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Curie–Weiss Type Models for General Spin Spaces and Quadratic Pressure in Ergodic Theory

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Abstract

We extend results on quadratic pressure and convergence of Gibbs measures from Leplaideur and Watbled (Bull Soc Math France 147(2):197–219, 2019) to some general models for spin spaces. We define the notion of equilibrium state for the quadratic pressure and show that under some conditions on the maxima for some auxiliary function, the Gibbs measure converges to a convex combination of eigen-measures for the Transfer Operator. This extension works for dynamical systems defined by infinite-to-one maps. As an example, we compute the equilibrium for the mean-field XY model as the number of particles goes to $+\infty$.

Keywords Thermodynamic formalism · Equilibrium states · Curie–Weiss model · Spin spaces · Curie–Weiss–Potts model · Gibbs measure · Phase transition · XY model

Mathematics Subject Classification 37A35 · 37A50 · 37A60 · 82B20 · 82B30 · 82C26

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1 Introduction

1.1 Background, Main Motivations, Open Questions

Thermodynamic formalism has been introduced in Dynamical Systems in the 70's mainly by Sinai, Ruelle and Bowen (see [2,23,24]). The main motivation was to exhibit special invariant measures representing systems at equilibrium. Along the years, vocabulary from Physics has been used but for notions that may differ from the original ones in Physics. This is in particular the case for the notion of phase transition.

The present paper is the continuation of a recent work [20] where a dictionary between some phase transitions in Statistical Mechanics and their interpretation in Ergodic Theory was given.

In this paper we shall call *Probabilistic Gibbs Measures* (PGM for short) the Gibbs measures used in Physics or in Probability Theory and usually defined on finite lattices. On the other hand, *Dynamical Gibbs Measures* (DGM) will denote invariant measures usually considered in Ergodic Theory for Dynamical Systems, thus defined on an infinite system.

With this distinction, authors in [20] introduced the notion of quadratic pressure for dynamical systems and showed that for a generalization of the Curie–Weiss model, the PGM converge as the number of sites goes to $+\infty$ to a convex combination of measures associated to the DGM which maximize the quadratic pressure.

The number of these DGM is the number of maxima for a special auxiliary function $\varphi_\beta : \mathbb{R} \to \mathbb{R}$ and it was shown that a phase transition occurs at β_c when this number changes at β_c.

The Curie–Weiss–Potts model is usually considered as an extension of the Curie–Weiss model, although phase transitions are quite different (see below). We refer to [9,12] for a complete description of Ising, Curie–Weiss, and Potts models, and to [10] for the Curie–Weiss–Potts model. Even if for the Curie–Weiss–Potts model a similar result to Theorem 3 (in the present paper) was given in [20], a total description of the phenomenon with respect to quadratic pressure has not yet been done.

It was thus a natural question to investigate if a dictionary between Curie–Weiss general models for spin spaces and Ergodic Theory could be defined. In the present paper, we consider more general interactions between sites for mean field interactions and show that the approach of [20] can actually be adapted to these *generalized spin spaces*. These are the purposes of Theorems 2 and 3.

In view to give an application to the XY-model (see Sect. 5), the statement is done for dynamical systems which are not necessarily finite-to-one.

For that goal, the notion of entropy needs to be made precise. This notion has already been investigated and we mention e.g.a series of works [1,6,7,21] and more recently [13]. We re-employ here the idea to define the entropy as the Fenchel-Legendre transform of the pressure function and to link it to a min–max problem. We remind that *pressure* here has to be understood within ergodic viewpoint and is the logarithm of the spectral radius of the transfer operator.

Some version of the transfer operator with infinite-to-one map has already been studied in [21]. We point out that in our case we have a more flexible operator as the transition depends on the two first coordinates (see below). Actually, we believe that it can easily be extended to the case where transitions only depend on finitely many coordinates, which is what should be a natural extension of the notion of subshift of finite type with infinite (and uncountable) alphabet. In other words, the transfer operator in [21] is the one for a full-shift of finite type whereas our is the one for more general irreducible subshift of finite type.

The main purpose of Theorem 1 is thus to show the existence of the spectral gap for the transfer operator (one of the main assumption to use [13]) and also to study the regularity of the spectral radius, in particular for the multi-dimensional case.

Theorem 3 is where we actually make links between Dynamical Gibbs Measures and Probabilistic Gibbs Measures. It deals with convergence of the PGM to a convex combination of eigen-measures for the Transfer Operator. One of the key points is the Laplace method. This is where the auxiliary function φ_β is involved.

We remind that the Laplace method deals with integrals of the form $\int_I f(t)e^{n\phi(t)}dt$ and gives an equivalent of this quantity as n goes to $+\infty$. This equivalent only involves values c_i where ϕ is maximal. For each c_i one gets an expression $\gamma(c_i, n)$ depending on how flat ϕ is close to c_i and also on how f behaves close to c_i.

If I is some subset of $\mathbb{R}^d$ with non-empty interior, we emphasize that there is a big difference between $d = 1$ and the higher-dimensional case. For $d = 1$ we can, in general, compare all the $\gamma(c_i, n)$, up to the condition that they are finitely many, and whatever the flatness at each c_i is (as long as it is not totally degenerate). This does not hold for $d > 1$, and we only get case-by-case results, unless all the maxima have a non-degenerate Hessian. The consequence in our problem is that we can (in general) precisely determine what is the convex combination for the limit of the PGM, only if all the maxima are non-degenerate.

Once we link a phase transition with the change of maxima for an auxiliary function, it becomes easy to understand (and classify) different kinds of phase transitions. In spirit, a change in the number of the maxima may occur in two different ways. Either one maximum splits and produces several maxima. This is what happens in the Curie–Weiss model and also for the XY-model (see Sect. 5). Or, one local maxima far from the global ones becomes a global one. This is what happens in the Curie–Weiss–Potts model.

Now that we have at our disposal a kind of dictionary between the mean field theory and ergodic theory, we can formulate and translate some problems from one area to the other one, and expect to use tools from the latter to solve the problem of the former. We address here two questions of that kind.

Non-ergodic interacting particle systems. Some examples have quite recently been given or discussed (see [17,22]). It is clear from the mathematical point of view that even if the underlying dynamical system is uniquely ergodic, the conformal measure (with respect to some potential that has to be made precise) may be different from the unique invariant measure as it is not necessarily invariant. From [20] and Theorem 3 we have shown that thermodynamical limits for PGM are the conformal measures and not the DGM. This could thus be a way to exhibit new examples of these “non-ergodic” uniquely invariant systems.

Conservation of Gibbsianness under local transformations (see [19]) and non-linear pressure. In [5] the concept of quadratic pressure is extended to define a non-linear thermodynamic formalism. An equivalent to Theorem 2 is stated and also the convergence (at logarithmic scale) of the partition function is proved. Nevertheless at that stage, there is no equivalent to the PGM extending the quadratic pressure to general non-linear pressure. Following [26], it appears reasonable to expect that results from [19] could be used to define the good notion of PGM in non-linear thermodynamic formalism.

1.2 Settings and Results

1.2.1 Shift with (Possibly) Infinite Alphabet

Let (E, d) be a compact metric space, let ρ be a Borel probability measure on E with full support. We assume that ρ satisfies the following assumption

(H): for any sufficiently small $\varepsilon > 0$, $x \mapsto \rho(B(x, \varepsilon))$ is continuous.

We consider a map $A : E \times E \to [0, 1]$ called the transition function, which satisfies the following properties:

(**A1**) A is continuous with values in $\{0, 1\}$.
(**A2**) A is Lipschitz continuous with respect to the second variable with Lipschitz constant $\mathrm{Lip}(A)$.
(**A3**) A generates some mixing in the following sense.

$$\exists N \in \mathbb{N},\ \forall n \geq N,\ \forall a, b \in E,\ \exists z_1, \ldots, z_{n-1} \in E \\ \text{such that } A(a, z_1)A(z_1, z_2) \cdots A(z_{n-1}, b) = 1. \tag{1}$$

Remark 1 The assumption (A1) yields that A is constant with value 0 or 1 on each connected component of $E \times E$. The assumption (A3) implies in particular that A is not identically null. □

We define $\Omega \subset E^{\mathbb{N}}$ in the following way:

$$\Omega := \left\{ x = x_0x_1x_2 \cdots \in E^{\mathbb{N}};\ \forall i \in \mathbb{N},\ A(x_i, x_{i+1}) > 0 \right\}.$$

The shift map $\sigma : \Omega \to \Omega$ is defined by

$$\sigma(x_0x_1x_2 \cdots) = x_1x_2 \cdots .$$

Note that if E is connected, e.g.$E = [0, 1]$, then $A \equiv 1$. If E is a finite set $\{1, \ldots k\}$, then Ω is the subshift of finite type with transition matrix having entries $A(i, j)$.

For $n \geq 1$, let Ω_n be the set of words $z_1 \cdots z_n$ with $\prod_{i=1}^{n-1} A(z_i, z_{i+1}) = 1$. For a and b in E, let $\Omega_{n-1}(a, b)$ be the set of words $z_1 \cdots z_{n-1}$ in Ω_{n-1} with $A(a, z_1) = A(z_{n-1}, b) = 1$. Assumption (A3) on A means that for every a, b in E, for every $n \geq N$, $\Omega_{n-1}(a, b) \neq \emptyset$. It implies in particular that for every a in E, there always exist u, v in E such that $A(a, u) = 1$ and $A(v, a) = 1$. We denote by $\Omega_n(b)$ the set of words $z_0 \cdots z_{n-1}$ in Ω_n with $A(z_{n-1}, b) = 1$.

We set $\mathbb{P} = \rho^{\otimes \mathbb{N}}$. The distance on Ω is defined by

$$d_\Omega(x, y) = \sum_{n=0}^{+\infty} \frac{d(x_n, y_n)}{2^{n+1}}.$$

We notice that for any a in Ω_n,

$$d_\Omega(ax, ay) = \frac{1}{2^n} d_\Omega(x, y)$$

and that

$$d_\Omega(\sigma^n x, \sigma^n y) = 2^n \left(d_\Omega(x, y) - \sum_{k=0}^{n-1} \frac{d(x_k, y_k)}{2^{k+1}} \right).$$

We denote by $\mathcal{C}^0(\Omega)$, respectively $\mathcal{C}^{+1}(\Omega)$, the set of continuous, respectively Lipschitz continuous, functions from Ω to $\mathbb{R}$, equipped respectively with the norms

$$\|\phi\|_\infty = \max_{x\in\Omega}|\phi(x)|, \qquad \|\phi\|_L = \|\phi\|_\infty + \mathrm{Lip}(\phi),$$

where $\mathrm{Lip}(\phi)$ stands for the Lipschitz constant of ϕ. We recall that the spaces $(\mathcal{C}^0(\Omega), \|\cdot\|_\infty)$ and $(\mathcal{C}^{+1}(\Omega), \|\cdot\|_L)$ are Banach spaces. We set $M(\Omega)$ the space of probability measures on Ω and recall that by the Riesz representation theorem, the map $\mu \mapsto (f \mapsto \int f\,d\mu)$ is a bijection between $M(\Omega)$ and

$$\{l \in \mathcal{C}^0(\Omega)^*;\, l(\mathbb{1}) = 1 \text{ and } l(f) \geq 0 \text{ whenever } f \geq 0\},$$

where $*$ means the topological dual space and $\mathbb{1}$ is the constant function $\equiv 1$.

A measure μ is σ-invariant if $\mu(\sigma^{-1}(B)) = \mu(B)$ for all Borel sets B. The set $M_\sigma(\Omega)$ is the space of σ-invariant probability measures on Ω. Both $M(\Omega)$ and $M_\sigma(\Omega)$ are convex and compact for the weak star topology.

The transfer operator associated to $\phi : \Omega \to \mathbb{R}$ (Lipschitz continuous) is the linear operator defined by

$$\mathcal{L}_\phi(f)(\omega) = \int_E e^{\phi(t\omega)} A(t, \omega_0) f(t\omega)\, d\rho(t).$$

Theorem 1 states several properties on the spectrum of the transfer operator. To properly state the theorem we need to introduce some more quantities.

The operator $\mathcal{L}_\phi$ acts on $\mathcal{C}^0(\Omega)$ and on $\mathcal{C}^{+1}(\Omega)$. The spectral radius of $\mathcal{L}_\phi$ on $\mathcal{C}^0(\Omega)$, denoted by r_ϕ, is a simple eigenvalue of the adjoint operator $\mathcal{L}_\phi^*$ acting on the space of Radon measures on Ω, and the conformal measure ν_ϕ is the unique probability eigen-measure associated to the eigenvalue r_ϕ. It is also a simple eigenvalue of $\mathcal{L}_\phi$ acting on $\mathcal{C}^{+1}(\Omega)$, with a positive eigenfunction G_ϕ such that the measure $\mu_\phi = G_\phi \nu_\phi$ is a probability measure. We call μ_ϕ the dynamical Gibbs measure (DGM for short) associated to ϕ.

If z belongs to $\mathbb{R}^q$ and ψ_i, $i = 1, \ldots q$ are in $\mathcal{C}^{+1}(\Omega)$ one sets $\overrightarrow{\psi} := (\psi_1, \ldots, \psi_q)$ and $z \cdot \overrightarrow{\psi} := \sum_{i=1}^q z_i \psi_i$. We denote by $||z||$ the Euclidean norm of z

$$||z||^2 = \sum_{i=1}^q z_i^2.$$

Definition 1.1 For fixed $\overrightarrow{\psi}$ and $z \in \mathbb{R}^q$, one sets

$$\mathcal{H}(z, \overrightarrow{\psi}) := \inf_{t\in\mathbb{R}^q} \left\{\log r_{t\cdot\overrightarrow{\psi}} - t \cdot z\right\},$$

where $r_{t\cdot\overrightarrow{\psi}}$ is the spectral radius for $\mathcal{L}_{t\cdot\overrightarrow{\psi}}$, and

$$I(\overrightarrow{\psi}) := \left\{\int \overrightarrow{\psi}\, d\mu,\ \mu \in M_\sigma(\Omega)\right\}.$$

Note that $I(\overrightarrow{\psi})$ is a closed convex subset of $\mathbb{R}^q$. Moreover, $z \mapsto \mathcal{H}(z, \overrightarrow{\psi})$ is upper semi-continuous, as it is an infimum of affine functions.

Theorem 1 *For any $\phi \in \mathcal{C}^{+1}(\Omega)$, r_ϕ is a simple single dominating eigenvalue. Moreover, for any $\overrightarrow{\psi}$ with $\psi_i \in \mathcal{C}^{+1}(\Omega)$, the map $\mathcal{P} : t \mapsto \log r_{t\cdot\overrightarrow{\psi}}$ is infinitely differentiable.*

Furthermore,

(1) $\mathcal{H}(z, \overrightarrow{\psi}) = -\infty$ *if* $z \notin I(\overrightarrow{\psi})$,
(2) $\mathcal{H}(z, \overrightarrow{\psi})$ *is finite if* $z = \nabla\mathcal{P}(\boldsymbol{t})$ *for some* $\boldsymbol{t} \in \mathbb{R}^q$.

We call *pressure function* for $\overrightarrow{\psi}$ the map $\boldsymbol{t} \mapsto \mathcal{P}(\boldsymbol{t})$. In the following $\nabla\mathcal{P}(\mathbb{R}^q)$ will denote the set of z such that $z = \nabla\mathcal{P}(\boldsymbol{t})$ for some $\boldsymbol{t} \in \mathbb{R}^q$.

1.2.2 Quadratic Pressure

For fixed $\overrightarrow{\psi} \in \mathcal{C}^{+1}(\Omega)^q$ and for $\beta \geq 0$, $\boldsymbol{t}$, z in $\mathbb{R}^q$, we set

$$\varphi_\beta(\boldsymbol{t}) := -\frac{\beta}{2}||\boldsymbol{t}||^2 + \log r_{\beta\boldsymbol{t}\cdot\overrightarrow{\psi}} \text{ and } \overline{\varphi}_\beta(z) := \mathcal{H}(z, \overrightarrow{\psi}) + \frac{\beta}{2}||z||^2.$$

Notation 1 *We set* $\mathcal{H}_{top} := \log r_{\mathbf{0}}$.

Definition 1.2 For μ in $M_\sigma(\Omega)$, the *entropy* is the quantity

$$\widehat{\mathcal{H}}(\mu) := \inf_{\overrightarrow{\psi} \in \mathcal{C}^{+1}(\Omega)^q} \mathcal{H}\left(\int \overrightarrow{\psi}\, d\mu, \overrightarrow{\psi}\right).$$

Let $\overrightarrow{\psi} \in \mathcal{C}^{+1}(\Omega)^q$ be fixed. The quantity

$$\mathcal{P}_2(\beta) = \sup_\mu \left\{\widehat{\mathcal{H}}(\mu) + \frac{\beta}{2}\left\|\int \overrightarrow{\psi}\, d\mu\right\|^2\right\}$$

is referred to as the quadratic pressure function for $\overrightarrow{\psi}$.

Remark 2 In [13] the authors define the entropy by setting

$$h_{\mathcal{X}}(\mu) := \inf_{A \in \mathcal{X}(\Omega)} \left(\log r_A - \int A\, d\mu\right) \text{ for } \mu \in M_\sigma(\Omega)$$

and the pressure by setting

$$\Pr(B) := \sup_{\mu \in M_\sigma(\Omega)} \left(h_{\mathcal{X}}(\mu) + \int B\, d\mu\right) \text{ for } B \in \mathcal{X}(\Omega),$$

where $\mathcal{X}(\Omega)$ is a suitable space of potentials $\Omega \to \mathbb{R}$. We notice that our definition of entropy is the same as theirs with $\mathcal{X}(\Omega) = \mathcal{C}^{+1}(\Omega)$, whereas our definition of pressure is linked to theirs by $\mathcal{P}(\boldsymbol{t}) = \Pr(\boldsymbol{t} \cdot \overrightarrow{\psi})$, where $\overrightarrow{\psi}$ is fixed in $\mathcal{C}^{+1}(\Omega)^q$. We point out that $\mu \mapsto \widehat{\mathcal{H}}(\mu)$ is upper semi-continuous as the infimum over a family of upper semi-continuous functions. □

From Theorem 1 we can use the work of Giulietti et al. [13, th.F]. We emphasize that the key point in their work is the spectral decomposition of the transfer operator, which is stated in Theorem 1. Then, we deduce the existence of the DGM which is the unique equilibrium state for ϕ.

Stated with our settings, we get that for any $\overrightarrow{\psi} \in \mathcal{C}^{+1}(\Omega)^q$ and for any $\boldsymbol{t}$ there is a unique invariant measure $\mu_{\boldsymbol{t}.\overrightarrow{\psi}}$ which maximizes $\widehat{\mathcal{H}}(\mu) + \int \boldsymbol{t} \cdot \overrightarrow{\psi}\, d\mu$. Moreover,

$$\mathcal{P}(\boldsymbol{t}) = \widehat{\mathcal{H}}(\mu_{\boldsymbol{t}\cdot\overrightarrow{\psi}}) + \int \boldsymbol{t} \cdot \overrightarrow{\psi}\, d\mu_{\boldsymbol{t}\cdot\overrightarrow{\psi}}. \tag{2}$$

Convexity for the multi-dimensional pressure function $\boldsymbol{t} \mapsto \mathcal{P}(\boldsymbol{t}) = \sup_\mu\{\widehat{\mathcal{H}}(\mu) + \int \boldsymbol{t} \cdot \overrightarrow{\boldsymbol{\psi}}\, d\mu\}$ and differentiability obtained from Theorem 1 yield that for every $\boldsymbol{t}$ and every i, $\dfrac{\partial \log r_{\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}}{\partial t_i} = \displaystyle\int \psi_i \, d\mu_{\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$.

Theorem 2 Equilibrium states for the quadratic pressure *For any $\overrightarrow{\boldsymbol{\psi}} \in \mathcal{C}^{+1}(\Omega)^q$ and for any $\beta \geq 0$ the invariant probability measures which maximize $\mathcal{P}_2(\beta)$ are the dynamical Gibbs measures $\mu_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$ where the $\boldsymbol{t}$'s are the maxima for φ_β.*

This result goes in the same direction as the ones from [5,20]. However, we point out some interesting difference here: in the higher-dimensional case, there may be infinitely many measures which maximize the quadratic pressure. This is actually the case for XY-model (see below Remark 6).

1.2.3 Generalized Curie–Weiss Hamiltonian

For $\phi \in \mathcal{C}^{+1}(\Omega)$, we remind that $S_n(\phi)$ stands for $\phi + \phi\circ\sigma + \cdots + \phi\circ\sigma^{n-1} = \sum_{k=0}^{n-1} \phi\circ\sigma^k$. With previous notations, $S_n(\overrightarrow{\boldsymbol{\psi}})$ is the vector with coordinates $S_n(\psi_i)$. Then, the Generalized Curie–Weiss Hamiltonian is defined for $\omega \in \Omega$ by

$$H_n(\omega) := -\frac{1}{2n}\|S_n(\overrightarrow{\boldsymbol{\psi}})(\omega)\|^2.$$

We define the probabilistic Gibbs measure (PGM for short) $\mu_{n,\beta}$ on Ω by

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta H_n(\omega)}}{Z_{n,\beta}}\mathbb{P}(d\omega) = \frac{e^{\frac{\beta}{2n}\|S_n(\overrightarrow{\boldsymbol{\psi}})(\omega)\|^2}}{Z_{n,\beta}}\mathbb{P}(d\omega), \tag{3}$$

where $Z_{n,\beta}$ is the suitable normalization factor.

If P_n, P are probability measures in $M(\Omega)$, we say that P_n converges weakly to P if $\int_\Omega f\, dP_n \to \int_\Omega f\, dP$ for each f in $\mathcal{C}^0(\Omega)$. As $\mathcal{C}^{+1}(\Omega)$ is dense in $\mathcal{C}^0(\Omega)$, this is equivalent to $\int_\Omega f\, dP_n \to \int_\Omega f\, dP$ for each f in $\mathcal{C}^{+1}(\Omega)$.

Theorem 3 Generalized Curie–Weiss model *One-dimensional case: if $q = 1$, then the PGM $\mu_{n,\beta}$ converges weakly to a convex combination of the conformal measures $\nu_{\beta t\psi}$ associated to the $\mu_{\beta t\psi}$'s from Theorem 2 as n goes to $+\infty$.*

*Higher-dimensional case: for $q > 1$, if φ_β attains its maximum only on non-degenerate points (*i.e., *$d^2\varphi_\beta$ is invertible at those points), then they are finitely many and the PGM $\mu_{n,\beta}$ converges weakly to a convex combination of the conformal measures $\nu_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$ associated to the $\mu_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$'s, where the $\boldsymbol{t}$'s are the maxima for φ_β.*

Remark 3 The classical Curie–Weiss–Potts model (see Theorem 2.1 of [10]) is obtained by taking $E = \{1, \ldots q\}$, $\psi_i = \mathbb{1}_{[i]}$ and $A(i, j) = 1$ for every pair (i, j). □

We point out that the mean-field XY model (see Sect. 5) is an example where φ_β attains its maximum on an infinite set of points, and for all of them the Hessian is degenerate.

1.3 Plan of the Paper

In Sect. 2 we prove Theorem 1. As we said above, the main ingredient is to define and study the spectrum of the Transfer Operator. We prove that this operator has a spectral gap and that the spectral radius is a simple isolated dominating eigenvalue. This allows to define the notion of conformal measure.

In Sect. 3 we prove Theorem 2. The main ingredient is to define two auxiliary functions, φ_β and $\overline{\varphi}_\beta$, to show that one is always bigger than the other one, but both have the same maxima (arising for the same points). The maximal value for $\overline{\varphi}_\beta$ equals the quadratic pressure but maxima for φ_β are easier to detect. Part of the difficulty in this section comes from our way to define the entropy for measure, as we want to deal with possibly infinite-to-one maps.

In Sect. 4 we prove Theorem 3. The main trick is the Hubbard–Stratonovich formula and then the Laplace method, as in [20].

In Sect. 5 we discuss an application to the mean-field XY model.

2 Proof of Theorem 1

2.1 Properties for the Transfer Operator with Infinite Alphabet

2.1.1 First Spectral Properties: $\mathcal{L}_\phi$ is Quasi-compact

The function A is continuous thus uniformly continuous (since $E \times E$ is compact) and with values in $\{0, 1\}$. Therefore, there exists $\varepsilon_A \in]0, 1[$ such that for any u, u', t and t' in E satisfying $d(u, u') < \varepsilon_A$ and $d(t, t') < \varepsilon_A$,

$$A(t, u) = A(t', u'). \tag{4}$$

Lemma 2.1 *$\mathcal{L}_\phi$ acts on $\mathcal{C}^0(\Omega)$ and on $\mathcal{C}^{+1}(\Omega)$.*

Proof Let f be in $\mathcal{C}^0(\Omega)$. The function $x \mapsto e^{\phi(tx)}A(t, x_0)f(tx)$ is continuous on Ω and

$$|e^{\phi(tx)}A(t, x_0)f(tx)| \leq e^{\|\phi\|_\infty}\|f\|_\infty,$$

thus by the dominated convergence theorem $\mathcal{L}_\phi(f)$ is continuous on Ω. Moreover $\mathcal{L}_\phi$ acts continuously on $\mathcal{C}^0(\Omega)$ with operator norm

$$\|\mathcal{L}_\phi\|_\infty \leq e^{\|\phi\|_\infty}. \tag{5}$$

Now let f be in $\mathcal{C}^{+1}(\Omega)$. Notice that for any t in E and x, y in Ω,

$$d_\Omega(tx, ty) = \frac{1}{2}d_\Omega(x, y),$$

so that the function $f_t : x \mapsto f(tx)$ is Lipschitz with $\text{Lip}(f_t) \leq \frac{1}{2}\text{Lip}(f)$. It is easy to show that if ϕ is Lipschitz, then e^ϕ is Lipschitz with

$$\text{Lip}(e^\phi) \leq e^{\|\phi\|_\infty}\text{Lip}(\phi).$$

Notice also that for any t in E, the map $A_t : x \mapsto A(t, x_0)$ is Lipschitz with

$$\text{Lip}(A_t) \leq 2\text{Lip}(A).$$

As the product of two Lipschitz functions f and g is Lipschitz with

$$\text{Lip}(fg) \leq \|f\|_\infty\text{Lip}(g) + \|g\|_\infty\text{Lip}(f),$$

we easily deduce that $\mathcal{L}_\phi(f)$ is Lipschitz and that $\mathcal{L}_\phi$ acts continuously on $\mathcal{C}^{+1}(\Omega)$. □

Lemma 2.2 *The spectral radius on $C^0(\Omega)$ satisfies* $\log r_\phi = \lim_{n\to+\infty} \frac{1}{n} \log ||\mathcal{L}^n_\phi(\mathbb{1})||_\infty$.

Proof We recall that $r_\phi := \lim_{n\to+\infty} \|\mathcal{L}^n_\phi\|^{1/n}_\infty = \inf_{n\geq 1} \|\mathcal{L}^n_\phi\|^{1/n}_\infty$. For any f in $C^0(\Omega)$, $x \in \Omega$,

$$|\mathcal{L}_\phi(f)(x)| \leq \int_E e^{\phi(tx)} A(t, x_0)|f(tx)|\, d\rho(t) \leq \|f\|_\infty \mathcal{L}_\phi(\mathbb{1})(x) \leq \|f\|_\infty \|\mathcal{L}_\phi(\mathbb{1})\|_\infty,$$

hence $\|\mathcal{L}_\phi\|_\infty = \|\mathcal{L}_\phi(\mathbb{1})\|_\infty$. For $n \in \mathbb{N}$,

$$\mathcal{L}^n_\phi(f)(x) = \int_{\Omega_n(x_0)} e^{S_n(\phi)(tx)} f(tx)\rho^{\otimes n}(dt),$$

where t inside the integral stands for $t = t_0 \cdots t_{n-1}$ and $\rho^{\otimes n}(dt) = \prod_{i=0}^{n-1} \rho(dt_i)$. Then $\|\mathcal{L}^n_\phi\|_\infty = \|\mathcal{L}^n_\phi(\mathbb{1})\|_\infty$ for the same reason as for $n = 1$, hence $r_\phi = \lim_{n\to+\infty} \|\mathcal{L}^n_\phi(\mathbb{1})\|^{1/n}_\infty$. □

As the measure ρ is of full support and A satisfies the hypothesis (A3) it is easy to show that for any x in Ω, $\mathcal{L}_\phi(\mathbb{1})(x)$ is strictly positive. One then has that

$$\widehat{\mathcal{L}}^*_\phi(\mu) := \frac{\mathcal{L}^*_\phi(\mu)}{\mathcal{L}^*_\phi(\mu)(\mathbb{1})}$$

is a probability measure for any $\mu \in M(\Omega)$. The map $\widehat{\mathcal{L}}^*_\phi : M(\Omega) \to M(\Omega)$ is continuous on the compact (for the weak*-topology) convex space $M(\Omega)$ therefore by the Schauder-Tychonoff theorem, there exists a probability measure ν_ϕ such that

$$\widehat{\mathcal{L}}^*_\phi(\nu_\phi) = \nu_\phi.$$

This measure is either called the *conformal measure* or the *eigen-measure*. In the following we set $\lambda_\phi := \mathcal{L}^*_\phi(\nu_\phi)(\mathbb{1}) = \int \mathcal{L}_\phi(\mathbb{1})\, d\nu_\phi$.

Proposition 2.3 *With previous notations $r_\phi = \lambda_\phi$. Moreover for any $x \in \Omega$,*

$$\log r_\phi = \lim_{n\to+\infty} \frac{1}{n} \log \mathcal{L}^n_\phi(\mathbb{1})(x). \tag{6}$$

Proof Lipschitz regularity for ϕ yields that the Bowen condition holds: for any n, for any x and y satisfying $x_i = y_i$ for $i = 0, \ldots n-1$,

$$|S_n(\phi)(x) - S_n(\phi)(y)| \leq D\,\mathrm{Lip}(\phi), \tag{7}$$

where $D = \mathrm{Diam}(\Omega) = \max\{d_\Omega(x, y); x, y \in \Omega\}$. Indeed if $x_i = y_i$ for $i = 0, \ldots n-1$ then $d_\Omega(\sigma^k x, \sigma^k y) = 2^k d_\Omega(x, y)$ for $0 \leq k \leq n$, so that

$$\begin{aligned}|S_n(\phi)(x) - S_n(\phi)(y)| &\leq \sum_{k=0}^{n-1} \mathrm{Lip}(\phi) 2^k d_\Omega(x, y) \leq 2^n \mathrm{Lip}(\phi) d_\Omega(x, y) \\ &= \mathrm{Lip}(\phi) d_\Omega(\sigma^n(x), \sigma^n(y)).\end{aligned}$$

Lemma 2.4 *For any x, y in Ω such that $d_\Omega(x, y) < \frac{\varepsilon_A}{2}$, for any $n \geq 2$,*

$$e^{-D\mathrm{Lip}(\phi)} \leq \frac{\mathcal{L}^n_\phi(\mathbb{1})(x)}{\mathcal{L}^n_\phi(\mathbb{1})(y)} \leq e^{D\mathrm{Lip}(\phi)}. \tag{8}$$

Proof If $d_\Omega(x,y) < \frac{\varepsilon_A}{2}$ then $d(x_0, y_0) < \varepsilon_A$ so that $\Omega_n(x_0) = \Omega_n(y_0)$. For any $a \in \Omega_n(x_0)$, (7) implies that

$$|S_n(\phi)(ax) - S_n(\phi)(ay)| \leq D\,\mathrm{Lip}(\phi),$$

therefore

$$e^{S_n(\phi)(ay)}e^{-D\mathrm{Lip}(\phi)} \leq e^{S_n(\phi)(ax)} \leq e^{S_n(\phi)(ay)}e^{D\mathrm{Lip}(\phi)}$$

and by integrating over $\Omega_n(x_0)$ one gets

$$\mathcal{L}_\phi^n(\mathbb{1})(y)e^{-D\mathrm{Lip}(\phi)} \leq \mathcal{L}_\phi^n(\mathbb{1})(x) \leq \mathcal{L}_\phi^n(\mathbb{1})(y)e^{D\mathrm{Lip}(\phi)}.$$

□

We pick N_1 sufficiently big such that Assumption (A3) holds, that is

$$\forall a, b \in E,\ \ \Omega_{N_1-1}(a,b) \neq \emptyset.$$

Then, we choose N_2 sufficiently big such that $2^{-N_2} < \dfrac{\varepsilon_A}{2\mathrm{Diam}(\Omega)}$.

Claim 1 *There exists $C = C(\phi) > 0$ such that for every $n > N_1 + N_2$, for every x and y in Ω,*

$$e^{-C} \leq \frac{\mathcal{L}_\phi^n(\mathbb{1})(x)}{\mathcal{L}_\phi^n(\mathbb{1})(y)} \leq e^C. \tag{9}$$

Proof of the Claim We pick x and y in Ω. We denote by $\mathbf{t}$ an element $(t_1, \ldots, t_{N_2})$ of Ω_{N_2} and by $\mathbf{u}$ and $\mathbf{v}$ some elements of Ω_{N_1}. We set $N := N_2 + N_1$ and $m := n - N$.

$$\begin{aligned}
\mathcal{L}_\phi^n(\mathbb{1})(x) &= \mathcal{L}_\phi^N \circ \mathcal{L}_\phi^m(\mathbb{1})(x)\\
&= \iint e^{S_N(\phi)(\mathbf{tu}x)}A(t_{N_2}, u_1)A(u_{N_1}, x_0)\mathcal{L}_\phi^m(\mathbb{1})(\mathbf{tu}x)\,d\rho^{\otimes N_2}(\mathbf{t})d\rho^{\otimes N_1}(\mathbf{u})\\
&= \iint e^{S_{N_2}(\phi)(\mathbf{tu}x)}e^{S_{N_1}(\phi)(\mathbf{u}x)}A(t_{N_2}, u_1)A(u_{N_1}, x_0)\mathcal{L}_\phi^m(\mathbb{1})(\mathbf{tu}x)\,d\rho^{\otimes N_2}(\mathbf{t})d\rho^{\otimes N_1}(\mathbf{u})
\end{aligned}$$

where we used the identity

$$S_{N_1+N_2}(\phi) = S_{N_2}(\phi) + S_{N_1}(\phi) \circ \sigma^{N_2}.$$

Let us set $m_A = \inf_{t\in E} \rho(B(t, \varepsilon_A))$, which is positive thanks to hypotheses **(H)**. As $\Omega_{N_1-1}(u_1, y_0) \neq \emptyset$ we can pick $\mathbf{u}$ in this set. If $\mathbf{v} \in \Omega_{N_1}$ is such that $d(v_1, u_1) < \varepsilon_A$ and $d(v_i, u_{i-1}) < \varepsilon_A$ for every $2 \leq i \leq N_1$, then $A(v_{N_1}, y_0) = 1$, that is $\mathbf{v} \in \Omega_{N_1}(y_0)$. We deduce that

$$\int_{\Omega_{N_1}} A(v_{N_1}, y_0)\mathbb{1}_{B(u_1,\varepsilon_A)}(v_1)d\rho^{\otimes N_1}(\mathbf{v}) \geq m_A^{N_1}.$$

Therefore

$$\begin{aligned}
\mathcal{L}_\phi^n(\mathbb{1})(x) \leq \frac{1}{m_A^{N_1}} \iiint & e^{S_{N_2}(\phi)(\mathbf{tu}x)}e^{S_{N_1}(\phi)(\mathbf{u}x)}A(t_{N_2}, u_1)A(u_{N_1}, x_0)\mathcal{L}_\phi^m(\mathbb{1})(\mathbf{tu}x)\\
& A(v_{N_1}, y_0)\mathbb{1}_{B(u_1,\varepsilon_A)}(v_1)d\rho^{\otimes N_1}(\mathbf{v})d\rho^{\otimes N_2}(\mathbf{t})d\rho^{\otimes N_1}(\mathbf{u}).
\end{aligned}$$

From (7) we deduce that

$$e^{S_{N_2}(\phi)(\mathbf{tu}x)} \leq e^{D\mathrm{Lip}(\phi)}e^{S_{N_2}(\phi)(\mathbf{tv}y)}.$$

As $d_\Omega(\mathbf{tu}x, \mathbf{tv}y) = 2^{-N_2} d_\Omega(\mathbf{u}x, \mathbf{v}y) < \frac{\varepsilon_A}{2}$ we know from (8) that

$$\mathcal{L}_\phi^m(\mathbb{1})(\mathbf{tu}x) \le e^{D\mathrm{Lip}(\phi)} \mathcal{L}_\phi^m(\mathbb{1})(\mathbf{tv}y).$$

As $e^{S_{N_1}(\phi)(\mathbf{u}x)} \le e^{2N_1\|\phi\|_\infty} e^{S_{N_1}(\phi)(\mathbf{v}y)}$ we deduce eventually that

$$\begin{aligned}
\mathcal{L}_\phi^n(\mathbb{1})(x) \le & \frac{e^{2D\mathrm{Lip}(\phi)+2N_1\|\phi\|_\infty}}{m_A^{N_1}} \iiint e^{S_{N_2}(\phi)(\mathbf{tv}y)} e^{S_{N_1}(\phi)(\mathbf{v}y)} \mathcal{L}_\phi^m(\mathbb{1})(\mathbf{tv}y) A(t_{N_2}, u_1) \\
& A(u_{N_1}, x_0) A(v_{N_1}, y_0) \mathbb{1}_{B(u_1,\varepsilon_A)}(v_1)\, d\rho^{\otimes N_1}(\mathbf{v})\, d\rho^{\otimes N_2}(\mathbf{t}) d\rho^{\otimes N_1}(\mathbf{u}) \\
\le & \frac{e^{2D\mathrm{Lip}(\phi)+2N_1\|\phi\|_\infty}}{m_A^{N_1}} \iint e^{S_N(\phi)(\mathbf{tv}y)} \mathcal{L}_\phi^m(\mathbb{1})(\mathbf{tv}y) \\
& A(t_{N_2}, v_1) A(v_{N_1}, y_0)\, d\rho^{\otimes N_2}(\mathbf{t}) d\rho^{\otimes N_1}(\mathbf{v}) \\
& \text{where we use } A(t_{N_2}, v_1) = A(t_{N_2}, u_1), \\
\le & \frac{e^{2D\mathrm{Lip}(\phi)+2N_1\|\phi\|_\infty}}{m_A^{N_1}} \mathcal{L}_\phi^n(\mathbb{1})(y).
\end{aligned}$$

Exchanging x and y we get the reverse inequality. □

We can now finish the proof of Proposition 2.3. First we recall that according to Lemma 2.2, $\log r_\phi = \lim_{n\to+\infty} \frac{1}{n} \log ||\mathcal{L}_\phi^n(\mathbb{1})||_\infty$. As Ω is compact there exists $x_n \in \Omega$ such that $\|\mathcal{L}_\phi^n(\mathbb{1})||_\infty = \mathcal{L}_\phi^n(\mathbb{1})(x_n)$. For any x in Ω and $n > N_1 + N_2$ we get from (9) that

$$e^{-C} \mathcal{L}_\phi^n(\mathbb{1})(x_n) \le \mathcal{L}_\phi^n(\mathbb{1})(x) \le e^C \mathcal{L}_\phi^n(\mathbb{1})(x_n), \tag{10}$$

therefore

$$-\frac{C}{n} + \frac{1}{n} \log \mathcal{L}_\phi^n(\mathbb{1})(x_n) \le \frac{1}{n} \log \mathcal{L}_\phi^n(\mathbb{1})(x) \le \frac{C}{n} + \frac{1}{n} \log \mathcal{L}_\phi^n(\mathbb{1})(x_n),$$

and taking the limit we get (6). Now integrating (10) we get

$$e^{-C} \mathcal{L}_\phi^n(\mathbb{1})(x_n) \le \int \mathcal{L}_\phi^n(\mathbb{1})(x)\, d\nu_\phi(x) \le e^C \mathcal{L}_\phi^n(\mathbb{1})(x_n),$$

and since $\lambda_\phi^n = \int \mathcal{L}_\phi^n(\mathbb{1})(x)\, d\nu_\phi(x)$ we get $\log \lambda_\phi = \log r_\phi$. □

We claim that we can apply the Ionescu-Tulcea & Marinescu Theorem[1] (see [16], see also [4], Theorem 4.2 or [11], Theorem 2.1) to get a spectral decomposition of the operator $\widetilde{\mathcal{L}}_\phi := \frac{1}{r_\phi} \mathcal{L}_\phi$. Indeed the spaces $\mathcal{C}^0(\Omega)$, $\mathcal{C}^{+1}(\Omega)$ satisfy the first hypotheses of the ITM Theorem, which is

(1) if $f_n \in \mathcal{C}^{+1}(\Omega)$, $f \in \mathcal{C}^0(\Omega)$, $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, and $\|f_n\|_L \le C$ for all n, then $f \in \mathcal{C}^{+1}(\Omega)$ and $\|f\|_L \le C$,

and $\widetilde{\mathcal{L}}_\phi$ satisfies the three following hypotheses:

(2) $\sup_{n\in\mathbb{N}}\{\|\widetilde{\mathcal{L}}_\phi^n(f)\|_\infty,\ f \in \mathcal{C}^{+1}(\Omega),\ \|f\|_L \le 1\} < +\infty$,

[1] ITM Theorem in short.

(3) there exists $a \in]0,1[$, $b > 0$ and $n_0 \geq 1$ such that for any $f \in \mathcal{C}^{+1}(\Omega)$,

$$\|\widetilde{\mathcal{L}}_\phi^{n_0}(f)\|_L \leq a\|f\|_L + b\|f\|_\infty,$$

(4) if V is a bounded subset of $(\mathcal{C}^{+1}(\Omega), \|\cdot\|_L)$, then $\widetilde{\mathcal{L}}_\phi^{n_0}(V)$ has compact closure in $(\mathcal{C}^0(\Omega), \|\cdot\|_\infty)$.

We sketch the proof of (3) and let the reader check the other conditions.
Proof of (3). A direct computation yields that for f Lipschitz continuous

$$\left|\mathcal{L}_\phi^n(f)(x) - \mathcal{L}_\phi^n(f)(y)\right| \leq \mathrm{Lip}(f)\frac{d_\Omega(x,y)}{2^n}\mathcal{L}_\phi^n(\mathbb{1})(x) + e^{n\|\phi\|_\infty}\|f\|_\infty(\mathrm{Lip}(\phi) + 2\mathrm{Lip}(A))d_\Omega(x,y).$$

From (9) we know that for any $n > N_1 + N_2$, for any x in Ω,

$$e^{-C} \leq \frac{\mathcal{L}_\phi^n(\mathbb{1})(x)}{\lambda_\phi^n} \leq e^C,$$

hence as $r_\phi = \lambda_\phi$ we have

$$\left|\widetilde{\mathcal{L}}_\phi^n(f)(x) - \widetilde{\mathcal{L}}_\phi^n(f)(y)\right| \leq \mathrm{Lip}(f)\frac{d_\Omega(x,y)}{2^n}e^C + \frac{e^{n\|\phi\|_\infty}}{r_\phi^n}\|f\|_\infty(\mathrm{Lip}(\phi) + 2\mathrm{Lip}(A))d_\Omega(x,y).$$

Therefore

$$\|\widetilde{\mathcal{L}}_\phi^n(f)\|_L = \mathrm{Lip}(\widetilde{\mathcal{L}}_\phi^n(f)) + \|\widetilde{\mathcal{L}}_\phi^n(f)\|_\infty \leq A_n\mathrm{Lip}(f) + B_n\|f\|_\infty \leq A_n\|f\|_L + B_n\|f\|_\infty$$

where $A_n = \dfrac{e^C}{2^n}$ and $B_n = \dfrac{e^{n\|\phi\|_\infty}}{r_\phi^n}(\mathrm{Lip}(\phi) + 2\mathrm{Lip}(A) + 1)$. Picking any a in $]0,1[$ and adjusting n such that $2^{-n}e^C < a$ one gets the result. □

In particular, and considering $\mathcal{L}_\phi$ as an operator on $\mathcal{C}^{+1}(\Omega)$, the proof of the ITM Theorem shows (see [4, Lem. 4.7], or [11, Lemma 2.4]) that r_ϕ is an eigenvalue for $\mathcal{L}_\phi$ associated to the function in $\mathcal{C}^{+1}(\Omega)$ defined by

$$G_\phi := \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{\mathcal{L}}_\phi^k(\mathbb{1}).$$

Furthermore, we have the following decomposition

$$\widetilde{\mathcal{L}}_\phi := \sum e^{i\theta_j}\Pi_j + \Psi$$

where the Π_j's are (finitely many) projectors with finite rank, the θ_j's are real numbers and Ψ has spectral radius strictly smaller than 1. Moreover,

$$\Pi_k\Pi_j = 0 \text{ if } j \neq k \text{ and } \Psi\Pi_j = \Pi_j\Psi = 0.$$

2.1.2 Second Decomposition of the Spectrum: r_ϕ is the Unique Eigenvalue with Maximal Modulus and Its Eigenspace is of Dimension One

For simplicity we set $\theta_0 = 0$. We shall see that mixing yields more precise results on the spectral decomposition of $\mathcal{L}_\phi$.

Lemma 2.5 *For any $x \in \Omega$, $\bigcup_{n\geq 0} \sigma^{-n}(\{x\})$ is dense in Ω.*

Proof Let x and y be in Ω, let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $2^{-n}\mathrm{Diam}(\Omega) < \varepsilon$, and let $z_i = y_i$ for every i in $\{0, \dots, n-1\}$, so that $d_\Omega(y, z) \leq \varepsilon$. According to assumption **(A3)** there exist $u_n, \dots, u_{n+N-2}$ in E such that $z := y_0 \cdots y_{n-1} u_n \cdots u_{n+N-2} x$ belongs to Ω. Then z belongs to $\sigma^{-(n+N-1)}(\{x\}) \cap B(y, \varepsilon)$.

Remark 4 Actually, we have proved a better result: for any ε, there exists $N' = N'(\varepsilon)$ such that for any y and x, $B(y, \varepsilon) \cap \sigma^{-N'}(\{x\}) \neq \emptyset$. □

Proposition 2.6 *The spectral radius r_ϕ is a simple single dominating eigenvalue. The rest of the spectrum for $\mathcal{L}_\phi$ is a compact set strictly inside the disk $\mathbb{D}(0, r_\phi)$.*

Proof Because of the first result on the spectrum of $\mathcal{L}_\phi$, it remains to prove that r_ϕ is simple and that any other eigenvalue has modulus strictly lower than r_ϕ. For that we use spectral properties of positive operators exposed in [18, chap. 1& 2]. We claim that the set K of non-negative Lipschitz functions is a *solid and reproducing cone*. Solid means it has non-empty interior and reproducing means

$$\mathcal{C}^{+1}(\Omega) = K - K.$$

It is easy to see that any positive Lipschitz function is in $\overset{\circ}{K}$.

• **Step one** We prove that for any $f \not\equiv 0 \in K$, there exists p such that $\mathcal{L}_\phi^p(f)$ belongs to $\overset{\circ}{K}$.

Let y be such that $f(y) > 0$. Let $\varepsilon > 0$ be such that $d_\Omega(y, y') \leq \varepsilon \implies f(y') > 0$. According to Remark 4, there exists $p \in \mathbb{N}$ such that for any $x \in \Omega$, $B(y, \varepsilon) \cap \sigma^{-p}(\{x\}) \neq \emptyset$. Then $\mathcal{L}_\phi^p(f)$ is positive. Indeed let $x \in \Omega$, and z in $B(y, \varepsilon) \cap \sigma^{-p}(\{x\})$. Let $\eta := \min(\varepsilon_A, \frac{\varepsilon}{2})$. We remind that ε_A has been fixed just above formula (4). The definition of ε_A yields that every t in $\prod_{i=0}^{p-1} B(z_i, \eta)$ belongs to $\Omega_p(x_0)$, therefore

$$\mathcal{L}_\phi^p(f)(x) = \int_{\Omega_p(x_0)} e^{S_p(\phi)(tx)} f(tx) \rho^{\otimes p}(dt) \geq \int_{\prod_{i=0}^{p-1} B(z_i,\eta)} e^{S_p(\phi)(tx)} f(tx) \rho^{\otimes p}(dt).$$

But if $t \in \prod_{i=0}^{p-1} B(z_i, \eta)$ then

$$d_\Omega(tx, y) \leq d_\Omega(tx, z) + d_\Omega(z, y) \leq \eta + \frac{\varepsilon}{2} \leq \varepsilon$$

hence $f(tx) > 0$. As any non-empty ball in E has positive ρ-measure we deduce that $\mathcal{L}_\phi^p(f)(x) > 0$.

• **Step two** End of the proof. We deduce from step one that $\mathcal{L}_\phi$ is strongly positive (see [18, Definitions 2.1.1]). Therefore it is u-positive for any $u \in \overset{\circ}{K}$. From Th. 2.10, 2.11 and 2.13

we deduce that r_ϕ is a simple eigenvalue and that every other eigenvalue λ of $\mathcal{L}_\phi$ satisfies the inequality $|\lambda| < r_\phi$. □

To re-employ notation from above, there is only one Π_0, no other Π_i's. Furthermore, using the fact that ν_ϕ is an eigenmeasure, one easily gets that for any $f \in \mathcal{C}^{+1}(\Omega)$,

$$\frac{1}{r_\phi}\mathcal{L}_\phi(f) = \underbrace{\left(\int f\, d\nu_\phi\right) \cdot G_\phi}_{=\Pi_0(f)} + \Psi(f). \tag{11}$$

Because $\Pi_0\Psi = \Psi\Pi_0 = 0$, we also immediately get

$$\forall\, n \geq 1,\ \frac{1}{r_\phi^n}\mathcal{L}_\phi^n(f) = \left(\int f\, d\nu_\phi\right) \cdot G_\phi + \Psi^n(f). \tag{12}$$

2.2 Gibbs Measure and Ergodic Properties

2.2.1 The Gibbs Measure and Its Main Properties

Let μ_ϕ be the measure defined by $d\mu_\phi := G_\phi d\nu_\phi$. We emphasize that by construction μ_ϕ is a probability measure. We shall use the following fact: for every $n \in \mathbb{N}$,

$$\mathcal{L}_\phi^n(f.g \circ \sigma^n) = g.\mathcal{L}_\phi^n(f). \tag{13}$$

Lemma 2.7 *The measure μ_ϕ is σ-invariant. It is called the* Dynamical Gibbs Measure (DGM in short) *associated to ϕ.*

Proof For f continuous

$$\begin{aligned}\int f \circ \sigma\, d\mu_\phi &= \int f \circ \sigma.G_\phi\, d\nu_\phi \\ &= \frac{1}{r_\phi}\int \mathcal{L}_\phi(f \circ \sigma.G_\phi)\, d\nu_\phi \\ &= \frac{1}{r_\phi}\int f.\mathcal{L}_\phi(G_\phi)\, d\nu_\phi = \int f\, d\mu_\phi,\end{aligned}$$

where we used $\mathcal{L}_\phi^*(\nu_\phi) = r_\phi.\nu_\phi$ to get the second equality and (13) to get the third. □

Proposition 2.8 *The measure μ_ϕ is mixing thus ergodic.*

Proof Let f and g be two functions in $\mathcal{C}^{+1}(\Omega)$. Then (12) yields

$$\begin{aligned}\int f.g \circ \sigma^n\, d\mu_\phi &= \int f.g \circ \sigma^n.G_\phi\, d\nu_\phi \\ &= \frac{1}{r_\phi^n}\int \mathcal{L}_\phi^n(fG_\phi.g \circ \sigma^n)\, d\nu_\phi \\ &= \frac{1}{r_\phi^n}\int \mathcal{L}_\phi^n(f.G_\phi).g\, d\nu_\phi \\ &= \int \left(\left(\int f.G_\phi\, d\nu_\phi\right).G_\phi + \Psi^n(f.G_\phi)\right).g\, d\nu_\phi.\end{aligned}$$

We have seen that the spectral radius of Ψ is strictly lower than 1. Therefore $\Psi^n(f.G_\phi)$ goes to 0 for the Lipschitz norm, thus for the continuous norm. This yields

$$\int f.g\circ\sigma^n\,d\mu_\phi \to_{n\to+\infty} \int f\,d\mu_\phi \int g\,d\mu_\phi,$$

and the proposition is proved. □

2.2.2 Further Properties

Lemma 2.9 *There exists $C(\phi)$ such that for every x, $e^{-C(\phi)} \le G_\phi(x) \le e^{C(\phi)}$.*

Proof By definition, $G_\phi \ge 0$. Let us prove by contradiction it is positive. Assume that $G_\phi(x) = 0$. Then, $\mathcal{L}_\phi(G_\phi) = r_\phi G_\phi$ shows that $G_\phi(tx) = 0$ for ρ-*a.e.* t in E such that $A(t, x_0) > 0$. As A and G_ϕ are continuous and ρ has full support, this yields that for every t such that $A(t, x_0) > 0$ $G_\phi(tx) = 0$. In other words, for every y in $\sigma^{-1}(\{x\})$, $G_\phi(y) = 0$. By induction we deduce that for every $n \in \mathbb{N}$, for every z in $\sigma^{-n}(\{x\})$, $G_\phi(z) = 0$. Now, the set $\cup_{n\ge0}\sigma^{-n}(\{x\})$ is dense, and G_ϕ is continuous everywhere and null on a dense set. It is thus null everywhere which is impossible because $\int G_\phi\,d\nu_\phi = 1$. This shows that G_ϕ is positive, thus bounded from below by some constant of the form $e^{-C(\phi)}$. Furthermore, Ω is compact and then G_ϕ is bounded from above. □

Lemma 2.9 immediately yields

Corollary 2.10 *Both measures μ_ϕ and ν_ϕ are equivalent.*

2.2.3 Regularity of the Spectral Radius

Proposition 2.11 *The map $P : \phi \mapsto \log r_\phi$ is convex on $\mathcal{C}^{+1}(\Omega)$.*

Proof Let us pick ϕ_1, ϕ_2 in $\mathcal{C}^{+1}(\Omega)$, and $\alpha \in [0, 1]$. Set $\phi := \alpha\phi_1 + (1-\alpha)\phi_2$. For $n \in \mathbb{N}$, $x \in \Omega$,

$$\begin{aligned}
\mathcal{L}^n_\phi(\mathbb{1})(x) &= \int_{E^n} e^{S_n(\phi)(tx)}\mathbb{1}_{\Omega_n(x_0)}(t)\rho^{\otimes n}(dt)\\
&= \int_{E^n} e^{\alpha S_n(\phi_1)(tx)}\mathbb{1}^{\alpha}_{\Omega_n(x_0)}(t)e^{(1-\alpha)S_n(\phi_2)(tx)}\mathbb{1}^{1-\alpha}_{\Omega_n(x_0)}(t)\rho^{\otimes n}(dt)\\
&\le \left(\int_{E^n} e^{S_n(\phi_1)(tx)}\mathbb{1}_{\Omega_n(x_0)}(t)\rho^{\otimes n}(dt)\right)^{\alpha}\left(\int_{E^n} e^{S_n(\phi_2)(tx)}\mathbb{1}_{\Omega_n(x_0)}(t)\rho^{\otimes n}(dt)\right)^{1-\alpha}
\end{aligned}$$

therefore

$$\frac{1}{n}\log\left(\mathcal{L}^n_\phi(\mathbb{1})(x)\right) \le \alpha\frac{1}{n}\log\left(\mathcal{L}^n_{\phi_1}(\mathbb{1})(x)\right) + (1-\alpha)\frac{1}{n}\log\left(\mathcal{L}^n_{\phi_2}(\mathbb{1})(x)\right).$$

We deduce from (6) that

$$\log r_{\alpha\phi_1+(1-\alpha)\phi_2} \le \alpha\log r_{\phi_1} + (1-\alpha)\log r_{\phi_2},$$

which proves the convexity of P. □

Let $\overrightarrow{\psi} = (\psi_1, \ldots, \psi_q) \in \mathcal{C}^{+1}(\Omega)^q$. We recall the definition

$$I(\overrightarrow{\psi}) := \left\{ \int \overrightarrow{\psi}\, d\mu,\ \mu \in M_\sigma(\Omega) \right\}.$$

By definition $I(\overrightarrow{\psi})$ is a convex and closed set.

Proposition 2.12 *The map $\mathcal{P} : \boldsymbol{t} \mapsto \log r_{\boldsymbol{t}\cdot\overrightarrow{\psi}}$ is convex and infinitely differentiable on $\mathbb{R}^q$ with*

$$\nabla\mathcal{P}(\boldsymbol{t}) = \int \overrightarrow{\psi}\, d\mu_{\boldsymbol{t}\cdot\overrightarrow{\psi}}. \tag{14}$$

For any $z = \nabla\mathcal{P}(\boldsymbol{t})$ in $\nabla\mathcal{P}(\mathbb{R}^q)$, $\mathcal{H}(z, \overrightarrow{\psi})$ is finite with

$$\mathcal{H}(\nabla\mathcal{P}(\boldsymbol{t}), \overrightarrow{\psi}) = \mathcal{P}(\boldsymbol{t}) - \boldsymbol{t}\cdot\nabla\mathcal{P}(\boldsymbol{t}) = \log r_{\boldsymbol{t}\cdot\overrightarrow{\psi}} - \int \boldsymbol{t}\cdot\overrightarrow{\psi}\, d\mu_{\boldsymbol{t}\cdot\overrightarrow{\psi}}. \tag{15}$$

If z does not belong to the closure $\overline{\nabla\mathcal{P}(\mathbb{R}^q)}$ of $\nabla\mathcal{P}(\mathbb{R}^q)$, in particular when $z \notin I(\overrightarrow{\psi})$, then $\mathcal{H}(z, \overrightarrow{\psi}) = -\infty$.

Proof The convexity of $\mathcal{P}$ follows from Proposition 2.11. The map Q with values in $\mathcal{L}(\mathcal{C}^{+1}(\Omega))$ defined on $\mathbb{R}^q$ by

$$Q(\boldsymbol{t}) = \mathcal{L}_{\boldsymbol{t}\cdot\overrightarrow{\psi}}$$

is infinitely differentiable with

$$\frac{\partial Q}{\partial t_k}(\boldsymbol{t})(g) = Q(\boldsymbol{t})(\psi_k g).$$

Adapting the proof of Thm. III.8 and Corollary III.11. of [14] we see that the map $\boldsymbol{t} \mapsto r_{\boldsymbol{t}\cdot\overrightarrow{\psi}}$ is infinitely differentiable with

$$\frac{\partial r_{\boldsymbol{t}\cdot\overrightarrow{\psi}}}{\partial t_k}(\boldsymbol{t}) = r_{\boldsymbol{t}\cdot\overrightarrow{\psi}} \int \psi_k\, d\mu_{\boldsymbol{t}\cdot\overrightarrow{\psi}},$$

from which we deduce (14).

The conjugate function $\mathcal{P}^*$ of $\mathcal{P}$, defined by

$$\mathcal{P}^*(z) = \sup_{\boldsymbol{t}\in\mathbb{R}^q} (\boldsymbol{t}\cdot z - \mathcal{P}(\boldsymbol{t})),$$

is convex on $\mathbb{R}^q$ with values in $]-\infty, +\infty]$. We refer for instance to [25], section 26, for the theory of conjugates of convex functions. In particular it is known that

$$\nabla\mathcal{P}(\mathbb{R}^q) \subset \mathrm{dom}\mathcal{P}^* \subset \overline{\nabla\mathcal{P}(\mathbb{R}^q)},$$

where $\mathrm{dom}\mathcal{P}^* = \{z \in \mathbb{R}^q,\ \mathcal{P}^*(z) < +\infty\}$, with

$$\mathcal{P}^*(\nabla\mathcal{P}(\boldsymbol{t})) = \boldsymbol{t}\cdot\nabla\mathcal{P}(\boldsymbol{t}) - \mathcal{P}(\boldsymbol{t}).$$

As $\mathcal{H}(\cdot, \overrightarrow{\psi}) = -\mathcal{P}^*$ the proof is finished. □

3 Proof of Theorem 2

3.1 Auxiliary Functions φ and $\overline{\varphi}_\beta$

We recall the definitions of φ_β and $\overline{\varphi}_\beta$ defined on $\mathbb{R}^q$:

$$\varphi_\beta(\boldsymbol{t}) = -\frac{\beta}{2}\|\boldsymbol{t}\|^2 + \log r_{\beta \boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}, \quad \overline{\varphi}_\beta(z) := \mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}}) + \frac{\beta}{2}\|z\|^2. \tag{16}$$

3.1.1 The Function φ_β

We remind the notation $\mathcal{H}_{top} := \log r_{\boldsymbol{0}}$.

Lemma 3.1 *For every $\beta > 0$ and every $\boldsymbol{t}$ satisfying $||\boldsymbol{t}|| > 4\|\overrightarrow{\boldsymbol{\psi}}\|_\infty$,*

$$\varphi_\beta(\boldsymbol{t}) < \mathcal{H}_{top} - \frac{\beta}{4}||\boldsymbol{t}||^2.$$

Proof Let us set $g(x) := \log r_{x\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$ with $x \in [0, 1]$. It is differentiable and Prop. 2.12 yields that for every x,

$$g'(x) = \beta\boldsymbol{t} \cdot \int \overrightarrow{\boldsymbol{\psi}}\, d\mu_{x\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}.$$

Then, we use the mean value theorem. There exists $\theta \in]0, 1[$ such that

$$\log r_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = g(1) = \mathcal{H}_{top} + g'(\theta) = \mathcal{H}_{top} + \beta\boldsymbol{t} \cdot \int \overrightarrow{\boldsymbol{\psi}}\, d\mu_{\theta\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}.$$

This yields

$$\varphi_\beta(\boldsymbol{t}) = \log r_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} - \frac{\beta}{2}||\boldsymbol{t}||^2 \leq \mathcal{H}_{top} + \beta\|\boldsymbol{t}\|\|\overrightarrow{\boldsymbol{\psi}}\|_\infty - \frac{\beta}{2}||\boldsymbol{t}||^2.$$

Now

$$\beta\|\boldsymbol{t}\|\|\overrightarrow{\boldsymbol{\psi}}\|_\infty - \frac{\beta}{2}||\boldsymbol{t}||^2 - \left(-\frac{\beta}{4}||\boldsymbol{t}||^2\right) = -\frac{\beta\|t\|}{4}\left(\|t\| - 4\|\overrightarrow{\boldsymbol{\psi}}\|_\infty\right)$$

and we get the result. □

We emphasize an immediate consequence of Lemma 3.1: all the maxima for φ_β are reached at critical points and inside the hypercube $[-K, K]^q$ if K is chosen greater than $4\|\overrightarrow{\boldsymbol{\psi}}\|_\infty$. Indeed $\varphi_\beta(\boldsymbol{0}) = \mathcal{H}_{top}$ and if $\boldsymbol{t}$ is outside the hypercube $[-K, K]^q$ with $K \geq 4\|\overrightarrow{\boldsymbol{\psi}}\|_\infty$ then $||\boldsymbol{t}|| > 4\|\overrightarrow{\boldsymbol{\psi}}\|_\infty$, which implies $\varphi_\beta(\boldsymbol{t}) < \mathcal{H}_{top}$.

3.1.2 The Function $\overline{\varphi}_\beta$

We also recall the definition

$$\mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}}) := \inf_{\boldsymbol{t}\in\mathbb{R}^q}\left\{\log r_{\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} - \boldsymbol{t}.\boldsymbol{z}\right\} = -\mathcal{P}^*(z), \text{ where } \mathcal{P}(\boldsymbol{t}) = \log r_{\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}. \tag{17}$$

From the theory of conjugate functions we know that $\mathcal{H}(\cdot, \overrightarrow{\boldsymbol{\psi}})$ is concave and upper semi-continous, with values in $[-\infty, +\infty[$.

We emphasize that Proposition 2.12 yields that $\overline{\varphi}_\beta = -\infty$ outside $\mathcal{I}(\overrightarrow{\boldsymbol{\psi}})$, and $\overline{\varphi}_\beta$ is finite on $\nabla\mathcal{P}(\mathbb{R}^q)$. Consequently all the maxima for $\overline{\varphi}_\beta$ are reached inside the hypercube $[-\|\overrightarrow{\boldsymbol{\psi}}\|_\infty, \|\overrightarrow{\boldsymbol{\psi}}\|_\infty]^q$, which contains $\mathcal{I}(\overrightarrow{\boldsymbol{\psi}})$.

Let us set

$$\widetilde{\mathcal{H}}(z) := \begin{cases} \sup_\mu \left\{\widehat{\mathcal{H}}(\mu),\ \int \overrightarrow{\boldsymbol{\psi}}\, d\mu = z\right\} \text{ if } z \in I(\overrightarrow{\boldsymbol{\psi}}), \\ -\infty \text{ if } z \notin I(\overrightarrow{\boldsymbol{\psi}}). \end{cases}$$

Then,

Lemma 3.2 *The function $\widetilde{\mathcal{H}}$ is upper semi-continuous on $\mathbb{R}^q$.*

Proof Let z be fixed in $\mathbb{R}^q$, let (z_n) be a sequence in $\mathbb{R}^q$ converging to z. If z is not in $\mathcal{I}(\overrightarrow{\boldsymbol{\psi}})$ then neither is z_n for n big enough hence

$$\limsup_{n\to+\infty} \widetilde{\mathcal{H}}(z_n) = -\infty = \widetilde{\mathcal{H}}(z).$$

Let us thus assume that z is in $\mathcal{I}(\overrightarrow{\boldsymbol{\psi}})$. If only a finite number of $z_n's$ belong to $\mathcal{I}(\overrightarrow{\boldsymbol{\psi}})$ then

$$\limsup_{n\to+\infty} \widetilde{\mathcal{H}}(z_n) = -\infty \leq \widetilde{\mathcal{H}}(z).$$

If an infinite number of $z_n's$ belong to $\mathcal{I}(\overrightarrow{\boldsymbol{\psi}})$ then to compute the limsup we can assume without loss of generality that every z_n is in $\mathcal{I}(\overrightarrow{\boldsymbol{\psi}})$. Let μ_n be an invariant measure such that $\displaystyle\int \overrightarrow{\boldsymbol{\psi}}\, d\mu_n = z_n$ and

$$\widehat{\mathcal{H}}(\mu_n) \geq \widetilde{\mathcal{H}}(z_n) - \frac{1}{n}. \tag{18}$$

Let μ be any accumulation point for (μ_n) for the weak* topology. For simplicity we shall write $\mu = \lim_{n\to+\infty} \mu_n$.

Then, $\displaystyle\int \overrightarrow{\boldsymbol{\psi}}\, d\mu = \lim_{n\to+\infty} \int \overrightarrow{\boldsymbol{\psi}}\, d\mu_n = \lim_{n\to+\infty} z_n = z$ and as the metric entropy is upper semi-continuous we get

$$\widetilde{\mathcal{H}}(z) \geq \widehat{\mathcal{H}}(\mu) \geq \limsup_{n\to+\infty} \widehat{\mathcal{H}}(\mu_n) \geq \limsup_{n\to+\infty} \left(\widetilde{\mathcal{H}}(z_n) - \frac{1}{n}\right) = \limsup_{n\to+\infty} \widetilde{\mathcal{H}}(z_n).$$

□

Proposition 3.3 *For every z in $\mathbb{R}^q$, $\widetilde{\mathcal{H}}(z) = \mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}})$.*

Proof For any $\boldsymbol{t} \in \mathbb{R}^q$ we have

$$\begin{aligned}\mathcal{P}(\boldsymbol{t}) &= \sup_\mu \left(\widehat{\mathcal{H}}(\mu) + \boldsymbol{t} \cdot \int \overrightarrow{\boldsymbol{\psi}}\, d\mu\right) \\ &= \sup_{z\in\mathbb{R}^q} \sup_{\mu, \int \overrightarrow{\boldsymbol{\psi}}\, d\mu = z} \left(\widehat{\mathcal{H}}(\mu) + \boldsymbol{t} \cdot z\right) \\ &= \sup_{z\in\mathbb{R}^q} \left(\widetilde{\mathcal{H}}(z) + \boldsymbol{t} \cdot z\right).\end{aligned}$$

In other words $(-\widetilde{\mathcal{H}})^* = \mathcal{P}$. It is easily seen that $\widetilde{\mathcal{H}}$ is concave. By Theorem 12.2 of [25] the biconjugate $(-\widetilde{\mathcal{H}})^{**}$ of $-\widetilde{\mathcal{H}}$ is equal to its closure. Then, Lemma 3.2 shows that $-\widetilde{\mathcal{H}}$ is closed convex therefore $-\widetilde{\mathcal{H}} = \mathcal{P}^*$. As by definition $\mathcal{P}^* = -\mathcal{H}(\cdot, \overrightarrow{\boldsymbol{\psi}})$, we deduce that $\widetilde{\mathcal{H}} = \mathcal{H}(\cdot, \overrightarrow{\boldsymbol{\psi}})$. □

Furthermore, note that the definition of $\mathcal{P}(\boldsymbol{t})$ and Proposition 3.3 yield

$$\log r_{\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = \mathcal{P}(\boldsymbol{t}) = \sup_{z}\{\mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}}) + \boldsymbol{t}\cdot z\}. \tag{19}$$

Finally we have:

Corollary 3.4 *For every $z \in \mathbb{R}^q$, $\overline{\varphi}_\beta(z) = \widetilde{\mathcal{H}}(z) + \frac{\beta}{2}||z||^2$.*

3.1.3 Maxima for $\boldsymbol{\varphi_\beta}$ and $\overline{\boldsymbol{\varphi}}_{\boldsymbol{\beta}}$

The main result of this Subsection is

Proposition 3.5 *Inequality $\varphi_\beta(z) \geq \overline{\varphi}_\beta(z)$ holds for any z in $\mathbb{R}^q$. Moreover $\varphi_\beta(z)$ is maximum if and only if $\overline{\varphi}_\beta(z)$ is maximum. Furthermore, if $\varphi_\beta(z)$ is maximum then $\varphi_\beta(z) = \overline{\varphi}_\beta(z)$.*

Proof • Step 1. $\varphi_\beta \geq \overline{\varphi}_\beta$. We use Equality (17) with $\boldsymbol{t} = \beta z$. This yields

$$\overline{\varphi}_\beta(z) = \mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}}) + \frac{\beta}{2}\|z\|^2 \leq \log r_{\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} - \boldsymbol{t}.z + \frac{\beta}{2}||z||^2 = \log r_{\beta z\cdot\overrightarrow{\boldsymbol{\psi}}} - \frac{\beta}{2}||z||^2 = \varphi_\beta(z).$$

• Step 2. $\overline{\varphi}_\beta(z)$ is maximal if and only if $\varphi_\beta(z)$ is maximal and maximal values do coincide. Let z be a maximum for φ_β. Then, it is a critical point for φ_β. As

$$\nabla\varphi_\beta(z) = \beta\nabla\mathcal{P}(\beta z) - \beta z$$

this yields $z = \nabla\mathcal{P}(\beta z)$. Using (15) we get

$$\mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}}) = \mathcal{H}(\nabla\mathcal{P}(\beta z), \overrightarrow{\boldsymbol{\psi}}) = \mathcal{P}(\beta z) - \beta z\cdot\nabla\mathcal{P}(\beta z) = \mathcal{P}(\beta z) - \beta\|z\|^2,$$

therefore

$$\mathcal{P}(\beta z) = \mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}}) + \beta\|z\|^2.$$

Using step 1 and this last equality we get

$$\overline{\varphi}_\beta(z) \leq \varphi_\beta(z) = \mathcal{P}(\beta z) - \frac{\beta}{2}\|z\|^2 = \mathcal{H}(z, \overrightarrow{\boldsymbol{\psi}}) + \beta\|z\|^2 - \frac{\beta}{2}\|z\|^2 = \overline{\varphi}_\beta(z),$$

which shows that $\overline{\varphi}_\beta(z) = \varphi_\beta(z)$.

On the other hand for any z',

$$\overline{\varphi}_\beta(z') \leq \varphi_\beta(z') \leq \varphi_\beta(z) = \overline{\varphi}_\beta(z),$$

which shows that z is also a maximum for $\overline{\varphi}_\beta$.

Conversely, if z is a maximum for $\overline{\varphi}_\beta$, let z' be any maximum for φ_β. We get

$$\overline{\varphi}_\beta(z) \geq \overline{\varphi}_\beta(z') = \varphi_\beta(z') \geq \varphi_\beta(z) \geq \overline{\varphi}_\beta(z).$$

This shows that z is also a maximum for φ_β, which finishes the proof. □

Corollary 3.6 *Maxima for $\overline{\varphi}_\beta$ are reached on $\nabla\mathcal{P}(\mathbb{R}^q)$.*

Proof Proposition 3.5 states that maxima for $\overline{\varphi}_\beta$ are maxima for φ_β. We have seen after Lemma 3.1 that all the maxima for φ_β are reached at critical points.

Now, $\boldsymbol{t}$ is a critical point for $\varphi_\beta(\boldsymbol{t}) = -\frac{\beta}{2}\|\boldsymbol{t}\|^2 + \log r_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$ means

$$\boldsymbol{t} = \nabla\mathcal{P}(\beta\boldsymbol{t}) \in \nabla\mathcal{P}(\mathbb{R}^q).$$

□

3.2 Measures Maximizing Quadratic Pressure

3.2.1 Another Expression for $\mathcal{P}_2(\beta)$

We remind that the metric entropy $\mu \mapsto \widehat{\mathcal{H}}(\mu)$ is upper semi-continuous (see Remark 2).

Therefore the function

$$F : \mu \mapsto \widehat{\mathcal{H}}(\mu) + \frac{\beta}{2}\left\| \int \overrightarrow{\psi}\, d\mu \right\|^2$$

is upper semi-continuous hence attains it supremum on the compact set $M_\sigma(\Omega)$.

$$\begin{aligned}
\mathcal{P}_2(\beta) &= \max_{\mu \in M_\sigma(\Omega)} F(\mu) \\
&= \max_{z\in\mathbb{R}^q} \max \left\{ \widehat{\mathcal{H}}(\mu) + \frac{\beta}{2}||z||^2,\ \int \overrightarrow{\psi}\, d\mu = z \right\} \\
&= \max_{z\in\mathbb{R}^q} \left(\widetilde{\mathcal{H}}(z) + \frac{\beta}{2}\|z\|^2 \right) \\
&= \max_{z\in\nabla\mathcal{P}(\mathbb{R}^q)} \overline{\varphi}_\beta(z) \\
&= \max_{z\in\mathbb{R}^q} \varphi_\beta(z)
\end{aligned}$$

where the last equality comes from Proposition 3.5 and the fourth equality comes from Corollaries 3.4 and 3.6.

3.2.2 Good DGM Maximize Quadratic Pressure

We note

$$M := \{z \in \mathbb{R}^q ; \overline{\varphi}_\beta(z) \text{ is maximal}\}.$$

Let $z \in M$. We saw in the proof of Proposition 3.5 that z is then a critical point for φ_β hence $z = \nabla\mathcal{P}(\beta z) = \int \overrightarrow{\psi}\, d\mu_{\beta z\cdot\overrightarrow{\psi}}$. From (2) we know that

$$\widehat{\mathcal{H}}(\mu_{\beta z\cdot\overrightarrow{\psi}}) = \mathcal{P}(\beta z) - \int \beta z \cdot \overrightarrow{\psi}\, d\mu_{\beta z\cdot\overrightarrow{\psi}}$$

hence from (15) we deduce that $\mathcal{H}(z, \overrightarrow{\psi}) = \widehat{\mathcal{H}}(\mu_{\beta z\cdot\overrightarrow{\psi}})$, thus

$$\overline{\varphi}_\beta(z) = F(\mu_{\beta z\cdot\overrightarrow{\psi}}).$$

Let μ be any measure, $z' := \int \overrightarrow{\psi}\, d\mu$. Then

$$F(\mu) = \widehat{\mathcal{H}}(\mu) + \frac{\beta}{2}||z'||^2 \le \widetilde{\mathcal{H}}(z') + \frac{\beta}{2}\|z'\|^2 = \overline{\varphi}_\beta(z') \le \overline{\varphi}_\beta(z).$$

Therefore, $\mu_{\beta z\cdot\overrightarrow{\psi}}$ maximizes F.

3.2.3 Maxima for Quadratic Pressure are Realized only by Good DGM

Conversely let μ maximizing the function F. Set $z := \int \overrightarrow{\psi}\, d\mu$. Then z is in M hence satisfies

$$z = \int \overrightarrow{\psi}\, d\mu_{\beta z\cdot\overrightarrow{\psi}} = \int \overrightarrow{\psi}\, d\mu,$$

and $\widehat{\mathcal{H}}(\mu) = \widetilde{\mathcal{H}}(z)$. Now using (2) we can write

$$\mathcal{P}(\beta z) = \widehat{\mathcal{H}}(\mu_{\beta z\cdot\overrightarrow{\psi}}) + \beta z \cdot \int \overrightarrow{\psi}\, d\mu_{\beta z\cdot\overrightarrow{\psi}} = \widetilde{\mathcal{H}}(z) + \beta z \cdot z = \widehat{\mathcal{H}}(\mu) + \beta z \cdot \int \overrightarrow{\psi}\, d\mu,$$

which means that μ is equal to $\mu_{\beta z\cdot\overrightarrow{\psi}}$ by uniqueness of the (linear) equilibrium state.

4 Proof of Theorem 3

4.1 A Useful Computation

Let $f : \Omega \to \mathbb{R}$ be continuous. We want to evaluate the limit of $\int f(\omega) d\mu_{n,\beta}(\omega)$ as $n \to +\infty$. In the first step we do the computation without the normalizing term $Z_{n,\beta}$ and estimate it in the second step. We recall the identity

$$e^{\|\boldsymbol{\xi}\|^2} = \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{R}^q} \exp\left(-\frac{1}{2}\|\boldsymbol{t}\|^2 + \sqrt{2}\boldsymbol{t}.\boldsymbol{\xi}\right) d\boldsymbol{t}. \tag{20}$$

Then we have

$$\begin{aligned}
Z_{n,\beta} \int_\Omega f(\omega) d\mu_{n,\beta}(\omega) &= \int_\Omega e^{\frac{\beta}{2n}} ||S_n(\overrightarrow{\psi})(\omega)||^2 f(\omega)\, d\mathbb{P}(\omega) \\
&= \frac{1}{(2\pi)^{q/2}} \int_\Omega \int_{\mathbb{R}^q} e^{-\frac{1}{2}\|\boldsymbol{t}\|^2} e^{\sqrt{\frac{\beta}{n}}\boldsymbol{t}.S_n(\overrightarrow{\psi})(\omega)} f(\omega)\, d\boldsymbol{t}\, d\mathbb{P}(\omega) \\
&= \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{R}^q} e^{-\frac{1}{2}\|\boldsymbol{t}\|^2} \int_\Omega \int_{\Omega_n(\omega_0)} e^{\sqrt{\frac{\beta}{n}}\boldsymbol{t}.S_n(\overrightarrow{\psi})(\alpha\omega)} f(\alpha\omega)\, d\rho^{\otimes n}(\alpha)\, d\mathbb{P}(\omega)\, d\boldsymbol{t} \\
&= \frac{1}{(2\pi)^{q/2}} \int_{\mathbb{R}^q} e^{-\frac{1}{2}\|\boldsymbol{t}\|^2} \int_\Omega \mathcal{L}^n_{\sqrt{\frac{\beta}{n}}\boldsymbol{t}\cdot\overrightarrow{\psi}}(f)(\omega)\, d\mathbb{P}(\omega)\, d\boldsymbol{t} \\
&= \left(\frac{\beta n}{2\pi}\right)^{q/2} \int_{\mathbb{R}^q} e^{-\frac{n\beta}{2}\|z\|^2} \int_\Omega \mathcal{L}^n_{\beta z\cdot\overrightarrow{\psi}}(f)(\omega)\, d\mathbb{P}(\omega)\, dz
\end{aligned}$$

where we made the change of variable $\beta z = \sqrt{\frac{\beta}{n}}\boldsymbol{t}$ to get the last equality.

We claim and prove just below that for fixed β, the part of the integral in z outside the hypercube $[-K, K]^q$ is negligible with respect to the other part.

$$\begin{aligned}
&\int_{\mathbb{R}^q\setminus[-K,K]^q} e^{-\frac{n\beta}{2}\|z\|^2} \int_\Omega \mathcal{L}^n_{\beta z\cdot\overrightarrow{\psi}}(f)(\omega)\, d\mathbb{P}(\omega)\, dz \\
&\quad\le \int_{\mathbb{R}^q\setminus[-K,K]^q} e^{-\frac{n\beta}{2}\|z\|^2} \|\mathcal{L}^n_{\beta z\cdot\overrightarrow{\psi}}\|_\infty \|f\|_\infty\, dz.
\end{aligned} \tag{21}$$

On the other hand, we remind (see Inequality (5)) $\|\mathcal{L}^n_{\beta z\cdot\vec{\psi}}\|_\infty \le e^{n\beta\|z\|\|\vec{\psi}\|_\infty}$, which yields

$$e^{-\frac{n\beta}{2}\|z\|^2}\|\mathcal{L}^n_{\beta z\cdot\vec{\psi}}\|_\infty \le e^{n\beta(\|z\|\|\vec{\psi}\|_\infty-\frac{1}{2}\|z\|^2)} \le e^{-n\beta\frac{\|z\|^2}{4}}$$

for $\|z\| > 4\|\vec{\psi}\|_\infty$.

Now, reporting this inequality in the right hand side term of (21) and assuming $K > 4||\vec{\psi}||_\infty$, one gets

$$\begin{aligned}\int_{\mathbb{R}^q\setminus[-K,K]^q} e^{-n\beta\frac{\|z\|^2}{4}}\,dz &\le \sum_{i=1}^q \int_{|z_i|>K} e^{-n\beta\frac{z_i^2}{4}} \prod_{j\ne i} e^{-n\beta\frac{z_j^2}{4}}\,dz\\ &= \left(\frac{4\pi}{n\beta}\right)^{\frac{q-1}{2}} \sum_{i=1}^q \int_{|z_i|>K} e^{-n\beta\frac{z_i^2}{4}}\,dz_i,\end{aligned}$$

and

$$\int_{|z_i|>K} e^{-n\beta\frac{z_i^2}{4}}\,dz_i \le \frac{4}{n\beta K} e^{-n\beta\frac{K^2}{4}}.$$

Returning to (21) we get

$$\int_{\mathbb{R}^q\setminus[-K,K]^q} e^{-\frac{n\beta}{2}\|z\|^2}\int_\Omega \mathcal{L}^n_{\beta z\cdot\vec{\psi}}(f)(\omega)\,d\mathbb{P}(\omega)\,dz = O\left(\frac{e^{-n\beta\frac{K^2}{4}}}{n^{\frac{q+1}{2}}}\right). \tag{22}$$

if K is greater than $4\|\vec{\psi}\|_\infty$.

Now, we recall that

$$\varphi_\beta(z) = -\frac{\beta}{2}\|z\|^2 + \log r_{\beta z.\vec{\psi}}$$

and that if f belongs to $\mathcal{C}^{+1}(\Omega)$ then

$$\mathcal{L}^n_{\beta z\cdot\vec{\psi}}(f)(\omega) = e^{n\log r_{\beta z\cdot\vec{\psi}}}\left[\left(\int_\Omega f\,d\nu_{\beta z\cdot\vec{\psi}}\right)G_{\beta z\cdot\vec{\psi}}(\omega) + \Psi^n_{\beta z\cdot\vec{\psi}}(f)(\omega)\right],$$

where the operator norm of $\Psi_{\beta z\cdot\vec{\psi}}$ acting on $\mathcal{C}^{+1}(\Omega)$ is strictly less than one. We write $\Psi_{\beta z\cdot\vec{\psi}} = e^{-\varepsilon(\beta,z)}T(\beta,z)$ where $\varepsilon(\beta,z)$ is the spectral gap of the operator $\mathcal{L}_{\beta z\cdot\vec{\psi}}$ and $||T(\beta,z)||_L = 1$. Then

$$\begin{aligned}&\int_{[-K,K]^q} e^{-\frac{n\beta}{2}\|z\|^2}\int_\Omega \mathcal{L}^n_{\beta z\cdot\vec{\psi}}(f)(\omega)\,d\mathbb{P}(\omega)\,dz\\ &= \int_{[-K,K]^q} e^{n\varphi_\beta(z)}\int_\Omega\left[\left(\int_\Omega f\,d\nu_{\beta z\cdot\vec{\psi}}\right)G_{\beta z\cdot\vec{\psi}}(\omega)+e^{-n\varepsilon(\beta,z)}T^n(\beta,z)(f)(\omega)\right]d\mathbb{P}(\omega)\,dz\end{aligned} \tag{23}$$

The spectral gap $\varepsilon(\beta,z)$ is lower semi-continuous in z hence, on the compact set $[-K,K]^q$, it attains its infimum $m(\beta)$, which is strictly positive. We set

$$\alpha(n,z,f) = \int_\Omega e^{-n\varepsilon(\beta,z)}T^n(\beta,z)(f)(\omega)\,d\mathbb{P}(\omega),$$

and notice that for any z in $[-K, K]^q$,

$$|\alpha(n, z, f)| \leq e^{-nm(\beta)} \|f\|_L.$$

Eventually we get

$$Z_{n,\beta} \int_\Omega f(\omega)\, d\mu_{n,\beta}(\omega) = \left(\frac{\beta n}{2\pi}\right)^{q/2} \int_{[-K,K]^q} e^{n\varphi_\beta(z)} \left[\left(\int_\Omega f\, d\nu_{\beta z\cdot\vec{\psi}}\right)\left(\int_\Omega G_{\beta z\cdot\vec{\psi}}\, d\mathbb{P}\right) + \alpha(n, z, f)\right] dz + O\left(\frac{e^{-n\beta\frac{K^2}{4}}}{n^{\frac{q+1}{2}}}\right) \tag{24}$$

The normalization term $Z_{n,\beta}$ is obtained taking $f \equiv 1$. This yields

$$\int_\Omega f(\omega)\, d\mu_{n,\beta}(\omega) = \frac{\int_{[-K,K]^q} e^{n\varphi_\beta(z)} \left[\left(\int_\Omega f\, d\nu_{\beta z\cdot\vec{\psi}}\right)\left(\int_\Omega G_{\beta z\cdot\vec{\psi}}\, d\mathbb{P}\right) + \alpha(n, z, f)\right] dz + O\left(\frac{e^{-n\beta\frac{K^2}{4}}}{n^{\frac{q+1}{2}}}\right)}{\int_{[-K,K]^q} e^{n\varphi_\beta(z)} \left[\int_\Omega G_{\beta z\cdot\vec{\psi}}\, d\mathbb{P} + \alpha(n, z, \mathbb{1})\right] dz + O\left(\frac{e^{-n\beta\frac{K^2}{4}}}{n^{\frac{q+1}{2}}}\right)}. \tag{25}$$

where $\alpha(n, z, f)$ and $\alpha(n, z, \mathbb{1})$ converge uniformly to 0 with respect to z when n tends to infinity.

4.2 The Case $q = 1$

In this case the function φ_β is analytic hence admits only finitely many maxima, and we can argue as in [20].

4.3 The Higher-Dimensional Case

We assume that all the maxima for φ_β are non-degenerate.

Lemma 4.1 *The function φ_β admits only finitely many maxima.*

Proof The proof is done by contradiction. Let us consider a sequence (z_n) of maxima for φ_β. We have seen that all the maxima are critical points and are in some compact set $[-K, K]^q$ (see Lemma 3.1 and discussion after).

Therefore, we may consider some accumulation point z for the z_n's. For simplicity we set $z = \lim_{n\to+\infty} z_n$ and we assume that $z_n \neq z_{n+1}$ holds for every n. Note that by continuity, z is also a critical point for φ_β and φ_β is maximal at z.

We remind that φ_β is $\mathcal{C}^\infty$. If we consider the restriction ${\varphi_\beta}_n$ of φ_β to each segment $[z_n, z_{n+1}]$, then ${\varphi_\beta}_n$ is $\mathcal{C}^\infty$ and ${\varphi_\beta}'_n(z_n) = {\varphi_\beta}'_n(z_{n+1}) = 0$. Hence, Rolle's theorem shows that there exists $z'_n \in [z_n, z_{n+1}]$ such that

$${\varphi_\beta}''_n(z'_n) = 0. \tag{26}$$

Set $\mathbf{u}_n := z_n - z_{n+1}$ and consider any accumulation point $\mathbf{u}$ for $\dfrac{z_n - z_{n+1}}{||z_n - z_{n+1}||}$. Equality (26) can be rewritten under the form

$$d^2\varphi_\beta(z'_n)(\mathbf{u}_n, \mathbf{u}_n) = 0,$$

which yields as $n \to +\infty$ $d^2\varphi_\beta(z)(\mathbf{u}, \mathbf{u}) = 0$. This means that z is a degenerate maximal point for φ_β, which is in contradiction with our assumption. □

Let $z_1, \dots, z_k$ be the points where φ_β attains its maximum. We recall that the Laplace method (see [27, Ch.IX Th.3]) states

$$\int_0 e^{n\varphi_\beta(z)} g(z)\, dz \sim_{n\to\infty} \frac{(2\pi)^{q/2} g(z_1) e^{n\varphi_\beta(z_1)}}{n^{q/2}\sqrt{|\det d^2\varphi_\beta(z_1)|}},$$

provided that φ_β admits no other critical point than z_1 in an open set O of $\mathbb{R}^q$, that $g(z_1) \neq 0$ and that the Hessian matrix $d^2\varphi_\beta(z_1)$ is negative definite (which holds by our assumption).

Remark 5 We emphasize the assumption $g(z_1) \neq 0$. □

We choose K such that $\varphi_\beta(z_1) + \beta\frac{K^2}{4} > 0$, and letting $n \to +\infty$ in (25), we get that for every f in $\mathcal{C}^{+1}(\Omega)$,

$$\lim_{n\to+\infty} \int_\Omega f(\omega)\, d\mu_{n,\beta}(\omega) = \frac{\displaystyle\sum_{j=1}^{k} \frac{\int G_{\beta z_j\cdot\overrightarrow{\psi}}\, d\mathbb{P}}{\sqrt{\det d^2\varphi_\beta(z_j)}} \int f\, d\nu_{\beta z_j\cdot\overrightarrow{\psi}}}{\displaystyle\sum_{j=1}^{k} \frac{\int G_{\beta z_j\cdot\overrightarrow{\psi}}\, d\mathbb{P}}{\sqrt{\det d^2\varphi_\beta(z_j)}}},$$

which finishes the proof of Theorem 3.

5 Application to the Mean-Field *XY* Model

5.1 The Cosine Potential

The mean-field XY model is a system of n globally coupled planar spins (or alternatively of n globally interacting particles constrained on a ring), with Hamiltonian

$$H_n = -\frac{1}{2n}\sum_{i,j=1}^{n} \cos(p_i - p_j),$$

where $p_i \in [0, 2\pi[$. We can interpret it as a generalized Curie–Weiss model by setting $E = \mathbb{T} = \{z \in \mathbb{R}^2, \|z\| = 1\}$, $\Omega = \mathbb{T}^{\mathbb{N}}$, and $\overrightarrow{\psi}(\omega) = \omega_0$. Indeed every ω_k in the word $\omega = \omega_0\omega_1\cdots$ of Ω is uniquely expressed as $\omega_k = (\cos\theta_k, \sin\theta_k)$ with θ_k in $[-\pi, \pi[$, and then

$$\|S_n(\overrightarrow{\psi})(\omega)\|^2 = \|\sum_{k=0}^{n-1}\omega_k\|^2 = \sum_{i,j=0}^{n-1}\langle\omega_i, \omega_j\rangle = \sum_{i,j=0}^{n-1}\cos(\omega_i - \omega_j).$$

We endow $\mathbb{T}$ with the usual distance on $\mathbb{R}^2$, and the Haar measure ρ given by

$$\int_{\mathbb{T}} h(z)\,\rho(dz) = \int_{-\pi}^{\pi} h(\mathbf{u}_\theta)\,\frac{d\theta}{2\pi}, \text{ where } \mathbf{u}_\theta = (\cos\theta, \sin\theta).$$

As $\overrightarrow{\boldsymbol{\psi}}$ only depends on the first coordinate, we see that for any $\boldsymbol{t}$ in $\mathbb{R}^2$ and any f in $\mathcal{C}^0(\Omega)$,

$$\mathcal{L}_{\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}(f)(\omega) = \int_{-\pi}^{\pi} e^{\boldsymbol{t}\cdot\mathbf{u}_\theta}\, f(\mathbf{u}_\theta\omega)\,\frac{d\theta}{2\pi},$$

so that the spectral radius of $\mathcal{L}_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$ is

$$r_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = \lambda_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = \int_{-\pi}^{\pi} e^{\beta\boldsymbol{t}\cdot\mathbf{u}_\theta}\,\frac{d\theta}{2\pi},$$

with eigenfunction $G_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = \mathbb{1}$, and $\nu_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = \mu_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}}$. We notice that $r_0 = 1$. If $\boldsymbol{t} \neq 0$, we denote by $|\boldsymbol{t}|$ its Euclidean norm and by $\theta_{\boldsymbol{t}}$ the unique element of $[-\pi, \pi[$ such that $\boldsymbol{t} = |\boldsymbol{t}|\mathbf{u}_{\theta_{\boldsymbol{t}}}$. Then

$$\begin{aligned} r_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} &= \int_{-\pi}^{\pi} e^{\beta|\boldsymbol{t}|\cos(\theta_{\boldsymbol{t}}-\theta)}\,\frac{d\theta}{2\pi} \\ &= \int_{\theta_{\boldsymbol{t}}-\pi}^{\theta_{\boldsymbol{t}}+\pi} e^{\beta|\boldsymbol{t}|\cos y}\,\frac{dy}{2\pi} \\ &= \int_{-\pi}^{\pi} e^{\beta|\boldsymbol{t}|\cos y}\,\frac{dy}{2\pi} \end{aligned}$$

because the integral does not depend on the interval of length 2π where we compute it. Eventually we have

$$r_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = \int_{0}^{\pi} e^{\beta|\boldsymbol{t}|\cos y}\,\frac{dy}{\pi} = I_0(\beta|\boldsymbol{t}|), \tag{27}$$

where I_0 is the modified Bessel function of order zero, and we get

$$\varphi_\beta(\boldsymbol{t}) = -\frac{\beta}{2}|\boldsymbol{t}|^2 + \log r_{\beta\boldsymbol{t}\cdot\overrightarrow{\boldsymbol{\psi}}} = -\frac{\beta}{2}|\boldsymbol{t}|^2 + \log I_0(\beta|\boldsymbol{t}|).$$

This shows that $\varphi_\beta(\boldsymbol{t})$ is constant on all the circles centered in 0.

Remark 6 Unless φ_β is maximal only at 0, which does not hold for every β as we will see below, we have here an example where all the maxima of the auxiliary function are degenerate. □

We set

$$\phi_\beta(x) := -\frac{\beta}{2}x^2 + \log I_0(\beta x) \text{ for } x \geq 0.$$

The equality (25) becomes

$$\int_\Omega f(\omega)\,d\mu_{n,\beta}(\omega) = \frac{\displaystyle\int_{B(0,K)} e^{n\varphi_\beta(z)}\left[\int_\Omega f\,d\mu_{\beta z\cdot\overrightarrow{\boldsymbol{\psi}}} + \alpha(n,z,f)\right]dz + O\left(\frac{e^{-n\beta\frac{K^2}{4}}}{n^{\frac{3}{2}}}\right)}{\displaystyle\int_{B(0,K)} e^{n\varphi_\beta(z)}\,dz + O\left(\frac{e^{-n\beta\frac{K^2}{4}}}{n^{\frac{3}{2}}}\right)}. \tag{28}$$

where we replaced for convenience the square $[-K, K]^2$ by the disk $B(0, K)$. We are thus led to study the asymptotic behaviour of the integral

$$I(n, f) = \int_{B(0,K)} e^{n\varphi_\beta(z)} \left[\int_\Omega f\, d\mu_{\beta z\cdot \vec{\psi}}\right] dz.$$

In polar coordinates we write $z = r\mathbf{u}_\theta$ and we get

$$I(n, f) = \int_0^K \int_{-\pi}^{\pi} e^{n\phi_\beta(r)} \left[\int_\Omega f\, d\mu_{\beta r\mathbf{u}_\theta\cdot \vec{\psi}}\right] r\, dr\, d\theta.$$

For $x \in \mathbb{R}_+$, we denote by η_x the mean value of DGM's defined by

$$\int_\Omega h\, d\eta_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\int_\Omega h\, d\mu_{x\mathbf{u}_\theta\cdot \vec{\psi}}\right] d\theta$$

for any bounded measurable h, so that

$$I(n, f) = 2\pi \int_0^K e^{n\phi_\beta(r)} \left(\int_\Omega f\, d\eta_{\beta r}\right) r\, dr,$$

which is then a one-dimensional Laplace integral. We study the maximum of the function ϕ_β on $\mathbb{R}_+$. First we notice that $0 \le I_0(\beta x) \le \beta x$ and $I_0(0) = 1$, hence

$$\phi_\beta(x) \le \beta x(1 - \frac{x}{2}) \text{ and } \phi_\beta(0) = 0,$$

from which we deduce that $\max_{\mathbb{R}_+} \phi_\beta = \max_{[0,2[} \phi_\beta$. Next we look for the critical points of ϕ_β on $[0, 2[$. We compute the first and second derivatives

$$\phi'_\beta(x) = \beta\left[\left(\frac{I'_0}{I_0}\right)(\beta x) - x\right] = \beta\left[\frac{\int_0^\pi e^{\beta x\cos\theta}\cos\theta\, d\theta}{\int_0^\pi e^{\beta x\cos\theta}\, d\theta} - x\right], \tag{29}$$

$$\begin{aligned}\phi''_\beta(x) &= \beta\left[\beta\left(\frac{I''_0 I_0 - I'^2_0}{I_0^2}\right)(\beta x) - 1\right]\\ &= \beta\left[\frac{\int_0^\pi e^{\beta x\cos\theta}\cos^2\theta\, d\theta}{\int_0^\pi e^{\beta x\cos\theta}\, d\theta} - \left(\frac{\int_0^\pi e^{\beta x\cos\theta}\cos\theta\, d\theta}{\int_0^\pi e^{\beta x\cos\theta}\, d\theta}\right)^2 - 1\right].\end{aligned} \tag{30}$$

We notice that $\phi'_\beta(x) \le \beta(1 - x)$, from which we deduce that $\max_{\mathbb{R}_+} \phi_\beta = \max_{[0,1]} \phi_\beta$. As $I'_0(0) = 0$ we know that $\phi'_\beta(0) = 0$. We compute

$$\phi''_\beta(0) = \beta\left[\beta I''_0(0) - 1\right] = \beta\left[\frac{\beta}{\pi}\int_0^\pi \cos^2\theta\, d\theta - 1\right] = \beta\left[\frac{\beta}{2} - 1\right].$$

We shall thus consider three cases: $\beta > 2$, $\beta = 2$, and $\beta < 2$. First we take a closer look at the critical points of ϕ_β. We recall that the Bessel function I_0 satisfies the differential equation (we refer for instance to [3] for information about Bessel functions)

$$I''_0(x) + \frac{1}{x}I'_0(x) - I_0(x) = 0 \tag{31}$$

so that

$$\left(\frac{I''_0}{I_0}\right)(\beta x) = 1 - \frac{1}{\beta x}\left(\frac{I'_0}{I_0}\right)(\beta x).$$

Replacing in (30) we get that for every x,

$$\phi''_\beta(x) = \beta^2 \left[\left(\frac{I''_0}{I_0}\right)(\beta x) - \left(\frac{I'_0}{I_0}\right)^2 (\beta x) - \frac{1}{\beta} \right]$$
$$= -\beta^2 \left[\left(\frac{I'_0}{I_0}\right)^2 (\beta x) + \frac{1}{\beta x}\left(\frac{I'_0}{I_0}\right)(\beta x) - 1 + \frac{1}{\beta} \right]. \quad (32)$$

Now from (29) we know that r is a critical point of ϕ_β if and only if

$$\left(\frac{I'_0}{I_0}\right)(\beta r) = r.$$

If r is such a point then replacing in (32) we get

$$\phi''_\beta(r) = -\beta^2 \left[r^2 + \frac{2}{\beta} - 1 \right]. \quad (33)$$

Case $\beta > 2$: In this case $\phi''_\beta(0) > 0$ hence 0 is not a maximum point. We claim that ϕ has a unique maximum and that it belongs to $]\sqrt{\frac{\beta-2}{\beta}}, 1]$.

We denote by $r_1 < \cdots < r_m$ the m points of $]0, 1]$ where ϕ_β attains its maximum M on $\mathbb{R}_+$. Then every r_k satisfies $\phi'_\beta(r_k) = 0$ and $\phi''_\beta(r_k) \leq 0$. Remember that every critical point r satisfies (33) which we rewrite

$$\phi''_\beta(r) = \beta^2 \left[\frac{\beta - 2}{\beta} - r^2 \right] = \beta^2 \left(\sqrt{\frac{\beta - 2}{\beta}} - r \right) \left(\sqrt{\frac{\beta - 2}{\beta}} + r \right). \quad (34)$$

As $\phi''_\beta(r_k) \leq 0$ we deduce that $r_1 \geq \sqrt{\frac{\beta-2}{\beta}}$. We observe that any critical point r strictly bigger than r_1 satisfies $\phi''_\beta(r) < 0$, which means r is a local maximum for ϕ_β. Now, if $m \geq 2$ then between r_1 and r_2 there must be a local minimum which is also a critical point. This yields a contradiction. Therefore, $m = 1$ and r_1 is the unique critical point for ϕ.

Let us show that $r_1 > \sqrt{\frac{\beta-2}{\beta}}$. Indeed if $r_1 = \sqrt{\frac{\beta-2}{\beta}}$ then $\phi'_\beta(r_1) = \phi''_\beta(r_1) = 0$. From (29) we know that

$$\forall\, x,\ \left(\frac{I'_0}{I_0}\right)(\beta x) = x + \frac{\phi'_\beta(x)}{\beta}.$$

Replacing in (32) we get that

$$\forall\, x,\ \phi''_\beta(x) = -(\phi'_\beta(x))^2 - \left(2\beta x + \frac{1}{x}\right)\phi'_\beta(x) - \beta^2 x^2 + \beta^2 - 2\beta \quad (35)$$

which is a differential equation satisfied by ϕ_β. When we differentiate this equality we get that

$$\forall\, x,\ \phi'''_\beta(x) = -2\phi'_\beta(x)\phi''_\beta(x) - \left(2\beta - \frac{1}{x^2}\right)\phi'_\beta(x) - \left(2\beta x + \frac{1}{x}\right)\phi''_\beta(x) - 2\beta^2 x. \quad (36)$$

We deduce that $\phi'''_\beta(r_1) = -2\beta^2 r_1$ is strictly negative, therefore r_1 can not be a maximum, which yields a contradiction. Hence $r_1 > \sqrt{\frac{\beta-2}{\beta}}$ holds and this finishes to prove the claim.

Now we are ready to conclude the case $\beta > 2$. We apply the Laplace method to the integral $I(n, f)$ and we find that

$$I(n, f) \sim 2\pi \frac{\sqrt{2\pi} e^{n\phi_\beta(r)} r \left(\int_\Omega f \, d\eta_{\beta r}\right)}{n^{1/2}\sqrt{|\phi''_\beta(r)|}},$$

hence

$$\int_\Omega f(\omega)\, d\mu_{n,\beta}(\omega) \sim \frac{I(n, f)}{I(n, \mathbb{1})} \sim \int_\Omega f \, d\eta_{\beta r}.$$

Case $\beta < 2$: In this case $\phi'_\beta(0) = 0$ and $\phi''_\beta(0) < 0$ hence 0 is a local maximum of ϕ_β. The equation (33) tells us that every critical point r satisfies

$$\phi''_\beta(r) = -\beta^2 \left[r^2 + \frac{2-\beta}{\beta} \right],$$

which is strictly negative, therefore every critical point is a local maximum. The same argument than above shows that ϕ_β attains its maximum at 0 and only at 0.

As the maximum is reached at 0, we cannot directly apply the Laplace method as it is emphasized in Remark 5. We then use the following lemma. It is a special version of the Laplace method and can be found in [8].

Lemma 5.1 *Let α and γ be two positive real numbers. Then, for any sequence (b_n) such that $nb_n^\alpha \to +\infty$*

$$\int_0^{b_n} x^\gamma e^{-nx^\alpha} dx \sim_{n\to+\infty} \frac{1}{\alpha\, n^{\frac{\gamma+1}{\alpha}}} \Gamma\left(\frac{\gamma+1}{\alpha}\right).$$

Proof Just set $u = nx^\alpha$. □

We apply Lemma 5.1 for $\int_0^{b_n} e^{n\phi_\beta(r)} \left(\int_\Omega f \, d\eta_{\beta r}\right) r\, dr$, with $b_n = 1/\sqrt[4]{n}$. Because $b_n \to 0$, we can get $\phi_\beta(r) = -\phi''_\beta(0) r^2 + O(r^3)$ on $[0, b_n]$. Note that for $r \in [0, b_n]$, by continuity for the eigen-measures $\nu_{\beta \boldsymbol{t}\cdot\vec{\boldsymbol{\psi}}}$ for operators $\mathcal{L}_{\boldsymbol{t}\cdot\vec{\boldsymbol{\psi}}}$ we also have.

$$r \int f \, d\eta_{\beta r} \sim_{n\to+\infty} r \int f \, d\eta_0$$

A computation shows that $\int_{b_n}^{b} e^{n\phi_\beta(r)} \left(\int_\Omega f \, d\eta_{\beta r}\right) r\, dr$ is of order less than $e^{-nb_n^2/2}$ if b is chosen small but positive such that $\phi_\beta(r) \leq -\phi''_\beta(0)\frac{r^2}{2}$ on $[0, b]$. Then, $nb_n^2 \to +\infty$ yields that this quantity is exponentially small (in n). Because 0 is the unique maximum for ϕ_β, $\int_{b_n}^{b} e^{n\phi_\beta(r)} \left(\int_\Omega f \, d\eta_{\beta r}\right) r\, dr$ is of order less than $e^{-n\varepsilon(b)}$ with $\varepsilon(b) > 0$.

Hence we get

$$I(n, f) \sim 2\pi \frac{\int_\Omega f \, d\eta_0}{n|\phi''_\beta(0)|} \Gamma(1) = 2\pi \frac{\int_\Omega f \, d\mu_0}{n|\phi''_\beta(0)|},$$

hence

$$\int_\Omega f(\omega)\, d\mu_{n,\beta}(\omega) \sim \frac{I(n, f)}{I(n, \mathbb{1})} \sim \int_\Omega f \, d\mu_0.$$

Case $\beta = 2$*:* In this case $\phi'_\beta(0) = \phi''_\beta(0) = 0$ and every critical point $r \neq 0$ is a local maximum since it satisfies

$$\phi''_\beta(r) = -\beta^2 r^2,$$

which is strictly negative. We deduce, as above, that there exists only one maximum, which may be 0 or not. If it is 0, then we conclude as before but we have to pick an higher order for the derivative (and use Lemma 5.1 with $\alpha \geq 4$). If it is not 0, then we conclude the computation as above.

We conclude that for every $\beta > 0$, the sequence of measures $(\mu_{n,\beta})_{n\in\mathbb{N}}$ weakly converges to $\eta_{\beta r}$ where r is the unique point at which ϕ_β reaches its maximum on $\mathbb{R}^+$.

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