



Interface Fluctuations in Non Equilibrium Stationary States: The SOS Approximation

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Abstract

We study the $2d$ stationary fluctuations of the interface in the SOS approximation of the non equilibrium stationary state found in De Masi et al. (J Stat Phys 175:203–221, 2019). We prove that the interface fluctuations are of order $N^{1/4}$, N the size of the system. We also prove that the scaling limit is a stationary Ornstein–Uhlenbeck process.

Keywords Non equilibrium stationary states · Interfaces · SOS model

1 Introduction

The non equilibrium stationary states (NESS) for diffusive systems in contact with reservoirs have been extensively studied, one of the main targets being to understand how the presence of a current affects what seen in thermal equilibrium. In particular it has been shown that fluctuations in NESS have a non local structure as opposite to what happens in thermal equilibrium. The theory of such phenomena is well developed, [1,5] but mathematical proofs are restricted to very special systems (SEP, [6], KMP, [8], chain of oscillators, [2] . . .).

The general structure of the NESS in the presence of phase transitions is a very difficult and open problem not only mathematically, also a theoretical understanding is lacking. However a breakthrough came recently from a paper by De Masi et al. [4], where they prove that the

Dedicated to Joel for his important contributions to the theory of phase transition and interfaces.

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NESS can be computed explicitly for a quite general class of Ginzburg–Landau stochastic models which include phase transitions.

The main point in [4] is that the NESS is still a Gibbs state but with the original Hamiltonian modified by adding a slowly varying chemical potential. Thus for boundary driven Ginzburg–Landau stochastic models the analysis of the NESS is reduced to an equilibrium Gibbsian problem and, at least in principle, very fine properties of their structure can be investigated which is unthinkable for general models.

In particular we can study cases where there are phase transitions and purpose of this paper is to give an indication that the $2d$ NESS interface is much more rigid than in thermal equilibrium.

The analysis in [4] includes a system where the Ising model is coupled to a Ginzburg–Landau process. In the corresponding NESS the distribution of the Ising spin is a Gibbs measure with the usual nearest neighbour ferromagnetic interaction plus a slowly varying external magnetic field.

In particular in the $2d$ square $\Lambda_N := [0, N] \times [-N, N] \cap \mathbb{Z}^2$ the NESS $\mu_N(\sigma)$ is

$$\mu_N(\sigma) = \frac{1}{Z_N} e^{-\beta H_N(\sigma)}, \quad \sigma = (\sigma(x) \in \{-1, 1\}, x \in \Lambda_N)$$

$$H_N(\sigma) = H^{\text{ising}}(\sigma) + \sum_{x \in \Lambda_N} \frac{bx \cdot e_2}{N} \sigma(x), \quad H^{\text{ising}}(\sigma) = \sum_{\substack{x, y \in \Lambda_N \\ |x-y|=1}} \mathbf{1}_{\sigma(x) \neq \sigma(y)} \quad e_2 = (0, 1)$$

where $b > 0$ is fixed by the chemical potentials at the boundaries.

We assume $\beta > \beta_c$, thus since the slowly varying external magnetic field $\frac{bx \cdot e_2}{N}$ is positive in the half upper plane and negative in the half lower plane, we expect the existence of an interface, namely a connected “open line” λ in the dual lattice which goes from left to right and which separates the region with the majority of spins equal to 1 to the one with the majority of spins equal to -1 .

The problem of the microscopic location of the interface has been much studied in equilibrium without external magnetic field and when the interface is determined by the boundary conditions: $+$ boundary conditions on $\Lambda_N^c \cap \{x \cdot e_2 \geq 0\}$ and $-$ boundary conditions on $\Lambda_N^c \cap \{x \cdot e_2 < 0\}$.

It is well known since the work initiated by Gallavotti [7], that in the $2d$ Ising model at thermal equilibrium the interface fluctuates by the order of \sqrt{N} , N the size of the system.

In this paper we argue that at low temperature (much below the critical value) and in the presence of a stationary current produced by reservoirs at the boundaries the interface is much more rigid as it fluctuates only by the order $N^{1/4}$.

We study the problem with a drastic simplification by considering the SOS approximation of the interface. Namely we consider the simplest case where the interface λ is a graph, namely λ is described by a function $s_x, x \in \{0, \dots, N\}$ with integers values in \mathbb{Z} . The corresponding Ising configurations are spins equal to -1 below s_x and $+1$ above s_x . Namely $\sigma(x, i) = 1$ if $i \geq s_x$ and $\sigma(x, i) = -1$ if $i \leq s_x$.

The interface is then made by a sequence of horizontal and vertical segments and the Ising energy of such configurations is $|\lambda|$. We normalise the energy by subtracting the energy of the flat interface so that the normalised energy is

$$\sum_{x=1}^N |s_x - s_{x-1}| = |\lambda| - N$$

i.e. the sum of the lengths of the vertical segments.

The energy due to the external magnetic field is normalised by subtracting the energy of the configuration when all s_x are equal to 0. This is (below we set $b = 1$)

$$2 \sum_{x=0}^N \sum_{i=1}^{|s_x|} \frac{i}{N} \approx \sum_{x=0}^N \frac{s_x^2}{N}$$

Thus we get the SOS Hamiltonian

$$H_N(s) = \frac{1}{N} \sum_{x=0}^N s_x^2 + \sum_{x=1}^N |s_x - s_{x-1}| \tag{1.1}$$

We prove that the stationary fluctuations of the interface in this SOS approximation scaled by $N^{1/4}$ convergence to a stationary Ornstein–Uhlenbeck process.

The problem addressed in this article is the behavior of the interface in the NESS and the aim is to argue that its fluctuations are more rigid than in thermal equilibrium as indicated by the SOS approximation. Thus in the SOS approximation we prove the $N^{1/4}$ behavior in the simplest setting of Sect. 2.

More general results similar to those in [9] presumably apply. We cannot use directly the results in [9] because their SOS models have an additional constraint (the interface is in the upper half plane). Our proofs have several points in common with [9], but since we work in a more specific setup with less constraints, they are considerably simpler and somehow more intuitive.

2 Model and Results

We consider $\Lambda_N = \{0, \dots, N\} \times \mathbb{Z}$ and denote the configuration of the interface with $\mathbf{s} = \{s_x \in \mathbb{Z}, x = 0, \dots, N\}$. The interface increments are denoted by $\eta_x = s_x - s_{x-1} \in \mathbb{Z}, x = 1, \dots, N$.

Let π a symmetric probability distribution on \mathbb{Z} aperiodic and such that

$$\sum_{\eta \in \mathbb{Z}} e^{a\eta} \pi(\eta) < +\infty \quad \forall |a| \leq a_0, \text{ for some } a_0 > 0 \tag{2.1}$$

We denote σ^2 the variance of π and as we shall see the result does not depend on the particular choice of π but only on the variance σ^2 .

For $s, \bar{s} \in \mathbb{Z}$ define the positive kernel

$$T_N(s, \bar{s}) = e^{-\frac{s^2 + \bar{s}^2}{2N}} \pi(s - \bar{s}). \tag{2.2}$$

Call $T_N f(s)$ the integral operator with kernel T_N . T_N is a symmetric positive operator in $\ell_2(\mathbb{Z})$, and it can be checked immediately that it is Hilbert–Schmidt, consequently compact. Then the Krein–Rutman theorem [11] applies, thus there is a strictly positive eigenfunction $h_N \in \ell_2(\mathbb{Z})$ and a strictly positive eigenvalue $\lambda_N > 0$:

$$\sum_{s'} T_N(s, s') h_N(s') = \lambda_N h_N(s), \quad \sum_s h_N^2(s) = 1, \tag{2.3}$$

The eigenvalue $\lambda_N < 1$, and $\lambda_N \rightarrow 1$ as $N \rightarrow \infty$, see Theorem 3.1.

We then observe that the Gibbs distribution ν_N with the Hamiltonian given in (1.1) and with the values at the boundaries distributed according to the measure $h_N(s)e^{\frac{s^2}{2N}}$ can be expressed in terms of the kernel T_N and the double-geometric distribution

$$\pi(\eta) = \frac{1}{V} e^{-|\eta|} \quad V = \sum_{\eta} e^{-|\eta|}$$

In fact

$$\begin{aligned} \nu_N(\mathbf{s}) &= \frac{1}{Z_N} h(s_0) e^{\frac{s_0^2}{2N}} e^{-\frac{1}{N} \sum_{x=0}^N s_x^2} \prod_{x=1}^N \frac{e^{-|s_x - s_{x-1}|}}{V} h(s_N) e^{\frac{s_N^2}{2N}} \\ &= \frac{1}{Z_N} h_N(s_0) e^{-\frac{1}{2N} \sum_{x=1}^N (s_x^2 + s_{x-1}^2)} \prod_{x=1}^N \pi(\eta_x) h_N(s_N) \\ &= \frac{1}{Z_N} h_N(s_0) \prod_{x=1}^N T_N(s_{x-1}, s_x) h_N(s_N) \end{aligned} \tag{2.4}$$

with Z_N the partition function.

Call

$$p_N(s, s') := \frac{h_N(s')}{\lambda_N h_N(s)} T_N(s, s') \tag{2.5}$$

p_N defines an irreducible positive-recurrent Markov chain on \mathbb{Z} with reversible measure given by $h_N^2(s)$. We call P_N the law of the Markov chain starting from the invariant measure $h_N^2(s)$.

Observe that $\nu_N(\mathbf{s})$ in (2.4) is the P_N -probability of the trajectory \mathbf{s} , indeed from (2.5) we get

$$\nu_N(\mathbf{s}) = \frac{1}{Z_N} h_N(s_0) \prod_{x=1}^N T_N(s_{x-1}, s_x) h_N(s_N) = \frac{\lambda_N^N}{Z_N} h_N^2(s_0) \prod_{x=1}^N p_N(s_{x-1}, s_x) \tag{2.6}$$

which proves that $Z_N = \lambda_N^N$ and that $\nu(\mathbf{s}) = P_N(\mathbf{s})$.

We define the rescaled variables

$$\tilde{S}^N(t) = \frac{S_{[tN^{1/2}]}}{N^{1/4}}, \quad t = 0, 1, \dots, N^{1/2}, \quad [\cdot] = \text{integer part}$$

then $\tilde{S}^N(t)$ is extended to $t \in [0, 1]$ by linear interpolation, in this way we can consider the induced distribution \mathcal{P}_N on the space of continuous function $C([0, 1])$.

We denote by \mathcal{E}_N the expectation with respect to \mathcal{P}_N .

Our main result is the following Theorem.

Theorem 2.1 *The process $\{\tilde{S}^N(t), t \in [0, 1]\}$ converges in law to the stationary Ornstein–Uhlenbeck process with variance $\sigma/2$. Moreover $\lim_{N \rightarrow \infty} \lambda_N^{\sqrt{N}} = e^{-\sigma/2}$.*

The paper is organized as follows: in Sect. 3 we give a priori estimates on the eigenfunctions h_N and on the eigenvalues λ_N , in Sect. 4 we prove convergence of the eigenfunctions h_N and identify the limit, in Sect. 5 we prove Theorem 2.1.

3 Estimates on the Eigenfunctions and the Eigenvalues

Theorem 3.1 *The operator T_N defined in (2.2) has a maximal positive eigenvalue λ_N and a positive normalized eigenvector $h_N(s) \in \ell^2(\mathbb{Z})$ as in (2.3) with the following properties:*

- (i) h_N is a symmetric function.
- (ii) $\|h_N\|_\infty \leq 1$ for all N .
- (iii) There exists c so that $1 - \frac{c}{\sqrt{N}} \leq \lambda_N < 1$.

Proof That $h_N(s)$ is positive follows by the Krein–Rutman theorem, [11], also λ_N is not degenerate, its eigenspace is one-dimensional. The symmetry follows from the symmetry of T_N , since $h_N(-s)$ is also eigenfunction for λ_N .

The ℓ_∞ bound follows from

$$\|h_N\|_\infty^2 = \sup_s h_N(s)^2 \leq \sum_s h_N(s)^2 = 1. \tag{3.1}$$

The upper bound in (iii) easily follows from

$$\lambda_N \leq \sum_{s,\bar{s}} \pi(s - \bar{s}) h_N(s) h_N(\bar{s}) \leq \frac{1}{2} \sum_{s,\bar{s}} \pi(s - \bar{s}) (h_N(s)^2 + h_N(\bar{s})^2) \leq 1$$

having used that $\sum_s h_N(s)^2 = 1$.

To prove the lower bound in (iii) we use the variational formula

$$\lambda_N = \sup_h \frac{\sum_{s,s'} T_N(s, s') h(s) h(s')}{\sum_s h(s)^2} \tag{3.2}$$

By choosing h with $\sum_s h(s)^2 = 1$, and using the inequality $e^{-x} \geq 1 - x$, we have a lower bound

$$\lambda_N \geq \sum_{s,\bar{s}} \pi(s - \bar{s}) h(s) h(\bar{s}) - \frac{1}{N} \sum_{s,\bar{s}} s^2 \pi(s - \bar{s}) h(s) h(\bar{s}) \tag{3.3}$$

Observe that, since $\sum_s h(s)^2 = 1$,

$$\begin{aligned} \frac{1}{N} \sum_{s,\bar{s}} s^2 \pi(s - \bar{s}) h(s) h(\bar{s}) &\leq \frac{1}{2N} \sum_{s,\bar{s}} s^2 \pi(s - \bar{s}) (h(s)^2 + h(\bar{s})^2) \\ &\leq \frac{1}{2N} \sum_s s^2 h(s)^2 + \frac{1}{2N} \sum_{\eta,\bar{s}} (\bar{s} + \eta)^2 \pi(\eta) h(\bar{s})^2 \\ &= \frac{1}{N} \sum_s s^2 h(s)^2 + \frac{\sigma^2}{2N} \end{aligned}$$

Thus

$$\lambda_N \geq \sum_{s,\bar{s}} \pi(s - \bar{s}) h(s) h(\bar{s}) - \frac{1}{N} \sum_s s^2 h(s)^2 - \frac{\sigma^2}{2N} \tag{3.4}$$

For $\alpha > 0$, we choose $h(s) = h_\alpha(s) := C_\alpha e^{-\alpha s^2/4}$, with $C_\alpha = \left(\sum_s e^{-\alpha s^2/2}\right)^{-1/2}$. Observe that for $\alpha \rightarrow 0$

$$\left| \sqrt{\alpha} \sum_s e^{-\alpha s^2/2} - \int e^{-r^2/2} dr \right| \leq C\alpha \quad \left| \sqrt{\alpha} \sum_s (\alpha s^2 e^{-\alpha s^2/2} - \int r^2 e^{-r^2/2} dr) \right| \leq C\alpha$$

Thus

$$\sum_s s^2 h_\alpha(s)^2 = \alpha^{-1} + O(\alpha) \text{ as } \alpha \rightarrow 0. \tag{3.5}$$

We next prove that

$$\sum_{s,s'} \pi(s - s') h_\alpha(s) h_\alpha(s') \geq 1 - \frac{\alpha \sigma^2}{4} \tag{3.6}$$

To prove (3.6) observe that $h_\alpha(s) h_\alpha(s + \tau) = h_\alpha(s)^2 e^{-\alpha \tau^2/4 - \alpha s \tau/2}$, then

$$\begin{aligned} \sum_{s,s'} \pi(s - s') h_\alpha(s) h_\alpha(s') &= \sum_s h_\alpha(s) \sum_\tau \pi(\tau) h_\alpha(s + \tau) \\ &= \sum_s h_\alpha(s)^2 \sum_\tau \pi(\tau) e^{-\alpha \tau^2/4} e^{-\alpha s \tau/2} \end{aligned}$$

Using again that $e^{-z} \geq 1 - z$ and the parity of h_α and of π we get

$$\begin{aligned} &\sum_s h_\alpha(s)^2 \sum_\tau \pi(\tau) e^{-\alpha \tau^2/4} e^{-\alpha s \tau/2} \\ &\geq \sum_s h_\alpha(s)^2 \sum_\tau \pi(\tau) \left(1 - \frac{\alpha}{4} \tau^2\right) \left(1 - \frac{\alpha s \tau}{2}\right) = 1 - \frac{\alpha \sigma^2}{4} \end{aligned}$$

which proves (3.6).

We choose $\alpha = N^{-1/2}$ and from (3.4), (3.5) and (3.6) we then get

$$\lambda_N \geq 1 - \frac{1}{\sqrt{N}} \left(\frac{\sigma^2}{4} + 1\right) - \frac{\sigma^2}{2N} + O(N^{-3/2}), \tag{3.7}$$

which gives the lower bound. □

Given s let s_x be the position at x of the random walk starting at s , namely $s_x = s + \sum_{k=1}^x \eta_k$ where $\{\eta_k\}_k$ are i.i.d. random variables with distribution π . By an abuse of notation we will denote by π also the probability distribution of the trajectories of the corresponding random walk and by \mathbb{E}_s the expectation with respect to the law of the random walk which starts from s .

We will use the local central limit theorem as stated in Theorem (2.1.1) in [12] [see in particular formula (2.5)]. There exists a constant c not depending on n such that for any s :

$$\left| \pi \left(\sum_{k=1}^n \eta_k = s \right) - \bar{p} \left(\sum_{k=1}^n \eta_k = s \right) \right| \leq \frac{c}{n^{3/2}} \tag{3.8}$$

where

$$\bar{p} \left(\sum_{k=1}^n \eta_k = s \right) = \frac{1}{\sqrt{2\pi\sigma^2 n}} e^{-\frac{s^2}{2\sigma^2 n}}$$

By iterating (2.3) n times we get

$$h_N(s) = \frac{1}{\lambda_N^n} \mathbb{E}_s \left(e^{-\frac{1}{2N} \sum_{x=0}^n s_x^2} h_N(s_n) \right) \tag{3.9}$$

Theorem 3.2 *There exist positive constants c, C (independent of N) such that*

$$h_N(s) \leq \frac{C}{N^{1/8}} \exp \left\{ -\frac{cs^2}{N^{1/2}} \right\} \tag{3.10}$$

Proof Below we will write $h(s)$ for the eigenfunction $h_N(s)$, and λ for λ_N .

Because of the symmetry of h , it is enough to consider $s > 0$. From (3.9) we get

$$h(s) \leq \frac{1}{\lambda^n} \left[\mathbb{E}_s \left(e^{-\frac{2}{2N} \sum_{x=0}^n s_x^2} \right) \right]^{1/2} \left[\mathbb{E}_s (h^2(s_n)) \right]^{1/2} \tag{3.11}$$

To estimate $\mathbb{E}_s (h^2(s_n))$ we use (3.8),

$$\begin{aligned} \mathbb{E}_s (h^2(s_n)) &= \sum_{s_n} \pi \left(\sum_{k=0}^n \eta_k = s_n - s \right) h^2(s_n) \\ &\leq \sum_{s_n} \bar{p} \left(\sum_{k=0}^n \eta_k = s_n - s \right) h^2(s_n) + \frac{c}{n^{3/2}} \sum_{s_n} h^2(s_n) \\ &\leq \left[\frac{1}{\sqrt{2\pi n \sigma^2}} + \frac{c}{n^{3/2}} \right] \sum_{s_n} h^2(s_n) \\ &\leq \frac{K}{\sqrt{n}} \sum_{s'} h^2(s') = \frac{K}{\sqrt{n}} \end{aligned} \tag{3.12}$$

where K is a constant independent of N .

Thus for $n = \sqrt{N}$ we get

$$h(s) \leq \frac{1}{\lambda \sqrt{N}} \frac{\sqrt{K}}{N^{1/8}} \left[\mathbb{E}_s \left(e^{-\frac{1}{N} \sum_{x=0}^n s_x^2} \right) \right]^{1/2} \tag{3.13}$$

For $\alpha \in (0, 1)$ we consider

$$z = \inf \{x : s_x \leq s(1 - \alpha)\} \tag{3.14}$$

and we split the expectation on the right hand side of (3.13)

$$\begin{aligned} \mathbb{E}_s \left(e^{-\frac{1}{N} \sum_{x=0}^n s_x^2} \right) &\leq \mathbb{E}_s \left(e^{-\frac{1}{N} \sum_{x=0}^{z-1} s_x^2} 1_{[z \leq n]} \right) + \mathbb{E}_s \left(e^{-\frac{1}{N} \sum_{x=0}^n s_x^2} 1_{[z > n]} \right) \\ &\leq \mathbb{E}_s \left(e^{-\frac{s^2(1-\alpha)^2}{N} z} 1_{[z \leq n]} \right) + e^{-\frac{s^2(1-\alpha)^2(n+1)}{N}} \end{aligned} \tag{3.15}$$

Calling $M_x := s_x - s$, and $\Lambda(a) = \log \mathbb{E}(e^{an})$ for $|a| \leq a_0$, see (2.1), we get that $e^{aM_x - x\Lambda(a)}$ is a martingale, so that

$$1 = \mathbb{E}_s \left(e^{aM_{z \wedge n} - z \wedge n \Lambda(a)} \right) \geq \mathbb{E}_s \left(e^{aM_z - z \Lambda(a)} 1_{[z \leq n]} \right) \tag{3.16}$$

Also $M_z \leq -\alpha s$ and thus, choosing $a < 0$, we have $aM_z \geq -\alpha a s$, so that:

$$\mathbb{E} \left(e^{-z \Lambda(a)} 1_{[z \leq n]} \right) \leq e^{\alpha a s}.$$

Since $\Lambda(a) = \frac{1}{2} \sigma^2 a^2 + O(a^4)$ choosing $a = -\frac{\sqrt{2}(1-\alpha)s}{\sigma N^{1/2}}$ we get

$$\mathbb{E}_s \left(e^{-\frac{(1-\alpha)^2 s^2}{N} z} 1_{[z \leq n]} \right) \leq e^{-\frac{\sqrt{2}\alpha(1-\alpha)s^2}{2\sigma N^{1/2}}}$$

Recalling (3.15), we have

$$\mathbb{E}_s \left(e^{-\frac{1}{N} \sum_{x=0}^n s_x^2} \right) \leq e^{-\frac{\sqrt{2\alpha}(1-\alpha)s^2}{2\sigma N^{1/2}}} + e^{-\frac{s^2(1-\alpha)^2(n+1)}{N}}$$

For $n = \sqrt{N}$ we thus get that there is a constant b so that

$$\left[\mathbb{E}_s \left(e^{-\frac{1}{N} \sum_{x=0}^n s_x^2} \right) \right]^{1/2} \leq e^{-\frac{bs^2}{N^{1/2}}} \tag{3.17}$$

From (iii) of Theorem 3.1 there is $B > 0$ so that $\lambda\sqrt{N} \geq B$, thus from (3.13) and (3.17) we get (3.10). \square

4 Convergence and Identification of the Limit

We start the section with a preliminary lemma.

Lemma 4.1 *There is $b > 0$ so that*

$$\sum_{s, \bar{s}} \pi(s - \bar{s}) (h_N(s) - h_N(\bar{s}))^2 \leq \frac{b}{N^{1/2}}. \tag{4.1}$$

Proof Using that $\sum_s h_n(s)^2 = 1$ we have

$$\begin{aligned} \sum_{s, \bar{s}} \pi(s - \bar{s}) (h_N(s) - h_N(\bar{s}))^2 &= 2 \sum_{s, \bar{s}} \pi(s - \bar{s}) h_N^2(s) - 2 \sum_{s, \bar{s}} \pi(s - \bar{s}) h_N(s) h_N(\bar{s}) \\ &= 2 - 2\lambda_N - 2 \sum_{s, \bar{s}} (1 - e^{-(s^2 + \bar{s}^2)/2N}) \pi(s - \bar{s}) h_N(s) h_N(\bar{s}) \end{aligned} \tag{4.2}$$

By (iii) of Theorem 3.1 $2(1 - \lambda_N) \leq \frac{2c}{\sqrt{N}}$. By using that $1 - e^x < x$ and that $\sum_s s^2 h_N(s) \leq c'$, by Theorem 3.2 we have

$$\begin{aligned} 2 \sum_{s, \bar{s}} (1 - e^{-(s^2 + \bar{s}^2)/2N}) \pi(s - \bar{s}) h_N(s) h_N(\bar{s}) &\leq \frac{1}{2N} \sum_{s, \bar{s}} (s^2 + \bar{s}^2) \pi(s - \bar{s}) [h_N^2(s) + h_N^2(\bar{s})] \\ &\leq \frac{\sigma^2}{2N} + \frac{c'}{2N} \end{aligned}$$

From this (4.1) follows. \square

Define for $r \in \mathbb{R}$

$$\tilde{h}_N^2(r) = N^{1/4} h_N^2([rN^{1/4}]), \quad [] = \text{integer part} \tag{4.3}$$

Proposition 4.2 *The following holds.*

- (1) *The sequence of measures $\tilde{h}_N^2(r)dr$ in \mathbb{R} is tight and any limit measure is absolutely continuous with respect to the Lebesgue measure.*
- (2) *The sequence of functions $\tilde{h}_N(r) := N^{1/8} h_N([rN^{1/4}])$ is sequentially compact in $L^2(\mathbb{R})$.*

Proof As a straightforward consequence of Theorem 3.2, we have that

$$\tilde{h}_N^2(r) \leq C e^{-c r^2} \tag{4.4}$$

It follows that for any ϵ there is k so that $\int_{|r| \leq k} \tilde{h}_N^2(r) dr \geq 1 - \epsilon$, which proves tightness of the sequence of probability measures $\tilde{h}_N^2(r) dr$ on \mathbb{R} . From (4.4) we also get that any limit measure must be absolutely continuous.

To prove that the sequence $(\tilde{h}_N(r))_{N \geq 1}$ is sequentially compact in $L^2(\mathbb{R})$ we prove below that there exists a constant C such that for any N and any $\delta > 0$:

$$\int (\tilde{h}_N(r + \delta) - \tilde{h}_N(r))^2 dr \leq C \delta^2 \tag{4.5}$$

Assume that $\pi(1) > 0$, then

$$\begin{aligned} \int (\tilde{h}_N(r + \delta) - \tilde{h}_N(r))^2 dr &= \sum_s (h_N(s + [\delta N^{1/4}]) - h_N(s))^2 \\ &= \sum_s \left(\sum_{i=1}^{[\delta N^{1/4}]} (h_N(s + i) - h_N(s + i - 1)) \right)^2 \\ &\leq \frac{[\delta N^{1/4}]}{\pi(1)} \sum_s \sum_{i=1}^{[\delta N^{1/4}]} \pi(1) (h_N(s + i) - h_N(s + i - 1))^2 \\ &\leq \frac{[\delta N^{1/4}]^2}{\pi(1)} \sum_{s, \bar{s}} \pi(s - \bar{s}) (h_N(s) - h_N(\bar{s}))^2 \leq \frac{c \delta^2}{\pi(1)} \end{aligned}$$

The condition $\pi(1) > 0$ can be relaxed easily by a slight modification of the above argument.

From (4.4) and (4.5), applying the Kolmogorov–Riesz compactness theorem (see e.g. [10]), we get that \tilde{h}_N is sequentially compact in $L^2(\mathbb{R})$. \square

We next identify the limit.

Proposition 4.3 Any limit point $u(r)$ of $\tilde{h}_N(r)$ in L^2 satisfies in weak form

$$u(r) = \frac{1}{\lambda} \mathbb{E}_r \left(e^{-\frac{1}{2} \int_0^1 B_s^2 ds} u(B_1) \right) \tag{4.6}$$

where B_s is a Brownian motion with variance σ^2 and with $B_0 = r$ furthermore $\lambda = \lim_{N \rightarrow \infty} \lambda_N^{\sqrt{N}}$ which exists.

The unique solution of (4.6) (up to a multiplicative constant) is $u(r) = \exp\{-r^2/2\sigma\}$ and $\lambda = e^{-\sigma/2}$.

Proof Given r call $r_N = [rN^{1/4}]$, iterating (2.3) \sqrt{N} times (assuming that \sqrt{N} is an integer) we get

$$\tilde{h}_N(r) = \frac{1}{\lambda_N^{\sqrt{N}}} \mathbb{E}_{r_N}^N \left(\exp \left\{ -\frac{1}{2\sqrt{N}} \sum_{x=0}^{\sqrt{N}} \frac{s_x^2}{N^{1/2}} \right\} \tilde{h}_N(N^{-1/4} s_{\sqrt{N}}) \right) \tag{4.7}$$

where $\mathbb{E}_{r_N}^N$ is the expectation w.r.t. the random walk which starts from r_N .

$$s_x = r_N + \sum_{k=1}^x \eta_k, \quad x = 1, \dots, \sqrt{N} \tag{4.8}$$

By the invariance principle,

$$\frac{s_{t\sqrt{N}} - r_N}{N^{1/4}} \longrightarrow \sigma B_t \quad t \in [0, 1] \tag{4.9}$$

in law, where B_t is a standard Brownian motion which starts from 0.

Take a subsequence along which \tilde{h}_N converges strongly in $L^2(\mathbb{R})$ and call $u(r)$ the limit point. Choosing a test function $\varphi \in L^2(\mathbb{R})$, and denoting $\pi_n(s) = \pi(\sum_{k=1}^n \eta_k = s)$, we get along that sequence

$$\begin{aligned} & N^{-1/4} \sum_{s'} \varphi(N^{-1/4}s') \mathbb{E}_{s'}^N \left(\exp \left\{ -\frac{1}{2\sqrt{N}} \sum_{x=0}^{\sqrt{N}} \left(\frac{s_x}{N^{1/4}} \right)^2 \right\} \left| \tilde{h}_N(N^{-1/4}s_{\sqrt{N}}) - u(N^{-1/4}s_{\sqrt{N}}) \right| \right) \\ & \leq N^{-1/4} \sum_{s'} \varphi(N^{-1/4}s') \mathbb{E}_{s'}^N \left(\left| \tilde{h}_N(N^{-1/4}s_{\sqrt{N}}) - u(N^{-1/4}s_{\sqrt{N}}) \right| \right) \\ & = N^{-1/4} \sum_{s,s'} \varphi(N^{-1/4}s') \pi_{\lfloor \sqrt{N} \rfloor}(s-s') \left(\left| \tilde{h}_N(N^{-1/4}s) - u(N^{-1/4}s) \right| \right) \\ & \leq N^{-1/4} \sum_{s'} \left| \varphi(N^{-1/4}s') \right| \left(\sum_s \pi_{\lfloor \sqrt{N} \rfloor}(s-s') \left| \tilde{h}_N(N^{-1/4}s) - u(N^{-1/4}s) \right|^2 \right)^{1/2} \\ & \leq \left(N^{-1/4} \sum_{s'} \left| \varphi(N^{-1/4}s') \right|^2 \right)^{1/2} \left(N^{-1/4} \sum_{s'} \sum_s \pi_{\lfloor \sqrt{N} \rfloor}(s-s') \left| \tilde{h}_N(N^{-1/4}s) - u(N^{-1/4}s) \right|^2 \right)^{1/2} \\ & = \left(N^{-1/4} \sum_{s'} \left| \varphi(N^{-1/4}s') \right|^2 \right)^{1/2} \left(N^{-1/4} \sum_s \left| \tilde{h}_N(N^{-1/4}s) - u(N^{-1/4}s) \right|^2 \right)^{1/2} \\ & \leq C \left\| \varphi \right\|_{L^2} \left\| \tilde{h}_N - u \right\|_{L^2} \xrightarrow{N \rightarrow \infty} 0. \end{aligned} \tag{4.10}$$

Since the exponential on the right hand side of (4.7) is a bounded functional of the random walk, from (4.9) we get (along the chosen sequence),

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{E}_{r_N}^N \left(\exp \left\{ -\frac{1}{2\sqrt{N}} \sum_{x=0}^{\sqrt{N}} \left(\frac{s_x}{N^{1/4}} \right)^2 \right\} u(N^{-1/4}s_{\sqrt{N}}) \right) \\ & = \lim_{N \rightarrow \infty} \mathbb{E}_0^N \left(\exp \left\{ -\frac{1}{2\sqrt{N}} \sum_{x=0}^{\sqrt{N}} \left(\frac{s_x + r_N}{N^{1/4}} \right)^2 \right\} u(N^{-1/4}s_{\sqrt{N}}) \right) \\ & = \mathbb{E}_0 \left(e^{-\frac{1}{2} \int_0^1 (\sigma B_s + r)^2 ds} u(\sigma B_1 + r) \right) \end{aligned} \tag{4.11}$$

where \mathbb{E}_0 is the expectation w.r.t. the law of a standard Brownian motion starting at 0 and the limits are intended in the weak L^2 sense.

Since \tilde{h}_N is converging strongly in L^2 (along the subsequence we have chosen) and the expectation on the right hand side of (4.7) has a finite limit, we get that the limit of $\lambda_N^{\sqrt{N}}$ must exist.

Observe that for a standard Brownian motion $\{B_s\}_{s \in [0,1]}$ we have that

$$\exp \left\{ -\frac{1}{2} \int_0^t (\sigma B_s + r)^2 ds - \int_0^t (\sigma B_s + r) dB_s \right\}, \quad \text{is a martingale.}$$

Furthermore by Ito's formula

$$-\sigma \int_0^1 (\sigma B_s + r) dB_s = -\frac{1}{2}(\sigma B_1 + r)^2 + \frac{r^2}{2} + \frac{\sigma^2}{2}$$

Thus

$$\begin{aligned} 1 &= \mathbb{E} \left(\exp \left\{ -\frac{1}{2} \int_0^1 (\sigma B_s + r)^2 ds - \int_0^1 (\sigma B_s + r) dB_s \right\} \right) \\ &= \mathbb{E} \left(\exp \left\{ -\frac{1}{2} \int_0^1 (\sigma B_s + r)^2 ds - \frac{1}{2\sigma} (\sigma B_1 + r)^2 + \frac{r^2}{2\sigma} + \frac{\sigma}{2} \right\} \right) \end{aligned}$$

that implies

$$e^{-\frac{r^2}{2\sigma}} = e^{\sigma/2} \mathbb{E} \left(e^{-\int_0^1 (\sigma B_s + r)^2 ds} e^{-\frac{1}{2\sigma} (\sigma B_1 + r)^2} \right)$$

Comparing with (4.6) we identify $u(r)$ and λ . □

We thus have the following corollary of Proposition 4.2 and Proposition 4.3.

Corollary 4.4 *The sequence of measures $\tilde{h}_N^2(r)dr$ in \mathbb{R} converges weakly to the Gaussian measure $g^2(r)dr$ where $g(r) = (\pi\sigma)^{-1/4} e^{-r^2/2\sigma}$.*

Moreover for any $\psi, \varphi \in C_b(\mathbb{R})$ and any $t \in [0, 1]$

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{\lambda_N \sqrt{N}} \frac{1}{N^{1/4}} \sum_s \tilde{h}_N(N^{-1/4}s) \psi(N^{-1/4}s) \\ &\mathbb{E}_s^N \left(\exp \left\{ -\frac{1}{2\sqrt{N}} \sum_{x=0}^{[t\sqrt{N}]} \frac{s_x^2}{N^{1/2}} \right\} \tilde{h}_N(N^{-1/4} s_{[t\sqrt{N}]}) \varphi(N^{-1/4} s_{[t\sqrt{N}]}) \right) \\ &= e^{\sigma/2} \int \psi(r) g(r) \mathbb{E}_r \left(e^{-\frac{1}{2} \int_0^t (\sigma B_s)^2 ds} \varphi(\sigma B_t) g(\sigma B_t) \right) dr \end{aligned} \tag{4.12}$$

where \mathbb{E}_r is the expectation w.r.t. the law of the Brownian motion starting at r .

Proof From Proposition 4.3 we have that any subsequence of $\tilde{h}_N(r)$ converges in $L^2(\mathbb{R})$ to $c e^{-r^2/2\sigma}$ but since $\|\tilde{h}_N^2\|_{L^2} = 1$ we get that c must be equal to $(\pi\sigma)^{-1/4}$. This together with (1) of Proposition 4.2 concludes the proof.

The proof of (4.12) is an adaptation of (4.10) and (4.11). □

5 Proof of Theorem 2.1

Recall that \mathcal{P}_N and \mathcal{E}_N denote respectively the law and the expectation in $C([0, 1])$ of the process $\tilde{S}_N(t) = N^{-1/4} s_{[tN^{1/2}]}$ induced by the law of the Markov chain with transition probabilities given in (2.5) and initial distribution the invariant measure $\tilde{h}_N^2(r)dr$.

Proposition 5.1 *The finite dimensional distributions of $\tilde{S}_N(t), t \in [0, 1]$, converge in law to those of the stationary Ornstein–Uhlenbeck.*

Proof For any k , any $0 \leq \tau_1 < \dots < \tau_k \leq 1$ and any collection of continuous bounded functions with compact support $\varphi_0, \varphi_1, \dots, \varphi_k$, setting $t_i = \tau_i - \tau_{i-1}$, $i = 1, \dots, k$, $\tau_0 = 0$ we have

$$\begin{aligned} & \mathcal{E}_N \left(\varphi_0(\tilde{S}_N(0)) \varphi_1(\tilde{S}_N(t_1)) \dots \varphi_k(\tilde{S}_N(t_k)) \right) \\ &= N^{-1/4} \sum_{r_0 \in N^{-1/4}\mathbb{Z}} \tilde{h}_N(r_0) \varphi(r_0) \lambda_N^{-k\sqrt{N}} \\ & \mathbb{E}_{r_0}^N \left(e^{-\frac{1}{2\sqrt{N}} \sum_{x=0}^{\lfloor t_1 \sqrt{N} \rfloor} \frac{s_x^2}{N^{1/2}}} \varphi_1(r_1) \mathbb{E}_{r_1}^N \left(e^{-\frac{1}{2\sqrt{N}} \sum_{x=0}^{\lfloor t_2 \sqrt{N} \rfloor} \frac{s_x^2}{N^{1/2}}} \varphi_2(r_2) \right. \right. \\ & \left. \left. \dots \mathbb{E}_{r_{k-1}}^N \left(e^{-\frac{1}{2\sqrt{N}} \sum_{x=0}^{\lfloor t_k \sqrt{N} \rfloor} \frac{s_x^2}{N^{1/2}}} \tilde{h}_N(r_k) \varphi_k(r_k) \right) \dots \right) \right) \end{aligned}$$

where $r_i = N^{-1/4} \left[r_{i-1} + \sum_{x=1}^{\lfloor t_i \sqrt{N} \rfloor} \eta_x \right]$. Then from a ripetute use of (4.12) we get

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathcal{E}_N \left(\varphi_0(\tilde{S}_N(0)) \varphi_1(\tilde{S}_N(t_1)) \varphi_2(\tilde{S}_N(t_2)) \dots \varphi_k(\tilde{S}_N(t_k)) \right) \\ &= e^{k\sigma/2} \int g(r_0) \varphi(r_0) \mathbb{E}_{r_0} \left(e^{-\int_0^{t_1} \sigma B_s} \varphi_1(\sigma B_{t_1}) \dots e^{-\int_0^{t_k} \sigma B_s} \varphi_k(B_{t_k}) g(B_{t_k}) \right) dr_0 \end{aligned}$$

□

To conclude the proof of Theorem 2.1 we need to show tightness of \mathcal{P}_N in $C([0, 1])$; this is a consequence of Proposition 5.2 below, see Theorem 12.3, Eq. (12.51) of [3].

Proposition 5.2 *There is C so that for all N ,*

$$\mathcal{E}_N \left((\tilde{S}_N(t) - \tilde{S}_N(0))^4 \right) \leq Ct^{3/2}. \tag{5.1}$$

Proof

$$\begin{aligned} & \mathcal{E}_N \left((\tilde{S}_N(t) - \tilde{S}_N(0))^4 \right) \\ &= \lambda_N^{-\sqrt{N}} \sum_s h_N(s) \mathbb{E}_s^N \left(e^{-\frac{1}{2\sqrt{N}} \sum_{x=0}^{\lfloor N^{1/2}t \rfloor} \frac{s_x^2}{N^{1/2}}} (\tilde{S}_N(t) - sN^{-1/4})^4 h_N(s_{\lfloor N^{1/2}t \rfloor}) \right) \\ &\leq \lambda_N^{-\sqrt{N}} \sum_s h_N(s) \mathbb{E}_s^N \left((\tilde{S}_N(t) - sN^{-1/4})^4 h_N(s_{\lfloor N^{1/2}t \rfloor}) \right) \\ &\leq C \lambda_N^{-\sqrt{N}} \sum_{s,s'} h_N(s) h_N(s') \pi_{\lfloor N^{1/2}t \rfloor}(s - s') \left| \frac{s - s'}{t^{1/2} N^{1/4}} \right|^4 t^2 \end{aligned} \tag{5.2}$$

where $\pi_n(s) = \pi \left(\sum_{k=1}^n \eta_k = s \right)$. By Proposition 2.4.6 in [12], if π is aperiodic with finite 4th moments, as in our case, we have the bound

$$\pi_n(s) \leq \frac{C}{n^{1/2}} \left(\frac{\sqrt{n}}{|s|} \right)^4, \quad \forall s \in \mathbb{Z}. \tag{5.3}$$

From this estimate it follows that the right hand side of (5.2) is bounded by

$$\leq t^2 C' \lambda_N^{-\sqrt{N}} \sum_{s,s'} h_N(s) h_N(s') \frac{1}{\sqrt{t} N^{1/2}} = C' t^{3/2} \lambda_N^{-\sqrt{N}} N^{-1/4} \left(\sum_s h_N(s) \right)^2,$$

By (3.10) we have that $\sum_s h_N(s) \leq N^{1/8}$, and the bound follows. □

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