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Equilibration and Diffusion for a Dynamical Lorentz Gas

Émilie Soret[1,2]

Abstract We consider a model of a *dynamical* Lorentz gaz: a single particle is moving in $\mathbb{R}^d$ through an array of fixed and soft scatterers each possessing an internal degree of freedom coupled to the particle. Assuming the initial velocity is sufficiently high and modelling the parameters of the scatterers as random variables, we describe the evolution of the kinetic energy of the particle by a Markov chain for which each step corresponds to a collision. We show that the momentum distribution of the particle approaches a Maxwell–Boltzmann distribution with effective temperature T such that $k_B T$ corresponds to an average of the scatterers' kinetic energy.

Keywords Mathematical physics · Probability · Lorentz gas · Diffusion processes · Equilibration · Stationary distribution

1 Introduction

We study a class of Hamiltonian systems referred to as *dynamical* Lorentz gases and introduced in [6,11]. These models describe the motion of a single particle through an array of independent scatterers, each possessing an internal degree of freedom to which the particle is locally coupled. In [6], it is argued that the particle momentum distribution $\rho(t)$ will converge, asymptotically in time, to a Maxwell–Boltzmann thermal equilibrium distribution characterized by a temperature that is determined by the energy distribution of the individual scatterers . This convergence holds in a suitable parameter range corresponding to a weak coupling limit and for an arbitrary distribution $\rho_0(t)$ of sufficiently large average mean speed. This result holds even if initially the scatterers are not in thermal equilibrium. In this paper, we provide a rigorous proof of this result and identify conditions on the parameter range for

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which it holds. The original, fully Hamiltonian model consists of a particle–scatterer system which obeys to the following laws of motion :

$$\begin{cases} \ddot{q}(t) = -\alpha \sum_{i \in \mathbb{Z}^d} \eta(Q_i(t)) \nabla \sigma(q(t) - r_i), \\ M \ddot{Q}_i(t) + U'(Q_i(t)) = -\tilde{\alpha} \eta'(Q_i(t)) \sigma(q(t) - r_i). \end{cases} \tag{1.1}$$

In these equations, $q(t) \in \mathbb{R}^d$, $d \geq 2$ represents the position of the particle at time t, and $Q_i \in \mathbb{R}$ is the displacement of the internal degree of freedom associated to a scatterer centred at a fixed point $r_i \in \mathbb{R}^d$, $i \in \mathbb{Z}^d$. One may think for example of

$$r_i = i(1)e_1 + \cdots + i(d)e_d, \tag{1.2}$$

with $e_1, \ldots, e_d$ a basis of $\mathbb{R}^d$, and $i(k)$ the k-th coordinate of i (see Fig. 1). The points r_i then lie on a lattice. Another example could be $r_i = i(1)e_1 + \cdots + i(d)e_d + \eta_i$, where the η_i are chosen in a unit cell of the lattice. In what follows the precise geometry of the scatterers will not play any role, as we will explain in details below.

We assume that the scatterers' centres are distributed on $\mathbb{R}^d$ with finite horizon, which means that whatever is the direction of the particle, it cannot move indefinitely without hitting a scatterer. This hypothesis is primordial for our model, indeed if the horizon is not finite, we cannot describe the complete model by a Markovian one as we will do in Sect. 2.

Both the potential U which controls the non-coupled dynamics of the degree of freedom of the scatterers and the coupling function η are supposed to be $\mathcal{C}^\infty(\mathbb{R})$. More particularly, we assumed that the coupling function η is linear and that U has a polynomial growth:

$$U(Q) \sim |Q|^r \quad \text{and} \quad |\eta(Q)| \sim Q, \quad r \geq 1. \tag{1.3}$$

Hence, the potential U is confining. The form factor σ appearing in the interaction term satisfies $\|\sigma\|_\infty \leq 1$. We assume that it is rotationally invariant and with compact support in the sphere $B(0, 1/2)$. The values $\alpha, \tilde{\alpha} > 0$ are coupling constants and M is the ratio of the mass associated to the internal degree of freedom over the mass of the particle. All these parameters are dimensionless.

The case $\tilde{\alpha} = 0$ is studied in [1,2,10]. It corresponds to a system in which the passage of the particle does not affect the evolution on the environment. We call such model *inert* (in opposition to *dynamical*). As there is no mechanism of dissipation of the energy of the particle, the kinetic energy of the latter grows with the time [10]. This phenomenon is called

Fig. 1 Example of path in $\mathbb{R}^2$. Here the points r_i lie on $\mathbb{Z}^2$. The path of the particle presents two recollisions
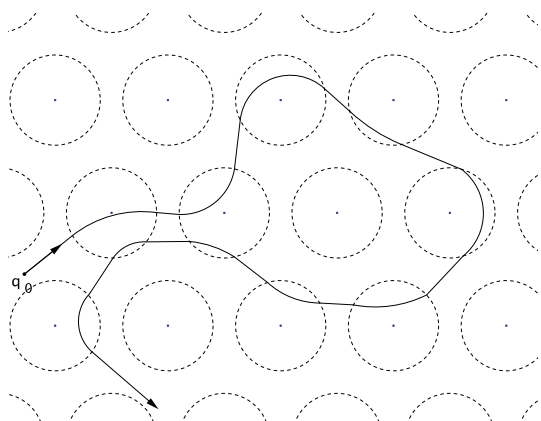
stochastic acceleration. In that case, the particle momentum distribution never approaches an equilibrium state, and a fortiori not a thermal equilibrium.

Another interesting case is $\tilde{\alpha} = \alpha$, which is the one here. The particle can be seen as a small perturbation of the environment, it leads to a loss of average energy which acts as a source of dynamical friction [3–5], and stochastic acceleration does not occur.

In the following, we assume that the two coupling constants are equal, $\alpha = \tilde{\alpha}$. As we consider a linear coupling, the law of motion (1.1) becomes

$$\begin{cases} \ddot{q}(t) = -\alpha \sum_{i \in \mathbb{Z}^d} Q_i(t) \nabla \sigma(q(t) - r_i) \\ M\ddot{Q}_i(t) + U'(Q_i(t)) = -\alpha \sigma(q(t) - r_i). \end{cases} \quad (1.4)$$

Let p be the momentum of the particle and let $P_i = M\dot{Q}_i$. Then the equations (1.4) are generated by the Hamiltonian

$$H(q, p, Q_i, P_i) = \frac{p^2}{2} + \sum_{i \in \mathbb{Z}^d} H_{\text{scatt}}(Q_i, P_i) + \alpha \sum_{i \in \mathbb{Z}^d} Q_i \sigma(q - r_i),$$

where H_{scatt} is the Hamiltonian associated to the dynamics of a single scatterer. The total energy of the system is conserved and

$$H_{\text{scatt}}(Q, P) = \frac{P^2}{2M} + U(Q). \quad (1.5)$$

In other words, the internal degree of freedom of each scatterer reacts to the passage of the particle while allowing the energy of the system to be conserved. Obviously, this evolution of the internal degree of freedom does not affect its spatial localisation, nor its interaction area.

Further more, we assume that the particle is always fast which means that it always crosses the interaction region in a time of order $\|p\|^{-1}$ (see [6]), where $\|\cdot\|$ is for the Euclidean norm in $\mathbb{R}^d$. We will assume moreover that its kinetic energy $\|p(t)\|^2/2$ is well above the typical interaction potential $\alpha Q\sigma(q(t) - r)$ that it encounters in any scattering event.

So defined, this dynamical system is too difficult to study with full mathematical rigour. Indeed, the particle can pass through a scatterer at least twice: recollisions between the particle and a same scatterer are possible (see Fig. 1), making it necessary to keep track of the evolution of the internal degree of freedom. In addition, the geometry of the scattering centers r_i induces additional difficulties. This means we are dealing with a very complex nonlinear infinite dimensional dynamical system. To simplify the problem, we will follow [6] and instead describe the trajectory of the particle by a Markov chain, eliminating both the complexity induced by recollisions and by the geometry of the family $(r_i)_{i \in \mathbb{Z}^d}$ provided the horizon is finite. In this manner, we concentrate on the essential dynamical phenomena induced by the individual scattering events.

Each step of this Markov chain, described in detail in Sect. 2, corresponds to a collision between the particle and a scatterer. We view the states of the scatterer that the particle successively visits as independent and identically distributed random variables and fix the distance between two consecutive scatterers met by the particle. In [6], it was shown, for a Markov chain which is a cut version of the one we consider in Sect. 2, that it captures the essential features of the behaviour of the original system very well, both in terms of the evolution of the momentum distribution of the particle and of its diffusive spatial displacement.

In this paper, our contribution is the obtaining of rigorous results on the asymptotic behaviour of the Markov chain model, in a suitable regime for α and M, described below,

and that corresponds to a weak-coupling limit. We will show that the momentum distribution of the particle then converges to a Maxwellian.

The rest of the paper is organized as follows. In Sect. 2 we introduce the Markov chain description of the particle's momentum (Proposition 2.1) and we identify precisely the parameters and the time scale for which we obtain, in Theorem 2.2, the weak coupling limit of the Markov chain. Once this limit obtained, we use in Sect. 3, the Fokker–Plank equation to compute the stationary distribution and we will show that it approaches the Maxwellian. The proof of Proposition 2.1 and Theorem 2.2 are given in Sect. 3.

2 The Markov Chain Description and Weak Coupling Limit

To each trajectory $(q(t), p(t))$, solution of the deterministic and Hamiltonian equations of motion (1.4), one can associate a sequence $(t_n, p_n, b_n, r_{i_n}, Q_n, P_n)_{n \in \mathbb{N}}$. Here, t_n is the instant the particle arrives on the n-th scattering region that it will encounter, $p_n = \dot{q}(t_n)$ is its incoming velocity; r_{i_n} is the n-th scattering center visited by the particle, Q_n and P_n are the initial states of the scatterer. Last, the impact parameter b_n models the approach of this scatterer (Fig. 2) and is, by its definition, orthogonal to the incoming velocity $\|p_n\|$.

The associated Markov chain is now constructed by first introducing randomness in the parameters b_n, Q_n, P_n, as follows. We assume that the sequence $(Q_n, P_n)_n$ is i.i.d with respect to a stationary distribution of the H_{scatt} (1.5):

$$\rho(Q_n, P_n) = \hat{\rho}(H_{\text{scatt}}(Q_n, P_n)) = \rho(Q_n, -P_n), \tag{2.1}$$

where ρ is assumed to be a probability density with compact support. Similarly, we assume that $(b_n)_n$ is a sequence of i.i.d random variables conditioned to be such that, for all $n \in \mathbb{N}$, the scalar product $b_n \cdot p_n = 0$. We denote, as above, by p_n the momentum of the particle before it encounters, at time t_n and with an impact parameter b_n, the n-th scatterer. The
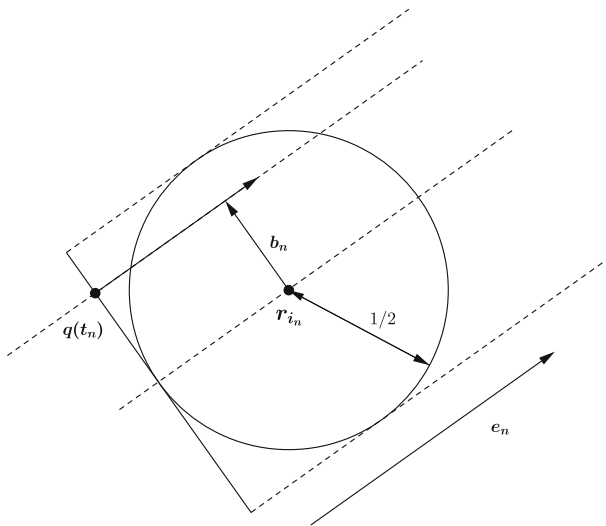

Fig. 2 Approach of the n-th scatterer: the particle arrives at the instant t_n on the n-th scatterer it encounter which is centered in r_{i_n}. At this instant, the particle has position $q(t_n)$ and velocity $p_n = \dot{q}(t_n)$

latter is in the initial state (Q_n, P_n). Then, after scattering, the particle leaves this scatterer with velocity p_{n+1} that it keeps until it encounters with a random impact parameter b_{n+1} a $n + 1$-th scatterer to which we attribute randomly an initial state (Q_{n+1}, P_{n+1}). Explicitly, the momentum change is determined by

$$p_{n+1} = p_n + R(p_n, b_n, Q_n, P_n), \qquad b_n \cdot p_n = 0, \qquad (2.2)$$

where

$$R(p, b, Q, P) = -\alpha \int_0^{t_+} dt \, Q(t) \nabla \sigma(q(t)) \qquad (2.3)$$

with t_+ the instant the particle exits the scatterer and with $(q(t), Q(t))$ the unique solution of

$$\begin{cases} \ddot{q}(t) = -\alpha Q(t) \nabla \sigma(q(t)) \\ M \ddot{Q}(t) + U'(Q(t)) = -\alpha \sigma(q(t)), \end{cases} \qquad (2.4)$$

with initial conditions

$$p(0) = p, \quad q(0) = b - \frac{p}{2\|p\|}, \quad Q(0) = Q, \quad P(0) = P \text{ and } \quad p \cdot b = 0.$$

Denoting by $\Delta p_n = p_{n+1} - p_n$ the transfer of momentum, one checks readily (see below also) that this transfer depends only on b_n, (Q_n, P_n) and $\|p_n\|$. Finally, we eliminate the geometry inherent to the initial problem and encoded in the positions of the scattering centres r_i by fixing the distance ℓ_* the particle travels between two consecutive scatterers. With an initial data (q_0, p_0), the position q_n and the momentum p_n of the particle at time t_n are therefore iteratively defined through the relations

$$\begin{cases} p_{n+1} = p_n + R(p_n, \kappa_n) \\ t_{n+1} = t_n + \dfrac{\ell_*}{\|p_{n+1}\|} \\ q_{n+1} = q_n + \dfrac{\ell_*}{\|p_{n+1}\|} p_{n+1}, \end{cases} \qquad (2.5)$$

where we denote by κ_n the triple (b_n, Q_n, P_n). Note that this is indeed a Markov chain which is completely determined by the initial data (q_0, p_0) and by the sequence of independent and identically distributed random variables $(\kappa_n)_n$. This provides a simplification of the full deterministic model (1.4).

In the following, for all functions depending on p and other variables such as α and M, we denote its expectation with respect to the random variable $\kappa = (Q, P, b)$ by

$$\overline{f(p, \alpha, M)} = \int_{\substack{\|b\| \leq 1/2 \\ b \cdot p = 0}} \frac{db}{C_d} \int_{\mathbb{R} \times \mathbb{R}} dQ \, dP \, \rho(Q, P) f(p, b, Q, P), \qquad (2.6)$$

with C_d the volume of the sphere of radius $1/2$ in $\mathbb{R}^{d-1}$. Moreover, we denote by

$$E_* = \overline{H_{scatt}} = \int dQ \, dP \, H_{\text{scatt}}(Q, P) \hat{\rho}(H_{\text{scatt}}(Q, P)). \qquad (2.7)$$

As the distribution ρ is with compact support, the average energy E_* in (2.7) is finite.

In (2.5), the first equation, which describes the momentum of the particle after each collision with a scatterer, is independent of the two others, thus, we can treat it separately.

The transfer of kinetic energy which takes place during a single collision for a particle of momentum $p \in \mathbb{R}^d$ is

$$\Delta E(\|p\|, \kappa) = \frac{1}{2} \left(\|p + R(p, \kappa)\|^2 - \|p\|^2 \right). \tag{2.8}$$

Using (2.3), this energy transfer becomes

$$\Delta E(\|p\|, \kappa) = \alpha \int_0^{t_+} ds \, \dot{Q}(s)\sigma(q(s)). \tag{2.9}$$

It is the Markov chain for the energy transfer that we shall study here:

$$E_{n+1} = E_n + \Delta E(\sqrt{2E_n}, \kappa_n), \tag{2.10}$$

with $\kappa_n = (Q_n, P_n, b_n)$ as before.

We change the parameters (α, M) into (α_*, M) where

$$\alpha_* = \alpha/\sqrt{M}$$

and keep track of the dependence on these parameters in the energy transfer.

The main result of this paper is Theorem 2.2. It is a weak coupling limit on the Markov chain (2.10). The statement is that after an appropriate rescaling of the time variable, the limit of the Markov chain describing the kinetic energy as $\alpha_* \to 0$ and $M \to +\infty$ is a diffusion process which is solution of a well posed martingale problem. But, before giving the full statement of this theorem, we need to expand the term of energy transfer of (2.10) in a high energy regime. This is the aim of Proposition 2.1.

To analyse the weak coupling of the Markov chain (2.10), we need the high energy behaviour of the energy transfer. It was shown in [6] that one can expand ΔE in inverse powers of $\|p\|$ for large E: for all $K \in \mathbb{N}_*$ and for all $(p, \kappa) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$,

$$\Delta E(\|p\|, \alpha_*, M, \kappa) = \sum_{\ell=0}^{K} \frac{\beta^{(\ell)}(\alpha_*, M, \kappa)}{\|p\|^{\ell}} + O\left(\|p\|^{-(K+1)}\right). \tag{2.11}$$

As the time spent by the particle in a scatterer is of order $\|p\|^{-1}$, it appears that, for all $\alpha > 0$ and for all κ,

$$\beta^{(0)}(\alpha_*, M, \kappa) = 0.$$

The leading coefficients of this expansion were computed in [6]. In the following proposition, we provide detailed control on the error terms, in particular their behaviour in M and in α_*.

Thus, we have the following result on the energy transfer (2.9).

Proposition 2.1 *For all $\alpha_* > 0$, $M \gg 1$, $\kappa = (P, Q, b)$ and $p \in \mathbb{R}^d$ such that $p \cdot b = 0$, we have*

$$\Delta E(\|p\|, \alpha_*, M, \kappa) = \alpha_* \left(\frac{\beta^{(1)}(\kappa)}{\|p\|} + O_0\left(M^{-1/2}\|p\|^{-2}\right) + O\left(M^{-3/2}\|p\|^{-4}\right) \right)$$

$$+ \alpha_*^2 \left(\frac{\delta\beta^{(2)}(\kappa)}{\|p\|^2} + O_0\left(M^{1/2}\|p\|^{-3}\right) + \frac{\delta\beta^{(4)}(\kappa)}{\|p\|^4} + O\left(M^{-1/2}\|p\|^{-5}\right) \right)$$

$$+ o(\alpha_*^2 \sqrt{M}), \tag{2.12}$$

where

$$\beta^{(1)}(\kappa) = \frac{P}{\sqrt{M}} L_0(\|b\|) \ and \ \overline{\beta^{(1)}} = 0, \tag{2.13}$$

$$\delta\beta^{(2)}(\kappa) = -\frac{L_0(\|b\|)^2}{2} \quad \text{and} \quad \overline{\delta\beta^{(2)}} = -\frac{\overline{L_0^2}}{2}, \tag{2.14}$$

$$\overline{\delta\beta^{(4)}} = \Sigma_1^2 \left(\frac{d-3}{2} + \frac{M\overline{U''}}{\overline{P^2}} \frac{\overline{\int_0^1 d\lambda \int_0^\lambda d\lambda' K_0(\lambda', \|b\|)\sigma(b+\lambda'e)}}{\overline{L_0^2}} \right), \quad e = \frac{p}{\|p\|} \tag{2.15}$$

with $\Sigma_1^2 = \overline{(\beta^{(1)})^2} = \frac{\overline{P^2}}{M}\overline{L_0^2}$, and

$$L_0(\mu, \|b\|) = \int_0^\mu d\lambda\, \sigma\left(b + \left(\lambda - \frac{1}{2}\right)e\right) \quad \text{and} \quad K_0(\|b\|) = \int_0^1 d\lambda\, L_0(\lambda, \|b\|).$$

We denote by $O_0(\|p\|^{-k})$ a term of order $O(\|p\|^{-k})$ of zero average with respect to the random variable κ, see (2.6).

The proof of Proposition 2.1 is given in Sect. 3.

Note that, the coefficients $\beta^{(1)}(\kappa)$, $\delta\beta^{(2)}(\kappa)$ and $\delta\beta^{(4)}(\kappa)$ only depend on κ and not on (α_*, M). By Proposition 2.1, we can write the transfer of the particle's kinetic energy as

$$\frac{\|p_{n+1}\|^2}{2} = \frac{\|p_n\|^2}{2} + \alpha_* \left(\frac{\beta^{(1)}(\kappa_n)}{\|p_n\|} + O_0(M^{-1/2}\|p_n\|^{-2}) + O(M^{-3/2}\|p_n\|^{-4}) \right)$$

$$+ \alpha_*^2 \left(\frac{\delta\beta^{(2)}(\kappa_n)}{\|p_n\|^2} + \frac{\delta\beta^{(4)}(\kappa_n)}{\|p_n\|^4} + O_0(M^{1/3}\|p_n\|^{-3}) + O(M^{-1/2}\|p_n\|^{-5}) \right)$$

$$+ o(\alpha_*^2\sqrt{M}). \tag{2.16}$$

We observe that the dominating term of order α_*^2 is negative. As the kinetic energy of the particle is a positive quantity, we have to avoid that this term leads to a negative energy. Moreover, even if the energy is positive it is not sufficient to describe its dynamics by a Markov chain. When the particle is in a low energy state, we do not have any information on its dynamics. It can spend a very long time in scatterer, or be trapped in indefinitely. In this specific case, a description by a Markov chain can not be done. The mechanisms of the dynamics at low energy is a complicated subject, so we will introduce a critical value $\xi_* \gg 0$ and force the Markov chain to remain higher than this critical value using a reflection principle over ξ_+.

This cut off in the description of the kinetic energy is a drawback in our model but it is necessary to have a Markov chain description. We proceed as follow: let $\xi_+ \gg 0$ and let F be a function such that for all $x \geq \xi_+$

$$F(x, \alpha_*, M, \kappa) = x + \alpha_* \left(\frac{\beta^{(1)}(\kappa)}{(2x)^{1/2}} + O_0(M^{-1/2}x^{-1}) + O(M^{-3/2}x^{-2}) \right)$$

$$+ \alpha_*^2 \left(\frac{\delta\beta^{(2)}(\kappa)}{x} + \frac{\delta\beta^{(4)}(\kappa)}{x^2} + O_0(M^{1/3}x^{-3/2}) + O(M^{-1/2}x^{-5/2}) \right)$$

$$+ o(\alpha_*^2\sqrt{M}), \tag{2.17}$$

and we will consider the Markov chain $(E_n(\alpha_*, M))_n$ with initial condition

$$\forall\alpha_* > 0 \text{ and } \forall M \gg 1, \quad E_0(\alpha_*, M) = \|p(0)\|^2/2 \gg \xi_+$$

and such that

$$
\begin{cases}
E_{n+1}(\alpha_*, M) = F(E_n, \alpha_*, M, \kappa_n) & \text{if } F(E_n, \alpha_*, M, \kappa_n) \geq \xi_+, \\
E_{n+1}(\alpha_*, M) = 2\xi_+ - F(E_n, \alpha_*, M, \kappa_n) & \text{if } F(E_n, \alpha_*, M, \kappa) < \xi_+,
\end{cases}
\tag{2.18}
$$

to shorten we write E_n instead of $E_n(\alpha_*, M)$ in the function F for a sake of notations. Hence described, the Markov chain $(E_n(\alpha_*, M))$ is a description of the kinetic energy of the particle.

We are now in position to state the weak coupling result we have for the Markov chain $(E_n(\alpha_*, M))_n$ defined by (2.18). We first construct a family of continuous stochastic processes depending on (α_*, M) and built from the Markov chain $(E_n(\alpha_*, M))_n$. For all $n \in \mathbb{N}$, let $\tau_n = \alpha_*^2 n$ be the time scale parameter. Then, for all $n \in \mathbb{N}$ and for all $\tau \in [\tau_n, \tau_{n+1})$, we define

$$
E(\alpha_*, M, \tau) = \frac{\tau_{n+1} - \tau}{\alpha_*^2} E_n(\alpha_*, M) + \frac{\tau - \tau_n}{\alpha_*^2} E_{n+1}(\alpha_*, M).
$$

Thus, for all $\alpha_* > 0$, $M \gg 1$, the process $(E(\alpha_*, M, \tau))_\tau$ is continuous. Note that τ corresponds to a number of collision and not to a time. The following result holds for the family $(E(\alpha_*, M, \tau), \tau \in [0, 1])_{(\alpha_*, M)}$.

Theorem 2.2 *The family of stochastic processes* $(E(\alpha_*, M, \tau), \tau \in [0, 1])_{(\alpha_*, M)}$ *converges weakly, for* $\alpha_* \to 0$ *and* $M \to +\infty$*, to the unique solution of the martingale problem associated to the operator* $(\mathcal{L}, \mathcal{D}_*)$ *with*

$$
\mathcal{D}_* = \left\{ f \in \mathcal{C}^\infty([\xi_+, +\infty)), \lim_{x \to \xi_+} f'(x) = 0 \right\},
\tag{2.19}
$$

$$
\mathcal{L}f(x) = \frac{1}{2} \frac{\Sigma_1^2}{2x} f''(x) - \left(\frac{\overline{\delta\beta_+^{(2)}}}{2x} + \frac{\overline{\delta\beta^{(4)}}}{4x^2} \right) f'(x).
\tag{2.20}
$$

Hence, $(\mathcal{L}, \mathcal{D}_*)$ is a core for the infinitesimal generator of the process generated by $\mathcal{L}$ with reflection over ξ_+. We refer to [7,8] for more details on the cores for infinitesimal generator. The proof of this theorem is given in Sect. 3.

3 Stationary Distribution

The probability distribution $\rho(x, \tau)$ of the limiting process satisfies the Fokker–Plank equation with a reflection condition in ξ_+. Denoting by J the probability current (or flow, see [9]), ρ is solution of

$$
\begin{cases}
\dfrac{\partial}{\partial \tau} \rho(x, \tau) = -\dfrac{\partial}{\partial x} J(x, \tau) \\
J(\xi_+, \tau) = 0, \qquad \forall \tau > 0,
\end{cases}
\tag{3.1}
$$

where

$$
J(x, \tau) = -\frac{\partial}{\partial x} (D_1(x)\rho(x, \tau)) + D_2(x)\rho(x, \tau)
\tag{3.2}
$$

$$
D_1(x) = \frac{1}{2} \frac{\Sigma_1^2}{2x}
\tag{3.3}
$$

$$
D_2(x) = \frac{\overline{\delta\beta^{(2)}}}{2x} + \frac{\overline{\delta\beta^4}}{4x^2}.
\tag{3.4}
$$

We are interested in the distribution of the momentum ρ_{eq} when the system has reach is equilibrium state. This distribution does not depend on the time any more, i.e $\partial\tau\rho_{eq} \equiv 0$. Hence, it remains to resolve

$$J(x) = k \tag{3.5}$$

where k is a constant. As $J(\xi_+) = 0$, the stationary distribution ρ_{eq} is solution of

$$\forall x \geq \xi_+, \quad \frac{\partial}{\partial x}\left(D_1(x)\rho_{eq}(x)\right) + D_2(x)\rho_{eq}(x) = 0. \tag{3.6}$$

It yields that

$$\rho_{eq}(x) = \frac{\mathcal{N}_0}{D_1(x)}\exp\left(\int_{\xi_+}^x \mathrm{d}y\, \frac{D_2(y)}{D_1(y)}\right)$$

$$= \mathcal{N}(\xi_+)x^{1+\frac{\overline{\delta\beta^{(4)}}}{\Sigma_1^2}}\exp\left(\frac{2\overline{\delta\beta^{(2)}}}{\Sigma_1^2}x\right) \tag{3.7}$$

$$= \mathcal{N}(\xi_+)x^{\frac{d-1}{2}}x^{\frac{C}{k_{scatt}}}\exp\left(-\frac{1}{k_{scatt}}x\right) \tag{3.8}$$

with

$$C = \overline{U''}\frac{\overline{\int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\lambda'\, K_0(\lambda')\sigma(b+\lambda'e)}}{\overline{L_0^2}}$$

and with $\mathcal{N}_0$ a normalisation constant and $\mathcal{N}(\xi_+)$ a normalisation constant depending on ξ_+ and $k_{scatt}^2 = \overline{P^2}/M$. The change of variable from $E = \|p\|^2/2$ to $\|p\|$, gives,

$$\hat{\rho}_{eq}(\|p\|) = \|p\|\hat{\rho}_{eq}(\|p\|^2/2)$$

$$= \tilde{\mathcal{N}}(\xi_+)\|p\|^d\|p\|^{2C/k_{scatt}^2}\exp\left(-\frac{1}{2k_{scatt}^2}\|p\|^2\right). \tag{3.9}$$

This distribution is associated to the number of collisions for which the final momentum of the particle is $\|p\|$ when the system is in an equilibrium state. In order to have a density which describes the state of the particle with respect to the time and not to the number of collisions, we remark that the mean time that the particle of velocity $\|p\|$ spends between two consecutive collisions is $\ell_*/\|p\|$. As a result, the probability density at the equilibrium $\hat{\rho}_{eq}(\|p\|)$, describing the time $\rho_{eq}(\|p\|)\mathrm{d}\|p\|$ that the particle spends with a momentum between $\|p\|$ and $\|p\| + \mathrm{d}\|p\|$, satisfies the relation

$$\hat{\rho}_{eq}(\|p\|) = \frac{\ell_*}{\|p\|}\rho_{eq}(\|p\|).$$

Finally, we have

$$\hat{\rho}_{eq}(\|p\|) = \mathcal{N}(\xi_+)\|p\|^{d-1+\frac{2C}{k_{scatt}^2}}\exp\left(-\frac{\|p\|^2}{2k_{scatt}^2}\right). \tag{3.10}$$

For high values of k_{scatt}, this stationary distribution of the particle's momentum (3.10), approaches the Maxwell–Boltzmann distribution with effective temperature T such that

$$k_B T = k_{scatt}^2,$$

with k_B the Boltzmann's constant.

We end with a remark on the differences between the results presented here and the ones of [6]. With our notations, the Markov chain description of [6] is equivalent to

$$\frac{\|p_{n+1}\|^2}{2} = \frac{\|p_n\|^2}{2} + \alpha_* \frac{\beta^{(1)}(\kappa_n)}{\|p_n\|} + \alpha_*^2 \left(\frac{\delta\beta^{(2)}(\kappa)}{\|p_n\|^4} + \frac{\delta\beta^{(4)}(\kappa)}{\|p_n\|^4} \right). \tag{3.11}$$

Despite some similarities at a first sight with (2.16) and between the convergence of these two chains to the same Maxwellian distribution, the time scaling of the convergence for (3.11) is not the same that the one needed here. Indeed, the chain (2.16) contains the big O terms which depend on inverse power of M. Considering these terms implies that a scaling only in α will give us a stationary distribution depending also on M, and hence, the convergence to the Maxwellian can not be observed.

Appendix: Proof of Proposition 2.1 and of Theorem 2.2

Proof of Proposition 2.1

First, we expand $q(s)$ in s with the initial condition $q = b - \frac{1}{2}e$. Since we always have $s \leq t_+$, which is of order $\|p\|^{-1}$, any term of order $O(s^k)$ is automatically a term of order $O(\|p\|^{-k})$. We have,

$$q(s) = q + ps - \alpha Q \nabla K_0(s\|p\|, \|b\|)\|p\|^{-2} - \alpha \frac{P}{M} \nabla K_1(s\|p\|, \|b\|)\|p\|^{-3} + O(s^4),$$

where, for $b \cdot e = 0$, we define

$$L_k(\mu, \|b\|) = \int_0^\mu d\lambda\, \sigma(b + (\lambda - \frac{1}{2})e)\lambda^k, \quad \text{and } K_k(\mu, \|b\|) = \int_0^\mu d\lambda\, L_k(\lambda, \|b\|).$$

Hence, we proceed as in [6] to write ΔE, defined in (2.9), as follow

$$\Delta E(\|p\|, \alpha, M, \kappa) = \Delta E_a(\|p\|, \alpha, M, \kappa) + $$
$$\Delta E_b(\|p\|, \alpha, M, \kappa) + \Delta E_c(\|p\|, \alpha, M, \kappa), \tag{4.1}$$

with

$$\Delta E_a(\|p\|, \alpha, M, \kappa) = \alpha \int_0^{t_+} ds\, \dot{Q}(s)\sigma(q + ps),$$

$$\Delta E_b(\|p\|, \alpha, M, \kappa) = -\frac{\alpha^2 Q}{\|p\|^2} \int_0^{t_+} ds\, \dot{Q}(s)\nabla K_0(s\|p\|, \|b\|) \cdot \nabla\sigma(q + ps),$$

$$\Delta E_c(\|p\|, \alpha, M, \kappa) = -\frac{\alpha^2 P}{M\|p\|^3} \int_0^{t_+} ds\, \dot{Q}(s)\nabla K_1(s\|p\|, \|b\|) \cdot \nabla\sigma(q + ps).$$

As we replace the coupling constant α by $\alpha_* = \alpha/\sqrt{M}$ and as the quantity $P^2/2M$ is the kinetic energy of the scatterers, we write, in what follows $P/\sqrt{M} = k_{scatt}$. Then, with these notations,

$$\Delta E_a(\|p\|, \alpha_*, M, \kappa) = \alpha_* \sqrt{M} \int_0^{t_+} ds\, \dot{Q}(s)\sigma(q + ps), \tag{4.2}$$

$$\Delta E_b(\|p\|, \alpha_*, M, \kappa) = -\frac{\alpha_*^2 M Q}{\|p\|^2} \int_0^{t_+} ds\, \dot{Q}(s)\nabla K_0(s\|p\|, \|b\|) \cdot \nabla\sigma(q + ps), \tag{4.3}$$

$$\Delta E_c(\|p\|, \alpha_*, M, \kappa) = -\frac{\alpha_*^2 k_{scatt}}{\sqrt{M}\|p\|^3} \int_0^{t_+} ds \, \dot{Q}(s) \nabla K_1(s\|p\|, \|b\|) \cdot \nabla \sigma(q + ps). \quad (4.4)$$

We need to expand in s each of these terms in order to determine the coefficients of order $\|p\|^{-\ell}$ in (2.11), $\ell \geq 1$. We have already noted that $\beta^{(0)} \equiv 0$ because t_+ is of order $\|p\|^{-1}$. Moreover, since $0 \leq s \leq t_+$ and t_+ is of order $\|p\|^{-1}$, to obtain the contributions $\beta_a^{(\ell)}$, $\beta_b^{(\ell)}$, $\beta_c^{(\ell)}$ from ΔE_a, ΔE_b and ΔE_c requires that we expand the integrand of the latter.

We first compute ΔE_a.

We expand $\dot{Q}(s)\sigma(q + ps)$ in s and we identify the terms which do not depend on α_* and those which are of order α_*. We have, for any $N \geq 1$,

$$\dot{Q}(s)\sigma(q + ps) = \sum_{n=0}^{N} \sum_{k=0}^{n} C_n^k \, Q^{(k+1)}(0)s^k \frac{d^{n-k}}{ds^{n-k}} \sigma(q + ps)s = 0 + O(s^{N+1}), \quad (4.5)$$

where, for all $n \leq N$, for all $k \leq n$, C_n^k is the binomial coefficient. To identify the terms of order α_* in ΔE_a, it remains to identify those that not depend on α_* in (4.5).

Here and in what follows, whenever a function f depends on α_*, and possibly on other variables such as $\|p\|$, M and κ, we shall write $f(\alpha_* = 0)$ for the values of the function f on the hyperplane $\{\alpha_* = 0\}$ and $\delta f := (f - f(\alpha_* = 0) - o(\alpha_*))/\alpha_*$ for the part of order α_* of the function f.

Then, in order to identify the terms which do not depend on α_* in (4.5), we have to compute $Q^{(\ell)}(0, \alpha_* = 0)$, $\ell \geq 1$. One observes that ΔE_b and ΔE_c do not contribute to the coefficients $\beta^{(1)}$ and $\beta^{(2)}$, consequently $\beta^{(1)} \equiv \beta_a^{(1)}$ and $\beta^{(2)} \equiv \beta_a^{(2)}$. Taking the terms of order s^0 in (4.5), we easily obtain, as $\dot{Q}(0) = k_{scatt}/\sqrt{M}$

$$\beta^{(1)}(\kappa, \alpha_* = 0) = k_{scatt} L_0(\|b\|),$$

with $L_0(\|b\|) = \int_0^1 d\lambda \sigma(b + (\lambda - \frac{1}{2})e)$. As $\beta^{(1)}(\kappa, \alpha_* = 0)$ is linear in P and hence of zero average in any stationary distribution ρ, $\overline{\beta^{(1)}(\alpha_* = 0)} = 0$. Moreover, $\overline{\beta^{(2)}(\alpha_* = 0)} = 0$, indeed, this term is obtained by the term of order s^1 in (4.5), we see that they behave as $\ddot{Q}(0, \alpha_* = 0)$ which is such that $\overline{\ddot{Q}(0, \alpha_* = 0)} = 0$, since ρ is stationary for the free dynamics of the scatterer generated by H_{scatt}.

Also, we note that $Q^{(3)}(0, \alpha_* = 0)$ is linear in P, which imply that $\overline{\beta_a^{(3)}(\alpha_* = 0)} = 0$. Finally, for all $k \geq 1$, we easily obtain

$$\sup_{\kappa} \left| Q^{(k+1)}(0, \alpha_* = 0) \right| \leq \Phi_k(k_{scatt})M^{-(k+1)/2}, \quad (4.6)$$

with $\Phi_k(k_{scatt})$ a polynomial function of k_{scatt} of degree inferior to k. Let

$$\Delta E_{a1} = \alpha_* \sqrt{M} \int_0^{t_+} ds \, \dot{Q}(s, \alpha_* = 0)\sigma(q + ps),$$

be the part of order α_* of ΔE_a. Then, by (4.6), it yields that

$$\Delta E_{a1}(\|p\|, \alpha_*, M, \kappa) = \alpha_* \left(\frac{\beta^{(1)}(\kappa, \alpha_* = 0)}{\|p\|} + O_0(M^{-1/2}\|p\|^{-2}) + O(M^{-3/2}\|p\|^{-4}) \right),$$

$$(4.7)$$

Ones observe that as the coefficients $\beta^{(2)}(\kappa, \alpha_* = 0)$ and $\beta^{(3)}(\kappa, \alpha_* = 0)$ are of zero average, they are contained in the term $O_0(M^{-1/2}\|p\|^{-2})$.

Now, we identify the terms of order α_* in (4.5), it remains to compute $\delta Q^{(k+1)}(0)$ for $k \geq 0$. First, only $\dot{Q}(0)$ contributes to the terms of order s^0 in (4.5) and hence to the coefficient $\beta^{(1)}(\kappa)$. As $\dot{Q}(0)$ does not depend on α_*, $\delta\beta^{(1)}(\kappa) = 0$ and

$$\beta^{(1)}(\kappa) = \beta^{(1)}(\kappa, \alpha_* = 0).$$

Thus, as $\dot{Q}(0)$ does not depend on α_*, $\delta\dot{Q}(0) = 0$.

Moreover, we have

$$\delta\ddot{Q}(0) = -\frac{1}{M}\sigma(q + ps)s = 0$$

and, by successive derivations of $Q(s)$, using (1.4), identifying the term on the hyperplane $\{\alpha_* = 0\}$ and isolating the quantities depending on inverse power of M from k_{scatt}, we obtain for $k \geq 2$

$$\overline{\delta Q^{(k+1)}(0)} = \frac{1}{M}\|p\|^{k-1} + \sum_{j=0}^{k-3} \frac{\tilde{\Phi}_k(k_{scatt})}{M^{(k-j+1)/2}}\|p\|^j, \tag{4.8}$$

where $\tilde{\Phi}_k(k_{scatt})$ is equal to k_{scatt}^m with $0 \leq m < k$. Then, combining with (4.5), and identifying the terms of order s which contribute to $\delta\beta_a^{(2)}(\kappa)$, it yields that, factorising by M^{-1} (which goes to α_*^2),

$$\delta\beta^{(2)}(\kappa) = -M^{-1/2}L_0(\|b\|)^2.$$

Ones can observe that, for all $k \geq 0$, $\overline{\delta Q^{(k+1)}(0)}$ has no term of order $\|p\|^{-(k-2)}$ and hence, no term of order s^2 in $\delta\left(\dot{Q}(s)\sigma(q + ps)\right)$. It yields that

$$\delta\beta_a^{(3)}(\kappa) = 0.$$

Moreover, controlling the terms of order more that α_*^2 and combining (4.2), (4.7), (4.5) and (4.8), we obtain

$$\Delta E_a(\|p\|, \alpha_*, M, \kappa) = \alpha_*\left(\frac{\beta^{(1)}(\kappa)}{\|p\|} + O_0\left(M^{-1/2}\|p\|^{-2}\right) + O\left(M^{-3/2}\|p\|^{-4}\right)\right)$$

$$+ \alpha_*^2\left(\frac{\delta\beta^{(2)}(\kappa)}{\|p\|^2} + \frac{\delta\beta_a^{(4)}(\kappa)}{\|p\|^4} + O\left(M^{-3/2}\|p\|^{-5}\right)\right) + o(\alpha_*^2\sqrt{M}), \tag{4.9}$$

where $\delta\beta_a^{(4)}(\kappa)$ not explicitly computed, but we can have the following estimate by identifying the terms of order s^3 in (4.5)

$$\sup_\kappa\left|\delta\beta_a^{(4)}\right| \leq \Phi_3(k_*)M^{-1},$$

and the error term is controlled by $o(\alpha_*^2\sqrt{M})$.

As ΔE_b already depends on α_*^2, we just expand its integrand in s and identifies the terms which do not depend on α_*. It remains to expand in s

$$\dot{Q}(s, \alpha_* = 0)\nabla K_0(s\|p\|, \|b\|) \cdot \nabla\sigma(q + ps).$$

We proceed similarly as for ΔE_a and we obtain

$$\dot{Q}(s)\nabla K_0(s\|p\|, \|b\|) \cdot \nabla\sigma(q + ps)$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{n} C_n^k Q^{(k+1)}(0) s^k \frac{\mathrm{d}^{n-k}}{\mathrm{d}s^{n-k}} \nabla K_0(s\|p\|, \|b\|) \cdot \nabla \sigma(q + ps)s = 0 + O(s^{N+1}), \quad (4.10)$$

where, for all $n \leq N$, for all $k \leq n$, $C_{k,n}$ is a constant. Then we identify the terms of order α_*^2 in ΔE_b, it remains to identify those that do not depend on α_* in (4.10). For that, we need $Q^{(k+1)}(0, \alpha_* = 0)$ already given in (4.6). As ΔE_b does not generate any contribution f order $\|p\|^{-1}$ and $\|p\|^{-2}$, combining (4.3), (4.10) and (4.6) gives us

$$\Delta E_b = \alpha_*^2 \left(O_0(\sqrt{M}\|p\|^{-3} + \frac{\beta_b^{(4)}(\kappa, \alpha_* = 0)}{\|p\|^{-4}} + O\left(M^{-1/2}\|p\|^{-5}\right) \right) + o(\alpha_*\sqrt{M}).$$
$$(4.11)$$

Finally, we proceed in the same way for ΔE_c. One observes that ΔE_c contributes only to the term of order $\|p\|^{-\ell}$, $\ell \geq 4$. We obtain

$$\Delta E_c = \alpha_*^2 \left(\frac{\beta_c^{(4)}(\kappa, \alpha = 0)}{\|p\|^{-4}} + O\left(M^{-1/2}\|p\|^{-5}\right) \right) + o(\alpha_*\sqrt{M}). \quad (4.12)$$

Moreover, it is shown in [6] that

$$\overline{\delta\beta_a^{(4)}} + \overline{\beta_b^{(4)}(\alpha = 0)} + \overline{\beta_c^{(4)}(\alpha = 0)}$$

$$= \Sigma_1^2 \left(\frac{d-3}{2} + \frac{M\overline{U''}}{P^2} \frac{\int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\lambda' K_0(\lambda', \|b\|)\sigma(b + \lambda'e)}{L_0^2} \right). \quad (4.13)$$

Finally, (4.9), (4.11), (4.12) and (4.13) yield the result.

3.1 Proof of Theorem 2.2

In this proof, we replace the notation of the Markov chain $(E_n(\alpha_*, M))_n$ describing the kinetic energy of the particle in (2.18) by $(\xi_n(\alpha_*, M))_n$.

The scheme of the proof of Theorem 2.2 is quite classical. We first give and show a result that will allow us to conclude to the existence of a converging sub-families. Then we will show that all these converging sub-families have the same limit. Thus, we can conclude that it is the whole family that converges to this limiting process. First of all, we need the following lemma.

Lemma 3.1 *For all* $\alpha_* > 0$ *and* $M \gg 1$, *let* $\Pi_{(\alpha_*, M)}$ *be the transition function of the Markov chain* $(\xi(\alpha_*, M, \tau_n))_n$ *and let* $\left(\Pi_{(\alpha_*, M)} - I\right)$ *be its generator.*

i) *Let* $(\mathcal{L}, \mathcal{D}_*)$ *the operator defined in* (2.19)-(2.20). *For all* $f \in \mathcal{D}_*$ *and for* $x \in [\xi_+, +\infty)$

$$\lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} \left| \mathcal{L}f(x) - \frac{1}{\alpha_*^2}(\Pi_{(\alpha_*, M)} - I)f(x) \right| = 0.$$

ii) *The family* $(\xi(\alpha_*, M, \tau), \ \tau \in [0, 1])_{(\alpha_*, M)}$ *is pre-compact.*

Proof For all $x \in [\xi_+, +\infty)$, consider the following coefficients

$$a_{(\alpha_*, M)}(x) = \int_{[\xi_+, +\infty)} (y - x)^2 \Pi_{(\alpha_*, M)}(x, \mathrm{d}y) \quad (4.14)$$

$$b_{(\alpha_*, M)}(x) = \int_{[\xi_+, +\infty)} (y - x) \Pi_{(\alpha_*, M)}(x, \mathrm{d}y)$$

and the operator

$$\mathcal{A}_{(\alpha_*,M)}f(x) = \frac{1}{2}a_{(\alpha_*,M)}(x)f''(x) + b_{(\alpha_*,M)}(x)f'(x). \tag{4.15}$$

As the Markov chain $(\xi_n(\alpha_*, M))_n$ is time-homogeneous, the coefficients $a_{(\alpha_*,M)}(x)$ and $b_{(\alpha_*,M)}(x)$ are respectively the momentum of order 1 and 2 of $\xi(\tau_1)$ conditionally to the event $\xi(0) = x$.

Therefore, for all $\alpha_* > 0$ and for all $M \gg 1$ and for all $x \in [\xi_+, +\infty)$, we have

$$\left| \mathcal{L}f(x) - \frac{1}{\alpha_*^2}(\Pi_{(\alpha_*,M)} - I)f(x) \right| \leq \left| \mathcal{L}f(x) - \frac{1}{\alpha_*^2}\mathcal{A}_{(\alpha_*,M)}f(x) \right|$$

$$+ \frac{1}{\alpha_*^2}\left| \mathcal{A}_{(\alpha_*,M)}f(x) - (\Pi_{(\alpha_*,M)} - I)f(x) \right|. \tag{4.16}$$

First, we will compute the limit for $\alpha_* \to 0$ and $M \to +\infty$ for all $x \in [\xi_+, +\infty)$ of $a_{(\alpha_*,M)}(x)$ and $b_{(\alpha_*,M)}(x)$. Using (4.14), we have

$$a_{(\alpha_*,M)}(x) = \mathbb{E}\Big[(F(\alpha_*, M, x, \kappa) - x)^2 \mathbb{1}_{F(\alpha_*,M,x,\kappa)>\xi_+}$$

$$+ (2\xi_+ - F(\alpha_*, M, x, \kappa) - x)^2 \mathbb{1}_{F(\alpha_*,M,x,\kappa)\leq\xi_+} \Big] \tag{4.17}$$

and

$$b_{(\alpha_*,M)}(x) = \mathbb{E}\Big[(F(\alpha_*, M, x, \kappa) - x) \mathbb{1}_{F(\alpha_*,M,x,\kappa)>\xi_+}$$

$$+ (2\xi_+ - F(\alpha_*, M, x, \kappa) - x) \mathbb{1}_{F(\alpha_*,M,x,\kappa)\leq\xi_+} \Big]. \tag{4.18}$$

As the step of the Markov chain is in α_*, for all $x > \xi_+$, there exists $\tilde{\alpha}_*$ such that for all $\alpha_* < \tilde{\alpha}_*$,

$$\mathbb{1}_{F(\alpha_*,M,x,\kappa)>\xi_+} = 1 \qquad \text{almost surely.} \tag{4.19}$$

Thus, by Proposition 2.1, for all $x > \xi_+$, there exists $\tilde{\alpha}_*$ such that for all $\alpha_* < \tilde{\alpha}_*$,

$$a_{(\alpha_*,M)}(x) = \mathbb{E}\left[(F(\alpha_*, M, x, \kappa) - x)^2 \right]$$

$$= \alpha_*^2 \Big(\frac{\Sigma_1^2}{2x} + O(M^{-1}x^{-2}) + O(M^{-3}x^{-4}) + O(M^{-1/2}x^{-3/2})$$

$$+; O(M^{-3/2}x^{-5/2}) + O(M^{-2}x^{-3}) \Big) + o(\alpha_*^2\sqrt{M})$$

Moreover, by Proposition 2.1, for all $x > \xi_+$, there exists $\tilde{\alpha}_*$ such that for all $\alpha_* < \tilde{\alpha}_*$,

$$b_{(\alpha_*,M)}(x) = \mathbb{E}\left(F(\alpha_*, Mx, \kappa) - x \right)$$

$$= \alpha_* O\left(M^{-3/2}x^{-2} \right) + \alpha_*^2 \left(\frac{\overline{\delta\beta^{(2)}}}{2x} + \frac{\overline{\delta\beta^{(4)}}}{4x^2} + O\left(M^{-1/2}x^{-5/2} \right) \right) + o(\alpha_*^2\sqrt{M}),$$

and

$$b(x) = \lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} \frac{1}{\alpha_*}b_{(\alpha_*,M)}(x) = \frac{\overline{\delta\beta^{(2)}}}{2x} + \frac{\overline{\delta\beta^{(4)}}}{4x^2}.$$

For all $x > \xi_+$, let

$$a(x) = \lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} a_{(\alpha_*,M)}(x)/\alpha_*^2 \text{ and } b(x) = \lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} b_{(\alpha_*,M)}(x)/\alpha_*^2,$$

then, using Proposition 2.1, we have

$$a(x) = \frac{\Sigma_1^2}{2x}, \tag{4.20}$$

and

$$b(x) = \frac{\overline{\delta\beta^{(2)}}}{2x} + \frac{\overline{\delta\beta^{(4)}}}{4x^2} \tag{4.21}$$

Now, let $x = \xi_+$, we can easily verify that

$$\lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} \frac{1}{\alpha_*^2} a_{(\alpha_*, M)}(\xi_+) = \frac{\Sigma_1^2}{2\xi_+} = a(\xi_+).$$

However, the drift coefficient does not satisfy

$$\lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} \frac{1}{\alpha_*^2} b_{(\alpha_*, M)}(\xi_+) = b(\xi_+),$$

indeed

$$b_{(\alpha_*, M)}(\xi_+) = \lim_{x \to \xi_+} b_{(\alpha_*, M)}(x) \left(2\mathbb{P}\left(F_{(\alpha_*, M)}(\xi_+, \kappa) \geq 0 \right) - 1 \right)$$
$$+ \xi_+ \left(1 - \mathbb{P}\left(F_{(\alpha_*, M)}(\xi_+, \kappa) \geq 0 \right) \right)$$

and

$$\lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} \frac{1}{\alpha_*^2} b_{(\alpha_*, M)}(\xi_+) \neq \lim_{x \to \xi_+} b(x).$$

Thus, for all $f \in \mathcal{D}_*$, and for all $w \geq \xi_+$,

$$\lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} \left| \mathcal{L}f(x) - \frac{1}{\alpha_*^2} \mathcal{A}_{(\alpha_*, M)} f(x) \right| = 0.$$

It remains to control the second term of (4.16). With a Taylor development of order 2 and using (4.14), we have that for all $x \geq \xi_+$, there exists $z \in (x, y)$ such that

$$\left(\Pi_{(\alpha_*, M)} - I \right) f(x) \leq \mathcal{A}_{(\alpha_*, M)} f(x) + \frac{1}{6} \int_{[\xi_+, +\infty)} f^{(3)}(z)(y - x)^3 \Pi_{(\alpha_*, M)}(x, dy). \tag{4.22}$$

Moreover, by (2.18), we have

$$\left| \int_{[\xi_+, +\infty)} f^{(3)}(z)(y - x)^3 \Pi_{(\alpha_*, M)}(x, dy) \right| \leq \|f^{(3)}\|_\infty \int_{[\xi_+, +\infty)} |y - x|^3 \Pi_{(\alpha_*, M)}(x, dy)$$
$$\leq \|f^{(3)}\|_\infty \Big(\mathbb{E}\left(|F_{(\alpha_*, M)}(x, \kappa) - x|^3 \mathbb{1}_{F_{(\alpha_*, M)}(x, \kappa) \geq \xi_+}\right)$$
$$+ \mathbb{E}\Big(|2\xi_+ + F_{(\alpha_*, M)}(x, \kappa) - x|^3 \mathbb{1}_{F_{(\alpha_*, M)}(x, \kappa) < \xi_+}\Big) \Big).$$

Using (4.19), we obtain

$$\left| \int_{[\xi_+, +\infty)} f^{(3)}(z)(y - x)^3 \Pi_{(\alpha_*, M)}(x, dy) \right| \leq \|f^{(3)}\|_\infty O(\alpha_*^3).$$

Combining the latter with (4.22), we thus obtain that for all $x \geq \xi_+$,

$$\left| \frac{1}{\alpha_*^2} (\Pi_{(\alpha_*,M)} - \mathrm{I}) f(x) - \frac{1}{\alpha_*^2} \mathcal{A}_{(\alpha_*,M)} f(x) \right| \underset{\substack{\alpha_* \to 0 \\ M \to +\infty}}{\longmapsto} 0,$$

which shows the first affirmation.

We, now, show the statement ii). In order to show the pre-compactness of the family $(\xi(\alpha_*, M, \tau), \ \tau \in [0,1])_{(\alpha_*, M)}$, we first remark that by (4.22) and (4.20), (4.21), for all $f \in \mathcal{D}_*$ there exists a constant $\widetilde{C}_f > 0$ such that,

$$\frac{1}{\alpha_*^2} \|(\Pi_{(\alpha_*,M)} - \mathrm{I}) f\|_\infty \leq \widetilde{C}_f. \tag{4.23}$$

It follows that for all $f \in \mathcal{D}_*$,

$$\mathbb{E}\left(f(\xi(\alpha_*, M, \tau_{n+1})) + \widetilde{C}_f \tau_{n+1} \mid \mathcal{F}_n^{\xi(\alpha_*,M)} \right) = f(\xi(\alpha_*, M, \tau_n)) + \widetilde{C}_f \tau_n$$
$$+ \ (\Pi_{(\alpha_*,M)} - \mathrm{I}) f(\xi(\alpha_*, M, \tau_n)) + \widetilde{C}_f \alpha_*^2 \tag{4.24}$$
$$\geq f(\xi(\alpha_*, M, \tau_n)) + \widetilde{C}_f \tau_n. \tag{4.25}$$

Then, the process $\left(f(\xi(\alpha_*, M, \tau_n) + \widetilde{C}_f \tau_n \right)_n$ is a sub-martingale. Next, each step of the Markov chain $(\xi(\alpha_*, M, \tau_n))_n$ has a size of order α_*. So, for all $\delta > 0$, there exists $\tilde{\alpha}_*$ such that, for all $\alpha_* < \tilde{\alpha}_*$, we have

$$\mathbb{P}\left(\left| \xi(\alpha_*, M, \tau_j) - \xi(\alpha_*, M, \tau_{j-1}) \right| > \delta \right) = 0,$$

and,

$$\lim_{\substack{\alpha_* \to 0 \\ M \to +\infty}} \sum_{j=1}^{\lfloor \frac{1}{\alpha_*^2} \rfloor} \mathbb{P}\left(\left| \xi(\alpha_*, M, \tau_j) - \xi(\alpha_*, M, \tau_{j-1}) \right| > \delta \right) = 0, \tag{4.26}$$

By the Theorem 1.4.11 of [12], these two last statements, combined with (4.25), imply the pre-compactness of the family $(\xi(\alpha_*, M, \tau), \ \tau \in [0,1])_{(\alpha_*, M)}$. □

Now, we are able to give a proof of Theorem 2.2.

Proof of Theorem 2.2 statement ii) of Lemma 3.1 allows us to establish the existence of decreasing sequences $\{\alpha_*(k)\}_k$ with positive values such that $\alpha_*(k) \to 0$ as $k \to +\infty$ and such that the sub-families $\left(\xi^{\alpha_*(k)}(\tau), \ \tau \in [0,1] \right)_{k \in \mathbb{N}}$ converge weakly to limiting processes $(\xi(\tau))_{\tau \in [0,1]}$. Consequently, these processes take values in $[\xi_+, +\infty)$. Then, Skorohod's Representation Theorem implies the existence of a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and of processes

$$\left(\widetilde{\xi}(\alpha_*(k), M, \tau), \ \tau \in [0,1] \right)_{k \in \mathbb{N}} \quad \text{and} \quad \left(\widetilde{\xi}(\tau) \right)_{\tau \in [0,1]}$$

respectively with same distribution

$$(\xi(\alpha_*(k), M, \tau), \ \tau \in [0,1])_{k \in \mathbb{N}} \quad \text{and} \quad (\xi(\tau))_{\tau \in [0,1]},$$

and such that

$$\sup_{\tau \in [0,1]} \left| \widetilde{\xi}(\tau) - \widetilde{\xi}(\alpha_*(k), M, \tau) \right| \longmapsto_{k \to +\infty} 0, \qquad \widetilde{\mathbb{P}}\text{-almost surely.}$$

Lemma 3.2 *For all $f \in \mathcal{D}_*$. We have the two following properties on the Skorohod's space.*

(i) *Let $0 \le u_1 < u_2 \le 1$, then,*

$$f\left(\tilde{\xi}(\alpha_*(k), M, \lfloor \frac{u_2}{\alpha_*(k)^2} \rfloor \alpha_*(k)^2\right) - f\left(\tilde{\xi}(\alpha_*(k), M, \lfloor \frac{u_1}{\alpha_*(k)^2} \rfloor \alpha_*(k)^2\right)$$
$$- \sum_{j=\lfloor \frac{u_1}{\alpha_*(k)^2} \rfloor}^{\lfloor \frac{u_2}{\alpha_*(k)^2} \rfloor - 1} (\Pi_{(\alpha_*(k), M)} - \mathrm{I}) f\left(\tilde{\xi}(\alpha_*(k), M, \tau_j)\right)$$

converges $\widetilde{\mathbb{P}}$ almost surely as $k \to +\infty$, to

$$f\left(\tilde{\xi}(u_2)\right) - f\left(\tilde{\xi}(u_1)\right) - \int_{u_1}^{u_2} \mathcal{L} f\left(\tilde{\xi}(s)\right) ds, \qquad \widetilde{\mathbb{P}}\text{-p.s.}$$

(ii) *The limit process $\left(\widetilde{M}(\tau)\right)_\tau$,*

$$\widetilde{M}(\tau) = f\left(\tilde{\xi}(\tau)\right) - \int_0^\tau \mathcal{L} f\left(\tilde{\xi}(s)\right) ds$$

is a $\mathcal{F}_\tau^{\tilde{\xi}}$-martingale.

Coming back to the initial probability space, this lemma assures that all processes $(\xi(\tau))_\tau$ are solutions of the martingale problem associated to the operator $(\mathcal{L}, \mathcal{D}_*)$ defined in (2.19) and (2.20). We easily check that the coefficients $a(x)$ and $b(x)$ of the operator $(\mathcal{L}, \mathcal{D}_*)$ are Lipschitz on the interval $[\xi_+, +\infty)$, hence, the martingale problem associated to this operator is well-posed.

As the coefficients of this martingale problem are Lipschitz on $[\xi_+, +\infty)$, it is well posed. As a result, the limit processes are all equal in distribution. In order to conclude this proof, we show that it is not only the sub-families that converge weakly to the solution of this martingale problem but the whole family $(\xi(\alpha_*, M, \tau), \tau \in [0, 1])_{(\alpha_*, M)}$. To this end, we will make a reductio ad absurdum. Suppose that there exist $\phi \in \mathcal{C}([0, 1], \mathbb{R})$, $\varepsilon > 0$ and a decreasing sequence $\{\alpha_*(k)\}_k$ of positive numbers which tends to 0 as $k \to +\infty$ such that, for all $k \in \mathbb{N}$,

$$|\mathbb{E}\left(\phi(\xi(\alpha_*(k), M, \cdot)\right) - \mathbb{E}\left(\phi(\xi(\cdot))\right)| > \varepsilon. \tag{4.27}$$

But the family $(\xi(\alpha_*(k), M, \tau), \tau \in [0, 1])_{k \in \mathbb{N}}$ is still pre-compact. Then, we again can extract a converging sub-sequence $\{\alpha_*(\varphi(k))\}_k$ such that $(\xi(\alpha_*(\varphi(k)), M, \tau), \tau \in [0, 1])_{k \in \mathbb{N}}$ converges weakly. According to the foregoing, the limit process of this family is $(\xi(\tau))_{\tau \in [0,1]}$. Then,

$$\lim_{k \to +\infty} \mathbb{E}\left(\phi(\xi(\alpha_*(\varphi(k)), M, \cdot)\right) = \mathbb{E}\left(\phi(\xi(\cdot))\right),$$

and (4.27) is absurd. Such a function ϕ does not exist and it concludes this proof. □

This appendix ends with the proof of Lemma 3.2 used in that of Theorem 2.2.

Proof of Lemma 3.2 i) The limiting processes $\tilde{\xi}$ are continuous in time as limit almost sure of $\tilde{\xi}(\alpha_*(k), M, \cdot)$ which are continuous in time. Then, for all $f \in \mathcal{D}_*$,

$$\left| f\left(\tilde{\xi}(\tau)\right) - f\left(\tilde{\xi}(\alpha_*(k)), M, \lfloor\frac{\tau}{\alpha_*(k)^2}\rfloor\alpha_*(k)^2\right)\right| \le ||f'||_\infty\left(\left|\tilde{\xi}(\tau) - \tilde{\xi}\left(\lfloor\frac{\tau}{\alpha_*(k)^2}\rfloor\alpha_*(k)^2\right)\right|\right.$$

$$+ \left.\left|\tilde{\xi}\left(\lfloor\frac{\tau}{\alpha_*(k)^2}\rfloor\alpha_*(k)^2\right) - \tilde{\xi}\left(\alpha_*(k), M, \lfloor\frac{\tau}{\alpha_*(k)^2}\alpha_*(k)^2\rfloor\right)\right|\right)$$

$$\xrightarrow[k\to+\infty]{} 0 \qquad \mathbb{P} - \text{a.s.}$$

It remains to control, for all $f \in \mathcal{D}_*$,

$$\int_{u_1}^{u_2} \mathcal{L}f\left(\tilde{\xi}(s)\right)ds - \sum_{j=\lfloor\frac{\tau}{\alpha_*(k)^2}\rfloor}^{\lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor-1} (\Pi_{(\alpha_*(k),M)} - I)f\left(\tilde{\xi}(\alpha_*(k), M, \tau_j)\right)$$

We have

$$\left|\int_{u_1}^{u_2} \mathcal{L}f\left(\tilde{\xi}(s)\right)ds - \sum_{j=\lfloor\frac{u_1}{\alpha_*(k)^2}\rfloor}^{\lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor-1} (\Pi_{(\alpha_*(k),M)} - I)f\left(\tilde{\xi}(\alpha_*(k), M, \tau_j)\right)\right|$$

$$\le \left|\int_{u_1}^{u_2} \mathcal{L}f\left(\tilde{\xi}(s)\right)ds - \alpha_*(k)^2\sum_{j=\lfloor\frac{u_1}{\alpha_*(k)^2}\rfloor}^{\lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor-1} \mathcal{L}f\left(\tilde{\xi}(\tau_j)\right)\right|$$

$$+ \alpha_*(k)^2\left|\sum_{j=\lfloor\frac{u_1}{\alpha_*(k)^2}\rfloor}^{\lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor-1} \mathcal{L}f\left(\tilde{\xi}(\tau_j)\right) - \sum_{j=\lfloor\frac{u_1}{\alpha_*(k)^2}\rfloor}^{\lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor-1} \mathcal{L}f\left(\tilde{\xi}(\alpha_*(k), M, \tau_j)\right)\right|$$

$$+ \alpha_*(k)^2\left|\sum_{j=\lfloor\frac{u_1}{\alpha_*(k)^2}\rfloor}^{\lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor-1} \mathcal{L}f\left(\tilde{\xi}(\alpha_*(k), M, \tau_j)\right)\right.$$

$$\left. - \frac{1}{\alpha_*(k)^2}\sum_{j=\lfloor\frac{u_1}{\alpha_*(k)^2}\rfloor}^{\lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor-1} (\Pi_{(\alpha_*(k),M)} - I)f\left(\tilde{\xi}(\alpha_*(k), M, \tau_j)\right)\right|.$$

The first term is an approximation of the integral and consequently tends to 0 as $k \to +\infty$. The convergence almost surely of $\tilde{\xi}(\alpha_*(k), M, \cdot)$ to $\tilde{\xi}(\cdot)$ implies that the second term tends to 0 too as $k \to +\infty$. The statement i) of Lemma 3.1 implies the convergence to 0 as $k \to +\infty$ of the last term. For all $f \in \mathcal{D}_*$, let the process $\left(\tilde{S}(\alpha_*(k), M, \tau_n)\right)_n$ be defined by

$$\tilde{S}(\alpha_*(k), M, \tau_n) = f\left(\tilde{\xi}(\alpha_*(k), M, \tau_n)\right) - \sum_{j=0}^{n-1}(\Pi_{(\alpha_*(k),M)} - I)f\left(\tilde{\xi}(\alpha_*(k), M, \tau_j)\right).$$

Furthermore, for all $f \in \mathcal{D}_*$, the process $(S(\alpha_*(k), M, t_n))_n$ is a $\mathcal{F}_n^{\xi(\alpha_*(k),M)}$- martingale. Then,

$$\left|\mathbb{E}\left[\left(\tilde{S}(\alpha_*(k), M, \lfloor\frac{u_2}{\alpha_*(k)^2}\rfloor\alpha_*(k)^2) - \tilde{S}(\alpha_*(k), M, \lfloor\frac{u_1}{\alpha_*(k)^2}\rfloor\alpha_*(k)^2)\right)\right.\right.$$

$$\left.\left.\phi\left(\tilde{\xi}(\alpha_*(k), M, s_1), \ldots, \tilde{\xi}(s_d)\right)\right]\right| = 0, \qquad (4.28)$$

and for all function $\phi \in \mathcal{C}_\infty([\xi_+, +\infty)^d)$ with compact support, $d \in \mathbb{N}^*$, and all subdivision

$$0 = s_1 < s_2 < \cdots < s_d = u_1.$$

As we are in the Skorohod' space, the almost surely convergence

$$\phi\left(\tilde{\xi}(\alpha_*(k), M, s_1), \ldots, \tilde{\xi}(\alpha_*(k), M, s_d)\right) \longmapsto_{k \to +\infty} \phi\left(\tilde{\tilde{\xi}}(s_1), \ldots, \tilde{\tilde{\xi}}(s_d)\right),$$

is satisfied. Then, by statement i) of this Lemma, the product in the expectation of (4.28) converges almost surely to

$$\left(\widetilde{M}(u_2) - \widetilde{M}(u_1)\right)\phi\left(\tilde{\xi}(s_1), \ldots, \tilde{\xi}(s_d)\right)$$

for all function $\phi \in \mathcal{C}^\infty([\xi_+, +\infty[^d)$ with compact support. Moreover, by (4.23), the product in the expectation of (4.28) is bounded by

$$[2\|f\|_\infty + \widetilde{C}_f(u_2 - u_1)]\|\phi\|_\infty.$$

Using (4.28) and the dominated convergence theorem, we obtain

$$\mathbb{E}\left[\left(\widetilde{M}(u_2) - \widetilde{M}(u_1)\right)\phi\left(\tilde{\xi}(s_1), \ldots, \tilde{\xi}(s_d)\right)\right] = 0.$$

Therefore process $\left(\widetilde{M}(\tau)\right)_{\tau \in [0,1]}$ is a $\mathcal{F}_\tau^{\tilde{\xi}}$-martingale.   □

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