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On a Homogenization Problem

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Abstract We study the average Green's function of stochastic, uniformly elliptic operators of divergence form on $Zd\mathbb{Z}^d$. When the randomness is independent and has small variance, we prove regularity of the Fourier transform of the self-energy. The proof relies on the Schur complement formula and the analysis of singular integral operators combined with a Steinhaus system.

Keywords Stochastic homogenization · Schur complement · Singular integral operators · Uniformly elliptic · Self-energy

1 Introduction and Statement

Let $\{\sigma_x(\omega); x \in \mathbb{Z}^d\}$ be i.i.d., $\mathbb{E}[\sigma_x] = 0$ and assume moreover,

$$\|\sigma_x\|_\infty \leq C. \tag{1.1}$$

Consider the finite difference random operator

$$L_\omega = -\Delta + \delta \nabla^* \sigma \nabla \tag{1.2}$$

$\nabla f(x) = (f(x+e_1) - f(x), f(x+e_2) - f(x), \dots f(x+e_d) - f(x))$. Here e_i are the unit lattice vectors and f is defined on $\mathbb{Z}^d$.

Consider the stochastic equation

$$L_\omega u_\omega = f. \tag{1.3}$$

Let $\langle \cdot \rangle$ denote the expectation. Formally we have

$$\mathbb{E}[u_\omega] \equiv \langle u_\omega \rangle = \langle L_\omega^{-1} \rangle f \quad \text{and} \quad A\langle u_\omega \rangle = f \tag{1.4}$$

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with

$$A = \langle L_\omega^{-1} \rangle^{-1}. \tag{1.5}$$

Since A is translation invariant it can be expressed as a multiplication operator $\hat{A}(\xi)$ in Fourier space. We prove the following result about the regularity of $\hat{A}(\xi)$.

Theorem *With the above notation, given $\varepsilon > 0$, there is $\delta_0 > 0$ such that for $|\delta| < \delta_0$, A has the form*

$$A = \nabla^*(1 + K_1)\nabla \tag{1.6}$$

with K_1 given by a convolution operator such that $\hat{K}_1(\xi)$ has $d - \varepsilon$ derivatives or

$$K_1(x - y) = O(\delta[1 + |x - y|]^{-(2d-\varepsilon)}) \tag{1.7}$$

for $x, y \in \mathbb{Z}^d$.

Remark (1) In the general case when σ_x defines an ergodic process, homogenization was developed by Kozlov [3] and Papanicolaou and Varadhan [5]. However, the regularity of K_1 is was not addressed in these papers. See [2] for a review of results in homogenization.

(2) This paper is closely related to an unpublished note of Sigal [6], where the exact same problem is considered. In [6] an asymptotic expansion for K_1 is given and (1.7) verified up to the leading order by applying the Feshbach-Schur formula. What we basically manage to do here is to control the full series. The argument is rather simple, but contains perhaps some novel ideas that may be of independent interest in the study of the averaged dynamics of stochastic PDE's.

(3) In Bach, Fröhlich, Sigal, [1] a multi-scale version of Feshbach-Schur were used the study an atom coupled to an electromagnetic field.

(4) In the context of homogenization, the same formalism was developed by J. Conlon, A. Naddaf in [4]. This paper proved some regularity of K_1 under certain mixing conditions.

(5) It is an open question whether the same strong regularity holds assuming that only $|\delta| < 1$.

(6) The author is grateful to T. Spencer for bringing the problem to his attention and a few preliminary discussions. Thanks also to the referee and W. Schlag for clarifying the exposition. He has also benefited from some comments of A. Gloria.

2 The Expansion

We briefly recall the derivation of the multi-linear expansion for K_1 established in [6]. Denote $b = \delta\sigma$, $P = \mathbb{E}$, $P^\perp = 1 - \mathbb{E}$. Using the Feshbach-Shur map to the block decomposition

$$\begin{pmatrix} (P, P) & (P, P^\perp) \\ (P^\perp, P) & (P^\perp, P^\perp) \end{pmatrix}$$

we obtain

$$PL^{-1}P = \left(PLP - PLP^\perp(P^\perp LP^\perp - io)^{-1}P^\perp LP\right)^{-1}$$

Since $PLP = -\Delta P$, $PLP^\perp = P\nabla^* b\nabla P^\perp$, $P^\perp LP = P^\perp\nabla^* b\nabla P$, we obtain

$$(-\Delta P - \nabla^* Pb\nabla(P^\perp LP^\perp)^{-1}\nabla^* b\nabla P)^{-1}. \tag{2.1}$$

Next, $P^\perp L P^\perp = (-\Delta)\big(1 + (-\Delta)^{-1}\nabla^* P^\perp b\nabla\big)P^\perp$ and we expand

$$(P^\perp L P^\perp)^{-1} = \Big[1 - (-\Delta)^{-1}\nabla^* P^\perp b\Big(\sum_{n\geq 0}(-1)^n (K P^\perp b)^n\Big)\nabla P^\perp\Big](-\Delta)^{-1} \tag{2.2}$$

where we denoted K the convolution singular operator

$$K = \nabla(-\Delta)^{-1}\nabla^*. \tag{2.3}$$

Substitution of (2.2) in (2.1) gives

$$\langle\nabla^* b\nabla(P^\perp L P^\perp)^{-1}\nabla^* b\nabla\rangle = \sum_{n\geq 1}(-1)^{n+1}\nabla^*\langle b(K P^\perp b)^n\rangle\nabla. \tag{2.4}$$

Hence

$$\langle L_\omega^{-1}\rangle = \big(-\Delta + (2.4)\big)^{-1}$$

and

$$A = -\Delta + (2.4) = \nabla^*\big(1 + K_1\big)\nabla$$

with

$$K_1 = \sum_{n\geq 1}(-1)^n\langle b(K P^\perp b)^n\rangle. \tag{2.5}$$

Remains to analyze the individual terms in (2.5). In doing so, without loss of generality, we treat K as a scalar singular integral operator.

3 A Deterministic Inequality

Our first ingredient in controlling the multi-linear terms in the series (2.5) is the following (deterministic) bound on composing singular integral and multiplication operators.

Lemma 1 *Let K be a (convolution) singular integral operator acting on $\mathbb{Z}^d$ and $\sigma_1, \ldots, \sigma_s \in \ell^\infty(\mathbb{Z}^d)$. Define the operator*

$$T = K\sigma_1 K\sigma_2 \cdots K\sigma_s. \tag{3.1}$$

Then T satisfies the pointwise bound

$$|T(x_0, x_s)| < |x_0 - x_s|^{-d+\varepsilon}(C\varepsilon^{-1})^s\prod_1^s\|\sigma_j\|_\infty \tag{3.2}$$

for all $\varepsilon > 0$.

Proof Firstly, recalling the well-known bound

$$\|K\|_{p\to p} < \frac{c}{p-1} \quad \text{for} \quad 1 < p \leq 2 \tag{3.3}$$

and normalizing $\|\sigma_j\|_\infty = 1$, we get

$$\|T\|_{p\to p} + \|T^*\|_{p\to p} < \Big(\frac{c}{p-1}\Big)^s. \tag{3.4}$$

In particular

$$\max_x \Big(\sum_y |T(x,y)|^p\Big)^{\frac{1}{p}} + \max_y \Big(\sum_x |T(x,y)|^p\Big)^{\frac{1}{p}} < \Big(\frac{c}{p-1}\Big)^s. \tag{3.5}$$

Next, write

$$T_s(x_0,x_s) = \sum_{x_1,\dots,x_{s-1}} K(x_0-x_1)\sigma_1(x_1)K(x_1-x_2)\sigma_2(x_2)\cdots K(x_{s-1}-x_s)\sigma_s(x_s). \tag{3.6}$$

We use a dyadic decomposition according to $\max_{0\le j<s}|x_j - x_{j+1}|$. Specify $R \gg 1$ and $0 \le i < s$ satisfying

$$|x_i - x_{i+1}| \sim R \tag{3.7}$$

$$\max_j |x_j - x_{j+1}| \lesssim R. \tag{3.8}$$

In particular $|x_0 - x_s| \lesssim sR$. The corresponding contribution to (3.6) may be bounded by

$$\sum_{\substack{x_i,x_{i+1}\\ |x_i-x_{i+1}|\sim R}} |T_i^{(*)}(x_0,x_i)|\,|K(x_i-x_{i+1})|\,|T_{s-1-i}^{(*)}(x_{i+1},x_s)| \tag{3.9}$$

with $T_i^{(*)}$ obtained from formula (3.6) with additional restriction (3.8). The bound (3.5) also holds for $T_i^{(*)}$. Since $|K(z)| < |z|^{-d}$ (where we denote $|\ | = |\ | + 1$), it follows from (3.5), (3.7), (3.8) and Hölder's inequality that

$$\begin{aligned} (3.9)\ &\le \Big(\frac{c}{p-1}\Big)^s \Big(\sum_{x_i,x_{i+1},|x_0-x_i|<sR,|x_0-x_{i+1}|<sR} 1\Big)^{1/p'} R^{-d} \quad \Big(p' = \frac{p}{p-1}\Big) \\ &< \Big(\frac{c}{p-1}\Big)^s (sR)^{2d(p-1)} R^{-d} < (C\varepsilon^{-1})^s R^{-d+\varepsilon} \end{aligned} \tag{3.10}$$

by taking p such that $2d(p-1) = \varepsilon$. Then

$$\sum_{0\le i<s} \sum_{R\in 2^{\mathbb{N}},\ R\gtrsim |x_0-x_s|/s} (3.10) < s^{d+1}(C\varepsilon^{-1})^s |x_0-x_s|^{-d+\varepsilon}$$

proving (3.2). □

4 Use of the Randomness

Returning to (2.5), the randomness and the projectors will allow us to further improve the pointwise bounds on $\langle b(KP^{\perp}b)^n\rangle$.

Write

$$b(KP^{\perp}b)^n(x_0,x_n) = \sum_{x_1,\dots,x_{n-1}\in\mathbb{Z}^d} b_{x_0}K(x_0,x_1)P^{\perp}b_{x_1}K(x_1,x_2)P^{\perp}b_{x_2}\dots b_{x_n}. \tag{4.1}$$

Note that evaluation of $\langle b(KP^{\perp}b)^n\rangle$ by summation over all diagrams would produce combinatorial factors growing more rapidly than C^n and hence we need to proceed differently.

Let again $R \gg 1$ and $0 \le j_0 < n$ s.t.

$$|x_{j_0} - x_{j_0+1}| \sim R \quad \text{and} \quad \max_{0 \le j < n} |x_j - x_{j+1}| \lesssim R. \tag{4.2}$$

We denote

$$S = \left\{ \begin{array}{l} (x_1, \dots, x_{n-1}) \in (\mathbb{Z}^d)^{n-1} \text{ and } \{x_0, \dots, x_{j_0}\} \cap \{x_{j_0+1}, \dots, x_n\} \neq \phi \\ \text{subject to (4.2)} \end{array} \right\} \tag{4.3}$$

$\mathbb{E}[(4.1)]$ only involves the irreducible graphs in (4.1), due to the presence of the projection operators $P^\perp$. This means that

$$\mathbb{E}[b_{x_0} P^\perp b_{x_1} P^\perp b_{x_2} \cdots P^\perp b_{x_n}] = 0$$

whenever there exists some $0 \le j < n$ such that $\{x_0, \dots, x_j\} \cap \{x_{j+1}, \dots, x_n\} = \phi$. From the preceding, it follows in particular that

$$\mathbb{E}[(4.1)] = \mathbb{E}[(4.4)]$$

defining

$$(4.4) = \sum_{(x_1, \dots, x_{n-1}) \in S} b_{x_0} K(x_0, x_1) P^\perp b_{x_1} \dots b_{x_n}.$$

Our goal is to prove

Lemma 2 *For all $\varepsilon > 0$, we have*

$$|\mathbb{E}[(4.4)]| < C_\varepsilon^n R^{-d+4\varepsilon} |x_0 - x_n|^{-d} \tag{4.5}$$

which clearly implies the Theorem.

For definition (4.3)

$$S = \bigcup_{\substack{0 \le j_1 \le j_0 \\ j_0 < j_2 \le n}} S_{j_1, j_2}$$

where

$$S_{j_1, j_2} = \{(x_1, \dots, x_{n-1}) \in (\mathbb{Z}^d)^{n-1} \text{ subject to (4.2) and } x_{j_1} = x_{j_2}\}. \tag{4.6}$$

Note that these sets S_{j_1, j_2} are not disjoint and we will show later how to make them disjoint at the cost of another factor C^n.

Consider the sum

$$\sum_{(x_1, \dots, x_{n-1}) \in S_{j_1, j_2}} b_{x_0} K(x_0, x_1) P^\perp b_{x_1} \cdots b_{x_n} = (4.7).$$

We claim that for all $\varepsilon > 0$

$$|(4.7)| < C_\varepsilon^n R^{-d+4\varepsilon} |x_0 - x_n|^{-d} \tag{4.8}$$

(thus without taking expectation).

To prove (4.8), factor (4.7) as

$$\begin{aligned} &(K P^\perp b)^{j_1} (x_0, x_{j_1}) (K P^\perp b)^{j_0 - j_1} (x_{j_1}, x_{j_0}) K(x_{j_0}, x_{j_0+1}) P^\perp b_{x_{j_0+1}}, \\ &\quad (K P^\perp b)^{j_2 - j_0} (x_{j_0+1}, x_{j_1}) (K P^\perp b)^{n - j_2 - 1} (x_{j_1}, x_n) \end{aligned} \tag{4.9}$$

with summation over $x_{j_0}, x_{j_0+1}, x_{j_1}$.

Using the deterministic bound implied by Lemma 1

$$|(KP^{\perp}b)^{\ell}(x,y)| < C_\varepsilon^{\ell}|x-y|^{-d+\varepsilon} \tag{4.10}$$

we may indeed estimate

$$\begin{aligned}|(4.7)| &< R^{-d}C_\varepsilon^n \sum_{x_{j_0},x_{j_0+1},x_{j_1}} |x_0-x_{j_1}|^{-d+\varepsilon}|x_{j_1}-x_{j_0}|^{-d+\varepsilon}|x_{j_0+1}-x_{j_1}|^{-d+\varepsilon}|x_{j_1}-x_n|^{-d+\varepsilon}\\ &< C_\varepsilon^n R^{-d+4\varepsilon}|x_0-x_n|^{-d}.\end{aligned}$$

Remains the disjointification issue for the sets S_{j_1,j_2}.

Our device to achieve this may have an independent interest. Define the disjoint sets

$$S'_{j_1,j_2} = S_{j_1,j_2}\Big\backslash\Big(\bigcup_{\substack{j<j_1\\ j_0<j'\le n}} S_{j,j'} \cup \bigcup_{j_0<j'<j_2} S_{j_1,j'}\Big). \tag{4.11}$$

Replacing S_{j_1,j_2} by S'_{j_1,j_2} in (4.7), we prove that the bound (4.8) is still valid.

Note that, by definition, $(x_1,\dots,x_{n-1}) \notin \bigcup_{\substack{j<j_1\\ j_0<j'\le n}} S_{j.j'}$ means that

$$\{x_0,\dots,x_{j_1-1}\}\cap\{x_{j_0+1},\dots,x_n\} = \phi. \tag{4.12}$$

n Thus we need to implement the condition (4.12) in the summation (4.7) at the cost of a factor bounded by C^n.

We introduce an additional set of variables $\bar\theta = (\theta_x)_{x\in\mathbb{Z}^d}$, $\theta_x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and consider the corresponding Steinhaus system. Denote $E = \{0,1,\dots,j_1-1\}$, $F = \{j_0+1,\dots,n\}$. Replace in (4.7)

$$\begin{cases} b_{x_j} \text{ by } b_{x_j}e^{i\theta_{x_j}} \text{ for } j\in E\\ b_{x_j} \text{ by } b_{x_j}e^{-i\theta_{x_j}} \text{ for } j\in F.\end{cases} \tag{4.13}$$

After this replacement, (4.7) becomes a Steinhaus polynomial in $\bar\theta$, i.e. we obtain

$$\sum_{(x_1,\dots,x_{n-1})\in S_{j_1,j_2}} e^{i(\sum_{j\in E}\theta_{x_j}-\sum_{k\in F}\theta_{x_k})} b_{x_0}K(x_0,x_1)P^{\perp}b_{x_1}\dots b_{x_n} \tag{4.14}$$

for which the estimate (4.8) still holds (uniformly in $\bar\theta$).

Next, performing a convolution with the Poisson kernel $P_t(\theta_x) = \sum_{n\in\mathbb{Z}} t^{|n|}e^{in\theta_x}$ in each θ_x (which is a contraction), gives

$$\begin{aligned}&\int (4.14)\prod_x P_t(\theta'_x-\theta_x)\frac{d\theta_x}{2\pi}\\ &\quad= \sum_{(x_1,\dots,x_{n-1})\in S_{j_1,j_2}} t^{w_{\bar x}} e^{i(\sum_{j\in E}\theta'_{x_j}-\sum_{k\in F}\theta'_{x_k})} b_{x_0}K(x_0,x_1)P^{\perp}\cdots b_{x_n}\end{aligned} \tag{4.15}$$

where $0\le t\le 1$ and

$$w_{\bar x} = \sum_x \Big|\, |\{j\in E; x_j = x\}| - |\{k\in F; x_k = x\}|\Big| \le |E|+|F| = D.$$

Note that the condition $\{x_j, j \in E\} \cap \{x_k; k \in F\} = \phi$ is equivalent to $w_{\bar{x}} = D$ and (4.14) obtained by projection of (4.15), viewed as polynomial t, on the top degree term. Our argument is then concluded by the standard Markov brothers' inequality.

Lemma 3 *Let $P(t)$ be a polynomial of degree $\leq D$. Then*

$$\max_{-1\leq t\leq 1} |P^{(k)}(t)| \leq \frac{D^2(D^2-1^2)(D^2-2^2)\cdots(D^2-(k-1)^2)}{1, 3, 5\ldots(2k-1)} \max_{-1\leq t\leq 1} |P(t)|. \quad (4.16)$$

Indeed, we conclude that for all $\bar{\theta}$

$$\Bigg| \sum_{\substack{(x_1,\ldots,x_{n-1})\in S_{j_1,j_2} \\ w_{\bar{x}}=D}} e^{i\left(\sum_{j\in E}\theta_{x_j} - \sum_{k\in F}\theta_{x_k}\right)} b_{x_0} K(x_0, x_1) P^{\perp} \ldots b_{x_n} \Bigg| < C^n. \quad (4.8)$$

and set then $\bar{\theta} = 0$.

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