\ &\leq CN^{-1/2}(\log N)^{1/2}(\mathcal{H}_N + 1) + C\end{aligned}$$

With (64) we obtain

$$\pm\,\mathcal{E}_L \leq CN^{-1/2}(\log N)^{1/2}\big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big) + CN^{-1/2}(\log N)^{3/2}\mathcal{N}_+ + C\,, \tag{68}$$

and therefore, with $\mathcal{N}_+ \leq N$

$$E_N - 2\pi N \leq C \inf_{\xi\in\mathcal{F}^{\leq N}_+:\|\xi\|=1} \big\langle\xi, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big)\xi\big\rangle + CN^{1/2}(\log N)^{3/2} + C\,.$$

From the result (3) of [13,14,16]

$$\begin{aligned}\inf_{\xi\in\mathcal{F}^{\leq N}_{\geq M/2}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{R}^{\text{eff}}_{N,\alpha}\xi\rangle - 2\pi &\geq \inf_{\xi\in\mathcal{F}^{\leq N}_+:\|\xi\|=1} \frac{1}{N}\langle\xi, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big)\xi\rangle \\ &\geq c\left(\frac{E_N}{N} - 2\pi\right) - \frac{C}{\sqrt{N}}(\log N)^{3/2} - CN^{-1} \to 0\end{aligned}$$

as $N \to \infty$. If we assume by contradiction that (67) does not hold true, then we can find a subsequence $N_j \to \infty$ with

$$\inf_{\xi\in\mathcal{F}^{\leq N_j}_{\geq M_j/2}:\|\xi\|=1} \frac{1}{N_j}\langle\xi, \mathcal{R}^{\text{eff}}_{N_j,\alpha}\xi\rangle - 2\pi \to 0$$

as $j \to \infty$ (here we used the notation $M_j = N_j^{1-\varepsilon}$). This implies that there exists a sequence $\tilde{\xi}_{N_j} \in \mathcal{F}_{\geq M_j/2}^{\leq N_j}$ with $\|\tilde{\xi}_{N_j}\| = 1$ for all $j \in \mathbb{N}$ such that

$$\lim_{j\to\infty} \frac{1}{N_j} \langle \tilde{\xi}_{N_j}, \mathcal{R}^{\text{eff}}_{N_j,\alpha} \tilde{\xi}_{N_j} \rangle = 2\pi \, .$$

On the other hand, using the relation $\mathcal{R}^{\text{eff}}_{N_j,\alpha} = e^{-A} \mathcal{G}_{N_j,\alpha} e^{A} - \mathcal{E}_{L,j}$ with $\mathcal{E}_{L,j}$ satisfying the bound (68) (with $\mathcal{N}_+ \leq N_j$), we obtain that there exist constants $c_1, c_2, C > 0$ such that

$$\begin{aligned} c_1 \langle \tilde{\xi}_{N_j}, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N_j\big) \tilde{\xi}_{N_j} \rangle - C N_j^{1/2} (\log N_j)^{3/2} \\ \leq \langle e^{A} \tilde{\xi}_{N_j}, \big(\mathcal{G}_{N_j,\alpha} - \; 2\pi N_j\big) e^{A} \tilde{\xi}_{N_j} \rangle \\ \leq c_2 \langle \tilde{\xi}_{N_j}, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N_j\big) \tilde{\xi}_{N_j} \rangle + C N_j^{1/2} (\log N_j)^{3/2} \end{aligned}$$

Hence for $\xi_{N_j} = e^{A} \tilde{\xi}_{N_j}$ we have

$$\lim_{N_j\to\infty} \frac{1}{N_j} \langle \xi_{N_j}, \mathcal{G}_{N_j,\alpha} \xi_{N_j} \rangle = 2\pi \, .$$

Let now $S := \{N_j : j \in \mathbb{N}\} \subset \mathbb{N}$ and denote by ξ_N a normalized minimizer of $\mathcal{G}_{N,\alpha}$ for all $N \in \mathbb{N} \setminus S$. Setting $\psi_N = U_N^* e^{B} \xi_N$, for all $N \in \mathbb{N}$, we obtain that $\|\psi_N\| = 1$ and that

$$\lim_{N\to\infty} \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle = \lim_{N\to\infty} \frac{1}{N} \langle \xi_N, \mathcal{G}_{N,\alpha} \xi_N \rangle = 2\pi \tag{69}$$

Eq. (69) shows that the sequence ψ_N is an approximate ground state of H_N. From (5), we conclude that ψ_N exhibits complete Bose–Einstein condensation in the zero-momentum mode φ_0, and in particular that there exists $\bar{\delta} > 0$ such that

$$|1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle| \leq C N^{-\bar{\delta}} \, .$$

Using Lemma 2, Proposition 2 and the rules (11), we observe that

$$\begin{aligned} \frac{1}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle &= \frac{1}{N} \langle e^{-B} U_N \psi_N, \mathcal{N}_+ e^{-B} U_N \psi_N \rangle \\ &\leq \frac{C}{N} \langle \psi_N, U_N^* (\mathcal{N}_+ + 1) U_N \psi_N \rangle \\ &= \frac{C}{N} + C \left[1 - \frac{1}{N} \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \right] \\ &= \frac{C}{N} + C \left[1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \right] \leq C N^{-\bar{\delta}} \end{aligned} \tag{70}$$

as $N \to \infty$.

On the other hand, for $N \in S = \{N_j : j \in \mathbb{N}\}$, we have $\xi_N = \chi(\mathcal{N}_+ \geq M/2) \xi_N$ and therefore

$$\frac{1}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle \geq \frac{M}{2N} = \frac{N^{-\varepsilon}}{2} \, .$$

Choosing $\varepsilon < \bar{\delta}$ and N large enough we get a contradiction with (70). This proves (67), (66) and therefore also

$$g_M \Big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \Big) g_M \geq c \mathcal{N}_+ g_M^2 \, . \tag{71}$$

Inserting (65) and (71) on the r.h.s. of (62), we obtain that

$$\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \geq c\mathcal{N}_+ - C(\log N)N^{2\varepsilon-2}(\mathcal{H}_N + 1) - C \tag{72}$$

for N large enough. With (64), (72) implies

$$\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \geq c\mathcal{N}_+ - C.$$

To conclude, we use the relation $e^{-A}\mathcal{G}_{N,\alpha}e^{A} = \mathcal{R}^{\text{eff}}_{N,\alpha} + \mathcal{E}_L$ and the bound (68). We have that for $\alpha \geq 5/2$ there exist $c, C > 0$ such that

$$\begin{aligned}\mathcal{G}_{N,\alpha} - 2\pi N &\geq ce^{A}\big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big)e^{-A} - CN^{-1/2}(\log N)^{3/2}e^{A}\mathcal{N}_+e^{A} - C \\ &\geq c\, e^{A}\mathcal{N}_+e^{-A} - C \geq c\mathcal{N}_+ - C\end{aligned}$$

where we used (72) and Proposition 2. □

We are now ready to show our main theorem.

Proof of Theorem 1 Let E_N be the ground state energy of H_N. Evaluating (42) and (43) on the vacuum $\Omega \in \mathcal{F}_+^{\leq N}$ and using (40), we obtain the upper bound

$$E_N \leq 2\pi N + C\log N\,.$$

Notice that we cannot reach the expected optimal upper bound $E_N \leq 2\pi N + C$ because of the logarithmic correction in $\hat{\omega}_N(0)$ (see (40)). In the lower bound, this logarithmic factor is compensated by the contribution arising from the off-diagonal quadratic term, extracted starting from (54). To obtain the same term for the upper bound, we would have to modify our trial state (diagonalizing the quadratic terms in $\mathcal{R}_{N,\alpha}$); this, however, would produce even larger contributions arising from the potential energy.

With Eq. (61) we also find the lower bound $E_N \geq 2\pi N - C$. This proves (6).

Let now $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\langle \psi_N, H_N\psi_N\rangle \leq 2\pi N + K\,. \tag{73}$$

We define the excitation vector $\xi_N = e^{-B}U_N\psi_N$. Then $\|\xi_N\| = 1$ and, recalling that $\mathcal{G}_{N,\alpha} = e^{-B}U_N H_N U_N^* e^{B}$ we have, with (61),

$$\big\langle\psi_N, (H_N - 2\pi N)\psi_N\big\rangle = \big\langle\xi_N, (\mathcal{G}_{N,\alpha} - 2\pi N)\xi_N\big\rangle \geq c\big\langle\xi_N, \mathcal{N}_+\xi_N\big\rangle - C. \tag{74}$$

From Eqs. (73) and (74) we conclude that

$$\langle\xi_N, \mathcal{N}_+\xi_N\rangle \leq C(1+K)\,.$$

If γ_N denotes the one-particle reduced density matrix associated with ψ_N, using Lemma 2 we obtain

$$\begin{aligned}1 - \langle\varphi_0, \gamma_N\varphi_0\rangle &= 1 - \frac{1}{N}\langle\psi_N, a^*(\varphi_0)a(\varphi_0)\psi_N\rangle \\ &= 1 - \frac{1}{N}\langle U_N^*e^{B}\xi_N, a^*(\varphi_0)a(\varphi_0)U_N^*e^{B}\xi_N\rangle \\ &= \frac{1}{N}\langle e^{B}\xi_N, \mathcal{N}_+e^{B}\xi_N\rangle \leq \frac{C}{N}\langle\xi_N, \mathcal{N}_+\xi_N\rangle \leq \frac{C(1+K)}{N}\end{aligned}$$

which concludes the proof of (8). □

6 Analysis of the Excitation Hamiltonian $\mathcal{R}_N$

In this section, we show Proposition 4, where we establish a lower bound for the operator $\mathcal{R}_{N,\alpha} = e^{-A}\mathcal{G}^{\text{eff}}_{N,\alpha}e^{A}$, with $\mathcal{G}^{\text{eff}}_{N,\alpha}$ as defined in (42) and with

$$A = \frac{1}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} \eta_r \big[b^*_{r+v}a^*_{-r}a_v - \text{h.c.}\big]. \tag{75}$$

We decompose

$$\mathcal{G}^{\text{eff}}_{N,\alpha} = \mathcal{O}_N + \mathcal{K} + \mathcal{Z}_N + \mathcal{C}_N + \mathcal{V}_N \tag{76}$$

with $\mathcal{K}$ and $\mathcal{V}_N$ as in (38), and with

$$\begin{aligned}
\mathcal{O}_N &= \frac{1}{2}\widehat{\omega}_N(0)(N-1)\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \big[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+\Big(1-\frac{\mathcal{N}_+}{N}\Big),\\
\mathcal{Z}_N &= \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)(b_pb_{-p} + \text{h.c.})\\
\mathcal{C}_N &= \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\Big[b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\Big].
\end{aligned} \tag{77}$$

We will analyze the conjugation of all terms on the r.h.s. of (76) in Sects. 6.2–6.6. The estimates emerging from these subsections will then be combined in Sect. 6.6 to conclude the proof of Proposition 4. Throughout the section, we will need Proposition 3 to control the growth of the expectation of the energy $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$ under the action of (75); the proof of Proposition 3 is contained in Sect. 6.1.

In this section, we will always assume that $V \in L^3(\mathbb{R}^2)$ is compactly supported, pointwise non-negative and spherically symmetric.

6.1 A Priori Bounds on the Energy

In this section, we show Proposition 3. To this end, we will need the following proposition.

Proposition 7 *Let $\mathcal{V}_N$ and A be defined in (38) and (44) respectively. Then, there exists a constant $C > 0$ such that*

$$[\,\mathcal{V}_N, A] = \frac{1}{N^{1/2}}\sum_{\substack{u,r,v\in\Lambda_+^*\\ u\neq -v}} \widehat{V}((u-r)/e^N)\eta_r\big[b^*_{u+v}a^*_{-u}a_v + h.c.\big] + \delta_{\mathcal{V}_N}$$

where

$$|\langle\xi, \delta_{\mathcal{V}_N}\xi\rangle| \leq C(\log N)^{1/2}N^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|^2 \tag{78}$$

for any $\alpha > 0$, for all $\xi \in \mathcal{F}_+^{\leq N}$, and $N \in \mathbb{N}$ large enough.

Proof We proceed as in [4, Prop. 8.1], computing $[a^*_{p+u}a^*_qa_pa_{q+u}, b^*_{r+v}a^*_{-r}a_v]$. We obtain

$$[\mathcal{V}_N, A] = \frac{1}{N^{1/2}}\sum^{*}_{u\in\Lambda^*,r,v\in\Lambda_+^*}\widehat{V}((u-r)/e^N)\eta_r b^*_{u+v}a^*_{-u}a_v + \Theta_1 + \Theta_2 + \Theta_3 + \text{h.c.}$$

with

$$\begin{aligned}\Theta_1 &:= \frac{1}{\sqrt{N}} \sum^*_{\substack{u\in\Lambda^*\\ r,p,v\in\Lambda^*_+}} \widehat{V}(u/e^N)\eta_r b^*_{p+u} a^*_{r+v-u} a^*_{-r} a_p a_v \,, \\ \Theta_2 &:= \frac{1}{\sqrt{N}} \sum^*_{\substack{u\in\Lambda^*\\ p,r,v\in\Lambda^*_+}} \widehat{V}(u/e^N)\eta_r b^*_{r+v} a^*_{p+u} a^*_{-r-u} a_p a_v \,, \\ \Theta_3 &:= -\frac{1}{\sqrt{N}} \sum^*_{u\in\Lambda^*,\, p,r,v\in\Lambda^*_+} \widehat{V}(u/e^N)\eta_r b^*_{r+v} a^*_{-r} a^*_{p+u} a_p a_{v+u} \,.\end{aligned} \tag{79}$$

and with $\sum^*$ running over all momenta, except choices for which the argument of a creation or annihilation operator vanishes. We conclude that $\delta_{\mathcal{V}_N} = \Theta_1 + \Theta_2 + \Theta_3 + \text{h.c.}$. Next, we show that each error term Θ_j, with $j = 1, 2, 3$, satisfies (78). To bound Θ_1 we switch to position space and apply Cauchy–Schwarz. We find

$$\begin{aligned}|\langle \xi, \Theta_1 \xi\rangle| &\le \frac{1}{\sqrt{N}} \int_{\Lambda^2} dxdy\; e^{2N} V(e^N(x-y)) \|\check{a}(\check{\eta}_y)\check{a}_y\check{a}_x\xi\| \|\check{a}_y\check{a}_x\xi\| \\ &\le C\|\eta\| \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|\check{a}_y\check{a}_x\xi\|^2 \\ &\le C N^{-\alpha} \|\mathcal{V}_N^{1/2}\xi\|^2 \,,\end{aligned}$$

for any $\xi \in \mathcal{F}_+^{\le N}$ The term Θ_3 can be controlled similarly. We find

$$\begin{aligned}|\langle \xi, \Theta_3 \xi\rangle| &= \left| \frac{1}{\sqrt{N}} \int_{\Lambda^2} dxdy\; e^{2N} V(e^N(x-y)) \langle \xi, \check{b}^*_x \check{a}^*(\check{\eta}_x) \check{a}^*_y \check{a}_x \check{a}_y \xi\rangle \right| \\ &\le C N^{-\alpha} \|\mathcal{V}_N^{1/2}\xi\|^2 \,.\end{aligned}$$

It remains to bound the term Θ_2 on the r.h.s. of (79). Passing to position space we obtain, by Cauchy–Schwarz,

$$\begin{aligned}|\langle \xi, \Theta_2 \xi\rangle| &= \left| \frac{1}{\sqrt{N}} \int_{\Lambda^3} dxdydz\; e^{2N} V(e^N(y-z)) \check{\eta}(x-z) \langle \xi, \check{b}^*_x \check{a}^*_y \check{a}^*_z \check{a}_x \check{a}_y \xi\rangle \right| \\ &\le C N^{-1/2} \int_{\Lambda^3} dxdydz\; e^{2N} V(e^N(y-z)) |\check{\eta}(x-z)| \|\check{a}_x\check{a}_y\check{a}_z\xi\| \|\check{a}_x\check{a}_y\xi\| \\ &\le C N^{-1/2} \|\mathcal{V}_N^{1/2}\mathcal{N}_+^{1/2}\xi\| \left[\int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(y-z)) |\check{\eta}(x-z)|^2 \|\check{a}_x\check{a}_y\xi\|^2 \right]^{1/2} ,\end{aligned}$$

To bound the term in the square bracket, we write it in first quantized form and, for any $2 < q < \infty$, we apply Hölder inequality and the Sobolev inequality $\|u\|_q \le C\sqrt{q}\, \|u\|_{H^1}$ to estimate (denoting by $1 < q' < 2$ the dual index to q),

$$\begin{aligned}\sum_{n=2}^{N}\sum_{i<j}^{n}\int\left[e^{2N}V(e^N\cdot)*|\check{\eta}|^2\right](x_i-x_j)\,|\xi^{(n)}(x_1,\dots,x_n)|^2dx_1\dots dx_n\\ \le Cq\|e^{2N}V(e^N\cdot)*|\check{\eta}|^2\|_{q'}\\ \times\sum_{n=2}^{N}n\sum_{i=1}^{n}\int\left[|\nabla_{x_i}\xi^{(n)}(x_1,\dots,x_n)|^2+|\xi^{(n)}(x_1,\dots,x_n)|^2\right]dx_1\dots dx_n\\ \le Cq\|\check{\eta}\|_{2q'}^2\|(\mathcal{K}+\mathcal{N}_+)^{1/2}\mathcal{N}_+^{1/2}\xi\|^2.\end{aligned} \tag{80}$$

With the bounds (25), (26),

$$\|\check{\eta}\|_{2q'}^2\le\|\check{\eta}\|_2^{2/q'}\|\check{\eta}\|_\infty^{2(q'-1)/q'}\le N^{-2\alpha/q'}N^{2(q'-1)/q'}$$

we conclude that

$$\begin{aligned}|\langle\xi,\Theta_2\xi\rangle|&\le Cq^{1/2}N^{-1/2}N^{-\alpha/q'}N^{1/q}\|\mathcal{V}_N^{1/2}\mathcal{N}_+^{1/2}\xi\|\|(\mathcal{K}+\mathcal{N}_+)^{1/2}\mathcal{N}_+^{1/2}\xi\|\\ &\le Cq^{1/2}N^{1/2}N^{-\alpha/q'}N^{1/q}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\end{aligned}$$

for any $2 < q < \infty$, if $1/q + 1/q' = 1$. Choosing $q = \log N$, we obtain that

$$|\langle\xi,\Theta_2\xi\rangle|\le C(\log N)^{1/2}N^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|^2.$$

□

Using Proposition 7, we can now show Proposition 3.

Proof of Proposition 3 The proof follows a strategy similar to [4, Lemma 8.2]. For fixed $\xi\in\mathcal{F}_+^{\le N}$ and $s\in[0;1]$, we define

$$f_\xi(s):=\langle\xi,e^{-sA}\mathcal{H}_Ne^{sA}\xi\rangle\,.$$

We compute

$$f'_\xi(s)=\langle\xi,e^{-sA}[\mathcal{K},A]e^{sA}\xi\rangle+\langle\xi,e^{-sA}[\mathcal{V}_N,A]e^{sA}\xi\rangle\,. \tag{81}$$

With Proposition 7, we have

$$[\mathcal{V}_N,A]=\frac{1}{\sqrt{N}}\sum_{u,v\in\Lambda_+^*,u\ne -v}(\widehat{V}(\cdot/e^N)*\eta)(u)\left[b^*_{u+v}a^*_{-u}a_v+\text{h.c.}\right]+\delta_{\mathcal{V}_N}$$

with $\delta_{\mathcal{V}_N}$ satisfying (78). Switching to position space and using Proposition 2 we find , using (25) to bound $\|\check{\eta}\|_\infty\le CN$,

$$\begin{aligned}&\left|\frac{1}{\sqrt{N}}\sum_{u,v\in\Lambda_+^*}(\widehat{V}(\cdot/e^N)*\eta)(u)\langle\xi,e^{-sA}b^*_{u+v}a^*_{-u}a_ve^{sA}\xi\rangle\right|\\ &=\left|\frac{1}{\sqrt{N}}\int_{\Lambda^2}dxdy\;e^{2N}V(e^N(x-y))\check{\eta}(x-y)\langle\xi,e^{-sA}\check{a}^*_x\check{a}^*_y\check{a}_ye^{sA}\xi\rangle\right|\\ &\le N^{1/2}\left[\int_{\Lambda^2}dxdy\;e^{2N}V(e^N(x-y))\|\check{a}_x\check{a}_ye^{sA}\xi\|^2\right]^{1/2}\\ &\quad\times\left[\int_{\Lambda^2}dxdy\;e^{2N}V(e^N(x-y))\|\check{a}_ye^{sA}\xi\|^2\right]^{1/2}\\ &\le CN^{1/2}\|\mathcal{V}_N^{1/2}e^{sA}\xi\|\|\mathcal{N}_+^{1/2}e^{sA}\xi\|\end{aligned} \tag{82}$$

Together with (78) we conclude that for any $\alpha > 1/2$

$$\Big|\langle \xi, e^{-sA}[\mathcal{V}_N, A]e^{sA}\xi\rangle\Big| \leq C\langle \xi, e^{-sA}\mathcal{H}_N e^{sA}\xi\rangle + CN\langle \xi, e^{-sA}(\mathcal{N}_+ + 1)e^{sA}\xi\rangle \tag{83}$$

if N is large enough. Next, we analyze the first term on the r.h.s. of (81). We compute

$$\begin{aligned} [\mathcal{K}, A] &= \frac{1}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} 2r^2\eta_r\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\ &\quad + \frac{2}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} r\cdot v\ \eta_r\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\ &=: \mathrm{T}_1 + \mathrm{T}_2\,. \end{aligned} \tag{84}$$

With (31), we write

$$\begin{aligned} \mathrm{T}_1 = &-\sqrt{N} \sum_{\substack{r,v\in\Lambda_+^* \\ r\neq -v}} (\widehat{V}(\cdot/e^N) * \widehat{f}_{N,\ell})(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\ &+ 2\sqrt{N} \sum_{r,v\in\Lambda_+^*} e^{2N}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\ =&: \mathrm{T}_{11} + \mathrm{T}_{12}\,. \end{aligned} \tag{85}$$

The contribution of T_{11} can be estimated similarly as in (82); switching to position space and using (20), we obtain

$$\begin{aligned} \big|\langle \xi_1, \mathrm{T}_{11}\,\xi_2\rangle\big| &\leq C\sqrt{N}\int dxdy e^{2N}V(e^N(x-y))f_\ell(e^N(x-y))\|\check{a}_x\check{a}_y\xi\|\|a_y\xi\| \\ &\leq C\sqrt{N}\Big[\int dxdy e^{2N}V(e^N(x-y))\|\check{a}_x\check{a}_y\xi\|^2\Big]^{1/2} \\ &\quad\times\Big[\int dxdy e^{2N}V(e^N(x-y))f_\ell(e^N(x-y))\|a_y\xi\|^2\Big]^{1/2} \\ &\leq C\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\,. \end{aligned} \tag{86}$$

for any $\xi \in \mathcal{F}_+^{\leq N}$. The second term in (85) can be controlled using that for any $\xi \in \mathcal{F}_+^{\leq N}$ and $2 \leq q < \infty$ we have

$$\begin{aligned} &N^{2\alpha}\int_{\Lambda^2} dxdy\,\chi(|x-y|\leq N^{-\alpha})\|\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\| \\ &\leq N^{2\alpha}\int_{\Lambda^2} dx\|\check{a}_x\xi\|\Big(\int dy\,\chi(|x-y|\leq N^{-\alpha})\Big)^{1-1/q}\Big(\int dy\|\check{a}_x\check{a}_y\xi\|^q\Big)^{1/q} \\ &\leq CN^{2\alpha/q}q^{1/2}\Big[\int dx\|\check{a}_x\xi\|^2\Big]^{1/2}\Big[\int dxdy\|\check{a}_x\nabla_y\check{a}_y\xi\|^2 + \int dxdy\|\check{a}_x\check{a}_y\xi\|^2\Big]^{1/2} \\ &\leq CN^{2\alpha/q}q^{1/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\left[\|\mathcal{K}^{1/2}(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|(\mathcal{N}_+ + 1)\xi\|\right]. \end{aligned} \tag{87}$$

Hence, choosing $q = \log N$,

$$\begin{aligned} &\left|\langle \xi, \mathrm{T}_{12}\xi\rangle\right| \\ &= \left|\sqrt{N}e^{2N}\lambda_\ell \int_{\Lambda^2} dxdy\, \chi(|x-y| \le N^{-\alpha}) f_{N,\ell}(x-y)\langle \xi, \check{b}_x^* \check{a}_y^* \check{a}_x \xi\rangle\right| \\ &\le CN^{2\alpha-1/2} \int_{\Lambda^2} dxdy\, \chi(|x-y|\le N^{-\alpha}) \|\check{a}_x \check{a}_y \xi\| \|\check{a}_x \xi\| \\ &\le C(\log N)^{1/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \left[\|\mathcal{K}^{1/2}\xi\| + \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\right], \end{aligned} \tag{88}$$

With (86) and (88) we conclude that

$$|\langle \xi, e^{-A}\mathrm{T}_1 e^{A}\xi\rangle| \le C(\log N)^{1/2} \|(\mathcal{H}_N + 1)^{1/2} e^{sA}\xi\| \|(\mathcal{N}_+ + 1)^{1/2} e^{sA}\xi\|. \tag{89}$$

for all $\xi \in \mathcal{F}_+^{\le N}$. As for the second term on the r.h.s. of (84) we have

$$\begin{aligned} &\left|\langle \xi, \mathrm{T}_2\xi\rangle\right| \\ &\le \frac{C}{\sqrt{N}} \Big[\sum_{r\in\Lambda_+^*} |r|^2 \|\mathcal{N}_+^{1/2} a_{-r}\xi\|^2\Big]^{1/2} \Big[\sum_{r,v\in\Lambda_+^*} |v|^2 \eta_r^2 \|a_v \xi\|^2\Big]^{1/2} \\ &\le CN^{-\alpha} \|\mathcal{K}^{1/2}\xi\|^2. \end{aligned} \tag{90}$$

for any $\xi \in \mathcal{F}_+^{\le N}$. With (89) and Proposition 2, we conclude that

$$|\langle \xi, e^{-sA}[\mathcal{K}, A] e^{sA}\xi\rangle| \le C\langle \xi, e^{-sA}\mathcal{H}_N e^{sA}\xi\rangle + C\log N \langle \xi, e^{-sA}\mathcal{N}_+ e^{sA}\xi\rangle.$$

Combining with Eq. (83) we obtain

$$|\langle \xi, e^{-sA}[\mathcal{H}_N, A] e^{sA}\xi\rangle| \le C\langle \xi, e^{-sA}\mathcal{H}_N e^{sA}\xi\rangle + CN \langle \xi, e^{-sA}\mathcal{N}_+ e^{sA}\xi\rangle.$$

With Proposition 2 we obtain the differential inequality

$$|f_\xi'(s)| \le Cf_\xi(s) + CN\langle \xi, (\mathcal{N}_+ + 1)\xi\rangle.$$

By Gronwall's Lemma, we find (45). □

6.2 Analysis of $e^{-A}\mathcal{O}_N e^{A}$

In this section we study the contribution to $\mathcal{R}_{N,\alpha}$ arising from the operator $\mathcal{O}_N$, defined in (77). To this end, it is convenient to use the following lemma.

Lemma 3 *Let A be defined in (44). Then, there exists a constant $C > 0$ such that*

$$\sum_{p\in\Lambda_+^*} F_p\, e^{-A} a_p^* a_p e^{A} = \sum_{p\in\Lambda_+^*} F_p\, a_p^* a_p + \mathcal{E}_F$$

where

$$|\langle \xi_1, \mathcal{E}_F \xi_2\rangle| \le CN^{-\alpha} \|F\|_\infty \|(\mathcal{N}_+ + 1)^{1/2}\xi_1\| \|(\mathcal{N}_+ + 1)^{1/2}\xi_2\|$$

for all $\alpha > 0$, $\xi_1, \xi_2 \in \mathcal{F}_+^{\le N}$, $F \in \ell^\infty(\Lambda_+^*)$, and $N \in \mathbb{N}$ large enough.

Proof The lemma is analogous to [4, Lemma 8.6]. We estimate

$$\begin{aligned}
&\Big| \sum_{p\in\Lambda_+^*} F_p(\langle\xi_1, e^{-A}a_p^*a_pe^A\xi_2\rangle - \langle\xi_1, a_p^*a_p\xi_2\rangle)\Big| \\
&\qquad = \Big| \int_0^1 ds \sum_{p\in\Lambda_+^*} F_p\langle\xi_1, e^{-sA}[a_p^*a_p, A]e^{sA}\xi_2\rangle\Big| \\
&\qquad \le \frac{1}{\sqrt{N}}\int_0^1 ds \sum_{r,v\in\Lambda_+^*} |F_{r+v}+F_{-r}-F_v||\eta_r||\langle e^{sA}\xi_1, b_{r+v}^*a_{-r}^*a_ve^{sA}\xi_2\rangle| \\
&\qquad \le C\|\eta\|\|F\|_\infty\|(\mathcal{N}_++1)^{1/2}\xi_1\|\|(\mathcal{N}_++1)^{1/2}\xi_2\|\,.
\end{aligned}$$

where we used Proposition 2. □

We consider now the action of e^A on the operator $\mathcal{O}_N$, as defined in (77).

Proposition 8 *Let A be defined in (44). Then there exists a constant $C > 0$ such that*

$$\begin{aligned}
e^{-A}\mathcal{O}_Ne^A &= \frac{1}{2}\widehat{\omega}_N(0)(N-1)\left(1-\frac{\mathcal{N}_+}{N}\right) + \big[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+(1-\mathcal{N}_+/N) \\
&\quad + \delta_{\mathcal{O}_N}
\end{aligned}$$

where

$$\pm\delta_{\mathcal{O}_N} \le CN^{1-\alpha}(\mathcal{N}_++1)$$

for all $\alpha > 0$, and $N \in \mathbb{N}$ large enough.

Proof The proof is very similar to [4, Prop. 8.7]. First of all, with Lemma 3 we can bound

$$\begin{aligned}
\pm\Big\{&e^{-A}\left[\frac{1}{2}\widehat{\omega}_N(0)(N-1)\left(1-\frac{\mathcal{N}_+}{N}\right) + \big[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+\right]e^A \\
&\quad - \left[\frac{1}{2}\widehat{\omega}_N(0)(N-1)\left(1-\frac{\mathcal{N}_+}{N}\right) + \big[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+\right]\Big\} \\
&\quad \le CN^{1-\alpha}(\mathcal{N}_++1)\,.
\end{aligned}$$

Moreover, for the contribution quadratic in $\mathcal{N}_+$, we can decompose

$$\begin{aligned}
&\Big\langle\xi, \Big[e^{-A}\mathcal{N}_+^2e^A - \mathcal{N}_+^2\Big]\xi\Big\rangle \\
&\quad = \Big\langle\xi_1, \Big[e^{-A}\mathcal{N}_+e^A - \mathcal{N}_+\Big]\xi\Big\rangle + \Big\langle\xi, \Big[e^{-A}\mathcal{N}_+e^A - \mathcal{N}_+\Big]\xi_2\Big\rangle
\end{aligned}$$

with $\xi_1 = e^{-A}\mathcal{N}_+e^A\xi$ and $\xi_2 = \mathcal{N}_+\xi$, and estimate, again with Lemma 3,

$$\begin{aligned}
&\Big|\Big\langle\xi, \Big[e^{-A}\mathcal{N}_+^2e^A - \mathcal{N}_+^2\Big]\xi\Big\rangle\Big| \\
&\quad \le CN^{-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\left[\|(\mathcal{N}_++1)^{1/2}\xi_1\| + \|(\mathcal{N}_++1)^{1/2}\xi_2\|\right]\,.
\end{aligned}$$

With Proposition 2, we have $\|(\mathcal{N}_++1)^{1/2}\xi_1\| \le C\|(\mathcal{N}_++1)^{3/2}\xi\|$. □

6.3 Contributions from $e^{-A}\mathcal{K}e^{A}$

In Sect. 6.6 we will analyse the contributions to $\mathcal{R}_{N,\alpha}$ arising from conjugation of the kinetic energy operator $\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p$. To this aim we will exploit properties of the commutator $[\mathcal{K}, A]$, collected in the following proposition.

Proposition 9 *Let A be defined as in (44) and $\widehat{\omega}_N(r)$ be defined in (39). Then there exists a constant $C > 0$ such that*

$$\begin{aligned}[\mathcal{K}, A] = &-\sqrt{N} \sum_{p,q\in\Lambda_+^*, p\neq -q} (\widehat{V}(\cdot/e^N) * \widehat{f}_{N,\ell})(p)(b_{p+q}^* a_{-p}^* a_q + h.c.)\\ &+ \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b_{p+q}^* a_{-p}^* a_q + h.c.\big] + \delta_{\mathcal{K}}\end{aligned}$$

where

$$\big|\langle\xi, \delta_{\mathcal{K}}\xi\rangle\big| \leq CN^{-1}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\| + CN^{-\alpha}\|\mathcal{K}^{1/2}\xi\|^2 \tag{91}$$

for all $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$, and $N \in \mathbb{N}$ large enough. Moreover, the operator

$$\Delta_{\mathcal{K}} = \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b_{p+q}^* a_{-p}^* a_q, A\big]$$

satisfies

$$\big|\langle\xi, \Delta_{\mathcal{K}}\xi\rangle\big| \leq CN^{-\alpha}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|^2 + CN^{-1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \tag{92}$$

for all $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$, and $N \in \mathbb{N}$ large enough.

Proof To show (91) we recall from Eqs. (84), (85) that

$$\begin{aligned}[\mathcal{K}, A] = &-\sqrt{N} \sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}} (\widehat{V}(\cdot/e^N) * \widehat{f}_{N,\ell})(r)\big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.}\big]\\ &+ 2\sqrt{N} \sum_{r,v\in\Lambda_+^*} e^{2N}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r)\big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.}\big]\\ &+ \frac{2}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} r\cdot v\,\eta_r\big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.}\big]\\ =& \,\mathrm{T}_{11} + \mathrm{T}_{12} + \mathrm{T}_2\,.\end{aligned}$$

with T_2 satisfying (90). Using the definition $\widehat{\omega}_N(p) = 2Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p)$ we write

$$\begin{aligned}\mathrm{T}_{12} =& \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\big]\\ &+ \frac{2}{\sqrt{N}} e^{2N}\lambda_\ell \sum_{p,q\in\Lambda_+^*, p\neq -q} (\widehat{\chi}_\ell * \eta)(p)\big[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\big]\\ =&\mathrm{T}_{121} + \mathrm{T}_{122}.\end{aligned}$$

Hence, $\delta_K = T_2 + T_{122}$. To bound T_{122} we switch to position space:

$$
\begin{aligned}
&|\langle \xi, \mathrm{T}_{122}\xi\rangle| \\
&\leq CN^{2\alpha-3/2}\int_{\Lambda^2} \chi_\ell(x-y)\check{\eta}(x-y)\|\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\| \\
&\leq CN^{2\alpha-3/2}\left[\int_{\Lambda^2} \chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|^2 dxdy\right]^{1/2}\left[\int_{\Lambda^2} |\check{\eta}(x-y)|^2\|\check{a}_x\xi\|^2 dxdy\right]^{1/2} \\
&\leq CN^{\alpha-3/2}\|\mathcal{N}_+^{1/2}\xi\|\left[\int_{\Lambda^2} \chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|^2 dxdy\right]^{1/2}.
\end{aligned}
$$

To bound the term in the parenthesis, we proceed similarly as in (80). We find

$$
\int_{\Lambda^2} \chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|^2 dxdy \leq Cq\|\chi_\ell\|_{q'}\|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\|^2 \leq CqN^{1-2\alpha/q'}\|\mathcal{K}^{1/2}\xi\|^2
$$

for any $q > 2$ and $1 < q' < 2$ with $1/q + 1/q' = 1$. Choosing $q = \log N$, we obtain

$$
|\langle \xi, \mathrm{T}_{122}\xi\rangle| \leq CN^{-1}(\log N)^{1/2}\|\mathcal{N}_+^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|
$$

With (90), this implies (91).

Let us now focus on (92). We have

$$
\begin{aligned}
&\frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b^*_{p+q}a^*_{-p}a_q, A\big] \\
&\qquad = \frac{1}{N} \sum_{\substack{r,p,q,v\in\Lambda_+^*,\\ p\neq -q, r\neq -v}} \widehat{\omega}_N(p)\eta_r\big[b^*_{p+q}a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v - a^*_v a_{-r}b_{r+v}\big].
\end{aligned}
$$

With the commutators from the proof of Proposition 8.8 in [4], we arrive at

$$
\frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b^*_{p+q}a^*_{-p}a_q, A\big] + \text{h.c.} = \sum_{j=1}^{12} \Upsilon_j + \text{h.c.}
$$

where

$$
\begin{aligned}
\Upsilon_1 &:= -\frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq v, r\neq -v}} \big(\widehat{\omega}_N(v-q)+\widehat{\omega}_N(v)\big)\eta_r b^*_{r+v} b^*_{-r} a^*_{q-v} a_q\,,\\
\Upsilon_2 &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ r\neq -v, r\neq -q}} \widehat{\omega}_N(r+q)\eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{r+q} a_q a_{r+v}\,,\\
\Upsilon_3 &:= \frac{1}{N} \sum_{\substack{r,v\in\Lambda_+^*,\\ r\neq -v}} \big(\widehat{\omega}_N(r+v)+\widehat{\omega}_N(r)\big)\eta_r (1-\mathcal{N}_+/N) a^*_v a_v,\\
\Upsilon_4 &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq v, r\neq -v}} \widehat{\omega}_N(r+v-q)\eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{q-r-v} a_{-r} a_q,\\
\Upsilon_5 &:= -\frac{1}{N^2} \sum_{\substack{p,q,r,v\in\Lambda_+^*,\\ p\neq -q, r\neq -v}} \widehat{\omega}_N(p)\eta_r a^*_v a^*_{p+q} a^*_{-p} a_{-r} a_{r+v} a_q\,,\\
\Upsilon_6 &:= -\frac{1}{N^2} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq r+v}} \widehat{\omega}_N(r+v)\eta_r a^*_v a^*_{q-r-v} a_{-r} a_q,\\
\Upsilon_7 &:= -\frac{1}{N^2} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq -r, r\neq -v}} \widehat{\omega}_N(r)\eta_r a^*_v a^*_{q+r} a_{r+v} a_q\,,\\
\Upsilon_8 &:= \frac{1}{N} \sum_{\substack{r,v,p\in\Lambda_+^*,\\ p\neq -r-v}} \widehat{\omega}_N(p)\eta_r b^*_{p+r+v} b^*_{-p} a^*_{-r} a_v\,,\\
\Upsilon_9 &:= \frac{1}{N} \sum_{\substack{p,r,v\in\Lambda_+^*,\\ p\neq r, r\neq -v}} \widehat{\omega}_N(p)\eta_r b^*_{p-r} b^*_{r+v} a^*_{-p} a_v\,,\\
\Upsilon_{10} &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq -r, r\neq -v}} \widehat{\omega}_N(r)\eta_r b^*_{q+r} a^*_v a_q b_{r+v}\,,\\
\Upsilon_{11} &:= -\frac{1}{N} \sum_{\substack{p,r,v\in\Lambda_+^*,\\ p\neq -v, r\neq -v}} \widehat{\omega}_N(p)\eta_r b^*_{p+v} a^*_{-p} a_{-r} b_{r+v}\,,\\
\Upsilon_{12} &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*\\ r\neq q-v, -v}} \widehat{\omega}_N(r+v)\eta_r b^*_{q-r-v} a^*_v a_{-r} b_q\,.
\end{aligned}
\tag{93}
$$

To conclude the proof of Proposition 9, we show that all operators in (93) satisfy (92). To study all these terms it is convenient to switch to position space. We recall that $\widehat{\omega}_N(p) =$

$g_N \widehat{\chi}(\ell p)$ with $|g_N| \leq C$ and $\ell = N^{-\alpha}$. Using (87) we find:

$$\begin{aligned}\big|\langle \xi, \Upsilon_1 \xi\rangle\big| &\leq C N^{2\alpha-1} \int_{\Lambda^2} dxdy\; \chi_\ell(x-y) \|\check{b}(\check{\eta}_x)\check{b}_x \check{a}_y \xi\| \left[\|\check{a}_x \xi\| + \|\check{a}_y \xi\|\right] \\ &\leq C N^{2\alpha-1} \|\eta\| \int_{\Lambda^2} dxdy\; \chi_\ell(x-y) \|\check{b}_x \check{a}_y (\mathcal{N}_+ + 1)^{1/2} \xi\| \|\check{a}_x \xi\| \\ &\leq C N^{-\alpha} (\log N)^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\| \,.\end{aligned}$$

The expectation of Υ_2 is bounded following the same strategy used to show (87). For any $2 \leq q < \infty$ we have

$$\begin{aligned}&\big|\langle \xi, \Upsilon_2 \xi\rangle\big| \\ &\leq C N^{2\alpha-1} \int_{\Lambda^3} dxdydz \chi_\ell(z-y) |\check{\eta}(z-x)| \|\check{a}_x \check{a}_y \xi\| \|\check{a}_z \check{a}_x \xi\| \\ &\leq C N^{2\alpha-1} \int_{\Lambda^2} dxdz |\check{\eta}(z-x)| \|\check{a}_z \check{a}_x \xi\| \\ &\quad \times \left(\int_\Lambda dy\; \chi(|z-y| \leq N^{-\alpha}) \right)^{1-1/q} \left(\int_\Lambda dy \|\check{a}_x \check{a}_y \xi\|^q \right)^{1/q} \\ &\leq C q^{1/2} N^{2\alpha/q-1} \|\eta\| \|(\mathcal{N}_+ + 1)\xi\| \left[\int_{\Lambda^2} dxdy \|\check{a}_x \nabla_y \check{a}_y \xi\|^2 + \int_{\Lambda^2} dxdy \|\check{a}_x \check{a}_y \xi\|^2 \right]^{1/2} \\ &\leq C N^{-\alpha} (\log N)^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\| \,,\end{aligned}$$

where in the last line we chose $q = \log N$. The term Υ_3 is of lower order; using that $\big|\sum_r \widehat{\omega}_N(r) \eta_r\big| \leq \|\widehat{\chi}(./N^\alpha)\|_2 \|\eta\|_2 \leq C$ and Cauchy–Schwarz, we easily obtain

$$\big|\langle \xi, \Upsilon_3 \xi\rangle\big| \leq C N^{-1} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \,.$$

The term Υ_4 can be estimated as Υ_1 using (87):

$$\begin{aligned}\big|\langle \xi, \Upsilon_4 \xi\rangle\big| &\leq C N^{2\alpha-1} \int_{\Lambda^2} dxdy\; \chi_\ell(x-y) \|\check{a}_x \check{a}_y \xi\| \|\check{a}(\check{\eta}_y) \check{a}_y \xi\| \\ &\leq C N^{2\alpha-1} \|\eta\| \int_{\Lambda^2} dxdy\; \chi_\ell(x-y) \|\check{a}_x \check{a}_y \xi\| \|\check{a}_y (\mathcal{N}_+ + 1)^{1/2} \xi\| \\ &\leq C N^{-\alpha} (\log N)^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\| \,.\end{aligned}$$

The term Υ_5 is bounded similarly to Υ_2; with $q = \log N$ we have

$$\begin{aligned}\big|\langle \xi, \Upsilon_5 \xi\rangle\big| &\leq C N^{2\alpha-2} \|\eta\| \int_{\Lambda^3} dxdydz\; \chi_\ell(y-z) \|\check{a}_x \check{a}_y \check{a}_z \xi\| \|\mathcal{N}_+^{1/2} \check{a}_x \check{a}_y \xi\| \\ &\leq C N^{2\alpha-3/2} \|\eta\| \int_{\Lambda^2} dxdy\; \|\check{a}_x \check{a}_y \xi\| \\ &\quad \times \left(\int_\Lambda dz\; \chi(|y-z| \leq N^{-\alpha}) \right)^{1-1/q} \left(\int_\Lambda dz\; \|\check{a}_x \check{a}_y \check{a}_z \xi\|^q \right)^{1/q} \\ &\leq C N^{-\alpha} (\log N)^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\| \,.\end{aligned}$$

The terms Υ_6 and Υ_7 are of smaller order and can be bounded with Cauchy–Schwarz; we have

$$\begin{aligned}\big|\langle\xi,\Upsilon_6\xi\rangle\big| &\leq CN^{2\alpha-2}\int_{\Lambda^2} dxdydz\,\chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|\|\check{a}(\check{\eta}_x)\check{a}_y\xi\| \\ &\leq CN^{\alpha-3/2}\left(\int_{\Lambda^2} dxdy\,\|\check{a}_x\check{a}_y\xi\|^2\right)^{1/2}\left(\int_{\Lambda^2} dxdy\,\chi(|x-y|\leq N^{-\alpha})\|\check{a}_y\xi\|^2\right)^{1/2} \\ &\leq CN^{-1}\|(\mathcal{N}_++1)^{1/2}\xi\|^2,\end{aligned}$$

and

$$\begin{aligned}\big|\langle\xi,\Upsilon_7\xi\rangle\big| &\leq CN^{2\alpha-2}\int_{\Lambda^3} dxdydz\,\chi_\ell(y-z)|\check{\eta}(z-x)|\|\check{a}_x\check{a}_y\xi\|^2 \\ &\leq CN^{2\alpha-2}\left(\int_{\Lambda^3} dxdydz\,\chi_\ell(y-z)\|\check{a}_x\check{a}_y\xi\|^2\right)^{1/2} \\ &\quad\times\left(\int_{\Lambda^3} dxdydz\,|\check{\eta}(z-x)|^2\|\check{a}_x\check{a}_y\xi\|^2\right)^{1/2} \\ &\leq CN^{-1}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\,.\end{aligned}$$

The terms $\Upsilon_8,\Upsilon_{11},\Upsilon_{12}$ are again bounded, as Υ_1, using (87). We find

$$\begin{aligned}\big|\langle\xi,\big(\Upsilon_8+\Upsilon_{11}+\Upsilon_{12}\big)\xi\rangle\big| &\leq CN^{2\alpha-1}\|\eta\|\int_{\Lambda^2} dxdy\,\chi_\ell(x-y)\|\mathcal{N}_+^{1/2}\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\| \\ &\leq CN^{-\alpha}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.\end{aligned}$$

It remains to bound Υ_9 and Υ_{10}. The term Υ_9 is bounded analogously to Υ_2:

$$\begin{aligned}&\big|\langle\xi,\Upsilon_9\xi\rangle\big| \\ &\leq CN^{2\alpha-1}\int_{\Lambda^3} dxdydz\,\chi_\ell(x-z)|\check{\eta}(x-y)|\|\check{a}_x\check{a}_y\check{a}_z\xi\|\|\check{a}_y\xi\| \\ &\leq CN^{2\alpha-1}\int_{\Lambda^2} dxdy\,|\check{\eta}(x-y)|\|\check{a}_y\xi\|\left(\int_{\Lambda} dz\,\chi(|y-z|\leq N^{-\alpha})\right)^{1-1/q} \\ &\quad\times\left(\int_{\Lambda} dz\,\|\check{a}_x\check{a}_y\check{a}_z\xi\|^q\right)^{1/q} \\ &\leq Cq^{1/2}N^{2\alpha/q-1}\left[\int_{\Lambda^2} dxdy\,|\check{\eta}(x-y)|^2\|\check{a}_y\xi\|^2\right]^{1/2}\left[\int_{\Lambda^3} dxdy\,\Big\|\|\check{a}_x\check{a}_y\check{a}_z\xi\|\Big\|^2_{L^q_z}\right]^{1/2} \\ &\leq CN^{-\alpha}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.\end{aligned}$$

As for Υ_{10}, we find

$$\big|\langle\xi,\Upsilon_{10}\xi\rangle\big| \leq CN^{2\alpha-1}\int_{\Lambda^3} dxdydz\,\chi_\ell(y-z)|\check{\eta}(x-z)|\|\check{a}_x\check{a}_y\xi\|^2$$

Proceeding as in (80), we obtain

$$\big|\langle\xi,\Upsilon_{10}\xi\rangle\big| \leq CqN^{2\alpha}\|\chi_\ell*|\check{\eta}|\|_{q'}\|\mathcal{K}^{1/2}\xi\|^2 \leq Cq\|\check{\eta}\|_{q'}\|\mathcal{K}^{1/2}\xi\|^2$$

for any $q>2$, and $q'<2$ with $1/q+1/q'=1$. Since, for an arbitrary $q'<2$, $\|\check{\eta}\|_{q'}\leq\|\check{\eta}\|_2=\|\eta\|_2\leq N^{-\alpha}$, we obtain

$$\big|\langle\xi,\Upsilon_{10}\xi\rangle\big| \leq CN^{-\alpha}\|\mathcal{K}^{1/2}\xi\|^2$$

We conclude that for any $\alpha > 1$

$$\left| \langle \xi, \sum_{j=1}^{12} \Upsilon_i \xi \rangle \right| \leq C N^{-\alpha} (\log N)^{1/2} \, \|(\mathcal{K}+1)^{1/2}\xi\|^2 + C N^{-1} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \, .$$

□

6.4 Analysis of $e^{-A} \mathcal{Z}_N e^A$

In this subsection, we consider contributions to $\mathcal{R}_{N,\alpha}$ arising from conjugation of $\mathcal{Z}_N$, as defined in (77).

Proposition 10 *Let A be defined in (44). Then, there exists a constant $C > 0$ such that*

$$e^A \mathcal{Z}_N e^{-A} = \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{\omega}_N(p) \left(b_p^* b_{-p}^* + b_p b_{-p} \right) + \delta_{\mathcal{Z}_N}$$

where

$$\pm \delta_{\mathcal{Z}_N} \leq C N^{1-\alpha} (\mathcal{H}_N + 1)$$

for all $\alpha > 0$, and $N \in \mathbb{N}$ large enough.

Proof We have

$$\begin{aligned} &\frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{\omega}_N(p) \left[e^{-A} \big(b_p^* b_{-p}^* + b_p b_{-p} \big) e^A - \big(b_p^* b_{-p}^* + b_p b_{-p} \big) \right] \\ &\qquad = \frac{1}{2} \int_0^1 ds \sum_{p \in \Lambda_+^*} \widehat{\omega}_N(p) \, e^{-sA} \big[b_p^* b_{-p}^* + b_p b_{-p}, A \big] e^{sA}. \end{aligned} \tag{94}$$

We compute

$$\begin{aligned} &\frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{\omega}_N(p) \big[b_p^* b_{-p}^*, b_{r+v}^* a_{-r}^* a_v - a_v^* a_{-r} b_{r+v} \big] \\ &\quad = -\widehat{\omega}_N(v) b_{r+v}^* b_{-v}^* b_{-r}^* + \widehat{\omega}_N(r) b_v^* \Big(b_r^* b_{r+v} - \frac{2}{N} a_r^* a_{r+v} \Big) \\ &\qquad + \widehat{\omega}_N(r+v) \Big(1 - \frac{\mathcal{N}_+}{N} \Big) b_{-r-v}^* a_v^* a_{-r} - \frac{1}{N} \sum_{p \in \Lambda^*} \widehat{\omega}_N(p) b_p^* a_{-p}^* a_v^* a_{-r} a_{r+v}. \end{aligned} \tag{95}$$

With (95) we write

$$\frac{1}{2} \sum_{p \in \Lambda^*} \widehat{\omega}_N(p) \big[b_p^* b_{-p}^* + b_p b_{-p}, A \big] = \sum_{j=1}^4 \Pi_j + \text{h.c.}$$

with

$$\begin{aligned}
\Pi_1 &= -\frac{1}{\sqrt{N}} \sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}} \widehat{\omega}_N(v)\eta_r b^*_{r+v} b^*_{-v} b^*_{-r}\,,\\
\Pi_2 &= \frac{1}{\sqrt{N}} \sum_{\substack{r,v\in\Lambda_+^*:\\ r\neq -v}} \widehat{\omega}_N(r)\eta_r b^*_v \Big(b^*_r b_{r+v} - \frac{2}{N} a^*_r a_{r+v}\Big)\,,\\
\Pi_3 &= \frac{1}{\sqrt{N}} \sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}} \widehat{\omega}_N(r+v)\eta_r \Big(1-\frac{\mathcal{N}_+}{N}\Big) b^*_{-r-v} a^*_v a_{-r}\,,\\
\Pi_4 &= -\frac{1}{N^{3/2}} \sum_{\substack{r,v,p\in\Lambda_+^*:\\ r\neq -v}} \widehat{\omega}_N(p)\eta_r b^*_p a^*_{-p} a^*_v a_{-r} a_{r+v}\,.
\end{aligned}$$

To bound the first term, we observe, with (52),

$$\begin{aligned}
|\langle \xi, \Pi_1 \xi\rangle| &\leq \frac{\|\eta\|}{\sqrt{N}} \|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{1/2}\xi\| \left[\sum_{v\in\Lambda_+^*} \frac{|\widehat{\omega}_N(v)|^2}{v^2}\right]^{1/2}\\
&\leq CN^{-\alpha}(\log N)^{1/2} \|\mathcal{K}^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{1/2}\xi\|\,.
\end{aligned}$$

The term Π_3 can be bounded similarly to Π_1, with (52). We find

$$\big|\langle \xi, \Pi_3\xi\rangle\big| \leq CN^{-\alpha}(\log N)^{1/2} \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|\,.$$

With $|\widehat{\omega}_N(r)| \leq C$, we similarly obtain

$$\begin{aligned}
|\langle \xi, \Pi_2\xi\rangle| &\leq N^{-1/2}\|\eta\| \|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{1/2}\xi\|\\
&\leq CN^{-\alpha} \|\mathcal{K}^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{1/2}\xi\|\,.
\end{aligned}$$

Finally, we estimate, using again (52),

$$\begin{aligned}
\big|\langle \xi, \Pi_4\xi\rangle\big| &\leq N^{-3/2} \Big(\sum_{r,v,p\in\Lambda_+^*} p^2 |\eta_r|^2 \|a_{-p}a_v(\mathcal{N}_+ +1)^{1/2}\xi\|^2\Big)^{1/2}\\
&\quad\times \Big(\sum_{r,v,p\in\Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2} \|a_{-r}a_{r+v}\xi\|^2\Big)^{1/2}\\
&\leq CN^{-3/2}\|\eta\|(\log N)^{1/2} \|\mathcal{K}^{1/2}(\mathcal{N}_+ +1)\xi\| \|(\mathcal{N}_+ +1)\xi\|\\
&\leq CN^{-\alpha}(\log N)^{1/2} \|\mathcal{K}^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{1/2}\xi\|\,.
\end{aligned}$$

With (94), we conclude that

$$\begin{aligned}
&\left| \frac{1}{2}\sum_{p\in\Lambda^*} \widehat{\omega}_N(p) \left[\langle \xi, e^{-A}\big(b^*_p b^*_{-p} + b_p b_{-p}\big) e^A \xi\rangle - \langle \xi, \big(b^*_p b^*_{-p} + b_p b_{-p}\big)\xi\rangle\right]\right|\\
&\leq CN^{-\alpha}(\log N)^{1/2} \int_0^1 ds\ \|\mathcal{K}^{1/2} e^{sA}\xi\| \|(\mathcal{N}_+ +1)^{1/2} e^{sA}\xi\|\,.
\end{aligned}$$

With Proposition 2, Lemma 3, we conclude that

$$\begin{aligned}
&\left|\frac{1}{2}\sum_{p\in\Lambda^*}\widehat{\omega}_N(p)\left[\langle\xi, e^{-A}\big(b_p^*b_{-p}^*+b_pb_{-p}\big)e^{A}\xi\rangle-\langle\xi,\big(b_p^*b_{-p}^*+b_pb_{-p}\big)\xi\rangle\right]\right|\\
&\qquad\leq CN^{-\alpha}(\log N)^{1/2}\left[\|\mathcal{H}_N^{1/2}\xi\|+N^{1/2}\|\mathcal{N}_+^{1/2}\xi\|\right]\|(\mathcal{N}_++1)^{1/2}\xi\|\\
&\qquad\leq CN^{1-\alpha}\|(\mathcal{H}_N+1)^{1/2}\xi\|^2\,.
\end{aligned}$$

□

6.5 Contributions from $e^{-A}\mathcal{C}_Ne^{A}$

In Sect. 6.6 we will analyse the contributions to $\mathcal{R}_{N,\alpha}$ arising from conjugation of the cubic operator $\mathcal{C}_N$ defined in (77). To this aim we will need some properties of the commutator $[\mathcal{C}_N, A]$, as established in the following proposition.

Proposition 11 *Let A be defined in (44). Then, there exists a constant $C > 0$ such that*

$$\big[\mathcal{C}_N, A\big]=2\sum_{r,v\in\Lambda_+^*}\big[\widehat{V}(r/e^N)\eta_r+\widehat{V}((r+v)/e^N)\eta_r\big]a_v^*a_v\Big(1-\frac{\mathcal{N}_+}{N}\Big)+\delta_{\mathcal{C}_N}$$

where

$$|\langle\xi,\delta_{\mathcal{C}_N}\xi\rangle|\leq\ CN^{3/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\| \tag{96}$$

for all $\alpha>0$, $\xi\in\mathcal{F}_+^{\leq N}$, and $N\in\mathbb{N}$ large enough.

Proof We consider the commutator

$$\big[\mathcal{C}_N, A\big]=\sum_{\substack{p,q\in\Lambda_+^*:p+q\neq0\\ r,v\in\Lambda_+^*}}\widehat{V}(p/e^N)\eta_r\big[b_{p+q}^*a_{-p}^*a_q,\, b_{r+v}^*a_{-r}^*a_v-a_v^*a_{-r}b_{r+v}\big]+\text{h.c.}\,.$$

As in the proof of Proposition 9, we use the commutators from the proof of Proposition 8.8 in [4] to conclude that

$$\big[\mathcal{C}_N, A\big]=2\sum_{r,v\in\Lambda_+^*}\big[\widehat{V}(r/e^N)\eta_r+\widehat{V}((r+v)/e^N)\eta_r\big]a_v^*a_v\frac{N-\mathcal{N}_+}{N}+\sum_{j=1}^{12}(\varXi_j+\text{h.c.})$$

where

$$\begin{aligned}
\Xi_1 &:= - \sum_{\substack{r,v,p\in\Lambda_+^*,\\ p\neq v}} \widehat{V}(p/e^N)\eta_r b^*_{r+v} b^*_{-r} a^*_{-p} a_{v-p},\\
\Xi_2 &:= \sum_{\substack{r,v,p\in\Lambda_+^*\\ r\neq -p}} \widehat{V}(p/e^N)\eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{-p} a_{-r-p} a_{r+v},\\
\Xi_3 &:= \sum_{\substack{r,v,p\in\Lambda_+^*:\\ r+v\neq p}} \widehat{V}(p/e^N)\eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{-p} a_{-r} a_{r+v-p},\\
\Xi_4 &:= -\frac{1}{N} \sum_{r,v,p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/e^N)\eta_r a^*_v a^*_{p+q} a^*_{-p} a_{-r} a_{r+v} a_q,\\
\Xi_5 &:= -\frac{1}{N} \sum_{\substack{r,v,q\in\Lambda_+^*:\\ r+v\neq q}} \widehat{V}((r+v)/e^N)\eta_r a^*_v a^*_{q-r-v} a_{-r} a_q,\\
\Xi_6 &:= -\frac{1}{N} \sum_{\substack{r,v,q\in\Lambda_+^*:\\ r\neq -q}} \widehat{V}(r/e^N)\eta_r a^*_v a^*_{q+r} a_{r+v} a_q\\
\Xi_7 &:= \sum_{\substack{r,v,p\in\Lambda_+^*:\\ r+v\neq -p}} \widehat{V}(p/e^N)\eta_r b^*_{p+r+v} b^*_{-p} a^*_{-r} a_v,\\
\Xi_8 &:= \sum_{\substack{r,v,p\in\Lambda_+^*:\\ r\neq -p}} \widehat{V}(p/e^N)\eta_r b^*_{p-r} b^*_{r+v} a^*_{-p} a_v,\\
\Xi_9 &:= - \sum_{\substack{r,v,q\in\Lambda_+^*:\\ q\neq v}} \widehat{V}(v/e^N)\eta_r b^*_{q-v} b^*_{r+v} a^*_{-r} a_q,\\
\Xi_{10} &:= \sum_{\substack{r,v,q\in\Lambda_+^*:\\ r\neq -q}} \widehat{V}(r/e^N)\eta_r b^*_{q+r} a^*_v a_q b_{r+v},\\
\Xi_{11} &:= - \sum_{\substack{r,v,p\in\Lambda_+^*:\\ p\neq -v}} \widehat{V}(p/e^N)\eta_r b^*_{p+v} a^*_{-p} a_{-r} b_{r+v},\\
\Xi_{12} &:= \sum_{\substack{r,v,q\in\Lambda_+^*:\\ q\neq r+v}} \widehat{V}((r+v)/e^N)\eta_r b^*_{q-r-v} a^*_v a_{-r} b_q\,.
\end{aligned}$$

To prove the proposition, we have to show that all terms Ξ_j, $j=1,\dots,12$, satisfy the bound (96). We bound Ξ_1 in position space, with Cauchy–Schwarz, by

$$\begin{aligned}
\big|\langle \xi, \Xi_1 \xi\rangle\big| &\leq C\int_{\Lambda^3} dxdydz e^{2N} V(e^N(x-y)) |\check{\eta}(x-z)| \|\check{a}_x \xi\| \|\check{a}_x \check{a}_y \check{a}_z \xi\|\\
&\leq C\left[\int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y)) \|\check{a}_x \check{a}_y \check{a}_z \xi\|^2\right]^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\times \left[\int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y)) |\check{\eta}(x-z)|^2 \|\check{a}_x \xi\|^2 \right]^{1/2} \\
&\leq C \|\eta\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \mathcal{N}_+^{1/2} \xi\| \\
&\leq C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|.
\end{aligned}$$

We can proceed similarly to control Ξ_9. We obtain

$$\left| \langle \xi, \Xi_9 \xi \rangle \right| \leq C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|.$$

The expectations of the terms Ξ_3 and Ξ_{12} can be bounded analogously:

$$\begin{aligned}
&\left| \langle \xi, \Xi_3 \xi \rangle \right| + \left| \langle \xi, \Xi_{12} \xi \rangle \right| \\
&\leq C \int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y)) (|\eta(x-z)| + |\eta(y-z)|) \|\check{a}_x \check{a}_y \xi\| \|\check{a}_x \check{a}_z \xi\| \\
&\leq C \left[\int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y)) \|\check{a}_x \check{a}_y \xi\|^2 (|\eta(x-z)|^2 + |\eta(y-z)|^2) \right]^{1/2} \\
&\quad \times \left[\int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y)) \|\check{a}_x \check{a}_z \xi\|^2 \right]^{1/2} \\
&\leq C \|\eta\| \|(\mathcal{N}_+ + 1) \xi\| \|\mathcal{V}_N^{1/2} \xi\| \\
&\leq C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|.
\end{aligned}$$

As for Ξ_4, we find

$$\begin{aligned}
|\langle \xi, \Xi_4 \xi \rangle| &= \left| \frac{1}{N} \int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \langle \xi, \check{a}_x^* \check{a}_y^* \check{a}_z^* \check{a}(\check{\eta}_x) \check{a}_x \check{a}_y \xi \rangle \right| \\
&\leq C N^{-1} \|\eta\| \int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \|\check{a}_x \check{a}_y \check{a}_z \xi\| \|\mathcal{N}_+^{1/2} \check{a}_x \check{a}_y \xi\| \\
&\leq C N^{-1} \|\eta\| \left[\int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \|\check{a}_x \check{a}_y \check{a}_z \xi\|^2 \right]^{1/2} \\
&\quad \times \left[\int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \|\mathcal{N}_+^{1/2} \check{a}_x \check{a}_y \xi\|^2 \right]^{1/2} \\
&\leq C N^{1/2-\alpha} \|\mathcal{V}_N^{1/2} \xi\| \|\mathcal{N}_+^{1/2} \xi\|\,.
\end{aligned}$$

The terms Ξ_5 and Ξ_6 can be bounded in momentum space, using (154). Hence,

$$\begin{aligned}
&|\langle \xi, \Xi_5 \xi \rangle| + |\langle \xi, \Xi_6 \xi \rangle| \\
&\leq C N^{-1} \sum_{r,v,q \in \Lambda_+^*} \left(\frac{\widehat{V}((v+r)/e^N)}{|v|} |\eta_r| |v| \|a_v a_{q-r-v} \xi\| \|a_{-r} a_q \xi\| \right. \\
&\quad \left. + \frac{\widehat{V}(r/e^N)}{|r+v|} |\eta_r| |r+v| \|a_{r+q} a_v \xi\| \|a_q a_{r+v} \xi\| \right) \\
&\leq C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|.
\end{aligned}$$

Similarly we have

$$\begin{aligned}|\langle \xi, \Xi_2 \xi\rangle| + |\langle \xi, \Xi_{10}\xi\rangle| &\leq \sum_{r,v,p\in\Lambda_+^*}\Big(\frac{\widehat{V}(p/e^N)}{|p|}|\eta_r||p|\|a_v a_{-p}\xi\|\|a_{r+v}a_{-r-p}\xi\|\\ &\quad + \frac{\widehat{V}(r/e^N)}{|r+v|}|\eta_r||r+v|\|a_q a_{r+v}\xi\|\|a_{r+q}a_v\xi\|\Big)\\ &\leq CN^{3/2-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

Next, we rewrite Ξ_7, Ξ_8 and Ξ_{11} as

$$\begin{aligned}\Xi_7 &= \int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\check{b}_x^*\check{b}_y^* a^*(\check{\eta}_x)\check{a}_x\,,\\ \Xi_8 &= \int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))\check{\eta}(z-x)\check{b}_x^*\check{b}_z^*\check{a}_y^*\check{a}_z\,,\\ \Xi_{11} &= -\int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\check{b}_x^*\check{a}_y^*\check{a}(\check{\eta}_x)\check{b}_x\,.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|\langle \xi, \Xi_7\xi\rangle| &\leq C\|\eta\|\int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\;\|\mathcal{N}_+^{1/2}\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\|\\ &\leq C\|\eta\|\|\mathcal{N}_+^{1/2}\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\\ &\leq CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\,,\end{aligned}$$

as well as

$$\begin{aligned}&|\langle \xi, \Xi_8\xi\rangle|\\ &\leq C\int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))|\check{\eta}(x-z)|\|\check{a}_x\check{a}_y\check{a}_z\xi\|\|\check{a}_z\xi\|\\ &\leq C\left[\int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))\|\check{a}_x\check{a}_y\check{a}_z\xi\|^2\right]^{1/2}\\ &\quad\times\left[\int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))|\eta(x-z)|^2\|\check{a}_z\xi\|^2\right]^{1/2}\\ &\leq CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|,\end{aligned}$$

and

$$\begin{aligned}|\langle \xi, \Xi_{11}\xi\rangle| &\leq C\|\eta\|\int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\;\|\check{a}_x\check{a}_y\xi\|\|\mathcal{N}_+^{1/2}\check{a}_x\xi\|\\ &\leq C\|\eta\|\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+\xi\| \leq CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|.\end{aligned}$$

Collecting all the bounds above, we arrive at (96). □

6.6 Proof of Proposition 4

With the results of Sects. 6.1–6.5, we can now show Proposition 4. We assume $\alpha > 2$. From Eq. (76), Propositions 8 and 10 we obtain that

$$\begin{aligned}\mathcal{R}_{N,\alpha} &= e^{-A}\mathcal{G}^{\text{eff}}_{N,\alpha}e^{A}\\ &= \frac{1}{2}\widehat{\omega}_N(0)(N-1)(1-\mathcal{N}_+/N) + \Big[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\Big]\mathcal{N}_+(1-\mathcal{N}_+/N)\\ &\quad + \frac{1}{2}\sum_{p\in\Lambda^*_+}\widehat{\omega}_N(p)\big[b^*_p b^*_{-p} + b_p b_{-p}\big] + \mathcal{K} + \mathcal{C}_N + \mathcal{V}_N\\ &\quad + \int_0^1 ds\; e^{-sA}\big[\mathcal{K} + \mathcal{C}_N + \mathcal{V}_N, A\big]e^{sA} + \mathcal{E}^{(1)}_{\mathcal{R}}\end{aligned}$$

with

$$\pm\mathcal{E}^{(1)}_{\mathcal{R}} \leq CN^{1-\alpha}(\mathcal{H}_N + 1).$$

From Propositions 7, 9 and 11, we can write, for N large enough,

$$\begin{aligned}&[\mathcal{K} + \mathcal{C}_N + \mathcal{V}_N, A]\\ &= \frac{1}{\sqrt{N}}\sum_{r,v\in\Lambda^*_+}\widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] - \sqrt{N}\sum_{\substack{r,v,\in\Lambda^*_+,\\ p\neq -q}}\widehat{V}(r/e^N)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big]\\ &\quad + 2\sum_{r,v\in\Lambda^*_+}\big[\widehat{V}(r/e^N)\eta_r + \widehat{V}((r+v)/e^N)\eta_r\big]a^*_v a_v(1-\mathcal{N}_+/N) + \mathcal{E}^{(2)}_{\mathcal{R}}\end{aligned}$$

where

$$\begin{aligned}|\langle\xi, \mathcal{E}^{(2)}_{\mathcal{R}}\xi\rangle| \leq\, & CN^{1/2-\alpha}(\log N)^{1/2}\|\mathcal{H}_N^{1/2}\xi\|^2 + CN^{3/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\\ & + CN^{-1}(\log N)^{1/2}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\,.\end{aligned}$$

for all $\xi \in \mathcal{F}^{\leq N}_+$. From Proposions 2, 3 and recalling the definition (77) of the operator $\mathcal{C}_N$, we deduce that

$$\begin{aligned}&\int_0^1 ds\; e^{-sA}[\mathcal{K} + \mathcal{C}_N + \mathcal{V}_N, A]e^{sA}\\ &= \int_0^1 ds\; e^{-sA}\Big[-\mathcal{C}_N + \frac{1}{\sqrt{N}}\sum_{r,v\in\Lambda^*_+}\widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big]\\ &\quad + 2\sum_{r,v\in\Lambda^*_+}\big[\widehat{V}(r/e^N)\eta_r + \widehat{V}((r+v)/e^N)\eta_r\big]a^*_v a_v\Big(1 - \frac{\mathcal{N}_+}{N}\Big)\Big]e^{sA} + \mathcal{E}^{(3)}_{\mathcal{R}}\end{aligned} \tag{97}$$

with

$$\pm\mathcal{E}^{(3)}_{\mathcal{R}} \leq C[N^{2-\alpha} + N^{-1/2}(\log N)^{1/2}](\mathcal{H}_N + 1)$$

for $N \in \mathbb{N}$ sufficiently large.

We now rewrite

$$
\begin{aligned}
&2 \sum_{r,v\in\Lambda_+^*} \big[\widehat{V}(r/e^N)\eta_r + \widehat{V}((r+v)/e^N)\eta_r\big] a_v^* a_v \Big(1-\frac{\mathcal{N}_+}{N}\Big) \\
&\quad = 4 \sum_{r,v\in\Lambda_+^*} \widehat{V}(r/e^N)\eta_r a_v^* a_v \Big(1-\frac{\mathcal{N}_+}{N}\Big) \\
&\qquad + 2 \sum_{r,v\in\Lambda_+^*} \big[\widehat{V}((r+v)/e^N) - \widehat{V}(r/e^N)\big]\eta_r a_v^* a_v \Big(1-\frac{\mathcal{N}_+}{N}\Big) := \mathrm{Q}_1 + \mathrm{Q}_2 .
\end{aligned} \tag{98}
$$

With Lemma 1, part (iii) we get

$$
\Big| 2 \sum_{r\in\Lambda^*} \widehat{V}(r/e^N)\eta_r - \big[2\widehat{\omega}_N(0) - 2N\widehat{V}(0)\big]\Big| \le \frac{C}{N}, \tag{99}
$$

and therefore, using Lemma 3 and (99)

$$
\begin{aligned}
&\pm \Big[e^{-sA}\mathrm{Q}_1 e^{sA} - 2\big[2\widehat{\omega}_N(0) - 2N\widehat{V}(0)\big] \sum_{v\in\Lambda_+^*} a_v^* a_v \Big(1-\frac{\mathcal{N}_+}{N}\Big)\Big] \\
&\quad \le CN^{1-\alpha}(\mathcal{N}_+ + 1) + \frac{C}{N}\mathcal{N}_+ .
\end{aligned} \tag{100}
$$

On the other hand it is easy to check that $e^{-sA}\mathrm{Q}_2 e^{sA}$ is an error term; to this aim we notice that

$$
\Big| \sum_{r\in\Lambda^*} \big[\widehat{V}(r/e^N)\eta_r - \widehat{V}((r+v)/e^N)\eta_r\big]\Big| \le CN|v|e^{-N} .
$$

Hence with Props. 2 and 3 we find

$$
\pm \big[e^{-sA}\mathrm{Q}_2 e^{sA}\big] \le CNe^{-N} e^{-sA}\mathcal{N}_+^{1/2}\mathcal{K}^{1/2}e^{sA} \le CN^2 e^{-N}(\mathcal{H}_N + 1) . \tag{101}
$$

To handle the second term on the second line of (97), we apply Proposition 9 and then Propositions 2 and 3

$$
\begin{aligned}
&\pm \Big(\frac{1}{\sqrt{N}} \int_0^1 ds \sum_{r,v\in\Lambda_+^*} \widehat{\omega}_N(r)\Big[e^{-sA} b_{r+v}^* a_{-r}^* a_v e^{sA} - b_{r+v}^* a_{-r}^* a_v\Big] + \text{h.c.}\Big) \\
&= \pm \Big(\frac{1}{\sqrt{N}} \int_0^1 ds \int_0^s dt \sum_{r,v\in\Lambda_+^*} \widehat{\omega}_N(r) e^{-tA}\Big[b_{r+v}^* a_{-r}^* a_v, A\Big] e^{tA}\Big) \\
&\le C\int_0^1 ds \int_0^s dt\, e^{-tA}\big(N^{-\alpha}(\log N)\,\mathcal{K} + N^{-1}(\mathcal{N}_+ + 1)\big)e^{tA} \\
&\le CN^{1-\alpha}\log N(\mathcal{H}_N + 1) .
\end{aligned} \tag{102}
$$

As for the first term on the second line of (97), we use again Proposition 11. Using (98), (100) and (101) we have

$$
\begin{aligned}
\int_0^1 ds\, e^{-sA}\mathcal{C}_N e^{sA} - \mathcal{C}_N &= \int_0^1 ds \int_0^s dt\, e^{-tA}[\mathcal{C}_N, A]e^{tA} \\
&= \big[2\widehat{\omega}_N(0) - 2N\widehat{V}(0)\big] \sum_{p\in\Lambda_+^*} a_p^* a_p \Big(1-\frac{\mathcal{N}_+}{N}\Big) + \mathcal{E}_{\mathcal{R}}^{(4)}
\end{aligned} \tag{103}
$$

with $\pm\mathcal{E}_{\mathcal{R}}^{(4)} \leq CN^{2-\alpha}(\mathcal{H}_N+1) + CN^{-1}(\mathcal{N}_+ + 1)$.

Inserting the bounds (100), (101), (102) and (103) into (97) we arrive at

$$\begin{aligned}\mathcal{R}_{N,\alpha} &= \frac{1}{2}(N-1)\,\widehat{\omega}_N(0)(1-\mathcal{N}_+/N) + \frac{1}{2}\widehat{\omega}_N(0)\,\mathcal{N}_+\,(1-\mathcal{N}_+/N)\\ &\quad + \widehat{\omega}_N(0)\sum_{p\in\Lambda_+^*} a_p^* a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\big[b_p^* b_{-p}^* + b_p b_{-p}\big]\\ &\quad + \frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda_+^*:\\ r\neq -v}}\widehat{\omega}_N(r)\big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.}\big] + \mathcal{H}_N + \mathcal{E}_{\mathcal{R}}\end{aligned}$$

with

$$\pm\mathcal{E}_{\mathcal{R}} \leq C[N^{2-\alpha} + N^{-1/2}(\log N)^{1/2}](\mathcal{H}_N + 1)$$

for $N\in\mathbb{N}$ sufficiently large.

Acknowledgements We are thankful to A. Olgiati for discussions on the two dimensional scattering equation.

Funding Open Access funding provided by University Zurich.

Appendix A: Analysis of $\mathcal{G}_{N,\alpha}$

The aim of this section is to show Proposition 1. From (12) and (37), we can decompose

$$\mathcal{G}_{N,\alpha} = e^{-B}\mathcal{L}_N e^{B} = \mathcal{G}_{N,\alpha}^{(0)} + \mathcal{G}_{N,\alpha}^{(2)} + \mathcal{G}_{N,\alpha}^{(3)} + \mathcal{G}_{N,\alpha}^{(4)}$$

with

$$\mathcal{G}_{N,\alpha}^{(j)} = e^{-B}\mathcal{L}_N^{(j)} e^{B}\,.$$

To analyse $\mathcal{G}_{N,\alpha}$ we will need precise informations on the action of the generalized Bogoliubov transformation e^B with B the antisymmetric operator defined in (33), which are summarized in Sect. 1. Then, in the Sects. 1–1 we prove separate bounds for the operators $\mathcal{G}_{N,\alpha}^{(j)}$, $j=0,2,3,4$, which we combine in Sect. 1 to prove Proposition 1.

The analysis in this section follows closely that of [4, Sect. 7] with some slight modifications due to the different scaling of the interaction potential and the fact that the kernel η_p of e^B is different from zero for all $p\in\Lambda_+^*$ (in [4] η_p is different from zero only for momenta larger than a sufficiently large cutoff of order one). Moreover, while in three dimensions it was sufficient to choose the function η_p appearing in the generalized Bogoliubov transformation with $\|\eta\|$ sufficiently small but of order one, we need here $\|\eta\|$ to be of order $N^{-\alpha}$ for some $\alpha>0$ large enough. As discussed in the introduction this is achieved by considering the Neumann problem for the scattering equation in (16) on a ball of radius $\ell = N^{-\alpha}$; as a consequence some terms depending on ℓ will be large, compared to the analogous terms in [4].

Appendix A.1: Generalized Bogoliubov Transformations

In this subsection we collect important properties about the action of unitary operators of the form e^B, as defined in (34). As shown in [2, Lemma 2.5 and 2.6], we have, if $\|\eta\|$ is sufficiently small,

$$\begin{aligned} e^{-B} b_p e^B &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \text{ad}_B^{(n)}(b_p) \\ e^{-B} b_p^* e^B &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \text{ad}_B^{(n)}(b_p^*) \end{aligned} \tag{104}$$

where the series converge absolutely. To confirm the expectation that generalized Bogoliubov transformation act similarly to standard Bogoliubov transformations, on states with few excitations, we define (for $\|\eta\|$ small enough) the remainder operators

$$d_q = \sum_{m\geq 0} \frac{1}{m!} \Big[\text{ad}_{-B}^{(m)}(b_q) - \eta_q^m b_{\alpha_m q}^{\sharp_m} \Big], \quad d_q^* = \sum_{m\geq 0} \frac{1}{m!} \Big[\text{ad}_{-B}^{(m)}(b_q^*) - \eta_q^m b_{\alpha_m q}^{\sharp_{m+1}} \Big] \tag{105}$$

where $q \in \Lambda_+^*$, $(\sharp_m, \alpha_m) = (\cdot, +1)$ if m is even and $(\sharp_m, \alpha_m) = (*, -1)$ if m is odd. It follows then from (104) that

$$e^{-B} b_q e^B = \gamma_q b_q + \sigma_q b_{-q}^* + d_q, \quad e^{-B} b_q^* e^B = \gamma_q b_q^* + \sigma_q b_{-q} + d_q^* \tag{106}$$

where we introduced the notation $\gamma_q = \cosh(\eta_q)$ and $\sigma_q = \sinh(\eta_q)$. It will also be useful to introduce remainder operators in position space. For $x \in \Lambda$, we define the operator valued distributions $\check{d}_x, \check{d}_x^*$ through

$$e^{-B} \check{b}_x e^B = b(\check{\gamma}_x) + b^*(\check{\sigma}_x) + \check{d}_x, \qquad e^{-B} \check{b}_x^* e^B = b^*(\check{\gamma}_x) + b(\check{\sigma}_x) + \check{d}_x^* \tag{107}$$

where $\check{\gamma}_x(y) = \sum_{q\in\Lambda^*} \cosh(\eta_q) e^{iq\cdot(x-y)}$ and $\check{\sigma}_x(y) = \sum_{q\in\Lambda^*} \sinh(\eta_q) e^{iq\cdot(x-y)}$. The next lemma is taken from [4, Lemma 3.4].

Lemma 4 *Let $\eta \in \ell^2(\Lambda_+^*)$, $n \in \mathbb{Z}$. For $p \in \Lambda_+^*$, let d_p be defined as in (106). If $\|\eta\|$ is small enough, there exists $C > 0$ such that*

$$\begin{aligned} \|(\mathcal{N}_+ + 1)^{n/2} d_p \xi\| &\leq \frac{C}{N} \Big[|\eta_p| \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|\eta\| \|b_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \Big], \\ \|(\mathcal{N}_+ + 1)^{n/2} d_p^* \xi\| &\leq \frac{C}{N} \|\eta\| \, \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| \end{aligned} \tag{108}$$

for all $p \in \Lambda_+^*$, $\xi \in \mathcal{F}_+^{\leq N}$. In position space, with $\check{d}_x$ defined as in (107), we find

$$\|(\mathcal{N}_+ + 1)^{n/2} \check{d}_x \xi\| \leq \frac{C}{N} \|\eta\| \Big[\|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|b_x (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| . \Big] \tag{109}$$

Furthermore, letting $\check{\bar{d}}_x = \check{d}_x + (\mathcal{N}_+/N)b^*(\check{\eta}_x)$*, we find*

$$\begin{aligned}
&\|(\mathcal{N}_+ + 1)^{n/2}\check{a}_y\check{\bar{d}}_x\xi\| \\
&\quad\leq \frac{C}{N}\Big[\,\|\eta\|^2\|(\mathcal{N}_+ + 1)^{(n+2)/2}\xi\| + \|\eta\||\check{\eta}(x-y)|\|(\mathcal{N}+1)^{(n+2)/2}\xi\| \\
&\qquad + \|\eta\|\|\check{a}_x(\mathcal{N}_+ + 1)^{(n+1)/2}\xi\| + \|\eta\|^2\|\check{a}_y(\mathcal{N}_+ + 1)^{(n+3)/2}\xi\| \\
&\qquad + \|\eta\|\|\check{a}_x\check{a}_y(\mathcal{N}+1)^{(n+2)/2}\xi\|\,\Big]
\end{aligned} \tag{110}$$

and, finally,

$$\begin{aligned}
&\|(\mathcal{N}_+ + 1)^{n/2}\check{d}_x\check{d}_y\xi\| \\
&\quad\leq \frac{C}{N^2}\Big[\,\|\eta\|^2\|(\mathcal{N}_+ + 1)^{(n+6)/2}\xi\| + \|\eta\||\check{\eta}(x-y)|\|(\mathcal{N}_+ + 1)^{(n+4)/2}\xi\| \\
&\qquad + \|\eta\|^2\|a_x(\mathcal{N}_+ + 1)^{(n+5)/2}\xi\| + \|\eta\|^2\|a_y(\mathcal{N}_+ + 1)^{(n+5)/2}\xi\| \\
&\qquad + \|\eta\|^2\,\|a_x a_y(\mathcal{N}_+ + 1)^{(n+4)/2}\xi\|\,\Big]
\end{aligned} \tag{111}$$

for all $\xi \in \mathcal{F}_+^{\leq n}$.

A first simple application of Lemma 4 is the following bound on the growth of the expectation of $\mathcal{N}_+$.

Lemma 5 *Assume B is defined as in (33), with* $\eta \in \ell^2(\Lambda^*)$ *and* $\eta_p = \eta_{-p}$ *for all* $p \in \Lambda_+^*$. *Then, there exists a constant* $C > 0$ *such that*

$$\left|\langle\xi, \left[e^{-B}\mathcal{N}_+e^B - \mathcal{N}_+\right]\xi\rangle\right| \leq \|\eta\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

for all $\xi \in \mathcal{F}_+^{\leq N}$.

Proof With (106) we write

$$\begin{aligned}
&e^{-B}\mathcal{N}_+e^B - \mathcal{N}_+ \\
&\quad= \int_0^1 e^{-sB}[\mathcal{N}_+, B]e^{sB}ds \\
&\quad= \int_0^1 \sum_{p\in\Lambda_+^*} \eta_p\, e^{-sB}(b_pb_{-p} + b_p^*b_{-p}^*)e^{sB}\,ds \\
&\quad= \int_0^1 \sum_{p\in\Lambda_+^*} \eta_p\,\Big[(\gamma_p^{(s)}b_p + \sigma_p^{(s)}b_{-p}^* + d_p^{(s)})(\gamma_p^{(s)}b_{-p} + \sigma_p^{(s)}b_{-p}^* + d_{-p}^{(s)}) + \text{h.c.}\Big]\,ds
\end{aligned}$$

with $\gamma_p^{(s)} = \cosh(s\eta_p)$, $\sigma_p^{(s)} = \sinh(s\eta_p)$. Using $|\gamma_p^{(s)}| \leq C$ and $|\sigma_p^{(s)}| \leq C|\eta_p|$, (108) in Lemma 4 we arrive at

$$\begin{aligned}
&\left|\langle\xi, \left[e^{-B}\mathcal{N}_+e^B - \mathcal{N}_+\right]\xi\rangle\right| \\
&\quad\leq C\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \sum_{p\in\Lambda_+^*} |\eta_p|\left[|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|b_p\xi\|\right] \leq C\|\eta\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2
\end{aligned}$$

□

Appendix A.2: Analysis of $\mathcal{G}^{(0)}_{N,\alpha} = e^{-B}\mathcal{L}^{(0)}_N e^B$

We define $\mathcal{E}^{(0)}_N$ so that

$$\mathcal{G}^{(0)}_{N,\alpha} = e^{-B}\mathcal{L}^{(0)}_N e^B = \frac{1}{2}\widehat{V}(0)(N+\mathcal{N}_+-1)(N-\mathcal{N}_+) + \mathcal{E}^{(0)}_{N,\alpha}\,.$$

where we recall from (13) that

$$\mathcal{L}^{(0)}_N = \frac{1}{2}\widehat{V}(0)(N-1+\mathcal{N}_+)(N-\mathcal{N}_+)\,.$$

Proposition 12 *Under the assumptions of Proposition 1, there exists a constant $C>0$ such that*

$$\pm\mathcal{E}^{(0)}_{N,\alpha} \le CN^{1-\alpha}(\mathcal{N}_+ + 1)$$

for all $\alpha > 0$ and $N \in \mathbb{N}$ large enough.

Proof The proof follows [4, Prop. 7.1].

We write

$$\mathcal{L}^{(0)}_N = \frac{N(N-1)}{2}\widehat{V}(0) + \frac{N}{2}\widehat{V}(0)\Big[\sum_{q\in\Lambda^*_+} b^*_q b_q - \mathcal{N}_+\Big].$$

Hence,

$$\mathcal{E}^{(0)}_N = \frac{N}{2}\widehat{V}(0)\sum_{q\in\Lambda^*_+}\Big[e^{-B}b^*_q b_q e^B - b^*_q b_q\Big] - \frac{N}{2}\widehat{V}(0)\Big[e^{-B}\mathcal{N}_+ e^B - \mathcal{N}_+\Big]\,.$$

To bound the first term we use (106), $|\gamma_q^2 - 1| \le C\eta_q^2$, $|\sigma_q| \le C|\eta_q|$, the first bound in (108), Cauchy–Schwarz and the estimate $\|\eta\| \le CN^{-\alpha}$. To bound the second term, we use Lemma 5. We conclude that

$$|\langle\xi, \mathcal{E}^{(0)}_N \xi\rangle| \le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

□

Appendix A.3: Analysis of $\mathcal{G}^{(2)}_{N,\alpha} = e^{-B}\mathcal{L}^{(2)}_N e^B$

We consider first conjugation of the kinetic energy operator.

Proposition 13 *Under the assumptions of Proposition 1, there exists $C>0$ such that*

$$\begin{aligned} e^{-B}\mathcal{K}e^B &= \mathcal{K} + \sum_{p\in\Lambda^*_+} p^2\eta_p(b_p b_{-p} + b^*_p b^*_{-p}) \\ &\quad + \sum_{p\in\Lambda^*_+} p^2\eta_p^2\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) + \mathcal{E}^{(K)}_{N,\alpha} \end{aligned} \tag{112}$$

where

$$|\langle\xi, \mathcal{E}^{(K)}_{N,\alpha}\xi\rangle| \le CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \tag{113}$$

for any $\alpha > 1$, $\xi \in \mathcal{F}^{\le N}_+$ and $N \in \mathbb{N}$ large enough.

Proof We proceed as in the proof of [4, Prop. 7.2]. We write

$$\begin{aligned}
&e^{-B}\mathcal{K}e^{B}-\mathcal{K}\\
&=\int_0^1 ds\sum_{p\in\Lambda_+^*}p^2\eta_p\Big[\big(\gamma_p^{(s)}b_p+\sigma_p^{(s)}b^*_{-p}\big)\big(\gamma_p^{(s)}b_{-p}+\sigma_p^{(s)}b^*_p\big)\;+\text{h.c.}\Big]\\
&\quad+\int_0^1 ds\sum_{p\in\Lambda_+^*}p^2\eta_p\big[\big(\gamma_p^{(s)}b_p+\sigma_p^{(s)}b^*_{-p}\big)d^{(s)}_{-p}+d^{(s)}_p\big(\gamma_p^{(s)}b_{-p}+\sigma_p^{(s)}b^*_p\big)+\text{h.c.}\big]\\
&\quad+\int_0^1 ds\sum_{p\in\Lambda_+^*}p^2\eta_p\big[d^{(s)}_pd^{(s)}_{-p}+\text{h.c.}\big]\\
&=:\text{G}_1+\text{G}_2+\text{G}_3
\end{aligned}\tag{114}$$

with $\gamma_p^{(s)}=\cosh(s\eta_p)$, $\sigma_p^{(s)}=\sinh(s\eta_p)$ and where $d_p^{(s)}$ is defined as in (105), with η_p replaced by $s\eta_p$. We find

$$\text{G}_1=\sum_{p\in\Lambda_+^*}p^2\eta_p\big(b_pb_{-p}+b^*_{-p}b^*_p\big)+\sum_{p\in\Lambda_+^*}p^2\eta_p^2\left(1-\frac{\mathcal{N}_+}{N}\right)+\mathcal{E}_1^K$$

with

$$\begin{aligned}
\mathcal{E}_1^K=&\,2\int_0^1 ds\sum_{p\in\Lambda_+^*}p^2\eta_p(\sigma_p^{(s)})^2\big(b_pb_{-p}+b^*_{-p}b^*_p\big)\\
&+\int_0^1 ds\sum_{p\in\Lambda_+^*}p^2\eta_p\gamma_p^{(s)}\sigma_p^{(s)}(4b^*_pb_p-2N^{-1}a^*_pa_p)\\
&+2\int_0^1 ds\sum_{p\in\Lambda_+^*}p^2\eta_p\left[(\gamma_p^{(s)}-1)\sigma_p^{(s)}+(\sigma_p^{(s)}-s\eta_p)\right]\left(1-\frac{\mathcal{N}_+}{N}\right).
\end{aligned}$$

Since $|\big((\gamma_p^{(s)})^2-1\big)|\le C\eta_p^2$, $(\sigma_p^{(s)})^2\le C\eta_p^2$, $p^2|\eta_p|\le C$, $\|\eta\|_\infty\le N^{-\alpha}$, we can estimate

$$\begin{aligned}
&|\langle\xi,\mathcal{E}_1^K\xi\rangle|\\
&\le C\sum_{p\in\Lambda_+^*}p^2|\eta_p|^3\|b_p\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|+C\sum_{p\in\Lambda_+^*}p^2\eta_p^2\|a_p\xi\|^2+C\sum_{p\in\Lambda_+^*}p^2\eta_p^4\|\xi\|^2\\
&\le C\|\eta\|\|(\mathcal{N}_++1)^{1/2}\xi\|^2\le CN^{-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2,
\end{aligned}\tag{115}$$

for any $\xi\in\mathcal{F}_+^{\le N}$. To bound the term G_3 in (114), we switch to position space:

$$\begin{aligned}
|\langle\xi,\text{G}_3\xi\rangle|\le CN\int_0^1 ds\int_{\Lambda^2}dxdy\left[e^{2N}V(e^N(x-y))+N^{2\alpha-1}\chi(|x-y|\le N^{-\alpha})\right]\\
\times\|(\mathcal{N}_++1)^{-1/2}\check{d}^{(s)}_x\check{d}^{(s)}_y\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|
\end{aligned}$$

With (111), we obtain

$$
\begin{aligned}
&|\langle \xi, \mathrm{G}_3 \xi\rangle| \\
&\leq C N^{1-\alpha} \int_{\Lambda^2} dx dy \left[e^{2N} V(e^N(x-y)) + N^{2\alpha-1} \chi(|x-y| \leq N^{-\alpha}) \right] \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \\
&\quad + C N^{-2\alpha} \int_{\Lambda^2} dx dy \left[e^{2N} V(e^N(x-y)) + N^{2\alpha-1} \chi(|x-y| \leq N^{-\alpha}) \right] \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \\
&\quad \times \left[\|\check{a}_x (\mathcal{N}_+ + 1)\xi\| + \|\check{a}_y (\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{1/2} \xi\| \right] \\
&\leq C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\| .
\end{aligned}
\tag{116}
$$

Finally, we consider G_2 in (114). We split it as $\mathrm{G}_2 = \mathrm{G}_{21} + \mathrm{G}_{22} + \mathrm{G}_{23} + \mathrm{G}_{24}$, with

$$
\begin{aligned}
\mathrm{G}_{21} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\gamma_p^{(s)} b_p d_{-p}^{(s)} + \text{h.c.} \right), \\
\mathrm{G}_{22} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\sigma_p^{(s)} b_{-p}^* d_{-p}^{(s)} + \text{h.c.} \right) \\
\mathrm{G}_{23} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\gamma_p^{(s)} d_p^{(s)} b_{-p} + \text{h.c.} \right), \\
\mathrm{G}_{24} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\sigma_p^{(s)} d_p^{(s)} b_p^* + \text{h.c.} \right) .
\end{aligned}
\tag{117}
$$

We consider G_{21} first. We write

$$
\mathrm{G}_{21} = - \sum_{p \in \Lambda_+^*} p^2 \eta_p^2 \frac{\mathcal{N}_+ + 1}{N} \frac{N - \mathcal{N}_+}{N} + \left[\mathcal{E}_2^K + \text{h.c.} \right]
$$

where $\mathcal{E}_2^K = \sum_{j=1}^3 \mathcal{E}_{2j}^K$, with

$$
\begin{aligned}
\mathcal{E}_{21}^K &= \frac{1}{2N} \sum_{p \in \Lambda_+^*} p^2 \eta_p^2 (\mathcal{N}_+ + 1) \big(b_p^* b_p - \frac{1}{N} a_p^* a_p \big), \\
\mathcal{E}_{22}^K &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p (\gamma_p^{(s)} - 1) b_p d_{-p}^{(s)} , \\
\mathcal{E}_{23}^K &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p b_p \overline{d}_{-p}^{(s)} .
\end{aligned}
\tag{118}
$$

and where we introduced the notation $\overline{d}_{-p}^{(s)} = d_{-p}^{(s)} + s \eta_p (\mathcal{N}_+/N) b_p^*$. With (29), we find

$$
|\langle \xi, \mathcal{E}_{21}^K \xi\rangle| \leq C \sum_{p \in \Lambda_+^*} \eta_p \|a_p \xi\|^2 \leq C N^{-\alpha} \|\mathcal{N}_+^{1/2} \xi\|^2 \tag{119}
$$

Using $|\gamma_p^{(s)}-1|\le C\eta_p^2$ and (108), we obtain

$$|\langle \xi, \mathcal{E}_{22}^K\xi\rangle| \le \sum_{p\in\Lambda_+^*} p^2|\eta_p|^3\|\mathcal{N}_+^{1/2}\xi\|\|d_{-p}^{(s)}\xi\| \le CN^{-3\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2. \tag{120}$$

To control the third term in (118), we use (30) and we switch to position space. We find

$$\begin{aligned}\mathcal{E}_{23}^K =& -N\int_0^1 ds\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))f_{N,\ell}(x-y)\check{b}_x\check{\bar{d}}_y^{(s)}\\ &+N\int_0^1 dse^{2N}\lambda_\ell\int_{\Lambda^2} dxdy\,\chi_\ell(x-y)f_{N,\ell}(x-y)\check{b}_x\check{\bar{d}}_y^{(s)}\\ =&;\mathcal{E}_{231}^K+\mathcal{E}_{232}^K\,.\end{aligned} \tag{121}$$

With (110) and $|\check{\eta}(x-y)|\le CN$, we obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{231}^K\xi\rangle| \le& N\int_0^1 ds\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\\ &\times\|(\mathcal{N}_++1)^{1/2}\xi\|\|(\mathcal{N}_++1)^{-1/2}\check{a}_x\check{\bar{d}}_y^{(s)}\xi\|\\ &\le CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2+CN^{1/2-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.\end{aligned} \tag{122}$$

As for $\mathcal{E}_{232}^K$, with (110) and Lemma 1 (recalling $\ell=N^{-\alpha}$), we find

$$\begin{aligned}|\langle \xi, \mathcal{E}_{232}^K\xi\rangle| \le& CN^{-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\\ &+\int_{\Lambda^2} dxdy\,\chi(|x-y|\le N^{-\alpha})\|(\mathcal{N}_++1)^{1/2}\xi\|\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\end{aligned} \tag{123}$$

To bound the last term on the r.h.s. of (123) we use Hölder's and Sobolev inequality $\|u\|_q\le Cq^{1/2}\|u\|_{H^1}$, valid for any $2\le q<\infty$. We find

$$\begin{aligned}&\int_{\Lambda^2} dxdy\,\chi(|x-y|\le N^{-\alpha})\|(\mathcal{N}_++1)^{1/2}\xi\|\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\\ &\quad\le C\|(\mathcal{N}_++1)^{1/2}\xi\|\int_\Lambda dx\left(\int_\Lambda dy\,\chi(|x-y|\le N^{-\alpha})\right)^{1-1/q}\left(\int_\Lambda dy\,\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|^q\right)^{1/q}\\ &\quad\le CN^{2\alpha/q-2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\int_\Lambda dx\left(\int_\Lambda dy\,\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|^q\right)^{1/q}\\ &\quad\le Cq^{1/2}N^{2\alpha/q-2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\\ &\quad\quad\times\left[\int_{\Lambda^2} dxdy\,\|\check{a}_x\nabla_y\check{a}_y\mathcal{N}_+^{1/2}\xi\|^2+\int_{\Lambda^2} dxdy\,\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|^2\right]^{1/2}\\ &\quad\le Cq^{1/2}N^{2\alpha/q-2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\Big[\|\mathcal{K}^{1/2}\mathcal{N}_+\xi\|+\|\mathcal{N}_+^{3/2}\xi\|\Big]\,.\end{aligned}$$

Choosing $q=\log N$, we get

$$\begin{aligned}&\int_{\Lambda^2} dxdy\,\chi(|x-y|\le N^{-\alpha})\|(\mathcal{N}_++1)^{1/2}\xi\|\|\check{a}_x\check{a}_y(\mathcal{N}_++1)^{1/2}\xi\|\\ &\quad\le CN^{1-2\alpha}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned} \tag{124}$$

Therefore, for any $\xi\in\mathcal{F}_+^{\le N}$,

$$|\langle \xi, \mathcal{E}_{232}^K\xi\rangle| \le\; N^{1-2\alpha}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|+N^{-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\,.$$

Combining the last bound with (119), (120) and (122), we conclude that

$$|\langle \xi, \mathcal{E}_2^K \xi\rangle| \le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|. \quad (125)$$

for any $\alpha > 1$, $N \in \mathbb{N}$ large enough, $\xi \in \mathcal{F}_+^{\le N}$.

The term G_{22} in (117) can be bounded using (108). We find

$$|\langle \xi, \mathrm{G}_{22}\xi\rangle| \le CN^{-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2. \quad (126)$$

We split $\mathrm{G}_{23} = \mathcal{E}_{31}^K + \mathcal{E}_{32}^K + \text{h.c.}$, with

$$\mathcal{E}_{31}^K = \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p\big(\gamma_p^{(s)} - 1\big)d_p^{(s)}b_{-p}, \quad \mathcal{E}_{32}^K = \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p d_p^{(s)}b_{-p}$$

With (108), we find

$$|\langle \xi, \mathcal{E}_{31}^K\xi\rangle| \le C\int_0^1 ds \sum_{p\in\Lambda_+^*} p^2|\eta_p|^3\|(d_p^{(s)})^*\xi\|\|b_{-p}\xi\|ds \le CN^{-3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

To estimate $\mathcal{E}_{32}^K$, we use (30) and we switch to position space. Proceeding as we did in (121), (122), (123), we obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{32}^K\xi\rangle| \le CN\int_0^1 ds \int_{\Lambda^2} dxdy\Big[e^{2N}V(e^N(x-y)) + N^{2\alpha-1}\chi(|x-y|\le N^{-\alpha})\Big]\\ \times\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}\check{d}_x^{(s)}\check{b}_y\xi\|.\end{aligned}$$

With (109) and (124) we find

$$\begin{aligned}|\langle \xi, \mathcal{E}_{32}^K\xi\rangle| &\le CN^{-\alpha}\int_{\Lambda^2} dxdy\Big[e^{2N}V(e^N(x-y)) + N^{2\alpha-1}\chi(|x-y|\le N^{-\alpha})\Big]\\ &\quad\times\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\big[\|\check{a}_y(\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x\check{a}_y(\mathcal{N}_+ + 1)^{1/2}\xi\|\big]\\ &\le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\\ &\quad + CN^{1-2\alpha}(\log N)^{1/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

Combining the bounds for $\mathcal{E}_{31}^K$ and $\mathcal{E}_{32}^K$, we conclude that, if $\alpha > 1$,

$$|\langle \xi, \mathrm{G}_{23}\xi\rangle| \le CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{H}_N^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \quad (127)$$

To bound G_{24} in (117), we use (108), the bounds (28) and $\|\eta\|_{H_1}^2 \le CN$, and the commutator (14):

$$\begin{aligned}&|\langle \xi, \mathrm{G}_{24}\xi\rangle|\\ &\le C\int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}d_p^{(s)}b_p^*\xi\|\\ &\le C\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\sum_{p\in\Lambda_+^*} p^2\eta_p^2\Big[|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N^{-1}\|\eta\|\|b_pb_p^*(\mathcal{N}_+ + 1)^{1/2}\xi\|\Big]\\ &\le CN^{-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned}$$

Together with (117), (125), (126) and (127), this implies that

$$\mathrm{G}_2 = -\sum_{p\in\Lambda_+^*} p^2\eta_p^2 \frac{\mathcal{N}_+ + 1}{N}\frac{N-\mathcal{N}_+}{N} + \mathcal{E}_4^K$$

with

$$|\langle \xi, \mathcal{E}_4^K \xi\rangle| \le CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 . \quad (128)$$

Combining (115), (116) and (128), we obtain (112) and (113). □

In the next proposition, we consider the conjugation of the operator

$$\mathcal{L}_N^{(2,V)} = N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\left[b_p^* b_p - \frac{1}{N}a_p^* a_p\right] + \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\left[b_p^* b_{-p}^* + b_p b_{-p}\right]$$

Proposition 14 *Under the assumptions of Proposition 1, there is a constant $C > 0$ such that*

$$\begin{aligned} e^{-B}\mathcal{L}_N^{(2,V)}e^B &= N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\eta_p\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+ - 1}{N}\Big) \\ &\quad + N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)a_p^* a_p\left(1-\frac{\mathcal{N}_+}{N}\right) \\ &\quad + \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\big(b_p b_{-p} + b_{-p}^* b_p^*\big) + \mathcal{E}_N^{(V)} \end{aligned} \quad (129)$$

where

$$|\langle \xi, \mathcal{E}_N^{(V)} \xi\rangle| \le CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 . \quad (130)$$

for any $\alpha > 1$, $\xi \in \mathcal{F}_+^{\le N}$ and $N \in \mathbb{N}$ large enough.

Proof We write

$$\begin{aligned} e^{-B}\mathcal{L}_N^{(2,V)}e^B &= N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)e^{-B}b_p^* b_p e^B - \sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)e^{-B}a_p^* a_p e^B \\ &\quad + \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)e^{-B}[b_p b_{-p} + b_p^* b_{-p}^*]e^B \\ &=: \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3 . \end{aligned} \quad (131)$$

With (106), we find

$$\begin{aligned} \mathrm{F}_1 &= N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\big[\gamma_p b_p^* + \sigma_p b_{-p}\big]\big[\gamma_p b_p + \sigma_p b_{-p}^*\big] \\ &\quad + N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\big[(\gamma_p b_p^* + \sigma_p b_{-p})d_p + d_p^*(\gamma_p b_p + \sigma_p b_{-p}^*) + d_p^* d_p\big] \end{aligned}$$

where $\gamma_p = \cosh\eta_p, \sigma_p = \sinh\eta_p$ and the operators d_p are defined in (105). Using $|1-\gamma_p| \le \eta_p^2$, $|\sigma_p| \le C|\eta_p|$ and using Lemma 4 for the terms on the second line, we find

$$\mathrm{F}_1 = N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)b_p^* b_p + \mathcal{E}_1^V \quad (132)$$

with $\pm\mathcal{E}_1^V \leq CN^{1-\alpha}(\mathcal{N}_+ + 1)$.

Let us now consider the second contribution on the r.h.s. of (131). We find

$$-\mathrm{F}_2 = \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)a_p^*a_p + \mathcal{E}_2^V \tag{133}$$

with

$$\mathcal{E}_2^V = \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\int_0^1 e^{-sB}(\eta_p b_{-p}b_p + \text{h.c.})e^{sB}ds.$$

With Lemma 2, we easily find $\pm\mathcal{E}_2^V \leq CN^{-\alpha}(\mathcal{N}_+ + 1)$.

Finally, we consider the last term on the r.h.s. of (131). With (106), we obtain

$$\begin{aligned}
\mathrm{F}_3 = &\frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\Big[\gamma_p b_p + \sigma_p b_{-p}^*\Big]\Big[\gamma_p b_{-p} + \sigma_p b_p^*\Big] + \text{h.c.}\\
&+\frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\Big[(\gamma_p b_p + \sigma_p b_{-p}^*)\,d_{-p} + d_p\,(\gamma_p b_{-p} + \sigma_p b_p^*)\Big] + \text{h.c.}\\
&+\frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)d_p d_{-p} + \text{h.c.}\\
=:& \,\mathrm{F}_{31} + \mathrm{F}_{32} + \mathrm{F}_{33}\,.
\end{aligned} \tag{134}$$

Using $|1-\gamma_p| \leq C\eta_p^2$, $|\sigma_p| \leq C|\eta_p|$, we obtain

$$\mathrm{F}_{31} = \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\big(b_p b_{-p} + b_{-p}^* b_p^*\big) + N\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\frac{N-\mathcal{N}_+}{N} + \mathcal{E}_3^V \tag{135}$$

with $\pm\mathcal{E}_3^V \leq CN^{1-\alpha}(\mathcal{N}_+ + 1)$. As for F_{32} in (134), we divide it into four parts

$$\begin{aligned}
\mathrm{F}_{32} = &\frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\Big[(\gamma_p b_p + \sigma_p b_{-p}^*)\,d_{-p} + d_p\,(\gamma_p b_{-p} + \sigma_p b_p^*)\Big] + \text{h.c.}\\
=:& \,\mathrm{F}_{321} + \mathrm{F}_{322} + \mathrm{F}_{323} + \mathrm{F}_{324}\,.
\end{aligned} \tag{136}$$

We start with F_{321}, which we write as

$$\mathrm{F}_{321} = -N\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{\mathcal{N}_+ + 1}{N}\right) + \mathcal{E}_4^V$$

where $\mathcal{E}_4^V = \mathcal{E}_{41}^V + \mathcal{E}_{42}^V + \mathcal{E}_{43}^V + \text{h.c.}$, with

$$\begin{aligned}
\mathcal{E}_{41}^V &= \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\,(\gamma_p - 1)b_p d_{-p}\,, \qquad \mathcal{E}_{42}^V = \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)b_p\overline{d}_{-p}\\
\mathcal{E}_{43}^V &= -\frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\frac{\mathcal{N}_+ + 1}{N}(b_p^*b_p - N^{-1}a_p^*a_p)
\end{aligned}$$

and with the notation $\overline{d}_{-p} = d_{-p} + N^{-1}\eta_p\mathcal{N}_+ b_p^*$. Since $|\gamma_p - 1| \leq C\eta_p^2$, $\|\eta\|_\infty \leq CN^{-\alpha}$, we find easily with (108) that

$$|\langle\xi, \mathcal{E}_{41}^V\xi\rangle| \leq CN^{1-3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

Moreover

$$|\langle \xi, \mathcal{E}_{43}^V \xi \rangle| \le CN \sum_{p\in\Lambda_+^*} \eta_p \|a_p\xi\|^2 \le CN^{1-\alpha}\|\mathcal{N}_+^{1/2}\xi\|^2 .$$

As for $\mathcal{E}_{42}^V$, we switch to position space and we use (110). We obtain

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{42}^V \xi \rangle| &\le CN \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{-1/2} \check{a}_x \check{\bar{d}}_y \xi\| \\
&\le CN^{1-\alpha} \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\quad \times \Big[\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x\xi\| + \|\check{a}_y\xi\| + N^{-1/2}\|\check{a}_x\check{a}_y\xi\| \Big] \\
&\le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| .
\end{aligned}$$

We conclude that

$$|\langle \xi, \mathcal{E}_4^V \xi \rangle| \le CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.$$

To bound the term F_{322} in (136), we use (108) and $|\sigma_p| \le C|\eta_p|$; we obtain

$$\begin{aligned}
|\langle \xi, \mathrm{F}_{322}\xi \rangle| &\le CN \sum_{p\in\Lambda_+^*} |\eta_p| \|b_{-p}\xi\| \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta\| \|b_{-p}\xi\| \right] \\
&\le CN^{1-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 .
\end{aligned}$$

Let us now consider the term F_{323} on the r.h.s. of (136). We write $\mathrm{F}_{323} = \mathcal{E}_{51}^V + \mathcal{E}_{52}^V + \text{h.c.}$, with

$$\mathcal{E}_{51}^V = \frac{N}{2} \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\,(\gamma_p - 1)\, d_p b_{-p} , \qquad \mathcal{E}_{52}^V = \frac{N}{2} \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\, d_p b_{-p} .$$

With $|\gamma_p - 1| \le C\eta_p^2$ and (108) we obtain

$$|\langle \xi, \mathcal{E}_{51}^V \xi \rangle| \le CN \sum_{p\in\Lambda_+^*} \eta_p^2 \|d_p^*\xi\| \|a_p\xi\| \le CN^{1-3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 .$$

We find, switching to position space and using (109),

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{52}^V \xi \rangle| \\
&\le CN \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{-1/2} \check{d}_x \check{a}_y \xi\| \\
&\le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \left[\|\check{a}_y\xi\| + N^{-1/2}\|\check{a}_x\check{a}_y\xi\| \right] \\
&\le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| .
\end{aligned}$$

Hence,

$$|\langle \xi, \mathrm{F}_{323}\xi \rangle| \le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|$$

To estimate the term F_{324} in (136) we use (108) and the bound

$$\begin{aligned}\sum_{p\in\Lambda_+^*} |\widehat{V}(p/e^N)||\eta_p| &\leq C \sum_{p\in\Lambda_+^*,\,|p|\leq e^N} \frac{1}{p^2} + C \sum_{p\in\Lambda_+^*,\,|p|>e^N} \frac{|\widehat{V}(p/e^N)|}{p^2} \\ &\leq CN + C\Big(\sum_{p\in\Lambda_+^*} |\widehat{V}(p/e^N)|^2\Big)^{1/2}\Big(\sum_{p\in\Lambda_+^*,\,|p|>e^N} \frac{1}{p^4}\Big)^{1/2} \\ &\leq CN\end{aligned}$$

We find

$$\begin{aligned}|\langle \xi, \mathrm{F}_{324}\xi\rangle| \leq\; & CN \sum_{p\in\Lambda_+^*} \big|\widehat{V}(p/e^N)\big||\eta_p|\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2} d_p\, b_p^*\xi\| \\ \leq\; & CN \sum_{p\in\Lambda_+^*} \big|\widehat{V}(p/e^N)\big||\eta_p|\|(\mathcal{N}_+ +1)^{1/2}\xi\| \\ & \times\Big[|\eta_p|\|(\mathcal{N}_+ +1)^{1/2}\xi\| + N^{-1}\|\eta\|\|b_p b_p^*(\mathcal{N}_+ +1)^{1/2}\xi\|\Big] \\ \leq\; & CN \sum_{p\in\Lambda_+^*} \big|\widehat{V}(p/e^N)\big||\eta_p|\|(\mathcal{N}_+ +1)^{1/2}\xi\| \\ & \times\big[|\eta_p|\|(\mathcal{N}_+ +1)^{1/2}\xi\| + N^{-1}\|\eta\|\|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta\|\|a_p\xi\|\big] \\ \leq\; & CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2\,.\end{aligned}$$

Combining the last bounds, we arrive at

$$\mathrm{F}_{32} = N\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{-\mathcal{N}_+ -1}{N}\right) + \mathcal{E}_6^V$$

with

$$|\langle \xi, \mathcal{E}_6^V\xi\rangle| \leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,. \quad (137)$$

To control the last contribution F_{33} in (134), we switch to position space. With (111) and (25) we obtain

$$\begin{aligned}|\langle \xi, \mathrm{F}_{33}\xi\rangle| &\leq CN\|(\mathcal{N}_+ +1)^{1/2}\xi\|\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x\check{d}_y\xi\| \\ &\leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-2\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,.\end{aligned}$$

The last equation, combined with (134), (135) and (137), implies that

$$\begin{aligned}\mathrm{F}_3 =\; & \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)(b_p b_{-p} + b_{-p}^* b_p^*) \\ & + N\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{N-\mathcal{N}_+ -1}{N}\right) + \mathcal{E}_7^V\end{aligned}$$

with

$$|\langle \xi, \mathcal{E}_7^V\xi\rangle| \leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,.$$

Together with (132) and with (133), and recalling that $b_p^* b_p - N^{-1}a_p^* a_p = a_p^* a_p(1-\mathcal{N}_+/N)$, we obtain (129) with (130). □

Appendix A.4: Analysis of $\mathcal{G}_{N,\alpha}^{(3)} = e^{-B}\mathcal{L}_N^{(3)}e^B$

We consider here the conjugation of the cubic term $\mathcal{L}_N^{(3)}$, defined in (13).

Proposition 15 *Under the assumptions of Proposition 1, there exists a constant $C > 0$ such that*

$$\mathcal{G}_{N,\alpha}^{(3)} = e^{-B}\mathcal{L}_N^{(3)}e^B = \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\left[b_{p+q}^* a_{-p}^* a_q + h.c.\right] + \mathcal{E}_N^{(3)}$$

where

$$|\langle\xi, \mathcal{E}_N^{(3)}\xi\rangle| \leq CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \quad (138)$$

for any $\alpha > 1$ and $N \in \mathbb{N}$ large enough.

Proof This proof is similar to the proof of [4, Prop. 7.5]. Expanding $e^{-B}a_{-p}^* a_q e^B$, we arrive at

$$\begin{aligned}
\mathcal{E}_N^{(3)} &= \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\big((\gamma_{p+q}-1)b_{p+q}^* + \sigma_{p+q}b_{-p-q} + d_{p+q}^*\big)\, a_{-p}^* a_q \\
&\quad + \sqrt{N}\sum_{p,q\in\Lambda_+^*,p+q\neq 0}\widehat{V}(p/e^N)\eta_p\, e^{-B}b_{p+q}^* e^B\int_0^1 ds\, e^{-sB}b_p b_q e^{sB} \\
&\quad + \sqrt{N}\sum_{p,q\in\Lambda_+^*,p+q\neq 0}\widehat{V}(p/e^N)\eta_q\, e^{-B}b_{p+q}^* e^B\int_0^1 ds\, e^{-sB}b_{-p}^* b_{-q}^* e^{sB} \\
&\quad + \text{h.c.} \\
&=: \mathcal{E}_1^{(3)} + \mathcal{E}_2^{(3)} + \mathcal{E}_3^{(3)} + \text{h.c.}
\end{aligned} \quad (139)$$

where, as usual, $\gamma_p = \cosh\eta(p)$, $\sigma_p = \sinh\eta(p)$ and d_p is as in (105). We consider $\mathcal{E}_1^{(3)}$. To this end, we write

$$\begin{aligned}
\mathcal{E}_1^{(3)} &= \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\big((\gamma_{p+q}-1)b_{p+q}^* + \sigma_{p+q}b_{-p-q} + d_{p+q}^*\big)\, a_{-p}^* a_q \\
&=: \mathcal{E}_{11}^{(3)} + \mathcal{E}_{12}^{(3)} + \mathcal{E}_{13}^{(3)}\,.
\end{aligned}$$

Since $|\gamma_{p+q} - 1| \leq |\eta_{p+q}|^2$ and $\|\eta\| \leq CN^{-\alpha}$, we find

$$|\langle\xi, \mathcal{E}_{11}^{(3)}\xi\rangle| \leq CN\|\eta\|^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq CN^{1-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,. \quad (140)$$

As for $\mathcal{E}_{12}^{(3)}$, we commute a_{-p}^* through b_{-p-q} (recall $q \neq 0$). With $|\sigma_{p+q}| \leq C|\eta_{p+q}|$, we obtain

$$|\langle\xi, \mathcal{E}_{12}^{(3)}\xi\rangle| \leq CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,. \quad (141)$$

We decompose now $\mathcal{E}_{13}^{(3)} = \mathcal{E}_{131}^{(3)} + \mathcal{E}_{132}^{(3)}$, with

$$\begin{aligned}
\mathcal{E}_{131}^{(3)} &= \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\,\bar{d}_{p+q}^* a_{-p}^* a_q \\
\mathcal{E}_{132}^{(3)} &= -\frac{(\mathcal{N}_+ + 1)}{N}\sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\eta_{p+q}\, b_{-p-q}a_{-p}^* a_q\,.
\end{aligned}$$

where we defined $d^*_{p+q} = \overline{d}^*_{p+q} - \frac{(\mathcal{N}_++1)}{N}\eta_{p+q}b_{-p-q}$. The term $\mathcal{E}^{(3)}_{132}$ is estimated similarly to $\mathcal{E}^{(3)}_{12}$, moving a^*_{-p} to the left of b_{-p-q}; we find $\pm\mathcal{E}^{(3)}_{132} \le CN^{1-\alpha}(\mathcal{N}_+ + 1)$. We bound $\mathcal{E}^{(3)}_{131}$ in position space. We find

$$\begin{aligned}
&|\langle\xi, \mathcal{E}^{(3)}_{131}\xi\rangle| \\
&\quad\le N^{1/2}\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|\check{a}_x\xi\|\|\check{a}_y\check{\overline{d}}_x\xi\| \\
&\quad\le CN^{1/2-\alpha}\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|\check{a}_x\xi\| \\
&\qquad\times\left[\,\|(\mathcal{N}_++1)\xi\| + N^{-1}\|\check{a}_x(\mathcal{N}_++1)^{1/2}\xi\| + \|\eta\|\|\check{a}_y(\mathcal{N}_++1)^{1/2}\xi\| + \|\check{a}_x\check{a}_y\xi\|\right] \\
&\quad\le CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,.
\end{aligned}$$

With (140) and (141) we obtain

$$|\langle\xi, \mathcal{E}^{(3)}_1\xi\rangle| \le CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\,. \tag{142}$$

Next, we focus on $\mathcal{E}^{(3)}_2$, defined in (139). With Eq. (106), we find

$$\begin{aligned}
\mathcal{E}^{(3)}_2 &= \sqrt{N}\sum_{p,q\in\Lambda^*_+,\,p+q\neq 0}\widehat{V}(p/e^N)\eta_p\, e^{-B}b^*_{p+q}e^B \\
&\qquad\times\int_0^1 ds\,\big(\gamma^{(s)}_p\gamma^{(s)}_q b_p b_q + \sigma^{(s)}_p\sigma^{(s)}_q b^*_{-p}b^*_{-q} + \gamma^{(s)}_p\sigma^{(s)}_q b^*_{-q}b_p + \sigma^{(s)}_p\gamma^{(s)}_q b^*_{-p}b_q\big) \\
&\quad + \sqrt{N}\sum_{p,q\in\Lambda^*_+,\,p+q\neq 0}\widehat{V}(p/e^N)\eta_p\, e^{-B}b^*_{p+q}e^B\int_0^1 ds\,\gamma^{(s)}_p\sigma^{(s)}_q[b_p, b^*_{-q}] \\
&\quad + \sqrt{N}\sum_{p,q\in\Lambda^*_+,\,p+q\neq 0}\widehat{V}(p/e^N)\eta_p\, e^{-B}b^*_{p+q}e^B \\
&\qquad\times\int_0^1 ds\,\Big[d^{(s)}_p\big(\gamma^{(s)}_q b_q + \sigma^{(s)}_q b^*_{-q}\big) + \big(\gamma^{(s)}_p b_p + \sigma^{(s)}_p b^*_{-p}\big)d^{(s)}_q + d^{(s)}_p d^{(s)}_q\Big] \\
&=: \mathcal{E}^{(3)}_{21} + \mathcal{E}^{(3)}_{22} + \mathcal{E}^{(3)}_{23}
\end{aligned} \tag{143}$$

with $\gamma^{(s)}_p = \cosh(s\eta_p)$, $\sigma^{(s)}_p = \sinh(s\eta_p)$ and $d^{(s)}_p$ defined as in (105), with η replaced by $s\eta$. With Lemma 2, we get

$$|\langle\xi, \mathcal{E}^{(3)}_{21}\xi\rangle| \le CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\,. \tag{144}$$

Since $[b_p, b^*_{-q}] = -a^*_{-q}a_p/N$ for $p \neq -q$, we find

$$|\langle\xi, \mathcal{E}^{(3)}_{22}\xi\rangle| \le CN^{-2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\,. \tag{145}$$

As for the third term on the r.h.s. of (143), we switch to position space. We find

$$\begin{aligned}
\mathcal{E}^{(3)}_{23} &= \sqrt{N}\int_{\Lambda^3} dxdydz\, e^{2N}V(e^N(x-z))\check{\eta}(y-z)\, e^{-B}\check{b}^*_x e^B \\
&\qquad\times\int_0^1 ds\,\Big[\check{d}^{(s)}_y\big(b(\check{\gamma}^{(s)}_x) + b^*(\check{\sigma}^{(s)}_x)\big) + \big(b(\check{\gamma}^{(s)}_y) + b^*(\check{\sigma}^{(s)}_y)\big)\check{d}^{(s)}_x + \check{d}^{(s)}_y\check{d}^{(s)}_x\Big].
\end{aligned}$$

Using the bounds (109), (110), (111) and Lemma 2 we arrive at

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{23}^{(3)} \xi \rangle| \\
&\leq C\sqrt{N} \int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-z)) |\check{\eta}(y-z)| \|\check{b}_x e^B \xi\| \int_0^1 ds \\
&\quad \times \Big[\|\check{d}_y^{(s)} \big(\check{b}_x + b(\check{r}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)})\big)\xi\| + \|\big(\check{b}_y + b(\check{r}_y^{(s)}) + b^*(\check{\sigma}_y^{(s)})\big)\check{d}_x^{(s)}\xi\| + \|\check{d}_x^{(s)}\check{d}_y^{(s)}\xi\| \Big] \\
&\leq C\sqrt{N} \int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-z)) |\check{\eta}(y-z)| \|\check{b}_x e^B \xi\| \Big[N^{-1} |\check{\eta}(x-y)| \|(\mathcal{N}_+ + 1)\xi\| \\
&\quad + \|\eta\| \|\check{b}_x \check{b}_y \xi\| + \|\eta\| \|(\mathcal{N}_+ + 1)\xi\| + \|\eta\| \|\check{b}_x (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta\| \|\check{b}_y (\mathcal{N}_+ + 1)^{1/2}\xi\| \Big] \\
&\leq C N^{1-\alpha} \|\mathcal{N}_+^{1/2} e^B \xi\| \|(\mathcal{N}_+ + 1)\xi\| \\
&\leq C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2
\end{aligned}$$

where $\check{r}$ indicates the function in $L^2(\Lambda)$ with Fourier coefficients $r_p = 1 - \gamma_p$, and the fact that $\|\check{\eta}\|, \|\check{r}\|, \|\check{\sigma}\| \leq CN^{-\alpha}$. Combined with (144) and (145), the last bound implies that

$$\pm \mathcal{E}_2^{(3)} \leq C N^{1-\alpha} (\mathcal{N}_+ + 1)\,. \tag{146}$$

To bound the last contribution on the r.h.s. of (139), it is convenient to bound (in absolute value) the expectation of its adjoint

$$\begin{aligned}
\mathcal{E}_3^{(3)*} &= \sqrt{N} \sum_{p,q \in \Lambda_+^*, p+q \neq 0} \widehat{V}(p/e^N) \eta_q \int_0^1 ds\, e^{-sB} b_{-q} e^{sB} \\
&\qquad \times \big(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b_p^* + d_{-p}^{(s)}\big)\big(\gamma_{p+q} b_{p+q} + \sigma_{p+q} b_{-p-q}^* + d_{p+q}\big) \\
&= \sqrt{N} \sum_{p,q \in \Lambda_+^*, p+q \neq 0} \widehat{V}(p/e^N) \eta_q \int_0^1 ds\, e^{-sB} b_{-q} e^{sB} \\
&\quad \times \Big[\gamma_p^{(s)} \gamma_{p+q} b_{-p} b_{p+q} + \sigma_p^{(s)} \sigma_{p+q} b_p^* b_{-p-q}^* + \gamma_p^{(s)} \sigma_{p+q} b_{-p-q}^* b_{-p} + \gamma_{p+q} \sigma_p^{(s)} b_p^* b_{p+q} \\
&\quad + d_{-p}^{(s)} \big(\gamma_{p+q} b_{p+q} + \sigma_{p+q} b_{-p-q}^*\big) + \big(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b_p^*\big) d_{p+q} + d_{-p}^{(s)} d_{p+q} \Big] \\
&\quad + \sqrt{N} \sum_{p,q \in \Lambda_+^*, p+q \neq 0} \widehat{V}(p/e^N) \eta_q \int_0^1 ds\, e^{-sB} b_{-q} e^{sB} \gamma_p^{(s)} \sigma_{p+q} [b_{-p}, b_{-p-q}^*] \\
&=: \mathcal{E}_{31}^{(3)} + \mathcal{E}_{32}^{(3)}\,.
\end{aligned}$$

Since $q \neq 0$, $[b_{-p}, b_{-p-q}^*] = -a_{-p-q}^* a_{-p}/N$. Thus, we can estimate

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{32}^{(3)} \xi \rangle| \\
&\quad \leq C N^{-1/2} \int_0^1 ds \sum_{p,q \in \Lambda_+^*, p+q \neq 0} |\eta_q| |\eta_{p+q}|\, \|a_{-p-q}\, e^{-sB} b_{-q}^* e^{sB} \xi\| \|a_{-p}\xi\| \\
&\quad \leq C \|\eta\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq C N^{-2\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.
\end{aligned} \tag{147}$$

To bound the expectation of $\mathcal{E}_{31}^{(3)}$, we switch to position space. We find

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{31}^{(3)} \xi \rangle| \\
&\quad \leq N^{1/2} \int_0^1 ds \int_{\Lambda^2} dx dy\, e^{2N} V(e^N(x-y)) \| b^*(\check{\eta}_x) e^{sB} \xi \| \Big[\| \check{b}_x \check{b}_y \xi \| \\
&\qquad + \|\eta\| \| \check{b}_x (\mathcal{N}_+ + 1)^{1/2} \xi \| + \|\eta\| \| \check{b}_y (\mathcal{N}_+ + 1)^{1/2} \xi \| + N^{-1} |\check{\eta}(x-y)| \| (\mathcal{N}_+ + 1) \xi \| \Big].
\end{aligned}$$

With Lemma 2, we conclude that

$$|\langle \xi, \mathcal{E}_{31}^{(3)} \xi \rangle| \leq C N^{1/2-\alpha} \| \mathcal{V}_N^{1/2} \xi \| \| (\mathcal{N}_+ + 1)^{1/2} \xi \| + C N^{1-\alpha} \| (\mathcal{N}_+ + 1)^{1/2} \xi \|^2 . \quad (148)$$

From (147) and (148) we obtain

$$|\langle \xi, \mathcal{E}_{3}^{(3)} \xi \rangle| \leq C N^{1/2-\alpha} \| \mathcal{V}_N^{1/2} \xi \| \| (\mathcal{N}_+ + 1)^{1/2} \xi \| + C N^{1-\alpha} \| (\mathcal{N}_+ + 1)^{1/2} \xi \|^2 .$$

Together with (139), (142) and (146), we arrive at (138). □

Appendix A.5: Analysis of $\mathcal{G}_{N,\alpha}^{(4)} = e^{-B} \mathcal{L}_N^{(4)} e^B$

Finally, we consider the conjugation of the quartic term $\mathcal{L}_N^{(4)}$. We define the error operator $\mathcal{E}_N^{(4)}$ through

$$\begin{aligned}
\mathcal{G}_{N,\alpha}^{(4)} = e^{-B} \mathcal{L}_N^{(4)} e^B = \mathcal{V}_N &+ \frac{1}{2} \sum_{\substack{q \in \Lambda_+^*, r \in \Lambda^* \\ r \neq -q}} \widehat{V}(r/e^N) \eta_{q+r} \eta_q \left(1 - \frac{\mathcal{N}_+}{N}\right) \left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\
&+ \frac{1}{2} \sum_{\substack{q \in \Lambda_+^*, r \in \Lambda^* : \\ r \neq -q}} \widehat{V}(r/e^N)\, \eta_{q+r} \left(b_q b_{-q} + b_q^* b_{-q}^* \right) + \mathcal{E}_N^{(4)}
\end{aligned}$$

Proposition 16 *Under the assumptions of Proposition 1 there exists a constant $C > 0$ such that*

$$|\langle \xi, \mathcal{E}_N^{(4)} \xi \rangle| \leq C N^{1/2-\alpha} \| \mathcal{V}_N^{1/2} \xi \| \| (\mathcal{N}_+ + 1)^{1/2} \xi \| + C N^{1-\alpha} \| (\mathcal{N}_+ + 1)^{1/2} \xi \|^2 \quad (149)$$

for any $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$ and $N \in \mathbb{N}$ large enough.

To show Proposition 16, we use the following lemma, whose proof can be obtained as in [4, Lemma 7.7].

Lemma 6 *Let $\eta \in \ell^2(\Lambda^*)$ as defined in (27). Then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\| (\mathcal{N}_+ + 1)^{n/2} e^{-B} \check{b}_x \check{b}_y e^B \xi \| \\
&\qquad \leq C \Big[N \| (\mathcal{N}_+ + 1)^{n/2} \xi \| + \| \check{a}_y (\mathcal{N}_+ + 1)^{(n+1)/2} \xi \| \\
&\qquad\quad + \| \check{a}_x (\mathcal{N}_+ + 1)^{(n+1)/2} \xi \| + \| \check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{n/2} \xi \| \Big]
\end{aligned}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$, $n \in \mathbb{Z}$.

Proof of Proposition 16 We follow the proof of [4, Prop. 7.6]. We write

$$\mathcal{G}_{N,\alpha}^{(4)} = \mathcal{V}_N + W_1 + W_2 + W_3 + W_4$$

with

$$\begin{aligned}
\mathrm{W}_1 &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds \left(e^{-sB} b_q b_{-q}\, e^{sB} + \text{h.c.}\right) \\
\mathrm{W}_2 &= \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq p,-q} \widehat{V}(r/e^N)\, \eta_{q+r} \int_0^1 ds \left(e^{-sB} b_{p+r}^* b_q^* e^{sB} a_{-q-r}^* a_p + \text{h.c.}\right) \\
\mathrm{W}_3 &= \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p-q} \widehat{V}(r/e^N)\eta_{q+r}\eta_p \\
&\quad \times \int_0^1 ds \int_0^s d\tau \left(e^{-sB} b_{p+r}^* b_q^* e^{sB} e^{-\tau B} b_{-p}^* b_{-q-r}^* e^{\tau B} + \text{h.c.}\right) \\
\mathrm{W}_4 &= \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p-q} \widehat{V}(r/e^N)\, \eta_{q+r}^2 \\
&\quad \times \int_0^1 ds \int_0^s d\tau \left(e^{-sB} b_{p+r}^* b_q^* e^{sB} e^{-\tau B} b_p b_{q+r} e^{\tau B} + \text{h.c.}\right).
\end{aligned} \tag{150}$$

Let us first consider the term W_1. With (106), we find

$$\begin{aligned}
\mathrm{W}_1 &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds (\gamma_q^{(s)})^2 (b_q b_{-q} + \text{h.c.}) \\
&\quad + \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, \gamma_q^{(s)} \sigma_q^{(s)} \left([b_q, b_q^*] + \text{h.c.}\right) \\
&\quad + \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, \gamma_q^{(s)} \left(b_q d_{-q}^{(s)} + \text{h.c.}\right) + \mathcal{E}_{10}^{(4)} \\
&=: \mathrm{W}_{11} + \mathrm{W}_{12} + \mathrm{W}_{13} + \mathcal{E}_{10}^{(4)}
\end{aligned} \tag{151}$$

where

$$\mathcal{E}_{10}^{(4)} = \mathcal{E}_{101}^{(4)} + \mathcal{E}_{102}^{(4)} + \mathcal{E}_{103}^{(4)} + \mathcal{E}_{104}^{(4)} + \mathcal{E}_{105}^{(4)} \tag{152}$$

with

$$\begin{aligned}
\mathcal{E}_{101}^{(4)} &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds \Big[2\gamma_q^{(s)}\sigma_q^{(s)} b_q^* b_q + (\sigma_q^{(s)})^2 b_{-q}^* b_q^* + \text{h.c.}\Big] \\
\mathcal{E}_{102}^{(4)} &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, \sigma_q^{(s)} \left(b_{-q}^* d_{-q}^{(s)} + \text{h.c.}\right) \\
\mathcal{E}_{103}^{(4)} &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, \sigma_q^{(s)} \left(d_q^{(s)} b_q^* + \text{h.c.}\right) \\
\mathcal{E}_{104}^{(4)} &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, \gamma_q^{(s)} \left(d_q^{(s)} b_{-q} + \text{h.c.}\right)
\end{aligned}$$

$$\mathcal{E}^{(4)}_{105} = \frac{1}{2} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds \big(d^{(s)}_q d^{(s)}_{-q} + \text{h.c.}\big)\,. \tag{153}$$

With

$$\frac{1}{N} \sup_{q\in\Lambda^*_+} \sum_{r\in\Lambda^*_+} |\widehat{V}(r/e^N)||\eta_{q+r}| \leq C \;<\infty \tag{154}$$

uniformly in $N\in\mathbb{N}$, we can estimate the first term in (153) by

$$|\langle \xi, \mathcal{E}^{(4)}_{101}\xi\rangle| \leq CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

Using (154) and (108) we also find

$$|\langle \xi, \mathcal{E}^{(4)}_{102}\xi\rangle| \leq CN^{1-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

For the third term in (153) we switch to position space and use (109):

$$\begin{aligned}
|\langle \xi, \mathcal{E}^{(4)}_{103}\xi\rangle| &\leq \frac{1}{2}\int dxdy e^{2N} V(e^N(x-y))|\check{\eta}(x-y)| \\
&\quad\times \int_0^1 ds\, \|(\mathcal{N}+1)^{-1/2}\check{d}_y b^*(\check{\sigma}^{(s)}_x)\xi\|\|(\mathcal{N}+1)^{1/2}\xi\| \\
&\leq C\|\check{\eta}\|_\infty\|\eta\| \int dxdy e^{2N} V(e^N(x-y))\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_0^1 ds \\
&\quad\times \Big[\|b^*(\check{\sigma}^{(s)}_x)\xi\| + \frac{1}{N}|\check{\eta}^{(s)}(x-y)|\|(\mathcal{N}+1)^{1/2}\xi\| + \frac{1}{\sqrt{N}}\|b^*(\check{\sigma}^{(s)}_x)\check{b}_y\xi\|\Big] \\
&\leq CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.
\end{aligned}$$

Consider now the fourth term in (153). We write $\mathcal{E}^{(4)}_{104} = \mathcal{E}^{(4)}_{1041} + \mathcal{E}^{(4)}_{1042}$, with

$$\begin{aligned}
\mathcal{E}^{(4)}_{1041} &= \frac{1}{2} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, (\gamma^{(s)}_q - 1)d^{(s)}_q b_{-q} \\
\mathcal{E}^{(4)}_{1042} &= \frac{1}{2} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, d^{(s)}_q b_{-q}
\end{aligned}$$

With $|\gamma^{(s)}_q - 1| \leq C|\eta_q|^2$, (154) and $\|d^*_q\xi\| \leq C\|\eta\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|$, we find

$$|\langle \xi, \mathcal{E}^{(4)}_{1041}\xi\rangle| \leq CN^{1-3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

As for $\mathcal{E}_{1042}^{(4)}$, we switch to position space. Using (25) and (109), we obtain

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{1042}^{(4)} \xi \rangle| \\
&\quad = \Big| \frac{1}{2} \int_0^1 ds \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \check{\eta}(x-y) \langle \xi, \check{d}_x^{(s)} \check{b}_y \xi \rangle \Big| \\
&\quad \leq CN \int_0^1 ds \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{-1/2} \check{d}_x^{(s)} \check{b}_y \xi\| \\
&\quad \leq CN \|\eta\| \int_0^1 ds \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \\
&\qquad \times N^{-1} \Big[\|\check{a}_y \mathcal{N}_+ \xi\| + \|\check{a}_x \check{a}_y \mathcal{N}_+^{1/2} \xi\| \Big] \\
&\quad \leq CN^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + CN^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|
\end{aligned}$$

Let us consider the last term in (153). Switching to position space and using (111) in Lemma 4 and again (25), we arrive at

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{105}^{(4)} \xi \rangle| \\
&\leq CN \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \int_0^1 ds \|(\mathcal{N}_+ + 1)^{-1/2} \check{d}_x^{(s)} \check{d}_y^{(s)} \xi\| \\
&\leq CN \|\eta\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \\
&\quad \times \big[\|(\mathcal{N}_+ + 1)^{1/2} \xi\| + \|\eta\| \|\check{a}_x \xi\| + \|\eta\| \|\check{a}_y \xi\| + N^{-1/2} \|\eta\| \|\check{a}_x \check{a}_y \xi\| \big] \\
&\leq CN^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + CN^{1/2-2\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\| .
\end{aligned}$$

Summarizing, we have shown that (152) can be bounded by

$$|\langle \xi, \mathcal{E}_{10}^{(4)} \xi \rangle| \leq CN^{1/2-\alpha} \|\mathcal{V}_N^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| + CN^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \tag{155}$$

for any $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$. Next, we come back to the terms $\mathrm{W}_{11}, \mathrm{W}_{12}, \mathrm{W}_{13}$ introduced in (151). Using (154) and $|\gamma_q^{(s)} - 1| \leq C\eta_q^2$, we can write

$$\mathrm{W}_{11} = \frac{1}{2} \sum_{q \in \Lambda_+^*, r \in \Lambda^* : r \neq -q} \widehat{V}(r/e^N) \eta_{q+r} (b_q b_{-q} + \text{h.c.}) + \mathcal{E}_{11}^{(4)} , \tag{156}$$

where $\mathcal{E}_{11}^{(4)}$ is such that

$$|\langle \xi, \mathcal{E}_{11}^{(4)} \xi \rangle| \leq CN^{1-2\alpha} \|(\mathcal{N}_+ + 1) \xi\|^2 .$$

Next, we can decompose the second term in (151) as

$$\mathrm{W}_{12} = \frac{1}{2} \sum_{q \in \Lambda_+^*, r \in \Lambda^* : r \neq -q} \widehat{V}(r/e^N) \eta_{q+r} \eta_q \left(1 - \frac{\mathcal{N}_+}{N} \right) + \mathcal{E}_{12}^{(4)} \tag{157}$$

where $\pm \mathcal{E}_{12}^{(4)} \leq CN^{-\alpha} \mathcal{N}_+ + N^{1-3\alpha}$.

The third term on the r.h.s. of (151) can be written as

$$\mathrm{W}_{13} = -\frac{1}{2} \sum_{q \in \Lambda_+^*, r \in \Lambda^* : r \neq -q} \widehat{V}(r/e^N) \eta_{q+r} \eta_q \left(1 - \frac{\mathcal{N}_+}{N} \right) \frac{\mathcal{N}_+ + 1}{N} + \mathcal{E}_{13}^{(4)} \tag{158}$$

where $\mathcal{E}_{13}^{(4)} = \mathcal{E}_{131}^{(4)} + \mathcal{E}_{132}^{(4)} + \mathcal{E}_{133}^{(4)} + \mathcal{E}_{134}^{(4)}$, with

$$\begin{aligned}
\mathcal{E}_{131}^{(4)} &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, (\gamma_q^{(s)} - 1) b_q d_{-q}^{(s)} + \text{h.c.} \\
\mathcal{E}_{132}^{(4)} &= \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \int_0^1 ds\, b_q \left[d_{-q}^{(s)} + s\eta_q \frac{\mathcal{N}_+}{N} b_q^* \right] + \text{h.c.} \\
\mathcal{E}_{133}^{(4)} &= -\frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q b_q^* b_q \frac{\mathcal{N}_+ + 1}{N} \\
\mathcal{E}_{134}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q a_q^* a_q \frac{\mathcal{N}_+ + 1}{N}\,.
\end{aligned}$$

With (154), we immediately find

$$\pm\mathcal{E}_{133}^{(4)} \leq CN^{1-\alpha}(\mathcal{N}_+ + 1), \qquad \pm\mathcal{E}_{134}^{(4)} \leq CN^{-\alpha}(\mathcal{N}_+ + 1)\,.$$

With $|\gamma_q^{(s)} - 1| \leq C\eta_q^2$, Lemma 4 and, again, (154), we also obtain

$$|\langle \xi, \mathcal{E}_{131}^{(4)} \xi \rangle| \leq CN^{1-3\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

Let us now consider $\mathcal{E}_{132}^{(4)}$. In position space, with $\check{\bar{d}}_y^{(s)} = d_y^{(s)} + (\mathcal{N}_+/N) b^*(\check{\eta}_y)$ and using (110), we obtain

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{132}^{(4)} \xi \rangle| &= \left| \frac{1}{2} \int_0^1 ds \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \check{\eta}(x-y) \langle \xi, \check{b}_x \check{\bar{d}}_y^{(s)} \xi \rangle \right| \\
&\leq CN^{1-\alpha} \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\quad \times \left[\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_y\xi\| + \|\check{a}_x\xi\| + N^{-1}\|\check{a}_x\check{a}_y \mathcal{N}_+^{1/2}\xi\| \right] \\
&\leq CN^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|\,.
\end{aligned}$$

It follows that

$$|\langle \xi, \mathcal{E}_{13}^{(4)} \rangle| \leq CN^{1/2-\alpha} \|\mathcal{V}_N^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.$$

With (155), (156), (157), (158), we obtain

$$\begin{aligned}
\mathrm{W}_1 = &\frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r} \big(b_q b_{-q} + \text{h.c.} \big) \\
&+ \frac{1}{2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q \left(1 - \frac{\mathcal{N}_+}{N} \right) \left(1 - \frac{\mathcal{N}_+ + 1}{N} \right) + \mathcal{E}_1^{(4)}
\end{aligned} \tag{159}$$

where

$$|\langle \xi, \mathcal{E}_1^{(4)} \xi \rangle| \leq CN^{1/2-\alpha} \|\mathcal{V}_N^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,,$$

Next, we control the term W_2, from (150). In position space, we find

$$\mathrm{W}_2 = \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \int_0^1 ds \big(e^{-sB} \check{b}_x^* \check{b}_y^* e^{sB} a^*(\check{\eta}_x) \check{a}_y + \text{h.c.} \big)$$

with $\check{\eta}_x(z) = \check{\eta}(x-z)$. By Cauchy–Schwarz, we have

$$|\langle \xi, \mathrm{W}_2 \xi\rangle| \leq \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \int_0^1 ds \\ \times \|(\mathcal{N}_+ + 1)^{1/2} e^{-sB} \check{b}_x \check{b}_y e^{sB} \xi\| \|(\mathcal{N}_+ + 1)^{-1/2} a^*(\check{\eta}_x) \check{a}_y \xi\| .$$

With

$$\|(\mathcal{N}_+ + 1)^{-1/2} a^*(\check{\eta}_x) \check{a}_y \xi\| \leq C\|\eta\| \|\check{a}_y \xi\| \leq C N^{-\alpha} \|\check{a}_y \xi\|$$

and using Lemma 6, we obtain

$$\begin{aligned} |\langle \xi, \mathrm{W}_2 \xi\rangle| &\leq C N^{-\alpha} \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|\check{a}_y \xi\| \\ &\quad \times \Big\{ N \|(\mathcal{N}_+ + 1)^{1/2} \xi\| + N \|\check{a}_x \xi\| + N \|\check{a}_y \xi\| + N^{1/2} \|\check{a}_x \check{a}_y \xi\| \Big\} \\ &\leq C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\| . \end{aligned} \tag{160}$$

Also for the term W_3 in (150), we switch to position space. We find

$$\mathrm{W}_3 = \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \\ \times \int_0^1 ds \int_0^s d\tau \left(e^{-sB} \check{b}_x^* \check{b}_y^* e^{sB}\, e^{-\tau B} b^*(\check{\eta}_x) b^*(\check{\eta}_y) e^{\tau B} + \text{h.c.} \right) .$$

and thus

$$|\langle \xi, \mathrm{W}_3 \xi\rangle| \leq \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \int_0^1 ds \int_0^s d\tau\, \|(\mathcal{N}_+ + 1)^{1/2} e^{-sB} \check{b}_x \check{b}_y e^{sB} \xi\| \\ \times \|(\mathcal{N}_+ + 1)^{-1/2} e^{-\tau B} b^*(\check{\eta}_x)\, b^*(\check{\eta}_y) e^{\tau B} \xi\| .$$

With Lemma 2, we find

$$\|(\mathcal{N}_+ + 1)^{-1/2} e^{-\tau B} b^*(\check{\eta}_x)\, b^*(\check{\eta}_y) e^{\tau B} \xi\| \leq C \|\eta\|^2 \|(\mathcal{N}_+ + 1)^{1/2} \xi\| .$$

Using Lemma 6, we conclude that

$$\begin{aligned} |\langle \xi, \mathrm{W}_3 \xi\rangle| &\leq C \|\eta\|^2 \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \\ &\quad \times \Big\{ N \|(\mathcal{N}_+ + 1)^{1/2} \xi\| + N \|\check{a}_x \xi\| + N \|\check{a}_y \xi\| + N^{1/2} \|\check{a}_x \check{a}_y \xi\| \Big\} \\ &\leq C N^{1-2\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + C N^{1/2-2\alpha} \|\mathcal{V}_N^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\|. \end{aligned} \tag{161}$$

The term W_4 in (150) can be bounded similarly. In position space, we find

$$\mathrm{W}_4 = \int dxdy\, e^{2N} V(e^N(x-y)) \\ \times \int_0^1 ds \int_0^s d\tau \left(e^{-sB} \check{b}_x^* \check{b}_y^*\, e^{sB}\, e^{-\tau B} b(\check{\eta^2_x}) \check{b}_y e^{\tau B} + \text{h.c.} \right)$$

with $\check{\eta^2}$ the function with Fourier coefficients η_q^2, for $q \in \Lambda^*$, and where $\check{\eta^2_x}(y) := \check{\eta^2}(x-y)$. Clearly $\|\check{\eta^2_x}\| \leq C\|\check{\eta}\|^2 \leq C N^{-2\alpha}$. With Cauchy–Schwarz and Lemma 2, we obtain

$$|\langle \xi, \mathrm{W}_4 \xi\rangle| \leq\; C N^{-2\alpha} \int_0^1 ds \int_0^s d\tau \int dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \check{b}_y \check{b}_x e^{sB} \xi\| \|\check{b}_y e^{\tau B} \xi\|.$$

Applying Lemma 6 and then Lemma 2, we obtain

$$\begin{aligned}|\langle \xi, \mathrm{W}_4 \xi\rangle| &\le CN^{-2\alpha} \int_0^1 ds \int_0^s d\tau \int dxdy\, e^{2N} V(e^N(x-y)) \|\check{b}_y e^{\tau B}\xi\| \\ &\quad \times \left\{ N\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N\|\check{a}_x\xi\| + N\|\check{a}_y\xi\| + N^{1/2}\|\check{a}_x\check{a}_y\xi\|\right\} \\ &\le CN^{1-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-2\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\,.\end{aligned}$$

From (159), (160), (161) and the last bound, we conclude that

$$\begin{aligned}\mathcal{G}_{N,\alpha}^{(4)} &= \mathcal{V}_N + \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\left(b_q b_{-q} + \text{h.c.}\right) \\ &\quad + \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q\left(1-\frac{\mathcal{N}_+}{N}\right)\left(1-\frac{\mathcal{N}_+ + 1}{N}\right) + \mathcal{E}_{N,\alpha}^{(4)}\end{aligned}$$

where $\mathcal{E}_{N,\alpha}^{(4)}$ satisfies (149). □

Appendix A.6: Proof of Proposition 1

With the results established in Sects. 1–1, we cam now show Proposition 1. Propositions 12, 13, 14, 15, 16, imply that

$$\begin{aligned}\mathcal{G}_{N,\alpha} &= \frac{\widehat{V}(0)}{2}(N+\mathcal{N}_+ - 1)(N-\mathcal{N}_+) \\ &\quad + \sum_{p\in\Lambda_+^*} \eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{\substack{r\in\Lambda^*\\ p+r\neq 0}} \widehat{V}(r/e^N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+ - 1}{N}\Big) \\ &\quad + \mathcal{K} + N\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)a_p^* a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) \\ &\quad + \sum_{p\in\Lambda_+^*}\Big[\,p^2\eta_p + \frac{N}{2}\widehat{V}(p/e^N) + \frac{1}{2}\sum_{r\in\Lambda^*:\, p+r\neq 0}\widehat{V}(r/e^N)\eta_{p+r}\,\Big](b_p^* b_{-p}^* + b_p b_{-p}) \\ &\quad + \sqrt{N}\sum_{p,q\in\Lambda_+^*:\, p+q\neq 0} \widehat{V}(p/e^N)\Big[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\Big] + \mathcal{V}_N + \mathcal{E}_1\end{aligned} \tag{162}$$

where

$$|\langle \xi, \mathcal{E}_1\xi\rangle| \le CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

for any $\alpha > 1$ and $\xi \in \mathcal{F}_+^{\le N}$. With (31), we find

$$\begin{aligned}&\sum_{p\in\Lambda_+^*}\eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{r\in\Lambda^*:\, p+r\neq 0}\widehat{V}(r/e^N)\eta_{p+r}\Big] \\ &\quad = \sum_{p\in\Lambda_+^*}\eta_p\Big[\,\frac{N}{2}\widehat{V}(p/e^N) + Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p) + e^{2N}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q - \frac{1}{2}\widehat{V}(p/e^N)\eta_0\,\Big]\end{aligned}$$

From Lemma 1 and estimating $\|\widehat{\chi}_\ell\| = \|\chi_\ell\| \leq CN^{-\alpha}$, $\|\eta\| \leq CN^{-\alpha}$ and $\|\widehat{\chi}_\ell * \eta\| = \|\chi_\ell \check{\eta}\| \leq \|\check{\eta}\| \leq CN^{-\alpha}$, we have

$$\Big|Ne^{2N}\lambda_\ell \sum_{p\in\Lambda_+^*} \eta_p \widehat{\chi}_\ell(p)\Big| \leq CN^{2\alpha}\|\widehat{\chi}_\ell\|\|\eta\| \leq C,$$

and

$$\Big|e^{2N}\lambda_\ell \sum_{p\in\Lambda_+^*,\, q\in\Lambda^*} \widehat{\chi}_\ell(p-q)\eta_q\eta_p\Big| \leq CN^{2\alpha-1}\|\hat{\chi}_\ell * \eta\|\|\eta\| \leq CN^{-1}.$$

Moreover, using (154) and the bound (32) we find

$$\Big|\frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\eta_0\Big| \leq CN^{1-2\alpha}.$$

We obtain

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*} \eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{\substack{r\in\Lambda^*\\ p+r\in\Lambda_+^*}} \widehat{V}(r/e^N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big)\\
&\quad= \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{N-\mathcal{N}_+-1}{N}\right) + \mathcal{E}_2
\end{aligned}$$

with $\pm\mathcal{E}_2 \leq C$ for all $\alpha \geq 1/2$. On the other hand, using (32) we have

$$\begin{aligned}
\frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p &= \frac{N}{2}\big(\widehat{V}(\cdot/e^N) * \eta\big)(0) - \frac{N}{2}\widehat{V}(0)\eta_0\\
&= \frac{N^2}{2}\Big(\int dx\, V(x) f_\ell(x) - \widehat{V}(0)\Big) + \tilde{\mathcal{E}}_2
\end{aligned}$$

with $\pm\tilde{\mathcal{E}}_2 \leq CN^{1-2\alpha}$. With the first bound in (41) we conclude that

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*} \eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{\substack{r\in\Lambda^*\\ p+r\in\Lambda_+^*}} \widehat{V}(r/e^N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big)\\
&= \frac{1}{2N}\left[\widehat{\omega}_N(0) - N\widehat{V}(0)\right](N-\mathcal{N}_+-1)(N-\mathcal{N}_+) + \mathcal{E}_3
\end{aligned} \tag{163}$$

where $\pm\mathcal{E}_3 \leq C$, if $\alpha \geq 1/2$. Using (31), we can also handle the fourth line of (162); we find

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\Big[p^2\eta_p + \frac{N}{2}\widehat{V}(p/e^N) + \frac{1}{2}\sum_{r\in\Lambda^*:\, p+r\in\Lambda_+^*} \widehat{V}(r/e^N)\eta_{p+r}\Big]\big(b_p^*b_{-p}^* + b_pb_{-p}\big)\\
&= \sum_{p\in\Lambda_+^*}\Big[Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p) + e^{2N}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q - \frac{1}{2}\widehat{V}(p/e^N)\eta_0\Big]\big(b_p^*b_{-p}^* + b_pb_{-p}\big).
\end{aligned} \tag{164}$$

The last two terms on the right hand side of (164) are error terms. With (32) and (154) we have

$$\begin{aligned}
&\Big| \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_0\big(b_p^*b_{-p}^* + b_pb_{-p}\big)\Big| \\
&\qquad \leq CN^{-2\alpha}\Big[\sum_{p\in\Lambda_+^*}\frac{|\widehat{V}(p/e^N)|^2}{p^2}\Big]^{1/2}\Big[\sum_{p\in\Lambda_+^*} p^2\|a_p\xi\|^2\Big]^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\| \\
&\qquad \leq CN^{1/2-2\alpha}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\,.
\end{aligned}$$

The second term on the right hand side of (164) can be bounded in position space:

$$\begin{aligned}
&\Big|\langle\xi,\ e^{2N}\lambda_\ell\sum_{p\in\Lambda_+^*}(\widehat{\chi}_\ell*\eta)(p)(b_p^*b_{-p}^*+b_pb_{-p})\xi\rangle\Big| \\
&\leq CN^{2\alpha-1}\|(\mathcal{N}_++1)^{1/2}\xi\|\int_{\Lambda^2}dxdy\ \chi_\ell(x-y)|\check{\eta}(x-y)|\|(\mathcal{N}_++1)^{-1/2}\check{b}_x\check{b}_y\xi\| \\
&\leq CN^{\alpha-1}\|(\mathcal{N}_++1)^{1/2}\xi\|\Big[\int_{\Lambda^2}dxdy\ \chi_\ell(x-y)\|(\mathcal{N}_++1)^{-1/2}\check{a}_x\check{a}_y\xi\|^2\Big]^{1/2}.
\end{aligned}$$

The term in parenthesis can be bounded similarly as in (80). Namely,

$$\int_{\Lambda^2}dxdy\ \chi_\ell(x-y)\|(\mathcal{N}_++1)^{-1/2}\check{a}_x\check{a}_y\xi\|^2 \leq CqN^{-2\alpha/q'}\|\mathcal{K}^{1/2}\xi\|^2$$

for any $q>2$ and $1<q'<2$ with $1/q+1/q'=1$. Choosing $q=\log N$, we get

$$\begin{aligned}
&\Big|\langle\xi, e^{2N}\lambda_\ell\sum_{p\in\Lambda_+^*}(\widehat{\chi}_\ell*\eta)(p)(b_p^*b_{-p}^*+b_pb_{-p})\xi\rangle\Big| \\
&\qquad\leq CN^{-1}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,,
\end{aligned}$$

and, from (164), we conclude that

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\Big[p^2\eta_p+\frac{N}{2}\widehat{V}(p/e^N)+\frac{1}{2}\sum_{\substack{r\in\Lambda^*:\\ p+r\in\Lambda_+^*}}\widehat{V}(r/e^N)\eta_{p+r}\Big]\big(b_p^*b_{-p}^*+b_pb_{-p}\big) \\
&\qquad=\sum_{p\in\Lambda_+^*}Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p)\big(b_p^*b_{-p}^*+b_pb_{-p}\big)+\mathcal{E}_4\,,
\end{aligned} \tag{165}$$

with

$$|\langle\xi,\mathcal{E}_4\xi\rangle|\leq CN^{-1}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.$$

if $\alpha>1$. Combining (162) with (163) and (165), and using the definition (39) we conclude that

$$\begin{aligned}
\mathcal{G}_{N,\alpha} &= \frac{1}{2}\widehat{\omega}_N(0)(N-1)\Big(1-\frac{\mathcal{N}_+}{N}\Big)+\Big[N\widehat{V}(0)-\frac{1}{2}\widehat{\omega}_N(0)\Big]\mathcal{N}_+\Big(1-\frac{\mathcal{N}_+}{N}\Big) \\
&\quad+N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)a_p^*a_p\left(1-\frac{\mathcal{N}_+}{N}\right)+\frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)(b_pb_{-p}+\text{h.c.}) \\
&\quad+\sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq0}\widehat{V}(p/e^N)\Big[b_{p+q}^*a_{-p}^*a_q+\text{h.c.}\Big] \\
&\quad+\mathcal{K}+\mathcal{V}_N+\mathcal{E}_5\,,
\end{aligned} \tag{166}$$

with

$$|\langle \xi, \mathcal{E}_5 \xi \rangle| \leq C N^{1/2-\alpha} \|\mathcal{H}_N^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| + C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + C N^{-1} (\log N)^{1/2} \|\mathcal{K}^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| + C \|\xi\|^2 ,$$

for any $\alpha > 1$. Observing that $|\widehat{V}(p/e^N) - \widehat{V}(0)| \leq C|p|e^{-N}$ in the second line on the r.h.s. of (166), we arrive at $\mathcal{G}_{N,\alpha} = \mathcal{G}^{\text{eff}}_{N,\alpha} + \mathcal{E}_{\mathcal{G}}$, with $\mathcal{G}^{\text{eff}}_{N,\alpha}$ defined as in (42) and with $\mathcal{E}_{\mathcal{G}}$ that satisfies (43).

Appendix B: Properties of the Scattering Function

Let V be a potential with finite range $R_0 > 0$ and scattering length $\mathfrak{a}$. For a fixed $R > R_0$, we study properties of the ground state f_R of the Neumann problem

$$\Big(-\Delta + \frac{1}{2} V(x) \Big) f_R(x) = \lambda_R \, f_R(x) \tag{167}$$

on the ball $|x| \leq R$, normalized so that $f_R(x) = 1$ for $|x| = R$. Lemma 1, parts (i)–(iv), follows by setting $R = e^N \ell$ in the following lemma.

Lemma 7 *Let $V \in L^3(\mathbb{R}^2)$ be non-negative, compactly supported and spherically symmetric, and denote its scattering length by $\mathfrak{a}$. Fix $R > 0$ sufficiently large and denote by f_R the Neumann ground state of (167). Set $w_R = 1 - f_R$. Then we have*

$$0 \leq f_R(x) \leq 1$$

Moreover, for R large enough there is a constant $C > 0$ independent of R such that

$$\left| \lambda_R - \frac{2}{R^2 \log(R/\mathfrak{a})} \left(1 + \frac{3}{4} \frac{1}{\log(R/\mathfrak{a})} \right) \right| \leq \frac{C}{R^2} \frac{1}{\log^3(R/\mathfrak{a})} . \tag{168}$$

and

$$\left| \int dx \, V(x) f_R(x) - \frac{4\pi}{\log(R/\mathfrak{a})} \right| \leq \frac{C}{\log^2(R/\mathfrak{a})} . \tag{169}$$

Finally, there exists a constant $C > 0$ such that

$$\begin{aligned} |w_R(x)| &\leq \chi(|x| \leq R_0) + C \frac{\log(|x|/R)}{\log(\mathfrak{a}/R)} \chi(R_0 \leq |x| \leq R) \\ |\nabla w_R(x)| &\leq \frac{C}{\log(R/\mathfrak{a})} \frac{\chi(|x| \leq R)}{|x| + 1} \end{aligned} \tag{170}$$

for R large enough.

To show Lemma 7 we adapt to the two dimensional case the strategy used in [8, Lemma A.1] for the three dimensional problem. We will use some well known properties of the zero energy scattering equation in two dimensions, summarized in the following lemma.

Lemma 8 *Let $V \in L^3(\mathbb{R}^2)$ non-negative, with supp $V \subset B_{R_0}(0)$ for an $R_0 > 0$. Let $\mathfrak{a} \leq R_0$ denote the scattering length of V. For $R > R_0$, let $\phi_R : \mathbb{R}^2 \to \mathbb{R}$ be the radial solution of the zero energy scattering equation*

$$\left[-\Delta + \frac{1}{2} V \right] \phi_R = 0 \tag{171}$$

normalized such that $\phi_R(x) = 1$ *for* $|x| = R$. *Then*

$$\phi_R(x) = \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})} \tag{172}$$

for all $|x| > R_0$. *Moreover,* $|x| \to \phi_R(x)$ *is monotonically increasing and there exists a constant* $C > 0$ *(depending only on* V*) such that*

$$\phi_R(x) \geq \phi_R(0) \geq \frac{C}{\log(R/\mathfrak{a})} \tag{173}$$

for all $x \in \mathbb{R}^2$. *Furthermore, there exists a constant* $C > 0$ *such that*

$$|\nabla \phi_R(x)| \leq \frac{C}{|\log(R/\mathfrak{a})|} \frac{1}{|x| + 1} \tag{174}$$

for all $x \in \mathbb{R}^2$.

Proof The existence of the solution of (171), the expression (172), the fact that $\phi_R(x) \geq 0$ and the monotonicity are standard (see, for example, Theorem C.1 and Lemma C.2 in [17]). The bound (173) for $\phi_R(0)$ follows from (172), comparing $\phi_R(0)$ with $\phi_R(x)$ at $|x| = R_0$, with Harnack's inequality (see [24, Theorem C.1.3]). Finally, (174) follows by rewriting (171) in integral form

$$\phi_R(x) = 1 - \frac{1}{4\pi} \int_{\mathbb{R}^2} \log\Big(R/|x-y|\Big) V(y) \phi_R(y) \mathrm{d}y \,.$$

For $|x| \leq R_0$, this leads (using that $\phi_R(y) \leq \log(R_0/\mathfrak{a})/\log(R/\mathfrak{a})$ for all $|y| \leq R_0$ and the local integrability of $|.|^{-3/2}$) to

$$|\nabla \phi_R(x)| \leq C \int \frac{V(y)\phi_R(y)}{|x-y|} dy \leq \frac{C\|V\|_3}{\log(R/\mathfrak{a})}$$

Combining with the bound for $|x| > R_0$ obtained differentiating (172), we obtain the desired estimate. □

Proof of Lemma 7 By standard arguments (see for example [17, proof of theorem C1]), $f_R(x)$ is spherically symmetric and non-negative. We now start by proving an upper bound for λ_R, consistent with (168). To this end, we calculate the energy of a suitable trial function. For $k \in \mathbb{R}$ we define

$$\psi_k(x) = J_0(k|x|) - \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} Y_0(k|x|) \,.$$

with J_0 and Y_0 the zero Bessel functions of first and second type, respectively. Note that

$$-\Delta \psi_k(x) = k^2 \psi_k(x) \,.$$

and $\psi_k(x) = 0$ if $|x| = a$. We define $k = k(R)$ to be the smallest positive real number satisfying $\partial_r \psi_R(x) = 0$ for $|x| = R$. One can check that

$$\left| k^2 - \frac{2}{R^2 \log(R/\mathfrak{a})} \left(1 + \frac{3}{4} \frac{1}{\log(R/\mathfrak{a})} \right) \right| \leq \frac{C}{R^2} \frac{1}{\log^3(R/\mathfrak{a})} \tag{175}$$

in the limit $R \to \infty$. To prove (175), we observe that

$$\partial_r \psi_k(x) \Big|_{|x|=R} = -k J_1(kR) + k \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} Y_1(kR) \,, \tag{176}$$

and we expand for $kR, k\mathfrak{a} \ll 1$ using (with γ the Euler constant)

$$\begin{aligned} &\left| J_0(r) - 1 + \frac{r^2}{4} \right| \leq Cr^4 , \qquad \left| J_1(r) - \frac{r}{2}\Big(1 - \frac{r^2}{8}\Big) \right| \leq Cr^5 , \\ &\left| Y_0(r) - \frac{2}{\pi} \log(re^\gamma/2) \right| \leq Cr^2 \log(r) , \\ &\left| Y_1(r) + \frac{2}{\pi}\frac{1}{r}\left(1 - \frac{r^2}{2}\Big(1 - \frac{r^2}{8}\Big) \log(re^\gamma/2) + \frac{r^2}{4}\right) \right| \leq Cr^3. \end{aligned} \tag{177}$$

With (177) one finds that (176)

$$\begin{aligned} &\partial_r \psi_R(x)\Big|_{|x|=R} \\ &= - \frac{1}{2kR \log(k\mathfrak{a}e^\gamma/2)} \\ &\quad \cdot \left\{ \frac{(kR)^4}{8} \log(R/\mathfrak{a}) - (kR)^2 \left[\log(R/\mathfrak{a}) - \frac{1}{2} \right] + 2 + \mathcal{O}((kR)^4 + (k\mathfrak{a})^2) \right\} \end{aligned} \tag{178}$$

The smallest solution of

$$\frac{(kR)^4}{8} \log(R/\mathfrak{a}) - (kR)^2 \left[\log(R/\mathfrak{a}) - \frac{1}{2} \right] + 2 = 0$$

is such that

$$(kR)^2 = \frac{2}{\log(R/\mathfrak{a})} \left[1 + \frac{3}{4 \log(R/\mathfrak{a})} \right] + \mathcal{O}(\log^{-3}(R/\mathfrak{a})) \tag{179}$$

in the limit of large R. Inserting in (178), we find that the r.h.s. changes sign around the value of k defined in (179). By the intermediate value theorem, we conclude that there is a $k = k(R) > 0$ satisfying (175), such that $\partial_r \psi_{k(R)}(x) = 0$ if $|x| = R$.

Now, let $\phi_R(x)$ be the solution of the zero energy scattering equation (171), with $\phi_R(x) = 1$ for $|x| = R$. We set

$$\Psi_R(x) := \psi_k(m_R(x)) = J_0(km_R(x)) - \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} Y_0(km_R(x)) , \tag{180}$$

with $k = k(R)$ satisfying (175) and

$$m_R(x) := \mathfrak{a} \exp\big(\log(R/\mathfrak{a})\phi_R(x)\big) .$$

With this choice we have $m_R(x) = |x|$ outside the range of the potential; hence $\Psi_R(x) = \psi_k(x)$ for $R_0 \leq |x| \leq R$. In particular, Ψ_R satisfies Neumann boundary conditions at $|x| = R$.

From (172), (173) and the monotonicity of ϕ_R, we get

$$C\mathfrak{a} \leq m_R(x) \leq R_0 \qquad \text{for all} \quad 0 \leq |x| \leq R_0 \tag{181}$$

and for a constant $C > 1$, independent of R. From (174) we also get

$$|\nabla m_R(x)| \leq C \qquad \text{for all} \quad 0 \leq |x| \leq R . \tag{182}$$

With the notation $\mathfrak{h} = -\Delta + \frac{1}{2}V$, we now evaluate $\langle \Psi_R, \mathfrak{h}\Psi_R \rangle$. To this end we note that

$$\langle \Psi_R, \mathfrak{h}\Psi_R \rangle = \int_{|x|<R_0} \overline{\Psi_R(x)} (\mathfrak{h}\Psi_R(x)) dx + k^2 \int_{|x|\geq R_0} |\Psi_R(x)|^2 \, dx . \tag{183}$$

Let us consider the region $|x| < R_0$. From (180) and (177) we find, first of all,

$$\left|\Psi_R(x) + \frac{\log(m_R(x)/\mathfrak{a})}{\log(k\mathfrak{a}e^\gamma/2)}\right| \leq C(km_R(x))^2 , \tag{184}$$

Next, we compute $-\Delta\Psi_R(x)$. With

$$J_0'(r) = -J_1(r) \qquad J_1'(r) = \frac{1}{2}\big(J_0(r) - J_2(r)\big)$$
$$Y_0'(r) = -Y_1(r) \qquad Y_1'(r) = \frac{1}{2}\big(Y_0(r) - Y_2(r)\big).$$

we obtain (here, we use the notation m_R' and m_R'' for the radial derivatives of the radial function m_R)

$$\begin{aligned}
-\Delta\Psi_R(x) =& -\partial_r^2\Psi_R(x) - \frac{1}{|x|}\partial_r\Psi_R(x)\\
=& -km_R''(x)\Big[-J_1(km_R(x)) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_1(km_R(x))\Big]\\
&- \frac{1}{2}k^2\big(m_R'(x)\big)^2\Big[J_2(km_R(x)) - \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_2(km_R(x))\Big]\\
&- \frac{1}{2}k^2\big(m_R'(x)\big)^2\Big[-J_0(km_R(x)) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_0(km_R(x))\Big]\\
&- \frac{km_R'(x)}{|x|}\Big[-J_1(km_R(x)) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_1(km_R(x))\Big].
\end{aligned}$$

We note that, using the scattering equation (171),

$$m_R'' - \frac{(m_R')^2}{m_R} + \frac{1}{|x|}m_R' = \frac{1}{2}Vm_R\,\phi_R\log(R/\mathfrak{a}) = \frac{1}{2}Vm_R\log(m_R/\mathfrak{a}). \tag{185}$$

Now we write

$$\begin{aligned}
&-\Delta\Psi_R(x)\\
&= \Big[-k\Big(m_R''(x) + \frac{m_R'(x)}{|x|}\Big)Y_1(km_R(x)) + \frac{k^2}{2}(m_R'(x))^2Y_2(km_R(x))\Big]\frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} + g_R(x)
\end{aligned} \tag{186}$$

where $g_R(x) = \sum_{i=1}^3 g_R^{(i)}(x)$ with

$$\begin{aligned}
g_R^{(1)}(x) &= k\left(m_R''(x) + \frac{m_R'(x)}{|x|}\right)J_1(km_R(x))\\
g_R^{(2)}(x) &= -\frac{1}{2}k^2(m_R'(x))^2J_2(km_R(x))\\
g_R^{(3)}(x) &= -\frac{1}{2}k^2(m_R'(x))^2\Big(-J_0(km_R(x) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_0(km_R(x))\Big) = \frac{k^2}{2}(m_R'(x))^2\Psi_R(x).
\end{aligned}$$

With (185), (177) and (181), (182), we find

$$|g_R^{(1)}(x)| \leq Ck^2\Big((m_R'(x))^2 + \frac{1}{2}V(x)m_R^2(x)\log(m_R(x)/\mathfrak{a})\Big) \leq Ck^2(1 + V(x)).$$

Next, with $|J_2(r) - r^2/8| \leq Cr^4$ we get

$$|g_R^{(2)}(x)| \leq Ck^4(m_R'(x))^2(m_R(x))^2 \leq Ck^4.$$

With (184), we can also bound

$$|g_R^{(3)}(x)| \le Ck^2 (m_R'(x))^2 \frac{\log(m_R(x)/\mathfrak{a})}{\log(k\mathfrak{a})} \le Ck^2 \log^{-1}(k\mathfrak{a})\,.$$

We conclude that $|g_R(r)| \le C(1+V(x))k^2$ for all $r \le R_0$ and R large enough. Finally, using Eq. (185), the expansion for $Y_1(r)$ in Eq. (177), and the bound

$$\Big| Y_2(r) + \frac{4}{\pi}\frac{1}{r^2} \Big| \le C\,,$$

we can rewrite the first term on the r.h.s. of (186) as

$$\begin{aligned} &\Big[-k\Big(m_R''(x) + \frac{m_R'(x)}{|x|}\Big) Y_1(km_R(x)) + \frac{k^2}{2}(m_R'(x))^2 Y_2(km_R(x))\Big] \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} \\ &= \frac{1}{\pi} V(x) \log(m_R(x)/\mathfrak{a}) \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} + h_R(x) \end{aligned} \tag{187}$$

with $|h_R(x)| \le C(1+V(x))k^2$ for all $r \le R_0$, R large enough. With the identities (186) and (187) we obtain

$$\Big| -\Delta\Psi_R(x) - \frac{1}{\pi}\frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} V(x)\log(m_R(x)/\mathfrak{a}) \Big| \le C(1+V(x))k^2\,,$$

for all $|x| \le R_0$ and for R sufficiently large. With (184), we conclude that, for $0 \le |x| \le R_0$,

$$\Big| (-\Delta + \frac{1}{2}V)\Psi_R(x) \Big| \le C(1+V(x))k^2\,. \tag{188}$$

With (183), (188) and the upper bound

$$|\Psi_R(r)| \le \frac{C}{|\log(k\mathfrak{a})|} \tag{189}$$

for all $|x| \le R_0$ (which follows from (184) and (181)), we get

$$\big\langle \Psi_R, \mathfrak{h}\Psi_R \big\rangle \le k^2 \big\langle \Psi_R, \Psi_R \big\rangle + \frac{Ck^2}{|\log(k\mathfrak{a})|} \int_{|x|\le R_0} (1+V(x))\,\mathrm{d}x\,.$$

On the other hand, Eq.(184), together with $m_R(x) = |x|$ for $|x| \ge R_0$, implies the lower bound

$$\big\langle \Psi_R, \Psi_R \big\rangle \ge \int_{R_0\le|x|\le R} |\Psi_R(x)|^2 \mathrm{d}x \ge \frac{C}{|\log(k\mathfrak{a})|^2} \int_{R_0 \le |x| \le R} \log^2(|x|/\mathfrak{a})\mathrm{d}x \ge CR^2\,.$$

Hence, with (175), we conclude that

$$\begin{aligned} \lambda_R &\le \frac{\big\langle \Psi_R, \mathfrak{h}\Psi_R \big\rangle}{\big\langle \Psi_R, \Psi_R \big\rangle} \le k^2 \left(1 + \frac{C\,|\log(k\mathfrak{a})|}{R^2}\right) \\ &\le \frac{2}{R^2 \log(R/\mathfrak{a})} \left(1 + \frac{3}{4}\frac{1}{\log(R/\mathfrak{a})} + \frac{C}{\log^2(R/\mathfrak{a})}\right) \end{aligned} \tag{190}$$

in agreement with (168).

To prove the lower bound for λ_R it is convenient to show some upper and lower bounds for f_R. We start by considering f_R outside the range of the potential. We denote $\varepsilon_R = \sqrt{\lambda_R}\,R$. Keeping into account the boundary conditions at $|x| = R$, we find, for $R_0 \le |x| \le R$,

$$f_R(x) = A_R\, J_0(\varepsilon_R\, |x|/R) + B_R\, Y_0(\varepsilon_R\, |x|/R)\,,$$

with

$$A_R = \left(J_0(\varepsilon_R) - J_1(\varepsilon_R)\,\frac{Y_0(\varepsilon_R)}{Y_1(\varepsilon_R)} \right)^{-1},$$

and

$$B_R = \left(Y_0(\varepsilon_R) - \frac{J_0(\varepsilon_R)}{J_1(\varepsilon_R)} Y_1(\varepsilon_R) \right)^{-1}.$$

From (190), we have $|\varepsilon_R| \le C\,|\log(R/\mathfrak{a})|^{-1/2}$. Thus, we can expand f_R for large R, using (177) and, for Y_0, the improved bound

$$\left| Y_0(r) - \frac{2}{\pi}\log(re^{\gamma}/2)\left(1 - \frac{1}{4}r^2\right)\right| \le C\,r^2,$$

we find

$$\begin{aligned} \left| A_R - 1 + \frac{\varepsilon_R^2}{4}\Big(2\log(\varepsilon_R e^{\gamma}/2) - 1\Big)\right| &\le C\varepsilon_R^4(\log\varepsilon_R)^2, \\ \left| B_R - \frac{\pi}{4}\varepsilon_R^2\left(1 - \frac{\varepsilon_R^2}{8}\right)\right| &\le C\varepsilon_R^6. \end{aligned} \tag{191}$$

which leads to

$$\begin{aligned} &\left| f_R(x) - 1 + \frac{\varepsilon_R^2}{4}\left(2\log(R/|x|) - 1 + \frac{x^2}{R^2}\right) - \frac{\varepsilon_R^4}{16}\log(R/|x|)\left(1 + \frac{2x^2}{R^2}\right)\right| \\ &\quad \le C\varepsilon_R^4(\log\varepsilon_R)^2. \end{aligned} \tag{192}$$

We can also compute the radial derivative

$$\partial_r f_R(x) = -\frac{\varepsilon_R}{R}\Big(A_R\,J_1(\varepsilon_R\,r/R) + B_R\,Y_1(\varepsilon_R\,r/R)\Big).$$

With the expansions (177) and (191) we conclude that for all $R_0 \le |x| < R$ we have

$$\left| \partial_r f_R(x) - \frac{\varepsilon_R^2}{2|x|}\left(1 - \frac{x^2}{R^2} + \frac{\varepsilon_R^2 x^2}{2R^2}\log(R/|x|)\right)\right| \le C\varepsilon_R^4\log\varepsilon_R. \tag{193}$$

The bound (193) shows that $\partial_r f_R(x)$ is positive, for, say, $R_0 < |x| < R/2$. Since $\partial_r f_R(x)$ must have its first zero at $|x| = R$, we conclude that f_R is increasing in $|x|$, on $R_0 \le |x| \le R$. From the normalization $f_R(x) = 1$, for $|x| = R$, we conclude therefore that $f_R(x) \le 1$, for all $R_0 \le |x| \le R$.

From (192) and (190) we obtain, on the other hand, the lower bound

$$\begin{aligned} f_R(x) &\ge 1 - \frac{\varepsilon_R^2}{2}\log(R/|x|) - C\varepsilon_R^4(\log\varepsilon_R)^2 \\ &\ge 1 - \frac{\log(R/|x|)}{\log(R/\mathfrak{a})}\left(1 + \frac{3}{4}\frac{1}{\log(R/\mathfrak{a})} + \frac{C}{\log^2(R/\mathfrak{a})}\right) - C\,\frac{(\log\log(R/\mathfrak{a}))^2}{\log^2(R/\mathfrak{a})} \\ &\ge \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})} - \frac{3}{4}\frac{\log(R/|x|)}{\log^2(R/\mathfrak{a})} - C\frac{\log(R/|x|)}{\log^3(R/a)} - C\,\frac{(\log\log(R/\mathfrak{a}))^2}{\log^2(R/\mathfrak{a})}, \end{aligned} \tag{194}$$

for R sufficiently large. Let $R_* = \max\{R_0, e\mathfrak{a}\}$. Then Eq. (194) implies in particular that, for R large enough,

$$f_R(x) \geq \frac{C}{\log(R/\mathfrak{a})}\,. \tag{195}$$

for all $R_* < |x| \leq R$.

Finally, we show that $f_R(x) \leq 1$ also for $|x| \leq R_0$. First of all, we observe that, by elliptic regularity, as stated for example in [12, Theorem 11.7, part iv)], there exists $0 < \alpha < 1$ and $C > 0$ such that

$$|f_R(x) - f_R(y)| \leq C\|(V - 2\lambda_R)f_R\|_2\,|x-y|^\alpha$$

With $\|Vf_R\|_2 \leq \|V\|_3\|f_R\|_6 \leq C\|f_R\|_{H^1} \leq C(1+\lambda_R)\|f_R\|_2$, we conclude that $0 \leq f_R(x) \leq 1 + C\|f\|_2$ for all $|x| \leq R_0$ (because we know that $f_R(x) \leq 1$ for $R_0 \leq |x| \leq R$). To improve this bound, we go back to the differential equation (167), to estimate

$$\Delta f_R = \frac{1}{2}Vf_R - \lambda_R f_R \geq -\lambda_R(1 + C\|f\|_2) \tag{196}$$

This implies that $f_R(x) + \lambda_R(1+C\|f\|_2)x^2/2$ is subharmonic. Using (192), we find $f_R(x) \leq 1 - C\varepsilon_R^2$ for $|x| = R_0$. From the maximum principle, we obtain therefore that

$$f_R(x) \leq 1 - C\varepsilon_R^2 + C\lambda_R(1 + C\|f_R\|_2) \tag{197}$$

for all $|x| \leq R_0$. In particular, this implies that $\|f_R\mathbf{1}_{|x|\leq R_0}\|_2 \leq C + C\lambda_R\|f_R\|_2$, and therefore that

$$\|f_R\mathbf{1}_{R_0\leq|x|\leq R}\|_2 \geq \|f_R\|_2(1 - C\lambda_R) - C$$

With $f_R(x) \leq 1$ for $R_0 \leq |x| \leq R$, we find, on the other hand, that $\|f_R\mathbf{1}_{R_0\leq|x|\leq R}\|_2 \leq CR$. We conclude therefore that $\|f_R\|_2 \leq CR$ and, from (197), that $f_R(x) \leq 1 - C\varepsilon_R^2 + C/R \leq 1$, for all $|x| \leq R_0$, if R is large enough.

We are now ready to prove the lower bound for λ_R. We use now that any function Φ satisfying Neumann boundary conditions at $|x| = R$ can be written as $\Phi(x) = q(x)\Psi_R(x)$, with $\Psi_R(x)$ the trial function used for the upper bound and $q > 0$ a function that satisfies Neumann boundary condition at $|x| = R$ as well. This is in particular true for the solution $f_R(x)$ of (167). In the following we write

$$f_R(x) = q_R(x)\Psi_R(x)$$

where q_R satisfies Neumann boundary conditions at $|x| = R$. From (184), we find $|\Psi_R(x)| \geq C/\log(ka)$. The bound $f_R(x) \leq 1$ implies therefore that there exists $c > 0$ such that

$$q_R(x) \leq C\log(k\mathfrak{a}) \qquad \forall\,|x| \leq R_0\,. \tag{198}$$

From the identity

$$\mathfrak{h}f_R = (\mathfrak{h}\Psi_R)q_R - (\Delta q_R)\Psi_R - 2\nabla q_R\nabla\Psi_R$$

we have

$$\int_{|x|\leq R} \mathrm{d}x\; f_R\mathfrak{h}f_R = \int_{|x|\leq R}\mathrm{d}x\;|\nabla q_R|^2\Psi_R^2 + \int_{|x|\leq R}\mathrm{d}x\;|q_R|^2\Psi_R\mathfrak{h}\Psi_R\,.$$

From (188) and (189), we have

$$\left|\Psi_R(x)(\mathfrak{h}\Psi_R)(x) - k^2\Psi_R^2(x)\right| \leq C\frac{k^2}{|\log ka|}(1 + V(x))\chi(|x| \leq R_0)\,.$$

Hence

$$\int_{|x|\le R} \mathrm{d}x\ f_R \mathfrak{h} f_R \ge\ k^2\|f_R\|^2 - \frac{Ck^2}{|\log k|}\int_{|x|\le R_0} \mathrm{d}x\ (1+V(x))|q_R(x)|^2 . \qquad (199)$$

With (198), we obtain

$$\int_{|x|\le R} \mathrm{d}x\ f_R \mathfrak{h} f_R \ge k^2\|f_R\|^2 - Ck^2\log(k\mathfrak{a}) .$$

With (195) (recalling that $R_* = \max\{R_0, e\mathfrak{a}\}$), we bound

$$\|f_R\|^2 \ge \int_{R_*\le|x|\le R} |f_R(x)|^2\,\mathrm{d}x \ge \frac{CR^2}{\log^2(R/\mathfrak{a})}$$

and, inserting in (199), we conclude that

$$\lambda_R = \frac{\langle f_R, \mathfrak{h} f_R\rangle}{\langle f_R, f_R\rangle} \ge k^2\left(1-\frac{C\log^3(R/\mathfrak{a})}{R^2}\right)$$
$$\ge \frac{2}{R^2\log(R/\mathfrak{a})}\left(1+\frac{3}{4}\frac{1}{\log(R/\mathfrak{a})}-\frac{C}{\log^2(R/\mathfrak{a})}\right),$$

where in the last inequality we used (175).

To prove (169) we use the scattering equation (167) to write

$$\int \mathrm{d}x\ V(x) f_R(x) = 2\int_{|x|\le R} \mathrm{d}x\ \Delta f_R(x) + 2\int_{|x|\le R} \mathrm{d}x \lambda_R f_R(x) .$$

Passing to polar coordinates, and using that $\Delta f_R(x) = |x|^{-1}\partial_r|x|\partial_r f_R(x)$, we find that the first term vanishes. Hence

$$\int \mathrm{d}x\ V(x) f_R(x) = 2\lambda_R \int \mathrm{d}x\ f_R(x) .$$

With the upper bound $f_R(r) \le 1$ and with (168), we find

$$\int \mathrm{d}x\ V(x) f_R(x) \le 2\pi R^2\lambda_R \le \frac{4\pi}{\log(R/\mathfrak{a})}\left(1+\frac{C}{\log(R/\mathfrak{a})}\right) .$$

To obtain a lower bound for the same integral we use that $f_R(r) \ge 0$ inside the range of the potential. Outside the range of V, we use (192). We find

$$\int \mathrm{d}x\ V(x) f_R(x) \ge 4\pi\lambda_R \int_{R_0}^{R} \mathrm{d}r\, r\,(1 - C\varepsilon_R^2\log(R/r)) \ge \frac{4\pi}{\log(R/\mathfrak{a})}\left(1-\frac{C}{\log(R/\mathfrak{a})}\right)$$

We conclude that

$$\left|\int \mathrm{d}x\ V(x) f_R(x) - \frac{4\pi}{\log(R/\mathfrak{a})}\right| \le \frac{C}{\log^2(R/\mathfrak{a})}.$$

Finally, we show the bounds in (170). For $r \in [R_0, R]$, from (192) we have

$$\left| w_R(x) - \frac{\log(R/|x|)}{\log(R/\mathfrak{a})}\right| \le \frac{C}{\log(R/\mathfrak{a})} . \qquad (200)$$

As for the derivative of w_R we use (193) to compute

$$|\partial_r f_R(x)| \le \frac{C}{|x|}\frac{1}{\log(R/\mathfrak{a})} .$$

Moreover $\partial_r f_R(x) = 0$ if $|x| = R$, by construction.

On the other hand, if $|x| \le R_0$, we have $w_R(x) = 1 - f_R(x) \le 1$. As for the derivative, we define $\widetilde{f}_R$ on $\mathbb{R}_+$ through $\widetilde{f}_R(r) = f_R(x)$, if $|x| = r$, and we use the representation

$$\widetilde{f}_R'(r) = \frac{1}{r}\int_0^r \mathrm{d}s \big(\widetilde{f}_R''(s)s + \widetilde{f}_R'(s)\big)\,.$$

With (167), we have (with $\widetilde{V}$ defined on $\mathbb{R}_+$ through $V(x) = \widetilde{V}(r)$, if $|x| = r$)

$$\widetilde{f}_R''(r) + \frac{1}{r}\widetilde{f}_R'(r) = \lambda_R \widetilde{f}_R(r) - \frac{1}{2}\widetilde{V}(r)\widetilde{f}_R(r)\,,$$

By (200), we can estimate $\widetilde{f}_R(R_0) \le C/\log(R/\mathfrak{a})$. From (196), we also recall that

$$\widetilde{f}_R(r) \le \widetilde{f}_R(R_0) + CR\lambda_R \le C/\log(R/\mathfrak{a})$$

for any $r < R_0$. We conclude therefore that

$$\begin{aligned}
|\widetilde{f}_R'(r)| &= \Big|\frac{1}{r}\int_0^r \mathrm{d}s s\big(\lambda_R \widetilde{f}_R(s) - \frac{1}{2}\widetilde{V}(s)\widetilde{f}_R(s)\big)\Big| \\
&\le \frac{\lambda_R}{r}\int_0^r r dr + \frac{C}{r\log(R/\mathfrak{a})}\int_0^r dr\, r\widetilde{V}(r) \\
&\le \frac{C}{\log(R/\mathfrak{a})} + C\,\|V\|_2 \frac{\log(R_0/\mathfrak{a})}{\log(R/\mathfrak{a})} \le \frac{C}{\log(R/\mathfrak{a})}\,.
\end{aligned}$$

□

References

1. Andersen, J.O.: Ground state pressure and energy density of an interacting homogeneous Bose gas in two dimensions. Eur. Phys. J. B **28**, 389 (2002)
2. Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B.: Complete Bose–Einstein condensation in the Gross–Pitaevskii regime. Commun. Math. Phys. **359**(3), 975–1026 (2018)
3. Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B.: Bogoliubov Theory in the Gross–Pitaevskii limit. Acta Math. **222**(2), 219–335 (2019)
4. Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B.: Optimal rate for Bose–Einstein condensation in the Gross–Pitaevskii regime. Commun. Math. Phys. **376**, 1311–1395 (2020)
5. Boßmann, L.: Derivation of the 2d Gross–Pitaevskii equation for strongly confined 3d bosons. Arch. Ration. Mech. Anal. **238**(2), 541–606 (2020)
6. Brennecke, C., Schlein, B.: Gross–Pitaevskii dynamics for Bose–Einstein condensates. Anal. PDE **12**(6), 1513–1596 (2019)
7. Brennecke, C., Schlein, B., Schraven, S.: Bose–Einstein Condensation with Optimal Rate for Trapped Bosons in the Gross–Pitaevskii Regime. Preprint arXiv:2102.11052 (2021)
8. Erdős, L., Schlein, B., Yau, H.-T.: Derivation of the Gross–Pitaevskii hierarchy for the dynamics of Bose–Einstein condensate. Commun. Pure Appl. Math. **59**(12), 1659–1741 (2006)
9. Fournais, S., Napiórkowski, M., Reuvers, R., Solovej, J.P.: Ground state energy of a dilute two-dimensional Bose gas from the Bogoliubov free energy functional. J. Math. Phys. **60**, 071903 (2019)
10. Jeblick, M., Leopold, N., Pickl, P.: Derivation of the time dependent Gross–Pitaevskii equation in two dimensions. Commun. Math. Phys. **372**(1), 1–69 (2019)
11. Lewin, M., Nam, P.T., Serfaty, S., Solovej, J.P.: Bogoliubov spectrum of interacting Bose gases. Commun. Pure Appl. Math. **68**(3), 413–471 (2014)
12. Lieb, E.H., Loss, M.: Analysis. AMS (2001)
13. Lieb, E.H., Seiringer, R.: Proof of Bose–Einstein condensation for dilute trapped gases. Phys. Rev. Lett. **88**, 170409 (2002)
14. Lieb, E.H., Seiringer, R.: Derivation of the Gross–Pitaevskii equation for rotating Bose gases. Commun. Math. Phys. **264**, 505–537 (2006)

15. Lieb, E.H., Yngvason, J.: The ground state energy of a dilute two-dimensional Bose gas. J. Stat. Phys. **103**, 509–526 (2001)
16. Lieb, E.H., Seiringer, R., Yngvason, J.: A rigorous derivation of the Gross–Pitaevskii energy functional for a two-dimensional bose gas. Commun. Math. Phys. **224**(1), 17–31 (2001)
17. Lieb, E.H., Seiringer, R., Solovej, J.P., Yngvason, J.: The Mathematics of the Bose Gas and Its Condensation. Oberwolfach Seminars, Birkhäuser, Basel (2005)
18. Mora, C., Castin, Y.: Ground state energy of the two-dimensional weakly interacting Bose gas: first correction beyond Bogoliubov theory. Phys. Rev. Lett. **102**, 180404 (2009)
19. Nam, P.T., Napiórkowski, M., Ricaud, J., Triay, A.: Optimal rate of condensation for trapped bosons in the Gross–Pitaevskii regime. To appear in Analysis & PDE
20. Pilati, S., Boronat, J., Casulleras, J., Giorgini, S.: Quantum Monte Carlo simulation of a two-dimensional Bose gas. Phys. Rev. A **71**, 023605 (2005)
21. Schick, M.: Two-dimensional system of hard-core bosons. Phys. Rev. A **3**, 1067 (1971)
22. Schnee, K., Yngvason, J.: Bosons in disc-shaped traps: from 3D to 2D. Commun. Math. Phys. **269**(3), 659–691 (2007)
23. Seiringer, R.: The excitation spectrum for weakly interacting bosons. Commun. Math. Phys. **306**, 565–578 (2011)
24. Simon, B.: Schrödinger semigroups. Bull. Am. Math. Soc. **7**(3), 447–526 (1982)