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The Nonlinear Schrödinger Equation on Z and R with Bounded Initial Data: Examples and Conjectures

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Abstract
We study the nonlinear Schrödinger equation (NLS) with bounded initial data which does not vanish at infinity. Examples include periodic, quasi-periodic and random initial data. On the lattice we prove that solutions are polynomially bounded in time for any bounded data. In the continuum, local existence is proved for real analytic data by a Newton iteration scheme. Global existence for NLS with a regularized nonlinearity follows by analyzing a local energy norm.

Keywords Propagation speed · Well-posedness · Local conservation laws · Newton iteration

1 Introduction

The aim of this note is to study nonlinear Schrödinger type equations on $\mathbb{Z}$ and on $\mathbb{R}$ with initial data that are bounded, and do not vanish at infinity. Examples include periodic, quasi-periodic and random data. We present some modest results describing the dynamics for such data. Although we shall phrase our results in one dimension, most of our methods can be adapted to higher dimensions.

(Dedicated to Joel Lebowitz with admiration for his inspiring leadership.)

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There is an extensive literature on space periodic data, with bounds on the Sobolev norms as a function of time [2,3,5]. More recently, localized perturbations of periodic initial data have been studied [7,12]. The periodic case appears naturally in nonlinear optics, describing periodic signal propagating through a fiber, and local perturbations would correspond to noise and modulation due to information carrying signal.

Much less is known about quasi-periodic and random cases. The quasi-periodic data arises when there is another periodic signal, not commensurate with the periodic signal. The nonlinearity then will produce a solution with arbitrary number of frequencies. In the case of the KdV equation, Damanik at al. [1,9] have proved the global existence of uniformly bounded almost periodic solutions for certain small amplitude quasi-periodic initial data. This remarkable result uses the integrability of KdV and the fact that the corresponding Schrödinger operator has only absolutely continuous spectrum. The case of global existence of large initial data has not yet been understood. For non-integrable NLS, Oh [19], proved the existence of solutions for quasi-periodic data for short time. He also proved that solutions exist for all time for limit-periodic data, [18]. The global existence of special space-time quasi-periodic solutions was recently established by Wang [21] by using Bourgain's semi-algebraic set methods together with a Newton iteration scheme. It is generally believed that typical solutions to non-integrable equations are not uniformly bounded in time unless the maximum is controlled by a conservation law. Note that in one dimension, the maximum of a solution of NLS with periodic data is bounded by the energy.

In this paper we consider the nonlinear Schrödinger equation

$$i\psi_t + \partial_{xx}\psi = |\psi|^2\psi, \quad \psi : \mathbb{R}^{1+1} \to \mathbb{C}, \quad \psi : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}, \quad \psi(0, x) = \psi_0(x), \ (1.1)$$

with data which are locally in H^s, with uniform bounds over the reals. The case when $\psi : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$ corresponds to a nonlinear Schrödinger equation on the one dimensional lattice with ∂_{xx} given by the discrete Laplacian. The case when $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ corresponds to a nonlinear Schrödinger equation in the continuum. What makes the NLS hard to analyze on $\mathbb{R}$ is the lack of a finite propagation speed. The speed of a signal is proportional to the derivative of the solution. In contrast to the nonlinear Schrödinger equation, the nonlinear wave equation,

$$u_{tt} - u_{xx} + u^3 = 0, \quad u : \mathbb{R}^{1+1} \to \mathbb{R}, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \qquad (1.2)$$

has a finite propagation speed. This means that at time t the solution $u(t, x)$ depends only on data in the backward light cone $\{x' : |x' - x| \le t\}$. The rest of the data can be set to 0. Hence the solution exists for all time by standard arguments. The following Proposition shows how finite propagation speed enables one to get bounds on the time evolution in one dimension.

Proposition 1 *If $u(t, x)$ is a solution to the cubic nonlinear wave equation in one dimension, (1.2) with initial conditions which are uniformly $C^1(\mathbb{R}) \times C^0(\mathbb{R})$, then*

$$|u(t, x)| \le Ct^{1/3}. \tag{1.3}$$

Proof For any $x_0 \in \mathbb{R}$ and $t \in [0, \infty)$, if $\chi(x)$ is a smooth cutoff function,

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 2, \end{cases} \tag{1.4}$$

if $v(t, x)$ is the solution to (1.2) with u_0 replaced with $\chi(\frac{x-x_0}{T})u_0$ and u_1 replaced with $\chi(\frac{x-x_0}{T})u_1$, then by finite propagation speed, $v(t, x_0) = u(t, x_0)$ for all $0 \le t \le T$. The

solution to (1.2) has the conserved energy

$$E(u, u_t) = \frac{1}{2} \int (\partial_x u(t, x))^2 dx + \frac{1}{2} \int (u_t(t, x))^2 dx + \frac{1}{4} \int u(t, x)^4 dx. \quad (1.5)$$

By direct computation,

$$E(\chi u_0, \chi u_1) \lesssim T(\|u_0\|_{L^\infty}^2 + \|\partial_x u_0\|_{L^\infty}^2 + \|u_1\|_{L^\infty}^2). \quad (1.6)$$

Then by the Sobolev embedding theorem,

$$\begin{aligned}
|u(t, x_0)|^3 = |v(t, x_0)|^3 &\lesssim \|\partial_x v(t)\|_{L^2} \|v(t)\|_{L^4}^2 \lesssim E(v, v_t) \\
&\lesssim T(\|u_0\|_{L^\infty}^2 + \|\partial_x u_0\|_{L^\infty}^2 + \|u_1\|_{L^\infty}^2).
\end{aligned} \quad (1.7)$$

This proves the proposition. □

Remark If the nonlinearity in (1.2) is replaced by u^{2p+1}, we would observe a growth rate bounded by $C_p t^{\frac{1}{p+2}}$.

For the solution to the NLS on the lattice, $\psi : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$, it is easy to prove that global solutions exist for all uniformly bounded initial data and that the solution grows at most like $Ct^{\frac{1}{2}}$. The nonlinear Schrödinger equation on the lattice may be thought of as having approximate finite propagation speed since derivatives are uniformly bounded. The bound $Ct^{\frac{1}{2}}$ may be proved for the nonlinear Schrödinger equation on the lattice with either focusing or defocusing nonlinearity. However, observe that the proof of Proposition 1 uses the fact that the wave equation is defocusing. For the defocusing, nonlinear Schrödinger equation on the lattice, we may improve the bounds to $Ct^{\frac{1}{4}}$. See Sec 2.

Remark The solution to the 1D linear Schrödinger with initial data $\psi_0(x) = \sum_j a_j e^{-(x-j)^2}$, $j \in \mathbb{Z}$ is given by

$$C \sum_j a_j \frac{e^{\frac{-(x-j)^2}{4it+1}}}{(4it+1)^{1/2}}.$$

It is easy to show that if the $|a_j|$ are uniformly bounded, then the sup norm of this solution is bounded by $Ct^{1/2}$. The phases in the exponential can be cancelled by the a_j so this is the best one can do. On the other hand, if the a_j are independent complex random variables such that $\mathbb{E}|a_j|^2 = 1$, then $\mathbb{E}|e^{it\Delta}\psi_0|^2(x) \leq C$. On the lattice, the upper bound $t^{1/2}$ follows from Proposition 2 and a $t^{1/2}$ lower bound is proved in the Appendix for particular a_j.

We also study a simplified model of NLS in which the nonlinearity is regularized. The Hamiltonian of the nonlinear interaction we consider has the form $\int |u_\phi(x, t)|^4 dx$ where u_ϕ denotes the convolution of u with a smooth positive function $\phi(x)$ of compact support. In this case we prove that solutions exist for all time and are polynomially bounded for uniformly smooth initial data. See Theorem 2 in Sec 3.

The global existence for NLS with bounded smooth initial data is still open. However in the last section we extend the local time results of T. Oh for quasi-periodic initial to the real analytic setting by developing a Newton iteration for the short time evolution. This is partly inspired by work of Greene and Jacobowitz [10] on analytic embedding. In particular we can include data of the form $\sum_j a_j e^{-(x-j)^2}$ where $j \in \mathbb{Z}$ and $|a_j| \leq 1$. The a_j may be random.

Remark Besides global existence a natural question would be to prove that the time average of the energy per unit volume of space is uniformly bounded and then to consider the possibility that the limit of the average exists as time goes to infinity. Partial answer is given in Section two, for the case of the NLS on the lattice.

1.1 Invariant Measures for NLS Equations

In many cases there exists an invariant measure which helps to control and describe the time evolution. The work of Lebowitz et al. [14] on the Gibbs measure for the focussing NLS on the circle is the foundational paper in this field. The construction of invariant measures for the NLS is directly related to the statistical mechanics problem with the NLS energy functional defining the theory. This equilibrium measure may also describe a soliton gas of NLS. This work was extended by Bourgain [3,4] who also studied the time evolution for rough data in the support of this measure. Recent works of Lebowitz et al. [13] as well as Carlen et al. [6] describe the rate of equilibration to the Gibbs measure when a suitable noise perturbation is added. More closely related to this note is Bourgain's work on NLS in the defocussing dynamics in a periodic box as the period goes to infinity [5]. He proves that weak limits of solutions as L goes to infinity converge to a unique distributional solution in $C(H^s)$, for $s < \frac{1}{2}$, which depends continuously on initial data in compact space time regions.

Let us now consider the NLS on a periodic lattice of length L. There is an equilibrium Gibbs measure given by

$$Z_L^{-1} \exp\left(-\sum_j^L |\nabla \phi|^2(j) \pm \sum_j^L |\phi(j)|^4 \right).$$

The + and - sign are the focussing and defocussing cases respectively. Let $\langle \cdot \rangle$ be the translation invariant expectation and define $\phi(t, j, \omega)$ to be the solution of the periodic NLS with random initial data distributed by the Gibbs measure. Then since the total energy is conserved and $\langle |\phi(t, j)|^4 \rangle$ is independent of j, we conclude that $\langle |\phi(t, 0)|^4 \rangle$ is uniformly bounded independently of L. This shows that averaging gives good control of the time evolution. We speculate that the time average of the local energy at 0 is bounded. In addition one would like to describe its time fluctuations. But this is far beyond the scope of this note. In [15], Lukkarinen and Spohn investigated the dynamics of the lattice NLS with initial data governed by the infinite volume Gibbs measure. They proved that as the coupling goes to 0, the time rescaled solution to NLS obeys a kinetic equation. Recent work of Mendl and Spohn [17] describes equilibrium time correlations on a one dimensional lattice. On the 3 dimensional lattice, Chatterjee and Kirkpatrick [8], studied the statistical mechanics with the focussing non linearity. They prove that as the density is varied, a first order phase transition occurs corresponding to soliton collapse.

Remark In [20] the detailed dynamics of a gas of solitons of the Ginzburg-Landau equation is studied. In particular, it is shown that there are solutions in which pairs of coupled solitons and separated from others is a possible class of (chaotic) solutions. The approach in this paper is based on the analysis of the infinite system of coupled ODE's corresponding to the internal degrees of freedom of the soliton pairs (center of mass, relative phase, amplitude). It should be pointed out that this system is not Hamiltonian, as it corresponds to the complex NLS.

2 Dynamics of NLS on the Lattice

The Schrödinger equation on the lattice has properties similar to the nonlinear wave equation. In this section let $-\Delta = \partial^* \partial$ be the finite difference Laplacian on $\mathbb{Z}$. Here

$$\partial f(x) = f(x+1) - f(x), \ and \ \partial^* f(x) = f(x-1) - f(x) \tag{2.1}$$

The lattice NLS is given by

$$i\frac{\partial}{\partial t}\psi(t,x) = i\dot{\psi}(t,x) = -\Delta\psi(t,x) + |\psi|^2\psi(t,x), \ x \in \mathbb{Z} \tag{2.2}$$

Proposition 2 *If $|\psi(0,x)| \leq A$ then there exists a constant C such that for any $x_0 \in \mathbb{Z}$ and $t_0 \geq 1$, we have $|\psi(t_0,x_0)| \leq CA t_0^{1/2}$, and*

$$\frac{1}{t_0} \sum_{|x-x_0|\leq t_0} |\psi(t_0,x)|^2 \lesssim A^2. \tag{2.3}$$

Proof The proof of local existence and uniqueness is an application of Picard iteration on the space $l^\infty(\mathbb{Z})$ since the linear operator Δ is bounded on $l^\infty(\mathbb{Z})$, the Taylor series for $e^{it\Delta}$ is uniformly bounded for times $0 \leq t \leq 1$.

To get a bound on the solution at arbitrary time t_0 near x_0, where t_0 is possibly large, we define

$$F(t,x) = \frac{[(x-x_0)^2 + 1]^{1/2}}{R(2t_0 - t + 1)}, \tag{2.4}$$

with $R \geq 1$, and let the local mass be given by

$$M(t) = \sum_x |\psi(t,x)|^2 e^{-F(x,t)} \tag{2.5}$$

By using ∂^* is the adjoint of ∂, and summation by parts, we have

$$dM(t)/dt = -\sum_x \left\{ |\psi(t,x)|^2 e^{-F(t,x)}(\dot{F}) + i[\partial\psi\bar{\psi} - \partial\bar{\psi}\psi]\partial e^{-F(x,t)} \right\} \tag{2.6}$$

The first term inside the braces is positive. To estimate the last term note that

$$\partial e^{-F(x,t)} \approx e^{-F(x,t)}\partial F \approx e^{-F(x,t)}/R(2t_0 - t + 1). \tag{2.7}$$

Since

$$|\partial\psi(x)\bar{\psi}(x) - \partial\bar{\psi}(x)\psi(x)| \leq |\psi(x+1)|^2 + |\psi(x)|^2 \tag{2.8}$$

and

$$\sum_x |\psi(x+1)|^2 e^{-F(x,t)} = \sum_x |\psi(x)|^2 e^{-F(x-1,t)}$$

$$\leq \sum_x |\psi(x)|^2 e^{-F(x,t)} \left(1 + \frac{1}{R(2t_0 - t + 1)}\right)$$

we get the inequality for $t_0 \geq t$

$$dM(t)/dt \leq 3M(t)/R(2t_0 - t + 1) \tag{2.9}$$

This implies $M(t_0) \leq \frac{3\ln(2)}{R} M(0)$, and Proposition 2 follows from

$$\sum_x e^{-\frac{[(x-x_0)^2+1]^{1/2}}{R(t_0+1)}} |\psi(t_0, x)|^2 \leq C \sum_x e^{-\frac{[(x-x_0)^2+1]^{1/2}}{R(2t_0+1)}} A^2 \leq C R A^2 t_0. \qquad (2.10)$$

Taking $R = 1$ completes the proof. $\qquad \square$

If we replace the local mass by the local energy $|\nabla \psi|^2(x) + |\psi(x)|^4$ in (2.5) we can get a improved estimate on the time growth $t^{1/4}$, compatible with Proposition 2. This result only holds for the defocussing case, and would not hold in the focussing case, unlike the proof of Proposition 2.

Proposition 3 *If $|\psi(0, x)| \leq A$, then for $R \geq 1$, $|\psi(t_0, x_0)| \leq C A t_0^{1/4}$, for any $x_0 \in \mathbb{Z}$, and when $t_0 \geq 1$,*

$$\frac{1}{t_0} \sum_{|x-x_0| \leq t_0} |\psi(t_0, x)|^4 \lesssim A^4. \qquad (2.11)$$

Proof Define the localized energy,

$$E(t) = \frac{1}{2} \sum_x |\psi(t, x+1) - \psi(t, x)|^2 e^{-F(t,x)} + \frac{1}{4} \sum_x |\psi(t, x)|^4 e^{-F(t,x)}. \qquad (2.12)$$

If we set $\langle v, w \rangle = Re\, v\bar{w}$, then by direct computation,

$$\begin{aligned}
\frac{dE}{dt} &= -\frac{1}{2} \sum_x |\psi(t, x+1) - \psi(t, x)|^2 e^{-F(t,x)} \dot{F}(t, x) - \frac{1}{4} \sum_x |\psi(t, x)|^4 e^{-F(t,x)} \dot{F}(t, x) \\
&\quad + \sum_x \langle \partial^* \partial \psi e^{-F}, \dot{\psi} \rangle + \sum_x \langle \partial \psi \partial^* e^{-F}, \dot{\psi} \rangle + \sum_x \langle e^{-F} |\psi|^2 \psi, \dot{\psi} \rangle \\
&\leq \sum_x \langle i\dot{\psi}, e^{-F} \dot{\psi} \rangle + \sum_x \langle \partial \psi \partial^* e^{-F}, \dot{\psi} \rangle = \sum_x \langle \partial \psi \partial^* e^{-F}, \dot{\psi} \rangle \\
&= \sum_x \langle \partial \psi \partial^* e^{-F}, i \Delta \psi(t, x) - i |\psi|^2 \psi(t, x) \rangle.
\end{aligned} \qquad (2.13)$$

Using the fact that ∂ and ∂^* are bounded operators on the lattice, (2.7) implies that

$$\begin{aligned}
\sum_x \langle \partial \psi \partial^* e^{-F}, -i |\psi|^2 \psi \rangle &\lesssim \sum_x \frac{1}{R(2t_0 + 1 - t)} e^{-F} |\psi|^4 \\
&\lesssim \frac{1}{R(2t_0 + 1 - t)} E(t).
\end{aligned} \qquad (2.14)$$

Also, since ∂ and ∂^* are bounded operators,

$$\begin{aligned}
\sum_x \langle \partial \psi \partial^* e^{-F}, i \Delta \psi(t, x) \rangle &\lesssim \sum_x \frac{1}{R(2t_0 + 1 - t)} e^{-F(t,x)} |\psi(t, x+1) - \psi(t, x)|^2 \\
&\lesssim \frac{1}{R(2t_0 + 1 - t)} E(t).
\end{aligned} \qquad (2.15)$$

In particular, this implies that $E(t_0) \lesssim_R E(0)$. Therefore,

$$\sum_x e^{-\frac{[(x-x_0)^2+1]^{1/2}}{R(t_0+1)}} |\psi(t_0, x)|^4 \leq C \sum_x e^{-\frac{[(x-x_0)^2+1]^{1/2}}{R(2t_0+1)}} A^4 \leq C R A^4 t_0. \qquad (2.16)$$

Taking $R = 1$ proves the Proposition. $\qquad \square$

Remark In Propositions 2 and 3 we can bound the time derivative $|\frac{\partial \psi}{\partial t}|$ by $O(t^{3/2})$ and $O(t^{3/4})$ respectively.

Remark This argument could be generalized to the defocussing equation

$$i \partial/\partial t\, \psi(t, x) = i \dot{\psi}(t, x) = -\Delta \psi(t, x) + |\psi|^p \psi(t, x), \quad x \in \mathbb{Z}. \tag{2.17}$$

In that case $|\psi(t, x)| \lesssim t^{\frac{1}{p+2}}$.

3 Regularized Nonlinearity

Next, we study the regularized nonlinear Schrödinger equation,

$$i u_t + u_{xx} = \mathcal{N}(u) = \phi * (|\phi * u|^2(\phi * u)), \quad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad u(0, x) = u_0, \tag{3.1}$$

where ϕ is a real valued, symmetric Schwartz function, and $*$ indicates the usual convolution

$$(f * g)(x) = \int f(y) g(x - y) dy. \tag{3.2}$$

Remark The operator $f \mapsto \phi * f$ is a smoothing operator. One important example of such an operator is the Fourier truncation operator. Since the solution to $i u_t + u_{xx} = 0$ travels at velocity ξ at frequency ξ, truncating the nonlinear term in Fourier space allows us to treat (3.1) using a finite propagation speed argument.

A solution to (3.1) conserves the quantities mass,

$$M(u(t)) = \int |u(t, x)|^2 dx, \tag{3.3}$$

and energy,

$$E(u(t)) = \frac{1}{2} \int |\partial_x u(t, x)|^2 dx + \frac{1}{4} \int |\phi * u|^4 dx. \tag{3.4}$$

Remark Both quantities could be infinite in this section.

3.1 Local Well-Posedness

Local well-posedness may be proved using perturbative arguments, for data lying in a large function space.

Theorem 1 *If u_0 lies in $C^2(\mathbb{R})$, then (1.1) has a local solution for $T(\phi, \|u_0\|_{L^\infty}, \|\nabla^2 u_0\|_{L^\infty}) > 0$. The solution u is bounded and uniformly continuous on $[0, T] \times \mathbb{R}$.*

Proof This proof would work equally well in the focussing or defocussing cases. We first prove

$$\|e^{it\Delta} u_0\|_{L^\infty} \lesssim (1 + t^{3/2})(\|u_0\|_{L^\infty} + \|\nabla u_0\|_{L^\infty} + \|\nabla^2 u_0\|_{L^\infty}), \tag{3.5}$$

by using stationary phase. Without loss of generality, it suffices to show that $e^{it\Delta} u_0$ is bounded at the origin by the right hand side of (3.5). Using the stationary phase kernel of $e^{it\Delta}$,

$$e^{it\Delta} u_0(0) = \frac{1}{Ct^{1/2}} \int e^{-i\frac{y^2}{4t}} u_0(y) dy. \tag{3.6}$$

Let χ be as in (1.4). Integrating by parts,

$$\frac{1}{Ct^{1/2}}\int e^{-i\frac{y^2}{4t}}(1-\chi(y))u_0(y)dy = \frac{1}{Ct^{1/2}}\int \frac{2it}{y}\frac{d}{dy}\left(e^{-i\frac{y^2}{4t}}\right)(1-\chi(y))u_0(y)dy$$

$$= O(t^{1/2}\|u_0\|_{L^\infty}) + Ct^{1/2}\int e^{-i\frac{y^2}{4t}}\frac{1}{y}(1-\chi(y))u_0'(y)dy. \tag{3.7}$$

Making another integration by parts argument shows that the second term on the right hand side of (3.7) is also bounded by the right hand side of (3.5).

Now then, by the fundamental theorem of calculus,

$$\chi(y)u_0(y) = \chi(y)u_0(0) + \chi(y)(u_0(y) - u_0(0)) = \chi(y)u_0(0) + \chi(y)\int_0^y u_0'(s)ds. \tag{3.8}$$

Since $\chi(y)$ is smooth and compactly supported, $\|\chi(y)u_0(0)\|_{H^1} \lesssim \|u_0\|_{L^\infty}$, and therefore by the Sobolev embedding theorem and the fact that $e^{it\Delta}$ is a unitary operator for L^2-based Sobolev spaces,

$$\|e^{it\Delta}(\chi(y)u_0(0))\|_{L^\infty} \lesssim \|u_0\|_{L^\infty}. \tag{3.9}$$

Finally, as in (3.7),

$$\frac{1}{Ct^{1/2}}\int e^{-i\frac{y^2}{4t}}\chi(y)(u_0(y) - u_0(0))dy$$

$$= \frac{1}{Ct^{1/2}}\int \frac{2it}{y}\frac{d}{dy}\left(e^{-i\frac{y^2}{4t}}\right)\chi(y)(u_0(y) - u_0(0))dy \tag{3.10}$$

$$= Ct^{1/2}\int \frac{d}{dy}\left(e^{-i\frac{y^2}{4t}}\right)\chi(y)\int_0^1 u_0'(sy)dsdy$$

The right hand side of (3.10) is also bounded by the right hand side of (3.5). This finally proves that (3.5) holds.

A similar argument may also be made for the Duhamel term

$$\int_0^t e^{i(t-\tau)\Delta}\mathcal{N}(u)d\tau. \tag{3.11}$$

Since $\phi * u$ is a smoothing operator, Theorem 1 also implies

$$\|e^{i(t-\tau)\Delta}\mathcal{N}(u)\|_{L^\infty} \lesssim_\phi (1 + |t-\tau|^{3/2})\|u\|_{L^\infty}^3. \tag{3.12}$$

Therefore, Theorem 1 holds by straightforward Picard iteration. To see this, let

$$X = \left\{u : \|u\|_{L^\infty_{t,x}([0,T]\times\mathbb{R})} \le 2C(\phi)(\|u_0\|_{L^\infty} + \|\nabla u_0\|_{L^\infty} + \|\nabla^2 u_0\|_{L^\infty})\right\}, \tag{3.13}$$

where $C(\phi)$ also depends on the implicit constant appearing in (3.5) when $T \le 1$. Then define the operator

$$\Phi(u(t)) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}\mathcal{N}(u)d\tau. \tag{3.14}$$

Then by (3.5) and (3.12), if $u \in X$,

$$\|\Phi u(t)\|_{L^\infty_{t,x}([0,T]\times\mathbf{R})} \le C(\|u_0\|_{L^\infty} + \|\nabla u_0\|_{L^\infty} + \|\nabla^2 u_0\|_{L^\infty})$$
$$+ 8C^3 T(\|u_0\|_{L^\infty} + \|\nabla u_0\|_{L^\infty} + \|\nabla^2 u_0\|_{L^\infty})^3. \tag{3.15}$$

Then for T sufficiently small,

$$\Phi : X \to X. \tag{3.16}$$

Moreover,

$$\|\Phi(u(t)) - \Phi(v(t))\|_{L_{t,x}^\infty([0,T]\times\mathbb{R})} \lesssim T\|u - v\|_{L_{t,x}^\infty([0,T]\times\mathbb{R})}(\|u\|_{L_{t,x}^\infty([0,T]\times\mathbb{R})}^2$$
$$+ \|v\|_{L_{t,x}^\infty([0,T]\times\mathbb{R})}^2), \tag{3.17}$$

which proves that for T sufficiently small, Φ is a contraction. □

3.2 Global Well-Posedness

The proof of global well-posedness for (3.1) relies on mass and energy conservation laws. Although it is not in general true that a function in $L^\infty(\mathbb{R})$ will lie in $L^2(\mathbb{R})$, it is true that such embeddings do hold on any bounded set. We therefore use local conservation laws to prove global well-posedness.

Theorem 2 *If u_0 lies in $C^4(\mathbb{R})$, then* (3.1) *has a global solution whose local energy (see* (3.18) *with $R = t^8$) is a quantity bounded by Ct^8 when t is large. Moreover, we show that* $|u(t, x)| \lesssim t^{8/3}$.

Proof Define the local energy

$$E(x_0, t) = \int \chi\left(\frac{x - x_0}{R}\right)^2 \left[\frac{1}{2}|u_x|^2 + \frac{1}{4}|\phi * u|^4 + \frac{1}{2}|u|^2\right]. \tag{3.18}$$

Remark Observe that we rely heavily on the fact that the local energy is positive definite. In the case of finite mass, (3.3)$< \infty$, we could prove global well-posedness for both the focussing and defocussing problems. Here, we only prove global well-posedness for the defocussing problem.

We may compute

$$\frac{d}{dt}E(x_0, t) = \int \chi\left(\frac{x - x_0}{R}\right)^2 \left[\langle u_x, u_{xt}\rangle + \langle|\phi * u|^2(\phi * u), \phi * u_t\rangle + \langle u, u_t\rangle\right] \tag{3.19}$$

$$= \int \chi\left(\frac{x - x_0}{R}\right)^2 \left[-\langle u_{xx}, u_t\rangle + \langle \mathcal{N}(u), u_t\rangle + \langle u, u_t\rangle\right]$$

$$- \frac{2}{R}\int \chi\left(\frac{x - x_0}{R}\right)\chi'\left(\frac{x - x_0}{R}\right)\langle u_x, u_t\rangle + \int \langle([\phi, \chi^2]u_t), |\phi * u|^2(\phi * u)\rangle. \tag{3.20}$$

Here we are using the inner product

$$\langle f, g\rangle = Re(f(x)\overline{g(x)}), \tag{3.21}$$

and the commutator is given by

$$[\phi, \chi^2]u = \int \left[\chi^2\left(\frac{y - x_0}{R}\right) - \chi^2\left(\frac{x - x_0}{R}\right)\right]\phi(x - y)u(y)dy. \tag{3.22}$$

Using (3.1) and the fact that $\langle u_t, iu_t \rangle = \langle u_t, -u_{xx} + \mathcal{N}(u) \rangle = 0$,

$$\frac{d}{dt} E(x_0, t) = \int \chi \left(\frac{x - x_0}{R} \right)^2 \langle u, u_t \rangle - \frac{2}{R} \int \chi \left(\frac{x - x_0}{R} \right) \chi' \left(\frac{x - x_0}{R} \right) \langle u_x, u_t \rangle$$

$$+ \int \langle ([\phi, \chi^2] u_t), |\phi * u|^2 (\phi * u) \rangle = A + B + C. \tag{3.23}$$

Observe that if $\chi(\frac{x-x_0}{R})$ were replaced by 1, the local energy (3.18) would be exactly equal to the mass in $\frac{1}{2} \times$ (3.3) plus the energy in (3.4). It is also clear that in the case that $\chi = 1$, $\chi'(x) = 0$ and $[\phi, \chi^2] = 0$, so the last two terms in (3.23) would drop out. The first term in (3.23) would also be zero due to the conservation of mass calculations. For the local energy, we will exploit the fact that a $\frac{1}{R}$ appears in $-\frac{2}{R} \int \chi(\frac{x-x_0}{R}) \chi'(\frac{x-x_0}{R}) \langle u_x, u_t \rangle$, and also in $\int \langle ([\phi, \chi^2] u_t), |\phi * u|^2 (\phi * u) \rangle$. Meanwhile, $E(x_0, 0) = O(R)$, so terms depending on the size of u itself will grow as R increases. We will show that for any $0 < R < \infty$, we are able to win the tug of war that develops between $\frac{1}{R}$ and the nonlinear terms for a time that increases as R increases. Taking $R \to \infty$ we shall prove global well-posedness and $|u(t, x)| \lesssim t^{8/3}$.

Term A: Since $u_t = iu_{xx} - i\mathcal{N}(u)$,

$$A = \int \chi \left(\frac{x - x_0}{R} \right)^2 \langle u, u_t \rangle = \int \chi \left(\frac{x - x_0}{R} \right)^2 \langle u, iu_{xx} - i\mathcal{N}(u) \rangle = A_1 + A_2. \tag{3.24}$$

Integrating by parts,

$$A_1 = \int \chi \left(\frac{x - x_0}{R} \right)^2 \langle u, iu_{xx} \rangle = -\frac{2}{R} \int \chi \left(\frac{x - x_0}{R} \right) \chi' \left(\frac{x - x_0}{R} \right) \langle u, iu_x \rangle$$

$$\lesssim \frac{1}{R} \left\| \chi \left(\frac{x - x_0}{R} \right) u \right\|_{L^2} \left\| \chi' \left(\frac{x - x_0}{R} \right) u_x \right\|_{L^2} \lesssim \frac{1}{R} E(x_0, t)^{1/2} \left\| \chi' \left(\frac{x - x_0}{R} \right) u_x \right\|_{L^2}. \tag{3.25}$$

By the definition of χ,

$$|\chi'(x)| \lesssim |\chi(x)| + |\chi(x - 1)| + |\chi(x + 1)|. \tag{3.26}$$

Therefore,

$$A_1 \lesssim \frac{1}{R} E(x_0, t)^{1/2} \left\| \chi' \left(\frac{x - x_0}{R} \right) u_x \right\|_{L^2} \lesssim \frac{1}{R} E(x_0, t)^{1/2} (\sup_{x_0} E(x_0, t)^{1/2}). \tag{3.27}$$

Next,

$$A_2 = -\int \chi^2 \left(\frac{x - x_0}{R} \right) \langle u, i\mathcal{N}(u) \rangle = \int \langle \phi * \chi^2 \left(\frac{x - x_0}{R} \right) u, i|\phi * u|^2 (\phi * u) \rangle$$

$$= \int \langle [\phi, \chi^2] u, i|\phi * u|^2 (\phi * u) \rangle. \tag{3.28}$$

The last equality follows from the fact that

$$\int \chi^2 \left(\frac{x - x_0}{R} \right) Re(-i|\phi * u|^4) = 0. \tag{3.29}$$

To analyze the commutator $[\phi, \chi^2]$, first consider $|x - x_0| > 4R$. In that case,

$$(3.22) = [\phi, \chi^2] = \int \chi^2 \left(\frac{y - x_0}{R}\right) \phi(x - y) u(y) dy$$

$$= \int \frac{1}{1 + |x - y|^2} (1 + |x - y|^2) \phi(x - y) \chi^2 \left(\frac{y - x_0}{R}\right) u(y) dy. \tag{3.30}$$

Since ϕ is a Schwartz function,

$$\||x - x_0|^2 \int \chi^2 \left(\frac{y - x_0}{R}\right) \phi(x - y) u(y) dy\|_{L^4(|x - x_0| \geq 4R)} \lesssim_\phi \|\chi \left(\frac{y - x_0}{R}\right) u\|_{L^2}. \tag{3.31}$$

Therefore,

$$\int_{|x - x_0| \geq 4R} \langle [\phi, \chi^2] u, i |\phi * u|^2 (\phi * u) \rangle dx$$

$$\lesssim_\phi \|\chi \left(\frac{x - x_0}{R}\right) u\|_{L^2} \|\frac{1}{|x - x_0|^2} |\phi * u|^3\|_{L^{4/3}(|x - x_0| \geq 4R)}$$

$$\lesssim E(x_0, t)^{1/2} \left(\sum_{|j| \geq 3} \frac{1}{j^2 R^2} \|\chi \left(\frac{x - x_0 - jR}{R}\right) (\phi * u)\|_{L^4}^3\right) \tag{3.32}$$

$$\lesssim \frac{1}{R} E(x_0, t)^{1/2} (\sup_{x_0} E(x_0, t)^{3/4}).$$

For $|x - x_0| \leq 4R$ observe that by the fundamental theorem of calculus,

$$|\chi^2 \left(\frac{y - x_0}{R}\right) - \chi^2 \left(\frac{x - x_0}{R}\right)| \lesssim \frac{1}{R} |x - y|. \tag{3.33}$$

Then,

$$\int_{|x - x_0| \leq 4R} \langle [\phi, \chi^2] u, i |\phi * u|^2 (\phi * u) \rangle dx$$

$$\lesssim \frac{1}{R} \|\phi * u\|_{L^4(|x - x_0| \leq 4R)}^3 \|\int |x - y| |\phi(x - y)| |u(y)| dy\|_{L^4(|x - x_0| \leq 4R)} \tag{3.34}$$

$$\lesssim \frac{1}{R} (\sup_{x_0} E(x_0, t)^{3/4}) \|\int |x - y| |\phi(x - y)| |u(y)| dy\|_{L^4(|x - x_0| \leq 4R)}.$$

Once again, since ϕ is a Schwartz function, by Young's inequality,

$$\|\int |x - y| |\phi(x - y)| |u(y)| dy\|_{L^4(|x - x_0| \leq 4R)}$$

$$\lesssim \|\int \frac{1}{1 + |x - y|^2} |u(y)| dy\|_{L^4(|x - x_0| \leq 4R)} \lesssim \left(\sup_{x_0} \|\chi \left(\frac{x - x_0}{R}\right) u\|_{L^2}\right). \tag{3.35}$$

Therefore,

$$A_2 \leq \int \langle [\phi, \chi^2] u, i |\phi * u|^2 (\phi * u) \rangle \lesssim \frac{1}{R} (\sup_{x_0} E(t, x_0)^{5/4}). \tag{3.36}$$

Term B: Split

$$-\frac{2}{R}\int \chi(\frac{x-x_0}{R})\chi'\left(\frac{x-x_0}{R}\right)\langle u_x, u_t\rangle$$

$$= -\frac{2}{R}\int \chi\left(\frac{x-x_0}{R}\right)\chi'\left(\frac{x-x_0}{R}\right)\langle u_x, iu_{xx}\rangle \tag{3.37}$$

$$+\frac{2}{R}\int \chi\left(\frac{x-x_0}{R}\right)\chi'\left(\frac{x-x_0}{R}\right)\langle u_x, i\mathcal{N}(u)\rangle = B_1 + B_2.$$

Decompose

$$B_1 = -\frac{2}{R}\int \chi\left(\frac{x-x_0}{R}\right)\chi'\left(\frac{x-x_0}{R}\right)\langle u_x, iu_{xx}\rangle$$

$$= -\frac{2}{R}\int \chi\left(\frac{x-x_0}{R}\right)\chi'\left(\frac{x-x_0}{R}\right)\langle u_x, ie^{it\partial_{xx}}u_{xx}(0)\rangle \tag{3.38}$$

$$-\frac{2}{R}\int \chi\left(\frac{x-x_0}{R}\right)\chi'\left(\frac{x-x_0}{R}\right)\langle u_x, i\partial_{xx}(u - e^{it\partial_{xx}}u(0))\rangle.$$

Using (3.5) and the fact that ∂_x commutes with $e^{it\partial_{xx}}$,

$$-\frac{2}{R}\int \chi\left(\frac{x-x_0}{R}\right)\chi'\left(\frac{x-x_0}{R}\right)\langle u_x, ie^{it\partial_{xx}}u_{xx}(0)\rangle$$

$$\lesssim \frac{1}{R^{1/2}}\|\chi\left(\frac{x-x_0}{R}\right)u_x\|_{L^2}\|e^{it\partial_{xx}}u_{xx}(0)\|_{L^\infty} \tag{3.39}$$

$$\lesssim (1+t^{3/2})\frac{1}{R^{1/2}}E(x_0, t)^{1/2}\|u_0\|_{C^4(\mathbb{R})}.$$

Next observe that by Duhamel's formula,

$$u - e^{it\partial_{xx}}u(0) = i\int_0^t e^{i(t-\tau)\partial_{xx}}\mathcal{N}(u)d\tau. \tag{3.40}$$

Next,

$$\chi'\left(\frac{x-x_0}{R}\right)\partial_{xx}\int_0^t e^{i(t-\tau)\partial_{xx}}\mathcal{N}(u)d\tau$$

$$= \chi'\left(\frac{x-x_0}{R}\right)\int_0^t e^{i(t-\tau)\partial_{xx}}\phi_{xx}*(|\phi*u|^2(\phi*u))d\tau. \tag{3.41}$$

Now then, by Strichartz estimates and the fact that ϕ is a Schwartz function, and thus in L^1,

$$\|\chi'\left(\frac{x-x_0}{R}\right)\int_0^t e^{i(t-\tau)\partial_{xx}}\phi_{xx}*\chi\left(\frac{x-x_0}{4R}\right)(|\phi*u|^2(\phi*u))d\tau\|_{L^2(\mathbb{R})}$$

$$\lesssim_\phi \left(\int_0^t \|\chi\left(\frac{x-x_0}{4R}\right)(|\phi*u|^2(\phi*u))\|_{L^{4/3}}^{8/7}d\tau\right)^{7/8} \lesssim \left(\int_0^t (\sup_{x_0} E(x_0, \tau))^{6/7}d\tau\right)^{7/8}. \tag{3.42}$$

Meanwhile, computing the kernel for a generic function f,

$$\chi'\left(\frac{x-x_0}{R}\right)\int_0^t e^{i(t-\tau)\partial_{xx}}\phi_{xx}*\left(1-\chi\left(\frac{x-x_0}{4R}\right)\right)f\,d\tau$$

$$=\chi'\left(\frac{x-x_0}{R}\right)\int_0^t\int\int e^{-i\frac{(x-y)^2}{4(t-\tau)}}\phi^{(2)}(y-z)\left(1-\chi\left(\frac{z-x_0}{4R}\right)\right)f(z)\,d\tau\,dy\,dz.$$

(3.43)

Since $\chi'(\frac{x-x_0}{R})$ is supported on $|x-x_0|\le 2R$ and $(1-\chi(\frac{z-x_0}{4R}))$ is supported on $|z-x_0|>4R$, $|x-z|>2R$ in (3.43). In the region where $|y-z|>|x-y|$, using the fact that ϕ is a Schwartz function and Young's inequality,

$$\left\|\chi'\left(\frac{x-x_0}{R}\right)\int_0^t\int\int_{|y-z|>|x-y|}e^{-i\frac{(x-y)^2}{4(t-\tau)}}\phi^{(2)}(y-z)\left(1-\chi\left(\frac{z-x_0}{4R}\right)\right)f(z)\,d\tau\,dy\,dz\right\|_{L^2}$$

$$\lesssim\left\|\chi'\left(\frac{x-x_0}{R}\right)\int_0^t\int\int_{|y-z|>|x-y|}\frac{1}{1+|x-y|^2}\cdot\frac{1}{1+|x-z|^2}\right.$$

$$\left.\times|\phi^{(2)}(y-z)(y-z)^4|\left(1-\chi\left(\frac{z-x_0}{4R}\right)\right)|f(z)|\,dz\,dy\,d\tau\right\|_{L^2}$$

$$\lesssim_\phi\sum_{j\neq 0}\frac{1}{1+j^2R^2}\int_0^t\left\|\chi\left(\frac{x-x_0-jR}{R}\right)f\right\|_{L^{4/3}}d\tau\lesssim\frac{1}{R^2}\int_0^t\left(\sup_{x_0}\left\|\chi\left(\frac{x-x_0}{R}\right)f\right\|_{L^{4/3}}\right).$$

(3.44)

Next, consider the case when $|x-y|>|y-z|$. In this case we use the fact that

$$e^{-i\frac{(x-y)^2}{4(t-\tau)}}=\frac{2i(t-\tau)}{(y-x)}\frac{d}{dy}\left(e^{-i\frac{(x-y)^2}{4(t-\tau)}}\right).$$

(3.45)

Then let ψ be a smooth, compactly supported function, $\psi(x)=1$ for $|x|\le 1$, and ψ supported on $|x|\le 2$. Using (3.45) and integrating by parts,

$$\chi'\left(\frac{x-x_0}{R}\right)\int_0^t\int\int\psi\left(\frac{y-z}{x-y}\right)e^{-i\frac{(x-y)^2}{4(t-\tau)}}\phi^{(2)}(y-z)\left(1-\chi\left(\frac{z-x_0}{4R}\right)\right)f(z)\,d\tau\,dy\,dz$$

$$=\chi'\left(\frac{x-x_0}{R}\right)\int_0^t\int\int\frac{2i(t-\tau)}{(y-x)^2}\psi\left(\frac{y-z}{x-y}\right)e^{-i\frac{(x-y)^2}{4(t-\tau)}}\phi^{(2)}(y-z)$$

$$\left(1-\chi\left(\frac{z-x_0}{4R}\right)\right)f(z)\,d\tau\,dy\,dz$$

$$-\chi'\left(\frac{x-x_0}{R}\right)\int_0^t\int\int\frac{2i(t-\tau)}{(y-x)}\psi'\left(\frac{y-z}{x-y}\right)\cdot\left(\frac{1}{x-y}+\frac{y-z}{(x-y)^2}\right)$$

$$\times e^{-i\frac{(x-y)^2}{4(t-\tau)}}\phi^{(2)}(y-z)\left(1-\chi\left(\frac{z-x_0}{4R}\right)\right)f(z)\,d\tau\,dy\,dz$$

$$-\chi'\left(\frac{x-x_0}{R}\right)\int_0^t\int\int\frac{2i(t-\tau)}{(y-x)}\psi\left(\frac{y-z}{x-y}\right)e^{-i\frac{(x-y)^2}{4(t-\tau)}}\phi^{(3)}(y-z)$$

$$\left(1-\chi\left(\frac{z-x_0}{4R}\right)\right)f(z)\,d\tau\,dy\,dz.$$

(3.46)

Applying (3.45) and integrating by parts three more times, and then making the argument in (3.42)–(3.44) with $x - y$ and $y - z$ reversed, implies that

$$\|(3.41)\|_{L^2} \lesssim_\phi \left(\int_0^t (\sup_{x_0} E(x_0, \tau))^{6/7} d\tau \right)^{7/8}$$
$$+ \frac{(1 + t^4)}{R^2} \int_0^t (\sup_{x_0} \|\chi\left(\frac{x - x_0}{R}\right)(\phi * u)\|_{L^4}^3) d\tau. \qquad (3.47)$$

Then consider

$$B_2 = \frac{2}{R} \int \chi\left(\frac{x - x_0}{R}\right) \chi'\left(\frac{x - x_0}{R}\right) \langle u_x, i\mathcal{N}(u)\rangle. \qquad (3.48)$$

We can compute

$$B_2 \lesssim \frac{1}{R} \|\chi\left(\frac{x - x_0}{R}\right) u_x\|_{L^2} \|\chi'\left(\frac{x - x_0}{R}\right) \phi * (|\phi * u|^2(\phi * u))\|_{L^2}$$
$$\lesssim \frac{1}{R} E(t, x_0)^{1/2} \|\chi'\left(\frac{x - x_0}{R}\right) \phi * (|\phi * u|^2(\phi * u))\|_{L^2}. \qquad (3.49)$$

To simplify notation let $f = |\phi * u|^3$. Since ϕ is a Schwartz function,

$$\|\chi'\left(\frac{x - x_0}{R}\right) \phi * f\|_{L^2}$$
$$\lesssim \|\int \chi'\left(\frac{x - x_0}{R}\right) |\phi(x - y)(x - y)^2| \frac{1}{|x - y|^2} |f(y)| dy\|_{L^2}$$
$$\lesssim_\phi \sum_j \frac{1}{1 + j^2 R^2} \|\chi\left(\frac{x - x_0 - jR}{R}\right) f\|_{L^{4/3}} \lesssim \left(\sup_{x_0} \|\chi\left(\frac{x - x_0}{R}\right)(\phi * u)\|_{L^4}^3\right). \qquad (3.50)$$

Term C: For this term as well, split $u_t = iu_{xx} - i\mathcal{N}(u)$.

$$\int \langle [\phi, \chi^2] u_t, |\phi * u|^2(\phi * u)\rangle = \int \langle [\phi, \chi^2] iu_{xx}, |\phi * u|^2(\phi * u)\rangle$$
$$- \int \langle [\phi, \chi^2] i\mathcal{N}(u), |\phi * u|^2(\phi * u)\rangle = C_1 + C_2. \qquad (3.51)$$

Once again we use the definition of the commutator in (3.5). Integrating by parts,

$$C_1 = \int \left[\chi^2\left(\frac{y - x_0}{R}\right) - \chi^2\left(\frac{x - x_0}{R}\right)\right] \phi(x - y)(iu_{xx}(t, y))(|\phi * u|^2(\overline{\phi * u}))(t, x) dy dx$$
$$= -\frac{2}{R} \int \chi\left(\frac{y - x_0}{R}\right) \chi'\left(\frac{y - x_0}{R}\right) \phi(x - y)(iu_x(t, y))(|\phi * u|^2(\overline{\phi * u}))(t, x) dy dx$$
$$- \int \left[\chi^2\left(\frac{x - x_0}{R}\right) - \chi^2\left(\frac{y - x_0}{R}\right)\right] \phi'(x - y)(iu_x(t, y))(|\phi * u|^2(\overline{\phi * u}))(t, x) dy dx. \qquad (3.52)$$

Since ϕ is a Schwartz function,

$$\frac{2}{R} \int \chi \left(\frac{y-x_0}{R}\right) \chi' \left(\frac{y-x_0}{R}\right) \phi(x-y)(iu_x(t,y))(|\phi * u|^2 (\overline{\phi * u}))(t,x) dy dx$$

$$\lesssim \frac{1}{R} \int |\chi \left(\frac{y-x_0}{R}\right)| |\chi' \left(\frac{y-x_0}{R}\right)| \frac{1}{1+|x-y|^2} \cdot (1+|x-y|^2)|\phi(x-y)|$$

$$\times |\phi * u(t,x)|^3 |u_x(t,y)| dy dx,$$

(3.53)

so by Young's inequality,

$$\lesssim_\phi \frac{1}{R} \sum_j \frac{1}{1+j^2 R^2} \|\chi \left(\frac{x-x_0-jR}{R}\right)|\phi * u\|_{L^4}^3 \|\chi \left(\frac{x-x_0}{R}\right) u_x\|_{L^2}$$

$$\lesssim \frac{1}{R}(\sup_{x_0} E(x_0,t)^{5/4}).$$

(3.54)

Next, consider the term

$$\int \left[\chi^2 \left(\frac{x-x_0}{R}\right) - \chi^2 \left(\frac{y-x_0}{R}\right)\right] \phi'(x-y)(iu_x(t,y))(|\phi * u|^2 (\overline{\phi * u}))(t,x) dy dx.$$

(3.55)

Integrating by parts again, if the derivative falls on $\chi^2(\frac{y-x_0}{R})$, then it is possible to proceed as above, only with u_x replaced by u. Therefore, it is left to estimate

$$\int \left[\chi^2 \left(\frac{x-x_0}{R}\right) - \chi^2 \left(\frac{y-x_0}{R}\right)\right] \phi''(x-y)(iu(t,y))(|\phi * u|^2 (\overline{\phi * u}))(t,x) dy dx.$$

(3.56)

To estimate this integral, it is useful to split the integral into three regions, $|x-x_0| \le 4R$ and $|y-x_0| \le 4R$, $|x-x_0| > 4R$, and $|y-x_0| > 4R$. When $|x-x_0| \le 4R$ and $|y-x_0| \le 4R$, then using (3.33) and Young's inequality,

$$\int \left[\chi^2 \left(\frac{x-x_0}{R}\right) - \chi^2 \left(\frac{y-x_0}{R}\right)\right] \phi''(x-y)(iu(t,y))(|\phi * u|^2 (\overline{\phi * u}))(t,x) dy dx$$

$$\lesssim_\phi \frac{1}{R} \int \int_{|x-x_0| \le 4R, |y-x_0| \le 4R} |x-y||\phi''(x-y)||u(t,y)||\phi * u(t,x)|^3 dx dy$$ (3.57)

$$\lesssim \sup_{x_0} \|\chi \left(\frac{y-x_0}{R}\right) u\|_{L^2} \|\chi \left(\frac{x-x_0}{R}\right)(\phi * u)\|_{L^4}^3 \lesssim \sup_{x_0} E(x_0,t)^{5/4}.$$

If $|x-x_0| > 4R$,

$$\chi^2 \left(\frac{x-x_0}{R}\right) - \chi^2 \left(\frac{y-x_0}{R}\right) = -\chi^2 \left(\frac{y-x_0}{R}\right).$$

(3.58)

Then,

$$\int_{|x-x_0|>4R} \chi^2\left(\frac{y-x_0}{R}\right)|\phi''(x-y)||u(t,y)||\phi*u(t,x)|^3 dxdy$$

$$= \int_{|x-x_0|>4R} \chi^2\left(\frac{y-x_0}{R}\right)|\phi''(x-y)(x-y)^2|\frac{1}{|x-y|^2}|u(t,y)||\phi*u(t,x)|^3 dxdy$$

$$\lesssim_\phi \sum_{j\neq 0}\frac{1}{j^2R^2}\|\chi\left(\frac{x-x_0-jR}{R}\right)u\|_{L^2}\|\chi(\frac{y-x_0}{R})|\phi*u|^3\|_{L^{4/3}} \lesssim \frac{1}{R^2}\sup_{x_0} E(x_0,t)^{5/4}.$$

(3.59)

A similar computation may be made for $|y-x_0|>4R$, since in that case,

$$\chi^2\left(\frac{x-x_0}{R}\right) - \chi^2\left(\frac{y-x_0}{R}\right) = \chi^2\left(\frac{x-x_0}{R}\right).$$

(3.60)

Therefore,

$$C_1 = \int \langle([\phi,\chi^2]iu_{xx},|\phi*u|^2(\phi*u)\rangle \lesssim \frac{1}{R}E(x_0,t)^{5/4}.$$

(3.61)

Finally, consider the contribution of C_2. If $|y-x_0|>4R$,

$$\int_{|y-x_0|>4R}\left[\chi^2\left(\frac{y-x_0}{R}\right) - \chi^2\left(\frac{x-x_0}{R}\right)\right]\phi(x-y)$$

$$\times (\phi*|\phi*u|^2(\phi*u)(t,x))(i|\phi*u|^2(\overline{\phi*u}))(t,y)dydx$$

$$= -\int_{|y-x_0|>4R}\chi^2\left(\frac{x-x_0}{R}\right)\tilde{\phi}(x-y)|\phi*u|^2(\phi*u)(t,x)(i|\phi*u|^2(\overline{\phi*u}))(t,y)dydx$$

(3.62)

$$\lesssim \|\chi\left(\frac{x-x_0}{R}\right)|\phi*u|^2(\phi*u)\|_{L^{4/3}}\|\int_{|y-x_0|>4R}\chi\left(\frac{x-x_0}{R}\right)|\tilde{\phi}(x-y)||\phi*u|^3 dy\|_{L^4}$$

$$\lesssim (\sup_{x_0} E(x_0,t)^{3/4})\|\int_{|y-x_0|>4R}\chi\left(\frac{x-x_0}{R}\right)|\tilde{\phi}(x-y)||\phi*u|^3 dy\|_{L^4}.$$

(3.63)

Because ϕ is a symmetric, real valued, Schwartz function,

$$\tilde{\phi}(x-y) = \int \phi(x-z)\phi(z-y)dz = \int \phi(x-z)\phi(y-z)dz = \int \phi(x-y-z)\phi(z)dz$$

(3.64)

is also a symmetric, real valued, Schwartz function. Therefore,

$$\|\chi\left(\frac{x-x_0}{R}\right)\int_{|y-x_0|>4R}|\tilde{\phi}(x-y)||\phi*u|^3 dy\|_{L^4}$$

$$= \|\chi\left(\frac{x-x_0}{R}\right)\int_{|y-x_0|>4R}|\tilde{\phi}(x-y)||x-y|^2\frac{1}{|x-y|^2}|\phi*u|^3 dy\|_{L^4}$$

(3.65)

$$\lesssim_\phi \sum_{j\neq 0}\frac{1}{j^2R^2}\|\chi\left(\frac{x-x_0-jR}{R}\right)|\phi*u|^3\|_{L^{4/3}} \lesssim \frac{1}{R^2}(\sup_{x_0} E(x_0,t)^{3/4}).$$

By symmetry of arguments, a similar estimate could also be obtained when $|x - x_0| > 4R$. Finally, consider the case when $|x - x_0| \leq 4R$ and $|y - x_0| \leq 4R$. By (3.33),

$$\int \int_{|x-x_0| \leq 4R, |y-x_0| \leq 4R} \left[\chi^2 \left(\frac{y - x_0}{R} \right) - \chi^2 \left(\frac{x - x_0}{R} \right) \right] \phi(x - y)$$
$$\times (\phi * |\phi * u|^2 (\phi * u))(t, x))(i|\phi * u|^2 (\overline{\phi * u}))(t, y) dy dx$$
$$\lesssim \frac{1}{R} \int_{|x-x_0| \leq 4R} \int_{|y-x_0| \leq 4R} |x - y| |\tilde{\phi}(x - y)| |\phi * u(t, x)|^3 |\phi * u(t, y)|^3 dx dy$$
$$\lesssim_\phi \frac{1}{R} (\sup_{x_0} E(t, x_0)^{3/2}). \tag{3.66}$$

Therefore,

$$C_2 = \int \langle ([\phi, \chi^2] i \phi * |\phi * u|^2 (\phi * u)), |\phi * u|^2 (\phi * u) \rangle \lesssim \frac{1}{R} E(x_0, t)^{3/2}. \tag{3.67}$$

Therefore, combining the above estimates on A+B+C we have proved

$$\frac{d}{dt} E(x_0, t) \lesssim \frac{1}{R} (\sup_{x_0} E(x_0, t)) + \frac{1}{R} (\sup_{x_0} E(x_0, t)^{5/4}) + (1 + t^{3/2}) \frac{E(x_0, t)^{1/2}}{R^{1/2}} \|u_0\|_{C^4}$$
$$+ \left(\int_0^t (\sup_{x_0} E(x_0, \tau))^{6/7} d\tau \right)^{7/8} + \frac{1 + t^4}{R^2} \int_0^t (\sup_{x_0} E(x_0, \tau)^{3/4}) d\tau$$
$$+ \frac{1}{R} (\sup_{x_0} E(x_0, t)^{3/2}). \tag{3.68}$$

By Hölder's inequality,

$$\sup_{x_0} E(x_0, 0) \lesssim R. \tag{3.69}$$

Making a bootstrap argument, suppose $[0, T]$ is an interval for which $\sup_{t, x_0} E(x_0, t) \lesssim R$, and also $T \leq R^{1/8}$. Then for any $t \in [0, T]$,

$$(3.68) \lesssim 1 + R^{1/4} + (1 + t^{3/2}) \|u_0\|_{C^4} + R^{3/4} t^{7/8} + \frac{1 + t^5}{R^{5/4}} + R^{1/2}. \tag{3.70}$$

Integrating the right hand side of (3.70) on the interval $[0, T]$,

$$\int_0^T 1 + R^{1/4} + (1 + t^{3/2}) \|u_0\|_{C^4} + R^{3/4} t^{7/8} + \frac{1 + t^5}{R^{5/4}} + R^{1/2} dt$$
$$\lesssim R^{63/64} + R^{5/32} \|u_0\|_{C^4} \ll R. \tag{3.71}$$

Making a bootstrap argument, this proves $\sup_{x_0} E(x_0, t) \lesssim R$ for all $t \in [0, T]$. This proves Theorem 2. Using (1.7), we obtain $|u(t, x)| \lesssim t^{8/3}$. $\qquad \square$

Remark If we assumed smooth initial data, we could combine the previous proof with Gronwall's inequality to prove bounds on higher order derivatives of the solution.

4 Real Analytic Local Well-Posedness

In this section we prove a local result for the one dimensional, nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi + |\psi|^2 \psi, \qquad \psi(0, x) = \psi_0. \tag{4.1}$$

T Oh [19] proved local well-posedness for almost periodic data. As a simple example consider

$$\psi_0(x) = \cos(x) + \cos(\sqrt{2}x). \tag{4.2}$$

Such an initial value problem cannot be solved directly using the usual methods of Strichartz estimates combined with the fact that the initial data is in an L^2-based space. Moreover, the fact that the data (4.2) consists of two periodic functions whose periods are not rational multiples of one another prevents the study of (4.1) on a torus.

In this note, we study (4.1) for initial data that is real analytic in a strip of width 3, that is, the set

$$\Omega = \{z : z = x + iy, |y| < 3\}. \tag{4.3}$$

For example, the initial data

$$\psi_0(x) = \sum_{n \in \mathbf{Z}} a_n e^{-(x-n)^2}, \quad |a_n| \leq 1 \tag{4.4}$$

satisfies this condition. In fact, (4.4) is an entire function since if $z = x + iy$,

$$\psi_0(z) = \sum_{n \in \mathbf{Z}} a_n e^{-(z-n)^2} = e^{y^2} \sum_{n \in \mathbf{Z}} a_n e^{-(x-n)^2} e^{-2iy(x-n)}, \tag{4.5}$$

so the sum (4.5) converges uniformly in any compact subset of $\mathbf{C}$, proving that $\psi_0(x)$ is an entire function.

We prove the local well-posedness result for real analytic data. We suspect that Gevrey or very smooth data will suffice to get the existence of local solutions by using Moser's regularization [16]. See Hörmander [11] for an abstract version of the Newton-Nash implicit function.

Theorem 3 *The initial value problem* (4.1) *is locally well-posed on some interval* $[-T(\psi_0), T(\psi_0)]$, *where u_0 is real analytic in the strip* (4.3).

Proof Define the formal iteration as follows. First let

$$\psi_1(t, x) = e^{it\Delta} \psi_0. \tag{4.6}$$

Next, define $\xi_2(t, x)$, to be the linear correction to ψ_1 given by

$$i \frac{\partial \xi_2}{\partial t} = -\Delta \xi_2 + 2|\psi_1|^2 \xi_2 + \psi_1^2 \bar{\xi}_2 + R_1, \quad \xi_2(0, x) = 0, \tag{4.7}$$

and

$$R_1 = -i \frac{\partial \psi_1}{\partial t} - \Delta \psi_1 + |\psi_1|^2 \psi_1 = |\psi_1|^2 \psi_1. \tag{4.8}$$

and set

$$\psi_2(t, x) = \psi_1(t, x) + \xi_2(t, x). \tag{4.9}$$

In general, iteratively define

$$\psi_{n+1} = \psi_n + \xi_{n+1}, \tag{4.10}$$

where

$$i \frac{\partial \xi_{n+1}}{\partial t} = -\Delta \xi_{n+1} + 2|\psi_n|^2 \xi_{n+1} + \psi_n^2 \bar{\xi}_{n+1} + R_n, \quad \xi_{n+1}(0, x) = 0, \tag{4.11}$$

and when $n > 1$,

$$R_n = -i\frac{\partial\psi_n}{\partial t} - \Delta\psi_n + |\psi_n|^2\psi_n = 2|\xi_n|^2\psi_{n-1} + \xi_n^2\bar{\psi}_{n-1} + |\xi_n|^2\xi_n. \qquad (4.12)$$

The rapid convergence of the Newton iteration relies on the fact that R_n is quadratic in ξ_n. For any $n > 1$, arguing by induction,

$$i\frac{\partial\psi_n}{\partial t} + \Delta\psi_n = i\frac{\partial}{\partial t}\left(\psi_1 + \sum_{m=2}^{n}\xi_m\right) + \Delta\left(\psi_1 + \sum_{m=2}^{n}\xi_m\right) = |\psi_n|^2\psi_n - R_{n+1}. \quad (4.13)$$

Therefore, if we can show that $\xi_n \to 0$ and $R_n \to 0$ in a suitable Banach space as $n \to \infty$, we are done.

It is convenient to rewrite (4.11) and (4.12) in matrix notation. Let

$$u_n = \begin{pmatrix} \xi_n \\ \bar{\xi}_n \end{pmatrix}, \quad b_n = \begin{pmatrix} R_n \\ -\bar{R}_n \end{pmatrix}, \quad M_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix}, \quad V_n(t, x) = \begin{pmatrix} 2|\psi_n|^2 & \psi_n^2 \\ -\bar{\psi}_n^2 & -2|\psi_n|^2 \end{pmatrix}. \tag{4.14}$$

Then (4.11) has the form

$$i\frac{\partial u_{n+1}}{\partial t} = M_n(t, x)u_{n+1} + b_n, \quad M_n = M_0 + V_n. \tag{4.15}$$

Let $\Phi(t, t_1)$ be the fundamental solution operator to this equation,

$$i\frac{\partial\Phi(t, t_1)}{\partial t} = M_n\Phi_n(t, t_1), \qquad \Phi_n(t_1, t_1) = Id. \tag{4.16}$$

Formally, $\Phi_n(t, t_1)$ can be expressed as a time ordered integral

$$\Phi_n(t, t_1) = \mathbf{T}\exp\left(-i\int_{t_1}^{t}M(s)ds\right). \tag{4.17}$$

Then the solution to (4.15) is given by

$$u_{n+1} = -i\Phi_n(t, 0)\int_0^t \Phi_n(s, 0)^{-1}b_n(s)ds. \tag{4.18}$$

Indeed, under (4.18), $u_{n+1}(0) = 0$, and

$$i\frac{\partial u_{n+1}}{\partial t} = b_n(t) + M_n u_{n+1}. \tag{4.19}$$

Lemma 1 *(Properties of Φ) Let*

$$|V_n(t)| = \sup_{s \le t, x}|V(s, x)|. \tag{4.20}$$

Then,

$$\|\Phi_n f\|_{L^2(\mathbb{R})} \le e^{|t||V_n(t)|}\|f\|_{L^2(\mathbb{R})}. \tag{4.21}$$

Proof Φ_n is the solution operator to (4.15) with $b_n = 0$. Standard local well-posedness arguments show that (4.15) has a solution. Next, (4.21) holds for the interval of existence via Gronwall's inequality. The estimates (4.20) and (4.21) combine to show that $\Phi_n(t, t_1)$ is well-defined for all $t, t_1 \in \mathbb{R}$.   □

Lemma 2 (More properties of Φ)

$$\left[\frac{d}{dx}, \Phi_n(t_2, t_1)\right] = -i \int_{t_1}^{t_2} \Phi_n(t_2, s) \frac{d}{dx} V_n(s, x) \Phi_n(s, t_1) ds, \tag{4.22}$$

and

$$[x, \Phi_n(t_2, t_1)] = 2i \int_{t_1}^{t_2} \Phi_n(t_2, s) \frac{d}{dx} \Phi_n(s, t_1) ds. \tag{4.23}$$

Proof Without loss of generality suppose $t_1 = 0$ and $t_2 = 1$. Then for any N,

$$\Phi_n(1, 0) = \Phi_n\left(1, 1 - \frac{1}{N}\right) \circ \cdots \circ \Phi_n\left(\frac{1}{N}, 0\right). \tag{4.24}$$

Therefore, by direct computation,

$$\left[\frac{d}{dx}, \Phi_n(1, 0)\right] = \sum_{j=1}^{N} \Phi_n\left(1, 1 - \frac{1}{N}\right) \circ \cdots \circ \left[\frac{d}{dx}, \Phi_n\left(\frac{j}{N}, \frac{j-1}{N}\right)\right] \circ \cdots \circ \Phi_n\left(\frac{1}{N}, 0\right). \tag{4.25}$$

Then compute,

$$\left[\frac{d}{dx}, \Phi_n\left(\frac{j}{N}, \frac{j-1}{N}\right)\right] \sim -\frac{i}{N} \frac{d}{dx} V_n\left(\frac{j-1}{N}, x\right). \tag{4.26}$$

Similarly,

$$\left[x, \Phi_n\left(\frac{j}{N}, \frac{j-1}{N}\right)\right] \sim 2i \frac{1}{N} \frac{d}{dx}. \tag{4.27}$$

This proves the lemma. $\qquad\square$

By taking the time derivative of $\Phi(t)^{-1} \Phi(t) = Id$,

$$\frac{d}{dt}(\Phi(t)^{-1}) = -i \Phi(t)^{-2} \Phi(t) M(t) = -i \Phi(t)^{-1} M(t), \qquad \Phi(t) = \mathbf{T} \exp(-i \int_0^t M(s) ds). \tag{4.28}$$

Next, if f is a real analytic function, making a Taylor expansion,

$$e^{a\frac{d}{dx}} f = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(x) = f(x + a). \tag{4.29}$$

Therefore, we have the identity,

$$e^{a\frac{d}{dx}} \Phi_{n,0} f = \Phi_{n,a} e^{a\frac{d}{dx}} f, \tag{4.30}$$

where $\Phi_{n,a}$ is the fundamental solution for $M_0 + V_n(t, x + a)$.

The norm that will be used to control the iteration is given by

$$\|f\|(r, p) = \sup_x \sum_{n \geq 0, 0 \leq q \leq p} |\frac{f^{(n+q)}(x)}{n!}| r^n. \tag{4.31}$$

The following general lemma relates different norms.

Lemma 3 *Let $r_{n+1} < r_n$ and set $\delta_n = r_n - r_{n+1}$. Then,*

$$\|f\|(r_{n+1}, p) \lesssim_p \|f\|(r_n, 0)\delta_n^{-p}. \tag{4.32}$$

Proof We consider the simplest form of the estimate

$$\|f\|(r_{n+1}, 1) \lesssim \|f\|(r_n, 0)\delta_n^{-1}. \tag{4.33}$$

$$\|f\|(r_{n+1}, 1) = \sup_x \sum_{m \geq 0} \frac{|f^{(m+1)}(x)|}{(m+1)!} r_n^{m+1} \left[(m+1)\frac{r_{n+1}^m}{r_n^{m+1}}\right]. \tag{4.34}$$

The inequality follows since the factor in brackets is uniformly bounded by δ_n^{-1}. The case for general p may proved the same way. $\qquad\square$

Remark It may be possible to refine the estimates to bound the implicit constant in (4.32) to q!, which would agree with Cauchy integral formula. However, it is not too important to determine the exact constant here.

The proof of the convergence of the Newton iteration relies on the bound for the operator $\Phi_n(t, s)$, where $|t|, |s| \leq 1$.

Lemma 4 *For $|t| \leq 1$ and any $r_{n+1} > 0$,*

$$\|\Phi_n(t, 0)b_n\|(r_{n+1}, 0) \lesssim e^{t|V_n(t)|}\|b_n\|(r_{n+1}, 3)(1 + \|V_n\|(r_{n+1}, 3) + \|V_n\|(r_{n+1}, 3)^2). \tag{4.35}$$

Proof As in the previous section we have

$$\|e^{it\Delta}\psi_0\|(r, 0) \lesssim (1 + t^{3/2})(\|\psi_0\|(r, 0) + \|\nabla\psi_0\|(r, 0) + \|\nabla^2\psi_0\|(r, 0)). \tag{4.36}$$

This takes care of the case when $V_n \equiv 0$. When V_n is not identically equal to zero, we use (4.30) and the commutator estimates in Lemma 2. Let χ_j be a partition of unity, where $\sum_j \chi(x - j) = \sum_j \chi_j(x) = 1$ for any $x \in \mathbf{R}$, and set

$$g_{n,j} = [1 + (x - j)^2]\Phi_n(t, 0)\chi_j b_n. \tag{4.37}$$

Therefore,

$$\|\Phi_n(t, 0)b_n\|(r_n, 0) = \|\sum_j (1 + (x - j))^{-2}g_{n,j}\|(r_n, 0) \leq \sup_j \|g_{n,j}\|(r_n, 0). \tag{4.38}$$

To simplify notation, let $f = \chi_j b_n$. Also, without loss of generality suppose that $j = 0$. Then,

$$x^2\Phi_n(t, 0)f = x\Phi_n(t, 0)xf + x\int_0^t \Phi_n(t, s)\frac{d}{dx}\Phi_n(s, 0)f\,ds$$

$$= x\Phi_n(t, 0)xf + x\int_0^t \Phi_n(t, 0)\left(\frac{d}{dx}f\right)ds$$

$$+ x\int_0^t \Phi_n(t, s)\int_0^s \Phi_n(s, s_1)\left(\frac{d}{dx}V_n(s_1, s)\right)\Phi_n(s_1, 0)f\,ds_1ds \tag{4.39}$$

$$= x\Phi_n(t, 0)xf + tx\Phi_n(t, 0)\left(\frac{d}{dx}f\right)$$

$$+ x\int_0^t \int_0^s \Phi_n(t, s_1)\left(\frac{d}{dx}V_n(s_1, s)\right)\Phi_n(s_1, 0)f\,ds_1ds.$$

Making a similar calculation,

$$x\Phi_n(t,0)xf = \Phi_n(t,0)x^2 f + t\Phi_n(t,0)\left(\frac{d}{dx}(xf)\right)$$
$$+ \int_0^t \int_0^s \Phi_n(t,s_1)\left(\frac{d}{dx}V_n(s_1,s)\right)\Phi_n(s_1,0)(xf)ds_1ds. \tag{4.40}$$

Then, remembering that $f = \chi_j b_n$, since $0 \leq t \leq 1$,

$$\|x\Phi_n(t,0)xf\|_{L^2} \leq e^{t|V_n(t)|}\|x^2 f\|_{L^2} + te^{t|V_n(t)|}\|f\|_{L^2}$$
$$+ te^{t|V_n(t)|}\|x\frac{d}{dx}f\|_{L^2} + e^{t|V_n|}|\frac{d}{dx}V_n|\|xf\|_{L^2} \tag{4.41}$$
$$\lesssim e^{|V_n(1)|}\left(1 + |\frac{d}{dx}V_n(1)|\right)\|b_n\|(r_{n+1},1).$$

By a similar calculation, since $0 \leq t \leq 1$,

$$\|t\Phi_n(t,0)\left(\frac{d}{dx}(xf)\right)\|_{L^2} \lesssim e^{|V_n(1)|}\left(1 + |\frac{d}{dx}V_n(1)|\right)\|b_n\|(r_{n+1},2). \tag{4.42}$$

Finally,

$$x\int_0^t \int_0^s \Phi_n(t,s_1)\left(\frac{d}{dx}V_n(s_1,s)\right)\Phi_n(s_1,0)f ds_1ds$$
$$= \int_0^t \int_0^s \Phi_n(t,s_1)\left(\frac{d}{dx}V_n(s_1,s)\right)x\Phi_n(s_1,0)f ds_1ds \tag{4.43}$$
$$+ \int_0^t \int_0^s \int_s^t \Phi_n(t,\tau)\frac{d}{dx}\Phi_n(\tau,s_1)\left(\frac{d}{dx}V_n(s_1,s)\right)\Phi_n(s_1,0)f ds_1ds.$$

Also, by (4.22),

$$\int_0^t \int_0^s \int_s^t \Phi_n(t,\tau)\frac{d}{dx}\Phi_n(\tau,s_1)\left(\frac{d}{dx}V_n(s_1,s)\right)\Phi_n(s_1,0)f ds_1ds$$
$$= \int_0^t \int_0^s \int_s^t \Phi_n(t,\tau)\Phi_n(\tau,s_1)\frac{d}{dx}\left[\left(\frac{d}{dx}V_n(s_1,s)\right)\Phi_n(s_1,0)f\right]ds_1ds d\tau \tag{4.44}$$
$$+ \int_0^t \int_0^s \int_s^t \Phi_n(t,\tau)\int_{s_1}^\tau \Phi_n(\tau,\tilde{\tau})\left(\frac{d}{dx}V_n(\tilde{\tau},x)\right)\Phi_n(\tilde{\tau},s_1)$$
$$\left(\frac{d}{dx}V_n(s_1,s)\right)\Phi_n(s_1,0)f ds_1ds d\tau d\tilde{\tau}.$$

The L^2 norm of the last term is bounded by

$$e^{t|V_n(t)|}\|V_n\|(r_{n+1},3)^2\|f\|_{L^2}. \tag{4.45}$$

The other remaining terms,

$$\int_0^t \int_0^s \Phi_n(t,s_1)\left(\frac{d}{dx}V_n(s_1,s)\right)x\Phi_n(s_1,0)f ds_1ds, \tag{4.46}$$

and

$$\int_0^t \int_0^s \int_s^t \Phi_n(t,\tau)\Phi_n(\tau,s_1)\frac{d}{dx}\left[\left(\frac{d}{dx}V_n(s_1,s)\right)\Phi_n(s_1,0)f\right]ds_1ds d\tau, \tag{4.47}$$

may be handled in a similar manner.

Finally, using the Sobolev embedding theorem,

$$\|g_{n,j}\|_{L^\infty} \lesssim \|g_{n,j}\|_{L^2} + \|\nabla g_{n,j}\|_{L^2}, \tag{4.48}$$

completes the proof of the lemma. $\qquad\square$

Newton iteration: The induction argument makes use of the fact that if b_n is real analytic, then u_{n+1}, given by (4.18), is also real analytic. Additionally, a product of two real analytic functions is real analytic, as well as the complex conjugate of a real analytic function is real analytic.

We start the induction by assuming

$$\epsilon_1 = \|u_1\|(r_1, 0) \ll 1, \tag{4.49}$$

and define

$$\epsilon_k = \|u_k\|(r_k, 0), \qquad k \geq 2. \tag{4.50}$$

It is always possible to assume (4.49) after rescaling. By Lemmas 3 and 4,

$$\epsilon_2 = \|u_2\|(r_2, 0) \leq C\|\psi_1\|(r_2, p)^3 \leq C[\|\psi_1\|(r_1, 0)\delta_1^{-p}]^3 = [C\epsilon_1\delta_1^{-p}]^3. \tag{4.51}$$

In general, the remainder b_n is quadratic in u_n by (4.12), and therefore, for any n,

$$\epsilon_{n+1} \leq C[C\epsilon_n\delta^{-p}]^2(C\epsilon_1\delta^{-p}). \tag{4.52}$$

If $\epsilon_1 > 0$ is sufficiently small, $p = 3$ and $\delta_n = n^{-2}$,

$$\epsilon_{n+1} \leq C\epsilon_n^2 n^4 \leq C^n(n!)^4\epsilon_1^{2^n} \to 0, \tag{4.53}$$

as $n \to \infty$. Thus, ψ_n converges as $n \to \infty$.

5 Appendix: Linear Schrödinger Time Evolution on $\mathbb{Z}$

Let $\psi(t, n) = e^{it\Delta}\psi_0(n)$ where $\psi_0 = \sum_j a_j\delta_j$ and Δ is the finite difference Laplacian.

Lemma A *We can choose* a_n, $|a_j| \leq 1$ *so that* $|\psi(t, 0)| \geq \delta t^{1/2}$, $\delta > 0$.

Proof The fundamental solution to the lattice Schrödinger equation can be expressed in terms of the integral

$$F_n(t) = (2\pi)^{-1} \int_0^{2\pi} e^{it\cos(\theta)+in\theta}d\theta, \ n \in \mathbb{Z},$$

which is closely related to the Bessel function. This integral has two saddle points θ_s, $\pi - \theta_s$ where $\sin(\theta_s) = n/t$, $\cos(\theta_s) = [1 - (n/t)^2]^{1/2}$. By classical stationary phase, if $|n| \leq t/2$

$$F_n(t) \approx [t\cos(\theta_s)]^{-1/2} \cos(\phi(t, n)(1 + O(1/t))$$

when n is even and

$$F(t, n) = i[t \cos(\theta_s)]^{-1/2} \sin(\phi(t, n)(1 + O(1/t)),$$

when n is odd. Here

$$\phi(t, n) = \pi/4 + t \cos(\theta_s) + n\theta_s .$$

Since $|\psi(t, 0)| = |\sum_n a_n F_n(t)|$ we will choose a_n so that the sum equals

$$\sum_{|n| \leq t/2} |F(t, n)| \geq \delta t^{1/2} .$$

To prove this lower bound note that if $|\cos(\phi(t, n))|$ is small when n is even then $|\sin(\phi(t, n+1))| \geq 1/4$ since $|\phi(t, n+1) - \phi(t, n)| \approx |\theta_s| = \arcsin n/t \leq \delta t^{1/2}$. □

References

1. Binder, I., Damanik, D., Goldstein, M., Lukic, M.: Almost periodicity in time of solutions of the KdV equation. Duke Math. J. **167**(14), 2633–2678 (2018)
2. Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations II. The KdV-equation. Geom. Funct. Anal. **3**(3), 209–262 (1993)
3. Bourgain, J.: Periodic nonlinear Schrödinger equation and invariant measures. Commun. Math. Phys. **166**(1), 1–26 (1994)
4. Bourgain, J.: Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. Commun. Math. Phys. **176**(2), 421–445 (1996)
5. Bourgain, J.: Invariant measures for NLS in infinite volume. Commun. Math. Phys. **210**(3), 605–620 (2000)
6. Carlen, E.A., Fröhlich, J., Lebowitz, J.: Exponential relaxation to equilibrium for a one-dimensional focusing non-linear schrödinger equation with noise. Commun. Math. Phys. **342**(1), 303–332 (2016)
7. Chaichenets, L., Hundertmark, D., Kunstmann, P., Pattakos, N.: On the existence of global solutions of the one-dimensional cubic NLS for initial data in the modulation space $M_{p,q}(\mathbb{R})$. J. Differ. Equ. **263**(8), 4429–4441 (2017)
8. Chatterjee, S., Kirkpatrick, K.: Probabilistic methods for discrete nonlinear schrödinger equations. Commun. Pure Appl. Math. **65**(5), 727–757 (2012)
9. Damanik, D., Goldstein, M.: On the existence and uniqueness of global solutions for the KdV equation with quasi-periodic initial data. J. Am. Math. Soc. **29**(3), 825–856 (2016)
10. Greene, R.E., Jacobowitz, H.: Analytic isometric embeddings. Ann. Math. **2**(93), 189–204 (1971)
11. Hörmander, Lars: On the Nash-Moser implicit function theorem. Ann. Acad. Sci. Fenn. Ser. A I Math., **10**, 255–259 (1985)
12. Kappeler, T., Schaad, B., Topalov, P.: Scattering-like phenomena of the periodic defocusing NLS equation. Math. Res. Lett. **24**(3), 803–826 (2017)
13. Lebowitz, J.L., Mounaix, P., Wang, W.-M.: Approach to equilibrium for the stochastic nls. Commun. Math. Phys. **321**(1), 69–84 (2013)
14. Lebowitz, J.L., Rose, H.A., Speer, E.R.: Statistical mechanics of the nonlinear Schrödinger equation. J. Stat. Phys. **50**(3–4), 657–687 (1988)
15. Lukkarinen, J., Spohn, H.: Weakly nonlinear schrödinger equation with random initial data. Invent. Math. **183**(1), 79–188 (2011)
16. Moser, J.: A new technique for the construction of solutions of nonlinear differential equations. Proc. Nat. Acad. Sci. USA **47**, 1824–1831 (1961)
17. Mendl, C.B., Spohn, H.: Low temperature dynamics of the one-dimensional discrete nonlinear schrödinger equation. J. Stat. Mech. **2015**(8), P08028 (2015)
18. Tadahiro, O.: Global existence for the defocusing nonlinear schrödinger equations with limit periodic initial data. Commun. Pure Appl. Anal. **14**(4), 1563–1580 (2015)
19. Tadahiro, O.: On nonlinear Schrödinger equations with almost periodic initial data. SIAM J. Math. Anal. **47**(2), 1253–1270 (2015)
20. Turaev, D., Zelik, S.: Analytical proof of space-time chaos in ginzburg-landau equations. Dyn. Syst. **28**(4), 1713–1751 (2010)

21. Wang, W.-M.: Infinite energy quasi-periodic solutions to nonlinear schrödinger equations on $\mathbb{R}$. arXiv preprint arXiv:1908.11627 (2019)

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