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Renormalization of Symmetric Bimodal Maps with Low Smoothness

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Abstract
This paper deals with the renormalization of symmetric bimodal maps with low smoothness. We prove the existence of the renormalization fixed point in the space C^{1+Lip} symmetric bimodal maps. Moreover, we show that the topological entropy of the renormalization operator defined on the space of C^{1+Lip} symmetric bimodal maps is infinite. Further we prove the existence of a continuum of fixed points of renormalization. Consequently, this proves the non-rigidity of the renormalization of symmetric bimodal maps.

Keywords Renormalization fixed point · Symmetric bimodal maps · Low smoothness · Non-rigidity

Mathematics Subject Classification 37E05 · 37E20

1 Introduction

Renormalization is a technique to analyze maps having the property that the first return map to a small part of the phase space resembles the original map itself. Period doubling renormalization operator was introduced by Feigenbaum [1,2] and by Coullet and Tresser [3], to study asymptotic small scale geometry of the attractor of one dimensional systems which are at the transition from simple to chaotic dynamics. Renormalization is a method to study microscopic geometrical properties of attractors. The geometric rigidity of the attractors is the center of attention in one dimensional theory. The smoothness of the maps plays a crucial role for rigidity. From the last four decades, a lot of mathematical theory have been developed for the renormalization theory in low-dimensional dynamics. Especially, Sullivan [4] showed the convergence of renormalizations. Moreover, all limits of renormalization are quadratic-

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like maps with a definite modulus. Hu [5] proved that the real polynomial map having the periodic points of all power of 2 is infinitely renormalizable. Further, McMullen [6] proved the exponential convergence towards the limit set of renormalization. Also, Martens [7] showed that the renormalization operator acting on the space of smooth unimodal maps with critical exponent greater than one, has periodic points of any combinatorial type. Lyubich considered the renormalization with bounded combinatorics in [8]. Then, the hyperbolicity of the renormalization fixed point in the space of $C^{2+\alpha}$ ($\alpha > 0$) unimodal maps, was shown by Davie [9]. Using the results of [8]; de Faria et al. [10] extended to more general types of renormalization in the space C^r, provided $r \geq 2+\alpha$ with α close to one. Later, Chandramouli et al. [11], proved that the period doubling renormalization converges to the analytic generic fixed point proving it to be globally unique in a class $C^{2+|\cdot|}$ which is bigger than $C^{2+\alpha}$ (for any positive $\alpha \leq 1$). Furthermore, they showed that the uniqueness is lost below C^2 space and other asymptotic behavior encountered. Recently, Kozlovski and van Strien [12] proved the existence of a period doubling infinitely renormalizable asymmetric unimodal map with non universal scaling laws.

In the context of circle diffeomorphisms, Herman [13] proved the rigidity result, using real variable techniques. Yoccoz [14] proved the other fundamental rigidity results by using conformal surgery, where Herman's theorem holds in the real-analytic category. Further, Khanin and Sinai [15] gave a proof of Herman's theorem which is based on the thermodynamic formalism and ergodic properties for the corresponding random variables. Later, Yampolsky [16] proved the rigidity of circle map with a critical point. Furthermore, the rigidity theory for circle maps with break type singularities have been developed by Khanin et al. [17–20], Cunha and Smania [21], Akhadkulov et al. [22]. In the context of interval maps, the rigidity phenomena is understood for $C^{2+\alpha}$ ($\alpha > 0$) smooth maps. Further, de Melo and Pinto [23] proved the rigidity of C^2 infinitely renormalizable unimodal maps with bounded combinatorial type. The measure-theoretical properties of real family of unimodal maps are studied by Lyubich et al. [24] proved that almost any real quadratic map has either an attracting cycle or an absolutely continuous invariant measure. Further, Avila et al. [25] extended these result for any non-trivial real analytic family of quasiquadratic maps. Bruin et al. [26] showed that almost every unicritical polynomial with even critical order greater than or equal to 2, admits a physical measure, which is either supported on an attracting periodic orbit, or is absolutely continuous, or is supported on the postcritical set. Further, Moreira and Smania [27] showed the rigidity of infinitely renormalizable Fibonacci unimodal maps with even critical order and having negative Schwarzian derivative. Bruin and Todd [28] proved the existence of wild attractor for a countably piecewise linear infinitely renormalizable Fibonacci unimodal map with infinite critical order.

In the context of two dimensional maps, de Carvalho et al. [29] showed the non-rigidity of Cantor attractors of Hénon-like maps. Further, Hazard et al. [30] discussed the unbounded geometry of Cantor attractor of strongly dissipative infinitely renormalizable Hénon-like map with stationary combinatorics. In case of Lorenz maps, Martens and Winckler [31,32] studied the hyperbolicity of Lorenz renormalization and also proved the non-existence of physical measures for Lorenz maps which are infinitely renormalizable.

With a relatively complete understanding of the period doubling renormalization of unimodal maps, recent research in dynamical systems has either focused on more complicated maps of the real line or other low dimensional maps. Jonker and Rand [33], and van Strien [34] used renormalization as a natural vehicle to decompose the non-wandering set in a hierarchical manner, for unimodal maps. The multimodal maps are interesting as generalizations of unimodal maps, as well as for their applications. For example, in the case of bimodal maps, they are essential to understand the non-invertible circle maps which have been used extensively to model the transitions to chaos in two frequency systems [35]. Mackay and van Zeijts [36] explained the period doubling renormalization of two parameter families of bimodal maps in the term of a horseshoe with a Cantor set of two dimensional unstable manifold. Also, they calculated the periodic points of renormalization up to period five. Veitch [37] presented some work on topological renormalization of C^0 bimodal maps with zero and positive entropy. Further, Smania developed a combinatorial theory for certain kind of multimodal maps and proved that for the same combinatorial type the renormalizations of infinitely renormalizable smooth multimodal maps are exponentially close [38,39]. Later, Smania [40] proved the hyperbolicity of renormalization for real analytic multimodal maps with bounded combinatorics.

In this work, we focus on the construction of renormalization fixed point for the family of symmetric bimodal maps with low smoothness (i.e., below C^2 space). First, we show that there exists a sequence of affine pieces which are nested and shrinking down to the critical points of the bimodal map corresponding to a pair of proper scaling data $s^* = (s_l^*, s_r^*)$. This helps us to show the existence of a fixed point f_{s^*} of the renormalization operator defined on the space of piece-wise affine infinitely renormalizable maps, which is denoted by W, corresponding to a pair of proper scaling data s^*. This gives us the following result.

Theorem 1 *There exists a map $f_{s^*} \in W$, where $s^* = (s_l^*, s_r^*)$ is characterized by*

$$Rf_{s^*} = f_{s^*}.$$

In particular, $W = \{f_{s^*}\}$.

Here, the renormalization operator R is a pair of period tripling renormalization operators R^l and R^r which are defined on piecewise affine period tripling infinitely renormalizable maps corresponding to a proper scaling data s_l and s_r, respectively.
The proof of theorem 1 mainly relies on the Propositions 1 and 2 presented in Sect. 2. We use Mathematica for some computational work to prove these propositions.
In the next Sect. 3, we explain the extension of the renormalization fixed point f_{s^*} to a C^{1+Lip} symmetric bimodal map g_{s^*}. Then, we have the following theorem,

Theorem 2 *There exists an infinitely renormalizable C^{1+Lip} symmetric bimodal map g_{s^*}, which is not C^2 map, such that*

$$Rg_{s^*} = g_{s^*}.$$

In Sect. 4, we describe the topological entropy of renormalization defined on the space of C^{1+Lip} symmetric bimodal maps. Then we obtain the following theorem,

Theorem 3 *The renormalization operator R acting on the space of C^{1+Lip} symmetric bimodal maps has unbounded topological entropy.*

Furthermore, we discuss the existence of another fixed point of renormalization by considering the small perturbation on the scaling data. Then, we get the following result,

Theorem 4 *There exists a continuum of fixed points of the renormalization operator acting on C^{1+Lip} symmetric bimodal maps.*

Consequently, this result leads to the non-rigidity of the Cantor attractors of infinitely renormalizable symmetric bimodal maps, whose smoothness is below C^2.
We recall some basic definitions. Let $I = [0, 1]$ be a closed interval.

A point $c \in I$ is said to be a *critical point* of a C^1 map $u : I \to I$ if $Du(c) = 0$.
The critical point c is called *non-flat critical point of order k* if u is C^{k+1} in a neighborhood of c and $Du(c) = D^2u(c) = \cdots = D^{k-1}u(c) = 0$ and $D^k u(c) \neq 0$.
Note that $D^k u(c)$ stands for $k^{th}-$derivative of u at c.
A unimodal map $\mathfrak{u} : I \to I$, which is a C^1 map having a unique non-flat critical point c, is called *period tripling renormalizable map* if there exists a proper subinterval $J \subset I$ with $c \in J$ such that
(1) J, $\mathfrak{u}(J)$ and $\mathfrak{u}^2(J)$ are pairwise disjoint,
(2) $\mathfrak{u}^3(J) \subset J$.
Then $\mathfrak{u}^3 : J \to J$ is called a pre-renormalization of $\mathfrak{u}$.
Where, u^n denotes n fold composition of u with itself.
Let $\mathscr{U}$ be the collection of unimodal maps and $\mathscr{U}_\infty (\subset \mathscr{U})$ be the collection of period tripling infinitely renormalizable unimodal maps.
An interval map f is *piece-wise monotone* if there exists a partition of I into finitely many subintervals on each of which the restriction of f is continuous and strictly monotonic.
A map f is called a *bimodal* map if three is the minimal number of such subintervals.

Definition 1 Let $f : I \to I$ be a map with two subsets J_l and J_r such that $J_l^{\circ} \cap J_r^{\circ} = \emptyset$. If $f|_{J_l}$ and $f|_{J_r}$ are unimodal maps which are concave up and concave down respectively, their *join*, denoted by $f|_{J_l} \oplus f|_{J_r}$, is a bimodal map whose graph is obtained by joining $\big(max(J_l), f(max(J_l))\big)$ and $\big(min(J_r), f(min(J_r))\big)$ by a C^{1+Lip} curve.
The notation I° stands for the interior of I.

Definition 2 A bimodal map $b : I \to I$, is a C^1 map having two critical points c_l and c_r, which is said to be *renormalizable* if there exists two disjoint intervals I_l containing c_l and I_r containing c_r such that

(i) $b^i(I_l) \cap b^j(I_l) = \emptyset$, for each $i \neq j$ and $i, j \in \{0, 1, 2\}$,
$b^i(I_r) \cap b^j(I_r) = \emptyset$, for each $i \neq j$ and $i, j \in \{0, 1, 2\}$,
(ii) $b^3(I_l) \subset I_l$ and $b^3(I_r) \subset I_r$,
(iii) The unimodal maps $\hat{b}_l : [0, b(0)] \to [0, b(0)]$ and $\hat{b}_r : [b(1), 1] \to [b(1), 1]$ are joined to generate a bimodal map $\hat{b}_l \oplus \hat{b}_r$. The unimodal maps $\hat{b}_l$ and $\hat{b}_r$ are defined as

$$\hat{b}_l(x) = h_1^{-1} b^3 h_1(x)$$

and

$$\hat{b}_r(x) = h_2^{-1} b^3 h_2(x)$$

where $h_1 : [0, b(0)] \to I_l$ and $h_2 : [b(1), 1] \to I_r$ are the affine orientation reversing homeomorphisms.

The renormalization of a bimodal map is illustrated in Fig. 1.
In the next section, we construct the renormalization operator defined on the space of piece-wise affine maps which are infinitely renormalizable maps.

2 Piece-Wise Affine Renormalizable Maps

A symmetric bimodal map $b : [0, 1] \to [0, 1]$ of the form $b(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, for $a_3 < 0$, is a C^1 map with the following conditions

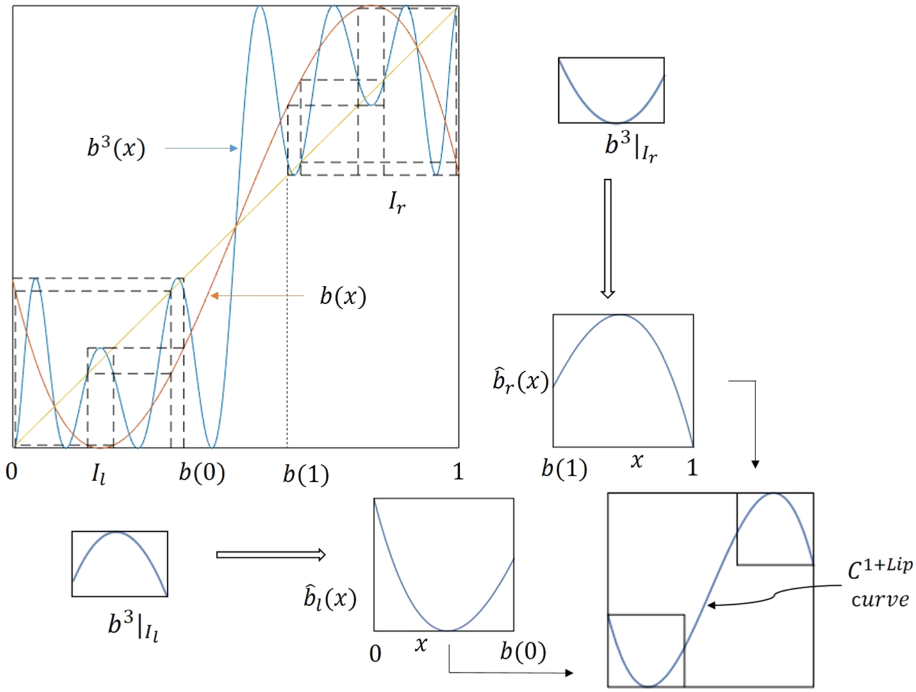


Fig. 1 Renormalization of a bimodal map

- $b(0) = 1 - b(1)$,
- $b(\frac{1}{2}) = \frac{1}{2}$,
- let c_l and c_r be the two critical points of $b(x)$, then $b(c_l) = 0$ and $b(c_r) = 1$.

Let us consider a one parameter family of symmetric bimodal maps $\mathscr{B}_c : [0, 1] \to [0, 1]$ which are increasing on the interval between the critical points and decreasing elsewhere. then, we obtained a family of bimodal maps as

$$\mathscr{B}_c(x) = \begin{cases} 1 - \frac{1-6c+9c^2-4c^3+6cx-6c^2x-3x^2+2x^3}{(1-2c)^3}, & \text{if } c \in \left[0, \frac{1}{4}\right] \\ 1 - \frac{4c^3-3c^2+6cx-6c^2x-3x^2+2x^3}{(2c-1)^3}, & \text{if } c \in \left[\frac{3}{4}, 1\right] \end{cases}$$
$$\equiv \begin{cases} b_c(x), & \text{if } c \in \left[0, \frac{1}{4}\right] \\ \tilde{b}_c(x), & \text{if } c \in \left[\frac{3}{4}, 1\right] \end{cases} \tag{1}$$

Note that the bimodal maps b_c and $\tilde{b}_c$ are identical maps.
Let us define an open set

$$\Delta^3 = \left\{ (s_0, s_1, s_2) \in \mathbb{R}^3 \; : \; s_0, s_1, s_2 > 0, \; \sum_{i=0}^{2} s_i < 1 \right\}.$$

Each element (s_0, s_1, s_2) of Δ^3 is called a scaling tri-factor. A pair of scaling tri-factors $(s_{0,l}, s_{1,l}, s_{2,l})$ and $(s_{0,r}, s_{1,r}, s_{2,r})$ induces two sets of affine maps $(F_{0,l}, F_{1,l}, F_{2,l})$ and $(F_{0,r}, F_{1,r}, F_{2,r})$ respectively. For each $i = 0, 1, 2$,

$$F_{i,l} : I_L = [0, b_c(0)] \longrightarrow I_L$$

are defined as

$$F_{0,l}(t) = b_c(0) - s_{0,l} \cdot t,$$
$$F_{1,l}(t) = b_c^2(0) - s_{1,l} \cdot t,$$
$$F_{2,l}(t) = s_{2,l} \cdot t$$

and

$$F_{i,r} : I_R = [\tilde{b}_c(1), 1] \longrightarrow I_R$$

are defined as

$$F_{0,r}(t) = \tilde{b}_c(1) + s_{0,r} \cdot (1-t),$$
$$F_{1,r}(t) = \tilde{b}_c^2(1) + s_{1,r} \cdot (1-t),$$
$$F_{2,r}(t) = 1 - s_{2,r} \cdot (1-t).$$

Note that ${I_L}^{\circ} \cap {I_R}^{\circ} = \phi$, for $c \in [0, \frac{3-\sqrt{3}}{6}]$.
The functions $s_l : \mathbb{N} \to \Delta^3$ and $s_r : \mathbb{N} \to \Delta^3$ are said to be a scaling data. We set scaling tri-factors
$s_l(n) = (s_{0,l}(n), s_{1,l}(n), s_{2,l}(n)) \in \Delta^3$ and $s_r(n) = (s_{0,r}(n), s_{1,r}(n), s_{2,r}(n)) \in \Delta^3$,
so that $s_l(n)$ and $s_r(n)$ induce the triplets of affine maps $(F_{0,l}(n)(t), F_{1,l}(n)(t), F_{2,l}(n)(t))$ and $(F_{0,r}(n)(t), F_{1,r}(n)(t), F_{2,r}(n)(t))$ as described above.
For $i = 0, 1, 2$, let us define the intervals

$$I_{i,l}^n = F_{1,l}(1) \circ F_{1,l}(2) \circ F_{1,l}(3) \circ \ldots.. \circ F_{1,l}(n-1) \circ F_{i,l}(n)([0, b_c(0)]).$$

Also,

$$I_{i,r}^n = F_{1,r}(1) \circ F_{1,r}(2) \circ F_{1,r}(3) \circ \ldots.. \circ F_{1,r}(n-1) \circ F_{i,r}(n)([\tilde{b}_c(1), 1]).$$

Definition 3 A scaling data $s_j \equiv \{s_j(n)\}$, for $j = l, r$, is said to be proper if, for each $n \in \mathbb{N}$,

$$d(s_j(n), \partial\Delta^3) \geq \epsilon, \quad \text{for some } \epsilon > 0.$$

Where $d(s_j(n), \partial\Delta^3)$ stands for the Euclidean distance between $s_j(n)$ and the closest boundary point of Δ^3.

A pair of proper scaling data $s_l : \mathbb{N} \to \Delta^3$ and $s_r : \mathbb{N} \to \Delta^3$, which is denoted by $s = (s_l, s_r)$, induce the sets $D_{s_l} = \bigcup\limits_{n\geq 1}(I_{0,l}^n \cup I_{2,l}^n)$ and $D_{s_r} = \bigcup\limits_{n\geq 1}(I_{0,r}^n \cup I_{2,r}^n)$, respectively. Consider a map

$$f_s : D_{s_l} \cup D_{s_r} \to [0, 1]$$

defined as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

where $f_{s_l}|_{I_{0,l}^n}$ and $f_{s_l}|_{I_{2,l}^n}$ are the affine extensions of $b_c|_{\partial I_{0,l}^n}$ and $b_c|_{\partial I_{2,l}^n}$ respectively. Similarly, $f_{s_r}|_{I_{0,r}^n}$ and $f_{s_r}|_{I_{2,r}^n}$ are the affine extensions of $b_c|_{\partial I_{0,r}^n}$ and $b_c|_{\partial I_{2,r}^n}$ respectively. These affine extensions are shown in Fig. 2.
The end points of the intervals at each level are labeled by

$$y_0 = 0,\ z_0 = b_c(0),\ I_{1,l}^0 = I_L = [0, b_c(0)]$$

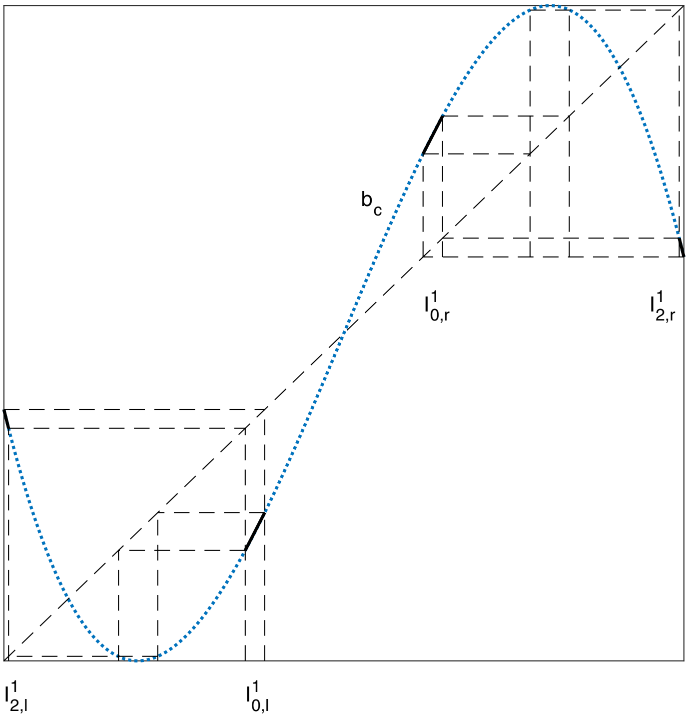


Fig. 2 Piece-wise affine extension

and for $n \geq 1$

$$\begin{aligned} x_n &= \partial I^n_{0,l} \backslash \partial I^{n-1}_{1,l} \\ y_{2n-1} &= max\{\partial I^{2n-1}_{1,l}\} \\ y_{2n} &= min\{\partial I^{2n}_{1,l}\} \\ z_{2n-1} &= min\{\partial I^{2n-1}_{1,l}\} \\ z_{2n} &= max\{\partial I^{2n}_{1,l}\} \\ w_n &= \partial I^n_{2,l} \backslash \partial I^{n-1}_{1,l}, \end{aligned}$$

where ∂I stands for the boundary of I. These points are illustrated in Fig. 3. Also, the end points of the intervals at each level are labeled by

$$z'_0 = \tilde{b}_c(1),\ \ y'_0 = 1,\ \ I^0_{1,r} = I_R = [\tilde{b}_c(1), 1]$$

and for $n \geq 1$

$$\begin{aligned} x'_n &= \partial I^n_{0,r} \backslash \partial I^{n-1}_{1,r} \\ y'_{2n-1} &= min\{\partial I^{2n-1}_{1,r}\} \\ y'_{2n} &= max\{\partial I^{2n}_{1,r}\} \\ z'_{2n-1} &= max\{\partial I^{2n-1}_{1,r}\} \\ z'_{2n} &= min\{\partial I^{2n}_{1,r}\} \end{aligned}$$

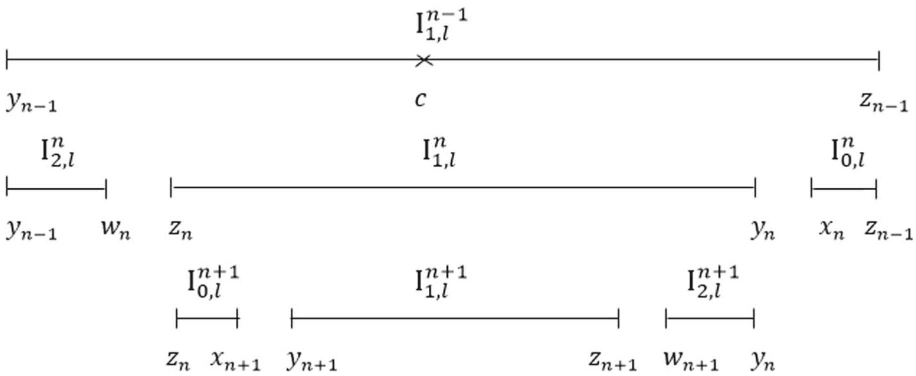


Fig. 3 Formation of interval $I_{1,l}^{n-1}$ into three sub-intervals $I_{2,l}^n$, $I_{1,l}^n$ and $I_{0,l}^n$

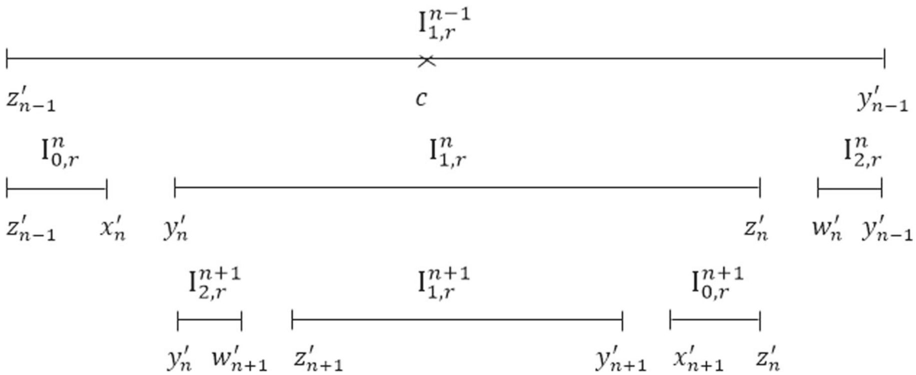


Fig. 4 Formation of interval $I_{1,r}^{n-1}$ into three sub-intervals $I_{0,r}^n$, $I_{1,r}^n$ and $I_{2,r}^n$

$$w_n' = \partial I_{2,r}^n \backslash \partial I_{1,r}^{n-1}.$$

These points are illustrated in Fig. 4.

Definition 4 For a given pair of proper scaling data $s_l, s_r : \mathbb{N} \to \Delta^3$, a map f_s is said to be *infinitely renormalizable* if for $n \geq 1$,

1(i) $[0, f_{s_l}(y_n)]$ is the maximal domain containing 0 on which $f_{s_l}^{3^n-1}$ is defined affinely, $[f_{s_l}^2(y_n), f_{s_l}(0)]$ is the maximal domain containing $f_{s_l}(0)$ on which $f_{s_l}^{3^n-2}$ is defined affinely,

(ii) $[f_{s_r}(y_n'), 1]$ is the maximal domain containing 1 on which $f_{s_r}^{3^n-1}$ is defined affinely and $[f_{s_r}(1), f_{s_r}^2(y_n')]$ is the maximal domain containing $f_{s_r}(1)$ on which $f_{s_r}^{3^n-2}$ is defined affinely,

2(i) $f_{s_l}^{3^n-1}([0, f_{s_l}(y_n)]) \quad = I_{1,l}^n$,

(ii) $f_{s_l}^{3^n-2}([f_{s_l}^2(y_n), f_{s_l}(0)]) = I_{1,l}^n$,

(iii) $f_{s_r}^{3^n-1}([f_{s_r}(y_n'), 1]) \quad = I_{1,r}^n$,

(iv) $f_{s_r}^{3^n-2}([f_{s_r}(1), f_{s_r}^2(y_n')]) = I_{1,r}^n$.

Define $W = \{f_s : f_s \text{ is infinitely renormalizable map}\}$.

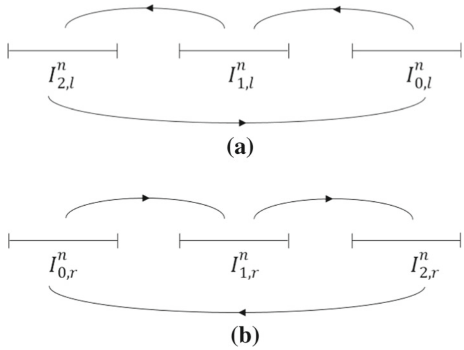


Fig. 5 The combinatorics: **a** corresponding to f_{s_l}, $(I^n_{1,l} \to I^n_{2,l} \to I^n_{0,l} \to I^n_{1,l})$ and **b** corresponding to f_{s_r}, $(I^n_{1,r} \to I^n_{2,r} \to I^n_{0,r} \to I^n_{1,r})$

Further using Definition 4, we write $W_l = \{f_{s_l} : f_{s_l}$ satisfies 1(i), 2(i) and 2(ii)$\}$ and $W_r = \{f_{s_r} : f_{s_r}$ satisfies 1(ii), 2(iii) and 2(iv)$\}$.
Note that W_l and W_r be the collection of the piece-wise affine period tripling infinitely renormalizable maps f_{s_l} on I_L and f_{s_r} on I_R, respectively.
The combinatorics for renormalization of f_{s_l} and f_{s_r} are shown in the following Fig. 5a, b.

2.1 Renormalization on $I_L = [0, b_c(0)]$

Let $f_{s_l} \in W_l$ be given by the proper scaling data $s_l : \mathbb{N} \to \Delta^3$ and define

$$\tilde{I}^n_{1,l} = [b_c^2(y_n), b_c(0)] = [f_{s_l}^2(y_n), f_{s_l}(0)],$$

and

$$\hat{I}^n_{1,l} = [0, b_c(y_n)] = [0, f_{s_l}(y_n)].$$

Let

$$h_{s_l,n} : [0, b_c(0)] \to I^n_{1,l}$$

be defined by

$$h_{s_l,n} = F_{1,l}(1) \circ F_{1,l}(2) \circ F_{1,l}(3) \circ ..... \circ F_{1,l}(n)$$

Furthermore, let

$$\tilde{h}_{s_l,n} : [0, b_c(0)] \to \tilde{I}^n_{1,l} \quad \text{and} \quad \hat{h}_{s_l,n} : [0, b_c(0)] \to \hat{I}^n_{1,l}$$

be the affine orientation preserving homeomorphisms. Then define

$$R^l_n f_{s_l} : h^{-1}_{s_l,n}(D_{s_l} \cap I^n_{1,l}) \to [0, b_c(0)]$$

by

$$R^l_n f_{s_l}(x) = \begin{cases} R^{l-}_n f_{s_l}(x), & \text{if } x \in h^{-1}_{s_l,n}(\underset{m \geq n+1}{\cup} I^m_{0,l}) \\ R^{l+}_n f_{s_l}(x), & \text{if } x \in h^{-1}_{s_l,n}(\underset{m \geq n+1}{\cup} I^m_{2,l}) \end{cases}$$

where,

$$R^{l-}_n f_{s_l} : h^{-1}_{s_l,n}(\underset{m \geq n+1}{\cup} I^m_{0,l}) \to [0, b_c(0)]$$

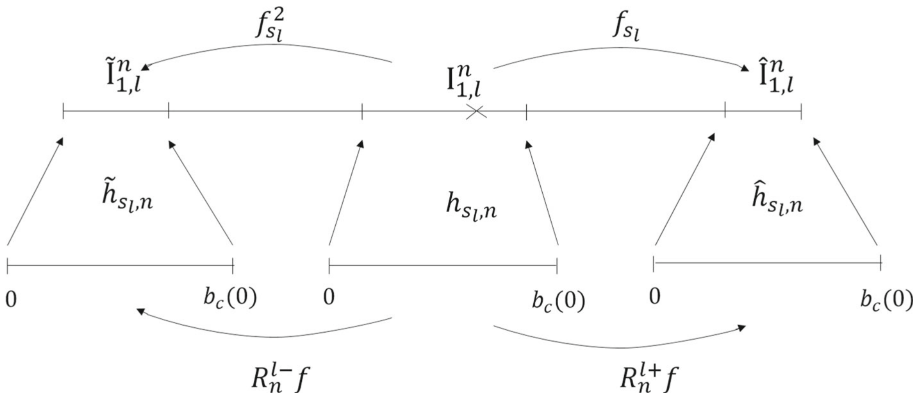


Fig. 6 Illustration of operators R_n^{l-} and R_n^{l+}

and

$$R_n^{l+} f_{s_l} : h_{s_l,n}^{-1}(\underset{m\geq n+1}{\cup} I_{2,l}^m) \to [0, b_c(0)]$$

are defined by

$$R_n^{l-} f_{s_l}(x) = \tilde{h}_{s_l,n}^{-1} \circ f_{s_l}^2 \circ h_{s_l,n}(x)$$
$$R_n^{l+} f_{s_l}(x) = \hat{h}_{s_l,n}^{-1} \circ f_{s_l} \circ h_{s_l,n}(x),$$

which are illustrated in Fig. 6.
Let $\sigma : (\Delta^3)^{\mathbb{N}} \to (\Delta^3)^{\mathbb{N}}$ be the shift map defined as

$$\sigma(s_l(1)s_l(2)s_l(3)s_l(4)....) = (s_l(2)s_l(3)s_l(4)....),$$

where $s_l(i) \in \Delta^3$ for all $i \in \mathbb{N}$.
Note that the operator R_n^l normalize the affine pieces $f_{s_l}^2(\underset{m\geq n+1}{\cup} I_{0,l}^m)$ and $f_{s_l}(\underset{m\geq n+1}{\cup} I_{2,l}^m)$ to I_L with the help of affine homeomorphism $\tilde{h}_{s_l,n}^{-1}$ and $\hat{h}_{s_l,n}^{-1}$, respectively.
This implies, $R_n^l f_{s_l}$ is a piecewise affine map associated with the scaling data $(s_l(n+1)s_l(n+2)s_l(n+3)\ldots)$. Thus,

$$R_n^l f_{s_l} = f_{s_l(n+1)s_l(n+2)s_l(n+3)\ldots}.$$

The above explanation leads the following lemma.

Lemma 1 *Let $s_l : \mathbb{N} \to \Delta^3$ be proper scaling data such that f_{s_l} is infinitely renormalizable. Then*

$$R_n^l f_{s_l} = f_{\sigma^n(s_l)}.$$

□

Let f_{s_l} be infinitely renormalization, then for $n \geq 0$, we have

$$f_{s_l}^{3^n} : D_{s_l} \cap I_{1,l}^n \to I_{1,l}^n$$

is well defined.

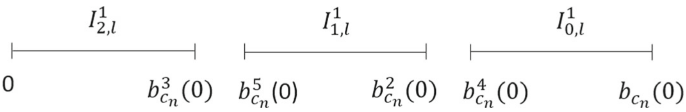


Fig. 7 Length of intervals

Define the renormalization $R^l : W_l \to W_l$ by

$$R^l f_{s_l} = h^{-1}_{s_l,1} \circ f^3_{s_l} \circ h_{s_l,1}.$$

The maps $f^{3^n-2}_{s_l} : \tilde{I}^n_{1,l} \to I^n_{1,l}$ and $f^{3^n-1}_{s_l} : \hat{I}^n_{1,l} \to I^n_{1,l}$ are the affine homeomorphisms whenever $f_{s_l} \in W_l$.

One can observe that, for each $n \in \mathbb{N}$, $\bigcup_{m \geq n+1} I^m_{0,l} \subset I^n_{1,l}$ and $\bigcup_{m \geq n+1} I^m_{2,l} \subset I^n_{1,l}$.

By the definition of R^l_n, the operator R^l_n is just normalizing the affine pieces, which are contained in $I^n_{1,l}$, to I_L. Also, $I^n_{1,l}$ are the renormalization intervals corresponding to n^{th} renormalization operator $(R^l)^n$. Then, we have the following lemma,

Lemma 2 *We have $(R^l)^n f_{s_l} : D_{\sigma^n(s_l)} \to [0, b_c(0)]$ and $(R^l)^n f_{s_l} = R^l_n f_{s_l}$.* □

Using Lemmas 1 and 2, now we are in a position to state the following proposition:

Proposition 1 *There exists a map $f_{s^*_l} \in W_l$, where s^*_l is characterized by*

$$R^l f_{s^*_l} = f_{s^*_l}.$$

Proof Consider $s_l : \mathbb{N} \to \Delta^3$ be proper scaling data such that f_{s_l} is an infinitely renormalizable. Let c_n be the critical point of $f_{\sigma^n(s_l)}$. Then
we have the following scaling ratios which are illustrated in Fig. 7

$$s_{0,l}(n) = \frac{b_{c_n}(0) - b^4_{c_n}(0)}{b_{c_n}(0)} \tag{2}$$

$$s_{1,l}(n) = \frac{b^2_{c_n}(0) - b^5_{c_n}(0)}{b_{c_n}(0)} \tag{3}$$

$$s_{2,l}(n) = \frac{b^3_{c_n}(0)}{b_{c_n}(0)} \tag{4}$$

$$c_{n+1} = \frac{b^2_{c_n}(0) - c_n}{s_{1,l}(n)} \equiv \mathscr{R}(c_n). \tag{5}$$

Since $(s_{0,l}(n), s_{1,l}(n), s_{2,l}(n)) \in \Delta^3$, this implies the following conditions

$$s_{0,l}(n),\ s_{1,l}(n),\ s_{2,l}(n) > 0 \tag{6}$$

$$s_{0,l}(n) + s_{1,l}(n) + s_{2,l}(n) < 1 \tag{7}$$

As the intervals $I^n_{i,l}$, for $i = 0, 1, 2$, are mutually disjoint, we denote the gap ratios as $g^n_{0,l}$ and $g^n_{1,l}$ which are in between $I^n_{0,l}$ & $I^n_{1,l}$ and $I^n_{1,l}$ & $I^n_{2,l}$ respectively. The gap ratios are defined as, for $n \in \mathbb{N}$,

$$g^n_{0,l} = \frac{b^4_{c_n}(0) - b^2_{c_n}(0)}{b_{c_n}(0)} \equiv G_{0,l}(c_n) > 0 \tag{8}$$

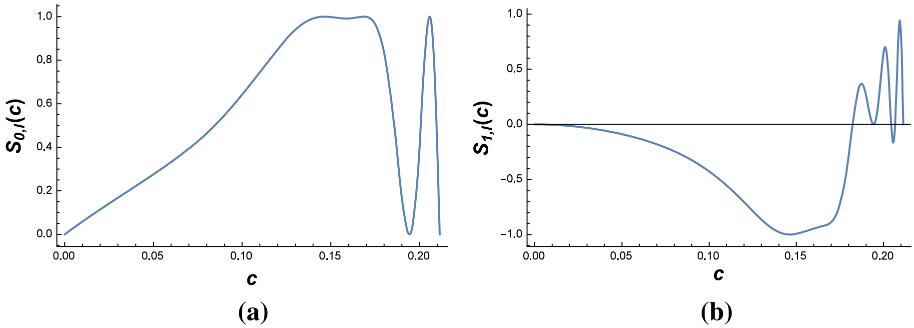


Fig. 8 **a** and **b** shows the graph of $S_{0,l}(c)$, and $S_{1,l}(c)$ respectively

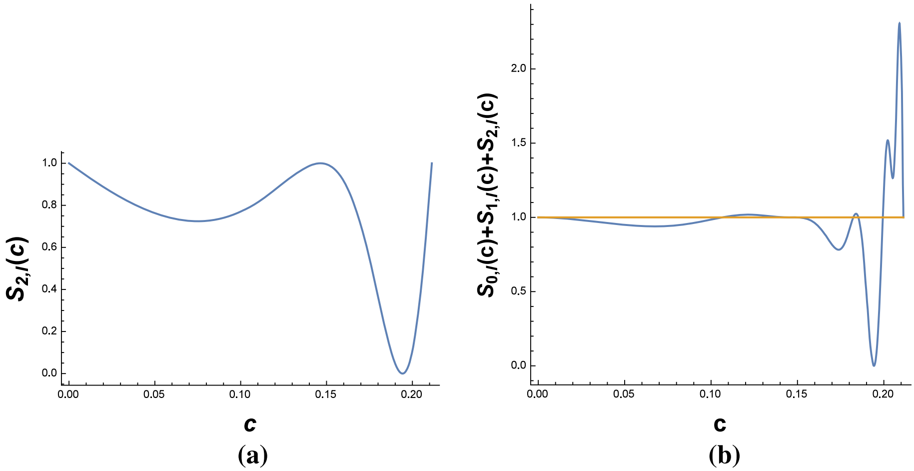


Fig. 9 **a** and **b** shows the graph of $S_{2,l}(c)$, and $(S_{0,l} + S_{1,l} + S_{2,l})(c)$ respectively

$$g^n_{1,l} = \frac{b^5_{c_n}(0) - b^3_{c_n}(0)}{b_{c_n}(0)} \equiv G_{1,l}(c_n) > 0 \tag{9}$$

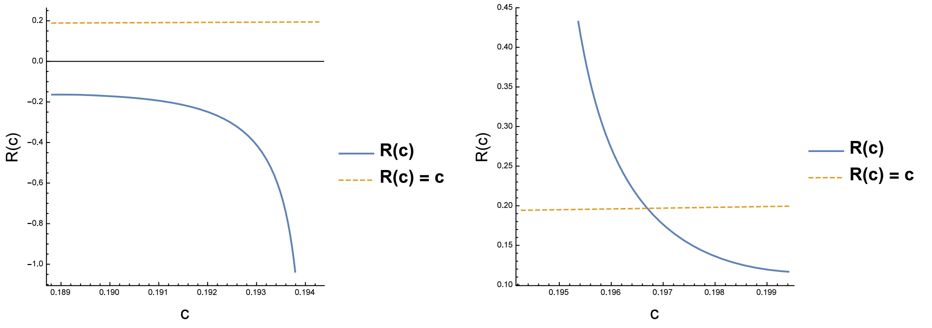
$$0 < c_n < \frac{3-\sqrt{3}}{6} \tag{10}$$

We use Mathematica for solving the Eqs. (2), (3) and (4), then we get the expressions for $s_{0,l}(n)$, $s_{1,l}(n)$ and $s_{2,l}(n)$.
Let $s_{i,l}(n) \equiv S_{i,l}(c_n)$ for $i = 0, 1, 2$. The graphs of $S_{i,l}(c)$ are shown in Figs. 8a, b and 9a. Note that the conditions (6), (8) and (9) give the condition (7)

$$0 < \sum_{i=0}^{2} s_{i,l}(n) < 1.$$

The conditions (6) together with (8) to (10) define the feasible domain F^l_d is to be:

$$F^l_d = \left\{ c \in \left(0, \frac{3-\sqrt{3}}{6}\right) : S_{i,l}(c) > 0 \text{ for } i = 0, 1, 2, G_{0,l}(c) > 0, G_{1,l}(c) > 0 \right\}. \tag{11}$$



(a) $\mathscr{R}$ has no fixed point in $F^l_{d_1}$. **(b)** $\mathscr{R}$ has only one fixed point in $F^l_{d_2}$.

Fig. 10 The graph of $\mathscr{R} : F^l_d \to \mathbb{R}$ and the diagonal $\mathscr{R}(c) = c$

To compute the feasible domain F^l_d, we need to find subinterval(s) of $\left(0, \frac{3-\sqrt{3}}{6}\right)$ which satisfies the conditions of (11). By using Mathematica software, we employ the following command to obtain the feasible domain

$$\mathrm{N[Reduce[}\left\{S_{0,l}(c) > 0, S_{1,l}(c) > 0,\ S_{2,l}(c) > 0,\ G_{0,l}(c) > 0,\ G_{1,l}(c) > 0, 0 < c < \frac{3-\sqrt{3}}{6}\right\}, c]].$$

This yields:

$$F^l_d = (0.188816...,\ \ 0.194271...) \cup (0.194271...,\ \ 0.199413...) \equiv F^l_{d_1} \cup F^l_{d_2}.$$

From the Eq. (5), the graphs of $\mathscr{R}(c)$ are plotted in the sub-domains $F^l_{d_1}$ and $F^l_{d_2}$ of F^l_d which are shown in Fig. 10.

The map $\mathscr{R} : F^l_d \to \mathbb{R}$ is expanding in the neighborhood of fixed point c^*_l which is illustrated in Fig. 10b. By Mathematica computations, we get an unstable fixed points $c^*_l =$ 0.196693... in F^l_d such that

$$\mathscr{R}(c^*_l) = c^*_l$$

corresponds to an infinitely renormalizable maps $f_{s^*_l}$. We observe that the map $f_{s_l{}^*}$ corresponding to c^*_l has the following property

$$\{c^*_l\} = \bigcap_{n \geq 1} I^n_{1,l}.$$

In other words, consider the scaling data $s_l{}^* : \mathbb{N} \to \Delta^3$ with

$$\begin{aligned} s_l{}^*(n) &= (s^*_{0,l}(n), s^*_{1,l}(n), s^*_{2,l}(n)) \\ &= \left(\frac{b_{c^*_l}(0) - b^4_{c^*_l}(0)}{b_{c^*_l}(0)}, \frac{b^2_{c^*_l}(0) - b^5_{c^*_l}(0)}{b_{c^*_l}(0)}, \frac{b^3_{c^*_l}(0)}{b_{c^*_l}(0)} \right). \end{aligned}$$

Then $\sigma(s^*_l) = s^*_l$ and using Lemma 1 we have

$$R^l f_{s^*_l} = f_{s^*_l}.$$

□

2.2 Renormalization on $I_R = [\tilde{b}_c(1), 1]$

In Sect. 2.1, the bimodal map $b_c(x)$ has two critical points $c \in I_L$ and $1 - c \in I_R$ and we define the piece-wise renormalization on I_L. In similar fashion, to define the renormalization on I_R with $c \in I_R$, from Eq. 1, we consider

$$\tilde{b}_c(x) = 1 - \frac{4c^3 - 3c^2 + 6cx - 6c^2x - 3x^2 + 2x^3}{(2c-1)^3}$$

where $x \in [0, 1]$ and $c \in [\frac{3}{4}, 1]$.

Note that ${I_L}^{\circ} \cap {I_R}^{\circ} = \phi$, for $c \in [\frac{3+\sqrt{3}}{6}, 1]$.

Let $f_{s_r} \in W_r$ be given by the proper scaling data $s_r : \mathbb{N} \to \Delta^3$ and define

$$\tilde{I}^n_{1,r} = [\tilde{b}_c(1), \tilde{b}^2_c(y'_n)] = [f_{s_r}(1), f^2_{s_r}(y'_n)],$$

and

$$\hat{I}^n_{1,r} = [\tilde{b}_c(y'_n), 1] = [f_{s_r}(y'_n), 1].$$

Let

$$h_{s_r,n} : [\tilde{b}_c(1), 1] \to I^n_{1,r}$$

be defined by

$$h_{s_r,n} = F_{1,r}(1) \circ F_{1,r}(2) \circ F_{1,r}(3) \circ ..... \circ F_{1,r}(n).$$

Furthermore, let

$$\tilde{h}_{s_r,n} : [\tilde{b}_c(1), 1] \to \tilde{I}^n_{1,r} \quad \text{and} \quad \hat{h}_{s_r,n} : [\tilde{b}_c(1), 1] \to \hat{I}^n_{1,r}$$

be the affine orientation preserving homeomorphisms. Then define

$$R^r_n f_{s_r} : h^{-1}_{s_r,n}(D_{s_r} \cap I^n_{1,r}) \to [\tilde{b}_c(1), 1]$$

by

$$R^r_n f_{s_r}(x) = \begin{cases} R^{r-}_n f_{s_r}(x), & \text{if } x \in h^{-1}_{s_r,n}(\bigcup_{m \geq n+1} I^m_{0,r}) \\ R^{r+}_n f_{s_r}(x), & \text{if } x \in h^{-1}_{s_r,n}(\bigcup_{m \geq n+1} I^m_{2,r}) \end{cases}$$

where,

$$R^{r-}_n f_{s_r} : h^{-1}_{s_r,n}(\bigcup_{m \geq n+1} I^m_{0,r}) \to [\tilde{b}_c(1), 1]$$

and

$$R^{r+}_n f_{s_r} : h^{-1}_{s_r,n}(\bigcup_{m \geq n+1} I^n_{2,r}) \to [\tilde{b}_c(1), 1]$$

are defined by

$$R^{r-}_n f_{s_r}(x) = \tilde{h}^{-1}_{s_r,n} \circ f^2_{s_r} \circ h_{s_r,n}(x)$$
$$R^{r+}_n f_{s_r}(x) = \hat{h}^{-1}_{s_r,n} \circ f_{s_r} \circ h_{s_r,n}(x),$$

which are illustrated in Fig. 11.

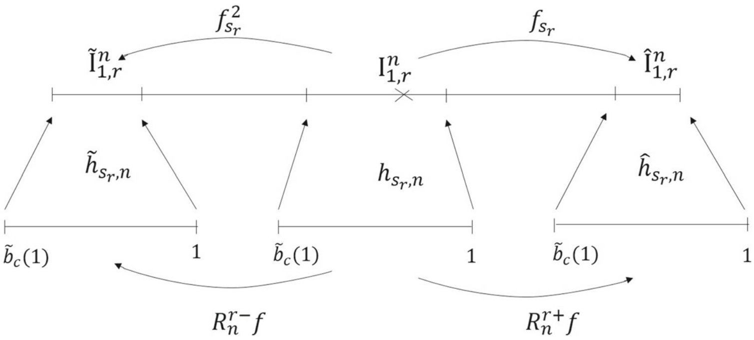


Fig. 11 Illustration of operators R_n^{r-} and R_n^{r+}

Let $\sigma : (\Delta^3)^{\mathbb{N}} \to (\Delta^3)^{\mathbb{N}}$ be the shift map which is defined as

$$\sigma(s_r(1)s_r(2)s_r(3)s_r(4)....) = (s_r(2)s_r(3)s_r(4)....),$$

where $s_r(i) \in \Delta^3$ for all $i \in \mathbb{N}$.

Lemma 3 *Let $s_r : \mathbb{N} \to \Delta^3$ be proper scaling data such that f_{s_r} is infinitely renormalizable. Then*

$$R_n^r f_{s_r} = f_{\sigma^n(s_r)}.$$

□

Let f_{s_r} be infinitely renormalization, then for $n \geq 0$, we have

$$f_{s_r}^{3^n} : D_{s_r} \cap I_{1,r}^n \to I_{1,r}^n$$

is well defined.
Define the renormalization $R^r : W_r \to W_r$ by

$$R^r f_{s_r} = h_{s_r,1}^{-1} \circ f_{s_r}^3 \circ h_{s_r,1}.$$

The maps $f_{s_r}^{3^n-2} : \tilde{I}_{1,r}^n \to I_{1,r}^n$ and $f_{s_r}^{3^n-1} : \hat{I}_{1,r}^n \to I_{1,r}^n$ are the affine homeomorphisms whenever $f_{s_r} \in W_r$. Then we have:

Lemma 4 *We have $(R^r)^n f_{s_r} : D_{\sigma^n(s_r)} \to [\tilde{b}_c(1), 1]$ and $(R^r)^n f_{s_r} = R_n^r f_{s_r}$.* □

From the above Lemmas 3 and 4, consequently we get

Proposition 2 *There exists a map $f_{s_r^*} \in W_r$, where s_r^* is characterized by*

$$R^r f_{s_r^*} = f_{s_r^*}.$$

Proof Consider $s_r : \mathbb{N} \to \Delta^3$ be proper scaling data such that f_{s_r} is an infinitely renormalizable. Let c_n be the critical point of $f_{\sigma^n(s_r)}$. Then
from Fig. 12, we have the following scaling ratios

$$s_{0,r}(n) = \frac{\tilde{b}_{c_n}^4(1) - \tilde{b}_{c_n}(1)}{1 - \tilde{b}_{c_n}(1)} \tag{12}$$

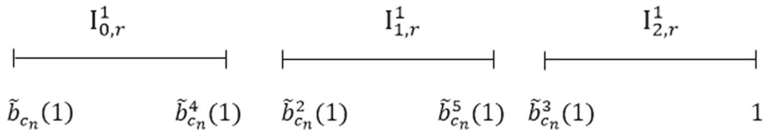
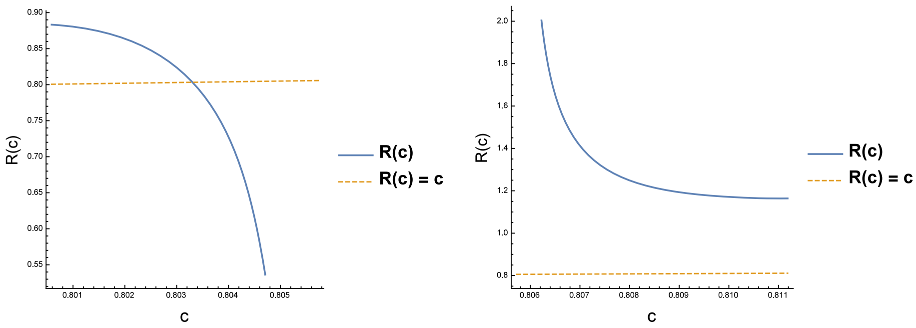


Fig. 12 Length of intervals



(a) $\mathscr{R}$ has only one fixed point in $F^r_{d_1}$. **(b)** $\mathscr{R}$ has no fixed point in $F^r_{d_2}$.

Fig. 13 The graph of $\mathscr{R} : F^r_d \to \mathbb{R}$ and the diagonal $\mathscr{R}(c) = c$

$$s_{1,r}(n) = \frac{\tilde{b}^5_{c_n}(1) - \tilde{b}^2_{c_n}(1)}{1 - \tilde{b}_{c_n}(1)} \tag{13}$$

$$s_{2,r}(n) = \frac{1 - \tilde{b}^3_{c_n}(1)}{1 - \tilde{b}_{c_n}(1)} \tag{14}$$

$$c_{n+1} = 1 - \frac{c_n - \tilde{b}^2_{c_n}(1)}{s_{1,r}(n)} \equiv \mathscr{R}(c_n). \tag{15}$$

Use the same argument as was given in Sect. 2.1, one can compute feasible domain F^r_d. Finally, we get

$$F^r_d = (0.800587..., \ 0.805729...) \cup (0.805729..., \ 0.811184...) \equiv F^r_{d_1} \cup F^r_{d_2}.$$

From the Eq. (15), the graphs of $\mathscr{R}(c)$ are plotted in the sub-domains $F^r_{d_1}$ and $F^r_{d_2}$ of F^r_d which are shown in Fig. 13.

The map $\mathscr{R} : F^r_d \to \mathbb{R}$ is expanding in the neighborhood of fixed point c^*_r which is illustrated in Fig. 13b. By Mathematica computations, we get an unstable fixed points $c^*_r = 0.803307...$ in F^r_d such that

$$\mathscr{R}(c^*_r) = c^*_r$$

corresponds to an infinitely renormalizable maps $f_{s^*_r}$. We observe that the map f_{s_r*} corresponding to c^*_r has the following property

$$\{c^*_r\} = \bigcap_{n \geq 1} I^n_{1,r}.$$

In other words, consider the scaling data $s_r{}^* : \mathbb{N} \to \Delta^3$ with

$$s_r{}^*(n) = (s^*_{0,r}(n), s^*_{1,r}(n), s^*_{2,r}(n))$$

$$= \left(\frac{\tilde{b}^4_{c_r^*}(1) - \tilde{b}_{c_r^*}(1)}{1 - \tilde{b}_{c_r^*}(1)}, \frac{\tilde{b}^5_{c_r^*}(1) - \tilde{b}^2_{c_r^*}(1)}{1 - \tilde{b}_{c_r^*}(1)}, \frac{1 - \tilde{b}^3_{c_r^*}(1)}{1 - \tilde{b}_{c_r^*}(1)} \right).$$

Then $\sigma(s_r^*) = s_r^*$ and using Lemma 3 we have

$$R^r f_{s_r^*} = f_{s_r^*}.$$

□

For a given pair of proper scaling data $s = (s_l, s_r)$, we defined a map

$$f_s : D_{s_l} \cup D_{s_r} \to [0, 1]$$

as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

Then, the renormalization of f_s is defined as

$$Rf_s(x) = \begin{cases} R^l f_{s_l}(x), & \text{if } x \in D_{s_l} \\ R^r f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

From Propositions 1 and 2, we conclude that the period tripling infinitely renormalizable maps $f_{s_l^*}$ and $f_{s_r^*}$ are fixed points of R^l and R^r corresponding to the proper scaling data s_l^* and s_r^*, respectively. Then, for a given pair of scaling data $s^* = (s_l^*, s_r^*)$, we have

$$\begin{aligned} Rf_{s^*}(x) &= \begin{cases} R^l f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ R^r f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= \begin{cases} f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= f_{s^*}(x) \end{aligned}$$

This will give us the following theorem,

Theorem 1 *There exists a map $f_{s^*} \in W$, where $s^* = (s_l^*, s_r^*)$ is characterized by*

$$Rf_{s^*} = f_{s^*}.$$

In particular, $W = \{f_{s^*}\}$.

Remark 1 The constructed map f_{s^*} with a pair of proper scaling data $s^* = (s_l^*, s_r^*)$ holds the following conditions,

(i) $s^*_{2,l} \leq (s^*_{1,l})^2$
(ii) $s^*_{2,r} \leq (s^*_{1,r})^2$

Note that for $i \in \{0, 1, 2\}$, the scaling ratios $s_{i,l}(n)$ are the expressions in the terms of c_n which are described in Eqs. (2)–(4). Therefore, one can easily compute $s^*_{0,l}$, $s^*_{1,l}$ and $s^*_{2,l}$ by substituting $c_n = c_l^*$ in the respective expressions. Then,

$$s^*_{2,l} = s_{2,l}(n)\big|_{c_n=c_l^*} \leq \left(s_{1,l}(n)\big|_{c_n=c_l^*}\right)^2 = \left(s^*_{1,l}\right)^2.$$

Similarly,

$$s^*_{2,r} = s_{2,r}(n)\big|_{c_n=c_r^*} \leq \left(s_{1,r}(n)\big|_{c_n=c_r^*}\right)^2 = \left(s^*_{1,r}\right)^2.$$

Remark 2 The invariant Cantor set of the map f_{s^*}, namely Λ_{s^*}, is next in complexity to the invariant doubling Cantor set, namely Λ_{σ^*}, of piece-wise affine period doubling infinitely renormalizable map f_{σ^*} [11] in the following sense,

(i) like the both Cantor sets Λ_{s^*} and Λ_{σ^*}, on each scale and everywhere the same scaling ratio s^* and σ^* are used respectively,
(ii) but unlike the doubling Cantor set Λ_{σ^*}, there are now a pair of three different ratios at each scale corresponding to s^*.

Furthermore, the geometry of the invariant Cantor set of f_{s^*} is different from the geometry of the invariant Cantor set of piece-wise affine period tripling renormalizable map because the Cantor set of f_{s^*} has 2−copy of Cantor set of [41].

3 C^{1+Lip} Extension of f_{s^*}

In Sect. 2, we have constructed a piece-wise affine infinitely renormalizable map f_{s^*} corresponding to the pair of scaling data $s^* = (s_l^*,\ s_r^*)$. Let us define a pair of scaling functions

$$S_l : [0,\ b_{c_l^*}(0)]^2 \to [0,\ b_{c_l^*}(0)]^2$$
$$S_r : [\tilde{b}_{c_r^*}(1),\ 1]^2 \to [\tilde{b}_{c_r^*}(1),\ 1]^2$$

as

$$S_l\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b^2_{c_l^*}(0) - s^*_{1,l}\cdot x \\ s^*_{2,l}\cdot y \end{pmatrix}; \qquad S_r\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{b}^2_{c_r^*}(1) + s^*_{1,r}\cdot(1-x) \\ 1 - s^*_{2,r}\cdot(1-y) \end{pmatrix}.$$

Let G be the graph of g_{s^*} which is an extension of f_{s^*} where $f_{s^*} : D_{s_l^*}\cup D_{s_r^*} \to [0,\ 1]$. Let G_l^1 and G_l^2 are the graphs of $g_{s^*}|_{[y_1,\ z_0]}$ which is an C^{1+Lip} extension of f_{s^*} on $D_{s_l^*}\cap[y_1,\ z_0]$ and $g_{s^*}|_{[y_0,\ z_1]}$ which is an C^{1+Lip} extension of f_{s^*} on $D_{s_l^*}\cap[y_0,\ z_1]$ respectively. Also, G_r^1 and G_r^2 are the graphs of $g_{s^*}|_{[z_0',\ y_1']}$ which is an C^{1+Lip} extension of f_{s^*} on $D_{s_r^*}\cap[z_0',\ y_1']$ and $g_{s^*}|_{[z_1',\ y_0']}$ which is an C^{1+Lip} extension of f_{s^*} on $D_{s_r^*}\cap[z_1',\ y_0']$ respectively which are shown in Fig. 14. Also, note that G_r^1 and G_r^2 are the reflections of G_l^1 and G_l^2 across the point $\left(\frac{1}{2},\ \frac{1}{2}\right)$ respectively. Define

$$G_l = \cup_{n\ge 0} S_l^n(G_l^1 \cup G_l^2) \ \text{ and } \ G_r = \cup_{n\ge 0} S_r^n(G_r^1 \cup G_r^2).$$

Then, G_l is the graph of a unimodal map $g_{s_l^*}$ which extends $f_{s_l^*}$ and G_r is the graph of a unimodal map $g_{s_r^*}$ which extends $f_{s_r^*}$. Consequently, G is the graph of $g_{s^*} = g_{s_l^*} \oplus g_{s_r^*}$. We claim that g_{s^*} is a C^{1+Lip} symmetric bimodal map.
Let $B_l^0 = [0,\ b_{c_l^*}(0)] \times [0,\ b_{c_l^*}(0)]$ and $B_r^0 = [\tilde{b}_{c_r^*}(1),\ 1] \times [\tilde{b}_{c_r^*}(1),\ 1]$.
For $n \in \mathbb{N}$, define

$$B_l^n = S_l^n(B_l^0) \qquad \text{and} \qquad B_r^n = S_r^n(B_r^0)$$

as

$$B_l^n = \begin{cases} [z_n,\ y_n] \times [0,\ \hat{y}_n], & \text{if } n \text{ is odd} \\ [y_n,\ z_n] \times [0,\ \hat{y}_n], & \text{if } n \text{ is even} \end{cases}$$

and

$$B_r^n = \begin{cases} [y_n',\ z_n'] \times [\hat{y}'_n,\ 1], & \text{if } n \text{ is odd} \\ [z_n',\ y_n'] \times [\hat{y}'_n,\ 1], & \text{if } n \text{ is even.} \end{cases}$$

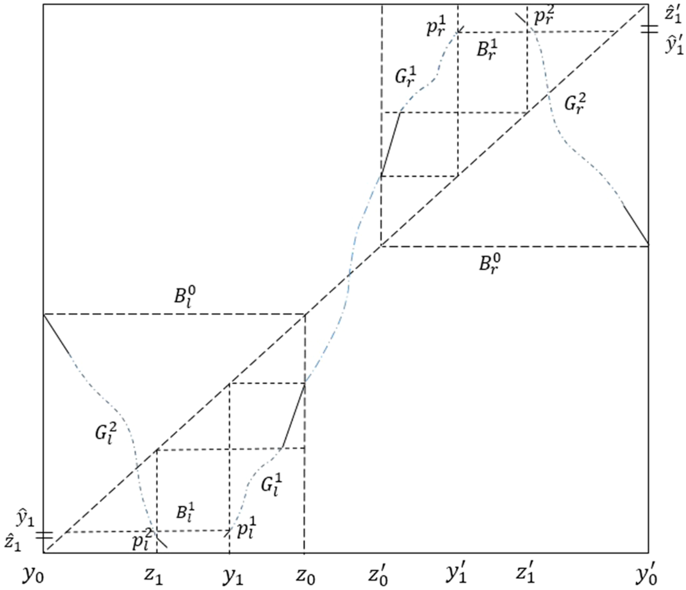


Fig. 14 Extension of f_{s^*}

Let p_l^n and p_r^n be the points on the graph of the bimodal map $b_{c_l^*}(x)$ and $b_{c_r^*}(x)$ respectively. For all $n \in \mathbb{N}$, p_l^n and p_r^n are defined as

$$p_l^n = \begin{cases} \begin{pmatrix} y_{\frac{n+1}{2}} \\ \hat{y}_{\frac{n+1}{2}} \end{pmatrix}, & \text{if } n \text{ is odd} \\ \begin{pmatrix} z_{\frac{n}{2}} \\ \hat{z}_{\frac{n}{2}} \end{pmatrix}, & \text{if } n \text{ is even} \end{cases}$$

$$p_r^n = \begin{cases} \begin{pmatrix} y'_{\frac{n+1}{2}} \\ \hat{y'}_{\frac{n+1}{2}} \end{pmatrix}, & \text{if } n \text{ is odd} \\ \begin{pmatrix} z'_{\frac{n}{2}} \\ \hat{z'}_{\frac{n}{2}} \end{pmatrix}, & \text{if } n \text{ is even} \end{cases}$$

where $\hat{y}_n = b_{c_l^*}(y_n)$, $\hat{z}_n = b_{c_l^*}(z_n)$, $\hat{y'}_n = \tilde{b}_{c_r^*}(y_n')$ and $\hat{z'}_n = \tilde{b}_{c_r^*}(z_n')$.

Then the above construction will lead to following proposition,

Proposition 3 *G is the graph of g_{s^*} which is a C^1 extension of f_{s^*}.*

Proof Since G_l^1 and G_l^2 are the graph of $f_{s_l^*}|_{[y_1,z_0]}$ and $f_{s_l^*}|_{[y_0,z_1]}$, respectively, and G_r^1 and G_r^2 are the graph of $f_{s_r^*}|_{[z_0',y_1']}$ and $f_{s_r^*}|_{[z_1',y_0']}$, respectively, we obtain $G_l^{2n+1} = S_l^n(G_l^1)$ and $G_l^{2n+2} = S_l^n(G_l^2)$ for each $n \in \mathbb{N}$. Note that G_l^n is the graph of a C^1 function defined

$$\begin{aligned} &\text{on } [z_{\frac{n-1}{2}},\ y_{\frac{n+1}{2}}] && \text{if } n \in 4\mathbb{N}-1, \\ &\text{on } [z_{\frac{n}{2}},\ y_{\frac{n}{2}-1}] && \text{if } n \in 4\mathbb{N}, \\ &\text{on } [y_{\frac{n+1}{2}},\ z_{\frac{n-1}{2}}] && \text{if } n \in 4\mathbb{N}+1, \\ \text{and } &\text{on } [y_{\frac{n}{2}-1},\ z_{\frac{n}{2}}] && \text{if } n \in 4\mathbb{N}+2. \end{aligned}$$

Also, we have $G_r^{2n+1} = S_r^n(G_r^1)$ and $G_r^{2n+2} = S_r^n(G_r^2)$ for each $n \in \mathbb{N}$. Note that G_r^n is the graph of a C^1 function defined

$$\begin{aligned} &\text{on } [y'_{\frac{n+1}{2}},\ z'_{\frac{n-1}{2}}] \quad \text{if } n \in 4\mathbb{N}-1,\\ &\text{on } [y'_{\frac{n}{2}-1},\ z'_{\frac{n}{2}}] \quad \text{if } n \in 4\mathbb{N},\\ &\text{on } [z'_{\frac{n-1}{2}},\ y'_{\frac{n+1}{2}}] \quad \text{if } n \in 4\mathbb{N}+1,\\ \text{and } &\text{on } [z'_{\frac{n}{2}},\ y'_{\frac{n}{2}-1}] \quad \text{if } n \in 4\mathbb{N}+2. \end{aligned}$$

To prove the proposition, we have to check continuous differentiability at the points p_l^n and p_r^n. Consider the neighborhoods $(y_1 - \epsilon, y_1 + \epsilon)$ around y_1 and $(z_1 - \epsilon, z_1 + \epsilon)$ around z_1, the slopes are given by an affine pieces of $f_{s_l^*}$ on the subintervals $(y_1 - \epsilon, y_1)$ and $(z_1, z_1 + \epsilon)$ and the slopes are given by the chosen C^1 extension on $(y_1, y_1 + \epsilon)$ and $(z_1 - \epsilon, z_1)$. This implies, G_l^1 and G_l^2 are C^1 at p_l^1 and p_l^2, respectively.
Let $\gamma_1 \subset G_l$ be the graph over the interval $(y_1 - \epsilon, y_1 + \epsilon)$ and $\gamma_2 \subset G_l$ be the graph over the interval $(z_1 - \epsilon, z_1 + \epsilon)$,
then the graph G_l locally around p_l^n is equal to $\begin{cases} S_l^{\frac{n-1}{2}}(\gamma_1) & \text{if } n \text{ is odd} \\ S_l^{\frac{n-2}{2}}(\gamma_2) & \text{if } n \text{ is even} \end{cases}$. This implies, for $n \in \mathbb{N}$, G_l^{2n-1} is C^1 at p_l^{2n-1} and G_l^{2n} is C^1 at p_l^{2n}.
Hence G_l is a graph of a C^1 function on $[0, b_{c_l^*}(0)] \setminus \{c_l^*\}$.
We note that the horizontal contraction of S_l is smaller than the vertical contraction. This implies that the slope of G_l^n tends to zero when n is large. Therefore, G_l is the graph of a C^1 function $g_{s_l^*}$ on $[0, b_{c_l^*}(0)]$.
In similar way, one can prove that G_r is the graph of a C^1 function $g_{s_r^*}$ on $[\tilde{b}_{c_r^*}(1), 1]$. Therefore, $G = G_l \oplus G_r$ is the graph of a C^1 bimodal map $g_{s^*} = g_{s_l^*} \oplus g_{s_r^*}$ which is a C^1 extension of f_{s^*}. □

Proposition 4 *Let g_{s^*} be the function whose graph is G then g_{s^*} is a C^{1+Lip} symmetric bimodal map.*

Proof As the function g_{s^*} is a C^1 extension of f_{s^*}. We have to show that, for $i \in \{l,\ r\}$, G_i^n is the graph of a C^{1+Lip} function

$$g_{s_i^*}^n : Dom(G_i^n) \to [0, 1]$$

with an uniform Lipschitz bound.
That is, for $n \geq 1$,

$$Lip((g_{s_i^*}^{n+1})') \leq Lip((g_{s_i^*}^n)')$$

let us assume that $g_{s_l^*}^n$ is C^{1+Lip} with Lipschitz constant λ_n for its derivatives. We show that $\lambda_{n+1} \leq \lambda_n$.
For given $\begin{pmatrix} u \\ v \end{pmatrix}$ on the graph of $g_{s_l^*}^n$, there is $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = S_l \begin{pmatrix} u \\ v \end{pmatrix}$ on the graph of $g_{s_l^*}^{n+1}$, this implies

$$g_{s_l^*}^{n+1}(\tilde{u}) = s_{2,l}^* \cdot g_{s_l^*}^n(u)$$

Since $u = \frac{b^2_{c_l^*}(0)-\tilde{u}}{s^*_{1,l}}$, we have

$$g^{n+1}_{s_l^*}(\tilde{u}) = s^*_{2,l} \cdot g^n_{s_l^*}\left(\frac{b^2_{c_l^*}(0)-\tilde{u}}{s^*_{1,l}}\right)$$

Differentiate both sides with respect to $\tilde{u}$, we get

$$\left(g^{n+1}_{s_l^*}\right)'(\tilde{u}) = -\frac{s^*_{2,l}}{s^*_{1,l}} \cdot \left(g^n_{s_l^*}\right)'\left(\frac{b^2_{c_l^*}(0)-\tilde{u}}{s^*_{1,l}}\right)$$

Therefore,

$$\begin{aligned}\left|\left(g^{n+1}_{s_l^*}\right)'(\tilde{u}_1) - \left(g^{n+1}_{s_l^*}\right)'(\tilde{u}_2)\right| &= \left|\frac{s^*_{2,l}}{s^*_{1,l}}\right| \cdot \left|\left(g^n_{s_l^*}\right)'\left(\frac{b^2_{c_l^*}(0)-\tilde{u}_1}{s^*_{1,l}}\right) - \left(g^n_{s_l^*}\right)'\left(\frac{b^2_{c_l^*}(0)-\tilde{u}_2}{s^*_{1,l}}\right)\right| \\ &\le \frac{s^*_{2,l}}{(s^*_{1,l})^2} \cdot \lambda\left(g^n_{s_l^*}\right)'|\tilde{u}_1 - \tilde{u}_2|\end{aligned}$$

From Remark 1, we have $(s^*_{1,l})^2 \ge s^*_{2,l}$. Then,

$$\lambda(g^{n+1}_{s_l^*})' \le \lambda(g^n_{s_l^*})' \le \lambda(g^1_{s_l^*})'.$$

Similarly, one can show that

$$\lambda(g^{n+1}_{s_r^*})' \le \lambda(g^n_{s_r^*})' \le \lambda(g^1_{s_r^*})'.$$

Therefore, choose $\lambda = max\{\lambda(g^1_{s_l^*})',\ \lambda(g^1_{s_r^*})'\}$ is the uniform Lipschitz bound. This completes the proof. □

Note that for a given pair of proper scaling data $s^* = (s_l^*, s_r^*)$, the piece-wise affine map f_{s^*} is infinitely renormalizable and g_{s^*} is a C^{1+Lip} extension of f_{s^*}. This implies g_{s^*} is also renormalizable map. Further, we observe that Rg_{s^*} is an extension of Rf_{s^*}. Therefore Rg_{s^*} is renormalizable. Hence, g_{s^*} is infinitely renormalizable map which is not a C^2 map. Then we have the following theorem,

Theorem 2 *There exists an infinitely renormalizable C^{1+Lip} symmetric bimodal map g_{s^*} such that*

$$Rg_{s^*} = g_{s^*}.$$

□

4 Topological Entropy of Renormalization

In this section, we calculate the topological entropy of the renormalization operator defined on the space of C^{1+Lip} bimodal maps.
Let us consider three pairs of C^{1+Lip} maps $\phi_i : [0, z_1] \cup [y_1, b_{c_l^*}(0)] \to [0, b_{c_l^*}(0)]$ and $\psi_i : [\tilde{b}_{c_r^*}(1), y_1'] \cup [z_1', 1] \to [\tilde{b}_{c_r^*}(1), 1]$, for $i = 0, 1, 2$, which extend f_{s^*}. Because of symmetricity, $\psi_i(x) = 1 - \phi_i(1-x)$. For a sequence $\alpha = \{\alpha_n\}_{n\ge1} \in \Sigma_3$,

where $\Sigma_3 = \{\{x_n\}_{n\geq 1} : x_n \in \{0,\ 1,\ 2\}\}$ is called full 3-Shift.
Now define

$$G_l^n(\alpha) = S_l^n(graph\ \phi_{\alpha_n}) \quad \text{and} \quad G_r^n(\alpha) = S_r^n(graph\ \psi_{\alpha_n}),$$

we have

$$G_l(\alpha) = \bigcup_{n\geq 1} G_l^n(\alpha) \quad \text{and} \quad G_r(\alpha) = \bigcup_{n\geq 1} G_r^n(\alpha).$$

Therefore, we conclude that $G(\alpha) = G_l(\alpha) \oplus G_r(\alpha)$ is the graph of a C^{1+Lip} bimodal map b_α by using the same facts of Sect. 3.
The shift map $\sigma : \Sigma_3 \to \Sigma_3$ is defined as

$$\sigma(\alpha_1\alpha_2\alpha_3\ldots) = (\alpha_2\alpha_3\alpha_4\ldots).$$

Proposition 5 *The restricted maps $b_\alpha^3 : [y_1,\ z_1] \to [y_1,\ z_1]$ and $b_\alpha^3 : [y_1',\ z_1'] \to [y_1',\ z_1']$ are the unimodal maps for all $\alpha \in \Sigma_3$. In particular, b_α is a renormalizable map and $Rb_\alpha = b_{\sigma(\alpha)}$.*

Proof We know that $b_\alpha : [y_1,\ z_1] \to I_{2,l}^1$ is a unimodal and onto, $b_\alpha : I_{2,l}^1 \to I_{0,l}^1$ is onto and affine and also $b_\alpha : I_{0,l}^1 \to [y_1,\ z_1]$ is onto and affine. Therefore b_α^3 is a unimodal map on $[y_1,\ z_1]$. Analogously, b_α^3 is a unimodal map on $[y_1',\ z_1']$. The above construction implies

$$Rb_\alpha = b_{\sigma(\alpha)}.$$

□

This gives us the following theorem.

Theorem 3 *The renormalization operator R acting on the space of C^{1+Lip} symmetric bimodal maps has unbounded topological entropy.*

Proof From the above construction, we conclude that $\alpha \longmapsto b_\alpha \in C^{1+Lip}$ is injective. The domain of R contains two copies, namely Λ_1 and Λ_2, of the full 3-shift. As topological entropy h_{top} is an invariant of topological conjugacy. Hence $h_{top}(R|_{\Lambda_1\cup\Lambda_2}) > \ln 3$. In fact, if we choose n different pairs of C^{1+Lip} maps, say, $\phi_0,\ \ \phi_1,\ \ \phi_2, \ldots \phi_{n-1}$ and $\psi_0,\ \ \psi_1,\ \ \psi_2, \ldots \psi_{n-1}$, which extends f_{s^*}, then it will be embedded two copies of the full $n-$ shift in the domain of R. Hence, the topological entropy of R on C^{1+Lip} symmetric bimodal maps is unbounded. □

5 An ϵ Perturbation of the Scaling Data

In this section, we use an ϵ perturbation on the construction of the scaling data as presented in Sect. 2, to obtain the following theorem

Theorem 4 *There exists a continuum of fixed points of the renormalization operator acting on C^{1+Lip} symmetric bimodal maps.*

Proof Consider an ϵ variation on scaling data and we modify the construction which is described in Sect. 2.

Let us define the neighborhoods N^l_ϵ and N^r_ϵ about the respective points $(b^3_c(0),\ b^4_c(0))$ and $(b^3_c(1),\ b^4_c(1))$ as

$$N^l_\epsilon(b^3_c(0),\ b^4_c(0)) = \{(b^3_c(0),\ \epsilon \cdot b^4_c(0))\ :\ \epsilon > 0\ \text{ and } \epsilon \text{ close to } 1\}$$
$$N^r_\epsilon(b^3_c(1),\ b^4_c(1)) = \{(b^3_c(1),\ \epsilon \cdot b^4_c(1))\ :\ \epsilon > 0\ \text{ and } \epsilon \text{ close to } 1\}$$

(i). The perturbed scaling data on I^l_0, then the scaling ratios are defined as

$$s_{2,l}(c,\epsilon) = \frac{b^3_c(0)}{b_c(0)}$$
$$s_{0,l}(c,\epsilon) = \frac{b_c(0) - \epsilon b^4_c(0)}{b_c(0)}$$
$$s_{1,l}(c,\epsilon) = \frac{b^2_c(0) - b_c(\epsilon b^4_c(0))}{b_c(0)},$$

where $c \in (0, \frac{3-\sqrt{3}}{6})$. Also, we define

$$\mathscr{R}(c,\epsilon) = \frac{b^2_c(0) - c}{s_{1,l}(c,\epsilon)}.$$

From Sect. 2.1, we know that the map $\mathscr{R}$ which is defined in Eq. 5, has unique fixed point c^*. Consequently, for a given ϵ close to 1, $\mathscr{R}(c,\epsilon)$ has only one unstable fixed point, namely c^*_ϵ. Therefore, we consider the perturbed scaling data $s^*_{l,\epsilon} : \mathbb{N} \to \Delta^3$ with

$$s^*_{l,\epsilon} = \left(\frac{b_{c^*_\epsilon}(0) - \epsilon b^4_{c^*_\epsilon}(0)}{b_{c^*_\epsilon}(0)},\ \frac{b^2_{c^*_\epsilon}(0) - b_{c^*_\epsilon}(\epsilon b^4_{c^*_\epsilon}(0))}{b_{c^*_\epsilon}(0)},\ \frac{b^3_{c^*_\epsilon}(0)}{b_{c^*_\epsilon}(0)} \right).$$

Then $\sigma(s^*_{l,\epsilon}) = s^*_{l,\epsilon}$ and using Lemma 1, we have

$$R^l f_{s^*_{l,\epsilon}} = f_{s^*_{l,\epsilon}}.$$

(ii). Considering the perturbed scaling data on I^r_0, one has the scaling data $s^*_{r,\epsilon} : \mathbb{N} \to \Delta^3$ with

$$s^*_{r,\epsilon} = \left(\frac{\epsilon b^4_{c^*_\epsilon}(1) - b_{c^*_\epsilon}(1)}{1 - b_{c^*_\epsilon}(1)},\ \frac{b_{c^*_\epsilon}(\epsilon b^4_{c^*_\epsilon}(1)) - b^2_{c^*_\epsilon}(1)}{1 - b_{c^*_\epsilon}(1)},\ \frac{1 - b^3_{c^*_\epsilon}(1)}{1 - b_{c^*_\epsilon}(1)} \right).$$

Then $\sigma(s^*_{r,\epsilon}) = s^*_{r,\epsilon}$ and using Lemma 3, we have

$$R^r f_{s^*_{r,\epsilon}} = f_{s^*_{r,\epsilon}}.$$

Moreover, $f_{s^*_{l,\epsilon}}$ and $f_{s^*_{r,\epsilon}}$ are the piece-wise affine maps which are infinitely renormalizable. For a given pair of proper scaling data $s^*_\epsilon = (s^*_{l,\epsilon},\ s^*_{r,\epsilon})$, we have

$$R f_{s^*_\epsilon} = f_{s^*_\epsilon}.$$

Now we use similar extension described in Sect. 3, then we get $g_{s^*_\epsilon}$ is the C^{1+Lip} extension of $f_{s^*_\epsilon}$. This implies that $g_{s^*_\epsilon}$ is a renormalizable map. As $R g_{s^*_\epsilon}$ is an extension of $R f_{s^*_\epsilon}$. Therefore $R g_{s^*_\epsilon}$ is renormalizable. Hence, for each ϵ close to 1, $g_{s^*_\epsilon}$ is a fixed point of the renormalization. This proves the existence of a continuum of fixed points of the renormalization. □

Remark 3 In particular, for two different perturbed scaling data $s_{\epsilon_1^*}$ and $s_{\epsilon_2^*}$, one can construct two infinitely renormalizable maps $g_{s_{\epsilon_1^*}}$ and $g_{s_{\epsilon_2^*}}$. Therefore, the respective Cantor attractors will have different scaling ratios. Consequently, it shows the non-rigidity for symmetric bimodal maps, whose smoothness is C^{1+Lip}.

6 Conclusions

In this paper, we have investigated the existence of fixed point of the renormalization operator which is defined on the space of symmetric bimodal maps with low smoothness. For a given pair of proper scaling data $s^* = (s_l^*, \ s_r^*)$, we have first constructed the piece-wise affine infinitely renormalizable map f_{s^*} which is the only fixed point of the renormalization. We observe that the geometry of invariant Cantor set is more complex than the geometry of the Cantor set of piece-wise affine period doubling renormalizable map [11]. Further, we have extended this fixed point f_{s^*} to a C^{1+Lip} symmetric bimodal map. Moreover, we proved that the renormalization operator acting on the space of C^{1+Lip} symmetric bimodal maps has infinite topological entropy. Finally, we proved the existence a continuum of fixed points of renormalization by considering a small perturbation on the scaling data. Consequently, it showed the non-rigidity of the Cantor attractors of infinitely renormalizable symmetric bimodal maps with low smoothness.

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