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Tightness of the Ising–Kac Model on the Two-Dimensional Torus

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Abstract We consider the sequence of Gibbs measures of Ising models with Kac interaction defined on a periodic two-dimensional discrete torus near criticality. Using the convergence of the Glauber dynamic proven by Mourrat and Weber (Commun Pure Appl Math 70:717–812, 2017) and a method by Tsatsoulis and Weber employed in (arXiv:1609.08447 2016), we show tightness for the sequence of Gibbs measures of the Ising–Kac model near criticality and characterise the law of the limit as the Φ_2^4 measure on the torus. Our result is very similar to the one obtained by Cassandro et al. (J Stat Phys 78(3):1131–1138, 1995) on $\mathbb{Z}^2$, but our strategy takes advantage of the dynamic, instead of correlation inequalities. In particular, our result covers the whole critical regime and does not require the large temperature/large mass/small coupling assumption present in earlier results.

1 Introduction

Let $N > 0$ be a positive integer and consider the (periodic) lattice $\Lambda_N = \{1 - N, \dots, N\}^2$. For $\gamma > 0$, let $\mathfrak{K} : \mathbb{R}^2 \to [0, 1]$ be a twice differentiable, non negative, isotropic function supported on a ball of radius 3 and define κ_γ for $x \in \Lambda_N$

$$\kappa_\gamma(x) \simeq \gamma^2 \mathfrak{K}(\gamma|x|), \qquad \sum_{z \in \Lambda_N \setminus \{0\}} \kappa_\gamma(z) = 1. \tag{1.1}$$

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Consider a spin system formed by a set of spins parametrized by the lattice Λ_N. Each spin can assume the value $+1$ or -1 representing two possible states of the magnetization and we will denote with $\Sigma_N = \{-1, 1\}^{\Lambda_N}$ the set of all possible configurations. The Ising–Kac model on the two-dimensional lattice with periodic boundary condition and external magnetization $b \in \mathbb{R}$ is given by the following Hamiltonian

$$\mathcal{H}^\gamma_{\beta,b}(\sigma) = \frac{\beta}{2} \sum_{x,y \in \Lambda_N} \kappa_\gamma(x-y)\sigma_x\sigma_y + b \sum_{x \in \Lambda_N} \sigma_x \qquad \sigma \in \Sigma_N. \tag{1.2}$$

The Gibbs measure over Σ_N associated to the potential (1.1), with inverse temperature β and external magnetic field b, is given by

$$\mathbf{P}^\gamma_{\beta,b}(\sigma) \stackrel{\text{def}}{=} \left(Z^N_{\gamma,\beta,b}\right)^{-1} \exp\left(\mathcal{H}^\gamma_{\beta,b}(\sigma)\right), \tag{1.3}$$

where $Z^N_{\gamma,\beta,b}$ is the partition function that makes (1.3) a probability measure. We will also denote with $\mathbf{E}^\gamma_{\beta,b}$ the expectation under $\mathbf{P}^\gamma_{\beta,b}$. For technical reason, we set $\kappa_\gamma(0) = 0$ and we remark that its precise value doesn't affect (1.3).

As in [12], we will let the inverse temperature β converge in a precise way as $\gamma \to 0$ to $\beta_c = 1$ the critical value of the mean-field system. The purpose of this paper is to prove the tightness of the magnetisation fluctuation field

$$\gamma^{-1}\left(\sigma_{\lfloor N\cdot \rfloor} - \mathbf{E}^\gamma_{\beta,b}\left[\sigma_{\lfloor N\cdot \rfloor}\right]\right) \tag{1.4}$$

in the strong topology of $\mathcal{S}'(\mathbb{T}^2)$ as both $N \to \infty$ and $\gamma \to 0$ in a precise way that we shall describe later.

Moreover, in case $b = 0$, we are also able to characterise the limit as the $\Phi^4(\mathbb{T}^2)$ measure, formally described by

$$\mathcal{Z}^{-1} \exp\left(-\int_{\mathbb{T}^2} \frac{1}{2}|\nabla\Phi(x)|^2 + \frac{1}{12}\Phi^{:4:}(x) - \frac{A}{2}\Phi^{:2:}(x) \, dx\right) d\Phi, \tag{1.5}$$

where $\Phi^{:4:}$ denotes the Wick renormalisation of the fourth power of the field. For a more detailed and formal definition, see for example [6,14].

The Ising–Kac model is a mean-field model with ferromagnetic long range potential that has been introduced in statistical mechanics for its simplicity and because it provides a framework to recover rigorously the van der Waals theory [9] of phase transition. It has been then developed by Lebowitz and Penrose in [11], see also [15] for more details. The model has already been useful to study the Φ^4_d theory, see [7], where a renormalisation group approach has been used to approximate Φ^4_d with generalised Ising models, and [16] with classical Ising spins.

The present work is mainly built upon the article of Weber and Mourrat [12], where the Glauber dynamic on a periodic two-dimensional lattice is shown to converge to the solution of the two-dimensional stochastic Allen–Cahn equation on the torus. Their approach however doesn't imply the tightness for the invariant measure of the model, which is treated in this article. The same result about convergence of Glauber dynamics in one space dimension had previously been proven in [2,5], via a coupling with a simpler model, the voter model. In a subsequent paper [18], Tsatsoulis and Weber show the exponential convergence to equilibrium for the dynamical Φ^4_2 model. In [3], the authors show the convergence of the 2d Ising–Kac model on $\mathbb{Z}^2$ to Φ^4_2 by proving the convergence of the discrete Schwinger functions. In particular they were the first to explain, to the best of our knowledge, the small

shift of the critical temperature for the Ising–Kac model with the renormalisation constants of the Wick powers. That result (see [3, Theorem 2]) is however restricted to temperatures satisfying a condition allowing to use Aizenman's correlation inequalities, which corresponds to large *negative* values of A in (1.5).

Our main result resembles the one obtained in [3], with some differences. We will work on a periodic lattice instead of $\mathbb{Z}^2$, which we think of as a discretisation of a 2D torus. This restriction is mainly due to our techniques for bounding the solutions globally in time and a posteriori doesn't appear to be strictly necessary since the limiting dynamic can be defined also on the whole 2D plane (see [13]). Moreover, as our main proof exploits the dynamical version of the model and not the correlation inequalities, we do not have the restriction on the temperature present in [3, Theorem 2], so that we cover arbitrary values $A \in \mathbb{R}$ in (1.5). A correlation inequality, the GHS inequality, is then employed in a subsequent corollary to partially extend the result to the case of arbitrary external magnetization b. Corollary 2.3 is the only place where we use a correlation inequality.

As remarked by one of the referees, one advantage of the strategy described in the present article is that it is robust enough to also imply tightness of the Gibbs measure of the Kac-Blume-Capel, using the result about the respective Glauber dynamic provided in [17]. In order to be able to identify the limiting law with the Φ_2^6 random field, thus generalising Theorem 2.4, one then also needs a suitable convergence to equilibrium result for the limiting Φ_2^6 dynamical model. While the result of [18] does not seem to generalised immediately (see Remark 6.2 there), this can be obtained by combining the strong Feller property obtained in [8, Sect. 5.3] with the fact that one has good control over the return times to bounded/compact sets, as well as the fact that the Φ_2^6 measure has full support, being equivalent to the free field.

The structure of the present article is as follows: our main result is Theorem 2.1 showing tightness of the fluctuations of local averages of the magnetic field in a distributional space. The proof of the theorem is based on the analysis of the dynamical Φ_2^4 model in [18, Sect. 3] and makes no use of correlation inequalities (not explicitly at least), avoids the restriction of [3, Eq. (1.8)] and exploits the regularisation provided by the time evolution of the Glauber dynamic. As a consequence of Theorem 2.1, we obtain in Corollaries 2.2 and 2.3 tightness in $\mathcal{S}'(\mathbb{T}^2)$ for the fluctuation fields (1.4).

In Theorem 2.4 we characterise the limit of each subsequence to be an invariant measure for the dynamical Φ_2^4 model constructed in [4]. Since it was shown in [4] that (1.5) is such a measure and in [18] that this invariant measure is unique, the result follows. For the proof, we make use of the uniform convergence to the invariant measure and the convergence of the Glauber dynamic in [12].

1.1 Notations

We shall consider spins arranged on a periodic lattice that we will think as embedded into a two-dimensional torus $\mathbb{T}^2 = [-1, 1]^2$. Let $\varepsilon = N^{-1}$ and $\Lambda_\varepsilon \stackrel{\text{def}}{=} \varepsilon \Lambda_N \subset \mathbb{T}^2$ the discretisation induced on $\mathbb{T}^2$. For $f, g : \Lambda_\varepsilon \to \mathbb{C}$ we define

$$\|f\|_{L^p(\Lambda_\varepsilon)}^p \stackrel{\text{def}}{=} \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 |f(x)|^p, \qquad \langle f, g \rangle_{\Lambda_\varepsilon} \stackrel{\text{def}}{=} \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 f(x) \overline{g(x)},$$

respectively the discrete L^p norm and the scalar product. We will use the discrete convolution

$$(f * g)(x) \stackrel{\text{def}}{=} \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 f(x - y) g(y), \qquad \text{for } x \in \Lambda_\varepsilon,$$

and, when there is no possibility of confusion, we will drop the set Λ_ε from the above definitions. We will make an extensive use of the Fourier transform

$$\widehat{f}(w) \overset{\text{def}}{=} \langle f, e_w \rangle_{\Lambda_\varepsilon}, \qquad e_w(x) = e^{i\pi w \cdot x} \qquad \text{for } w \in \Lambda_N.$$

It will sometimes be convenient to also set $e_w = 0$ for $w \in \mathbb{Z}^2 \backslash \Lambda_N$. With this notation, the Fourier inversion formula reads

$$f(x) = \frac{1}{4} \sum_{w \in \mathbb{Z}^2} \widehat{f}(w) e_w(x) \qquad \text{for } x \in \Lambda_\varepsilon. \tag{1.6}$$

We shall use the same notation $\mathsf{Ext}(f)$ as in [12] to denote the extension of f to the continuous torus $\mathbb{T}^2$ via (1.6) applied to $x \in \mathbb{T}^2$. We recall furthermore the fact that the operator Ext doesn't commute with the operation of taking the product. (Of course we could have used extensions that do commute, but Ext behaves nicely with respect to the scale of Besov spaces.) We will measure the regularity of a function $g : \mathbb{T}^2 \to \mathbb{R}$ (or $g : \Lambda_\varepsilon \to \mathbb{R}$) with the Besov norm, defined for $\nu \in \mathbb{R}$, and $p, q \in [1, \infty]$ as

$$\|g\|_{\mathcal{B}_{p,q}^\nu} = \begin{cases} \left(\sum_{k \geq -1} 2^{\nu k q} \|\delta_k g\|_{L^p(\mathbb{T}^2)}^q \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k \geq -1} 2^{\nu k} \|\delta_k g\|_{L^p(\mathbb{T}^2)} & \text{if } q = \infty \end{cases} \tag{1.7}$$

[see (4.1) below for the definition of the Paley–Littlewood projection δ_k] and we will denote by $\mathcal{B}_{p,q}^\nu$ the completion of the set of smooth test functions over the torus equipped with the corresponding Besov norm. We shall denote by $\mathcal{C}^\nu$ the (separable) Besov space $\mathcal{B}_{\infty,\infty}^\nu$. In particular, the parameter $\nu \in \mathbb{R}$ represents the regularity of a function and the space $\mathcal{B}_{p,q}^\nu$ contains distributions if $\nu < 0$. It will be useful to consider, for $g : \Lambda_\varepsilon \to \mathbb{R}$, a discrete version of the Besov norm, that we shall denote by $\|g\|_{\mathcal{B}_{p,q}^\nu(\Lambda_\varepsilon)}$ (resp. $\|g\|_{\mathcal{C}^\nu(\Lambda_\varepsilon)}$),

$$\|f\|_{\mathcal{B}_{p,q}^\nu(\Lambda_\varepsilon^d)} \overset{\text{def}}{=} \begin{cases} \left(\sum_{k \geq -1} 2^{\nu k q} \|\delta_k \mathsf{Ext}(f)\|_{L^p(\Lambda_\varepsilon^d)}^q \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k \geq -1} 2^{\nu k} \|\delta_k \mathsf{Ext}(f)\|_{L^p(\Lambda_\varepsilon^d)} & \text{if } q = \infty \end{cases}$$

see Sect. 4 for a more precise description and for some useful properties of this norm used in the article.

2 Definitions and Statements of the Theorem

Assume for the moment that $b = 0$, which is also the case studied in [12] and consider for $x \in \Lambda_N$

$$h_\gamma(x) \overset{\text{def}}{=} \sum_{z \in \Lambda_N} \kappa_\gamma(x - z) \sigma_z, \tag{2.1}$$

where the kernel is the same as in (1.2).

Following [2,12], we define the magnetisation fluctuation field over the lattice Λ_ε as $X_\gamma(z) = \gamma^{-1} h_\gamma(\varepsilon^{-1} z)$. We will consider a dynamic of Glauber type on Σ_N in order to gain insight into the properties of the fluctuations. In order for this dynamic to converge to a non-trivial limit, we will enforce the relation between the scalings ε and γ given by (3.10).

The dynamic can be described informally as follows. Each site $x \in \Lambda_N$ is assigned an independent exponential clock with rate 1. When the clock rings, the corresponding spin changes sign with probability

$$c_\gamma(z, \sigma) = \frac{1}{2}\left(1 - \sigma_z \tanh\left(\beta h_\gamma(z)\right)\right),\tag{2.2}$$

and remains unchanged otherwise. More formally, the generator of this dynamic is given by

$$\mathscr{L}_\gamma f(\sigma) = \sum_{z \in \Lambda_N} c_\gamma(z, \sigma)\left(f(\sigma^{\{z\}}) - f(\sigma)\right),\tag{2.3}$$

for $f : \Sigma_N \to \mathbb{R}$, where

$$\sigma_y^{\{z\}} = \begin{cases} -\sigma_z & \text{if } y = z, \\ \sigma_y & \text{if } y \neq z. \end{cases}$$

The probabilities $c_\gamma(z, \sigma)$ are chosen precisely in such a way that $\mathbf{P}_{\beta,0}^\gamma$ is invariant for this Markov process. We shall use the notations $\sigma_x(s)$ and $h_\gamma(s, x)$ to refer to the process at (microscopic) space $x \in \Lambda_N$ and time $s \in \mathbb{R}_+$. We will use the notation $\mathbb{P}_{\beta,0}^\gamma$ (resp. $\mathbb{E}_{\beta,0}^\gamma$) to refer to the probability (resp. expectation) of the process started with an initial condition drawn from $\mathbf{P}_{\beta,0}^\gamma$.

In order to rewrite the process in macroscopic coordinates, we speed up the generator $\mathscr{L}_\gamma$ by a factor α^{-1} and we will abuse the notation writing

$$X_\gamma(s, x) \overset{\text{def}}{=} \delta^{-1} h_\gamma(\alpha^{-1}s, \varepsilon^{-1}x),\tag{2.4}$$

in (macroscopic) space $x \in \Lambda_\varepsilon$ and time $s \in \mathbb{R}_+$. In [12, Theorem 3.2] it is proven that, if the parameters $\delta, \alpha, \varepsilon$ and the inverse temperature β are chosen such that

$$\delta = \gamma, \quad \alpha = \gamma^2, \quad \varepsilon = \gamma^2, \quad \beta - 1 = \alpha\left(\mathfrak{c}_\gamma + A\right),\tag{2.5}$$

where $\mathfrak{c}_\gamma$ is described in (2.8) below, and if the sequence of initial conditions satisfies $X_\gamma(0) \to X_\gamma^0$ in $\mathcal{C}^{-\nu}$, then the law of X_γ on $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$, converges in distribution to the solution of the stochastic quantisation equation

$$\partial_t X = \Delta X - \frac{1}{3}X^{:3:} + AX + \sqrt{2}\xi, \quad X(0, \cdot) = X^0 \in \mathcal{C}^{-\nu}\tag{2.6}$$

where $X_\gamma(\cdot, 0) \to X_0$ in $\mathcal{C}^{-\nu}$ and ξ denotes space-time white noise. The expression $X^{:3:}$ stands for a renormalised power defined as in [4], where the relevant notion of "solution" to (2.6) is also given. The solution theory of (2.6) will be briefly summarised in Sect. 2.1. The use of the renormalised powers is necessary since the solution belongs to a distributional space.

For $x \in \Lambda_\varepsilon$, let $K_\gamma(x) = \varepsilon^{-2}\kappa_\gamma(\varepsilon^{-1}x)$, the macroscopic version of the kernel K_γ, and define the discrete Laplacian $\Delta_\gamma f = \varepsilon^{-2}\gamma^2(K_\gamma * f - f)$. Under the Glauber dynamic, the process X_γ satisfies on $[0, T] \times \Lambda_\varepsilon$

$$\begin{aligned}
X_\gamma(t, x) = {} & X_\gamma(0, x) \\
& + \int_0^t \Delta_\gamma X_\gamma(s, x) - \frac{1}{3}\left(X_\gamma^3(s, x) - \mathfrak{c}_\gamma X_\gamma(s, x)\right) \\
& + AX_\gamma(s, x) + \mathcal{O}(\gamma^2 X_\gamma^5(s, x))\, ds + M_\gamma(t, x)
\end{aligned}\tag{2.7}$$

where $M_\gamma(t, x)$ is a martingale and $\mathfrak{c}_\gamma$ is the logarithmically diverging constant

$$\mathfrak{c}_\gamma = \frac{1}{4}\sum_{\omega \in \Lambda_N \setminus \{0\}} \frac{|\hat{K}_\gamma(\omega)|^2}{\varepsilon^{-2}\gamma^2(1 - \hat{K}_\gamma(\omega))}.\tag{2.8}$$

The next theorem is the main result of the paper. Recall the definition of the Besov norm given in Sect. 1.1. We think of $X_\gamma(z)$ as being a random function on $\mathbb{T}^2$, having been extended to $\mathbb{T}^2$ with the Ext operator.

Theorem 2.1 *Assume $b = 0$. Then for all positive $\nu > 0$ and for all $q > 0$*

$$\limsup_{\gamma \to 0} \mathbf{E}^\gamma_{\beta,0} \left[\|X_\gamma\|^q_{\mathcal{C}^{-\nu}} \right] < \infty.$$

In particular, the laws of X_γ form a tight set of probability measures on $\mathcal{C}^{-\nu}$.

From the above theorem it is possible to deduce

Corollary 2.2 *Assume $b = 0$. Then the law of the field $\left(\gamma^{-1} \sigma_{\lfloor \varepsilon^{-1} x \rfloor} \right)_{x \in \mathbb{T}^2}$ is tight in $\mathcal{S}'(\mathbb{T}^2)$ under $\mathbf{P}^\gamma_{\beta,0}$.*

Proof We will actually prove the tightness in the stronger norm of $H^{-k}(\mathbb{T}^2)$, for k sufficiently big. Let $\varphi \in \mathcal{S}(\mathbb{T}^2)$ and consider

$$\left\langle \gamma^{-1} \sigma_{\lfloor \varepsilon^{-1} \cdot \rfloor}, \varphi \right\rangle_{\mathbb{T}^2} = \sum_{x \in \Lambda_N} \varepsilon^2 \left(\gamma^{-1} \sigma_x \right) \bar\varphi(\varepsilon x)$$

where $\bar\varphi(\varepsilon x) = \epsilon^{-2} \int_{|y|_\infty \leq 2^{-1}\varepsilon} \varphi(\varepsilon x + y) \, dy$. Using the differentialbility of φ, we replace $\bar\varphi$ with $\kappa_\gamma * \bar\varphi$ at the cost of

$$\varepsilon^2 \gamma^{-2} \sup_{i_1, i_2 \in \{1,2\}} \left\| \partial_{i_1} \partial_{i_2} \varphi \right\|_{L^\infty(\mathbb{T}^2)},$$

and this is $\mathcal{O}(\gamma^2)$, as $\gamma \to 0$, if k is sufficiently big. Therefore (recall the form of the extension (1.6) of X_γ to the continuous torus)

$$\left\langle \gamma^{-1} \sigma_{\lfloor \varepsilon^{-1} \cdot \rfloor}, \varphi \right\rangle_{\mathbb{T}^2} = \left\langle X_\gamma, \varphi \right\rangle_{\mathbb{T}^2} + \mathcal{O}(\gamma)$$

the corollary follows from Theorem 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As remarked in the proof, the topology with respect to which the convergence in Corollary 2.2 is proved is not the optimal norm. Indeed we expect the result to hold also with respect to the norm of $\mathcal{C}^{-\nu}$. In the proof of Corollary 2.2 we didn't only show the tightness of the sequence of random variable, but we also proved that the limit of $\left\langle \gamma^{-1} \sigma_{\lfloor \varepsilon^{-1} \cdot \rfloor}, \varphi \right\rangle_{\mathbb{T}^2}$ coincide with $\lim_{\gamma \to 0} \left\langle X_\gamma, \varphi \right\rangle_{\mathbb{T}^2}$ for all φ sufficiently smooth.

We now show how to extend the previous result to the case $b \neq 0$. It is clear that, by symmetry it is sufficient to assume $b \geq 0$. In the case of ferromagnetic pair potential $\kappa_\gamma \geq 0$ with positive external magnetisation $b \geq 0$, one has

$$\mathbf{E}^\gamma_{\beta,b} \left[\sigma_x; \sigma_y \right] \leq \mathbf{E}^\gamma_{\beta,0} \left[\sigma_x; \sigma_y \right] \tag{2.9}$$

where $\mathbf{E}^\gamma_{\beta,b} \left[\sigma_x; \sigma_y \right]$ is the covariance between the spins. This follows from the fact that $\frac{d}{db} \mathbf{E}^\gamma_{\beta,b} \left[\sigma_x; \sigma_y \right] \leq 0$, which is an immediate consequence of the GHS inequality (see for instance [10] for a proof), valid for $\kappa_\gamma \geq 0$ and $b \geq 0$.

Corollary 2.3 *Consider any map $\gamma \mapsto b_\gamma \geq 0$ and denote by $m_\gamma(b) = \mathbf{E}^\gamma_{\beta,b}[\sigma_x]$ the mean of the spin σ_x, which is independent of $x \in \Lambda_N$. Then the law of the field*

$$\tilde{X}_\gamma(x) = \gamma^{-1} \left(\sigma_{\lfloor \varepsilon^{-1} x \rfloor} - m_\gamma(b_\gamma) \right), \qquad x \in \mathbb{T}^2, \tag{2.10}$$

is tight in $\mathcal{S}'(\mathbb{T}^2)$ under $\mathbf{P}^\gamma_{\beta,b_\gamma}$.

Proof Fixing a test function ϕ and replacing $\bar{\varphi}$ with $\kappa_\gamma * \bar{\varphi}$ as in Corollary 2.2, we have

$$\left\langle \tilde{X}_\gamma, \varphi \right\rangle = \sum_{x \in \Lambda_N} \varepsilon^2 \gamma^{-1} \big(\sigma_x - m_\gamma(b_\gamma) \big) (\kappa_\gamma * \bar{\varphi})(\varepsilon x) + \mathsf{Err},$$

where Err converges to 0 in probability as $\gamma \to 0$. Decompose $\varphi = \varphi^+ - \varphi^-$ into its positive and negative part. For each of them, using the correlation inequality (2.9), we have that

$$\mathbf{E}^\gamma_{\beta, b_\gamma} \left| \varepsilon^2 \sum_{x \in \Lambda_N} \frac{\sigma_x - m_\gamma(b_\gamma)}{\gamma} (\kappa_\gamma * \overline{\varphi}^\pm)(\varepsilon x) \right|^2 \leq \mathbf{E}^\gamma_{\beta, 0} \left| \varepsilon^2 \sum_{x \in \Lambda_N} (\gamma^{-1} \sigma_x) \, (\kappa_\gamma * \overline{\varphi}^\pm)(\varepsilon x) \right|^2.$$

Using Theorem 2.1 we see that, for $\nu \in (0, 1)$, this quantity is bounded uniformly by a fixed multiple of $\left\| \varphi^\pm \right\|^2_{\mathcal{B}^\nu_{1,1}} \mathbf{E}^\gamma_{\beta, 0} \left[\left\| X_\gamma \right\|^2_{\mathcal{C}^{-\nu}} \right]$, up to an error of order $\mathcal{O}(\gamma)$. In order to conclude, we observe that

$$\left\| \varphi^\pm \right\|_{\mathcal{B}^\nu_{1,1}} \lesssim \left\| \varphi^\pm \right\|_{L^1} + \left\| \varphi^\pm \right\|^\nu_{\mathrm{Lip}} \left\| \varphi^\pm \right\|^{1-\nu}_{L^1} \lesssim \left\| \varphi \right\|_{L^1} + \left\| \nabla \varphi \right\|_{L^\infty}$$

where the first inequality is (4.6), generalised to Lipschitz functions. $\square$

The next theorem shows that in the symmetric case $b = 0$, the limit of these measures is given by the Φ^4_2 measures, as already suggested in [12].

Theorem 2.4 *Assume $b = 0$. Then any limiting law of the sequence $\{X_\gamma\}_\gamma$ is invariant for the dynamic (2.6), and hence, by [18] and [4, Remark 4.3], coincides with the $\Phi^4(\mathbb{T}^2)$ measure.*

Proof In order to compare the law of a (discrete) random field X_γ with fields on the torus $\mathbb{T}^2$, we will use the extension operator Ext defined after (1.6). For the sake of precision we will explicitly write $\mathsf{Ext}(X_\gamma(t))$ where the process X_γ has been extended to the whole torus.

We will use the Glauber dynamic and the solution of the stochastic quantisation equation (2.6) introduced in the previous section: the idea is to exploit the exponential convergence to the invariant measure of the solution of the SPDE (2.6) proved in [18] and the convergence of the Glauber dynamic of the Kac–Ising model in [12].

By [12, Theorem 3.2], we know that if for $0 < \kappa < \nu$ small enough the sequence of initial conditions $\mathsf{Ext}(X^0_\gamma)$ is bounded in $\mathcal{C}^{-\nu+\kappa}$ and converges to a limit X^0 in $\mathcal{C}^{-\nu}$ as $\gamma \to 0$, one has

$$\mathsf{Ext}\left(X_\gamma \right) \xrightarrow{\mathcal{L}} X \qquad \text{in } \mathcal{D}\left([0, T]; \mathcal{C}^{-\nu}\right), \tag{2.11}$$

where X solves (2.6) starting from X^0. In the above equation we took into account the fact that X_γ is defined on the discrete lattice and therefore has to be extended with the operator Ext to be comparable with X.

We first want to show that (2.11) holds true when instead of a deterministic sequence $\mathsf{Ext} X^0_\gamma \to X^0$ in $\mathcal{C}^{-\nu}$, we have the convergence in law of the initial conditions $\mathcal{L}(\mathsf{Ext} X^0_\gamma) \to \mathcal{L}(X^0)$ in the topology of $\mathcal{C}^{-\nu}$. In order to do this call $\mathfrak{L}_\gamma$ (resp. $\mathfrak{L}_0$) the laws at time zero of the processes $\mathsf{Ext} X_\gamma$ (resp. X) and assume that $\mathfrak{L}_\gamma \to \mathfrak{L}_0$. Consider then a bounded continuous function $G : \mathcal{D}\left([0, T]; \mathcal{C}^{-\nu}\right) \to \mathbb{R}$: we want to show that

$$\lim_{\gamma \to 0} \left| \mathbb{E}\big[G(\mathsf{Ext}(X_\gamma)) \big| X^0_\gamma \sim \mathfrak{L}_\gamma \big] - \mathbb{E}\big[G(X) \big| X^0 \sim \mathfrak{L}_0 \big] \right| = 0.$$

Conditioning over the initial conditions we can define

$$f_G^\gamma(X_\gamma^0) := \mathbb{E}\left[G(\mathsf{Ext}(X_\gamma)) \Big| X_\gamma(0) = X_\gamma^0\right]$$
$$f_G(X^0) := \mathbb{E}\left[G(X) \Big| X(0) = X^0\right].$$

The result [12, Theorem 3.2] implies that $f_G^\gamma(X_\gamma^0) \to f_G(X^0)$ whenever $\mathsf{Ext}X_\gamma^0 \to X^0$ in $\mathcal{C}^{-\nu}$. Since $\mathcal{C}^{-\nu}$ is separable, we can apply the Skorokhod's representation theorem to deduce that there is a probability space $(\tilde{\mathbb{P}}, \tilde{\mathcal{F}}, \tilde{\Omega})$ where all the processes $\mathsf{Ext}(X_\gamma^0)$ and X^0 can be realised and the sequence $\mathsf{Ext}(X_\gamma^0)(\tilde{\omega})$ converge to $X^0(\tilde{\omega})$ in $\mathcal{C}^{-\nu}$ for $\tilde{\mathbb{P}}$-a.e. $\tilde{\omega} \in \tilde{\Omega}$.

An application of the dominated convergence theorem then shows that, as $\gamma \to 0$

$$\left|\mathbb{E}\left[G(\mathsf{Ext}(X_\gamma)) \big| X_\gamma^0 \sim \mathfrak{L}_\gamma\right] - \mathbb{E}\left[G(X) \big| X^0 \sim \mathfrak{L}_0\right]\right|$$
$$\leq \int \left|f_G^\gamma(X_\gamma^0(\tilde{\omega})) - f_G(X^0(\tilde{\omega}))\right| \tilde{\mathbb{P}}(d\tilde{\omega}) \to 0, \tag{2.12}$$

so that we can assume (2.11) to hold even when the initial datum is convergent in law.

By Theorem 2.1 we know that, if at time 0 the configuration $\sigma(0) \in \Sigma_N$ is distributed according to $\mathbf{P}_{\beta,0}^\gamma$, then the law of $X_\gamma^0(x) = \gamma^{-1}\kappa_\gamma * \sigma_{\lfloor \varepsilon^{-1}x \rfloor}(0)$ is tight, and therefore there exists a subsequence γ_k for $k \geq 0$ and a measure μ^* on $\mathcal{C}^{-\nu}$ such that the law of $\mathsf{Ext}X_{\gamma_k}^0$ converges to μ^*. In the following calculations we will tacitly assume $\gamma \to 0$ along the sequence γ_k to avoid the subscript. We will show that, if μ if the unique invariant measure of (2.6) then $\mu^* = \mu$.

Let $F : \mathcal{C}^{-\nu} \to \mathbb{R}$ be a bounded and continuous function, then, by the invariance of the Gibbs measure under the Glauber dynamic, for $t \geq 0$

$$\mathbf{E}_{\beta,0}^\gamma\left[F(\mathsf{Ext}X_\gamma^0)\right] = \mathbb{E}_{\beta,0}^\gamma\left[F(\mathsf{Ext}X_\gamma(t))\right].$$

Recall that the evaluation map, that associates to a process in $\mathcal{D}\left([0, T]; \mathcal{C}^{-\nu}\right)$ its value at a given time, is not continuous with respect to the Skorokhod topology, however the integral map $G : u \mapsto \int_0^T F(u(s))\, ds$ is continuous in its argument in virtue of the the continuity and boundedness of F. Hence for any fixed T we have

$$\mathbf{E}_{\beta,0}^\gamma\left[F(\mathsf{Ext}X_\gamma^0)\right] = \mathbb{E}_{\beta,0}^\gamma\left[T^{-1}\int_0^T F\left(\mathsf{Ext}X_\gamma(s)\right)\, ds\right]$$

and

$$\lim_{\gamma \to 0}\left|\mathbb{E}_{\beta,0}^\gamma\left[\int_0^T F\left(\mathsf{Ext}X_\gamma(s)\right)\, ds\right] - \mathbb{E}\left[\int_0^T F\left(X(s)\right)\, ds \Big| X(0) \sim \mu^*\right]\right| = 0.$$

By the uniform convergence to equilibrium of the stochastic quantisation equation [18, Corollary 6.6] there exist constants $c, C > 0$

$$\left|\mathbb{E}[F\left(X(s)\right) | X(0) \sim \mu^*] - \mu[F]\right| \leq C |F|_\infty e^{-cs}.$$

From the above inequality it follows that

$$\left|T^{-1}\int_0^T \mathbb{E}[F\left(X(s)\right) | X(0) \sim \mu^*] - \mu[F]ds\right| \lesssim T^{-1} |F|_\infty$$

and letting T be large enough the last difference can be made arbitrarily small. From the above estimates we can see that, for arbitrary $T > 0$,

$$\limsup_{\gamma \to 0} \left| \mathbf{E}_{\beta,0}^{\gamma} \left[F(\mathsf{Ext} X_{\gamma}^0) \right] - \mu[F] \right| \leq C |F|_{\infty} T^{-1}$$

and the result follows. □

Remark 2.5 For $b_{\gamma} = b$ constant, we actually expect the limiting points to vanish under the scaling (2.10). On the other hand, for $b_{\gamma} = b\gamma$, one can follow an argument virtually identical to the one given in this article to show that the limit is given by the law of the Φ_2^4 measure with external magnetic field b.

2.1 Solution of the Limiting Equation

Before the proof of the main theorem, let us briefly explain the construction of the solution in [4] to the following SPDE

$$dX = \left(\Delta X + \sum_{j=1}^{n} a_{2j-1} X^{2j-1} \right) dt + \sqrt{2} dW.$$

As in (2.6), the powers in the above SPDE have to be renormalised in order to find a nontrivial solution. The precise way the process is renormalised follows [12,17]. Consider at first $Z(t)$ the solution of the stochastic heat equation

$$dZ = \Delta Z dt + \sqrt{2} dW \qquad Z(\cdot, 0) = 0,$$

and therefore in two dimension Z belongs to $\mathcal{C}([0, T]; \mathcal{C}^{-\nu})$ a.s. for any $\nu > 0$. Consider the Galerkin approximation

$$dZ_{\varepsilon} = \Delta Z_{\varepsilon} dt + \sqrt{2} dW_{\varepsilon} \qquad Z_{\varepsilon}(\cdot, 0) = 0, \tag{2.13}$$

where dW_{ε} is the $L^2(\mathbb{T}^2)$-projection of the space time white noise on the span of the Fourier modes generated by $\{e_{\omega}\}_{|\omega| \leq N}$.

From the above SPDE we see that Z_{ε} has a representation in terms of the stochastic convolution with the heat kernel $P_t = e^{t\Delta}$

$$Z_{\varepsilon}(t, x) = \sqrt{2} \int_0^t P_{t-s} d\hat{W}(s, x).$$

Define the renormalisation constant

$$\mathfrak{c}_{\varepsilon} := \lim_{t \to \infty} \mathbb{E} \left[Z_{\varepsilon}^2(t, 0) - \frac{t}{2} \right] = \sum_{\omega \in \Lambda_N \setminus \{0\}} \frac{1}{4\pi^2 |\omega|^2}.$$

In order to renormalise the process $Z_{\varepsilon}(t)$ at finite time, it is more convenient to use its time dependent version

$$\mathfrak{c}_{\varepsilon}(t) \stackrel{\text{def}}{=} \mathbb{E} \left[Z_{\varepsilon}^2(t, 0) \right] = \frac{1}{2} \sum_{\omega \in \Lambda_N} \int_0^t e^{-2(t-s)\pi^2 |\omega|^2} ds = \frac{t}{2} + \sum_{\omega \in \Lambda_N \setminus \{0\}} \frac{1 - e^{-2t\pi^2 |\omega|^2}}{4\pi^2 |\omega|^2}$$

We therefore define the renormalised powers of the process Z_{ε} as

$$Z_{\varepsilon}^{:n:}(t) \stackrel{\text{def}}{=} H_n(Z_{\varepsilon}(t), \mathfrak{c}_{\varepsilon}(t)).$$

By [4, Lemma 3.2], the process $Z_\varepsilon^{:n:}(t)$ is Cauchy in $L^p \left(\mathcal{C}([0, T]; \mathcal{C}^{-\nu}), \mathbb{P} \right)$ for every $p \geq 1$ and we will be referring to its limit as $Z^{:n:}$. To be precise, the result in [4, Lemma 3.2] is proven for a fixed time, but the extension to the whole process is immediate.

By the variation of constants formula, the solution to (2.13) started from the initial condition $X_\varepsilon^0 \stackrel{\text{def}}{=} \Pi_\varepsilon X^0$ is given by $\tilde{Z}_\varepsilon(t) = e^{\Delta t} X_\varepsilon^0 + Z_\epsilon(t)$. To extend the definition of the renormalised powers to the process $\tilde{Z}_\varepsilon(t)$ one uses the following property of the Hermite polynomial

$$H_n(a + b, c) = \sum_{j=0}^{n} \binom{n}{j} b^{n-j} H_j(a, c)$$

and let

$$\tilde{Z}_\varepsilon^{:n:}(t) = \sum_{j=0}^{n} \binom{n}{j} \left(e^{\Delta t} X_\varepsilon^0 \right)^{n-j} Z_\varepsilon^{:j:}(t).$$

The above random variable is well defined because $e^{\Delta t} X_\varepsilon^0$ is a smooth function and the product with $Z_\varepsilon^{:j:}(t)$ is in $\mathcal{C}^{-\nu}$ for any $t > 0$ (see for instance [13, Corollary 3.2] or Theorems 2.82 and 2.85 in [1] for a proof).

We then set $X_\varepsilon(t)$ the Galerkin approximation of $X(t)$ solving

$$\begin{cases} dX_\varepsilon(t) & = \left(\Delta X_\varepsilon(t) + \sum_{j=1}^{n} a_{2j-1} H_n(X_\varepsilon(t), \mathfrak{c}_\varepsilon) \right) dt + \sqrt{2} dW_\varepsilon(t) \\ X_\varepsilon(0) & = \Pi_\varepsilon X^0 \in \mathcal{C}^{-\nu} \end{cases} \tag{2.14}$$

Since H_n is a polynomial in both variables, it is possible to replace $\mathfrak{c}_\varepsilon$ in the above formula with $\mathfrak{c}_\varepsilon(t)$ provided one compensates it in the coefficient of the polynomial.

$$\sum_{j=1}^{n} a_{2j-1} H_n(X_\varepsilon(t), \mathfrak{c}_\varepsilon) = \sum_{j=1}^{n} a_{2j-1}(t, \varepsilon) H_n(X_\varepsilon(t), \mathfrak{c}_\varepsilon(t))$$

with new coefficients $a_{2j-1}(t)$ depending polynomially only on the old coefficients and on the difference $\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t)$. From the definitions of $\mathfrak{c}_\varepsilon$ and $\mathfrak{c}_\varepsilon(t)$ one can see that their difference is diverging logarithmically as $t \to 0$ and therefore each power of $a_{2j-1}(t)$ is integrable in $[0, T]$. Hence we can rewrite (2.14) as

$$\begin{cases} dX_\varepsilon(t) & = \left(\Delta X_\varepsilon(t) + \sum_{j=1}^{n} a_{2j-1}(t, \varepsilon) H_n(X_\varepsilon(t), \mathfrak{c}_\varepsilon(t)) \right) dt + \sqrt{2} dW_\varepsilon(t) \\ X_\varepsilon(0) & = \Pi_\varepsilon X^0 \in \mathcal{C}^{-\nu} \end{cases} \tag{2.15}$$

We decompose $X_\varepsilon(t) = \tilde{Z}_\varepsilon(t) + V_\varepsilon(t)$ where, for a.e. realisation of Z_ε, the process V_ε solves the PDE

$$\begin{cases} \partial_t V_\varepsilon(t) & = \Delta V_\varepsilon(t) + \sum_{j=1}^{n} a_{2j-1}(t, \varepsilon) H_n \left(\tilde{Z}_\varepsilon(t) + V_\varepsilon(t), \mathfrak{c}_\varepsilon(t) \right) \\ V_\varepsilon(0) & = 0 \end{cases} \tag{2.16}$$

where

$$H_n \left(\tilde{Z}_\varepsilon(t) + V_\varepsilon(t), \mathfrak{c}_\varepsilon(t) \right) = \sum_{j=0}^{n} \binom{n}{j} V_\varepsilon^{n-j}(t) \tilde{Z}_\varepsilon^{:j:}(t).$$

The last product is again well-posed thanks to the fact that $V_\varepsilon(t) \in \mathcal{C}^{2-\nu-\kappa}$ for any $\kappa > 0$, from the regularizing properties of the parabolic Eq. (2.16).

As proven in [4], the processes V_ε converge in $\mathcal{C}([0, T]; \mathcal{C}^{2-\nu})$ to the solution of

$$\begin{cases} dV & = \Delta V(t) + \sum_{j=1}^n a_{2j-1}(t) \sum_{j=0}^n \binom{n}{j} V^{n-j}(t) \tilde{Z}^{:j:}(t) \\ V(0) & = 0 \end{cases} \tag{2.17}$$

where $\tilde{Z}^{:j:}(t) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} P_t X^0 + Z_\varepsilon^{:j:}(t)$. For all $\kappa > 0$, the solution of the above PDE is unique and $V \in \mathcal{C}([0, T]; \mathcal{C}^{2-\nu-\kappa})$ and only depends on the realisation of the process Z via the tuple $(\tilde{Z}, \ldots, \tilde{Z}^{:2n-1:}) \in L^\infty([0, T]; \mathcal{C}^{-\nu})^n$. We summarise it with the following proposition, essentially proven in [4].

Theorem 2.6 *For all $\nu > 0$ and $T > 0$ there exists a locally Lipschitz continuous function*

$$\mathcal{S}_T : L^\infty([0, T]; \mathcal{C}^{-\nu})^n \to \mathcal{C}([0, T]; \mathcal{C}^{2-\nu-\kappa})$$

that associates to $(\tilde{Z}, \tilde{Z}^{:2:} \ldots, \tilde{Z}^{:2n-1:})$ the solution of (2.17).

Using the definitions above, we now outline the skeleton of the proof in [12]. First of all we want to remark that we made the decision of absorbing the initial conditions in the process $\tilde{Z}_\varepsilon$, instead we could have started (2.16) from $\Pi_\varepsilon X_\gamma^0$ and defined a similar solution map $\mathcal{S}_T^{X^0}$. Consider Z_γ the solution to the linearised part of (2.7) satisfying

$$Z_\gamma(t, x) = \int_0^t \Delta_\gamma Z_\gamma(s, x) ds + M_\gamma(t, x), \tag{2.18}$$

which is an approximation to the stochastic heat equation. In [4] and [12] the authors provided a useful definition of $Z_\gamma^{:n:}$ the renormalized powers of Z_γ that we will not introduce here. In this article will only use the fact that for any $T > 0$, $q > 0$, $\nu > 0$, $j \geq 0$ and $\lambda > 0$,

$$\limsup_{\gamma \to 0} \mathbb{E}\left[\sup_{s \in [0,T]} s^\lambda \left\| H_j\left(Z_\gamma(s, \cdot), \mathfrak{c}_\gamma\right) \right\|_{\mathcal{C}^{-\nu}}^q \right] < \infty, \tag{2.19}$$

which follows from [12, Eq. (3.15), Propositions 5.3 and 5.4].

In [12, Sect. 6] it is proven that processes $\left(Z_\gamma, Z_\gamma^{:2:}, \ldots, Z_\gamma^{:2n-1:}\right)$ jointly converge in law to $\left(Z, Z^{:2:}, \ldots, Z^{:2n-1:}\right)$. Using the decomposition $X_\gamma = Z_\gamma + V_\gamma$, it is possible to see that V_γ satisfies an equation similar to (2.16), with initial condition X_γ^0 and

$$\left\| V_\gamma - \mathcal{S}_T^{X_\gamma^0}\left(Z_\gamma, Z_\gamma^{:2:}, \ldots, Z_\gamma^{:2n-1:}\right) \right\|_{L^\infty([0,T]; \mathcal{C}^{-\nu})} \to 0.$$

Therefore, by the continuity of $\mathcal{S}_T$, we have that, as $\gamma \to 0$,

$$X_\gamma \sim Z_\gamma + \mathcal{S}_T^{X_\gamma^0}\left(Z_\gamma, Z_\gamma^{:2:}, \ldots, Z_\gamma^{:2n-1:}\right) \xrightarrow{\mathcal{L}} Z + \mathcal{S}_T^{X^0}\left(Z, Z^{:2:}, \ldots, Z^{:2n-1:}\right) = X,$$

as required.

3 Proof of Theorem 2.1

We are now going to prove the statements used in Sect. 2 and in particular Theorem 2.1. We first obtain a very suboptimal bound on X_γ which can be used as a starting point for the derivation of sharper bounds.

Proposition 3.1 *Let $p \geq 2$ an even integer, and $\lambda \in [0, 1]$ then there exists $C(p, \lambda) > 0$ such that*

$$\mathbb{E}\big[\, \|X_\gamma(t, \cdot)\|^p_{L^p(\Lambda_\varepsilon)} \,\big] \leq C \left(\mathbb{E}\big[\, \|X_\gamma(0, \cdot)\|^p_{L^p(\Lambda_\varepsilon)} \,\big]^{1-\lambda} t^{-\frac{p}{2}\lambda} \right) \vee \gamma^{-\frac{p}{2}}.$$

In particular, if we start the process from the invariant measure, we obtain that there exists $C = C(p) > 0$ such that for all $t \geq 0$

$$\mathbb{E}^\gamma_{\beta,0}\big[\, \|X_\gamma\|^p_{L^p(\Lambda_\varepsilon)} \,\big] = \mathbb{E}^\gamma_{\beta,0}\big[\, \|X_\gamma(t, \cdot)\|^p_{L^p(\Lambda_\varepsilon)} \,\big] \leq C\gamma^{-\frac{p}{2}} \tag{3.1}$$

Proof In the following proof we will denote with C a generic constant whose value depends on p and might change from line to line. Recall the action of the generator of the Glauber dynamic (2.3):

$$\mathscr{L}_\gamma h^p_\gamma(t, x) = \sum_{z \in \Lambda_N} c_\gamma(z, \sigma(t)) \Big(\big(h_\gamma(t, x) - 2\sigma_z(t)\kappa_\gamma(z - x) \big)^p - h^p_\gamma(t, x) \Big)$$

$$\leq p\big(-h_\gamma + \kappa_\gamma * \tanh(\beta h_\gamma) \big)(t, x) h^{p-1}_\gamma(t, x) + C \big(|h_\gamma(t, x)| + \gamma^2 \big)^{p-2} \gamma^2.$$

The second inequality is a consequence of the fact that $\|\kappa_\gamma\|_\infty \lesssim \gamma^2$ and $\|\kappa_\gamma\|_{L^1} \lesssim 1$. We can take the average over $x \in \Lambda_N$ to obtain

$$\mathscr{L}_\gamma \|h_\gamma(t)\|^p_{L^p(\Lambda_N)} \leq p \Big\langle h^{p-1}_\gamma(t), \kappa_\gamma * \tanh(\beta h_\gamma(t)) \Big\rangle_{\Lambda_N} - p \|h_\gamma(t)\|^p_{L^p(\Lambda_N)}$$

$$+ C\gamma^2 \|h_\gamma(t)\|^{p-2}_{L^{p-2}(\Lambda_N)} + C\gamma^{2p-2}.$$

We use the fact that p is even and the hyperbolic tangent is monotone to bound

$$\Big\langle h^{p-1}_\gamma(t), \kappa_\gamma * \tanh(\beta h_\gamma(t)) \Big\rangle_{\Lambda_N} = \Big\langle h^{p-1}_\gamma(t), \tanh(\beta h_\gamma(t)) \Big\rangle_{\Lambda_N}$$

$$+ \frac{1}{2} \sum_{x,y \in \Lambda_N} \kappa_\gamma(x - y) \big(h^{p-1}_\gamma(t, x) - h^{p-1}_\gamma(t, y) \big) \big(\tanh(\beta h_\gamma(t, y)) - \tanh(\beta h_\gamma(t, x)) \big)$$

$$\leq \Big\langle h^{p-1}_\gamma(t), \tanh(\beta h_\gamma(t)) \Big\rangle_{\Lambda_N}.$$

Moreover, it is easy to see that there exists a constant $c_0 > 0$ such that

$$\frac{\tanh(\beta h)}{h} \leq \beta - c_0 h^2 \qquad \text{for } h \in [1, 1].$$

Since $|h_\gamma(t, x)| \leq 1$ and $\beta = 1 + \gamma^2(c_\gamma + A)$, we can bound $\mathscr{L}_\gamma \|h_\gamma(t)\|^p_{L^p(\Lambda_N)}$ by

$$p[\beta - 1] \|h_\gamma(t)\|^p_{L^p(\Lambda_N)} - c_0 p \|h_\gamma(t)\|^{p+2}_{L^{p+2}(\Lambda_N)} + C\gamma^2 \|h_\gamma(t)\|^{p-2}_{L^{p-2}(\Lambda_N)} + C\gamma^{2p-2}$$

$$\leq C(\gamma^2 c_\gamma)^{\frac{p+2}{2}} - \frac{c_0}{2} p \|h_\gamma(t)\|^{p+2}_{L^{p+2}(\Lambda_N)} + C\gamma^{\frac{p+2}{2}} + C\gamma^{2p-2}$$

$$\leq -\frac{c_0}{2} p \|h_\gamma(t)\|^{p+2}_{L^{p+2}(\Lambda_N)} + C\gamma^{\frac{p+2}{2}},$$

where we used the fact that $|A| \leq c_\gamma \leq \gamma^{-1}$ for γ small enough and the generalised Young inequality in the last line. Therefore, taking the expectation

$$\mathbb{E}\big[\, \|h_\gamma(\alpha^{-1}t)\|^p_{L^p(\Lambda_N)} \,\big]$$

$$\leq \mathbb{E}\big[\, \|h_\gamma(0)\|^p_{L^p(\Lambda_N)} \,\big] + \alpha^{-1} \int_0^t -\frac{c_0}{2} p \mathbb{E}\big[\, \|h_\gamma(\alpha^{-1}s)\|^{p+2}_{L^p(\Lambda_N)} \,\big] + C\gamma^{\frac{p+2}{2}} \, ds$$

and multiplying both sides by γ^{-p} and applying Jensen's inequality we obtain

$$\mathbb{E}\big[\,\|X_\gamma(t)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] = \mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] + \int_0^t \mathbb{E}\big[\mathscr{L}_\gamma\,\|X_\gamma(s)\|^p_{L^p(\Lambda_\varepsilon)}\,\big]\,ds$$

$$\leq \mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] - \frac{c_0}{2}p\gamma^2 \int_0^t \mathbb{E}\big[\,\|X_\gamma(s)\|^p_{L^p(\Lambda_\varepsilon)}\,\big]^{\frac{p+2}{p}}\,ds + C\gamma^{\frac{2-p}{2}}.$$

From the comparison test in Lemma 4.10 we have that

$$\mathbb{E}\big[\,\|X_\gamma(t)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] \lesssim \frac{\mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p}\,\big]}{\big(1 + Ct\mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p(\Lambda_\varepsilon)}\,\big]^{\frac{2}{p}}\big)^{\frac{p}{2}}} \vee \gamma^{-\frac{p}{2}},$$

and the result follows. $\qquad\qquad\square$

Remark 3.2 Despite its simplicity, Proposition 3.1 has the advantage of making the proof of [12, Theorem 6.1] simpler, avoiding the need for the stopping time $\tau_{\gamma,\mathfrak{m}}$ and providing sufficient control over [12, Eq. (6.7)].

Proposition 3.3 *Recall the definitions given in Sect. 2 of the processes X_γ, Z_γ and $V_\gamma :=$ $X_\gamma - Z_\gamma$. We want to remark that V_γ is not similar to of V_ε, because of the initial condition [see (2.17)]. Let $p \geq 2$ an even integer. Then there exist $v_0 > 0$, $\lambda_{j,i} > 0$ for $i = 1, 2$ and $j = 0, 1, 2$ such that for all $0 < v < v_0$ and $0 \leq s \leq t \leq T$*

$$\|V_\gamma(t, \cdot)\|^p_{L^p(\Lambda_\varepsilon)} - \|V_\gamma(s, \cdot)\|^p_{L^p(\Lambda_\varepsilon)}$$

$$+ C_1 \int_s^t \|V_\gamma(r, \cdot)\|^{p+2}_{L^p(\Lambda_\varepsilon)}\,dr + C_1 \int_s^t \big\langle V_\gamma^{p-1}(r), (-\Delta_\gamma)V_\gamma(r)\big\rangle_{\Lambda_\varepsilon}\,dr$$

$$\leq C_2 \int_s^t \sum_{j=0}^3 \sum_{i=1,2} \|H_j(Z_\gamma(r, \cdot), \mathfrak{c}_\gamma)\|^{\lambda_{j,i}}_{\mathcal{C}^{-v}(\Lambda_\varepsilon)}\,dr + \int_s^t \mathsf{Err}(r)\,dr \tag{3.2}$$

where, for every $q > 0$

$$\sup_{0 \leq r \leq T} \mathbb{E}^\gamma_{\beta,0}\big[\mathsf{Err}^q(r)\big]^{\frac{1}{q}} \lesssim C_3(p, q, T)\gamma^{\frac{p-2}{6} - 2v\frac{(p-2)}{3}}. \tag{3.3}$$

Proof The proof follows the argument provided in [18, Proposition 3.7] for the dynamical Φ_2^4 model, but applied to the model of [12], with the important difference that in our case all the operators are discrete operators. Without loss of generality, we will prove (3.2) starting at time $s = 0$ from $V_\gamma^0 = X_\gamma^0$.

In the following calculations, since there is no possibility of confusion, we will use L^p instead of $L^p(\Lambda_\varepsilon)$, and $\langle\cdot, \cdot\rangle$ instead of $\langle\cdot, \cdot\rangle_{\Lambda_\varepsilon}$. From (2.7) and (2.18) we see that $V_\gamma(t, x)$ satisfies, for $x \in \Lambda_\varepsilon, t \geq 0$

$$V_\gamma(t, x)$$

$$= V_\gamma^0(x) + \int_0^t \Delta_\gamma V_\gamma(s, x)ds + \int_0^t \gamma^{-2}K_\gamma * \big(\gamma^{-1}\tanh(\beta\gamma X_\gamma(s, x)) - X_\gamma(s, x)\big)\,ds$$

and in particular $V_\gamma(t, x)$ is continuous and weakly differentiable in time, for all $\gamma > 0$. Recall that $\beta = 1 + \gamma^2(\mathfrak{c}_\gamma + A)$ and expand the hyperbolic tangent up to third order

$$\tanh(\beta\gamma X_\gamma(s)) = \gamma X_\gamma(s) + \gamma^3(\mathfrak{c}_\gamma + A)X_\gamma(s) - \frac{\gamma^3}{3}X_\gamma^3(s)$$

$$+ \gamma^3(\beta - 1)\mathcal{O}\big(X_\gamma^3(s)\big) + \mathcal{O}\big(\gamma^5 X_\gamma^5(s)\big).$$

With the above formula the derivative of the discrete L^p norm of V_γ is calculated

$$\left\| V_\gamma(t) \right\|_{L^p}^p = \left\| V_\gamma^0 \right\|_{L^p}^p + p \int_0^t \left\langle V_\gamma^{p-1}, \Delta_\gamma V_\gamma \right\rangle(s)\, ds + \frac{1}{3}D(s) + B(s)\, ds \qquad (3.4)$$

where

$$D(s) = - \left\langle K_\gamma * V_\gamma^{p-1}(s), X_\gamma^3(s) - 3(\mathfrak{c}_\gamma + A)X_\gamma(s) \right\rangle$$

and $B(s)$ is produced by the remainder of the Taylor expansion of the hyperbolic tangent

$$B(s) \le C\gamma^2 \left\langle |V_\gamma^{p-1}|(s), \mathfrak{c}_\gamma|X_\gamma|^3(s) + |X_\gamma|^5(s) \right\rangle. \qquad (3.5)$$

where we used the fact that $|A| \le \mathfrak{c}_\gamma$ for γ small enough. We will first replace $D(s)$ with

$$\begin{aligned}
D_1(s) &:= -\left\langle V_\gamma^{p-1}(s), X_\gamma^3(s) - 3\mathfrak{c}_\gamma X_\gamma(s) \right\rangle + 3A \left\langle V_\gamma^{p-1}(s), X_\gamma(s) \right\rangle \\
&\le -\left\| V_\gamma^{p+2}(s) \right\|_{L^1} + 3\left| \left\langle V_\gamma^{p+1}(s), Z_\gamma(s) \right\rangle \right| + 3\left| \left\langle V_\gamma^p(s), H_2(Z_\gamma(s), \mathfrak{c}_\gamma) \right\rangle \right| \\
&\quad + \left| \left\langle V_\gamma^{p-1}(s), H_3(Z_\gamma(s), \mathfrak{c}_\gamma) \right\rangle \right| + 3A \left\| V_\gamma^p(s) \right\|_{L^1} + 3A \left| \left\langle V_\gamma^{p-1}(s), Z_\gamma(s) \right\rangle \right|
\end{aligned}$$
$$(3.6)$$

Let

$$L_s \stackrel{\text{def}}{=} \left\| V_\gamma(s) \right\|_{L^{p+2}}^{p+2}, \qquad K_s \stackrel{\text{def}}{=} \left\langle V_\gamma^{p-1}(s), \Delta_\gamma V_\gamma(s) \right\rangle.$$

Those terms are the good terms of (3.4), and the idea is now to bound all the other errors $|D(s) - D_1(s)|$ with expression containing L_s and K_s. In the following calculations we assume γ to be small enough such that $|A| \le \mathfrak{c}_\gamma$. The cost of replacing $D(s)$ with $D_1(s)$ is given by

$$\begin{aligned}
|D(s) - D_1(s)| &\le \sum_{x,y \in \Lambda_\varepsilon} \varepsilon^4 K_\gamma(x-y) \left| V_\gamma^{p-1}(s,y) - V_\gamma^{p-1}(s,x) \right| \\
&\quad \times \left| \left(X_\gamma^3(s,y) - X_\gamma^3(s,x) \right) - 3(\mathfrak{c}_\gamma + A)\left(X_\gamma(s,y) - X_\gamma(s,x) \right) \right| \\
&\le 3 \sum_{x,y \in \Lambda_\varepsilon} \varepsilon^4 K_\gamma(x-y) \left| V_\gamma^{p-1}(s,y) - V_\gamma^{p-1}(s,x) \right| \\
&\quad \times \left(|V_\gamma(s,y) - V_\gamma(s,x)| + |Z_\gamma(s,y) - Z_\gamma(s,x)| \right) \left(2\mathfrak{c}_\gamma + X_\gamma^2(s,x) \right).
\end{aligned}$$

Denote with

$$\begin{aligned}
D_2 &= 3 \sum_{x,y \in \Lambda_\varepsilon} \varepsilon^4 K_\gamma(x-y) \left| V_\gamma^{p-1}(s,y) - V_\gamma^{p-1}(s,x) \right| \\
&\quad \times |V_\gamma(s,y) - V_\gamma(s,x)| \left(2\mathfrak{c}_\gamma + X_\gamma^2(s,x) \right) \\
D_3 &= 3 \sum_{x,y \in \Lambda_\varepsilon} \varepsilon^4 K_\gamma(x-y) \left| V_\gamma^{p-1}(s,y) - V_\gamma^{p-1}(s,x) \right| \\
&\quad \times |Z_\gamma(s,y) - Z_\gamma(s,x)| \left(2\mathfrak{c}_\gamma + X_\gamma^2(s,x) \right).
\end{aligned}$$

We will now bound D_3 with a small multiple of L_s and K_s plus an error in (3.3), the term D_2 can be bounded in a similar way.

By $a^n - b^n = (a - b)(a^{n-1} + \cdots + b^{n-1})$ and the generalized Young inequality

$$a^{p-1} - b^{p-1} = (a - b)(a^{p-2} + \cdots + b^{p-2})$$
$$\leq |a^{p-1} - b^{p-1}| \frac{|a - b|}{2\lambda} + (|a|^{p-1} + |b|^{p-1})\lambda 2^{p-2}$$

Therefore, applying the previous inequality to each summands of D_3 and choosing $\lambda = c_1^{-1}(\gamma^{-1}\varepsilon)^2 \left| Z_\gamma(s, y) - Z_\gamma(s, x) \right| \left(2\mathfrak{c}_\gamma + X_\gamma^2(s, x) \right)$ we have that

$$D_3 \leq c_1 K_s + C c_1^{-1}(\varepsilon^2\gamma^{-2}) \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 |V_\gamma^{p-1}(s, x)||Z_\gamma(s, x)| \left(\mathfrak{c}_\gamma + X_\gamma^2(s, x) \right)$$
$$\leq c_1 K_s + c_1 L_s + C c_1^{-1} \left\| \varepsilon^2\gamma^{-2}|Z_\gamma(s)| \left(\mathfrak{c}_\gamma + X_\gamma^2(s) \right) \right\|_{L^{\frac{p-2}{3}}}^{(p-2)/3} \tag{3.7}$$

where $c_1 > 0$ can be chosen to be for instance $c_1 = 1/8$. The last term will be part of the error (3.3). Recall that $\varepsilon = \gamma^2$ and the last term of (3.7) is bounded in expectation using Proposition 3.1, Lemma 4.2 and (2.19)

$$\mathbb{E}_{\beta,0}^\gamma \left[\left\| \varepsilon^2\gamma^{-2}|Z_\gamma(s)| \left(\mathfrak{c}_\gamma + X_\gamma^2(s) \right) \right\|_{L^{\frac{p-2}{3}}}^{(p-2)/3} \right]$$
$$\leq \mathbb{E}_{\beta,0}^\gamma \left[\|Z_\gamma(s)\|_{\mathcal{C}^{-\nu}(\mathbb{T}^2)}^{2(p-2)/3} \right]^{1/2} \left((\gamma^2\mathfrak{c}_\gamma)^{\frac{2(p-2)}{3}} + \mathbb{E}_{\beta,0}^\gamma \left[\|\gamma X_\gamma(s)\|_{L^{2(p-2)/3}}^{2(p-2)/3} \right] \right)^{1/2}$$
$$\leq C(T)\gamma^{\frac{p-2}{6} - 2\nu \frac{(p-2)}{3}} \tag{3.8}$$

which is negligible if ν is small enough and $p > 2$. It is immediate to generalize (3.8) to any power, as in (3.3).

We will then bound the term $B(t)$ in (3.5) with Proposition 3.1. Using Young's inequality we have that

$$B(s) \leq C\gamma^2 \left\langle |V_\gamma^{p-1}|(s), \mathfrak{c}_\gamma + \left| Z_\gamma^2(s) + 2V_\gamma(s)Z_\gamma(s) + V_\gamma^2(s) \right| |X_\gamma|^3(s) \right\rangle$$
$$\leq \frac{1}{24} \|V_\gamma(s)\|_{L^{p+2}}^{p+2} + C\mathfrak{c}_\gamma^{\frac{p+2}{3}} \left\| (\gamma^{\frac{2}{3}} X_\gamma(s))^{p+2} \right\|_{L^1}$$
$$+ C \left\| Z_\gamma^{\frac{2p+4}{3}} (\gamma^2 X_\gamma^3(s))^{\frac{p+2}{3}} \right\|_{L^1} + \left\| Z_\gamma^{\frac{p+2}{2}} (\gamma^2 X_\gamma^3(s))^{\frac{p+2}{2}} \right\|_{L^1} + \left\| (\gamma^2 X_\gamma^3(s))^{p+2} \right\|_{L^1}.$$

The constant $1/24$ has been arbitrarily chosen in order to control $B(s)$ with a small multiple of L_s plus a quantity that will be part of the error in (3.3) and can be bounded in expectation, as we did in (3.8), by $C(T)\gamma^{\frac{p+2}{6} - 2\nu \frac{2p+4}{3}}$, which is negligible for ν small enough.

We are now in the setting of [18, Eq. (3.13)], namely the discrete process V_γ satisfies

$$\|V_\gamma(t)\|_{L^p}^p - p \int_0^t \frac{5}{6}K_s + \frac{5}{24}L_s \, ds$$
$$\leq \left\| V_\gamma^0 \right\|_{L^p}^p + \frac{p}{3} \int_0^t \sum_{j=0}^2 \binom{3}{j} \left\langle V_\gamma^{p-1+j}(s), H_{3-j}(Z_\gamma(s), \mathfrak{c}_\gamma) \right\rangle \, ds$$
$$+ A \int_0^t \left| \left\langle V_\gamma^{p-1}(s), (V_\gamma(s) + Z_\gamma(s)) \right\rangle \right| \, ds + \int_0^t \mathsf{Err}(s) ds, \tag{3.9}$$

where $\mathbb{E}^{\gamma}_{\beta,0}[|\mathsf{Err}(s)|^q]^{\frac{1}{q}} \leq C(T,p,q)\gamma^{\frac{p-2}{6}-2\nu\frac{(p-2)}{3}}$ for any positive q. We will now show that, for ν small enough and $j = 0, 1, 2$, there exist $\lambda_{j,1}, \lambda_{j,1} > 0$

$$\left\langle V_{\gamma}^{p-1+j}, H_{3-j}(Z_{\gamma}(s), \mathfrak{c}_{\gamma}) \right\rangle$$
$$\lesssim \left(L_s^{\frac{p-1+j}{p+2}-\nu\frac{p}{p+2}} K_s^{\nu} + L_s^{\frac{p-1+j}{p+2}} \right) \left\| H_{3-j}(Z_{\gamma}(s), \mathfrak{c}_{\gamma}) \right\|_{\mathcal{C}^{-\nu}(\Lambda_{\varepsilon})}$$
$$\leq \frac{1}{7} K_s + \frac{1}{30} L_s + C \sum_{i=1,2} \left\| H_{3-j}(Z_{\gamma}(s), \mathfrak{c}_{\gamma}) \right\|_{\mathcal{C}^{-\nu}(\Lambda_{\varepsilon})}^{\lambda_i} \tag{3.10}$$

where the last line follows from the Young inequality for ν sufficiently small. In a similar way,

$$A \left| \left\langle V_{\gamma}^{p-1}(s), (V_{\gamma}(s) + Z_{\gamma}(s)) \right\rangle \right| \leq \frac{1}{7} K_s + \frac{1}{30} L_s + C(A) \left(1 + \left\| Z_{\gamma}(s) \right\|_{\mathcal{C}^{-\nu}(\Lambda_{\varepsilon})}^{\lambda_i} \right) \tag{3.11}$$

Recall that all the norms appearing the proof so far are norms on the discrete lattice. The same proof of [18, Prop. 3.7] can be used to prove (3.10) and (3.11), provided the same inequalities hold in the discrete setting.

We are going to prove (3.10), (3.11) being essentially the same. Using the duality for discrete Besov spaces proved in Proposition 4.6,

$$\left\langle V_{\gamma}^{p-1+j}(s), H_{3-j}(Z_{\gamma}(s), \mathfrak{c}_{\gamma}) \right\rangle_{\Lambda_{\varepsilon}} \leq \left\| V_{\gamma}^{p-1+j}(s) \right\|_{\mathcal{B}_{1,1}^{\nu}(\Lambda_{\varepsilon})} \left\| H_{3-j}(Z_{\gamma}(s), \mathfrak{c}_{\gamma}) \right\|_{\mathcal{C}^{-\nu}(\Lambda_{\varepsilon})}.$$

We then control $\left\| V_{\gamma}^{p-1+j}(s) \right\|_{\mathcal{B}_{1,1}^{\nu}(\Lambda_{\varepsilon})}$ with Lemma 4.7. From (4.7) applied to $f(x) = V_{\gamma}^{p-1+j}(s, x)$

$$\| f \|_{\mathcal{B}_{1,1}^{\nu}(\Lambda_{\varepsilon})} \lesssim \| f \|_{L^1(\Lambda_{\varepsilon})}^{1-2\nu} \left(\sum_{x,y\in\Lambda_{\varepsilon}} \varepsilon^4 K_{\gamma}(x-y)\varepsilon^{-1}\gamma|f(x)-f(y)| \right)^{2\nu} + \| f \|_{L^1(\Lambda_{\varepsilon})}.$$

We will now estimate the term inside the brackets. For p even and $j \in \mathbb{N}$, we have

$$\left| a^{p-1+j} - b^{p-1+j} \right|^{\frac{p-1}{p-1+j}} \leq \left| a^{p-1} - b^{p-1} \right|$$

the above equation follows easily from the Minkowski inequality if one assumes a and b to have the same sign. If the a and b have different signs, the inequality follows by the fact that p is an even integer and hence the right-hand-side is equal to $|a|^{p-1} + |b|^{p-1}$. Therefore from the generalized Young inequality for $\lambda > 0$

$$\left| a^{p-1+j} - b^{p-1+j} \right| \leq \left| a^{p-1} - b^{p-1} \right| \left| a^{p-1} - b^{p-1} \right|^{\frac{j}{p-1}}$$
$$\leq \lambda \left| a^{p-1} - b^{p-1} \right| |a-b| + \frac{C}{\lambda} \left(|a|^{p-2+2j} + |b|^{p-2+2j} \right),$$

we have for every $\lambda > 0$

$$\sum_{x,y\in\Lambda_{\varepsilon}} \varepsilon^2 K_{\gamma}(x-y)\varepsilon^{-1}\gamma \left| V_{\gamma}^{p-1+j}(s,x) - V_{\gamma}^{p-1+j}(s,y) \right|$$
$$\lesssim \lambda \left\langle V_{\gamma}^{p-1}(s), \Delta_{\gamma} V_{\gamma}(s) \right\rangle + \frac{1}{\lambda} \left\| V_{\gamma}^{p+2+2j}(s) \right\|_{L^1}$$

and optimizing in λ we get (3.10). Finally we can combine (3.9) and (3.10) to conclude the proof.

We remark that the right-hand-side of (3.10) is slightly different from [18] since we have to use Δ_γ, the discrete (long range) Laplacian, which is a good approximation of the continuous Laplacian only on low frequencies. $\square$

We now turn to the proof of Theorem 2.1, which follows the lines of [18, Corollary 3.10].

Proof of Theorem 2.1 By the monotonicity of L^q norms it is sufficient to prove the statement of Theorem 2.1 for q large enough. In the following proof C will denote a constant possibly changing from line to line. The Gibbs measure $\mathbf{P}_{\beta,0}^\gamma$ is an invariant measure for the Glauber dynamic. Let $T \geq 0$ and consider

$$\mathbf{E}_{\beta,0}^\gamma\big[\, \|X_\gamma\|_{\mathcal{C}^{-\nu}}^q \,\big] = \frac{2}{T} \int_{T/2}^T \mathbb{E}_{\beta,0}^\gamma\big[\, \|X_\gamma(s)\|_{\mathcal{C}^{-\nu}}^q \,\big] ds. \tag{3.12}$$

From the definition of V_γ we can write

$$\|X_\gamma(s)\|_{\mathcal{C}^{-\nu}} \leq \|Z_\gamma(s)\|_{\mathcal{C}^{-\nu}} + \|V_\gamma(s)\|_{\mathcal{C}^{-\nu}}$$

By (2.19) proven in [12, Proposition 5.4], we have that

$$\mathbb{E}_{\beta,0}^\gamma\left[\sup_{s\in[T/4,T]} \|H_j(Z_\gamma(s), \mathfrak{c}_\gamma)\|_{\mathcal{C}^{-\nu}}^q\right] \leq C(T, q, j) \tag{3.13}$$

where the proportionality constant may depend on T and q. From the definition of the discrete Besov norm it follows that (3.13) holds true also when we replace the Besov norm with the discrete Besov Norm. By Proposition 4.4 and Lemma 4.2, for any $q > d/\nu$ and $\kappa > 0$ there exists $C(p, \kappa)$

$$\|V_\gamma(s)\|_{\mathcal{C}^{-\nu}} \leq \|\mathsf{Ext}V_\gamma(s)\|_{L^q(\mathbb{T}^2)}$$

$$\lesssim \|V_\gamma(s)\|_{L^q(\Lambda_\varepsilon)} + \varepsilon^{-\kappa} \|V_\gamma(s)\|_{L^{2q-2}(\Lambda_\varepsilon)}^{1-\frac{1}{q}} \left\{ \sum_{\substack{|x-y|=\varepsilon \\ x,y\in\Lambda_\varepsilon}} \varepsilon^2 (V_\gamma(s,y) - V_\gamma(s,x))^2 \right\}^{\frac{1}{2q}} \tag{3.14}$$

where the proportionality constant depends on q and κ.
In Proposition 3.3, using (3.13) and (3.3) we obtain that

$$\mathbb{E}_{\beta,0}^\gamma\left[\|V_\gamma(t)\|_{L^q(\Lambda_\varepsilon)}^q\right] + C_1 \int_s^t \mathbb{E}_{\beta,0}^\gamma\left[\|V_\gamma(r)\|_{L^q(\Lambda_\varepsilon)}^q\right]^{\frac{q+2}{p}} dr$$

$$+ C_1 \int_s^t \mathbb{E}_{\beta,0}^\gamma\left[\left\langle V_\gamma^{q-1}(r), (-\Delta_\gamma)V_\gamma(r)\right\rangle_{\Lambda_\varepsilon}\right] dr$$

$$\leq \mathbb{E}_{\beta,0}^\gamma\left[\|V_\gamma(s)\|_{L^q(\Lambda_\varepsilon)}^q\right] + C(q, T) \tag{3.15}$$

From Lemma 4.10, applied to $\mathbb{E}_{\beta,0}^\gamma\left[\|V_\gamma(t)\|_{L^q(\Lambda_\varepsilon)}^q\right]$ we have that there exists $C(q, T)$ such that for all $T/4 \leq s \leq t \leq T$ we have

$$\mathbb{E}_{\beta,0}^\gamma\left[\|V_\gamma(t)\|_{L^q(\Lambda_\varepsilon)}^q\right] \lesssim C(q, T) \left(|t-s|^{-\frac{q}{2}} \vee 1\right).$$

Let us choose $s = T/4$ and $t \in [T/2, T]$: from the above inequality we have that

$$\mathbb{E}^{\gamma}_{\beta,0} \left[\left\| V_\gamma(t) \right\|^p_{L^p(\Lambda_\varepsilon)} \right] \leq C(p, T). \tag{3.16}$$

At this point we only need to provide a bound for

$$\sum_{\substack{|x-y|=\varepsilon \\ x,y \in \Lambda_\varepsilon}} \varepsilon^2 (V_\gamma(s, y) - V_\gamma(s, x))^2 \lesssim \varepsilon^2 \sum_{\omega \in \Lambda_N} |\omega|^2 |\hat{V}_\gamma(s)|^2.$$

By (4.9), the operator Δ_γ approximates the discrete Laplacian only for low frequencies $|\omega| \leq \gamma^{-1}$

$$|\widehat{\Delta_\gamma V_\gamma}(s)(\omega)| = \gamma^{-2}(1 - \hat{K}_\gamma(\omega))|\hat{V}_\gamma(s)(\omega)| \geq c|\omega|^2 |\hat{V}_\gamma(s)(\omega)|.$$

On the other hand, for high frequencies $\gamma^{-1} \leq |\omega| \leq \gamma^{-2}$, we have

$$|\widehat{\Delta_\gamma V_\gamma}(s)(\omega)| \geq \gamma^{-2}(1 - \hat{K}_\gamma(\omega))|\hat{V}_\gamma(s)(\omega)| \gtrsim \gamma^{-2}|\hat{V}_\gamma(s)(\omega)|,$$

hence for all $\omega \in \Lambda_N$,

$$|\omega|^2 |\hat{V}_\gamma(s)|^2 \leq \gamma^{-2}(|\omega|^2 \wedge \gamma^{-2})|\hat{V}_\gamma(s)(\omega)|^2 \leq \gamma^{-4}(1 - \hat{K}_\gamma(\omega))|\hat{V}_\gamma(s)(\omega)|^2$$

and therefore

$$\sum_{\omega \in \Lambda_N} |\omega|^2 |\hat{V}_\gamma(s)|^2 \leq \gamma^{-2} \left\langle V_\gamma(s), (-\Delta_\gamma) V_\gamma(s) \right\rangle_{\Lambda_\varepsilon}.$$

Equation (3.15) holds for any positive even integer $q \geq 2$ and $T/4 \leq s \leq t \leq T$, if we choose $s = T/2$, $t = T$ and $q = 2$ we can conclude that

$$\int_{T/2}^{T} \sum_{|x-y|=\varepsilon} \epsilon^2 (V_\gamma(s, y) - V_\gamma(s, x))^2 ds$$

$$\leq \varepsilon^2 \gamma^{-2} \int_{T/2}^{T} \mathbb{E}^{\gamma}_{\beta,0} \left[\left\langle V_\gamma(r), (-\Delta_\gamma) V_\gamma(r) \right\rangle_{\Lambda_\varepsilon} \right] dr \leq C(T). \tag{3.17}$$

It is sufficient now to control the right-hand-side of (3.12) with (3.14). Again choose $s = T/2$, $t = T$ in (3.15): by (3.13), (3.16) and (3.17)

$$\mathbb{E}^{\gamma}_{\beta,0} \left[\left\| X_\gamma \right\|^q_{C^{-\nu}} \right] \leq \frac{2}{T} \int_{T/2}^{T} \mathbb{E}^{\gamma}_{\beta,0} \left[\left\| Z_\gamma(s) \right\|^q_{C^{-\nu}} \right] + \mathbb{E}^{\gamma}_{\beta,0} \left[\left\| V_\gamma(s) \right\|^q_{C^{-\nu}} \right] ds$$

$$\leq C(T, q, \kappa) \int_{T/2}^{T} \mathbb{E}^{\gamma}_{\beta,0} \left[\left\| Z_\gamma(s) \right\|^q_{C^{-\nu}} \right] + \mathbb{E}^{\gamma}_{\beta,0} \left\| V_\gamma(s) \right\|^q_{L^q} ds$$

$$+ C(T, q, \kappa) \, \mathbb{E}^{\gamma}_{\beta,0} \int_{T/2}^{T} \left\| V_\gamma(s) \right\|^{q-1}_{L^{2q-1}} \varepsilon^{-q\kappa} \left\{ \sum_{\substack{|x-y|=\varepsilon \\ x,y \in \Lambda_\varepsilon}} \varepsilon^2 (V_\gamma(s, y) - V_\gamma(s, x))^2 \right\}^{\frac{1}{2}} ds$$

$$\leq C(T, q, \kappa) \left(1 + \varepsilon^{-q\kappa} \varepsilon \gamma^{-1} \left\{ \int_{T/2}^{T} \mathbb{E}^{\gamma}_{\beta,0} \left[\left\langle V_\gamma(r), (-\Delta_\gamma) V_\gamma(r) \right\rangle_{\Lambda_\varepsilon} \right] dr \right\}^{1/2} \right),$$

where in the last line we applied the Cauchy-Schwarz inequality. The claim follows by choosing $\kappa > 0$ small enough. $\qquad \square$

4 Bounds for Discrete Besov Spaces

In this section we collect and prove some results in the context of discrete Besov spaces which are difficult to find. Let us first define the Besov norm on the discrete torus as follows. The definitions and proofs are based upon [12,13] and [1]. In [1, Proposition 2.10] it is proven the existence of continuous functions $\tilde{\chi}, \chi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\text{supp}(\tilde{\chi}) \subseteq B_0(4/3), \qquad \text{supp}(\chi) \subseteq B_0(8/3) \setminus B_0(3/4)$$

and such that, setting

$$\chi_{-1} \stackrel{\text{def}}{=} \tilde{\chi}, \qquad \chi_k(\cdot) \stackrel{\text{def}}{=} \chi(2^{-k} \cdot) \qquad \text{for } (k \geq 0)$$

one has $\tilde{\chi}(r) + \sum_{k=0}^{\infty} \chi(2^{-k} r) = 1$ for all $r \in \mathbb{R}^d$.

For $g : \mathbb{T}^d \to \mathbb{R}$ define the projection onto the k-th Paley-Littlewood block as

$$\delta_k g(x) = 2^{-d} \sum_{\omega \in \mathbb{Z}^d} \chi_k(\omega) \, \widehat{g}(\omega) e_\omega(x) \tag{4.1}$$

for $x \in \mathbb{T}^d$ and $k \geq -1$. [The factor 2^{-d} is such that $\sum_k \delta_k g = g$, see also (1.6).] Recall that the continuous Besov norm given in (1.7) is then defined in terms of these projections.

We now define a version of the Besov norm for functions defined in the discrete lattice. This is obtained by not only extending the function with the extension operator of Sect. 1.1, but also performing the L^p norm in (1.7) on the discrete space Λ_ε^d instead of $\mathbb{T}^d$. Let $f : \Lambda_\varepsilon \to \mathbb{R}$, for $\nu \in \mathbb{R}$, $p, q \in [1, \infty]$, with $\|\cdot\|_{\mathcal{B}_{p,q}^\nu(\Lambda_\varepsilon^d)}$ we define

$$\|f\|_{\mathcal{B}_{p,q}^\nu(\Lambda_\varepsilon^d)} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{k \geq -1} 2^{\nu k q} \, \|\delta_k \mathsf{Ext}(f)\|_{L^p(\Lambda_\varepsilon^d)}^q \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k \geq -1} 2^{\nu k} \, \|\delta_k \mathsf{Ext}(f)\|_{L^p(\Lambda_\varepsilon^d)} & \text{if } q = \infty \end{cases} \tag{4.2}$$

It is clear, from the definitions of $\mathsf{Ext}(g)$, (4.1) and (4.3), that for $x \in \Lambda_\varepsilon^d$

$$\delta_k \mathsf{Ext}(f)(x) = 2^{-d} \sum_{\omega \in \Lambda_N^d} \chi_k(\omega) \, \widehat{f}(\omega) e_\omega(x) = \eta_k^N * f(x) \qquad \text{for } x \in \Lambda_\varepsilon$$

where $\eta_k^N(x)$ is defined, for $k \geq -1$ and $x \in \mathbb{T}^d$, by

$$\eta_k^N(x) \stackrel{\text{def}}{=} 2^{-d} \sum_{\omega \in \Lambda_N^d} \chi_k(\omega) e_\omega(x), \tag{4.3}$$

where we abused the notations omitting N from the definition of η_k^N.

Remark 4.1 We could have easily avoided the definition of a discrete version of the Besov norm. There is only one point where such definition is really needed, and this is in Proposition 3.3 and Lemma 4.7, where we need to control the Besov norm with a combination of discrete L^p norms.

The next lemma is a minor generalisation of [12, Lemma B.6].

Lemma 4.2 *For $p \in [1, \infty]$ and $\kappa > 0$, there exists a constant C such that for all $f : \Lambda_\varepsilon \to \mathbb{R}$,*

$$\|\mathsf{Ext}(f)\|_{L^p(\mathbb{T}^2)} \leq C \log^2(\varepsilon^{-1}) \, \|f\|_{L^p(\Lambda_\varepsilon)} \tag{4.4}$$

$$\|\mathsf{Ext}(f)\|_{L^p(\mathbb{T}^2)} \leq C \, \|f\|_{L^p(\Lambda_\varepsilon)}$$

$$+ \, C\varepsilon^{-\kappa} \, \|f\|_{L^{2p-2}(\Lambda_\varepsilon)}^{1-\frac{1}{p}} \left\{ \sum_{\substack{|x-y|=\varepsilon \\ x,y \in \Lambda_\varepsilon}} \varepsilon^2 (f(y) - f(x))^2 \right\}^{\frac{1}{2p}}. \qquad (4.5)$$

The same lemma holds true in any dimension, with a factor $C(d) \log^d(\varepsilon^{-1})$ in (4.4).

Proof We first show (4.4). Recall that from the definition of the extension operator $\mathsf{Ext} f(x) = f(x)$ for $x \in \Lambda_\varepsilon^2$, and

$$\mathsf{Ext}(f)(x) = \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 f(z) \prod_{j=1,2} \frac{\sin\left(\pi \varepsilon^{-1}(x_j - z_j)\right)}{2 \sin\left(\frac{\pi}{2}(x_j - z_j)\right)} \qquad x \in \mathbb{T}^2.$$

Using the inequality $\sin(2\varepsilon^{-1}a)/\sin(a) \lesssim \varepsilon^{-1} \wedge |a|^{-1}$ we can bound

$$|\mathsf{Ext}(f)(x)| \lesssim \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |f(z)| \prod_{j=1,2} \varepsilon^{-1} \wedge |z_i - x_i|^{-1}.$$

For $x \in \mathbb{T}^2$, denote with $[x]_\varepsilon$ the closest point to x in Λ_ε. We can then rewrite the above inequality as

$$|\mathsf{Ext}(f)(x)| \lesssim \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |f(z + [x]_\varepsilon)| \prod_{j=1,2} \varepsilon^{-1} \wedge |z_i + [x]_{\varepsilon,i} - x_i|^{-1}.$$

we observe now that if $|z_i| \leq \varepsilon$, then $|z_i + [x]_{\varepsilon,i} - x_i|^{-1} \gtrsim \varepsilon^{-1}$, while if $|z_i| > \varepsilon$ we have that $|z_i + [x]_{\varepsilon,i} - x_i|^{-1} \lesssim |z_i|^{-1} \lesssim \varepsilon^{-1}$, hence

$$|\mathsf{Ext}(f)(x)| \lesssim \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 |f(z + [x]_\varepsilon)| \prod_{j=1,2} \varepsilon^{-1} \wedge |z_i|^{-1},$$

and taking the $L^p(\mathbb{T}^2, dx)$ norm yields

$$\left(\int_{\mathbb{T}^2} |f([x]_\varepsilon)|^p dx \right)^{\frac{1}{p}} \left(1 + 2 \sum_{1 \leq k \leq \varepsilon^{-1}} k^{-1} \right)^2 \lesssim \|f\|_{L^p(\Lambda_\varepsilon)} \log^2(\varepsilon^{-1}),$$

as claimed. The inequality (4.5) is a consequence of Hölder's inequality

$$\|\mathsf{Ext}(f)\|_{L^p(\mathbb{T}^2)}^p \lesssim \|f\|_{L^p(\Lambda_\varepsilon)}^p + \int_{|y| \leq \varepsilon/2} |\mathsf{Ext} f(x+y) - f(x)|^p d^2 y$$

$$\leq \|f\|_{L^p(\Lambda_\varepsilon)}^p + \|\mathsf{Ext} f\|_{L^{2p-2}(\mathbb{T}^2)}^{p-1} \left(\sum_{x \in \Lambda_\varepsilon} \int_{|y| \leq \varepsilon/2} |f(x+y) - f(x)|^2 d^2 y \right)^{\frac{1}{2}}$$

$$\leq \|f\|_{L^p(\Lambda_\varepsilon)}^p + \|\mathsf{Ext} f\|_{L^{2p-2}(\mathbb{T}^2)}^{p-1} \left(\sum_{x \in \Lambda_\varepsilon} \int_{|y| \leq \varepsilon/2} |f(x+y) - f(x)|^2 d^2 y \right)^{\frac{1}{2}}$$

$$\leq \|f\|_{L^p(\Lambda_\varepsilon)}^p + \varepsilon \|\mathsf{Ext} f\|_{L^{2p-2}(\mathbb{T}^2)}^{p-1} \|\mathsf{Ext} f\|_{\dot{H}^1(\mathbb{T}^2)}$$

where we denoted by $\|\mathsf{Ext}\,f\|_{\dot{H}^1(\mathbb{T}^2)}$ the homogeneous Sobolev seminorm. From the definition of the extension operator it is easy to see that

$$\|\mathsf{Ext}\,f\|^2_{\dot{H}^1(\mathbb{T}^2)} = \sum_{\omega\in\Lambda_N} |\omega|^2 |\hat{f}(\omega)|^2 \lesssim \sum_{\substack{|x-y|=\varepsilon \\ x,y\in\Lambda_\varepsilon}} \varepsilon^2 \frac{(f(y)-f(x))^2}{\varepsilon^2},$$

and an application of (4.4) yields (4.5). □

Let $d \in \mathbb{N}^+$ and Λ_ε^d a discretisation of the d-dimensional torus $\mathbb{T}^d = [-1,1]^d$.

Lemma 4.3 *Let* $\chi : \mathbb{R}^d \to \mathbb{R}$ *be a smooth function with compact support. For every* $p \in [0,\infty]$ *we have* $\sup_{\lambda\in(0,1)} \lambda^{d\left(1-\frac{1}{p}\right)} \left\|\sum_{w\in\Lambda_N} \chi(\lambda w)e_w\right\|_{L^p(\Lambda_\varepsilon^d)} < \infty.$

The above result is proven in [12, Lemma B.1] for the continuous $L^{(\mathbb{T}^d)}$ norm, the generalisation to Λ_ε^d follows trivially from the same argument. See also [13] for a proof in the case of a continuous Fourier transform. We quote in the next proposition a useful embedding between Besov and L^p spaces proven, for instance, in [1, Proposition 2.39].

Proposition 4.4 *For any* $\nu > 0$, *and* $p \geq \frac{d}{\nu}$ *there exists* $C > 0$

$$\|f\|_{\mathcal{B}^{-\nu}_{\infty,\infty}(\mathbb{T}^d)} \leq C \|f\|_{L^p(\mathbb{T}^d)}$$

We also mention some classical estimates for Besov norms restated in case of discrete Besov spaces. The proofs are omitted since they follow closely their continuous counterparts.

Proposition 4.5 (Product estimates for discrete Besov spaces) *Let* $\beta < 0 < \alpha$ *and* $p,q \in [1,\infty]$. *There exists* $C > 0$ *such that, uniformly over* $\varepsilon \in (0,1]$,

$$\|f\|_{\mathcal{B}^\beta_{p,q}(\Lambda_\varepsilon)} \leq C \|f\|_{\mathcal{B}^\alpha_{p,q}(\Lambda_\varepsilon)} \|g\|_{\mathcal{B}^\beta_{p,q}(\Lambda_\varepsilon)}.$$

Proposition 4.6 (Duality for discrete Besov spaces) *Let* $\alpha \in \mathbb{R}$, $p,q,p',q' \geq 1$ *with* $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. *There exists* $C > 0$ *such that, uniformly over* $\varepsilon \in (0,1]$,

$$\langle f,g\rangle_{\Lambda_\varepsilon} \leq C \|f\|_{\mathcal{B}^\alpha_{p,q}(\Lambda_\varepsilon)} \|g\|_{\mathcal{B}^{-\alpha}_{p',q'}(\Lambda_\varepsilon)}.$$

The next proposition is the main technical tool of the paper, and it allows to control the discrete Besov norm with the same discrete Laplacian of the dynamic. Recall that, in the case of continuous Besov spaces, for a differentiable function f and $\nu \in (0,1)$, one has [13, Proposition 3.8]

$$\|f\|_{\mathcal{B}^\nu_{1,1}} \lesssim \|f\|_{L^1} + \|f\|^{1-\nu}_{L^1} \|\nabla f\|^\nu_{L^1}. \tag{4.6}$$

We have the following analogue of this result.

Lemma 4.7 *For* $f : \Lambda_\varepsilon \to \mathbb{R}$ *and* $\nu \in (0,1/2)$

$$\|f\|_{\mathcal{B}^\nu_{1,1}(\Lambda_\varepsilon)} \lesssim \|f\|^{1-2\nu}_{L^1(\Lambda_\varepsilon)} \left(\sum_{x,y\in\Lambda_\varepsilon} \varepsilon^4 K_\gamma(x-y)\varepsilon^{-1}\gamma|f(x)-f(y)|\right)^{2\nu} + \|f\|_{L^1(\Lambda_\varepsilon)} \tag{4.7}$$

where the constant is independent of ε *or* f.

Remark 4.8 Here we write $x - y$ for the shortest element in the corresponding equivalence class (viewing Λ_ε as a quotient of $\mathbb{Z}_\varepsilon^2$ with $\mathbb{Z}_\varepsilon = \varepsilon\mathbb{Z}$). In cases where this might be ambiguous one has $K_\gamma(x - y) = 0$ for all possible interpretations anyway. Another equivalent interpretation is that one of the variables runs over Λ_ε and the other one runs over all of $\mathbb{Z}_\varepsilon$, f being identified with its periodic continuation.

Compare (4.7) with (4.6). The factor 2 in front of ν depends on the scale at which Δ_γ changes its behaviour, and this is not the best result that is possible to obtain.

Proof Rewrite the definition of $\|f\|_{\mathcal{B}_{1,1}^\nu(\Lambda_\varepsilon)}$ in (4.2) as

$$\|f\|_{\mathcal{B}_{1,1}^\nu(\Lambda_\varepsilon)} = \sum_{k \geq -1} 2^{\nu k} \left\| \eta_k^N * f \right\|_{L^1(\Lambda_\varepsilon)} \tag{4.8}$$

where η_k^N are the projections on the Paley-Littlewood blocks defined in (4.3). In the discrete case the summation over k extends up to a multiple of $\log(\varepsilon^{-1})$. In the proof, since there is no possibility of confusion, we will use L^p instead of $L^p(\Lambda_\varepsilon)$. We will divide the sum into

$$\sum_{-1 \leq k \leq L} 2^{\nu k} \left\| \eta_k^N * f \right\|_{L^1} + \sum_{L < k \leq -\log_2(\varepsilon)} 2^{\nu k} \left\| \eta_k^N * f \right\|_{L^1}$$

where L will be chosen later. We bound the first part with

$$\sum_{-1 \leq k \leq L} 2^{\nu k} \left\| \eta_k^N * f \right\|_{L^1} \leq \sum_{-1 \leq k \leq L} 2^{\nu k} \sup_{k' \leq L} \left\| \eta_{k'}^N \right\|_{L^1} \|f\|_{L^1} \lesssim 2^{\nu L} \|f\|_{L^1}. \tag{4.9}$$

In order to control the second summation we will now prove, for $k \geq 0$, the inequality

$$\left\| \eta_k^N * f \right\|_{L^1} \lesssim \left(2^{-k} \vee \varepsilon\gamma^{-1} \right) \sum_{x, y \in \Lambda_\varepsilon} \varepsilon^4 K_\gamma(x - y) \frac{|f(y) - f(x)|}{\varepsilon\gamma^{-1}}. \tag{4.10}$$

If $k \geq 0$ the projection kernel η_k^N has mean zero and therefore

$$\left\| \eta_k^N * f \right\|_{L^1} = \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 \left| \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 \eta_k^N(-y) \Big(f(x + y) - f(x) \Big) \right|.$$

At this point the treatment differs from the proof of [13, Proposition 3.8], because of the particular form of the Laplacian. The definition of K_γ (in particular the continuity of $\mathfrak{K}$) implies that there exists $b_0 > 0$ such that

$$\inf_{|w| \leq b_0\varepsilon\gamma^{-1}} \sum_{z \in \mathbb{Z}_\varepsilon} \varepsilon^2 \big(K_\gamma(z) \wedge K_\gamma(w - z) \big) \geq 1/2.$$

If $|y| \leq b_0\varepsilon\gamma^{-1}$, then

$$\left| f(x + y) - f(x) \right| \leq 2 \Big(\sum_{z \in \mathbb{Z}_\varepsilon} \varepsilon^2 \big(K_\gamma(z) \wedge K_\gamma(y - z) \big) \Big) \Big| f(x + y) - f(x) \Big|$$

$$\leq 2\varepsilon^2 \sum_{z \in \mathbb{Z}_\varepsilon} K_\gamma(y - z) \Big| f(x + y) - f(x + z) \Big| + K_\gamma(z) \Big| f(x + z) - f(x) \Big|.$$

If $|y| \geq b_0\varepsilon\gamma^{-1}$ on the other hand, then there exists a path $\{y_0, y_1, \ldots, y_n\}$ in $\mathbb{Z}_\varepsilon$ of length n proportional to $|y|\gamma\varepsilon^{-1}$ connecting $y_0 = 0$ with $y_n = y$ and such that $|y_{j+1} - y_j| \leq b_0\varepsilon\gamma^{-1}$

for $j = 0, \ldots, n - 1$. We can then apply the above inequality to every step of the path. Combining these bounds, we obtain

$$\left\| \eta_k^N * f \right\|_{L^1} \lesssim \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 |\eta_k^N(-y)| \{|y| \gamma \varepsilon^{-1} \vee 1\} \sum_{x \in \Lambda_\varepsilon, z \in \mathbb{Z}_\varepsilon} \varepsilon^4 K_\gamma(z) |f(x+z) - f(x)|,$$

(4.11)

and (4.10) follows from the fact that

$$\sum_{y \in \Lambda_\varepsilon} \varepsilon^2 |\eta_k^N(y)| \{|y| \varepsilon^{-1} \gamma \vee \gamma\} \lesssim \left\| \eta_k^N \right\|_{L^1} + \varepsilon^{-1} \gamma 2^{-k} \sum_{y \in \Lambda_\varepsilon} 2^k |y| |\eta_k^N(y)| \lesssim 1 \vee \varepsilon^{-1} \gamma 2^{-k}.$$

Summing over k yields

$$\sum_{L < k \leq \log_2(\varepsilon^{-1})} 2^{\nu k} \left\| \eta_k^N * f \right\|_{L^1}$$

$$\lesssim \sum_{L < k \leq \log_2(\varepsilon^{-1})} 2^{\nu k} \{\varepsilon \gamma^{-1} \vee 2^{-k}\} \sum_{x \in \Lambda_\varepsilon, z \in \mathbb{Z}_\varepsilon} \frac{\varepsilon^4 K_\gamma(z)}{\varepsilon \gamma^{-1}} |f(x+z) - f(x)|.$$

At this point we use the fact that $\varepsilon \gamma^{-1} \vee 2^{-k} \leq 2^{-\frac{k}{2}}$ for $k \leq -\log_2(\varepsilon) = -2 \log_2(\varepsilon \gamma^{-1})$ and, recalling (4.8) and (4.9), we obtain

$$\|f\|_{\mathcal{B}_{1,1}^\nu(\Lambda_\varepsilon)} \lesssim 2^{\nu L} \|f\|_{L^1} + 2^{\left(\nu - \frac{1}{2}\right) L} \sum_{x \in \Lambda_\varepsilon, z \in \mathbb{Z}_\varepsilon} \varepsilon^3 \gamma K_\gamma(z) |f(x+z) - f(x)|.$$

The claim now follows by optimising this expression over L. (The second term in (4.7) comes from the fact that we had to impose $L > 1$.) $\qquad \square$

The next proposition quantifies the decay in the Fourier space of the kernel K_γ used in the article. The proof is given in [12, Lemma 8.2]

Proposition 4.9 (Estimates on the kernel) *For $\omega \in \Lambda_N$ and γ small enough the following inequalities hold*

- *There exists a positive constant $C > 0$ such that,*

$$|\hat{K}_\gamma(\omega)| \leq 1 \wedge \frac{\gamma^{-2}}{|\omega|^2}$$

- *There exists a positive constant $c > 0$ such that, for $|\omega| \geq \gamma^{-1}$*

$$1 - \hat{K}_\gamma(\omega) \geq c \left(|\gamma \omega|^2 \wedge 1\right)$$

For completeness, we state the following simple but crucial comparison test which can be found in this specific form in [18].

Lemma 4.10 (Comparison test) *Let $\lambda > 1$ and $f : [0, T] \to \mathbb{R}^+$ differentiable satisfying for $t \in [0, T]$*

$$f'(t) + 2c_1 (f(t))^\lambda \leq c_2.$$

Then for $t \in [0, T]$

$$f(t) \leq \frac{f(0)}{\left(1 + c_1(\lambda - 1)t f(0)^{\lambda - 1}\right)^{\frac{1}{\lambda - 1}}} \vee \left(\frac{c_2}{c_1}\right)^{\frac{1}{\lambda}} \leq (c_1(\lambda - 1)t)^{-\frac{1}{\lambda - 1}} \vee \left(\frac{c_2}{c_1}\right)^{\frac{1}{\lambda}}.$$

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