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Free Self-decomposability and Unimodality of the Fuss–Catalan Distributions

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Abstract
We study properties of the Fuss–Catalan distributions $\mu(p, r)$, $p \geq 1, 0 < r \leq p$: free infinite divisibility, free self-decomposability, free regularity and unimodality. We show that the Fuss–Catalan distribution $\mu(p, r)$ is freely self-decomposable if and only if $1 \leq p = r \leq 2$. We verify numerically the following phase-transition conjecture: For every $p > 1$ there exists $r_0(p)$, with $p-1 < r_0(p) < p$, such that the Fuss–Catalan distribution $\mu(p, r)$ is unimodal if and only if either $r = p$ or $0 < r \leq r_0(p)$. We prove rigorously some partial results.

Keywords Fuss–Catalan distributions · Free cumulant transform · Voiculescu transform · Free Lévy measures · Free cumulants · Free infinite divisibility · Free self-decomposability · Free L_1 · Free regularity · Unimodality

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1 Introduction

The *two-parameter Fuss–Catalan numbers* (called also *Raney numbers*) are defined by

$$A_k(p,r) := \frac{r}{k!}\prod_{i=1}^{k-1}(kp+r-i) = \frac{r}{kp+r}\binom{kp+r}{k},$$

where p, r are real parameters. If p, r are natural numbers then $A_k(p,r)$ may admit several combinatorial interpretations in terms of plane trees, lattice paths or noncrossing partitions, see [12]. In particular, $A_k(2,1)$ is the famous Catalan sequence, see [23]:

$$A_0(2,1) = 1,\ A_1(2,1) = 1,\ A_2(2,1) = 2,\ A_3(2,1) = 5,$$
$$A_4(2,1) = 14,\ A_5(2,1) = 42,\ A_6(2,1) = 132\cdots.$$

It is known that the sequence $A_k(p,r)$ is positive definite if and only if either $p \geq 1$, $0 < r \leq p$ or $p \leq 0$, $p-1 \geq r$ or else if $r = 0$, see [16–19] for various proofs. The corresponding probability measure we will call the *Fuss–Catalan distribution* and denote $\mu(p,r)$, so that

$$A_k(p,r) = \int_{\mathbb{R}} x^k \mu(p,r)(\mathrm{d}x).$$

It is easy to observe that $\mu(p,0) = \delta_0$ and that $\mu(1-p,-r)$ is the reflection of $\mu(p,r)$. Therefore we will confine ourselves to the case $p \geq 1, 0 < r \leq p$. Then $\mu(p,r)$ is absolutely continuous (except $\mu(1,1) = \delta_1$) and the support is $\left[0, p^p(p-1)^{1-p}\right]$ for $p > 1$ and $[0,1]$ for $p = 1, 0 < r < 1$.

The Fuss–Catalan numbers and distributions play an important role in the context of random matrix theory. They appear as the singular values distributions of the product of independent, large sized non-hermitian Gaussian random matrices. In [10], Forrester and Liu proved that the two-parameters Fuss–Catalan numbers show up as the moment sequence for the spectral density of square of scaled singular values as the matrix size goes to infinity. Moreover, Forrester et al. stated in [11] that the Fuss–Catalan distributions have densities for equilibrium problems based on the eigenvalues of products of random matrices and related statistical systems. On the other hand, in [20] the Fuss–Catalan distributions were also applied in quantum information theory as the random constructed states: the level density of mixed quantum states associated with a graph [8] and states obtained by projection onto the maximally entangled states of a multi-partite system [28]. In this context, in [20], this class was analyzed by free probabilistic transformation.

Forrester and Liu [10] found the following implicit formula for the density function $W_{p,r}(x)$ of $\mu(p,r)$:

Proposition 1.1 (Proposition 2.1 in [10]) *For $p > 1$ put*

$$\rho(\varphi) := \frac{(\sin(p\varphi))^p}{\sin(\varphi)(\sin((p-1)\varphi))^{p-1}}, \quad 0 < \varphi < \frac{\pi}{p}. \tag{1}$$

Then $\rho(\varphi)$ is a decreasing function which maps $(0,\pi/p)$ onto $\left(0, p^p(p-1)^{1-p}\right)$ and we have

$$W_{p,r}(\rho(\varphi)) = \frac{(\sin((p-1)\varphi))^{p-r-1}\sin(\varphi)\sin(r\varphi)}{\pi(\sin(p\varphi))^{p-r}}, \quad 0 < \varphi < \frac{\pi}{p}. \tag{2}$$

For $p = 1$, $0 < r < 1$ the density function $W_{1,r}(x)$ is given in [17, formula (5.2)] (or see Proposition 4.7).

We have therefore

$$A_k(p,r) = \int_0^{p^p(p-1)^{1-p}} x^k W_{p,r}(x)\,\mathrm{d}x \tag{3}$$

for $k \geq 0$, $p > 1$, $0 < r \leq p$. This formula is still valid for $r > p > 1$, however in this case $W_{p,r}(x)$ is negative on some subinterval of $\left(0, p^p(p-1)^{1-p}\right)$. Another description of the density function $W_{p,r}$, in terms of the Meijer functions, for rational $p > 1$, was provided in [19].

It was proved in [17] that the free cumulant sequence $\{r_k(\mu(p,r))\}_{k=1}^{\infty}$ of $\mu(p,r)$ is given by

$$r_k(\mu(p,r)) = A_k(p-r,r). \tag{4}$$

Consequently, if $0 < r \leq \min\{p-1, p/2\}$ then $\mu(p,r)$ is freely infinitely divisible, i.e infinitely divisible with respect to the additive free convolution $\boxplus$. Here we will show in addition that $\mu(p,p)$ is freely infinitely divisible for $1 \leq p \leq 2$.

In this paper we investigate free self-decomposability and unimodality. These two concepts are very relevant.

Let us briefly explain the importance of modes of a distribution. It goes without saying that it is very easy to calculate modes from the observed data. It is also easy to imagine that the number and position of modes will be a tool for estimating the distribution. However, if the distribution is not given explicitly, but, for example, through some analytic transformations, like the characteristic function (i.e. the Fourier transform), the Stielties transform or the free R-transform, modality of that distribution becomes difficult to study. In this context a result of Yamazato [27] is useful: *all self-decomposable distributions are unimodal.* Hasebe and Thorbjørnsen [14] proved its free analog: *all freely self-decomposable distributions are unimodal.* From this property and the fact that the class of self-decomposable distributions in both classical and free case are closed under classical and free convolutions, respectively, we can find classical or free convolution semigroup of unimodal distributions.

We also investigate the class of free regular distribution. The distributions in this class are free infinitely divisible, supported on $[0, \infty)$ and closed under free additive convolution. The supports of the corresponding free convolution semigroup are concentrated on $[0, \infty)$. This type support information are quite important in quantum information theory. See e.g. [9].

1.1 Main Results

In this paper we study the Fuss-Catalan distributions $\mu(p,p)$ and $\mu(p,r)$ in the framework of the free probability theory. In particular we are interested in free infinite divisibility, free self-decomposability, free L_1 property, free regularity and also unimodality. First we briefly recall these concepts. In Sect. 3 we concentrate on the distributions $\mu(p,p)$. We will prove:

Theorem 1.2 *For the Fuss–Catalan distribution $\mu(p,p)$ we have the following:*

1. *$\mu(p,p)$ is freely infinitely divisible if and only if $1 \leq p \leq 2$.*
2. *If $1 \leq p \leq 2$ then $\mu(p,p)$ is freely self-decomposable, more precisely, it is in the free L_1 class.*
3. *$\mu(p,p)$ is not free regular for any $1 < p \leq 2$.*
4. *$\mu(p,p)$ is unimodal for all $p \geq 1$.*

In Sect. 4 we obtain some results Fuss–Catalan distribution $\mu(p,r)$ in general:

Theorem 1.3 *Suppose that $p \geq 1$ and $0 < r \leq p$. For the Fuss-Catalan distribution $\mu(p, r)$ we have the following:*

(1) *$\mu(p, r)$ is freely infinitely divisible if and only if either $0 < r \leq \min\{p/2, p-1\}$ or $1 \leq p = r \leq 2$.*
(2) *$\mu(p, r)$ is freely self-decomposable if and only if $1 \leq p = r \leq 2$.*
(3) *$\mu(p, r)$ is free regular if and only if either $0 < r \leq \min\{p/2, p-1\}$ or $p = r = 1$.*

Furthermore, we study the unimodality for the Fuss–Catalan distributions $\mu(p, p-1)$, $\mu(2r, r)$, $\mu(1, r)$ and $\mu(2, r)$. In particular, in Proposition 4.9, we find $r_1 \in (1, 2)$ such that $\mu(2, r)$ is unimodal for $0 < r < r_1$. We have calculated numerically the value of r_1. For other parameters (p, r) we verify numerically a phase-transition conjecture.

In this paper we will denote by $\mathcal{P}(I)$ and $\mathcal{B}(I)$ the family of all Borel probability measures and the class of all Borel sets on $I \subseteq \mathbb{R}$. We will denote $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) the set of complex numbers with strictly positive (resp. strictly negative) imaginary part.

2 Preliminaries in the Free Probability Theory

2.1 Freely Infinitely Divisible Distributions

The notion of the free infinite divisibility is an important research area. One reason is the Berovici–Pata map, which is a bijection which maps classical infinitely divisible distributions onto free infinitely divisible ones.

A probability measure μ on $\mathbb{R}$ is called *freely infinitely divisible* if for any $n \in \mathbb{N}$ there exists a probability measure $\mu_n \in \mathcal{P}(\mathbb{R})$ such that

$$\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}},$$

where $\boxplus$ denotes the *free additive convolution* which can be defined as the distribution of sum of freely independent selfadjoint operators. In this case, $\mu_n \in \mathcal{P}(\mathbb{R})$ is uniquely determined for each $n \in \mathbb{N}$. The freely infinite divisible distributions can be characterized as those admitting a Lévy–Khintchine representation in terms of R-transform which is the free analog of the cumulant transform $C_\mu(z) := \log(\widehat{\mu}(z))$, where $\widehat{\mu}$ is the characteristic function of μ. This was originally established by Bercovici and Voiculescu in [7]. To explain it, we gather analytic tools for free additive convolution $\boxplus$. In order to define the *R-transform* (or *free cumulant transform*) R_μ of a (Borel-) probability measure μ on $\mathbb{R}$ first we need to define its Cauchy–Stieltjes transform G_μ:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(\mathrm{d}t), \qquad (z \in \mathbb{C}^+).$$

Note in particular that $\mathrm{Im}(G_\mu(z)) < 0$ for any z in $\mathbb{C}^+$, and hence we may consider the reciprocal Cauchy transform $F_\mu \colon \mathbb{C}^+ \to \mathbb{C}^+$ given by $F_\mu(z) = 1/G_\mu(z)$. For any probability measure μ on $\mathbb{R}$ and any λ in $(0, \infty)$ there exist positive numbers α, β and M such that F_μ is univalent on the set $\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ \mid \mathrm{Im}(z) > \beta, |\mathrm{Re}(z)| < \alpha \mathrm{Im}(z)\}$ and such that $F_\mu(\Gamma_{\alpha,\beta}) \supset \Gamma_{\lambda,M}$. Therefore the right inverse F_μ^{-1} of F_μ exists on $\Gamma_{\lambda,M}$, and the free cumulant transform R_μ is defined by

$$R_\mu(w) = wF_\mu^{-1}(1/w) - 1, \quad \text{for all } w \text{ such that } 1/w \in \Gamma_{\lambda,M}.$$

The name refers to the fact that R_μ linearizes free additive convolution (cf. [7]). Variants of R_μ (with the same linearizing property) are the R-transform $\mathcal{R}_\mu$ and the Voiculescu transform φ_μ related by the following equalities:

$$R_\mu(w) = w\mathcal{R}_\mu(w) = w\varphi_\mu(\tfrac{1}{w}).$$

The free version of the Lévy–Khintchine representation now amounts to the statement that a probability measure μ on $\mathbb{R}$ is freely infinitely divisible if and only if there exist $a \geq 0$, $\eta \in \mathbb{R}$ and a Lévy measure[1] ν such that

$$R_\mu(w) = aw^2 + \eta w + \int_{\mathbb{R}} \left(\frac{1}{1-wx} - 1 - wx\mathbf{1}_{[-1,1]}(x) \right) \nu(\mathrm{d}x) \qquad (w \in \mathbb{C}^-). \tag{5}$$

The triplet (a, η, ν) is uniquely determined and referred to as the *free characteristic triplet* for μ, and ν is referred to as the *free Lévy measure* for μ. In terms of the Voiculescu transform φ_μ the free Lévy–Khintchine representation takes the form:

$$\varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\, \sigma(\mathrm{d}t), \qquad (z \in \mathbb{C}^+), \tag{6}$$

where the *free generating pair* (γ, σ) is uniquely determined and related to the free characteristic triplet by the formulas[4]:

$$\begin{cases} \nu(\mathrm{d}t) = \frac{1+t^2}{t^2} \cdot \mathbf{1}_{\mathbb{R}\setminus\{0\}}(t)\, \sigma(\mathrm{d}t), \\ \eta = \gamma + \int_{\mathbb{R}} t\Big(\mathbf{1}_{[-1,1]}(t) - \frac{1}{1+t^2} \Big)\, \nu(\mathrm{d}t), \\ a = \sigma(\{0\}). \end{cases}$$

In particular σ is a finite measure. The right hand side of (6) gives rise to an analytic function defined on all of $\mathbb{C}^+$, and in fact the property that φ_μ can be extended analytically to all of $\mathbb{C}^+$ also characterizes the measures in the class of freely infinitely divisible distributions. More precisely Bercovici and Voiculescu proved in [7] the following fundamental result:

Theorem 2.1 *A probability measure μ on $\mathbb{R}$ is freely infinitely divisible if and only if the Voiculescu transform φ_μ has an analytic extension defined on $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$.*

Recently free infinite divisibility has been proved for: normal distribution, some of the Boolean-stable distributions, some of the beta distributions and some of the gamma distributions, including the chi-square distribution, see [2,5,13].

2.2 Freely Self-decomposable Distributions

A probability measure μ on $\mathbb{R}$ is called *freely self-decomposable* if for any $c \in (0, 1)$ there exists a probability measure $\rho_c \in \mathcal{P}(\mathbb{R})$ such that

$$\mu = D_c(\mu) \boxplus \rho_c, \tag{7}$$

where D_c is dilation, that is, $D_c(\mu)(B) := \mu(c^{-1}B)$ for any $c > 0$ and $B \in \mathcal{B}(\mathbb{R})$. It is known that the measures ρ_c are freely infinitely divisible too.

The concept has six characterizations at least: (I) in terms of the free Lévy measure, (II) limit theorem, (III) stochastic integral representation, (IV) self-similarity, (V) the free cumulant sequence and (VI) analytic functions. From (I) to (IV) are proved in [4] based on the Bercovici–Pata bijection and classical results (see page 2 in [26]). (V) and (VI) are proved in [15]. In this paper we will apply (I) and (V).

[1] A (Borel-) measure ν on $\mathbb{R}$ is called a Lévy measure, if $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\}\, \nu(\mathrm{d}x) < \infty$.

2.2.1 Free Lévy Measure

In this section we characterize the class of classical and freely self-decomposable distributions via its Lévy measure. Firstly we define the concept of unimodality. A measure ρ is said to be *unimodal with mode* $a \in \mathbb{R}$ if

$$\rho(dx) = c\delta_a + f(x)\,\mathrm{d}x,$$

where dx is the Lebesgue measure, $c \geq 0$ and $f(x)$ is a function which is non-decreasing on $(-\infty, a)$ and non-increasing on (a, ∞). If μ is classically or freely self-decomposable, then its corresponding Lévy measure is absolutely continuous with respect to Lebesgue measure and classical/free Lévy measure ν_μ has following form:

$$\nu_\mu(\mathrm{d}x) = \frac{k(x)}{|x|}\,\mathrm{d}x,$$

where the measure $k(x)\,\mathrm{d}x$ is unimodal with mode 0, see [4] for details.

2.2.2 Free Cummulant Sequence

If μ is a compactly supported probability measure on $\mathbb{R}$ then the free cumulant transform R_μ can be extended analytically to an open neighborhood of 0 and $R_\mu(0) = 0$. Thus $R_\mu(z)$ admits a power series expansion:

$$R_\mu(z) = \sum_{n=1}^{\infty} r_n(\mu) z^n$$

in a disc around 0. The coefficients $\{r_n(\mu)\}_{n\geq 1}$ are called the *free cumulants of* μ (see e.g. [6]). They can be also computed from moments of μ via Möbius inversion, see [6,21]. Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is said to be *conditionally positive definite* if the infinite matrix $\{a_{i+j}\}_{i,j=1}^{\infty}$ is positive definite, see [21]. It is equivalent to positive definiteness of the sequence $\{a_{n+2}\}_{n=0}^{\infty}$.

Proposition 2.2 [15] *Let μ be a Borel probability measure on $\mathbb{R}$ with moments of all orders, and let $\{r_n(\mu)\}_{n=1}^{\infty}$ be the free cumulant sequence of μ. Then:*

(i) *If μ is freely self-decomposable then $\{nr_n(\mu)\}_{n=1}^{\infty}$ is conditionally positive definite.*
(ii) *Suppose further that μ has compact support. Then μ is freely self-decomposable if and only if $\{nr_n(\mu)\}_{n=1}^{\infty}$ is conditionally positive definite.*

Remark 2.3 Suppose that μ is compactly supported. It is well known that μ is freely infinitely divisible if and only if $\{r_n(\mu)\}_{n=1}^{\infty}$ is conditionally positive definite (see e.g. [21, Theorem 13.16]). Observe the following implication:

$$\{nr_n(\mu)\}_{n=1}^{\infty} \text{ is conditionally positive definite } \implies \{r_n(\mu)\}_{n=1}^{\infty} \text{ is conditionally positive definite.}$$

Indeed, the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is conditionally positive definite since $\frac{1}{n}$ is the $(n-1)$-th moment of the uniform distribution on (0, 1) and the pointwise product of two conditionally positive definite sequences is again conditionally positive definite.

2.3 Free Regular Distributions

A freely infinitely divisible distribution μ on $[0, \infty)$ is said to be *free regular* if the measure $\mu^{\boxplus t}$ is also a probability measure on $[0, \infty)$ for all $t > 0$. For example, the Marchenko–Pastur law $\Pi_{p,\theta}$ is free regular, where

$$\Pi_{p,\theta}(dx) := \max\{1-p, 0\}\delta_0 + \frac{\sqrt{(\theta(1+\sqrt{p})^2 - x)(x - \theta(1-\sqrt{p})^2)}}{2\pi\theta x}\mathbf{1}_{(\theta(1-\sqrt{p})^2, \theta(1+\sqrt{p})^2)}(x)dx,$$

for $p, \theta > 0$ since $\Pi_{p,\theta}^{\boxplus t} = \Pi_{pt,\theta} \in \mathcal{P}([0, \infty))$ for all $t > 0$. In [3] a characterization of free regular measures is given via R-transform as follows:

Theorem 2.4 [3, Theorem 4.2] *Let μ be a freely infinitely divisible distribution on $[0, \infty)$. Then μ is free regular if and only if its free cumulant transform is represented as*

$$R_\mu(z) = \eta' z + \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1\right)\nu(dx), \qquad (z \in \mathbb{C}^-),$$

for some $\eta' \geq 0$ and ν is the free Lévy measure with $\int_{(0,\infty)} \min\{1, x^2\}\nu(dx) < \infty$ and $\nu((-\infty, 0]) = 0$.

Futhermore, free regular measures are characterized as free subordinators, see Theorem 4.2 in [3].

3 Fuss–Catalan Distributions $\mu(p, p)$

In this section we discuss the Fuss–Catalan distributions $\mu(p, p)$, $p \geq 1$. First we study free infinite divisibility. In Sect. 3.2 we obtain a result for free self-decomposability of $\mu(p, p)$. Then we provide free Lévy–Khintchine representation of $\mu(p, p)$ via the Gauss hypergeometric functions. In Sect. 3.4 we introduce concept of the free L_1 class and prove that $\mu(p, p)$ is in the free L_1 class for all $1 \leq p \leq 2$. In Sect. 3.5 we investigate free regularity for $\mu(p, p)$. Finally we prove that all the distributions $\mu(p, p)$, $p \geq 1$, are unimodal.

3.1 Free Infinite Divisibility for $\mu(p, p)$

By (4) the free cumulants of $\mu(p, p)$ are given by $r_n(\mu(p, p)) = A_n(0, p) = \binom{p}{n}$, $n \geq 1$. Therefore first we are going to study the sequence $\left\{\binom{p}{n+2}\right\}_{n=0}^{\infty}$.

Proposition 3.1 *If $-1 < p < 2$, $p \neq 0, 1$, then the sequence $\left\{\binom{p}{n+2}\right\}_{n=0}^{\infty}$ admits the following integral representation:*

$$\binom{p}{n+2} = \frac{\sin(p\pi)}{\pi}\int_{-1}^{0} x^n \cdot x \left(\frac{1+x}{-x}\right)^p dx.$$

This sequence is positive definite if and only if $p \in [-1, 0] \cup [1, 2]$.

Proof Substituting $x \to -y$, using the properties of the Beta function and applying Euler's reflection formula:

$$\Gamma(1-p)\Gamma(p) = \frac{\pi}{\sin(p\pi)}, \qquad p \notin \mathbb{Z},$$

we get

$$\begin{aligned}
&\frac{\sin(p\pi)}{\pi}\int_{-1}^{0} x^n \cdot x\left(\frac{1+x}{-x}\right)^p dx \\
&\quad = (-1)^{n+1}\frac{\sin(p\pi)}{\pi}\int_0^1 y^{n+1-p}(1-y)^p dy \\
&\quad = (-1)^{n+1}\frac{\sin(p\pi)}{\pi}\frac{\Gamma(n+2-p)\Gamma(p+1)}{\Gamma(n+3)} \\
&\quad = (-1)^{n+1}\frac{\sin(p\pi)}{\pi}\frac{(n+1-p)\ldots(n+1-p)(2-p)(1-p)\Gamma(1-p)\times p\Gamma(p)}{(n+2)!} \\
&\quad = \frac{p(p-1)(p-2)\ldots(p-n-1)}{(n+2)!} = \binom{p}{n+2}.
\end{aligned}$$

If $p \in (-1,0)\cup(1,2)$, $-1 < x < 0$ then $x\sin(p\pi) > 0$. Therefore the sequence $a_n(p) := \binom{p}{n+2}$ is positive definite for $p \in (-1,0)\cup(1,2)$. We have also $a_n(-1) = (-1)^n$, $a_n(0) = a_n(1) = 0$ for $n \geq 0$, $a_0(2) = 1$ and $a_n(2) = 0$ for $n \geq 1$, so the sequences $a_n(-1)$, $a_n(0)$, $a_n(1)$, $a_n(2)$ are positive definite too. On the other hand, if the sequence $a_n(p)$ is positive definite, then $a_0(p) = p(p-1)/2 \geq 0$ and

$$a_0(p)a_2(p) - a_1(p)^2 = \frac{1}{144}(2-p)(1+p)p^2(p-1)^2 \geq 0,$$

which implies that $p \in [-1,0]\cup[1,2]$. □

According to Remark 2.3 and Proposition 3.1, we have

Theorem 3.2 *The Fuss–Catalan distribution $\mu(p,p)$ is freely infinitely divisible if and only if $1 \leq p \leq 2$.*

Note that this case was overlooked in Corollary 7.1 in [18].

3.2 Free Self-decomposability for $\mu(p,p)$

In this section we will prove free self-decomposability for $\mu(p,p)$, $1 \leq p \leq 2$. We need the following

Proposition 3.3 *If $0 < p < 2$, $p \neq 1$, then the sequence $\left\{(n+2)\binom{p}{n+2}\right\}_{n=0}^{\infty}$ admits the following integral representation:*

$$(n+2)\binom{p}{n+2} = -\frac{p\sin(p\pi)}{\pi}\int_{-1}^{0} x^n \left(\frac{1+x}{-x}\right)^{p-1} dx. \tag{8}$$

This sequence is positive definite if and only if $p \in \{0\}\cup[1,2]$.

Proof Similarly as before, we have

$$
\begin{aligned}
&-\frac{p\sin(p\pi)}{\pi}\int_{-1}^{0} x^n\left(\frac{1+x}{-x}\right)^{p-1}dx\\
&=(-1)^{n+1}\frac{p\sin(p\pi)}{\pi}\int_0^1 y^{n+1-p}(1-y)^{p-1}\,dy\\
&=(-1)^{n+1}\frac{p\sin(p\pi)}{\pi}\frac{\Gamma(n+2-p)\Gamma(p)}{\Gamma(n+2)}\\
&=(-1)^{n+1}\frac{p\sin(p\pi)}{\pi}\frac{(n+1-p)(n-p)\dots(1-p)\Gamma(1-p)\Gamma(p)}{(n+1)!}\\
&=\frac{p(p-1)\dots(p-n-1)}{(n+1)!}=(n+2)\binom{p}{n+2}.
\end{aligned}
$$

From (8) we see that the sequence $b_n(p) := (n+2)\binom{p}{n+2}$ is positive definite for $1 < p < 2$. We have also $b_n(0) = b_n(1) = 0$ for $n \geq 0$, $b_0(2) = 2$ and $b_n(2) = 0$ for $n \geq 1$, so that the sequences $b_n(0)$, $b_n(1)$ and $b_n(2)$ are positive definite too. On the other hand, if the sequence $b_n(p)$ is positive definite, then $b_0(p) = p(p-1) \geq 0$ and

$$b_0(p)b_2(p) - b_1(p)^2 = \frac{1}{12}p^3(p-1)^2(2-p) \geq 0,$$

which implies that $p \in \{0\} \cup [1, 2]$. □

Combining Proposition 2.2 with Proposition 3.3 we obtain

Theorem 3.4 *The Fuss–Catalan distribution $\mu(p, p)$ is freely self-decomposable if and only if $1 \leq p \leq 2$.*

3.3 Free Cumulant Transform and Nonregularity of $\mu(p, p)$

From the binomial expansion, the free cumulant transform $R_{\mu(p,p)}$ of $\mu(p, p)$ is given by

$$R_{\mu(p,p)}(z) = \sum_{n=1}^{\infty} A_n(0, p)z^n = (1+z)^p - 1.$$

Since $\mu(p, p)$ is freely infinitely divisible (even freely self-decomposable) for $1 \leq p \leq 2$, its free cumulant transform should be written by the free Lévy–Khintchine representation (5). We give a free characteristic triplet of $\mu(p, p)$ for $1 \leq p \leq 2$. In particular $R_{\mu(1,1)}(z) = z$ and $R_{\mu(2,2)}(z) = 2z + z^2$, so in these cases the free Lévy–Khintchine triples are $(0, 1, 0)$ and $(1, 2, 0)$.

Now we assume that $1 < p < 2$. We will apply the *Gauss hypergeometric series* which is defined by

$${}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where a, b, c are real parameters, $c \neq 0, -1, -2, \dots$ and $(a)_n$ denotes the *Pochhammer symbol*: $(a)_n := a(a+1)\dots(a+n-1)$, $(a)_0 := 1$, see [1,22]. The series is absolutely convergent for $|z| < 1$. The function ${}_2F_1(a, b, c; z)$ is the unique solution $f(z)$ which is analytic at $z = 0$ with $f(0) = 1$, of the following equation:

$$z(1-z)f''(z) + (c - (a+b+1)z)\,f'(z) - abf(z) = 0. \tag{9}$$

Moreover, if $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|z| < 1$ then we have the following integral representation:

$$ {}_2F_1(a, b, c; z) = \frac{1}{B(b, c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx, \tag{10} $$

where $B(t, s)$ is the Beta function. We give an explicit formula for a special choice of a, b, c.

Lemma 3.5 *For $s > 0$, $s \neq 1, 2$ we have*

$$ {}_2F_1(1, s, 3; z) = -\frac{2\left((s-2)z + 1 - (1-z)^{2-s}\right)}{(s-2)(s-1)z^2}. \tag{11} $$

Proof Let $s > 0$, $s \neq 1, 2$ and denote by $f_s(z)$ the right hand side of (11). This function is analytic at $z = 0$ and it is easy to check that $f_s(0) = 1$. Then we have

$$ f_s'(z) = \frac{2\left(2(1-z)^{2-s} + (2-s)z(1-z)^{1-s} - 3(2-s)z + 2\right)}{(s-2)(s-1)z^3}, $$
$$ f_s''(z) = \frac{-2\left(6(1-z)^{2-s} + 4(2-s)z(1-z)^{1-s} + (2-s)(1-s)z^2(1-z)^{-s} - 6(2-s)z + 6\right)}{(s-2)(s-1)z^4}, $$

and one can check that the differential equation (9) satisfied. This concludes the proof. □

For $1 < p < 2$, we get the free Lévy–Khintchine representation of $\mu(p, p)$.

Proposition 3.6 *For $1 < p < 2$, $z \in \mathbb{C}^-$, we have*

$$ \begin{aligned} R_{\mu(p,p)}(z) = pz + \int_{\mathbb{R}} &\left(\frac{1}{1-zx} - 1 - zx\mathbf{1}_{[-1,1]}(x)\right) \\ &\times \left(-\frac{\sin(p\pi)}{\pi|x|}\right)\left(\frac{1+x}{-x}\right)^p \mathbf{1}_{(-1,0)}(x) dx, \end{aligned} \tag{12} $$

Proof For $1 < p < 2$ and for z in a disc around 0, by applying (10) and Lemma 3.5, we have

$$ \begin{aligned} &\int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 - zx\mathbf{1}_{[-1,1]}(x)\right) \times \left(-\frac{\sin(p\pi)}{\pi|x|}\right)\left(\frac{1+x}{-x}\right)^p \mathbf{1}_{(-1,0)}(x) dx \\ &= z^2\left(-\frac{\sin(p\pi)}{\pi}\right)\int_0^1 x^{1-p}(1-x)^p(1+zx)^{-1} dx \\ &= z^2\left(-\frac{\sin(p\pi)}{\pi}\right) B(2-p, 1+p) {}_2F_1(1, 2-p, 3; -z) \\ &= z^2\left(-\frac{\sin(p\pi)}{\pi}\right)\left(-\frac{\pi(p-1)p}{2\sin(p\pi)}\right)\left(\frac{2((1+z)^p - pz - 1)}{(p-1)pz^2}\right) \\ &= (1+z)^p - pz - 1 \\ &= R_{\mu(p,p)}(z) - pz. \end{aligned} $$

Therefore the free cumulant transform of $\mu(p, p)$ has the representation (12) on a neighbourhood 0 for $1 < p < 2$. Since $\mu(p, p)$ is freely infinitely divisible for all $1 < p < 2$, its free cumulant transform has an analytic continuation to $\mathbb{C}^-$ and therefore the formula (12) holds for all $z \in \mathbb{C}^-$ by using the identity theorem of complex analytic functions. □

Remark 3.7 By Proposition 3.6, the free Lévy measure $\nu_{\mu(p,p)}$ of $\mu(p, p)$ is of the form

$$ \nu_{\mu(p,p)}(dx) = \frac{k_p(x)}{|x|} dx, $$

where

$$k_p(x) = \left(-\frac{\sin(p\pi)}{\pi}\right)\left(\frac{1+x}{-x}\right)^p \mathbf{1}_{(-1,0)}(x)dx.$$

Note that for $1 < p < 2$ the function k_p is non-decreasing on $(-1, 0)$, which is an another proof of free self-decomposability of $\mu(p, p)$. Moreover, the free Lévy measure $\nu_{\mu(p,p)}$ for $1 < p < 2$ has compact support so that it is not in free counterpart of so-called Thorin class. Note also that, surprisingly, the support of $\mu(p, p)$ is contained in the positive half-line, while the support of the Lévy measure $\nu_{\mu(p,p)}$ is contained in negative half-line.

As an immediate consequence of Proposition 3.6 and Theorem 2.4 we get

Theorem 3.8 *If* $1 < p \leq 2$ *then the Fuss–Catalan distribution* $\mu(p, p)$ *is not free regular.*

Since $\mu(2, 2)^{\boxplus t}$ corresponds to the Wigner's semicircle law with mean $2t$ and variance t, the measure $\mu(2, 2)^{\boxplus t}$ is not probability measure on $[0, \infty)$ for $0 < t < 1$, and therefore $\mu(2, 2)$ is not also free regular. The measure $\mu(1, 1) = \delta_1$ is free regular.

3.4 $\mu(p, p)$ is in the Class Free L_1

Now we define the following special subclass of all freely self-decomposable distributions.

Definition 1 A probability measure μ on $\mathbb{R}$ is said to be in the class free L_1 if μ is freely self-decomposable and for every $c \in (0, 1)$ the measure $\rho_c := \rho_c(\mu) \in \mathcal{P}(\mathbb{R})$ in (7) is also freely self-decomposable.

According to Sect. 3.1, the measure $\mu(p, p)$ is freely self-decomposable for $1 \leq p \leq 2$. Therefore for any $c \in (0, 1)$ there exists $\rho_{p,c} \in \mathcal{P}(\mathbb{R})$ such that

$$\mu(p, p) = D_c(\mu(p, p)) \boxplus \rho_{p,c}. \tag{13}$$

Now we will prove that $\rho_{p,c}$ is also freely self-decomposable for $1 \leq p \leq 2$.

Theorem 3.9 *The Fuss–Catalan distribution* $\mu(p, p)$ *is in the class free* L_1 *for all* $1 \leq p \leq 2$.

Proof For any $p \in [1, 2]$, $c \in (0, 1)$, we show that $\rho_{p,c}$ in (13) is freely self-decomposable. First we consider the cases $p = 1, 2$. If $p = 1$ then

$$R_{\rho_{1,c}}(z) = R_{\mu(1,1)}(z) - R_{D_c(\mu(1,1))}(z) = z - cz = (1 - c)z,$$

for all $z \in \mathbb{C}^-$. Therefore $\rho_{1,c}$ is freely self-decomposable. When $p = 2$ we have

$$R_{\rho_{2,c}}(z) = R_{\mu(2,2)}(z) - R_{D_c(\mu(2,2))}(z) = 2(1 - c)z + (1 - c^2)z^2,$$

for all $z \in \mathbb{C}^-$. Therefore $\rho_{2,c}$ is also freely self-decomposable.

Now assume that $p \in (1, 2)$. By (12), we have that the free Lévy measure $\nu_{\rho_{p,c}}$ of $\rho_{p,c}$ is given by

$$-\frac{\sin(p\pi)}{\pi|x|}\left\{\left(\frac{1+x}{-x}\right)^p \mathbf{1}_{(-1,0)}(x) - \left(\frac{c+x}{-x}\right)^p \mathbf{1}_{(-c,0)}\right\}dx.$$

Define

$$k_{p,c}(x) := -\frac{\sin(p\pi)}{\pi}\left\{\left(\frac{1+x}{-x}\right)^p \mathbf{1}_{(-1,0)}(x) - \left(\frac{c+x}{-x}\right)^p \mathbf{1}_{(-c,0)}\right\},$$

that is, $\nu_{\rho_{p,c}}(dx) = \frac{k_{p,c}(x)}{|x|}dx$. It is enough to show that $k_{p,c}(x)$ is non-decreasing on $(-1, 0)$. Put

$$u_{p,c}(x) := -\frac{\pi}{\sin(p\pi)}k_{p,c}(x).$$

Then

$$u'_{p,c}(x) = \frac{p(1+x)^{p-1}}{(-x)^{p+1}} > 0$$

for $-1 < x \le -c$ and

$$u'_{p,c}(x) = \frac{p\{(1+x)^{p-1} - c(c+x)^{p-1}\}}{(-x)^{p+1}} > 0$$

for $-c < x < 0$. Hence $u_{p,c}(x)$ is non-decreasing on $(-1, 0)$, and therefore so is $k_{p,c}(x)$. Thus $\rho_{p,c}$ is freely self-decomposable for $p \in (1, 2)$ too by [4] or Sect. 2.2.1. □

3.5 Unimodality for $\mu(p, p)$

Unimodality is a remarkable property of classical and freely self-decomposable distributions, namely, every classical or freely self-decomposable distribution is unimodal (see [14,27]). Since $\mu(p, p)$, $p > 0$, is absolutely continuous with respect to Lebesgue measure, we consider only unimodality of the density function $W_{p,p}(x)$. According to Proposition 1.1 we have

$$W_{p,p}(\rho(\varphi)) = \frac{(\csc((p-1)\varphi))\sin(\varphi)\sin(r\varphi)}{\pi}, \quad 0 < \varphi < \frac{\pi}{p}. \tag{14}$$

Proposition 3.10 *The Fuss–Catalan distribution $\mu(p, p)$ is unimodal for all $p \ge 1$.*

Proof If $1 \le p \le 2$ then $\mu(p, p)$ is freely self-decomposable, therefore, by [14, Theorem 1], it is unimodal. Now assume that $p > 2$. Denote by $g(\varphi)$ the right hand side of (14). Then

$$\frac{dg(\varphi)}{d\varphi} = \frac{\left(\sin^2(p\varphi) - p\sin^2(\varphi)\right)\csc^2(\varphi(1-p))}{\pi}, \quad 0 < \varphi < \frac{\pi}{p}.$$

Here we consider $h(\varphi) = \sin(p\varphi) - \sqrt{p}\sin(\varphi)$. $h'(\varphi) = p\cos(p\varphi) - \sqrt{p}\cos(\varphi)$ and $h''(\varphi) = -p^2\sin(p\varphi) + \sqrt{p}\sin(\varphi)$. $h'(0) = p - \sqrt{p} > 0$, $h'(\frac{\pi}{p}) = -p + \sqrt{p}\cos(\frac{\pi}{p}) < 0$ and $h''(\varphi) < 0$ on $0 < \varphi < \frac{\pi}{p}$ because $\sin(p\varphi) \ge \sin(\varphi)$ on $0 < \varphi < \frac{\pi}{p}$. So this means that $\sin(p\varphi) = \sqrt{p}\sin(\varphi)$ has only one solution in $0 < \varphi < \frac{\pi}{p}$ for $p > 2$ and so does $\frac{dg(\varphi)}{d\varphi}$. As a result we obtain that $\mu(p, p)$ is unimodal. □

4 General Fuss–Catalan Distributions $\mu(p, r)$

In this section, we discuss the general Fuss–Catalan distribution $\mu(p, r)$ and give a proof of Theorem 1.3. In Sect. 4.1, we study free infinite divisibility for $\mu(p, r)$. In Sect. 4.2, we obtain a Lévy–Khintchine representation of $\mu(p, r)$ for $r < p$. In Sect. 4.3, we investigate free self-decomposability for $\mu(p, r)$ via its Lévy measure. In Sect. 4.4, we discuss free regularity of $\mu(p, r)$. In Sect. 4.5, we study unimodality for four special families of Fuss–Catalan distributions.

4.1 Free Infinite Divisibility for $\mu(p, r)$

The characterization of those $\mu(p,r)$ which are freely infinitely divisible in [18, Corollary 7.1], was not quite correct and overlooked the distributions $\mu(p,p)$, $1 \le p \le 2$. Here we provide complete description and proof.

Theorem 4.1 *Suppose that $p \ge 1$, $0 < r \le p$. The Fuss–Catalan distribution $\mu(p,r)$ is freely infinitely divisible if and only if either $0 < r \le \min\{p/2, p-1\}$ or $1 \le p = r \le 2$.*

Proof From Corollary 5.1 in [17] and Theorem 3.2 we know that if either $0 < r \le \min\{p/2, p-1\}$ or $1 \le p = r \le 2$ then $\mu(p,r)$ is freely infinitely divisible.

On the other hand, by Theorem 3.2, $\mu(p,p)$ is not freely infinitely divisible for $p > 2$. Similarly, for $\mu(p, p-1)$ the free cumulants are $A_n(1, p-1) = \binom{n-2+p}{n}$. Since

$$A_2(1, p-1)A_4(1, p-1) - A_3(1, p-1)^2 = \frac{1}{144}p^2(p-1)^2(p+1)(2-p),$$

$\mu(p, p-1)$ is not freely infinitely divisible for $p > 2$.

Now assume that $p - 1 < r < p$. We are going to show that there exists even $n \ge 2$ such that $A_n(p-r, r) \le 0$. We have

$$\begin{aligned} r_n(p,r) = A_n(p-r,r) &= \frac{r}{n(p-r)+r}\binom{n(p-r)+r}{n} \\ &= \frac{r}{n!}\prod_{i=1}^{n-1}(n(p-r)+r-i) = \frac{r}{n!}\prod_{i=1}^{n-1}(n(p-r-1)+r+i). \end{aligned}$$

Putting $u := r + 1 - p$ we have $0 < u < 1$ and $A_n(p-r,r) \le 0$ if and only if $nu - r > 1$ and the floor $\lfloor nu - r \rfloor$ is odd. Consider the set

$$G := \{(nu - r) \mod 2 : n \in \mathbb{N},\ n \text{ even}, nu > r + 1\} \subseteq [0, 2).$$

If u is rational then G is a coset of a finite subgroup of the group $\big([0,2), 0, +_{\text{mod } 2}\big)$, otherwise G is a dense subset of $[0,2)$. It is easy to see in the former case that for some even n we have $(nu - r) \bmod 2 \in [1, 2)$, which implies $A_n(p-r,r) \le 0$. In the latter we can find n such that n is even, $1 < (nu - r) \bmod 2 < 2$ and then $A_n(p-r,r) < 0$.

Finally, assume that $p > 2$ and $p - 1 < r < p/2$. Then $p - r > 1$ and the free cumulants $A_n(p-r,r)$ admit integral representation (3), with $p - r$ instead of p. Since $r > p - r$, $W_{p-r,r}(x)$ is negative on some interval, and so is $x^2 W_{p-r,r}(x)$. This implies, that the sequence $A_{n+2}(p-r,r)$ is not positive definite (c.f. Lemma 7.1 in [18]). □

4.2 Free Cumulant Transform of $\mu(p, r)$

In this section, we give the free Lévy–Khintchine representation of $\mu(p,r)$ to discuss free self-decomposability and free regularity for $\mu(p,r)$.

Proposition 4.2 *If $0 < r \le \min\{p/2, p-1\}$ then the free cumulant transform $R_{\mu(p,r)}$ has an analytic continuation to $\mathbb{C}^-$ and we have*

$$R_{\mu(p,r)}(z) = \int_0^{(p-r)^{p-r}(p-r-1)^{1-(p-r)}} \left(\frac{1}{1-zx} - 1\right) W_{p-r,r}(x)dx \qquad (15)$$

for all $z \in \mathbb{C}^-$, where $W_{p-r,r}(t)$ is the density function of $\mu(p-r,r)$ with respect to Lebesgue measure on $\mathbb{R}$.

Proof Suppose that $p \geq 1$ and $0 < r < p$. Then the function $W_{p-r,r}(t)$ is the probability density function of the probability measure $\mu(p-r,r)$. Since $W_{p-r,r}(t)$ has a support concentrated on $[0, (p-r)^{p-r}(p-r-1)^{1-(p-r)}]$, we can take $\epsilon > 0$ such that $|z| < \epsilon$ and $|zt| < 1$ for all t in the support of $W_{p-r,r}(t)$. If it is necessary, then we replace $\epsilon > 0$ such that the free cumulant transform $R_{\mu(p,r)}(z)$ extends the power series at $z = 0$. Then for $|z| < \epsilon$, the free cumulant transform $R_{\mu(p,r)}$ is written by

$$\begin{aligned}R_{\mu(p,r)}(z) &= \sum_{n=1}^{\infty} A_n(p-r,r)z^n = \sum_{n=1}^{\infty}\left(\int_0^{(p-r)^{p-r}(p-r-1)^{1-(p-r)}} x^n W_{p-r,r}(x)dx\right)z^n \\ &= \int_0^{(p-r)^{p-r}(p-r-1)^{1-(p-r)}} \sum_{n=1}^{\infty}(zx)^n W_{p-r,r}(x)dx \\ &= \int_0^{(p-r)^{p-r}(p-r-1)^{1-(p-r)}} \frac{zx}{1-zx} W_{p-r,r}(x)dx \\ &= \int_0^{(p-r)^{p-r}(p-r-1)^{1-(p-r)}} \left(\frac{1}{1-zx}-1\right) W_{p-r,r}(x)dx.\end{aligned}$$

In particular, if $0 < r \leq \min\{p/2, p-1\}$, then $\mu(p,r)$ is freely infinitely divisible by Theorem 4.1. Hence the free cumulant transform $R_{\mu(p,r)}$ has an analytic continuation to $\mathbb{C}^-$ and therefore the formula (15) holds for all $z \in \mathbb{C}^-$ by using the identity theorem of complex analytic functions. □

From the above proposition, the free Lévy measure $\nu_{\mu(p,r)}$ of $\mu(p,r)$ is given by

$$\nu_{\mu(p,r)}(dx) = \frac{k_{p,r}(x)}{x}dx, \qquad k_{p,r}(x) := xW_{p-r,r}(x), \tag{16}$$

for all $p \geq 1$ and $0 < r \leq \min\{p/2, p-1\}$.

4.3 Free Self-decomposability for $\mu(p, r)$

In order to check free self-decomposability of $\mu(p,r)$ we should check whether or not $xW_{p-r,r}(x)dx$ is unimodal with mode 0.

Theorem 4.3 *Suppose that $p \geq 1$, $0 < r \leq p$. The Fuss–Catalan distribution $\mu(p,r)$ is freely self-decomposable if and only if $1 \leq p = r \leq 2$.*

Proof In view of Theorem 3.4 and Theorem 4.1 it suffices to check the case $0 < r \leq \min\{p/2, p-1\}$. By [10, Corollary 2.5], we have

$$k_{p,r}(x) \sim \frac{1}{\pi}\sin\left(\frac{r\pi}{p-r}\right)x^{\frac{r}{p-r}},$$

as $x \to 0^+$, where $k_{p,r}$ was defined in (16). Since $0 < r \leq \min\{p/2, p-1\}$, we have $p-r \geq r$. Hence $k'_{p,r}(x) \geq 0$ for $x \in (0,\epsilon)$, where $\epsilon > 0$ is sufficiently small. This implies that $k_{p,r}(x)dx$ can not be unimodal with mode 0. From remarks in Sect. 2.2.1, we conclude that $\mu(p,r)$ is not freely self-decomposable for $0 < r \leq \min\{p/2, p-1\}$. □

4.4 Free Regularity for $\mu(p, r)$

In this section, we investigate free regularity for $\mu(p, r)$. We should check the free Lévy measure of $\mu(p, r)$.

Theorem 4.4 *Suppose that $p \geq 1$, $0 < r \leq p$. The Fuss–Catalan distribution $\mu(p, r)$ is free regular if and only if either $0 < r \leq \min\{p/2, p-1\}$ or $p = r = 1$.*

Proof We may assume that $p \geq 1, 0 < r \leq p$ and either $0 < r \leq \min\{p/2, p-1\}$ or $p = r$ and $1 \leq p \leq 2$ holds since $\mu(p, r)$ has to be freely infinitely divisible. We have already proved that $\mu(p, r)$ is not free regular when $p = r$ and $1 < p \leq 2$ and is free regular when $p = r = 1$. Suppose that $0 < r \leq \min\{p/2, p-1\}$ holds. Note that $\nu_{\mu(p,r)}(dx) = W_{p-r,r}(x)dx$. Since the support of $W_{p-r,r}$ is concentrated on $[0, (p-r)^{p-r}(p-r-1)^{1-(p-r)}]$ and $\nu_{\mu(p,r)}(dx)$ is Lebesgue absolute continuous, we have $\nu_{\mu(p,r)}((-\infty, 0]) = 0$. By Proposition 4.2 and Theorem 2.4, the distribution $\mu(p, r)$ is free regular. Therefore $\mu(p, r)$ is free regular if and only if either $0 < r \leq \min\{p/2, p-1\}$ or $p = r = 1$. □

4.5 Unimodality for $\mu(p, r)$: 4 Cases

The Fuss-Catalan distributions $\mu(p, r)$ are absolutely continuous for $p > 1$, $0 < r \leq p$, therefore we have to verify unimodality of the density function $W_{p,r}(x)$. Equivalently, it is sufficient to check whether the right hand side of (2) is an unimodal function for $0 < \phi < \pi/p$. This turns out to be quite difficult in full generality. We know already from Proposition 3.10 that all $\mu(p, p)$, $p \geq 1$, are unimodal. Here we will study some further special cases.

4.5.1 Case I: $\mu(p, p-1)$

The Fuss–Catalan distributions $\mu(p, p-1)$, $p > 1$, are not freely self-decomposable, but we will show that they are unimodal.

Proposition 4.5 *The Fuss–Catalan distribution $\mu(p, p-1)$ is unimodal for all $p \geq 1$.*

Proof If $p = 1$, then $\mu(p, p-1) = \delta_0$, and therefore it is unimodal. Assume that $p > 1$. Then the density function is

$$W_{p,p-1}(\rho(\varphi)) = \frac{1}{\pi(\cot((p-1)\varphi) + \cot\varphi)}, \qquad 0 < \varphi < \frac{\pi}{p},$$

with $\rho(\varphi)$ given by (1). We consider a function $g_p(\varphi)$ defined by $g_p(\varphi) := \pi W_{p,p-1}(\rho(\varphi))$ for $0 < \varphi < \frac{\pi}{p}$. Then

$$g_p'(\varphi) = \frac{(p-1)\sin^2\varphi + \sin^2((p-1)\varphi)}{(\cot((p-1)\varphi) + \cot\varphi)^2 \sin^2((p-1)\varphi)\sin^2\varphi}.$$

If $p > 1$, then we have that $g_p'(\varphi) > 0$ for all $0 < \varphi < \frac{\pi}{p}$. Hence $g_p(\varphi)$ is non-decreasing on $(0, \frac{\pi}{p})$ and therefore $\mu(p, p-1)$ is unimodal with mode $\rho(\pi/p) = p^p(p-1)^{1-p}$. □

4.5.2 Case II: $\mu(2r, r)$

We show that $\mu(2r, r)$ is unimodal for all $r \geq 1$ in this section. Note that $\mu(2r, r)$ does not have a singular part with respect to Lebesgue measure for $r \geq 1$. Therefore we consider only probability density of $\mu(2r, r)$ to study unimodality. The probability density function of $\mu(2r, r)$ is given by

$$W_{2r,r}(\rho(\varphi)) = \frac{\sin^{r-1}((2r-1)\varphi)\sin\varphi\sin(r\varphi)}{\pi\sin^{r}(2r\varphi)}.$$

Proposition 4.6 *The Fuss–Catalan distribution $\mu(2r, r)$ is unimodal for all $r \geq 1$.*

Proof If $r = 1$, then $\mu(2r, r) = \Pi_{1,1}$ is unimodal. Assume that $r > 1$. Let $g_r(\varphi) := \pi W_{2r,r}(\rho(\varphi))$. We have

$$g_r'(\varphi) = \frac{G_r(\varphi)}{\sin^{r+1}(2r\varphi)},$$

where

$$\begin{aligned} G_r(\varphi) := \sin^{r-2}((2r-1)\varphi)\sin(2r\varphi)\Big[(2r^2-3r+1)\sin\varphi\sin(r\varphi)\cos((2r-1)\varphi) \\ + \sin((2r-1)\varphi)(r\sin\varphi\cos(r\varphi)+\cos\varphi\sin(r\varphi))\Big] \\ - 2r\cos(2r\varphi)\sin^{r-1}((2r-1)\varphi)\sin\varphi\sin(r\varphi) \end{aligned}$$

Since $2r^2 - 3r + 1 > 0$ and $r\varphi, (2r-1)\varphi \in (0, \pi/2)$ for all $\varphi \in (0, \frac{\pi}{2r}) \subset (0, \pi/2)$ and $r > 1$, we have

$$(2r^2-3r+1)\sin^{r-2}((2r-1)\varphi)\sin(2r\varphi)\sin\varphi\sin(r\varphi)\cos((2r-1)\varphi) \geq 0.$$

Therefore we have

$$\begin{aligned} G_r(\varphi) &\geq \sin^{r-1}((2r-1)\varphi)\sin(2r\varphi)(r\sin\varphi\cos(r\varphi)+\cos\varphi\sin(r\varphi)) \\ &\quad - 2r\cos(2r\varphi)\sin^{r-1}((2r-1)\varphi)\sin\varphi\sin(r\varphi) \\ &\geq r\sin^{r-1}((2r-1)\varphi)\sin(2r\varphi)\sin\varphi\cos(r\varphi) \\ &\quad - 2r\cos(2r\varphi)\sin^{r-1}((2r-1)\varphi)\sin\varphi\sin(r\varphi) \\ &= r\sin\varphi\sin^{r-1}((2r-1)\varphi)\Big[\sin(2r\varphi)\cos(r\varphi) - 2\cos(2r\varphi)\sin(r\varphi)\Big] \\ &= 2r\sin\varphi\sin^{r-1}((2r-1)\varphi)\sin^3(r\varphi) \\ &\geq 0, \end{aligned}$$

for all $\varphi \in (0, \frac{\pi}{2r})$. Hence $g_r(\varphi)$ is non-decreasing on $(0, \frac{\pi}{2r})$, and therefore $\mu(2r, r)$ is unimodal for all $r > 1$. □

4.5.3 Case III: $\mu(1, r)$

The density function of $\mu(1, r)$ can not be written by (2), but we have obtained a density formula of $\mu(1, r)$ in [17].

Proposition 4.7 *Suppose that $0 < r < 1$. We have*

$$W_{1,r}(x) = \frac{\sin(r\pi)}{\pi}x^{r-1}(1-x)^{-r}\mathbf{1}_{(0,1)}(x). \tag{17}$$

Furthermore $\mu(1, r)$ is not unimodal for $0 < r < 1$.

Proof Equation (17) was proved in [17, formula (5.2)]. Now it is elementary to check that for $0 < r < 1$ the function $W_{1,r}(z)$ is decreasing on $x \in (0, 1-r)$ and increasing for $x \in (1-r, 1)$, hence is not unimodal. □

Since $W_{1,p-1}(x)dx$ is the free Lévy measure of $\mu(p, p-1)$, $(1 < p < 2)$ and it is written by (17), we get an explicit free Lévy–Khintchine representation of $\mu(p, p-1)$. From the form of free cumulants of $\mu(p, p-1)$, its free cumulant transform $R_{\mu(p,p-1)}$ is written by

$$R_{\mu(p,p-1)}(z) = \sum_{n=1}^{\infty} A_n(1, p-1)z^n = \frac{1}{(1-z)^{p-1}} - 1.$$

Similarly as in Proposition 3.6, we get the following formula.

Corollary 4.8 *For $1 < p < 2$, we have*

$$\begin{aligned} R_{\mu(p,p-1)}(z) &= (p-1)z + \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 - zx\mathbf{1}_{[-1,1]}(x) \right) \\ &\quad \times \left(-\frac{\sin(p\pi)}{\pi} \right) x^{p-2}(1-x)^{1-p}\mathbf{1}_{(0,1)}(x)dx \qquad (18) \\ &= \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 \right) \times \left(-\frac{\sin(p\pi)}{\pi} \right) x^{p-2}(1-x)^{1-p}\mathbf{1}_{(0,1)}(x)dx, \end{aligned}$$

for all $z \in \mathbb{C}^-$.

Proof For all z in a neighborhood of 0, by applying the integral representation (10) and Lemma 3.5 again, we have

$$\begin{aligned} &\int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 - zx\mathbf{1}_{[-1,1]}(x) \right) \times \left(-\frac{\sin(p\pi)}{\pi} \right) x^{p-2}(1-x)^{1-p}\mathbf{1}_{(0,1)}(x)dx \\ &= z^2 \left(-\frac{\sin(p\pi)}{\pi} \right) \int_0^1 \frac{x^p(1-x)^{1-p}}{1-zx} dx \\ &= z^2 \left(-\frac{\sin(p\pi)}{\pi} \right) B(p+1, 2-p)\, {}_2F_1(1, p+1, 3; z) \\ &= z^2 \left(-\frac{\sin(p\pi)}{\pi} \right) \left(-\frac{\pi(p-1)p}{2\sin(p\pi)} \right) \left(-\frac{2(pz - z + 1 - (1-z)^{1-p})}{(p-1)pz^2} \right) \\ &= -pz + z - 1 + \frac{1}{(1-z)^{p-1}} \\ &= (1-p)z + R_{\mu(p,p-1)}(z). \end{aligned}$$

Therefore the free cumulant transform of $\mu(p, p-1)$ has the representation (18) on the neighborhood of 0 for $1 < p < 2$. Since $\mu(p, p-1)$ is freely infinitely divisible for all $1 < p < 2$, its free cumulant transform has an analytic continuation to $\mathbb{C}^-$ and therefore the formula of (18) hold for all $z \in \mathbb{C}^-$ by using the identity theorem of complex analytic functions. □

4.5.4 Case IV: $\mu(2, r)$

For $p = 2$ there is a more simple formula available for the density function $W_{2,r}(x)$, namely

$$W_{2,r}(x) = \frac{\sin\left(r \cdot \arccos\sqrt{x/4}\right)}{\pi x^{1-r/2}}, \qquad (19)$$

$x \in (0, 4)$, see formula (33) in [19]. In particular,

$$W_{2,1/2}(x) = \frac{\sqrt{2-\sqrt{x}}}{2\pi x^{3/4}}, \qquad W_{2,1}(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \tag{20}$$

$$W_{2,3/2}(x) = \frac{(1-\sqrt{x})\sqrt{2-\sqrt{x}}}{2\pi x^{1/4}}, \qquad W_{2,2}(x) = \frac{1}{2\pi}\sqrt{x(4-x)}. \tag{21}$$

From (20), (21) it is easy to check that $\mu(2, 1/2)$, $\mu(2, 1)$, $\mu(2, 3/2)$ and $\mu(2, 2)$ are unimodal. Now we will prove the following

Proposition 4.9 *There exists $r_1 \in (\frac{3}{2}, 2)$ such that the Fuss–Catalan distribution $\mu(2, r)$ is unimodal for all $0 < r < r_1$. The constant r_1 is the unique solution in $\frac{3}{2} < r < 2$ of the equation*

$$r \sin\left(\frac{3r+3}{2r+4}\pi\right) + 2\sin\left(\frac{3r}{2r+4}\pi\right)\cos\left(\frac{3}{2r+4}\pi\right) = 0. \tag{22}$$

One can find numerically that $r_1 = 1.6756\ldots$.

Proof First we will prove that if $0 < r < 1$ then $\mu(2, r)$ is unimodal. Substitute $x \mapsto 4t^2$ in (19). Then

$$w_r(t) := W_{2,r}(4t^2) = \frac{\sin(r \cdot \arccos t)}{\pi (2t)^{2-r}},$$

$t \in (0, 1)$, and

$$w_r'(t) = -\frac{rt\cos(r \cdot \arccos t) + (2-r)\sqrt{1-t^2}\sin(r \cdot \arccos t)}{\pi 2^{2-r} t^{3-r}\sqrt{1-t^2}}.$$

If $0 < r < 1$, $0 < t < 1$ then $0 < r \cdot \arccos t < r\pi/2 < \pi/2$, so both the summands in the numerator are positive. Consequently, $w_r(t)$ is decreasing on $t \in (0, 1)$.

Next, we show the existance and uniqueless of a solution $r_1 \in (\frac{3}{2}, 2)$ of the Eq. (22). Let $A(r)$ be the function of LHS of the Eq. (22). It is easy to check that $A(r) > 0$ for all $r \in (1, \frac{3}{2})$ and $A(2) < 0$. Moreover $A'(r) < 0$ for all $\frac{3}{2} < r < 2$. Hence there exists a unique solution $r_1 \in (\frac{3}{2}, 2)$ such that $A(r_1) = 0$ by the intermediate value theorem.

Assume $1 < r < r_1$. We consider a function $g_r(\varphi)$ defined by $g_r(\varphi) := \pi \cdot 2^{2-r} W_{2,r}(\rho(\varphi))$ for $0 < \varphi < \frac{\pi}{2}$. Then

$$g_r'(\varphi) = \frac{r\cos((r+1)\varphi) + 2\sin(r\varphi)\sin\varphi}{(\cos\varphi)^{3-r}}$$

It is sufficient to check the positivity of the function $h_r(\varphi) := r\cos((r+1)\varphi) + 2\sin(r\varphi)\sin\varphi$ to see the positivity of $g_r'(\varphi)$ since $(\cos\varphi)^{3-r} > 0$ for all $0 < \varphi < \frac{\pi}{2}$. Since $\sin(r\varphi)\sin(\varphi) > 0$ for all $\varphi \in (0, \frac{\pi}{2})$, we have that $h_r(\varphi) > 0$ for all $\varphi \in (0, \frac{\pi}{2(r+1)})$. Next we show that $h_r'(\varphi) < 0$ for all $\varphi \in (\frac{\pi}{2(r+1)}, \frac{3\pi}{2(r+2)})$. To show this, we divide two cases of the region of φ.

Case I $\frac{\pi}{2(r+1)} < \varphi \leq \frac{\pi}{r+1}$: It is clear that

$$\begin{aligned} h_r'(\varphi) &= -r(r+1)\sin((r+1)\varphi) + 2r\cos(r\varphi)\sin\varphi + 2\sin(r\varphi)\cos\varphi \\ &= -(r-1)(r\sin((r+1)\varphi) + 2\sin(r\varphi)\cos\varphi) \\ &< 0. \end{aligned}$$

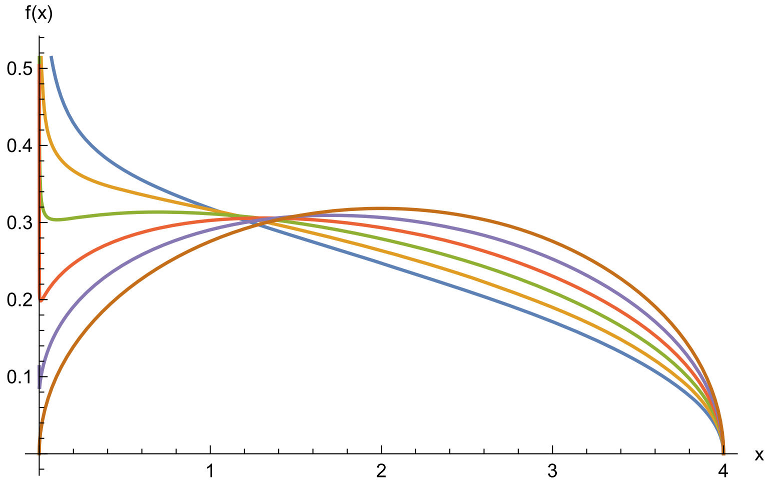

Fig. 1 Density function $W_{2,r}$ for $r = 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2$

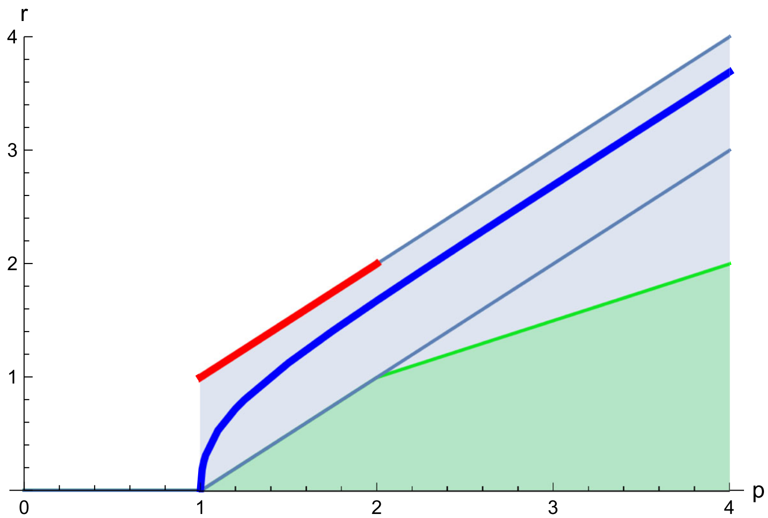

Fig. 2 Phase transition

Case II $\frac{\pi}{r+1} < \varphi < \frac{3\pi}{2(r+2)}$: Since $A(r) > 0$ for all $1 < r < r_1$, we have

$$\begin{aligned} h_r'(\varphi) &< -(r-1)\left[r\sin\left(\frac{3r+3}{2r+4}\pi\right) + 2\sin\left(\frac{3r}{2r+4}\pi\right)\cos\left(\frac{3}{2r+4}\pi\right)\right] \\ &= -(r-1)A(r) \\ &< 0. \end{aligned}$$

Due to the above evaluation, we obtain that $h_r'(\varphi) < 0$ for all $\frac{\pi}{2(r+1)} < \varphi < \frac{3\pi}{2(r+2)}$. In addition, for all $\frac{\pi}{r+1} < \varphi < \frac{\pi}{2}$, we have

$$h_r''(\varphi) = -(r-1)\left[r(r+1)\cos((r+1)\varphi) + 2r\cos(r\varphi)\cos\varphi - 2\sin(r\varphi)\sin\varphi\right] > 0.$$

Moreover $h_r'(\frac{\pi}{2}) = -(r-1)r\sin(\frac{(r+1)\pi}{2}) > 0$ for all $1 < r < r_1$. Hence there exists a unique solution $\varphi_0 \in (\frac{3\pi}{2(r+2)}, \frac{\pi}{2})$ such that $h_r'(\varphi_0) = 0$ by the intermediate value theorem. More strongly, we have that $h_r'(\varphi) < 0$ for all $\varphi \in (\frac{3\pi}{2(r+2)}, \varphi_0)$ and $h_r'(\varphi) > 0$ for all $\varphi \in (\varphi_0, \frac{\pi}{2})$. The equality $h_r'(\varphi_0) = 0$ implies that

$$\begin{aligned}(\cos\varphi_0)h_r(\varphi_0) &= r\cos((r+1)\varphi_0)\cos\varphi_0 + 2\sin(r\varphi_0)\cos\varphi_0\sin\varphi_0 \\ &= r\cos((r+1)\varphi_0)\cos\varphi_0 - r\sin((r+1)\varphi_0)\sin\varphi_0 \\ &= r\cos((r+2)\varphi_0) > 0,\end{aligned}$$

since $\frac{3\pi}{2} < (r+2)\varphi_0 < \frac{(r+2)\pi}{2} < 2\pi$. Thus $h_r(\varphi_0) > 0$, and therefore $h_r(\varphi) > 0$ for all $\varphi \in (\frac{\pi}{2(r+1)}, \frac{\pi}{2})$. Hence $g_r(\varphi)$ is non-decreasing on $(0, \frac{\pi}{2})$. This means that $\mu(2, r)$ is unimodal for all $1 < r < r_1$. □

4.6 Phase Transition

The results of Sects. 4.5.3 and 4.5.4, as well as some numerical experiments, suggest, that for every $p > 1$ the Fuss–Catalan distributions admit the following phase transition:

Conjecture 4.10 *For every $p > 1$ there exists $r_0(p)$, with $p - 1 < r_0(p) < p$, such that the Fuss–Catalan distribution $\mu(p, r)$ is unimodal if and only if either $r = p$ or $0 < r \le r_0(p)$.*

As an example we present on Fig. 1 graphs of $W_{2,r}(x)$ for $r = 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2$, the left parts of the graphs appear in this order from the top to the bottom. Note that if $0 < r < 2$ then $\lim_{x\to 0^+} W_{2,r}(x) = +\infty$ and $\lim_{x\to 0^+} W_{2,2}(x) = 0$. We see that $W_{2,1.5}(x)$, $W_{2,1.6}(x)$ are unimodal, $W_{2,1.7}(x)$, $W_{2,1.8}(x)$, $W_{2,1.9}(x)$ are not unimodal and $W_{2,2}(x)$ is again unimodal. We have found numerically that for $p = 2$ the phase transition is at $r_0(2) = 1.6756\ldots$, which suggests that $r_0(2)$ coincides with r_1 from Proposition 4.9.

On Fig. 2 we represent the Fuss–Catalan distributions $\mu(p, r)$, $p \ge 1$, $0 < r \le p$, on the (p, r)-plane. The middle thick blue line represents $r_0(p)$ which was found experimentally. Top thick red line segment corresponds to the freely self-decomposable distributions $\mu(p, r)$ and green area corresponds free regular infinitely divisible distributions $\mu(p, r)$. The union of the red and green areas corresponds to the freely infinitely divisible Fuss–Catalan distributions.

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