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Graph-Counting Polynomials for Oriented Graphs

David Ruelle[1]

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Abstract

If $\mathcal{F}$ is a set of subgraphs F of a finite graph E we define a graph-counting polynomial $p_\mathcal{F}(z) = \sum_{F\in\mathcal{F}} z^{|F|}$ In the present note we consider oriented graphs and discuss some cases where $\mathcal{F}$ consists of unbranched subgraphs E. We find several situations where something can be said about the location of the zeros of $p_\mathcal{F}$.

Let $\mathcal{F}$ be a set of subgraphs F of a finite graph E. We denote by $|F|$ the number of edges of F and define a polynomial

$$p_\mathcal{F}(z) = \sum_{F\in\mathcal{F}} z^{|F|}$$

(graph-counting polynomial associated with $\mathcal{F}$). The case of unoriented graphs has been discussed earlier (see [4–6] and [1–3]); here we mostly consider oriented graphs.

We shall find that for suitable $\mathcal{F}$ we can restrict the location of the zeros of $p_\mathcal{F}$ (for instance to the imaginary axis). The proofs will be based on the following fact:

Lemma (Asano-Ruelle). *Let K_1, K_2 be closed subsets of the complex plane* $\mathbf{C}$ *such that $K_1, K_2 \not\ni 0$ and assume that*

$$A + Bz_1 + Cz_2 + Dz_1z_2 \neq 0 \qquad \text{when} \qquad z_1 \notin K_1, z_2 \notin K_2$$

Then

$$A + Dz \neq 0 \qquad \text{when} \qquad z \notin -K_1K_2$$

where $-K_1K_2$ is minus the set of products of an element of K_1 and an element of K_2. (The replacement of $A + Bz_1 + Cz_2 + Dz_1z_2$ by $A + Dz$ is called Asano contraction and denoted $(z_1, z_2) \to z$).

For a proof see for instance the Appendix A of [6]. The results given below follow rather directly from this lemma.

1 Definitions: Subgraphs of an Oriented Graph

We say that a pair (V, E) of finite sets is an *oriented graph* if $V \neq \emptyset$ and we are given two maps $x', x'' : E \to V$ such that $x'(e) \neq x''(e)$ for all $e \in E$. The elements x of V

✉ David Ruelle
ruelle@ihes.fr

1 Mathematics Department, Rutgers University and IHES, 91440 Bures sur Yvette, France

are *vertices*, and the elements e of E are *oriented edges* with endpoints $x'(e), x''(e)$; e is outgoing at $x'(e)$ and ingoing at $x''(e)$ and we write $e : x'(e) \to x''(e)$. We allow different edges e_1, e_2 such that $e_1 : x' \to x''$ and $e_2 : x' \to x''$ or $e_2 : x'' \to x'$.

We say that (V, E) is *bipartite* if we are given a partition of V into nonempty sets V_1 and V_2 such that for each $e \in E$ the points $x'(e)$ and $x''(e)$ are in different sets of the partition $\{V_1, V_2\}$.

We call a subset F of E a *subgraph* of (V, E). We say that F is connected if for each partition $\{F_1, F_2\}$ of F there is an $x \in V$ which is an endpoint of both some $e_1 \in F_1$ and some $e_2 \in F_2$. A subgraph is thus a union of connected components F_j in a unique way.

We say that F is an *unbranched* subgraph if, for each $x \in V$,

$$|\{e \in F : x'(e) = x\}| \leq 1 \qquad \text{and} \qquad |\{e \in F : x''(e) = x\}| \leq 1$$

We say that an unbranched subgraph is a *loop subgraph* if for each $x \in V$ we have

$$|\{e \in F : x'(e) = x\}| = |\{e \in F : x''(e) = x\}|$$

We denote by $U(E)$, resp. $L(E)$, the set of unbranched subgraphs, resp. loop subgraphs of the oriented graph (V, E). If F is an unbranched subgraph, we can write F as a disjoint union of connected components F_j which are either loops (i.e., $F_i \in L(E)$) or, if they are not loops, have different endpoints x'_j and x''_j such that $x'_j \to \cdots (e) \cdots \to x''_j$ for each $e \in F_j$.

Introduce now complex variables z'_e, z''_e, write $Z' = (z'_e)_{e \in E}$, $Z'' = (z''_e)_{e \in E}$ and, for each $x \in V$, let

$$p_x(Z', Z'') = \left(a'(x) + \sum_{e:x'(e)=x} z'_e\right)\left(a''(x) + \sum_{e:x''(e)=x} z''_e\right) \tag{1}$$

with some choice of the $a'(x), a''(x) \in \mathbf{C}$. For small $\epsilon > 0$ we also write

$$p^\epsilon_x(Z', Z'') = \left(a'(x) + \sum_{e:x'(e)=x} (z'_e + \epsilon)\right)\left(a''(x) + \sum_{e:x''(e)=x} (z''_e + \epsilon)\right)$$

and

$$\tilde{p}_x = 1 + p_x, \quad \tilde{p}^\epsilon_x = 1 + p^\epsilon_x$$

Choosing between p_x and $\tilde{p}_x$ for each x and applying Asano contractions $(z'_e, z''_e) \to z_e$ for all $e \in E$ to the polynomials

$$\prod_{x \in V} (p_x(Z', Z'') \text{ or } \tilde{p}_x(Z', Z'')), \quad \prod_{x \in V} (p^\epsilon_x(Z', Z'') \text{ or } \tilde{p}^\epsilon_x(Z', Z'')) \tag{2}$$

we obtain polynomials

$$P(Z), \quad P^\epsilon(Z)$$

where $Z = (z_e)_{e \in E}$ and

$$\lim_{\epsilon \to 0} P^\epsilon(Z) = P(Z)$$

We shall obtain examples of $p(z) = p_{\mathcal{F}}(z)$ by taking all components z_e of Z equal to z.

2 Unbranched Subgraphs of an Oriented Graph

If there are only p_x factors in (2) and we assume $a'(x)a''(x) = 1$ for all x, we have

$$P(Z) = \sum_{F \in U(E)} \prod_{F_j \subset F}^{\text{conn}} [a''(x'_j)a'(x''_j)]_{F_j \text{not loop}} \prod_{e \in F_j} z_e \tag{3}$$

where the product is over the connected components F_j of F and x'_j, x''_j are the endpoints of F_j if j is not a loop; $[a''(x'_j)a'(x''_j)]$ is replaced by 1 if F_j is a loop.

If we take $a'(x) = a''(x) = 1$ and set all z_e equal to z we obtain the *unbranched subgraph counting polynomial*

$$p_{\text{unbranched}}(z) = \sum_{F \in U(E)} z^{|F|} \tag{4}$$

Proposition 2.1 *The zeros of the unbranched subgraph counting polynomial (4) are all real and strictly negative.*

To prove this let $\alpha', \alpha'' \in (-\pi/2, \pi/2)$. Assuming

$$\text{Re}(z'_e + \epsilon)e^{-i\alpha'} > 0 \quad , \quad \text{Re}(z''_e + \epsilon)e^{-i\alpha''} > 0$$

for all $e \in E$ and all $\epsilon > 0$, we have $\prod_{x \in V} p_x^\epsilon(Z', Z'') \neq 0$ and therefore by the Asano-Ruelle Lemma $P^\epsilon(Z) \neq 0$ if $e^{-i(\alpha'+\alpha'')}z_e$ is in a neighborhood of the positive real axis and $-\pi < \alpha' + \alpha'' < \pi$. Let $p(z)$ and $p^\epsilon(z)$ be obtained by taking all z_e equal to z in $P(Z)$ and $P^\epsilon(Z)$. Then $p^\epsilon(z) \neq 0$ if $\arg z \in (-\pi, \pi)$. Using Hurwitz's theorem we let $\epsilon \to 0$ in $p^\epsilon(z)$ and find that either $p(z)$ vanishes identically or $p(z) \neq 0$ if $\arg z \in (-\pi, \pi)$. Clearly $p(0) \neq 0$ because $\emptyset \subset U(E)$ and we obtain thus $p_{\text{unbranched}}(z) = p(z) \neq 0$ if z is not real strictly negative. □

[In fact since $a'(x) = a''(x) = 1$ we could have done without ϵ in the present situation].

Let now $\deg_2$ be the max over $x \in V$ of the number d'_x of outgoing edges at x times the max over x of the number d''_x of ingoing edges at x. Then $p_x(Z', Z'') \neq 0$ if $|z'_e| < 1/d'_x$ and $|z''_e| < 1/d''_x$ for all $e \in E$, so that $P(Z) \neq 0$ if all $z_e < 1/\deg_2$. Finally $p(z) \neq 0$ if $|z| < 1/\deg_2$, i.e., the zeros of $p(z)$ are negative and bounded above by $-1/\deg_2$.

Remark 2.2 Given $V_0 \subset V$ let $a'(x) = a''(x) = 1$ if $x \in V_0$ and $a'(x) = a''(x) = 0$ if $x \notin V_0$. The polynomial p counts then unbranched polynomials going through all $x \notin V_0$. The proof of the Proposition 2.1 still applies (but now ϵ is indeed needed) and one finds that the zeros of p are real less then or equal zero if p does not vanish identically.

The following result is relevant to Sect. 3.2 below.

Proposition 2.3 *Let (V, E) be bipartite corresponding to a partition $\{V_1, V_2\}$ of V and let $U_{\text{even}}(E)$ consist of the unbranched subgraphs F such that $|F_j|$ is even for each connected component F_j of F. We define*

$$p_{\text{unbranched even}}(z) = \sum_{F \in U_{\text{even}}(E)} z^{|F|} \tag{5}$$

Assume that the odd connected subgraphs in $U(E)$ come in pairs $(G, \bar{G})$ connecting the same vertices and both G, $\bar{G}$ are loops or not loops. If x', x'' (resp. $\bar{x}', \bar{x}''$) are the endpoints *of non-loop* G *(resp.* $\bar{G}$*) we also assume* $x' = \bar{x}''$, $x'' = \bar{x}'$. *Under these conditions the zeros of the polynomial* (5) *are all purely imaginary.*

To prove this let us in Eq. (1) *take* $a'(x) = (1+i)/\sqrt{2}$ *if* $x \in V_1$, $a'(x) = (1-i)/\sqrt{2}$ *if* $x \in V_2$ *and let* $a''(x)$ *be the complex conjugate* $a'(x)^*$ *of* $a'(x)$ *in all cases. We use thus the* $p_x(Z', Z'')$, $p_x^\epsilon(Z', Z'')$ *corresponding to those* $a'(x)$, $a''(x)$.

We obtain polynomials $P(Z)$, *resp.* $P^\epsilon(Z)$, *by Asano contractions of* $\prod p_x(Z', Z'')$, *resp.* $\prod p_x^\epsilon(Z', Z'')$, *and* (3) *gives*

$$P(Z) = \sum_{F \in U(E)} \prod_{F_j \subset F}^{\text{conn}} \gamma(F_j) \prod_{e \in F_j} z_e \tag{6}$$

with $\gamma(F_j) = a''(x_j')a'(x_j'')$ *if* F_j *is not a loop, and* $\gamma(F_j) = 1$ *if* F_j *is a loop. If* $|F_j|$ *is even,* x_j' *and* x_j'' *are both in either* V_1 *or* V_2, *so that* $\gamma(F_j) = 1$. *If* $|F_j|$ *is odd,* x_j' *and* x_j'' *are in different sets of the partition* (V_1, V_2), *so that* $a''(x_j')a'(x_j'') = ((1 \pm i)/\sqrt{2})^2$ *and* $\gamma(F_j) = \pm i$. *Choose now a pair* $(G, \bar{G})$ *with odd* $|G| = |\bar{G}|$ *then* $\gamma(G) + \gamma(\bar{G}) = 0$ *so that the terms in* $p(z)$ *corresponding to* F *containing a connected component* G *or* $\bar{G}$ *cancel. This holds for all pairs* $(G, \bar{G})$ *with odd* $|G| = |\bar{G}|$ *and therefore*

$$p(z) = \sum_{F \in U(E)} \prod_{F_j \subset F}^{\text{conn}} \gamma(F_j) z^{|F_j|} = \sum_{F \in U_{\text{even}}(E)} z^{|F|} = p_{\text{unbranched even}}(z)$$

With our choice of a', a'' *we see that if* $\alpha', \alpha'' \in (-\pi/4, \pi/4)$ *and*

$$\text{Re}(z_e' + \epsilon)e^{-i\alpha'} > 0 \quad , \quad \text{Re}(z_e'' + \epsilon)e^{-i\alpha''} > 0$$

for all $e \in E$, *we have* $\prod_{x \in V} p_x^\epsilon(Z', Z'') \neq 0$. *Therefore by the Asano-Ruelle Lemma* $P^\epsilon(Z) \neq 0$ *if*

$$-\pi/2 < \alpha' + \alpha'' < \pi/2 \quad \text{and} \quad (\forall e \in E)\ (z_e + \epsilon')e^{-i(\alpha' + \alpha'')} > 0$$

for some $\epsilon' > 0$. *We take all* z_e *equal to* z *and use Hurwitz's theorem to let* $\epsilon \to 0$. *Since* $\emptyset \in U_{\text{even}}$, p *does not vanish identically and we obtain* $p(z) \neq 0$ *if* $Re(z) > 0$, *or by symmetry if* $Re(z) \neq 0$. □

3 Oriented Subgraphs of a Non-oriented Graph

Let (V, E_0) be a non-oriented graph. There are different ways to associate an oriented graph with (V, E). Here we define the oriented graph $(V, \tilde{E}_0)$ where each non-oriented edge $e \in E_0$ with endpoints $x_1, x_2 \in V$ is replaced by two oriented edges $e', e'' \in \tilde{E}_0$ such that $x'(e') = x_1$, $x''(e') = x_2$ and $x'(e'') = x_2$, $x''(e'') = x_1$. We have thus $|\tilde{E}_0| = 2|E_0|$. The subgraphs $\tilde{F}$ of $(V, \tilde{E}_0)$, i.e., the subsets of $\tilde{E}_0$ may be called oriented subgraphs of (V, E_0).

3.1 Unbranched Subgraphs of a Non-oriented Graph

From Proposition 2.1 we know that the polynomial counting oriented unbranched subgraphs of (V, E_0), i.e.,

$$p_{\text{oriented unbranched}}(z) = \sum_{\tilde{F} \in U(\tilde{E}_0)} z^{|\tilde{F}|} = \sum_{\tilde{F} \in U(\tilde{E}_0)} \prod_{\tilde{F}_j \subset \tilde{F}}^{\text{conn}} z^{|\tilde{F}_j|}$$

has all its zeros real strictly negative. Note that without orientation

$$p_{\text{unbranched}}(z) = \sum_{F \in U(E_0)} z^{|F|} = \sum_{F \in U(E_0)} \prod_{F_j \subset F}^{\text{conn}} z^{|F_j|}$$

and it is known (see [5]) that this has all its zeros with real part strictly negative. [The set $U(E_0)$ of unbranched subgraphs of E_0 and the connected components of a non-oriented F are defined in the obvious manner].

Let us now assume that (V, E_0) has only simple edges between vertices. It is interesting to compare the connected components F_j of some non-oriented unbranched subgraph F with the possible corresponding oriented connected components $\tilde{F}_{j\alpha}$ of an unbranched oriented subgraph $\tilde{F}$ such that, for each edge e of F, one or both of the corresponding edges e', e'' belongs to $\tilde{F}$. If $|F_j| = 1$ then $F_j = e$ for some non-oriented $e \in E_0$ with endpoints x_1, x_2 and there are two oriented edges $e', e'' \in \tilde{E}_0$ corresponding to e. Then, corresponding to F_j there are three possible connected components $\tilde{F}_{j\alpha} \subset \tilde{E}_0$, namely $\{e'\}, \{e''\}, \{e', e''\}$, and $|\tilde{F}_{j\alpha}|$ is 1 or 2. If $|F_j| > 1$, there correspond to F_j two oriented $\tilde{F}_{j\alpha}$. We obtain thus for $\tilde{E}_0$ the polynomial

$$p_{\text{oriented unbranched}}(z) = \sum_{F \in U(E_0)} \prod_{F_j : |F_j| = 1}^{\text{conn}} (2z + z^2) \prod_{F_j : |F_j| > 1}^{\text{conn}} (2z^{|F_j|})$$

3.2 Even Oriented Unbranched Subgraphs of a Non-oriented Graph

For a bipartite graph E_0 we obtain pairs $(G, \bar{G})$ of subgraphs of $\tilde{E}_0$ as in Proposition 2.3 by orientation reversal so that

$$p_{\text{oriented unbranched even}}(z) = \sum_{\tilde{F} \in U_{\text{even}}(\tilde{E}_0)} z^{|\tilde{F}|}$$

has all its zeros purely imaginary by Proposition 2.3.

Let (V, E_0) have only simple edges between vertices. We define

$$U'(E_0) = \{F : \text{ for all connected components } F_j \text{ of } F \text{ either } |F_j| = 1 \text{ or } |F_j| \text{ iseven}\}$$

Then we have

$$p_{\text{oriented unbranched even}}(z) = \sum_{F \in U'(E_0)} z^{2|\{|j:|F_j|=1\}|} \cdot \prod_{j:|F_j|>1} (2z)^{|F_j|}$$

for the unbranched even subgraph counting polynomial of $\tilde{E}_0$.

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