J Stat Phys (2018) 170:286–350
https://doi.org/10.1007/s10955-017-1934-4

A Mechanical Model of Brownian Motion for One Massive Particle Including Slow Light Particles

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Received: 16 November 2016 / Accepted: 29 November 2017 / Published online: 6 December 2017
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Abstract We provide a connection between Brownian motion and a classical mechanical system. Precisely, we consider a system of one massive particle interacting with an ideal gas, evolved according to non-random mechanical principles, via interaction potentials, without any assumption requiring that the initial velocities of the environmental particles should be restricted to be "fast enough". We prove the convergence of the (position, velocity)-process of the massive particle under a certain scaling limit, such that the mass of the environmental particles converges to 0 while the density and the velocities of them go to infinity, and give the precise expression of the limiting process, a diffusion process.

Keywords Infinite particle systems · Classical mechanics · Markov processes · Diffusion · Convergence · Brownian motion

Mathematics Subject Classification 70F45 · 34F05 · 60B10 · 60J60

1 Introduction

In this paper, we consider a mechanical model consisting of a massive particle in an ideal gas (i.e., a Rayleigh gas model), and present a mathematical proof that under certain conditions, when the mass m of a gas particle converges to 0, while the number density and velocity distribution of the gas scale like $m^{-\frac{1}{2}}$, the motion of the massive particle converges to a Brownian motion (i.e., a Langevin process).

Financially supported by Grant-in-Aid (No. 17K05290), Japan Society for the Promotion of Science.

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1.1 Preliminary

Brownian motion, first observed by Brown in 1827, is a well-known physical phenomenon concerning the dynamics of a small particle immersed into a fluid in equilibrium, e.g., a grain of pollen in a glass of water [14]. It is an interesting problem in mathematical physics to describe the Brownian motion phenomenology by classical mechanical models.

The first physical explanation of Brownian motion was given by Einstein: the motion is coming about as a result of the repeated collisions of the massive particle with the numerous much smaller but faster fluid atoms. In more mathematical terms the explanation is often presented in the following rough way: since the massive particle is collided by a big number of very light water particles, if we could assume that the interactions from each light particle at each time are independent, then by the central limit theorem for the sum of *i.i.d.* random variables, this will give in a suitable limit the Brownian motion.

However, this assumption of independence can hardly be justified, even in a model where only interactions through collisions are considered, since there exists the possibility of re-collisions. This becomes a more evident and significant drawback when considering the model of interactions caused by potentials. Therefore, the actual motion of the massive particle can not be explained as resulting from a sum of *i.i.d.* random variables, it is not even a Markov process.

So in order to study this phenomenon more precisely, one needs to construct some model which is consistent with the mentioned dependence on the past. In such a model, a massive particle interacts with a gas of infinitely many light particles, with the dynamics fully deterministic and Newtonian, as long as the initial condition is given. The only source of randomness is from the initial configuration of the light particles. The problem we will be concerned with is to describe the motion of the massive particle in the Brownian limit, where the mass m of the light particles goes to 0, while the density and the velocities of them have order $m^{-1/2}$. The scaling is done in such a way that the variance of the momentum transfer stays of order 1. See the introductions of [12, 13] for a more detailed explanation with respect to the reason of this scaling. We notice that one expects that the non-Markovian character of the dynamics disappears when $m \to 0$, since at least when the initial velocities of the light particles are fast enough, the interactions should be short enough.

This type of model, called a mechanical model of Brownian motion, was first introduced and studied by Holley [9], for the case where the whole system is in dimension $d = 1$, and the interactions are given by collisions. This model was later extended by, e.g., Dürr et al. [6–8], Calderoni et al. [2], to the case of higher dimensional spaces. Szász and Tóth [15] also considered some related problem. We notice that in all these papers, the interactions were just of the collisions type.

Kusuoka and Liang [12] and Liang [13] considered this type of problem with interactions given by compactly supported smooth potentials, under certain conditions. In particular, [12, 13] assumed that all light particles are sufficiently fast, precisely, all light particles have initial velocities not less than $m^{-1/2}(2C_0 + 1)$, with C_0 a positive constant determined by the potential functions. See (1.2) for the definition of C_0. This technical condition was used in an essential way in [12, 13]. In detail, this assumption ensured that light particles cross the valid range of interaction in a bounded time, and never reenter the valid range. Therefore, when considering the behavior of the light particles, we could use the approximation that the massive particle is frozen. This is our freezing approximation which will be explained later in details.

However, in a physically more relevant model, there also exist light particles with initial kinetic energies less than $2C_0+1$, equivalently, with initial velocities less than $m^{-1/2}(2C_0+$

1). As explained, this could not be covered by [12,13]. Indeed, in a continuous interaction potential model with possibly not sufficiently fast light particles, the effective interaction time durations between particles might be unbounded. This is heuristically clear by considering the much simpler model with the massive particle frozen—if the initial relative position $x - X$ of a light particle (with the massive particle frozen) is parallel to its initial velocity v but with opposite directions, and with its energy $\frac{m}{2}|v|^2 + U(x - X)$ equal to the maximum of the potential U (which is assumed to be finite in the present paper), then after its first hit, the light particle will stop at the position that attains the maximum of the potential, hence the effective interaction time duration would be infinity.

Certainly, in a repulsive interaction potential model as discussed in the present paper, as long as $x - X$ is not totally parallel to v, the particle could leave the valid range at some finite time. However, the observation above suggests that the effective interaction duration could nevertheless be very long. In our model the situation is even more complicated since the massive particle is also evolving.

In this paper, we consider this continuous repulsive interaction potential model with possibly not "sufficiently fast" light particles, which, as discussed above, means that the effective interaction durations are not bounded. We notice that our model does not include the Lennard-Jones potential model, especially, does not include the Weeks–Chandler–Andersen potential model (see, for example, [5] for its formulation, the potential of which is repulsive and compactly supported), in which the potential diverges to ∞ as the inter-particle distance converges to 0. On the other hand, we notice that the difficulties of our present model and the Lennard-Jones model are different—as long as we assume that the initial energies of the light particles are bounded below (which is assumed in the present paper), in the Lennard–Jones model, the effective interaction time duration is always finite (but it has its own difficulty that the second derivative of the potential function is unbounded, which means that the nearness of the positions does not imply the nearness of the forces, see also Problem 3 of Sect. 4); whereas in the present model, the difficulty is the possibility of unbounded interaction time durations.

There are a lot of further papers related to our topic, in the sense of "deriving Brownian motion from dynamics involving a dependence on the past" (or "re-collisions" for the collisional interactions). For example, Chernov and Dolgopyat [4] considered a model with only one massive particle and one light particle but with full re-collisions, Caprino et al. [3] considered a model with the mean-field approximation from the beginning, and with a different scaling. See also the references therein. Simulation results on this topic have also been discussed, for example, Kim and Karniadakis [11] and the references therein.

However, in the literature, to the best knowledge of the author, there are not so many papers concerning our problem of "deriving Brownian motion from a Hamiltonian dynamics consisting of massive particle(s) with infinitely many ideal gas light particles", except the ones [2,6–9,12,13] quoted before. Especially, for the case where interactions are given by potentials, [12,13] are the only ones that the author knows. As for a potential model with possibly slow light particles (hence possibly long interaction durations as explained), we believe that the present paper is the first progress. We notice that the Markov approximation method used in, e.g. [6,15], is not applicable to smooth potential interaction models. We prove our convergence in this paper with the help of the martingale problem theory. This framework of proof was also used in [2,12,13]. Precisely, we first prove the tightness of the considered family of probabilities, and then prove that any cluster point of it must be the unique diffusion described. See also Sect. 2.1 for more detailed explanations.

We remark that our model (i.e., with smooth potential interactions and possibly low initial energy light particles) has the following evident difficulty when compared with hard core models or with potential models with sufficiently fast light particles: the system is strongly non-Markovian, due to the extensions in times of the interactions. Since we are not assuming that the initial velocities of light particles are at least $m^{-1/2}(2C_0+1)$, which was the case for [12,13], the interacting times could be arbitrarily long, depending on the initial states of the considered light particles. For hard core models, although still non-Markovian because of the possibility of re-collisions, each interaction happens in an instant; while for smooth potential interaction models, if the initial velocities of the light particles are at least $m^{-1/2}(2C_0 + 1)$ as in [12,13], the interacting times are bounded. However, in our model, as explained, this could not be the case.

One more evident difference between these models concerns the velocities of the light particles after interactions: in the hard core model, after each collision, the gas particle changes its velocity a lot—almost reflecting—, since the masses of the light particles and of the massive particle(s) are too different; in the model of smooth potential interaction but with light particles which are fast enough, as discussed in [12,13], each light particle simply "almost passes through" (this was also one of the main heuristic ideas of [12,13]); whereas in our model, the velocities of light particles after interactions could be in any direction varying from "almost passing through" to "almost reflecting".

Finally, we remark that, for the case where there is only one massive particle (as in this paper and in part of [12,13]), our limiting process for smooth potential interaction model coincides with the one for the hard core model, which was given by [6]. See [13, Remark 1] for a detailed explanation. (For the case where there are at least two massive particles, we could not make the comparison since the limiting process for the hard core model is unknown).

1.2 Description of the Model and Statement of the Result

Let us now describe our model in details. We consider a dynamical system that consists of one massive particle immersed into an environment of infinitely many light particles with mass $m > 0$ (we will take the limit $m \to 0$ later on). Without loss of generality, we assume that the mass of the massive particle is equal to 1. The initial condition of the environment is given by $\widetilde{\omega} \in Conf(\mathbf{R}^d \times \mathbf{R}^d)$. Here $Conf(*)$ stands for the set of all non-empty closed subsets of $*$ which have no cluster point. Also, $(x, v) \in \widetilde{\omega}$ means that there exists an environmental particle with position x and velocity v at time 0. The distribution of $\widetilde{\omega}$ will be given later. As soon as the initial condition of the system is given, our system is totally deterministic, Newtonian, with the Hamiltonian given by $\frac{1}{2}|V|^2 + \sum_{(x,v)} \frac{m}{2}|v|^2 + \sum_{(x,v)} U(X - x)$. Here (X, V) is the state (i.e., the position and the velocity) of the massive particles, and (x, v) is the state of each light particle. So the interaction between the massive particle and light particles is given by a potential function U. We assume that $U \in C_0^\infty(\mathbf{R}^d)$, the set of smooth functions on $\mathbf{R}^d$ with compact supports, and concentrate ourselves to the case where U is spherical-symmetric and gives us a repulsive force, so we are assuming the following:

U1. $d > 1$, $U \in C_0^\infty(\mathbf{R}^d)$, and there exists a constant $R_U > 0$ and a smooth function $h : [0, \infty) \to [0, \infty)$ such that $U(x) = h(|x|)$ for any $x \in \mathbf{R}^d$, $U(x) = 0$ if $|x| \geq R_U$, and $h'(a) < 0$ for any $a \in (0, R_U)$. Also, we assume that $h''(0) < 0$.

For any $a, b \in \mathbf{R}^d$, let (a, b) and $a \cdot b$ denote their inner product, and when $a \neq \mathbf{0}$, let $\pi_a^\perp b$ denote the component of b that is perpendicular to a, i.e., $\pi_a^\perp b := b - (b, \frac{a}{|a|})\frac{a}{|a|}$. The spherical-symmetry of U in the assumption U1 ensures that the freezing-approximation particles can evolve only in the directions of v and $\pi_v^\perp(x - X)$, so helps us to estimate the interacting time durations (see, for example, the proof of Lemma 3.13), see also Lemma 4.2 for the benefit of the spherical-symmetry of U; and the repulsive property in the assumption

U1 ensures that at least the freezing-approximation particles leave the valid range in a finite time for sure as long as $\pi_v^\perp(x - X) \neq 0$.

For any initial condition $\tilde{\omega}$ and time $t \in [0, \infty)$, let $(X^{(m)}(t, \tilde{\omega}), V^{(m)}(t, \tilde{\omega}))$ denote the state of the massive particle at time t, and for any $(x, v) \in \tilde{\omega}$, let $(x^{(m)}(t, x, v, \tilde{\omega}), v^{(m)}(t, x, v, \tilde{\omega}))$ denote the state at time t of the light particle which had state (x, v) at time 0. So our dynamical system is given by the following infinite system of ordinary differential equations:

$$\begin{cases} \frac{d}{dt}X^{(m)}(t, \widetilde{\omega}) = V^{(m)}(t, \widetilde{\omega}), \\ \frac{d}{dt}V^{(m)}(t, \widetilde{\omega}) = -\int_{\mathbf{R}^d \times \mathbf{R}^d} \nabla U\big(X^{(m)}(t, \widetilde{\omega}) - x^{(m)}(t, x, v, \widetilde{\omega})\big)\mu_{\widetilde{\omega}}(dx, dv), \\ \big(X^{(m)}(0, \widetilde{\omega}), V^{(m)}(0, \widetilde{\omega})\big) = (X_0, V_0), \\ \frac{d}{dt}x^{(m)}(t, x, v, \widetilde{\omega}) = v^{(m)}(t, x, v, \widetilde{\omega}), \\ m\frac{d}{dt}v^{(m)}(t, x, v, \widetilde{\omega}) = -\nabla U\big(x^{(m)}(t, x, v, \widetilde{\omega}) - X^{(m)}(t, \widetilde{\omega})\big), \\ \big(x^{(m)}(0, x, v, \widetilde{\omega}), v^{(m)}(0, x, v, \widetilde{\omega})\big) = (x, v), \qquad (x, v) \in \widetilde{\omega}. \end{cases} \tag{1.1}$$

Here $\mu_{\widetilde{\omega}}(\,\cdot\,)$ is the counting measure determined by $\widetilde{\omega}$: $\mu_{\widetilde{\omega}}(A) = \sharp(\widetilde{\omega} \cap A)$ for any $A \in \mathcal{B}(\mathbf{R}^d \times \mathbf{R}^d)$, the set of Borel subsets of $\mathbf{R}^d \times \mathbf{R}^d$. ($\sharp(\,\cdot\,)$ thus denoting the number of points in the argument). When there is no risk of confusion, we will omit the superscript (m) and the parameter $\widetilde{\omega}$. So $X(t)$ stands for $X^{(m)}(t, \widetilde{\omega})$, *etc.* .

The only randomness of our model comes from the distribution of the environmental initial condition $\widetilde{\omega}$, which is given by the following. Let $\rho : [0, \infty) \times \mathbf{R}^d \to [0, \infty)$ be a measurable function such that $\sup_{z \in \mathbf{R}^d} \rho(u, z) \to 0$ rapidly as $u \to \infty$ (see conditions A1, A2 and A3 below for detailed assumptions with respect to ρ). Let $\widetilde{\lambda_m}$ be the non-atomic Radon measure on $\mathbf{R}^d \times \mathbf{R}^d$ given by

$$\widetilde{\lambda_m}(dx, dv) = m^{\frac{d-1}{2}} \rho\left(\frac{m}{2}|v|^2, x - X_0\right) dxdv,$$

and let $\widetilde{P_m}(d\widetilde{\omega})$ be the Poisson point process with the intensity measure $\widetilde{\lambda_m}$. So $\widetilde{P_m}$ is a probability measure on $\widetilde{\Omega} = Conf(\mathbf{R}^d \times \mathbf{R}^d)$. We assume that the distribution of $\widetilde{\omega}$ is given by $\widetilde{P_m}$. (See, e.g., [10] for more details with respect to Poisson point processes).

A1. There exists a constant $\overline{v} > 0$ such that $\rho(u, z) = 0$ for any $u < \frac{1}{2}\overline{v}^2$ and $z \in \mathbf{R}^d$.
A2. $\rho(u, -z) = \rho(u, z)$ for any $z \in \mathbf{R}^d$ and $u \in [0, \infty)$. Also, there exist a function $\rho_0 : [0, \infty) \to [0, \infty)$ and a constant $R_1 > 0$ such that $\rho(u, z) = \rho_0(u)$ as long as $|z| \geq R_1$ for any $u \in [0, \infty)$.
A3. $\int_{\mathbf{R}^d} (1 + |v|^3)\rho_{max}(\frac{1}{2}|v|^2)dv < \infty$. Here $\rho_{max}(u) := \sup_{z \in \mathbf{R}^d} \rho(u, z)$, $u \in [0, \infty)$.

(A1) means that all light particles have initial velocities not smaller than $m^{-1/2}\overline{v}$, equivalently, we are assuming that the kinetic energies of all light particles are bounded below. This enables us to estimate the interacting time durations of those light particles with initial skews $|\pi_v^\perp(x - X(\cdot))|$ which are not too small (see Proposition 2.1). We emphasize again that $\overline{v}$ could be any positive number in our model. (The case with $\overline{v} = 0$ is more complicated, see Problem 1 of Sect. 4).

(A2) is satisfied, for example, if $\rho(u, z)$ is a function of u and $U(z)$, i.e., if the initial distribution of the light particles depends on the kinetic energies and the potentials of the light particles. In particular, different from [12,13], our model is also consistent with a model such that the initial distribution of the light particles is not affected by the existence of the massive particle (i.e., the case where $\rho(u, z)$ does not depend on z). The second half of (A2) implies that even if the initial position of the massive particle does affect the initial distribution of light particles, it affects only a bounded neighborhood of the massive particle, so since the velocities of all light particles are of order $m^{-\frac{1}{2}}$, after a time period that is short enough (precisely, for $t \geq m^{\frac{1}{2}} a_m$, where a_m is defined in Sect. 2.5), the distribution of the incoming light particles is almost independent of the massive particle. Also, since we have only one massive particle in our model, the first half of (A2) (i.e., the even property of $\rho(u, z)$ with respect to z) is enough to ensure that the mean of the forces of our freezing-approximation φ is 0 (see Lemma 3.33). We remark that, in the case where there are more than one massive particles as in [12,13], to ensure that the mean of the forces of the freezing-approximation φ is 0, one needed to assume that ρ is a function of $\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})$, i.e., the initial distribution of the light particles depends on the total energy (the summation of kinetic energy and the potentials) of the light particles.

(A3) assumes that $\rho(u, z)$ decreases rapidly enough when $u \to \infty$, uniformly with respect to z. So we are assuming that there are not too many extremely fast light particles.

We notice that our intensity function of the present model includes that of [12,13] as a special case. To see this, write the function ρ of [12,13] as $\overline{\rho}$ to distinguish the notations. Then the model of [12,13] is given by $\rho(u, z) := \overline{\rho}(u + U(z - X_0))$. Now our (A2) is satisfied trivially, and our (A3) is a direct consequence of (A2) of [12,13]. Also, the assumption (A1) of [12,13] ensures that (A1) of our present paper is satisfied with $\overline{v}$ given by $2C_0 + 1$. Here

$$C_0 := \sqrt{2R_U \|\nabla U\|_\infty}. \tag{1.2}$$

We emphasize again that the constant $2C_0 + 1$ was used in an essential way in [12,13]: they proved the fact that if the initial speed of a light particle is not smaller than $(2C_0 + 1)m^{-1/2}$, then until any given $T > 0$, the period of time that it could be in the valid range of the interaction with the massive particle is bounded by $3C_0^{-1}(R_U + |X_0| + nT + 1)m^{1/2}$ (see, for example, [12, Proposition 3.6.5]). In other words, [12,13] assumed from the beginning that the initial velocities of light particles are fast enough such that the interactions with the massive particle could not "stop" these light particles. In particular, when describing the behavior of light particles, [12,13] could use the so-called freezing-approximation $\varphi(t, x, v; X)$ and $\psi(t, x, v; X)$ (see (1.3) and (1.4) for their definitions, and see Lemma 3.20 for the meaning of our expression "approximation"). See also Sect. 2 for more explanations with respect to this freezing approximation.

This idea of freezing-approximation is also used in this paper, with more precise estimates – as explained, since we are not assuming in this paper that the initial velocities are that fast, it does not hold in our model that the interacting times are bounded. Our idea to tackle this problem will be explained in Sect. 2.

Let

$$E = \left\{(x, v) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash \{0\});\ x \cdot v = 0\right\},$$
$$E_v = \left\{x \in \mathbf{R}^d;\ x \cdot v = 0\right\}, \qquad v \in \mathbf{R}^d \backslash \{0\},$$

and let $\nu(dx, dv)$ be the measure on E given by $\nu(dx, dv) = |v|\widetilde{\nu}(dx; v)dv$, where $\widetilde{\nu}(dx; v)$ is the Lebesgue measure on E_v. E is used in the domain of the ray representation Ψ defined in (2.1) later.

By a slight modification of [12, Sect. 3.3, pp. 751–758] (the only modification needed is the definition of G_n – let $G_n := \{(t, x, v) \in \mathbf{R} \times E;\ |x| < R_0, |t| < T + m^{1/2}\overline{v}^{-1} R_0\}$ now. Here $R_0 \geq R_U + |X_0| + nT + 1$ as defined in (2.4)), we have that under (U1) (A1) and (A3), for any $m > 0$, (1.1) has a unique solution for $\widetilde{P_m}$-almost every $\widetilde{\omega}$.

Remark 1.1 If we define $G_n := \{(t, x, v) \in \mathbf{R} \times E; |x| < R_0, |t| < T + m^{1/2}|v|^{-1}R_0\}$, and let λ be as defined in (2.2), then with $S_{d-1} := \int_{\mathbf{R}^{d-1}} 1_{\{|x|\le 1\}}dx$, we have by (A3) that

$$\begin{aligned}
&\int_{\mathbf{R}\times E} 1_{G_n}(u, x, v)(1 + |v|^2)\lambda(du, dx, dv) \\
&\le m^{-1}\int_{\mathbf{R}^d}(1 + |v|^2)|v|\rho_{max}\Big(\frac{1}{2}|v|^2\Big)dv \int_{E_v} 1_{\{|x|<R_0\}}\widetilde{\nu}(dx; v)\int_{\mathbf{R}} 1_{\{|u|<T+m^{1/2}|v|^{-1}R_0\}}du \\
&= m^{-1}(2R_0)^{d-1}S_{d-1}\int_{\mathbf{R}^d}(1 + |v|^2)\big(T|v| + m^{1/2}R_0\big)\rho_{max}\Big(\frac{1}{2}|v|^2\Big)dv \\
&< \infty
\end{aligned}$$

for any fixed m, T and n. Therefore, by checking the proof of [12, Sect. 3.3, pp. 751–758] carefully, we can get the uniqueness of the solution of (1.1) for $\widetilde{P_m}$-almost every $\widetilde{\omega}$ and any $m > 0$ by assuming (A3) and (U1) (with $U \in C_0^2(\mathbf{R}^d)$ instead of $U \in C_0^\infty(\mathbf{R}^d)$) only. This ensures the existence of the solution of our Problem 1 presented in Sect. 4. □

We are interested in the limit behavior of the massive particle when $m \to 0$. In order to formulate our main result, we first prepare several notations, which are the same as in [12]. First, for any $X \in \mathbf{R}^d$ and $(x, v) \in \mathbf{R}^{2d}$, let $\varphi(t, x, v; X) = (\varphi^0(t, x, v; X), \varphi^1(t, x, v; X))$ denote the solution of the following system of ordinary differential equations:

$$\begin{cases} \dfrac{d}{dt}\varphi^0(t, x, v; X) = \varphi^1(t, x, v; X) \\ \dfrac{d}{dt}\varphi^1(t, x, v; X) = -\nabla U\big(\varphi^0(t, x, v; X) - X\big) \\ (\varphi^0\big(0, x, v; X), \varphi^1(0, x, v; X)\big) = (x, v). \end{cases} \tag{1.3}$$

We notice that (1.3) is the same as the second half of (1.1) with $m = 1$, except that the quantity $X^{(1)}(t)$ of (1.1) is substituted by X in (1.3).

As in [12,13], for any $X \in \mathbf{R}^d$ and $(x, v) \in E$, we have that

$$\psi(t, x, v; X) := (\psi^0(t, x, v; X), \psi^1(t, x, v; X)) := \lim_{s\to\infty}\varphi(t + s, x - sv, v; X) \tag{1.4}$$

is well-defined. Indeed, $\psi(t, x, v; X) = \varphi(t+s, x-sv, v; X)$ for any $s \ge (-t)\vee(\frac{R_U+|X|+1}{|v|})$. $\psi(t, x, v; X)$ with proper X is our freezing-approximation mentioned above.

Also, for any $(x, v) \in E$, $X, V \in \mathbf{R}^d$ and $a \in \mathbf{R}$, let $z(t; x, v, X, V, a)$ denote the solution of

$$\begin{cases} \frac{d^2}{dt^2}z(t) = -\nabla^2 U(\psi^0(t, x, v, X) - X)\Big(z(t) - (t + a)V\Big), \\ \lim_{t\to-\infty} z(t) = \lim_{t\to-\infty}\frac{d}{dt}z(t) = 0. \end{cases} \tag{1.5}$$

So $z(t; x, v, X, V, a)$ is a linear function of V.

Our limiting diffusion generator L on function over $\mathbf{R}^{2d}$ is given by the following:

$$L = \frac{1}{2}\sum_{k,l=1}^d a_{kl}\frac{\partial^2}{\partial V_k \partial V_l} + \sum_{k,l=1}^d b_{kl}V_l\frac{\partial}{\partial V_k} + \sum_{k=1}^d V_k\frac{\partial}{\partial X_k}, \tag{1.6}$$

with

$$\begin{aligned}
a_{kl} = &\int_E\Big(\int_{-\infty}^{\infty}\nabla_k U\big(\psi^0(t, x, v; X) - X\big)dt\Big) \\
&\times\Big(\int_{-\infty}^{\infty}\nabla_l U\big(\psi^0(t, x, v; X) - X\big)dt\Big)\rho_0\Big(\frac{1}{2}|v|^2\Big)\nu(dx, dv),
\end{aligned}$$

and $b_{kl} : \mathbf{R}^d \to R, k, l = 1, \cdots, d$, are C^∞-functions determined by the following relation:

$$-\int_E \left(\int_{-\infty}^{\infty} \nabla^2 U\big(\psi^0(t, x, v, X) - X\big) z(t, x, v, X, V, -t) dt \right) \rho_0\Big(\frac{1}{2}|v|^2\Big) \nu(dx, dv)$$
$$= \sum_{l=1}^{d} b_{kl} V_l^{\ell}.$$

The coefficients a and b correspond to the 0-order and the 1-order approximations, respectively, of our freezing-approximation. We notice that as proved in [13, pp. 248–249], a and b are indeed independent of X, since there is only one massive particle in our model. We express them in the present way, since the heuristic meanings of the present formulations are more clear. One more advantage of this formulation is to maintain consistency of the notations with [12,13], which discussed the case with more than one massive particles. We also remark that the integrals with respect to t in the definitions of a_{kl} and b_{kl} are finite by Proposition 2.1 (2).

Finally, our metric on $C([0, \infty); \mathbf{R}^{2d})$ is given by

$$dist(w_1, w_2) := \sum_{k=1}^{\infty} 2^{-k} \Big(1 \wedge \max_{t \in [0,k]} |w_1(t) - w_2(t)|\Big), \qquad w_1, w_2 \in \mathbf{R}^{2d}.$$

Our main result is the following.

Theorem 1.2 *Assume* (U1) *and* (A1)–(A3). *Also, assume that* $d > 2(1 + \|h''\|_\infty)(-h''(0))^{-1/2} + 1$. *Then the distribution of* $\{(X^{(m)}(t), V^{(m)}(t)); t \geq 0\}$ *under* $\widetilde{P_m}$ *converges weakly as* $m \to 0$ *to the diffusion process with generator* L *in* $(C([0, \infty); \mathbf{R}^{2d}), dist)$.

Remark 1.3 We remark that the assumption $d > 2(1 + \|h''\|_\infty)(-h''(0))^{-1/2} + 1$ in Theorem 1.2 implies that $d > 5$. This assumption is closely related to our estimate of the interacting time durations (Proposition 2.1). It might be possible to weaken this condition if one could estimate the effective interaction time durations more accurately. On the other hand, as explained below, we need at least $d > 3$ to apply our method of this paper.

On the one hand, we need $\alpha < \frac{\sqrt{\varepsilon_1}}{2(1+\|h''\|_\infty)}$ (here ε_1 is a lower bound of $-\frac{h'(y)}{y}$ in a neighbor of 0, which is approximately equal to $-h''(0)$, and α is a constant that we introduce in Sect. 2 to handle the singularity) in our estimate of the effective interaction durations of the light particles satisfying $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$ (see Proposition 2.1 (4)), since we need our approximation error, which is dominated by $m^{\frac{1}{2}} |x - \pi_v^\perp X(\tilde{r})|^{-(1+\|h''\|_\infty)\varepsilon_1^{-\frac{1}{2}}}$, to be of order $o(1)$ as long as $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$ (see the proof of Lemma 3.24 for details). On the other hand, we need $\alpha(d-1) - \frac{1}{2} > 0$ to ensure that the total force from singular light particles (i.e., those light particles with their initial states satisfying $|x - \pi_v^\perp X(\cdot)| < m^\alpha$) to be negligible: since the initial velocities are of order $m^{-\frac{1}{2}}$, and the density of light particles is of order $m^{-\frac{1}{2}}$, we get that the number of light particles satisfying this initial condition is of order $m^{\alpha(d-1)-1}$; also, the force from each of these light particles is of order $m^{\frac{1}{2}}$, so the total force from singular light particles is of order $m^{\alpha(d-1)-\frac{1}{2}}$.

Combining these, we need $\frac{\sqrt{\varepsilon_1}}{2(1+\|h''\|_\infty)} > \frac{1}{2(d-1)}$, equivalently, at least $d > (1 + \|h''\|_\infty)(-h''(0))^{-1/2} + 1$ to apply our method of this paper.

Moreover, for the case $d \leq 3$, as the following calculation shows, although we can prove that the diffusion coefficient a_{kl} of our limiting generator L is finite, we are not even able to prove that the drift coefficient is finite.

By Proposition 2.1 (2) below, we have that $|\int_{-\infty}^{\infty} \nabla_k U(\psi^0(t, x, v; X) - X)dt| = |\psi^1(-\infty, x, v, X) - \psi^1(\infty, x, v, X)| 1_{\{|x|\leq R_0\}} \leq 2|v| 1_{\{|x|\leq R_0\}}$. So $|a_{kl}| \leq \int_E 4|v|^2 1_{\{|x|\leq R_0\}} \rho_0(\frac{1}{2}|v|^2)\nu(dx, dv) \leq 4(2R_0)^{d-1} \int_{\mathbf{R}^d} |v|^3 \rho_0(\frac{1}{2}|v|^2) dv < \infty$.

However, for the drift coefficient b_{kl}, our estimation of this paper (see Lemma 3.38 and Proposition 2.1 (2) below) only ensures that the term $\int_{-\infty}^{\infty} \nabla^2 U(\psi^0(t, x, v, X) - X) z(t, x, v, X, V, -t) dt$ in the definition of b_{kl} is dominated by a constant multiplies $\left(\log \frac{1}{|x-\pi_v^{\perp} X|}\right)^2 |x - \pi_v^{\perp} X|^{-(1+C_1)\varepsilon_1^{-\frac{1}{2}}}$, which is integrable with respect to $\rho_0(\frac{1}{2}|v|^2)\nu(dx, dv)$ on E only if $d > (1 + C_1)\varepsilon_1^{-\frac{1}{2}} + 1 (\geq 3)$.

We close this remark by emphasizing that we do not mean that our limiting generator is not well-defined for $d \leq 3$, we just mean that we are not sure whether it is well-defined for $d \leq 3$ by our estimate in this paper. □

The paper is organized as follows. In Sect. 2, we explain the main ideas of this paper. In Sect. 3, we present the proof of Theorem 1.2. In Sect. 4, we give a brief summary and several concluding remarks.

2 Several Notations and Basic Ideas

In this section, we define several notations and explain the main ideas of this paper. In Sect. 2.1, we try to explain the main ideas with minimum mathematics. The mathematical formulations of these ideas are given in Sects. 2.2–2.5. Finally, in Sect. 2.6, we give a brief summary.

2.1 Explanation of the Main Ideas

For any $n \in \mathbf{N}$, let $\sigma_n := \inf\{t > 0; |V_t| \geq n\}$. We notice that in order to prove Theorem 1.2, a result with respect to $t \in [0, \infty)$, it suffices to prove the assertion for $t \in [0, T \wedge \sigma_n]$ for any $T > 0$ and $n \in \mathbf{N}$. Choose and fix any $T > 0$ and $n \in \mathbf{N}$ from now on.

Since most of the basic ideas of [12,13] except the boundedness of the interacting time durations (which, as explained in Sect. 1, does not hold in our model) is also used in the present paper, let us start with the explanation of these common ideas.

The first idea that needs to be mentioned is the ray representation defined in (2.1) below. Since until time $T \wedge \sigma_n$, the massive particle is in $\{X \in \mathbf{R}^d; |X| \leq |X_0| + nT\}$, a bounded domain, the effective interaction range for light particles is also bounded. Therefore, a light particle with its initial position far enough from the origin keeps a uniform motion until its first entrance to this bounded effective range. This is the base of our idea of the ray representation : if the effective interaction time durations are bounded, then for any $t \in [0, T \wedge \sigma_n]$ and $(r, x, v) \in \mathbf{R} \times E$, a light particle with initial state (position, velocity) $\Psi(r, x, -m^{\frac{1}{2}} v) = (x - m^{\frac{1}{2}} r v, m^{-\frac{1}{2}} v)$ is in the effective interaction range if and only if $r \approx t$; in other words, at each time t, the massive particle get forces from only those light particles with initial states $\Psi(r, x, -m^{\frac{1}{2}} v)$ with $r \approx t$. Although the effective interaction time durations are not bounded in the present paper, this idea of ray representation is still useful in order to give the entrance time of a light particle to the valid interaction range. The model after application of this ray representation is given by (2.3) below.

Freezing approximation is also an important idea. As explained, for the case with bounded effective interaction durations as in [12,13], when considering the force caused by a marked light particle, the movement of the massive particle during the effective interaction time duration of this light particle is very small, so the evolution of this light particle can be approximated by the evolution of a particle with the massive particle frozen. This is our freezing approximation φ and ψ. Also, the first order approximation error (i.e., $z(\cdot)$ defined by (1.5), see Lemma 3.21 below) gives us the drift term in our limiting process. In the present paper, although the effective interaction durations are not bounded, by presenting accurate estimates of the effective interaction time durations (see t_1 defined in (2.11)), we prove that this idea of freezing approximation is still applicable.

The common framework of the present paper and [12,13] is as follows: with the help of the explained ray representation and freezing approximation, we prove that the velocity of the massive particle can be re-expressed as the sum of a martingale term, a smooth term and a negligible term, with each of these terms tight. Precisely, the families of the distributions of these stochastic processes are tight in $\wp(D([0,T];\mathbf{R}^d))$, the set of probability measures on the Skorohod space $D([0,T];\mathbf{R}^d)$ (see Proposition 3.31 for the formulation). See Sect. 3.4.1 for a brief review of the Skorohod space and the basic facts that are used to get the tightness, and see [1] for more details. In particular, we get that the family of the distributions of position/velocity processes is tight in $\wp(C([0,T];\mathbf{R}^d))$, the set of probability measures on $C([0,T];\mathbf{R}^d)$ (the set of continuous $\mathbf{R}^d$-valued functions on $[0,T]$). We recall the well-known fact that on a separable metric space, the tightness of a family of probabilities is equivalent to the sequentially compactness of its closure, i.e., any infinite countable subset of it has a subsequence that converges weakly. We then prove the convergence of the drift term (see Sect. 3.5) and the convergence of the quadratic variation of the martingale term (see (3.24)) as $m\to 0$. This implies that any of the cluster point(s) of the considered distributions of position/velocity processes as $m\to 0$ is the unique solution of the martingale problem L, i.e., for any $f\in C_0^\infty(\mathbf{R}^d\times\mathbf{R}^d)$, after taking the limit $m\to 0$, the distribution of $\left\{f(X(t\wedge\sigma_n),V(t\wedge\sigma_n))-f(X_0,V_0)-\int_0^{t\wedge\sigma_n}Lf(X(s),V(s))ds;\ t\in[0,T]\right\}$ under P_m is a martingale. So by the one-to-one correspondence between diffusion and solution of the corresponding martingale problem, we get the expected convergence of the position/velocity processes. See [12, Sect. 5] for the detailed calculation.

We would like to emphasize that, although both our stochastic process $\{V^{(m)}(t);t\in[0,T]\}$ before taking limit and the corresponding expected limiting process are continuous, its martingale part ($M(t)$ defined by (3.20)) is not continuous.

We now explain the original main ideas of the present paper. As explained, the biggest difference between the present paper and [12,13], which is also the biggest difficulty of the present paper, is that the effective interaction time durations are not bounded in our model, even for the freezing approximation. However, we prove in this paper (see Proposition 2.1 (2) and (2.11)) that this unboundedness happens only in a order of $\log|\pi_v^\perp(x-X)|$, the logarithm of the initial skew between the particles. This log order estimate of the effective interaction time durations is one of the main ideas of this paper, and is used in an essential way; when applying Gronwall's inequality to estimate the approximation error between the states of the light particle and its freezing approximation, we get an estimate of exponential order with respect to the effective interaction time duration (see, e.g., Lemma 3.20). So a log-order estimate of the effective interaction time is necessary. Also, we remark that as mentioned in Remark 1.3, opposite to [12,13], since the effective interaction time durations are not bounded in the present model, the approximation errors of our freezing approximation (for example, the quantity $\kappa_1(t)$ defined in (3.10)) could not be of order $m^{\frac{1}{2}}$ uniformly with respect to $(x,v)\in E$. Nevertheless, we get the expected convergence by an accurate calculation (see also Sect. 2.4, especially (2.12)).

The estimation of the effective interaction time duration for a light particle is more complicated, and the method used by [12,13] is not applicable in our model. In [12,13], as in the case for the freezing approximation, since the initial velocity of a light particle were not less than $m^{-\frac{1}{2}}(2C_0+1)$, the light particle keeps a speed not less than $m^{-\frac{1}{2}}C_0$ in the direction of its initial velocity, so the light particle passes the valid interaction range in a short time (see [12, Propositions 3.6.1 and 3.6.5], which is formulated as Proposition 3.22 (1) in this paper). However, as explained, for a light particle with not sufficiently high initial energy, this could not be the case – even for a freezing approximation, the velocity of the particle might be 0 at some point. Therefore, since the massive particle is also evolving, it is hopeless to track the evolution of the light particle during its effective interaction duration directly.

This difficulty is solved in the following way, with the help of the freezing approximation: since the effective interaction time duration of the freezing approximation is of log order as explained (see Sect. 3.1), as long as the light particle is not too singular (i.e., if $|x-\pi_v^{\perp}X(\tilde{r})| \geq m^{\alpha}$, where α is a constant restricted by (2.9) and (2.10), see also the paragraph following it), the approximation error of our freezing-approximation (which is discussed in Sect. 3.2) could be small enough, so with the help of the information with respect to the behavior of the freezing approximation particle after its exiting time from the valid interaction range, we get that the light particle could not be in the valid interaction range after this time, too (see Sect. 3.3).

The introduction of α is also one of our main ideas of this paper: instead of trying to "track" all of the light particles, we use the fact that the total effect from those "singular" light particles (i.e., those with $|x-\pi_v^{\perp}X(\tilde{r})| < m^{\alpha}$) is small enough.

Also, as explained right after the assumption A2, the system needs a short time before the density of the incoming light particles getting almost no affect from the massive particle. Since the efficient interaction time durations are not bounded in the present paper, the definition of this short time is certainly different from that of [12,13], and is valid for only those non-singular light particles. The explicit definition is given in Sect. 2.5.

In Sects. 2.2–2.5, we present the mathematical formulations of the ideas that we just explained: we formulate the ray representation and present our model after having applied the ray representation in Sect. 2.2; we give the explicit definition of the necessary constants in Sects. 2.3 and 2.5; and we formulate our estimations of the effective interaction durations of the light particles and the freezing approximation particles in Sect. 2.4.

2.2 Ray Representation

We first formulate the ray representation explained in Sect. 2.1. Our ray representation Ψ is defined as follows:

$$\begin{aligned} \Psi:\ & \mathbf{R}\times E \to \mathbf{R}^d \times (\mathbf{R}^d\backslash\{0\}), \\ & (s,(x,v)) \mapsto \Psi(s,(x,v)) = (x-sv, v). \end{aligned} \tag{2.1}$$

We remark that in this new space $\mathbf{R}\times E$, v is still the initial velocity of the light particle, while x is not the initial position of it anymore: now x is only the component of its initial position that is perpendicular to the velocity. Also, s plays an important role as explained: it is approximately the time that the light particle with initial condition $\Psi(s,x,v)$ interacts with the massive particle.

We now apply the ray representation (2.1) to our model. See [12] for detailed calculation. Let $\Omega = Conf(\mathbf{R}\times E)$, let $\lambda(dr,dx,dv)$ be the measure on Ω given by

$$\lambda(dr, dx, dv) = \lambda_m(dr, dx, dv) = m^{-1}\rho\left(\frac{1}{2}|v|^2, x - m^{-1/2}rv - X_0\right) dr\nu(dx, dv), \tag{2.2}$$

(with ρ and ν as in Sect. 1), and let $P_m(d\omega) = P_{\lambda_m}(d\omega)$ be the Poisson point process on $Conf(\mathbf{R} \times E)$ with intensity function $\lambda_m(dr, dx, dv)$. Then we can convert our problem with respect to $Conf(\mathbf{R}^d \times \mathbf{R}^d)$ to a problem with respect to $Conf(\mathbf{R} \times E)$. Our $\omega \in \Omega$ has distribution P_m, and for each initial condition ω, we are considering the following system of infinite ODEs (we omit the superscription (m) for the sake of simplicity):

$$\begin{cases} \frac{d}{dt}X(t,\omega) = X(t,\omega), \\ \frac{d}{dt}V(t,\omega) = -\int_{\mathbf{R}\times E} \nabla U\left(X(t,\omega) - x\left(t, \Psi\left(r, x, m^{-\frac{1}{2}}v\right), \omega\right)\right) \mu_\omega(dr, dx, dv), \\ \big(X(0,\omega), V(0,\omega)\big) = (X_0, V_0), \\ \frac{d}{dt}x\big(t, \Psi(r,x,v), \omega\big) = v\big(t, \Psi(r,x,v), \omega\big), \\ m\frac{d}{dt}v(t, \Psi(r,x,v), \omega) = -\nabla U\big(x(t, \Psi(r,x,v),\omega) - X(t,\omega)\big), \\ \big(x(0, \Psi(r,x,v),\omega), v(0, \Psi(r,x,v),\omega)\big) = \Psi(r,x,v), \qquad (r,x,v) \in \omega. \end{cases} \tag{2.3}$$

2.3 Time for Freezing and Definition of the "Singularity"

As explained, we are going to approximate $x(s, \Psi(r, x, m^{-1/2}v))$ by $\varphi^0(m^{-\frac{1}{2}}s, \Psi(m^{-\frac{1}{2}}r, x, v; X))$ or $\psi^0(m^{-\frac{1}{2}}(s-r), x, v; X)$ with some proper X. In this section, let us first explain a little bit more about the freezing time of our freezing approximation: we are going to use different X in our freezing approximations (equivalently, different time to freeze the massive particle) for different purposes. To be precise, we take $X = X_0$ when proving that the force during the very first time duration (i.e., the duration $s \in [0, m^{\frac{1}{2}}a_m]$, where a_m is defined in Sect. 2.5 below) is negligible; we take $X = X(s)$ when proving the convergence in the last step of our proof; and we take $X = X(\tilde{r})$ with $\tilde{r}$ given by (2.5) below for estimating the effective interaction duration of the light particle (see Proposition 2.1 (4)) and in order to get a measurable approximation (see Lemma 3.19 below. The measurability is necessary for estimating the variation of the corresponding term under Poisson point process measure). See (3.18) for the concrete usage of these freezing-times in the decomposition of $V(t \wedge \sigma_n)$.

Let

$$\begin{aligned} R_0 &:= (R_U \vee R_1) + |X_0| + nT + 1, \\ \tau &:= (\overline{v} \wedge C_0)^{-1} R_0. \end{aligned} \tag{2.4}$$

As will be proved later (Proposition 2.1 (3) or Proposition 3.22 (3)), if $r \geq t + m^{1/2}\tau$, then $|x(u, \Psi(r, x, m^{-1/2}v)) - X(u)| \geq R_U$ for all $u \in [0, t]$, which means that this light particle does not enter the valid range until time t. So the behavior of the massive particle at time t is $\mathcal{F}_{(-\infty, t+m^{1/2}\tau]\times E}$-measurable (Lemma 3.19). For any $r \in \mathbf{R}$, $\tilde{r}$ is defined by

$$\tilde{r} := ((r - m^{1/2}\tau) \vee 0) \wedge T \wedge \sigma_n. \tag{2.5}$$

Next, let us present our definition of singularity. Write $C_1 := \|h''\|_\infty$. Then

$$|\nabla^2 U(x)y| \le C_1|y|, \qquad \text{for any } x, y \in \mathbf{R}^d, \tag{2.6}$$
$$|\nabla U(y_1) - \nabla U(y_2)| \le C_1|y_1 - y_2|, \qquad \text{for any } y_1, y_2 \in \mathbf{R}^d. \tag{2.7}$$

The proof is easy, and is given in Appendix.

Recall that by assumption, $h''(0) < 0$ and $d > 2(1 + C_1)(-h''(0))^{-1/2} + 1$. So there exists a constant $\varepsilon_1 \in (0, -h''(0))$ such that

$$d > 2(1 + C_1)\varepsilon_1^{-1/2} + 1. \tag{2.8}$$

Therefore, there exists a constant $\alpha > 0$ such that

$$\alpha > \frac{1}{d-1}, \qquad (d > 1 \text{ by assumption (U1)}) \tag{2.9}$$
$$\alpha < \frac{\varepsilon_1^{1/2}}{2(1 + C_1)}. \tag{2.10}$$

(so, in particular, $\alpha \le \frac{1}{4}$). These conditions are chosen such that α satisfies the following conditions: On the one hand, α is big enough ((2.9)) such that the number of "singular" light particles (i.e., with $|x - \pi_v^\perp X(\tilde{r})| < m^\alpha$) converges to 0. See, for example, Lemmas 3.41, 3.42 and 3.45. On the other hand, α is small enough ((2.10)) such that those light particles with $|x - \pi_v^\perp X(\tilde{r})| \ge m^\alpha$ leave the valid interaction range in a certain time and the error of our freezing-approximation is small enough (Proposition 2.1 (4) and Lemma 3.20).

2.4 Estimation of the Effective Interaction Time Duration

Since our interaction force is repulsive, for a freezing approximation, $|x - \pi_v^\perp X|$ is a lower bound of the inter-particle distance (see, e.g., Lemma 3.11), and plays an important role when estimating the effective interaction time duration.

Let $\varepsilon_2 > 0$ be a constant such that $h(y) \le \frac{1}{36}\overline{v}^2$ for any $y \ge R_U - \varepsilon_2$. Since $h'(0) = 0$ by (U1) and $-\varepsilon_1 > h''(0)$ by definition, there exists a constant $\varepsilon_3 > 0$ such that $h'(y) \le -\varepsilon_1 y$ for any $y \in (0, \varepsilon_3)$. Let $\varepsilon_4 > 0$ be a constant such that $-h'(y) \ge \varepsilon_4$ for any $y \in [\varepsilon_3, \frac{3}{4}R_U \vee (R_U - \varepsilon_2)]$, and let $C_2 := 9\tau + 2\varepsilon_4^{-1}(2C_0 + \sqrt{2\|U\|_\infty})$. Finally, for any $y \ge 0$ and $v \in \mathbf{R}^d$ satisfying $|v| \ge \overline{v}$, let

$$t_1(v, y) := \begin{cases} 2\tau, & \text{if } \quad |v| \ge 2C_0 + 1, \\ C_2 + 2\varepsilon_1^{-1/2}\left(\log \frac{2\varepsilon_3}{y}\right)1_{\left\{y \le \frac{R_U}{2} \wedge (2\varepsilon_3)\right\}}, & \text{if } \quad |v| \in [\overline{v}, 2C_0 + 1). \end{cases} \tag{2.11}$$

Here $\log \frac{1}{y}$ is understood to be ∞ for $y = 0$, i.e., $t_1(v, y) = \infty$ if $|v| \in [\overline{v}, 2C_0 + 1)$ and $y = 0$.

We shall prove in Proposition 3.22 that if $|x - \pi_v^\perp X(\tilde{r})| \ge m^\alpha$, then the light particle with initial condition $\Psi(r, x, m^{-1/2}v)$ could be in the valid range at time $m^{1/2}r + t$ only if $t \in [-\tau, t_1(v, |x - \pi_v^\perp X(\tilde{r})|)]$. As claimed, this is also one of our main ideas of this paper: although the effective interaction time durations of these light particles (with $|x - \pi_v^\perp X(\tilde{r})| \ge m^\alpha$) are not of order $m^{1/2}$, which was the case for $|v| \ge 2C_0 + 1$ as proven by [12], they are dominated by $m^{1/2}(t_1(v, |x - \pi_v^\perp X(\tilde{r})|) + \tau)$. See Proposition 2.1 for the precise statement. On the other hand, since t_1 is at most of order log, with the help of the general result

$$\int_{(0,R]} |\log r|^k r^a dr < \infty, \qquad \text{for any } R > 0, k \in \mathbf{N} \text{ and } a > -1, \tag{2.12}$$

this does not cause any essential problem after taking the integral. Several more estimates of this type are given in Lemma 4.5.

We summarize our key estimates for the efficient interaction time durations as follows:

Proposition 2.1 *For any* $(x, v) \in E$*,* $r \in \mathbf{R}$*,* $s \in [0, T \wedge \sigma_n]$ *and* $X \in \mathbf{R}^d$ *satisfying* $|X| \leq |X_0| + nT$*, we have the following:*

1. $|\nabla U(\varphi^0(s, x - rv, v; X) - X)| \leq \|\nabla U\|_\infty 1_{\{|x-\pi_v^\perp X| \leq R_U+1\}} 1_{\{r \in [-\tau, s+\tau]\}}$.
2. $|\nabla^k U(\psi^0(s, x, v; X) - X)| \leq \|\nabla^k U\|_\infty 1_{\{|x-\pi_v^\perp X| \leq R_U+1\}} 1_{[-\tau, t_1(v, |x-\pi_v^\perp X|)]}(s)$ *for any* $k \in \mathbf{N}$.
3. $|\nabla U(x(s, \Psi(r, x, m^{-1/2}v)) - X(s))| \leq \|\nabla U\|_\infty 1_{\{r \in [-m^{1/2}\tau, s+m^{1/2}\tau]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})| \leq R_U+1\}}$.
4. *Assume that* $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$ *and* α *satisfies* (2.10) *in addition, then*

$$\begin{aligned}&|\nabla U\left(x\big(s, \Psi(r, x, m^{-1/2}v)\big) - X(s)\big)|\\&\leq \|\nabla U\|_\infty 1_{\{|x-\pi_v^\perp X(\tilde{r})| \leq R_U+1\}} 1_{\{m^{-1/2}(s-r) \in [-\tau, t_1(v, |x-\pi_v^\perp X(\tilde{r})|)]\}} 1_{\{r \geq -m^{1/2}\tau\}}.\end{aligned}$$

Proposition 2.1 is proved in Sects. 3.1–3.3. Precisely, since $|X| \leq |X_0| + nT$, by (3.2), we get (1) as a direct consequence of Lemma 3.1 (1) (2) and Proposition 3.3 (1); and we get (2) as a direct consequence of Lemma 3.2 and Proposition 3.4. Finally, (3) and (4) are direct consequences of Lemma 3.18 and Proposition 3.22.

2.5 A Short Time Duration Right After Starting

Let

$$a_m := C_2 + 2\varepsilon_1^{-1/2} \log \frac{2\varepsilon_3}{m^\alpha} + (\tau \vee 1). \tag{2.13}$$

We remark that as long as $m \leq (2\varepsilon_3)^{1/\alpha}$, we have that

$$y \in \mathbf{R}^d, |y| \geq m^\alpha \Rightarrow t_1(v, |y|) \leq a_m - \tau. \tag{2.14}$$

a_m is used in the decomposition of $V(t)$, since the situations for $s \in (0, m^{1/2}a_m]$ and $s > m^{1/2}a_m$ are different (see (3.18)). We notice that $m a_m^6 e^{(1+C_1)a_m} \to 0$ as $m \to 0$ by (2.10). As will be proved in Lemmas 3.35 and 3.36, this ensures that the forces during the time interval $[0, m^{1/2}a_m]$ are negligible when $m \to 0$. Also, by Lemma 2.2 below, we get that our freezing time $\tilde{r}$ defined by (2.5) is given by $\tilde{r} = r - m^{1/2}\tau$ if $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$, $s \in [0, T \wedge \sigma_n]$ and $s \geq m^{1/2}a_m$.

Lemma 2.2 *Assume that* $\nabla U\Big(x(s, \Psi(r, x, m^{-1/2}v)) - X(s)\Big) \neq 0$ *or* $\nabla U(X(\tilde{r}) - \psi^0(m^{-1/2}(s - r), x, v; X(\tilde{r}))) \neq 0$*. Also, assume that* $m \leq (2\varepsilon_3)^{1/\alpha}$*,* $s \in [0, T \wedge \sigma_n]$*,* $s \geq m^{1/2}a_m$ *and* $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$*. Then* $r \in [m^{1/2}\tau, s + m^{1/2}\tau]$ *and* $\tilde{r} = r - m^{1/2}\tau$*.*

Proof Since $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$ by assumption, the first assumption combined with Proposition 2.1 (2) and (4), implies that $m^{-1/2}(s - r) \in [-\tau, t_1(v, |x - \pi_v^\perp X(\tilde{r})|)]$, equivalently, $r \in [s - m^{1/2}t_1(v, |x - \pi_v^\perp X(\tilde{r})|), s + m^{1/2}\tau]$. This combined with (2.14) and our assumption $s \in [m^{1/2}a_m, T \wedge \sigma_n]$ implies that for any $m \leq (2\varepsilon_3)^{1/\alpha}$, we have $r \in [m^{1/2}\tau, s + m^{1/2}\tau]$, so $r - m^{1/2}\tau \in [0, s] \subset [0, T \wedge \sigma_n]$, hence $\tilde{r} = r - m^{1/2}\tau$. □

2.6 Summary

In summary, we first apply the ray representation to specify the first entrance of each light particle to its valid interaction range, then use a_m, α and freezing-approximation to decompose

$V(t)$ in the following way: first we ignore the time period $[0, m^{1/2}a_m]$, since the density of the incoming light particles during this period might depend on the massive particle; then we ignore those light particles without enough skew (*i.e.*, those with $|x - \pi_v^\perp X(\tilde{r})| < m^\alpha$), since the behavior of this type of light particles is too difficult to be tracked; finally, after these approximations, we apply our freezing-approximation. The first approximation explained above is fine heuristically because the time period is short enough, and the second one is fine because the total number of such "singular" light particles is small enough. As a result, we are able to re-express $V(t)$ as a sum of a martingale term, a smooth term and a negligible term, and get the desired tightness (see Sect. 3.4.2 for the concrete decomposition, and see Proposition 3.31 for the result). Finally, with the help of this decomposition, we prove the convergence of the smooth term (see Sect. 3.5). As explained, by the well-known martingale problem theorem, this implies our Theorem 1.2.

The concrete proof of Theorem 1.2 is given in Sect. 3.

3 Proof of Theorem 1.2

We present the proof of Theorem 1.2 is this section. In Sect. 3.1, we discuss the behaviors of φ and ψ. In Sect. 3.2, we estimate the approximation error of our freezing approximation. With the help of these results, we discuss the behavior of the light particles in Sect. 3.3. In detail, φ is used for the discussion with respect to $x(t, \Psi(r, x, m^{-1/2}v))$ with $|r| \le m^{1/2}\tau$, and ψ is used for those with $r \ge m^{1/2}\tau$. In Sect. 3.4, we prove our key decomposition: we rewrite $V(t)$ as a sum of a martingale term, a smooth term and a negligible term, with the concrete expressions of the martingale term and the smooth term given. In particular, we get the tightness. Finally, in Sect. 3.5, with the help of the results of Sect. 3.4, we prove the desired convergence, by proving that a certain part of the smooth term is actually also negligible.

3.1 Some Discussion for φ and ψ

In this section, we discuss the behaviors of φ and ψ. Precisely, we prove that they are far from the valid range after certain times (see Propositions 3.3 and 3.4).

To begin with, Lemmas 3.1 and 3.2 consider the problem of until when does the particle keep a uniform motion. For any $X \in \mathbf{R}^d$, let $R_X := R_U + |X| + 1$ and $s_1 := s_1(v, X) := |v|^{-1}R_X$. So $s_1 \le \tau$ if $|v| \ge \overline{v}$ and $|X| \le |X_0| + nT$.

Lemma 3.1 *For any* $r \in \mathbf{R}$, $t \ge 0$ *and* $(x, v) \in E$, *we have that* $\varphi(t, x - rv, v; X) = (x - rv + tv, v)$ *and* $|\varphi^0(t, x - rv, v; X) - X| \ge R_U + 1$ *if* (1) $|x - \pi_v^\perp X| \ge R_U + 1$ *or* (2) $r \ge t + s_1$.

Proof First we consider the case $|x - \pi_v^\perp X| \ge R_U + 1$. Since $x \cdot v = 0$, we have that $|x - rv + sv - X| \ge |x - \pi_v^\perp X| \ge R_U + 1$ for any $s \ge 0$, so the particle keeps a uniform motion. Therefore, $\varphi(t, x - rv, v; X) = (x - rv + tv, v)$, hence $|\varphi^0(t, x - rv, v; X) - X| = |x - rv + tv - X| \ge |x - \pi_v^\perp X| \ge R_U + 1$ for any $t \ge 0$.

The assertion for $r \ge t + s_1$ is similar. In this case, for any $s \in [0, t]$, we have that $r - s \ge s_1$, hence $r - s + (X, \frac{v}{|v|})\frac{1}{|v|} \ge s_1 - \frac{|X|}{|v|} = \frac{R_U + 1}{|v|}$, so $|x - rv + sv - X| \ge |r - s + (X, \frac{v}{|v|})\frac{1}{|v|}| \cdot |v| \ge R_U + 1$. Therefore, the particle keeps a uniform motion during the time interval $[0, t]$. So $\varphi(t, x - rv, v; X) = (x - rv + tv, v)$, and $|\varphi^0(t, x - rv, v; X) - X| = |x - rv + tv - X| \ge R_U + 1$. □

Since $\psi^0(t, x, v; X) = \lim_{s\to\infty} \varphi(t+s, x-sv, v; X)$ by definition, we get the following as a corollary of Lemma 3.1.

Lemma 3.2 *For any* $(x, v) \in E$, *we have that* $\psi(t, x, v; X) = (x + tv, v)$ *and* $|\psi^0(t, x, v; X) - X| \geq R_U + 1$ *if* (1) $|x - \pi_v^\perp X| \geq R_U + 1$ *or* (2) $t \leq -s_1$. □

We next consider the case where the conditions of Lemmas 3.1 or 3.2 are not satisfied. For any $X \in \mathbf{R}^d$ and $(x, v) \in \mathbf{R}^{2d}$, let

$$d^\varphi(t, x, v; X) := |\varphi^0(t, x, v; X) - X|,$$
$$v^{d,\varphi}(t, x, v; X) := \frac{d}{dt} d^\varphi(t, x, v; X).$$

Finally, let $C_3 := 5s_1 + 2\varepsilon_4^{-1}(2C_0 + \sqrt{2\|U\|_\infty})$, where C_0 is as defined in (1.2), and for any $X \in \mathbf{R}^d$ and $(x, v) \in E$ satisfying $|v| \geq \overline{v}$, let

$$t_2(x, v, X) := \begin{cases} 2C_0^{-1} R_X, & \text{if} \quad |v| \geq 2C_0, \\ 8s_1, & \text{if} \quad |v| \in [\overline{v}, 2C_0) \text{ and } |x - \pi_v^\perp X| \geq \frac{R_U}{2}, \\ C_3 + 2\varepsilon_1^{-1/2} \log^+ \frac{2\varepsilon_3}{|x-\pi_v^\perp X|}, & \text{if} \quad |v| \in [\overline{v}, 2C_0) \text{ and } 0 \leq |x - \pi_v^\perp X| < \frac{R_U}{2}. \end{cases} \tag{3.1}$$

Here $\log^+ y := (\log y) \vee 0$ for any $y > 0$. The definition of $t_2(x, v, X)$ is similar to that of $t_1(v, *)$. We introduce this notation to make our statements of this section easier to be understood. We remark that

$$t_1(v, |x - \pi_v^\perp X|) \geq t_2(x, v, X) \qquad \text{if} \quad |X| \leq |X_0| + nT, \tag{3.2}$$

and

$$t_1(v, |x - \pi_v^\perp X|) \geq t_2(x, v, X) + \tau, \quad \text{if} \quad |v| \in [\overline{v}, 2C_0 + 1), |X| \leq |X_0| + nT. \tag{3.3}$$

Our main result of this section is the following:

Proposition 3.3 *For any* $X \in \mathbf{R}^d$, $r \in \mathbf{R}$ *and* $(x, v) \in E$ *satisfying* $|v| \geq \overline{v}$, *we have that* $d^\varphi(t, x - rv, v; X) \geq R_U + 1$ *and* $v^{d,\varphi}(t, x - rv, v; X) > 0$ *if at least one of the following conditions is satisfied:*

1. $r \leq -s_1$ *and* $t \geq 0$,
2. $|r| \leq s_1$ *and* $t \geq t_2(x, v, X)$,
3. $r \geq s_1$ *and* $t + s_1 - r \geq t_2(x, v, X)$.

Similarly, for any $X \in \mathbf{R}^d$ and $(x, v) \in E$, let

$$d^\psi(t, x, v; X) := |\psi^0(t, x, v; X) - X|,$$
$$v^{d,\psi}(t, x, v; X) := \frac{d}{dt} d^\psi(t, x, v; X).$$

Then we have the following.

Proposition 3.4 *For any* $X \in \mathbf{R}^d$ *and* $(x, v) \in E$ *satisfying* $|v| \geq \overline{v}$, *we have that* $|\psi^0(t, x, v; X) - X| \geq R_U + 1$ *and* $v^{d,\psi}(t, x, v; X) > 0$ *for any* $t \geq t_2(x, v, X) - s_1$.

Proof This follows easily by Proposition 3.3 (2) or (3) since $\psi(t, x, v; X) = \varphi(t + s_1, x - s_1 v, v; X)$ for any $t \geq -s_1$. □

We prove Proposition 3.3 in the rest of Sect. 3.1. First, we have the following by a direct calculation:

Lemma 3.5 *For any $x, v, X \in \mathbf{R}^d$, we have the following:*

(1) $\frac{d}{dt}\Big(\frac{1}{2}|\varphi^1(t,x,v;X)|^2 + h(d^\varphi(t,x,v;X))\Big) = 0,$

(2) $\frac{d}{dt}\Big((\varphi^0(t,x,v;X) - X, \varphi^1(t,x,v;X))^2 - d^\varphi(t,x,v;X)^2|\varphi^1(t,x,v;X)|^2\Big) = 0.$

□

Lemma 3.6 *For any $x, v, X \in \mathbf{R}^d$, we have the following:*

(1) $v^{d,\varphi}(t,x,v;X) = \Big(\varphi^1(t,x,v;X), \frac{\varphi^0(t,x,v;X)-X}{d^\varphi(t,x,v;X)}\Big)$ *for any $t \geq 0$,*

(2) $\frac{d}{dt}v^{d,\varphi}(t,x,v;X) = \frac{1}{d^\varphi(t,x,v;X)}\Big|\pi^\perp_{\varphi^0(t,x,v;X)-X}\varphi^1(t,x,v;X)\Big|^2 - h'(d^\varphi(t,x,v;X))$ *for any $t \geq 0$,*

(3) $v^{d,\varphi}(t,x,v;X)$ *is monotone non-decreasing with respect to $t \geq 0$,*

(4) $|v^{d,\varphi}(t,x,v;X)|^2 = -\frac{|v|^2|\pi_v^\perp(x-X)|^2}{d^\varphi(t,x,v;X)^2} - 2h(d^\varphi(t,x,v;X)) + |v|^2 + 2h(|x-X|)$ *for any $t \geq 0$.*

Proof (1) and (2) are easy by direct calculation. (3) is a direct consequence of (2) since $-h' \geq 0$. For the last assertion, we have that

$$\begin{aligned}
&d^\varphi(t,x,v;X)^2\Big(|v^{d,\varphi}(t,x,v;X)|^2 - |\varphi^1(t,x,v;X)|^2\Big)\\
&\quad= \Big(\varphi^1(t,x,v;X), \varphi^0(t,x,v;X) - X\Big)^2 - d^\varphi(t,x,v;X)^2|\varphi^1(t,x,v;X)|^2\\
&\quad= \Big(\varphi^1(0,x,v;X), \varphi^0(0,x,v;X) - X\Big)^2 - d^\varphi(0,x,v;X)^2|\varphi^1(0,x,v;X)|^2\\
&\quad= (v, x-X)^2 - |x-X|^2|v|^2\\
&\quad= -|v|^2|\pi_v^\perp(x-X)|^2,
\end{aligned}$$

where we used (1) of this lemma when passing to the second line, we used Lemma 3.5 (2) when passing to the third line, and we used the fact that $(a,b)^2 - |a|^2|b|^2 = -|b|^2|\pi_b^\perp a|^2$ for any $a, b \in \mathbf{R}^d$ when passing to the last line. Since $|\varphi^1(t,x,v;X)|^2 = |v|^2 + 2h(|x - X|) - 2h(d^\varphi(t,x,v;X))$ by Lemma 3.5 (1), we get our assertion (4). □

Also, by the definition of ψ, we get the following properties with respect to ψ, as a corollary of Lemmas 3.5 and 3.6. Lemma 3.7 will be used later.

Lemma 3.7 *For any $(x, v) \in E$ and $X \in \mathbf{R}^d$, we have the following:*

1. $\frac{d}{dt}\Big(\frac{1}{2}|\psi^1(t,x,v;X)|^2 + h(d^\psi(t,x,v;X))\Big) = 0,$
2. $v^{d,\psi}(t,x,v;X) = \Big(\psi^1(t,x,v;X), \frac{\psi^0(t,x,v;X)-X}{d^\psi(t,x,v;X)}\Big)$ *for any $t \in \mathbf{R}$,*
3. $|v^{d,\psi}(t,x,v;X)|^2 = -\frac{|v|^2|x-\pi_v^\perp X|^2}{d^\psi(t,x,v;X)^2} - 2h(d^\psi(t,x,v;X)) + |v|^2$ *for any $t \in \mathbf{R}$.*

Proof of Proposition 3.3 (1) and (3) We first prove the first assertion of Proposition 3.3. Suppose $r \leq -s_1$. Then we have that $|x - rv + sv - X| \geq (s-r)|v| - |X| \geq s_1|v| - |X| \geq R_U + 1$ for any $s \geq 0$, so the particle keeps a uniform motion. Therefore, for any $t \geq 0$, we have that $\varphi(t, x - rv, v; X) = (x - rv + tv, v)$. So $|d^\varphi(t, x - rv, v; X) = |x - rv + tv - X| \geq R_U + 1$, and by Lemma 3.6 (1),

$$\begin{aligned}
&v^{d,\varphi}(t, x - rv, v; X)\\
&\quad= \left(v, \frac{x - rv + tv - X}{|x - rv + tv - X|}\right) = \frac{1}{|x - rv + tv - X|}\Big((t-r)|v|^2 - (v, X)\Big)\\
&\quad\geq \frac{1}{|x - rv + tv - X|}\left(\frac{R_X}{|v|}|v|^2 - |X|\cdot|v|\right) \geq \frac{|v|}{|x - rv + tv - X|}(R_X - |X|) > 0.
\end{aligned}$$

This completes the proof of Proposition 3.3 (1).

For the third assertion of Proposition 3.3, we just notice that if $r \geq s_1$, then $\varphi(t, x - rv, v; X) = \varphi(t - r + s_1, x - s_1 v, v; X)$ for any $t \geq r - s_1$. Indeed, suppose $r \geq s_1$, then for any $s \in [0, r - s_1]$, we have that $|x - rv + sv - X| \geq |s - r| \cdot |v| - |X| \geq R_U + 1$, so the particle keeps a uniform motion during the time interval $[0, r - s_1]$, hence $\varphi(r - s_1, x - rv, v; X) = (x - rv + (r - s_1)v, v) = (x - s_1 v, v)$. Therefore, for any $t \geq r - s_1$, we have that $\varphi(t, x - rv, v; X) = \varphi(t - r + s_1, \varphi(r - s_1, x - rv, v; X); X) = \varphi(t - r + s_1, x - s_1 v, v; X)$.

Hence the third assertion of Proposition 3.3 is an easy corollary of the second assertion. □

We prove the second assertion of Proposition 3.3 in the rest of Sect. 3.1. First, we notice the following.

Lemma 3.8 *Let* $|r| \leq s_1$. *Then* $\xi_1 := \inf\{t \geq 0; d^{\varphi}(t, x - rv, v; X) < R_U + \frac{1}{2}\} \in [0, 2s_1] \cup \{\infty\}$.

Proof Suppose that $\xi_1 \notin [0, 2s_1]$. Then $d^{\varphi}(t, x - rv, v; X) \geq R_U + \frac{1}{2}$ for all $t \in [0, 2s_1]$, hence the particle keeps a uniform motion during this time period. So for any $t \in [0, 2s_1]$, we have that $\varphi(t, x - rv, v; X) = (x - rv + tv, v)$, hence

$$\left(R_U + \frac{1}{2}\right)^2 \leq |x - rv + tv - X|^2 = |x - \pi_v^{\perp} X|^2 + \left(t - r - \left(X, \frac{v}{|v|}\right)\frac{1}{|v|}\right)^2 |v|^2. \tag{3.4}$$

If $r + \left(X, \frac{v}{|v|}\right)\frac{1}{|v|} \geq 0$, then $r + \left(X, \frac{v}{|v|}\right)\frac{1}{|v|} \in [0, 2s_1]$, so applying (3.4) to $t = r + \left(X, \frac{v}{|v|}\right)\frac{1}{|v|}$, we get that $|x - \pi_v^{\perp} X| \geq R_U + \frac{1}{2}$, hence $|x - rv + tv - X| \geq R_U + \frac{1}{2}$ for any $t \geq 0$, which means that $\xi_1 = \infty$. If $r + \left(X, \frac{v}{|v|}\right)\frac{1}{|v|} < 0$, then applying (3.4) to $t = 0$, we get that $|x - \pi_v^{\perp} X|^2 + \left(0 - r - \left(X, \frac{v}{|v|}\right)\frac{1}{|v|}\right)^2 |v|^2 \geq (R_U + \frac{1}{2})^2$, hence again, $|x - rv + tv - X| > R_U + \frac{1}{2}$ for any $t \geq 0$, which means that $\xi_1 = \infty$. □

As a corollary of Lemma 3.8, we prove in Lemma 3.9 that in order to prove Proposition 3.3 (2), it suffices to prove that

$$d^{\varphi}(t, x - rv, v; X) \geq R_U + 1, \qquad \text{if } t \geq t_2(x, v, X). \tag{3.5}$$

Lemma 3.9 *Assume the same conditions as in Proposition 3.3 (2), i.e., we assume that* $r \in \mathbf{R}$, $(x, v) \in E$, $|v| \geq \overline{v}$ *and* $|r| \leq s_1$. *Also, assume that* $d^{\varphi}(t_2(x, v, X), x - rv, v; X) \geq R_U + 1$. *Then we get that* $v^{d,\varphi}(t, x - rv, v; X) > 0$ *for any* $t \geq t_2(x, v, X)$.

Proof First we notice that $t_2(x, v; X) \geq 2s_1$ by the definition of $t_2(x, v; X)$. Let ξ_1 be as in Lemma 3.8. Then by Lemma 3.8, we get that $\xi_1 \in [0, 2s_1] \cup \{\infty\}$.

If $\xi_1 \in [0, 2s_1]$, then $\xi_1 \leq 2s_1 \leq t_2(x, v; X)$. Also, $d^{\varphi}(\xi_1, x - rv, v, X) \leq R_U + \frac{1}{2}$ by definition, and $d^{\varphi}(t_2(x, v, X), x - rv, v, X) \geq R_U + 1$ by assumption. So by the mean-value theorem, there exists a $\tilde{t} \in [\xi_1, t_2(x, v, X)]$ such that $v^{d,\varphi}(\tilde{t}, x - rv, v; X) > 0$. This combined with Lemma 3.6 (3) implies our assertion.

If $\xi_1 = \infty$, then $d^{\varphi}(t, x - rv, v; X) \geq R_U + \frac{1}{2}$ for all $t \geq 0$, so the particle keeps a uniform motion, hence $\varphi(t, x - rv, v; X) = (x - rv + tv, v)$ for $t \geq 0$. Therefore, for any $t \geq t_2(x, v, X)$, since $|r| \leq s_1$, we get that $t - r \geq s_1$, hence

$$\begin{aligned}&\left(\varphi^1(t,x-rv,v;X),\varphi^0(t,x-rv,v;X)-X\right)\\&=(v,x-rv+tv-X)=(t-r)|v|^2-(v,X)\geq|v|^{-1}R_X|v|^2-|v|\cdot|X|\\&=|v|(R_X-|X|)>0,\end{aligned}$$

so by Lemma 3.6 (1), we get that $v^{d,\varphi}(t,x-rv,v;X)>0$. □

By Lemma 3.9, in order to prove Proposition 3.3 (2), it suffices to prove (3.5) under the conditions of Proposition 3.3 (2). We prove this from now on. We first prepare the following general result for later use.

Lemma 3.10 *Let g be a function satisfying the following conditions: $g\in C^2([0,\infty))$, $g(0)>0$, $g'(0)\geq 0$, and $g''(t)=l(t)g(t)$, for some $l(t)\geq 0$ for any $t\geq 0$. Then $g(t)$ is monotone non-decreasing with respect to $t\geq 0$.*

Proof Let $\xi_2:=\inf\{t>0;\, g'(t)<0\}$. It suffices to prove that $\xi_2=+\infty$. Suppose $\xi_2<\infty$. Then for any $n\in\mathbf{N}$, there exists a $t_n\in[\xi_2,\xi_2+\frac{1}{n}]$ such that $g'(t_n)<0$. On the other hand, we have that $g'(t)\geq 0$ for any $t\in[0,\xi_2]$, so for any $t\in[0,\xi_2]$, we have that $g(t)\geq g(0)>0$. By the continuity of g, this implies that there exists a $N\in\mathbf{N}$ such that for any $t\in[0,\xi_2+\frac{1}{N}]$, we have $g(t)>0$, hence $g''(t)=l(t)g(t)\geq 0$. Therefore,

$$0>g'(t_N)=\int_0^{t_N}g''(s)ds+g'(0)\geq 0,$$

which yields a contradiction. So $\xi_2=+\infty$. □

Lemma 3.11 *Suppose that $(x,v)\in E$ and $X\in\mathbf{R}^d$ satisfy $|x-\pi_v^\perp X|\neq 0$. Then for any $r\in\mathbf{R}$, we have that $(\varphi^0(t,x-rv,v;X)-X)\cdot\frac{x-\pi_v^\perp X}{|x-\pi_v^\perp X|}$ is monotone non-decreasing with respect to $t\geq 0$.*

Proof Let $g(t):=(\varphi^0(t,x-rv,v;X)-X)\cdot\frac{x-\pi_v^\perp X}{|x-\pi_v^\perp X|}$. Then $g(0)=(x-rv-X)\cdot\frac{x-\pi_v^\perp X}{|x-\pi_v^\perp X|}=|x-\pi_v^\perp X|>0$, $g'(0)=v\cdot\frac{x-\pi_v^\perp X}{|x-\pi_v^\perp X|}=0$. Also, $g''(t)=-\frac{h'(d^\varphi(t,x-rv,v;X))}{d^\varphi(t,x-rv,v;X)}g(t)$ and $-\frac{h'(d^\varphi(t,x-rv,v;X))}{d^\varphi(t,x-rv,v;X)}\geq 0$ for any $t\geq 0$. So we get our assertion by Lemma 3.10. □

Now we are ready to prove that (3.5) holds under the conditions of Proposition 3.3 (2). We first prove it for the case $|v|\geq 2C_0$. For this case, we have the following as a consequence of [12, Proposition 3.2.2].

Lemma 3.12 *For any $(x,v)\in E$ with $|v|\geq 2C_0$, we have that*

$$d^\varphi(t,x-rv,v;X)\geq R_U+1$$

if $t\geq 2C_0^{-1}R_X$ and $|r|\leq s_1$.

Proof Suppose that $(x,v)\in E$ and $|v|\geq 2C_0$. Then by [12, Proposition 3.2.2], we have for any $r\in\mathbf{R}$ and $s\geq 0$ that

$$\varphi^1(s,x-rv,v;X)\cdot\frac{v}{|v|}\geq C_0.$$

Therefore, for any $t \geq 2C_0^{-1}R_X$, since $|r| \leq s_1 = |v|^{-1}R_X$ by assumption, we get that

$$\begin{aligned} d^{\varphi}(t, x - rv, v; X) &= \Big|\varphi^0(t, x - rv, v; X) - X\Big| \\ &\geq \Big(\varphi^0(t, x - rv, v; X) - X\Big) \cdot \frac{v}{|v|} \\ &= \int_0^t \varphi^1(s, x - rv, v; X) \cdot \frac{v}{|v|} ds + \Big(\varphi^0(0, x - rv, v; X) - X\Big) \cdot \frac{v}{|v|} \\ &\geq C_0 t + (x - rv - X) \cdot \frac{v}{|v|} \geq 2R_X - |r| \cdot |v| - |X| \\ &\geq 2R_X - R_X - |X| = R_U + 1. \end{aligned}$$

□

From now on, we consider those particles with $|v| < 2C_0$ and $|r| \leq s_1$. In Lemma 3.13 below, we consider those particles that are skewed enough from the beginning (i.e., with $|x - \pi_v^{\perp} X| \geq \frac{1}{2} R_U$), and prove that they leave the valid interacting range quickly.

Lemma 3.13 *Suppose that* $r \in \mathbf{R}$, $(x, v) \in E$, $|x - \pi_v^{\perp} X| \geq \frac{1}{2} R_U$ *and* $|r| \leq s_1$. *Then*

$$|\varphi^0(t, x - rv, v; X) - X| \geq R_U + 1$$

for any $t \geq 8s_1$.

Proof Let $e_1 := \frac{x - \pi_v^{\perp} X}{|x - \pi_v^{\perp} X|}$ and $e_2 := \frac{v}{|v|}$. Define

$$\xi_3 := \inf\left\{t \geq 0; \varphi^1(t, x - rv, v; X) \cdot e_2 \leq \frac{|v|}{2}\right\}.$$

If $\xi_3 \geq 4s_1$, then $\varphi^1(4s_1, x - rv, v; X) \cdot e_2 \geq \frac{|v|}{2}$, and

$$\begin{aligned} &\Big(\varphi^0(4s_1, x - rv, v; X) - X\Big) \cdot e_2 \\ &= \Big(\varphi^0(0, x - rv, v; X) - X\Big) \cdot e_2 + \int_0^{4s_1} \varphi^1(s, x - rv, v; X) \cdot e_2 ds \\ &\geq (x - rv - X) \cdot \frac{v}{|v|} + \frac{|v|}{2} \cdot 4|v|^{-1} R_X \\ &\geq -s_1|v| - |X| + 2R_X = R_U + 1. \end{aligned}$$

In particular, by applying Lemma 3.10 to $g(t) := \Big(\varphi^0(t + 4s_1, x - rv, v; X) - X\Big) \cdot e_2$, we get that $\Big(\varphi^0(t, x - rv, v; X) - X\Big) \cdot e_2$ is monotone nondecreasing with respect to $t \geq 4s_1$. Combining these, we get that $|\varphi^0(t, x - rv, v; X) - X| \geq R_U + 1$ for any $t \geq 4s_1$.

We next deal with the case $\xi_3 < 4s_1$. We notice that

$$-\nabla U\Big(\varphi^0(t, x - rv, v; X) - X\Big) \cdot e_1 \geq \frac{1}{2} \nabla U\Big(\varphi^0(t, x - rv, v; X) - X\Big) \cdot e_2, \quad \text{for any } t \geq 0. \tag{3.6}$$

Indeed, if $\nabla U(\varphi^0(t, x - rv, v; X) - X) = 0$, then (3.6) is trivial, so it suffices to prove it for the case $\nabla U(\varphi^0(t, x - rv, v; X) - X) \neq 0$. In this case, we have that $|\varphi^0(t, x - rv, v; X) - X| \leq R_U$. On the other hand, we have by Lemma 3.11 and our assumption that $(\varphi^0(t, x - rv, v; X) - X) \cdot e_1 \geq (x - rv - X) \cdot e_1 = |x - \pi_v^{\perp} X| \geq \frac{R_U}{2}$. Therefore,

$$\Big(\varphi^0(t, x - rv, v; X) - X\Big) \cdot e_1 \geq -\frac{1}{2}\Big(\varphi^0(t, x - rv, v; X) - X\Big) \cdot e_2.$$

Since $-\frac{h'(d^\varphi(t,x-rv,v;X))}{d^\varphi(t,x-rv,v;X)}$ is always non-negative, this implies that

$$\begin{aligned}&-\nabla U\big(\varphi^0(t,x-rv,v;X)-X\big)\cdot e_1\\&=-\frac{h'\big(d^\varphi(t,x-rv,v;X)\big)}{d^\varphi(t,x-rv,v;X)}\big(\varphi^0(t,x-rv,v;X)-X\big)\cdot e_1\\&\geq\frac{1}{2}\frac{h'\big(d^\varphi(t,x-rv,v;X)\big)}{d^\varphi(t,x-rv,v;X)}\big(\varphi^0(t,x-rv,v;X)-X\big)\cdot e_2\\&=\frac{1}{2}\nabla U\big(\varphi^0(t,x-rv,v;X)-X\big)\cdot e_2.\end{aligned}$$

This completes the proof of (3.6).

We notice that $\varphi^1(0,x-rv,v;X)\cdot e_1=0$ and $\varphi^1(0,x-rv,v;X)\cdot e_2=|v|$. Also, $\varphi^1(\xi_3,x-rv,v;X)\cdot e_2=\frac{|v|}{2}$. Therefore, by (3.6), we have that

$$\begin{aligned}\varphi^1(\xi_3,x-rv,v;X)\cdot e_1&=\int_0^{\xi_3}-\nabla U\big(\varphi^0(t,x-rv,v;X)-X\big)\cdot e_1dt\\&\geq\frac{1}{2}\int_0^{\xi_3}\nabla U\big(\varphi^0(t,x-rv,v;X)-X\big)\cdot e_2dt\\&=\frac{1}{2}\Big(\varphi^1(0,x-rv,v;X)\cdot e_2-\varphi^1(\xi_3,x-rv,v;X)\cdot e_2\Big)\\&=\frac{1}{2}\Big(|v|-\frac{|v|}{2}\Big)=\frac{|v|}{4}.\end{aligned}\tag{3.7}$$

Also, for any $s\geq\xi_3$, since $(\varphi^0(s,x-rv,v;X)-X)\cdot e_1>0$ by Lemma 3.11, we have that $-\nabla U(\varphi^0(s,x-rv,v;X)-X)\cdot e_1\geq 0$. Therefore, for any $t\geq\xi_3$, we have by (3.7) that

$$\begin{aligned}&\varphi^1(t,x-rv,v;X)\cdot e_1\\&=\varphi^1(\xi_3,x-rv,v;X)\cdot e_1+\int_{\xi_3}^t\Big(-\nabla U(\varphi^0(s,x-rv,v;X)-X)\cdot e_1\Big)ds\\&\geq\varphi^1(\xi_3,x-rv,v;X)\cdot e_1\geq\frac{|v|}{4}.\end{aligned}\tag{3.8}$$

Also, we have $\Big(\varphi^0(\xi_3,x-rv,v;X)-X\Big)\cdot e_1\geq 0$ by Lemma 3.11. Combining this with (3.8), we get that

$$\begin{aligned}\Big(\varphi^0(t,x-rv,v;X)-X\Big)\cdot e_1&\geq\Big(\varphi^0(\xi_3,x-rv,v;X)-X\Big)\cdot e_1+\int_{\xi_3}^t\frac{|v|}{4}ds\\&\geq\frac{|v|}{4}(t-\xi_3)\end{aligned}$$

for any $t\geq\xi_3$. Also, recall that we are assuming $\xi_3<4s_1$ now. Therefore, for any $t\geq 8s_1$, we have that $|\varphi^0(t,x-rv,v;X)-X|\geq\Big(\varphi^0(t,x-rv,v;X)-X\Big)\cdot e_1\geq\frac{|v|}{4}(t-\xi_3)\geq\frac{|v|}{4}\cdot 4|v|^{-1}R_X=R_X\geq R_U+1$. □

Finally, we deal with the case $|x-\pi_v^\perp X|\leq\frac{R_U}{2}$ and $|v|\in[\overline{v},2C_0]$. Divide $d^\varphi(t,x-rv,v;X)<R_U+1$ into three parts: $d^\varphi(t,x-rv,v;X)\in[\frac{3}{4}R_U\vee(R_U-\varepsilon_2),R_U+1)$, $d^\varphi(t,x-rv,v;X)\in[\varepsilon_3,\frac{3}{4}R_U\vee(R_U-\varepsilon_2))$ and $d^\varphi(t,x-rv,v;X)<\varepsilon_3$. We notice that $\frac{d^2}{dt^2}d^\varphi(t,x-rv,v;X)\geq 0$ by Lemma 3.6 (2), hence d^φ is convex, so the particle passes through the first two domains at most twice, and could be in the last domain at most once. Therefore, by Lemmas 3.8 and 3.9, in order to prove Proposition 3.3 (2) for $|x-\pi_v^\perp X| \le \frac{R_U}{2}$ and $|v| \in [\overline{v}, 2C_0]$, it suffices to prove the following lemma:

Lemma 3.14 *For any $r \in \mathbf{R}$, $(x, v) \in E$ and $X \in \mathbf{R}^d$, we have the following:*

1. *If $|x - \pi_v^\perp X| \le \frac{R_U}{2}$ and $|v| \ge \overline{v}$, then*

$$\int_{[0,\infty)} 1_{\left\{d^\varphi(t,x-rv,v;X)\in\left[\frac{3}{4}R_U\vee(R_U-\varepsilon_2),R_U+1\right)\right\}}dt \le 2\sqrt{2}|v|^{-1}(R_U+1).$$

2. *If $|x - \pi_v^\perp X| \le \frac{R_U}{2}$, then*

$$\int_{[0,\infty)} 1_{\left\{d^\varphi(t,x-rv,v;X)\in\left[\varepsilon_3,\frac{3}{4}R_U\vee(R_U-\varepsilon_2)\right)\right\}}dt \le 2\varepsilon_4^{-1}\left(|v|+\sqrt{2\|U\|_\infty}\right).$$

3. *If $0 < |x - \pi_v^\perp X| \le \frac{R_U}{2}$, then*

$$\int_{[0,\infty)} 1_{\{d^\varphi(t,x-rv,v;X)<\varepsilon_3\}}dt \le 2{\varepsilon_1}^{-1/2}\log^+\left(\frac{2\varepsilon_3}{|x-\pi_v^\perp X|}\right).$$

Proof (1) For any $t \ge 0$ satisfying $d^\varphi(t, x-rv, v; X) \ge \frac{3}{4}R_U \vee (R_U - \varepsilon_2)$, we have by Lemma 3.6 (4) that

$$\begin{aligned}
&|v^{d,\varphi}(t,x-rv,v;X)|^2 \\
&= -|v|^2\frac{|x-\pi_v^\perp X|^2}{d^\varphi(t,x-rv,v;X)^2} - 2h(d^\varphi(t,x-rv,v;X)) + |v|^2 + 2h(|x-rv-X|) \\
&\ge -|v|^2\frac{(R_U/2)^2}{(3R_U/4)^2} - \frac{1}{18}\overline{v}^2 + |v|^2 = \frac{5}{9}|v|^2 - \frac{1}{18}\overline{v}^2 \ge \frac{1}{2}|v|^2.
\end{aligned}$$

Therefore, $\int_{[0,\infty)} 1_{\{d^\varphi(t,x-rv,v;X)\in[\frac{3}{4}R_U\vee(R_U-\varepsilon_2),R_U+1)\}}dt \le 2\frac{R_U+1}{\frac{1}{\sqrt{2}}|v|} = 2\sqrt{2}|v|^{-1}(R_U+1)$.

(2) For any $t \ge 0$ satisfying $d^\varphi(t, x-rv, v; X) \in (\varepsilon_3, \frac{3}{4}R_U \vee (R_U - \varepsilon_2))$, we have by Lemma 3.6 (2) and the definition of ε_4 that

$$\begin{aligned}
&\frac{d}{dt}v^{d,\varphi}(t,x-rv,v;X) \\
&= \frac{1}{d^\varphi(t,x-rv,v;X)}\left|\pi^\perp_{\varphi^0(t,x-rv,v;X)-X}\varphi^1(t,x-rv,v;X)\right|^2 - h'(d^\varphi(t,x-rv,v;X)) \\
&\ge \varepsilon_4.
\end{aligned}$$

Since $|v^{d,\varphi}(t,x-rv,v;X)| \le |v| + \sqrt{2\|U\|_\infty}$ by Lemma 3.6 (4), we get our assertion (2).

(3) Write the minimum of $\{d^\varphi(t,x-rv,v;X); t \ge 0\}$ as $\overline{r}$. It suffices to consider the case where $\overline{r} < \varepsilon_3$. By Lemma 3.6 (2), we have that $\frac{d}{dt}v^{d,\varphi}(t,x-rv,v;X) \ge -h'(d^\varphi(t,x-rv,v;X))$, so if $d^\varphi(t,x-rv,v;X) < \varepsilon_3$, then $\frac{d}{dt}v^{d,\varphi}(t,x-rv,v;X) > \varepsilon_1 d^\varphi(t,x-rv,v;X)$. Since

$$\begin{aligned}
&\frac{d}{dt}\left(|v^{d,\varphi}(t,x-rv,v;X)|^2 - \varepsilon_1 d^\varphi(t,x-rv,v;X)^2\right) \\
&\quad = 2v^{d,\varphi}(t,x-rv,v;X)\left(\frac{d}{dt}v^{d,\varphi}(t,x-rv,v;X) - \varepsilon_1 d^\varphi(t,x-rv,v;X)\right),
\end{aligned}$$

this implies that the derivatives of $|v^{d,\varphi}(t,x-rv,v;X)|^2 - \varepsilon_1 d^\varphi(t,x-rv,v;X)^2$ and $d^\varphi(t,x-rv,v;X)$ have the same sign in $\left\{d^\varphi(t,x-rv,v;X) < \varepsilon_3\right\}$. So $|v^{d,\varphi}(t,x-$

$rv, v; X)|^2 - \varepsilon_1 d^\varphi(t, x - rv, v; X)^2$ attains its minimum in $\left\{d^\varphi(t, x - rv, v; X) < \varepsilon_3\right\}$ at $\overline{r}$, too. So in this domain, we have that

$$|v^{d,\varphi}(t, x - rv, v; X)|^2 - \varepsilon_1 d^\varphi(t, x - rv, v; X)^2 \geq -\varepsilon_1 \overline{r}^2,$$

hence

$$|v^{d,\varphi}(t, x - rv, v; X)| \geq \sqrt{\varepsilon_1 \left(d^\varphi(t, x - rv, v; X)^2 - \overline{r}^2\right)}.$$

Therefore, if $\overline{r} \leq \varepsilon_3$, we get that

$$\int_{[0,\infty)} 1_{\{d^\varphi(t,x-rv,v;X)\leq\varepsilon_3\}} dt \leq 2\int_{\overline{r}}^{\varepsilon_3} \frac{1}{\sqrt{\varepsilon_1}\sqrt{a^2 - \overline{r}^2}} da$$
$$= \frac{2}{\sqrt{\varepsilon_1}} \log\left(\varepsilon_3 + \sqrt{\varepsilon_3^2 - \overline{r}^2}\right) - \frac{2}{\sqrt{\varepsilon_1}} \log \overline{r} \leq \frac{2}{\sqrt{\varepsilon_1}} \log\left(\frac{2\varepsilon_3}{\overline{r}}\right).$$

Since $\overline{r} \geq |x - \pi_v^\perp X|$ by Lemma 3.11, this implies our assertion. □

Proof of Proposition 3.3 (2) We first check that (3.5) holds: for the case with $|v| \geq 2C_0$ this is proved by Lemma 3.12; for the case with $|v| \in [\overline{v}, 2C_0)$ and $|x - \pi_v^\perp X| \geq \frac{1}{2} R_U$ it is proved by Lemma 3.13; and for the case with $|v| \in [\overline{v}, 2C_0)$ and $|x - \pi_v^\perp X| \in (0, \frac{1}{2} R_U]$ it is proved by Lemmas 3.8 and 3.14.

This combined with Lemma 3.9 completes the proof of Proposition 3.3 (2). □

3.2 Error Estimate of the Freezing-Approximation

We use $\psi^0(m^{-1/2}(t-r), x, v; X)$ with proper X as an approximation of $x(t, \Psi(r, x, m^{-1/2}v))$ for r large enough, and use $\varphi^0(m^{-1/2}t, \Psi(m^{-1/2}r, x, v); X_0)$ as an approximation of it for $|r|$ small. In this section, we discuss the error estimates of these approximations. In Sect. 3.3, we will use the first order estimates of both of them, combined with the results of Sect. 3.1, to prove that the light particles with $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$ are out of the valid range after certain times. Also, we discuss the second order estimate for the approximation by $\psi(\cdot, x, v; X)$, which is necessary for formulating the limiting process.

We first quote the well-known Gronwall's Lemma in the following form:

Lemma 3.15 (Gronwall) *Suppose that a* $\mathbf{R}$*-valued continuous function* $g(\cdot)$ *satisfies*

$$0 \leq g(t) \leq \beta_1(t) + \beta_2 \int_0^t g(s)ds, \quad 0 \leq t \leq \overline{t},$$

with $\overline{t} > 0$, $\beta_2 \geq 0$ *and* $\beta_1 : [0, \overline{t}] \to \mathbf{R}$ *integrable. Then*

$$g(t) \leq \beta_1(t) + \beta_2 \int_0^t \beta_1(s) e^{\beta_2(t-s)} ds, \quad 0 \leq t \leq \overline{t}.$$

In particular, if $\beta_1(t) = \tilde{\beta}_1(t)t$ *with some non-decreasing* $\tilde{\beta}_1(\cdot)$*, then*

$$g(t) \leq \frac{\tilde{\beta}_1(t)}{\beta_2}(e^{\beta_2 t} - 1), \quad 0 \leq t \leq \overline{t}.$$

Proof The first assertion is the well-known Gronwall's Lemma itself. For the second half, it suffices to notice that $\tilde{\beta}_1(\cdot)$ is non-decreasing and that $\int_0^t s e^{\beta_2(t-s)} ds = e^{\beta_2 t}\Big[\Big(-\frac{1}{\beta_2} t e^{-\beta_2 t} - \frac{1}{\beta_2^2} e^{-\beta_2 t}\Big) + \frac{1}{\beta_2^2}\Big]$. □

As an easy corollary of Lemma 3.15, we get the following variation of Gronwall's Lemma, which is used several times in this paper.

Lemma 3.16 *Let* $y(t)$ *be a* $\mathbf{R}^d$*-valued function defined on* $\mathbf{R}$ *satisfying the following: there exist a non-decreasing non-negative function* $\beta_1(\cdot)$ *and constants* $\beta_2, \bar{t}_1, \bar{t}_2 > 0$ *such that*

$$\left|\frac{d^2}{dt^2}y(t)\right| \le \beta_1(t) + \beta_2|y(t)|, \qquad \text{for all } t \in [-\bar{t}_1, \bar{t}_2],$$

and $y(-\bar{t}_1) = \frac{d}{dt}y(-\bar{t}_1) = 0$. *Then we have that*

$$|y(t)| \vee \left|\frac{d}{dt}y(t)\right| \le \frac{2\beta_1(t)}{1+\beta_2}\Big(e^{\frac{1}{2}(1+\beta_2)(t+\bar{t}_1)} - 1\Big), \qquad \text{for all } t \in [-\bar{t}_1, \bar{t}_2].$$

Proof Let $g_1(t) := \Big(|y(t)|^2 + |y'(t)|^2\Big)^{1/2}$. Then by assumption, we get that

$$\begin{aligned}|g_1'(t)| &= \frac{|y(t)\cdot y'(t) + y'(t)\cdot y''(t)|}{(|y(t)|^2+|y'(t)|^2)^{1/2}} \le \frac{|y'(t)|\Big(|y(t)| + \beta_1(t) + \beta_2|y(t)|\Big)}{(|y(t)|^2+|y'(t)|^2)^{1/2}}\\ &\le \beta_1(t) + (1+\beta_2)\frac{1}{2}(|y(t)|^2+|y'(t)|^2)^{1/2}\\ &= \beta_1(t) + \frac{1}{2}(1+\beta_2)g_1(t), \quad t \in [-\bar{t}_1, \bar{t}_2].\end{aligned}$$

Here when passing to the second line, we used the general fact that $\frac{ab}{(a^2+b^2)^{1/2}} \le \frac{1}{2}(a^2+b^2)^{1/2}$ for any $a, b \in \mathbf{R}$. Let $g_2(t) := g_1(t-\bar{t}_1)$. Then $g_2(0) = 0$ and $|g_2'(t)| \le \beta_1(t-\bar{t}_1) + \frac{1}{2}(1+\beta_2)g_2(t)$ for any $t \in [0, \bar{t}_1+\bar{t}_2]$. Therefore, since $\beta_1(\cdot)$ is non-decreasing, we get that

$$g_2(t) \le \beta_1(t-\bar{t}_1)t + \frac{1}{2}(1+\beta_2)\int_0^t g_2(s)ds, \qquad t \in [0, \bar{t}_1+\bar{t}_2].$$

By Lemma 3.15, this implies that

$$g_2(t) \le \frac{2\beta_1(t-\bar{t}_1)}{1+\beta_2}\Big(e^{\frac{1}{2}(1+\beta_2)t} - 1\Big), \qquad t \in [0, \bar{t}_1+\bar{t}_2],$$

which implies our assertion. □

Now we are ready to consider the difference between $x(t, \Psi(r, x, m^{-1/2}v))$ and $\varphi^0(m^{-1/2}t, \Psi(m^{-1/2}r, x, v); X_0)$.

Lemma 3.17 *Assume that* $0 \le m^{1/2}s \le T \wedge \sigma_n$. *Then we have that*

$$\begin{aligned}&\Big|x(m^{1/2}s, \Psi(r, x, m^{-1/2}v)) - \varphi^0(s, \Psi(m^{-1/2}r, x, v); X_0)\Big|\\ &\qquad\vee\Big|m^{1/2}v(m^{1/2}s, \Psi(r, x, m^{-1/2}v)) - \varphi^1(s, \Psi(m^{-1/2}r, x, v); X_0)\Big|\\ &\le \frac{2C_1nm^{1/2}s}{1+C_1}\Big(e^{\frac{1}{2}(1+C_1)s} - 1\Big).\end{aligned}$$

Proof Let $y(s) := x(m^{1/2}s, \Psi(r, x, m^{-1/2}v)) - \varphi^0(s, \Psi(m^{-1/2}r, x, v); X_0)$. Then we have that $y(0) = (x - rm^{-1/2}v) - (x - rm^{-1/2}v) = 0$, and $y'(0) = m^{1/2}v(0, \Psi(r, x, m^{-1/2}v)) -$

$\varphi^1(0, \Psi(m^{-1/2}r, x, v); X_0) = v - v = 0$. Also, $\frac{d^2}{ds^2} y(s) = -\nabla U(x(m^{1/2}s, \Psi(r, x, m^{-1/2}v)) - X(m^{1/2}s)) + \nabla U(\varphi^0(s, \Psi(m^{-1/2}r, x, v); X_0) - X_0)$, so by (2.7), we get that

$$\left|\frac{d^2}{ds^2} y(s)\right| \le C_1\Big(|y(s)| + |X(m^{1/2}s) - X_0|\Big) \le C_1|y(s)| + C_1 n m^{1/2} s.$$

By Lemma 3.16, this implies our assertion. □

For r large enough, we are going to approximate $x(t, \Psi(r, x, m^{-1/2}v))$ by the scattering $\psi^0(\cdot, x, v, X(\cdot))$ with some proper $X(\cdot)$ (see (3.10) and Lemma 3.20 below). In order to apply Lemma 3.16 to estimate the approximation error of this approximation, we need to prove first that they have the same value for t small enough. We do this in Lemma 3.18 below.

Lemma 3.18 *For any $(x, v) \in E$ with $|v| \ge \overline{v}$, we have that*

$$\big(x\big(s, \Psi(r, x, m^{-1/2}v)\big), v(s, \Psi(r, x, m^{-1/2}v))\big) = \Psi(r - s, x, m^{-1/2}v) \tag{3.9}$$

and

$$|x\big(s, \Psi(r, x, m^{-1/2}v)\big) - X(s)| \ge R_U + 1$$

if $s \in [0, T \wedge \sigma_n]$ and $r \in (-\infty, -m^{1/2}\tau] \cup [s + m^{1/2}\tau, \infty)$.

Proof Since $s \in [0, T \wedge \sigma_n]$, we have that $|X(u)| \le |X_0| + nT$ for any $u \in [0, s]$. If $r \ge s + m^{1/2}\tau$ or $r \le -m^{1/2}\tau$, then for any $u \in [0, s]$, we have that $|u - r| \ge m^{1/2}\tau \ge m^{1/2}\overline{v}^{-1}R_0$, hence

$$\begin{aligned}
&|x - rm^{-1/2}v + um^{-1/2}v - X(u)| \\
&\quad \ge m^{-1/2}|u - r||v| - |X_0| - nT \ge m^{-1/2}m^{1/2}\overline{v}^{-1}R_0\overline{v} - |X_0| - nT \\
&\quad = R_0 - |X_0| - nT = R_U + 1,
\end{aligned}$$

so the light particle keeps a uniform motion during the time interval $[0, s]$. Therefore, (3.9) holds, hence $|x(s, \Psi(r, x, m^{-1/2}v)) - X(s)| = |x - m^{-1/2}(r - s)v - X(s)| \ge R_U + 1$. □

For any $t \ge 0$, let $\mathcal{F}_t := \mathcal{F}_{(-\infty, t+m^{1/2}\tau]\times E}$. Then we get the following as a corollary of Lemma 3.18. This is used later.

Lemma 3.19 *For any $t \in [0, T \wedge \sigma_n]$, we have that $X(t)$ is $\mathcal{F}_t$-measurable.* □

Choose any $c_m \to 0$ (when $m \to 0$). We are going to approximate $x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v))$ by the scattering $\psi^0(t, x, v; X(s - c_m))$ (see (3.10) and Lemma 3.20 below). Let

$$\kappa_1(t) := x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) - \psi^0(t, x, v; X(s - c_m)). \tag{3.10}$$

We prove in Lemma 3.20 below that the first order error $\kappa_1(t)$ is small enough. Lemma 3.20 with $c_m = \tau m^{1/2}$ is enough for proving that $x(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v))$ is out of the valid interaction range for $t \notin [-\tau, t_1(v, |x - \pi_v^{\perp} X(\tilde{r})|)]$ and $r \ge m^{1/2}\tau$. However, in Sect. 3.4, Lemma 3.20 with general c_m as in our present form is necessary in order to estimate the second order error of our freezing-approximation.

Lemma 3.20 *Assume that $(x, v) \in E$, $|v| \ge \overline{v}$ and $s - c_m, s - m^{1/2}\tau \in [0, T \wedge \sigma_n]$. Then we have the following:*

1. $\kappa_1(t) = 0$ *if* $0 \le m^{1/2}t + s \le T \wedge \sigma_n$ *and* $t \le -\tau$.

2. $$\frac{d^2}{dt^2}\kappa_1(t) = -\nabla U(\kappa_1(t) + \psi^0(t, x, v; X(s - c_m)) - X(m^{1/2}t + s)) + \nabla U(\psi^0(t, x, v; X(s - c_m)) - X(s - c_m)). \tag{3.11}$$

3. *for any* $b_m \ge \tau$, *we have that*
$$|\kappa_1(t)| \vee |\kappa_1'(t)| \le \frac{2C_1 n}{1 + C_1}(m^{1/2}b_m + |c_m|)\left(e^{\frac{1}{2}(1+C_1)(t+\tau)} - 1\right)$$
for any $t \in [-\tau, b_m]$ *satisfying* $0 \le m^{1/2}t + s \le T \wedge \sigma_n$.

We remark that b_m and c_m in Lemmas 3.20 and 3.21 may depend on (x, v) and s.

Proof Under the given assumption, we notice that $|X(s - c_m)| \le |X_0| + nT$, so for any $t \le -\tau$ satisfying $0 \le m^{1/2}t + s \le T \wedge \sigma_n$, we have by Lemma 3.18 and Lemma 3.2 (2) that $x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) = \psi^0(t, x, v; X(s - c_m)) = x + tv$, hence $\kappa_1(t) = 0$.

The second assertion is trivial by definition.

For the last assertion, we notice that since $t \in [-\tau, b_m]$, $s - c_m, m^{1/2}t + s \in [0, T \wedge \sigma_n]$ and $b_m \ge \tau$, we have that
$$|X(m^{1/2}t + s) - X(s - c_m)| \le n|m^{1/2}t + c_m| \le n(m^{1/2}b_m + |c_m|).$$
Therefore, by (3.11) and (2.7), we get that
$$\left|\frac{d^2}{dt^2}\kappa_1(t)\right| \le C_1\left(|\kappa_1(t)| + n(m^{1/2}b_m + |c_m|)\right)$$
for any $t \in [-\tau, b_m]$ satisfying $0 \le m^{1/2}t + s \le T \wedge \sigma_n$. Also, $\kappa_1(-\tau) = \kappa_1'(-\tau) = 0$ by the first assertion of this lemma. These combined with Lemma 3.16 imply our third assertion. □

Lemma 3.20 is enough for our proof of the tightness. However, to find the explicit expression of the limiting process (see Sect. 3.5 for details), we need the second order error estimate, too. Let
$$\kappa_2(t) := \kappa_1(t) - m^{1/2}z\left(t, x, v; X(s - c_m), V(s - c_m), m^{-1/2}c_m\right).$$
Lemma 3.21 below gives an estimate of $\kappa_2(t)$.

The following estimate, which will be used several times in this paper, is trivial by a direct calculation.
$$\frac{2C_1}{1 + C_1}(e^b - 1) + 1 \le 2e^b, \qquad \text{for all } b > 0. \tag{3.12}$$

Using this we shall prove the following:

Lemma 3.21 *Assume that* $(x, v) \in E$, $|v| \ge \overline{v}$, $b_m \ge \tau$, $-b_m \le m^{-1/2}c_m \le \tau$, $-\tau \le t \le b_m$ *and* $s - c_m, s + m^{1/2}t, s - m^{1/2}\tau \in [0, T \wedge \sigma_n]$. *Then for any* $a \in [0, 1]$, *we have that*
$$|\kappa_2(t)| \le \frac{8n^2}{1 + C_1}(2C_1 \vee 1)(\|\nabla^3 U\|_\infty \vee 1)e^{(1+\frac{a}{2})(1+C_1)(t+\tau)}\left(m^{1/2}b_m + |c_m|\right)^{1+a} + \frac{2C_1}{1 + C_1}\left(e^{\frac{1}{2}(1+C_1)(t+\tau)} - 1\right)\int_{s-m^{1/2}\tau}^{(s+m^{1/2}t)\vee(s-c_m)} |V(u) - V(s - c_m)|du.$$

Proof First we notice that $\kappa_2(-\tau) = \frac{d}{dt}\kappa_2(-\tau) = 0$ by Lemmas 3.20 (1), 3.2 (2) and the definition of z. Also, by definition and a simple calculation, we get that

$$\begin{aligned}\frac{d^2}{dt^2}\kappa_2(t) = -\int_0^1 d\theta\Big\{\nabla^2 U\Big(\psi^0(t,x,v;X(s-c_m)) - X(s-c_m)\\ +\theta\big[\kappa_1(t) - (X(m^{1/2}t+s) - X(s-c_m))\big]\Big)\\ -\nabla^2 U(\psi^0(t,x,v;X(s-c_m)) - X(s-c_m))\Big\}\\ \cdot\big[\kappa_1(t) - X(m^{1/2}t+s) + X(s-c_m)\big]\\ -\nabla^2 U(\psi^0(t,x,v;X(s-c_m)) - X(s-c_m))\\ \cdot\big\{\kappa_2(t) - \big[X(m^{1/2}t+s) - X(s-c_m) - (m^{1/2}t+c_m)V(s-c_m)\big]\big\}.\end{aligned}$$

We have that $|X(m^{1/2}t+s) - X(s-c_m)| \le n|m^{1/2}t+c_m|$ and

$$\begin{aligned}&\Big|X(m^{1/2}t+s) - X(s-c_m) - (m^{1/2}t+c_m)V(s-c_m)\Big|\\ &= \Big|\int_{s-c_m}^{m^{1/2}t+s}(V(u) - V(s-c_m))du\Big| \le \int_{s-m^{1/2}\tau}^{(s+m^{1/2}t)\vee(s-c_m)}|V(u) - V(s-c_m)|du.\end{aligned}$$

Combining these with (2.6), for any $a \in [0,1]$, we get that

$$\begin{aligned}\Big|\frac{d^2}{dt^2}\kappa_2(t)\Big| &\le (2C_1)^{1-a}\|\nabla^3 U\|_\infty^a\Big(|\kappa_1(t)| + n|m^{1/2}t+c_m|\Big)^{1+a}\\ &\quad + C_1\left(|\kappa_2(t)| + \int_{s-m^{1/2}\tau}^{(s+m^{1/2}t)\vee(s-c_m)}|V(u) - V(s-c_m)|du\right)\end{aligned}$$

for any $t \in [-\tau, b_m]$. This combined with Lemma 3.20 and (3.12) implies that

$$\begin{aligned}\Big|\frac{d^2}{dt^2}\kappa_2(t)\Big| &\le (2C_1\vee 1)(\|\nabla^3 U\|_\infty \vee 1)\Big(2n(m^{1/2}b_m + |c_m|)e^{\frac{1}{2}(1+C_1)(t+\tau)}\Big)^{1+a}\\ &\quad + C_1\int_{s-m^{1/2}\tau}^{(s+m^{1/2}t)\vee(s-c_m)}|V(r) - V(s-c_m)|dr + C_1|\kappa_2(t)|.\end{aligned}$$

This combined with Lemma 3.16 implies our assertion. □

3.3 Behavior of $x(t,x,v)$

In this section, we study the sojourn time of the light particle in the valid interaction range. Precisely, we consider the question about when does $\nabla U(x(m^{1/2}t+r, \Psi(r,x,m^{-1/2}v)) - X(m^{1/2}t+r)) \ne 0$ hold. Our main result of this section is the following.

Proposition 3.22 *There exists a $m_1 \ge 0$ such that for any $m \in (0, m_1]$, $(x,v) \in E$ satisfying $|v| \ge \bar{v}$ and $r, t \in \mathbf{R}$ satisfying $m^{1/2}t + r \in [0, T \wedge \sigma_n]$, we have that*

$$\Big|x(m^{1/2}t+r, \Psi(r,x,m^{-1/2}v)) - X(m^{1/2}t+r)\Big| \ge R_U$$

if at least one of the following conditions is satisfied:

1. $|v| \ge 2C_0 + 1$ *and* $t \notin [-\tau, 2\tau]$,
2. $r \le -m^{1/2}\tau$,
3. $t \le -\tau$,

4. $|x-\pi_v^{\perp}X(\tilde{r})| \geq R_U + \frac{1}{2}$,
5. *α satisfies (2.10), $|x-\pi_v^{\perp}X(\tilde{r})| \geq m^{\alpha}$ and $t \geq t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|)$.*

We prove Proposition 3.22 in Sect. 3.3. Since the notation τ in [12] is equal to $C_0^{-1}R_0$, which is dominated by our τ in this paper, we get the first assertion of Proposition 3.22 as a trivial corollary of [12, Proposition 3.6.5]. The assertions (2) and (3) are trivial by Lemma 3.18. We prove the assertions (4) and (5) of Proposition 3.22 in the rest of Sect. 3.3.

Proposition 3.22 (4) is included in Lemma 3.23 below.

Lemma 3.23 *Assume that $r, t \in \mathbf{R}$ satisfy $m^{1/2}t + r \in [0, T \wedge \sigma_n]$. Let $(x, v) \in E$ with $|v| \geq \overline{v}$. Also, assume that $|x-\pi_v^{\perp}X(\tilde{r})| \geq R_U + \frac{1}{2}$. Then as long as $m \leq (4n\tau)^{-2} \wedge (n^{-2}\overline{v}^2)$, we have that*

$$(x(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)), v(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v))) = (x + tv, m^{-1/2}v)$$

and

$$|x\left(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)\right) - X(m^{1/2}t + r)| \geq R_U.$$

We remark that by Lemma 3.23, at least until $T \wedge \sigma_n$, those light particles with $|x - \pi_v^{\perp}X(\tilde{r})| \geq R_U + \frac{1}{2}$ keep uniform motions, and never enter the valid interaction range.

Proof of Lemma 3.23 If $r \leq -m^{1/2}\tau$ or $t \leq -\tau$, then our assertion is nothing but Lemma 3.18. From now on, we assume that $r \geq -m^{1/2}\tau$ and $t \geq -\tau$. Therefore, $r - m^{1/2}\tau \leq r + m^{1/2}t \leq T \wedge \sigma_n$, hence $\tilde{r} = (r - m^{1/2}\tau) \vee 0$ and $m^{1/2}t + r \geq (r - m^{1/2}\tau) \vee 0 = \tilde{r}$.

We first deal with the case $t \in [-\tau, \tau]$. First we notice that

$$\left(x\left(\tilde{r}, \Psi(r, x, m^{-1/2}v)\right), v\left(\tilde{r}, \Psi(r, x, m^{-1/2}v)\right)\right) = \left(x - (\tau \wedge m^{-1/2}r)v, m^{-1/2}v\right). \quad (3.13)$$

Indeed, if $r \leq m^{1/2}\tau$, then $\tilde{r} = 0$, hence $\left(x(\tilde{r}, \Psi(r, x, m^{-1/2}v)), v(\tilde{r}, \Psi(r, x, m^{-1/2}v))\right) = (x - m^{-1/2}rv, m^{-1/2}v)$; if $r > m^{1/2}\tau$, then $\tilde{r} = r - m^{1/2}\tau$, hence by Lemma 3.18, we get that $\left(x(\tilde{r}, \Psi(r, x, m^{-1/2}v)), v(\tilde{r}, \Psi(r, x, m^{-1/2}v))\right) = \Psi(r - \tilde{r}, x, m^{-1/2}v) = (x - \tau v, m^{-1/2}v)$.

Also, for any $u \in [\tilde{r}, m^{1/2}t + r]$, since $t \leq \tau$ by assumption and $\tilde{r}, u \in [0, T \wedge \sigma_n]$, we get that $|X(\tilde{r}) - X(u)| \leq n(u - \tilde{r}) \leq n(m^{1/2}t + r - (r - m^{1/2}\tau)) \leq 2nm^{1/2}\tau$. Therefore, since $m \leq (4n\tau)^{-2}$ by assumption, we get that

$$\begin{aligned}
&|x - m^{-1/2}rv + um^{-1/2}v - X(u)| \geq |x - \pi_v^{\perp}X(\tilde{r})| - |X(\tilde{r}) - X(u)| \\
&\geq R_U + \frac{1}{2} - 2nm^{1/2}\tau \geq R_U, \qquad u \in [\tilde{r}, m^{1/2}t + r].
\end{aligned} \quad (3.14)$$

So the particle keeps a uniform motion during the time interval $u \in [\tilde{r}, m^{1/2}t+r]$. Combining this with (3.13), we get that

$$\begin{aligned}
&v\left(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)\right) = v\left(\tilde{r}, \Psi(r, x, m^{-1/2}v)\right) = m^{-1/2}v, \\
&x\left(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)\right) \\
&\quad = x - (\tau \wedge m^{-1/2}r)v + (m^{1/2}t + r - \tilde{r})m^{-1/2}v \\
&\quad = x + \left(m^{1/2}t + r - \tilde{r} - (m^{1/2}\tau) \wedge r\right)m^{-1/2}v = x + tv.
\end{aligned}$$

This combined with (3.14) (with $u = m^{1/2}t + r$) implies our assertion, and completes our proof for the case $t \in [-\tau, \tau]$.

In particular, if $r + m^{1/2}\tau \leq T \wedge \sigma_n$, then $(x(m^{1/2}\tau + r, \Psi(r, x, m^{-1/2}v)), v(m^{1/2}\tau + r, \Psi(r, x, m^{-1/2}v))) = (x + \tau v, m^{-1/2}v)$. We prove in the following that this implies our assertions for $t \geq \tau$. Indeed, let $\xi_4 := \inf\{t > \tau; |x(m^{1/2}t+r, \Psi(r, x, m^{-1/2}v)) - X(m^{1/2}t+r)| < R_U\}$. Then $|x(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)) - X(m^{1/2}t + r)| \geq R_U$ for any $t \in [\tau, \xi_4]$, hence $(x(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)), v(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v))) = (x + tv, m^{-1/2}v)$ for any $t \in [\tau, \xi_4]$. Therefore, it suffices to prove that $m^{1/2}\xi_4 + r \geq T \wedge \sigma_n$. Suppose not. Then for any $t \in [\tau, \xi_4]$, since $m \leq n^{-2}\overline{v}^2$ by assumption, we have that $(m^{-1/2}v - V(m^{1/2}t+r)) \cdot \frac{v}{|v|} \geq m^{-1/2}|v| - n \geq 0$. Therefore,

$$\begin{aligned}
R_U &= \left|x\left(m^{1/2}\xi_4 + r, \Psi(r, x, m^{-1/2}v)\right) - X(m^{1/2}\xi_4 + r)\right| \\
&\geq \left(x(m^{1/2}\xi_4 + r, \Psi(r, x, m^{-1/2}v)) - X(m^{1/2}\xi_4 + r)\right) \cdot \frac{v}{|v|} \\
&= \Big\{x\left(m^{1/2}\tau + r, \Psi(r, x, m^{-1/2}v)\right) - X(m^{1/2}\tau + r) \\
&\qquad + m^{1/2}\int_\tau^{\xi_4} \left(v\left(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)\right) - V(m^{1/2}t + r)\right)dt\Big\} \cdot \frac{v}{|v|} \\
&= \left\{x + \tau v - X(m^{1/2}\tau + r) + m^{1/2}\int_\tau^{\xi_4} \left(m^{-1/2}v - V(m^{1/2}t + r)\right)dt\right\} \cdot \frac{v}{|v|} \\
&\geq \left\{x + \tau v - X(m^{1/2}\tau + r)\right\} \cdot \frac{v}{|v|} \geq \tau|v| - (|X_0| + nT) \\
&\geq R_0 - (|X_0| + nT) = R_U + 1,
\end{aligned}$$

which yields a contradiction. □

Finally, we prove the assertion (5) of Proposition 3.22. We notice that $t_1(v, |x - \pi_v^\perp X(\tilde{r})|) \geq 2\tau$ by definition, so by (1) (2) and (4) of Proposition 3.22, we only need to consider those light particles with $|v| \in [\overline{v}, 2C_0 + 1)$, $r \geq -m^{1/2}\tau$ and $|x - \pi_v^\perp X(\tilde{r})| \in [m^\alpha, R_U + \frac{1}{2}]$. In other words, it suffices to prove the following lemma:

Lemma 3.24 *Under (2.10), there exists a $m_2 > 0$ such that for any $m \in (0, m_2]$, we have that*

$$|x\left(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v)\right) - X(m^{1/2}t + r)| \geq R_U$$

as long as $(x, v) \in E$, $|v| \in [\overline{v}, 2C_0 + 1)$, $|x - \pi_v^\perp X(\tilde{r})| \in [m^\alpha, R_U + \frac{1}{2}]$, $t \geq t_1(v, |x - \pi_v^\perp X(\tilde{r})|)$, $r \geq -m^{1/2}\tau$ and $m^{1/2}t + r \in [0, T \wedge \sigma_n]$.

We prove Lemma 3.24 by using the results of Sects. 3.1 and 3.2. As explained before, we approximate $x(m^{1/2}t + r, \Psi(r, x, m^{-1/2}v))$ by $\varphi(t + m^{-1/2}r, \Psi(m^{-1/2}r, x, v); X_0)$ if $|r| \leq m^{1/2}\tau$, and by $\psi(t, x, v; X(\tilde{r}))$ if $r > m^{1/2}\tau$. Let $a(t, r, x, v)$ denote this approximation, i.e., for any $(x, v) \in E$, $r \geq -m^{1/2}\tau$ and $t \in \mathbf{R}$, define $a(t, r, x, v) = (a^0(t, r, x, v), a^1(t, r, x, v))$ by

$$a(t, r, x, v) := \begin{cases} \varphi\left(t + m^{-1/2}r, \Psi(m^{-1/2}r, x, v); X_0\right), & \text{if} \quad |r| \leq m^{1/2}\tau, \\ \psi\left(t, x, v; X(\tilde{r})\right), & \text{if} \quad r > m^{1/2}\tau. \end{cases}$$

Also, let $d^a(t, r, x, v) := |a^0(t, r, x, v) - X(\tilde{r})|$ and $v^{d,a}(t, r, x, v) := \frac{d}{dt}d^a(t, r, x, v) = \left(a^1(t, r, x, v), \frac{a^0(t,r,x,v) - X(\tilde{r})}{|a^0(t,r,x,v) - X(\tilde{r})|}\right)$. Then we have the following.

Lemma 3.25 *Suppose that $(x, v) \in E$, $r \geq -m^{1/2}\tau$ and $|v| \geq \overline{v}$. Then we have the following:*

1. $|a^0\left(t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|), r, x, v\right) - X(\tilde{r})| \geq R_U + 1$,
2. $v^{d,a}\left(t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|), r, x, v\right) > 0$.

Proof Let $s_1 := |v|^{-1}R_{X(\tilde{r})}$. Then $s_1 \leq \tau$. With the help of (3.3), we get our assertions for $r \in [-m^{1/2}\tau, -m^{1/2}s_1]$ by Proposition 3.3 (1); for $r \in (-m^{1/2}s_1, m^{1/2}s_1]$ by Proposition 3.3 (2); for $r \in (m^{1/2}s_1, m^{1/2}\tau]$ by Proposition 3.3 (3); and for $r > m^{1/2}\tau$ by Proposition 3.4. □

In order to prove Lemma 3.24, we first prove that the light particle is out of the valid interaction range at time $m^{1/2}t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|) + r$.

Lemma 3.26 *Assume* (2.10). *Then there exists a* $m_3 > 0$ *such that for any* $m \in (0, m_3]$, *we have that*

$$\left|x\left(m^{1/2}t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|) + r, \Psi(r, x, m^{-1/2}v)\right) - X\left(m^{1/2}t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|) + r\right)\right| \geq R_U + \frac{1}{2}$$

as long as $(x, v) \in E$, $|v| \in [\overline{v}, 2C_0 + 1)$, $|x-\pi_v^{\perp}X(\tilde{r})| \geq m^{\alpha}$, $r \geq -m^{1/2}\tau$ *and* $m^{1/2}t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|) + r \in [0, T \wedge \sigma_n]$.

Proof To simplify notations, write $t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|)$ as t_0 in the proof of this lemma. By Lemma 3.17 and Lemma 3.20 (3) for $c_m = m^{1/2}\tau$, we get that

$$\begin{aligned}&|x\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - a^0(t_0, r, x, v)|\\ &\leq \frac{2C_1}{1+C_1}nm^{1/2}(t_0+\tau)\left(e^{\frac{1}{2}(1+C_1)(t_0+\tau)} - 1\right).\end{aligned}$$

Also, since $m^{1/2}t_0 + r, \tilde{r} \in [0, T \wedge \sigma_n]$, we have that $|X(m^{1/2}t_0 + r) - X(\tilde{r})| \leq n|m^{1/2}t_0 + r - \tilde{r}| \leq nm^{1/2}(t_0 + \tau)$. Combining these with Lemma 3.25 (1), we get that

$$\begin{aligned}&|x\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - X(m^{1/2}t_0 + r)|\\ &\geq |a^0(t_0, r, x, v) - X(\tilde{r})| - |x\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - a^0(t_0, r, x, v)|\\ &\quad -|X(m^{1/2}t_0 + r) - X(\tilde{r})|\\ &\geq R_U + 1 - 2nm^{1/2}(t_0+\tau)e^{\frac{1}{2}(1+C_1)(t_0+\tau)}.\end{aligned}$$

So it suffices to choose $m_3 > 0$ such that $2nm^{1/2}(t_0+\tau)e^{\frac{1}{2}(1+C_1)(t_0+\tau)} \leq \frac{1}{2}$ for any $m \in (0, m_3]$. We have that $|x-\pi_v^{\perp}X(\tilde{r})| \geq m^{\alpha}$ by assumption, so as long as $m \leq (2\varepsilon_3)^{1/\alpha}$, we have that $t_0 = t_1(v, |x-\pi_v^{\perp}X(\tilde{r})|) \leq C_2 + 2\varepsilon_1^{-1/2}\log\frac{2\varepsilon_3}{m^{\alpha}}$. Therefore,

$$\begin{aligned}&2nm^{1/2}(t_0+\tau)e^{\frac{1}{2}(1+C_1)(t_0+\tau)}\\ &\leq 2ne^{\frac{1}{2}(1+C_1)(C_2+\tau+2\varepsilon_1^{-1/2}\log(2\varepsilon_3))}\left(C_2 + \tau + 2\varepsilon_1^{-1/2}\log\frac{2\varepsilon_3}{m^{\alpha}}\right)m^{\frac{1}{2}-\frac{1}{2}(1+C_1)2\varepsilon_1^{-1/2}\alpha}.\end{aligned}$$

Since the right hand side above converges to 0 as long as $(1+C_1)\varepsilon_1^{-1/2}\alpha < \frac{1}{2}$, we get our assertion. □

Notice: we do not need $|x-\pi_v^{\perp}X(s-am^{1/2})| \leq R_U + \frac{1}{2}$ in Lemma 3.26. However, this condition is necessary for Lemma 3.27 below.

In order to prove that the particle could not enter the valid interaction range after $m^{1/2}t_1(v, |x-\pi_v^\perp X(\tilde{r})|)+r$, either, we also need to discuss its velocity at this time. We first notice the following general estimate:

$$\left|v_1 \cdot \frac{x_1}{|x_1|} - v_2 \cdot \frac{x_2}{|x_2|}\right| \leq |v_1 - v_2| + 2\frac{|v_2|}{|x_2|}|x_1 - x_2|, \quad \text{for any } x_1, x_2, v_1, v_2 \in \mathbf{R}^d. \tag{3.15}$$

Indeed, since $\left|\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}\right| = \left|\frac{x_1-x_2}{|x_2|} + \left(\frac{1}{|x_1|} - \frac{1}{|x_2|}\right)x_1\right| \leq \frac{2|x_1-x_2|}{|x_2|}$, we get that

$$\begin{aligned}\left|v_1 \cdot \frac{x_1}{|x_1|} - v_2 \cdot \frac{x_2}{|x_2|}\right| &= \left|(v_1 - v_2) \cdot \frac{x_1}{|x_1|} + v_2 \cdot \left(\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}\right)\right| \\ &\leq |v_1 - v_2| + |v_2|\left|\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}\right| \leq |v_1 - v_2| + \frac{2|v_2|}{|x_2|}|x_1 - x_2|.\end{aligned}$$

Let

$$v^{d,2}(t, \Psi(r, x, m^{-1/2}v)) := \left(v(t, \Psi(r, x, m^{-1/2}v)), \frac{x(t, \Psi(r, x, m^{-1/2}v)) - X(t)}{|x(t, \Psi(r, x, m^{-1/2}v)) - X(t)|}\right).$$

We notice that $v^{d,2}(t, \Psi(r, x, m^{-1/2}v))$ is not equal to $\frac{d}{dt}|x(t, \Psi(r, x, m^{-1/2}v)) - X(t)|$. Then we have the following.

Lemma 3.27 *Assume* (2.10). *Then there exist constants* $m_4, C_4 > 0$ *such that for any* $m \in (0, m_4]$, *we have that*

$$v^{d,2}(m^{1/2}t_1(v, |x - \pi_v^\perp X(\tilde{r})|) + r, \Psi(r, x, m^{-1/2}v)) \geq C_4 m^{-1/2}$$

as long as $(x, v) \in E$, $|x - \pi_v^\perp X(\tilde{r})| \in [m^\alpha, R_U + \frac{1}{2}]$, $r \geq -m^{1/2}\tau$ *and* $m^{1/2}t_1(v, |x - \pi_v^\perp X(\tilde{r})|) + r \in [0, T \wedge \sigma_n]$.

Proof To simplify notations, we write $t_1(v, |x - \pi_v^\perp X(\tilde{r})|)$ as t_0 in the proof of this lemma, too. By Lemmas 3.17 and 3.20 (3) for $c_m = m^{1/2}\tau$, we have that

$$\begin{aligned}&|m^{1/2}v\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - a^1(t_0, r, x, v)| \\ &\vee|x\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - a^0(t_0, r, x, v)| \\ &\leq \frac{2C_1 n}{1 + C_1} m^{1/2}(t_0 + \tau)\left(e^{\frac{1}{2}(1+C_1)(t_0+\tau)} - 1\right).\end{aligned}$$

Also, by Lemma 3.5 (1) and Lemma 3.7 (1), we have that $|a^1(t_0, r, x, v)| \leq |v| + \sqrt{2\|U\|_\infty}$. Since $m^{1/2}t_0 + r, \tilde{r} \in [0, T \wedge \sigma_n]$, we have $|X(m^{1/2}t_0 + r) - X(\tilde{r})| \leq n|m^{1/2}t_0 + r - \tilde{r}| \leq nm^{1/2}(t_0 + \tau)$. These combined with (3.15) and Lemma 3.25 (1) imply that

$$\begin{aligned}&|m^{1/2}v^{d,2}\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - v^{d,a}(t_0, r, x, v)| \\ &\leq |m^{1/2}v\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - a^1(t_0, r, x, v)| \\ &\quad + \frac{2(|v| + \sqrt{2\|U\|_\infty})}{R_U + 1}\Big(|x\left(m^{1/2}t_0 + r, \Psi(r, x, m^{-1/2}v)\right) - a^0(t_0, r, x, v)| \\ &\quad + |X(m^{1/2}t_0 + r) - X(\tilde{r})|\Big) \\ &\leq \left(1 + \frac{2(|v| + \sqrt{2\|U\|_\infty})}{R_U + 1}\right)\frac{2C_1 n}{1 + C_1} m^{1/2}(t_0 + \tau)\left(e^{\frac{1}{2}(1+C_1)(t_0+\tau)} - 1\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{2(|v|+\sqrt{2\|U\|_\infty})}{R_U+1}nm^{1/2}(t_0+\tau)\\
&\le 4(2|v|+2\sqrt{2\|U\|_\infty}+1)nm^{1/2}t_0e^{\frac{1}{2}(1+C_1)(t_0+\tau)}.
\end{aligned}
\tag{3.16}
$$

On the other hand, let $C_5 := \left(1-\frac{(R_U+\frac{1}{2})^2}{(R_U+1)^2}\right)^{1/2}$, then by Lemmas 3.6 (4), 3.7 (3) and 3.25 (1), we get that

$$
|v^{d,a}(t_0,r,x,v)|^2 \ge -\frac{|v|^2|x-\pi_v^\perp X(\tilde r)|^2}{d^a(t_0,r,x,v)^2}+|v|^2 \ge \left(1-\frac{\left(R_U+\frac{1}{2}\right)^2}{(R_U+1)^2}\right)|v|^2 = C_5^2|v|^2.
$$

By Lemma 3.25 (2), this implies that $v^{d,a}(t_0,r,x,v) \ge C_5|v|$. Combining this with (3.16), we get that

$$
\begin{aligned}
&m^{1/2}v^{d,2}\big(m^{1/2}t_0+r,\Psi(r,x,m^{-1/2}v)\big)\\
&\ge C_5|v|-4\big(2|v|+2\sqrt{2\|U\|_\infty}+1\big)nm^{1/2}t_0e^{\frac{1}{2}(1+C_1)(t_0+\tau)}\\
&=\big(C_5-8nm^{1/2}t_0e^{\frac{1}{2}(1+C_1)(t_0+\tau)}\big)|v|-4\big(2\sqrt{2\|U\|_\infty}+1\big)nm^{1/2}t_0e^{\frac{1}{2}(1+C_1)(t_0+\tau)}.
\end{aligned}
\tag{3.17}
$$

Since $|x-\pi_v^\perp X(\tilde r)| \ge m^\alpha$ by assumption, we have that $t_0 = t_1(v,|x-\pi_v^\perp X(\tilde r)|) \le C_2+2\varepsilon_1^{-1/2}\log\frac{2\varepsilon_3}{m^\alpha}$. Since $\frac{1}{2}-(1+C_1)\varepsilon_1^{-1/2}\alpha > 0$ by assumption, we get that $m^{1/2}t_0e^{\frac{1}{2}(1+C_1)(t_0+\tau)} \to 0$ as $m\to 0$. Also, $|v| \ge \overline{v}$. Therefore, there exists a $m_4 \in (0,1]$ such that the right hand side of (3.17) with $m \in (0,m_4]$ is dominated below by a strictly positive constant. This gives us our assertion. □

Now, we are ready to prove that those light particles with $|x-\pi_v^\perp X(\tilde r)| \ge m^\alpha$ could never enter the valid range after $m^{1/2}t_1(v,|x-\pi_v^\perp X(\tilde r)|)+r$.

Proof of Lemma 3.24 As we did before, we shall write $t_1(v,|x-\pi_v^\perp X(\tilde r)|)$ as t_0 to simplify notations. Let $\xi_5 := \inf\{t > t_0 : |x(m^{1/2}t+r,\Psi(r,x,m^{-1/2}v))-X(m^{1/2}t+r)| < R_U\}$. It suffices to prove that $m^{1/2}\xi_5+r > T\wedge\sigma_n$.

Let $e_3 := \frac{x(m^{1/2}t_0+r,\Psi(r,x,m^{-1/2}v))-X(m^{1/2}t_0+r)}{|x(m^{1/2}t_0+r,\Psi(r,x,m^{-1/2}v))-X(m^{1/2}t_0+r)|}$. Then by Lemma 3.27 we have that $v(m^{1/2}t_0+r,\Psi(r,x,m^{-1/2}v))\cdot e_3 \ge C_4m^{-1/2}$. We notice that the particle keeps a uniform motion during the time interval $[m^{1/2}t_0+r, m^{1/2}\xi_5+r]$, hence $v(m^{1/2}t+r,\Psi(r,x,m^{-1/2}v)) = v(m^{1/2}t_0+r,\Psi(r,x,m^{-1/2}v))$ for any $t\in[t_0,\xi_5]$. Suppose that $m^{1/2}\xi_5+r \le T\wedge\sigma_n$. Then $0 \le m^{1/2}t_0+r \le m^{1/2}\xi_5+r \le T\wedge\sigma_n$, hence $|X(m^{1/2}t_0+r)-X(m^{1/2}\xi_5+r)| \le nm^{1/2}(\xi_5-t_0)$. Combining these with Lemma 3.26, we get that if $m \le m_3\wedge m_4$, then

$$
\begin{aligned}
R_U &= |x\big(m^{1/2}\xi_5+r,\Psi(r,x,m^{-1/2}v)\big)-X\big(m^{1/2}\xi_5+r\big)|\\
&\ge \Big(x\big(m^{1/2}\xi_5+r,\Psi(r,x,m^{-1/2}v)\big)-X(m^{1/2}\xi_5+r)\Big)\cdot e_3\\
&= |x\big(m^{1/2}t_0+r,\Psi(r,x,m^{-1/2}v)\big)-X(m^{1/2}t_0+r)|\\
&\quad +m^{1/2}(\xi_5-t_0)v\big(m^{1/2}t_0+r,\Psi(r,x,m^{-1/2}v)\big)\cdot e_3\\
&\quad +\Big(X(m^{1/2}t_0+r)-X(m^{1/2}\xi_5+r)\Big)\cdot e_3\\
&\ge R_U+\frac{1}{2}+m^{1/2}\big(\xi_5-t_0\big)(C_4m^{-1/2}-n\big).
\end{aligned}
$$

If $m \le n^{-2}C_4^2$ in addition, the right hand side above is greater than or equal to $R_U + \frac{1}{2}$, which yields a contradiction. □

3.4 Tightness and Decomposition of V(t)

In Sects. 3.4 and 3.5, we re-express $V(t)$ as a sum of a martingale term, a smooth term and a negligible term. The heuristic meanings of these quantities are the same as those discussed in [12, 13]. Precisely, the martingale term is approximately the variance of the force after our freezing-approximation (see the term $I^3(t)$ defined below). We remark that, when there are more than one massive particles, the mean of the force after our freezing-approximation (the term $I^5(t)$ defined below) gives us approximately the resulting interaction between the massive particles – since there is only one massive particle in our present model, this term is approximately 0 in our case (see [12, Lemma 4.3.3] and the proof of Lemma 3.37). The smooth part (the term $I^4(t)$ defined below) is given approximately by the first order of our approximation error.

Also, the discussion with respect to $I^6(t)$ and $I^9(t)$ defined below consists of two steps: we first prove in Sect. 3.4 that they can also be regarded as a part of the smooth term; and with the help of the results of Sect. 3.4, we prove in Sect. 3.5 that they are actually negligible.

Necessary estimates of these quantities are also provided in Sect. 3.4. In particular, we get the desired tightness with the help of the Skorohod space (see Sect. 3.4.1 for a brief review of Skorohod space).

Section 3.4 is organized as follows. In Sect. 3.4.1, we review several basic facts with respect to the Skorohod space and the tightness of the probability measures on it; in Sect. 3.4.2, we present our key decomposition of $V(t)$ and the tightness (Proposition 3.31), the main result of Sect. 3.4; Sects. 3.4.3 and 3.4.4 give the proof of Proposition 3.31 except for the part with respect to the martingale term, which will be presented in Sect. 3.4.5.

3.4.1 Basic Facts for Tightness

We briefly review several basic facts with respect to tightness. As claimed in Sect. 2, the martingale part of $V(t)$ is not continuous, so we consider the problem in the framework of the Skorohod space. We begin by recalling some basic facts with respect to the Skorohod space $(D([0, T]; \mathbf{R}^d), d^0)$ and the tightness of the probability measures on it (see Billingsley [1] for more details).

For any $T > 0$, $D([0, T]; \mathbf{R}^d)$ denotes the Skorohod space:

$$D([0,T];\mathbf{R}^d) = \Big\{ w : [0,T] \to \mathbf{R}^d; \quad w(t) = w(t+) := \lim_{s \downarrow t} w(s),\, t \in [0,T), \\ \text{and } w(t-) := \lim_{s \uparrow t} w(s) \text{ exists}, t \in (0,T] \Big\},$$

with the metric $d^0 = d^0_T$ given by $d^0(w, \widetilde{w}) = \inf_{\lambda \in \Lambda} \Big\{ \|\lambda\|^0 \vee \|w - \widetilde{w} \circ \lambda\|_\infty \Big\}$ for any $w, \widetilde{w} \in D([0, T]; \mathbf{R}^d)$, where $\Lambda = \Big\{ \lambda : [0, T] \to [0, T];$ continuous, non-decreasing, $\lambda(0) = 0, \lambda(T) = T \Big\}$, $\|w\|_\infty = \sup_{0 \le t \le T} |w(t)|$, and $\|\lambda\|^0 = \sup_{0 \le s < t \le T} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|$ for any $\lambda \in \Lambda$.

It is well-known that $(D([0, T]; \mathbf{R}^d), d^0)$ is a complete metric space, $C([0, T]; \mathbf{R}^d)$ is closed in $(D([0, T]; \mathbf{R}^d), d^0)$, and the Skorohod topology relativized to $C([0, T]; \mathbf{R}^d)$ coincides with the uniform topology there (see, e.g., [1]). Let $\wp(D([0,T];\mathbf{R}^d))$ denote the space of all probabilities on $D([0,T];\mathbf{R}^d)$.

Our base for the proof of tightness is Theorem 3.28 below, which is essentially a corollary of [1, Theorem 13.2]. See [12, Theorem3.4.1] for its proof.

Theorem 3.28 ([12]) *Let $(\Omega_n, \mathcal{F}_n, Q_n)$, $n \in \mathbf{N}$, be probability spaces, and let $X_n : \Omega_n \to D([0,T];\mathbf{R}^d)$, $n \in \mathbf{N}$, be measurable. Let $\mu_{X_n} = Q_n \circ X_n^{-1}$. Suppose that there exist constants $\varepsilon, \beta, \gamma, C > 0$ such that*

(1) $E^{Q_n}\big[\|X_n(\,\cdot\,)\|_\infty^\varepsilon\big] \le C$,
(2) $E^{Q_n}\Big[|X_n(r) - X_n(s)|^\beta |X_n(s) - X_n(t)|^\beta\Big] \le C|t-r|^{1+\varepsilon}$ *for any* $0 \le r \le s \le t \le 1$,
(3) $E^{Q_n}\Big[|X_n(s) - X_n(t)|^\varepsilon\Big] \le C|t-s|^\gamma$ *for any* $0 \le s \le t \le 1$,

for any $n \in \mathbf{N}$. Then $\{\mu_{X_n}\}_{n=1}^\infty$ is tight in $\wp(D([0,T];\mathbf{R}^d))$. □

As easy consequences of Theorem 3.28, we have Lemmas 3.29 and 3.30 below, which were also used in [12, 13].

Lemma 3.29 ([12]) *Let $(\Omega_n, \{\mathcal{F}_n(t)\}_{t\in[0,T]}, Q_n)$, $n \in \mathbf{N}$, be filtered probability spaces, and let $f_n : [0,T] \times \Omega_n \to \mathbf{R}$ be $\{\mathcal{F}_n(t)\}_{t\in[0,T]}$-adapted, $n \in \mathbf{N}$. If*

$$\sup_{n\in\mathbf{N}} \sup_{s\in[0,T]} E^{Q_n}\Big[|f_n(s)|^2\Big] < \infty,$$

then $\Big\{$the distribution of $\{\int_0^t f_n(s)ds\}_{t\in[0,T]}$ under Q_n; $n \in \mathbf{N}\Big\}$ is tight in $\wp(C([0,T];\mathbf{R}^d))$. □

Lemma 3.30 ([12]) *Let $(\Omega_n, \{\mathcal{F}_n(t)\}_{t\in[0,T]}, Q_n)$, $n \in \mathbf{N}$, be filtered probability spaces, and let $\{M_n(t)\}_t$ be $(\{\mathcal{F}_n(t)\}_{t\in[0,T]}, Q_n)$-martingales. If there exists a constant $C > 0$ such that*

$$E^{Q_n}\Big[|M_n(t) - M_n(s)|^2 \Big| \mathcal{F}_s\Big] \le C(t-s), \qquad 0 \le s < t \le T,$$

then $\Big\{$the distribution of $\{M_n(t)\}_{t\in[0,T]}$ under Q_n; $n \in \mathbf{N}\Big\}$ is tight in $\wp(D([0,T];\mathbf{R}^d))$. □

3.4.2 First Decomposition—The Result

We present our key decomposition of $V(t)$ in this section (see Proposition 3.31). Rewrite $V(t\wedge\sigma_n) = -\int_0^{t\wedge\sigma_n} ds \int_{\mathbf{R}\times E} \nabla U(X(s) - x(s, \Psi(r,x,m^{-1/2}v)))\mu_\omega(dr,dx,dv)$ as

$$V(t\wedge\sigma_n) = I^1(t) + \cdots I^{11}(t), \tag{3.18}$$

with

$$\begin{aligned} I^1(t) = &-\int_0^{t\wedge\sigma_n} 1_{[0,m^{1/2}a_m]}(s)ds \int_{\mathbf{R}\times E} \Big\{\nabla U\big(X(s) - x(s,\Psi(r,x,m^{-1/2}v))\big) \\ &-\nabla U\big(X_0 - \varphi^0\big(m^{-1/2}s, \Psi(m^{-1/2}r,x,v;X_0)\big)\big)\Big\}\mu_\omega(dr,dx,dv), \\ I^2(t) = &-\int_0^{t\wedge\sigma_n} 1_{[0,m^{1/2}a_m]}(s)ds \int_{\mathbf{R}\times E} \nabla U\big(X_0 - \varphi^0\big(m^{-1/2}s, \Psi(m^{-1/2}r,x,v;X_0)\big)\big)\mu_\omega(dr,dx,dv), \\ I^3(t) = &-\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E} \nabla U\big(X(\tilde r) - \psi^0\big(m^{-1/2}(s-r),x,v;X(\tilde r)\big)\big)(\mu_\omega - \lambda)(dr,dx,dv), \end{aligned}$$

$$I^4(t) = -\int_0^{t\wedge\sigma_n} ds \int_{\mathbf{R}\times E} f_1(s,r,x,v)\overline{\lambda}(dr,dx,dv),$$

$$I^5(t) = -\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E} \nabla U(X(s) - \psi^0(m^{-1/2}(s-r),x,v;X(s))\lambda(dr,dx,dv),$$

$$I^6(t) = -\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E, |x-\pi_v^\perp X(\tilde{r})|\geq m^\alpha} \Big(f_3(s,r,x,v) - f_2(s,r,x,v)\Big)(\mu_\omega - \lambda)(dr,dx,dv),$$

$$I^7(t) = -\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E, |x-\pi_v^\perp X(\tilde{r})|\geq m^\alpha} f_2(s,r,x,v)(\mu_\omega - \lambda)(dr,dx,dv),$$

$$I^8(t) = -\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E, |x-\pi_v^\perp X(\tilde{r})|< m^\alpha} f_3(s,r,x,v)(\mu_\omega - \lambda)(dr,dx,dv),$$

$$I^9(t) = -\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E} \Big(f_4(s,r,x,v) - f_1(s,r,x,v)\Big)\overline{\lambda}(dr,dx,dv),$$

$$I^{10}(t) = -\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E} f_4(s,r,x,v)\Big(\lambda(dr,dx,dv) - \overline{\lambda}(dr,dx,dv)\Big),$$

$$I^{11}(t) = \int_0^{t\wedge\sigma_n} 1_{[0,m^{1/2}a_m]}(s)ds \int_{\mathbf{R}\times E} f_1(s,r,x,v)\overline{\lambda}(dr,dx,dv).$$

Here

$$\overline{\lambda}(dr,dx,dv) := m^{-1}\rho_0\left(\frac{1}{2}|v|^2\right) dr\nu(dx,dv),$$

a_m is as defined in (2.13), and

$$\begin{aligned}
&f_1(s,r,x,v)\\
&:= -\nabla^2 U\big(X(s) - \psi^0(m^{-1/2}(s-r),x,v;X(s))\\
&\quad \cdot m^{1/2} z(m^{-1/2}(s-r),x,v,X(s),V(s),-m^{-1/2}(s-r)),\\
&f_2(s,r,x,v)\\
&:= \nabla^2 U(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;X(\tilde{r})))\\
&\quad \Big(-m^{1/2} z(m^{-1/2}(s-r),x,v;X(\tilde{r}),V(\tilde{r}),-m^{-1/2}(\tilde{r}-r)) + (s-\tilde{r})V(\tilde{r})\Big),\\
&f_3(s,r,x,v)\\
&:= \nabla U(X(s) - x(s,\Psi(r,x,m^{-1/2}v))) - \nabla U(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;X(\tilde{r}))),\\
&f_4(s,r,x,v)\\
&:= \nabla U(X(s) - x(s,\Psi(r,x,m^{-1/2}v))) - \nabla U(X(s) - \psi^0(m^{-1/2}(s-r),x,v;X(s)).
\end{aligned}$$

$I^1(t)$ and $I^2(t)$ in the decomposition above correspond to the force during the time interval right after starting (i.e., $s \leq m^{1/2}a_m$). They are negligible by virtue of the shortness of the time (see Lemmas 3.35 and 3.36). This blank of time, after necessary treatment, is refilled at the last step (see the definition of $I^4(t)$ and $I^{11}(t)$). As explained, the term $I^3(t)$ gives us approximately the martingale term. The discussion with respect to $I^3(t)$ is a little bit complicated, and will be given in Sect. 3.4.5. The term $I^4(t)$ gives us our drift term. Indeed, it is trivial that

$$\begin{aligned}
I^4(t) = -\int_0^{t\wedge\sigma_n} ds \int_{\mathbf{R}\times E} &\nabla^2 U(X(s) - \psi^0(u,x,v;X(s))z(u,x,v,X(s),V(s),-u)\\
&\times\rho_0\Big(\frac{1}{2}|v|^2\Big)du\nu(dx,dv),
\end{aligned} \tag{3.19}$$

which depends on m only via $X(\cdot)$. The necessary estimate for the tightness of $I^4(t)$ is essentially the same as that for the fact that I^{11} is negligible (see Lemma 3.43 (1) and (2)). The others, as will be proven in Sects. 3.4.3, 3.4.4 and 3.5, are approximately 0. To be precise, we prove in Sect. 3.4.3 that $I^1(t)$, $I^2(t)$, $I^5(t)$, $I^7(t)$, $I^8(t)$, $I^{10}(t)$ and $I^{11}(t)$ converge to 0 fast enough. Also, as announced, the terms $I^6(t)$ and $I^9(t)$ are discussed in two steps: we prove in Sect. 3.4.4 that they can also be considered as a part of our smooth part, which is enough to prove our tightness of the distribution of $\{(X(t \wedge \sigma_n), V(t \wedge \sigma_n)); t \in [0, T]\}$ under P_m for $m \in (0, 1]$; then, in Sect. 3.5, with the help of these results, we prove that they are also negligible.

In order to formulate our main result of Sect. 3.4, we prepare several notations. Let

$$N((0,t] \times A) := \mu_\omega((m^{1/2}\tau, m^{1/2}\tau + t] \times A), \qquad A \subset E.$$

Since $U(x - rv - X_0) = 0$ for all $r \geq m^{1/2}\tau$, we get that N is a Poisson point process with intensity $\overline{\lambda}(dt, dx, dv)$. Let

$$\overline{N}(dt, dx, dv) := N(dt, dx, dv) - \overline{\lambda}(dt, dx, dv).$$

Finally, let

$$\begin{aligned} M(t) := -\int_{(0,t]\times E} \overline{N}(dr, dx, dv) \int_{\left[0, m^{1/2}\left(t_1(v, |x - \pi_v^{\perp} X(r\wedge\sigma_n)|) + \tau\right)\right]} \\ \times \nabla U\left(X(r \wedge \sigma_n) - \psi^0\left(m^{-1/2}s - \tau, x, v; X(r \wedge \sigma_n)\right)\right) ds. \end{aligned} \tag{3.20}$$

$M(t)$ is the martingale part of our decomposition, which is, as explained before, approximately equal to $I^3(t)$.

Now we are ready to formulate our main result of Sect. 3.4.

Proposition 3.31 *There exists an $\{\mathcal{F}_t\}_t$-adapted process $\eta(t)$ and a constant C_6 such that*

1. $$V(t \wedge \sigma_n) - V(0) = M(t \wedge \sigma_n) + I^4(t) + I^6(t) + I^9(t) + \eta(t), \tag{3.21}$$

2. *$I^4(t)$, $I^6(t)$ and $I^9(t)$ are $\{\mathcal{F}_t\}_t$-adapted, C^1-class with respect to t, and*

$$\sup_{m\in(0,1]} \sup_{t\in[0,T]} E^{P_m}\left[\left|\frac{d}{dt} I^k(t)\right|^2\right] \leq C_6, \qquad k \in \{4, 6, 9\}, \tag{3.22}$$

3. *$M(\cdot)$ is a càdlàg $\{\mathcal{F}_t\}_t$-martingale with its jumps satisfying $|\Delta M(t)| \leq C_6 m^{1/2}$, and*

$$E^{P_m}\left[|M(t) - M(s)|^2 \Big| \mathcal{F}_s\right] \leq C_6 |t - s| \tag{3.23}$$

for any $0 \leq s < t \leq T$ and $m \in (0, 1]$, also, for any $k, l \in \{1, \cdots, d\}$ and any bounded $g : [0, \infty) \times \Omega \to \mathbf{R}$ such that $g(s, \cdot)$ is $\mathcal{F}_s$-measurable, we have that

$$E^{P_m}\left[\sup_{t\in[0,T]} \left|\int_{(0,t\wedge\sigma_n]} g(s)\left(d[M^k, M^l]_s - a_{kl} ds\right)\right|^2\right] \leq C_6 \|g\|_\infty^2 m. \tag{3.24}$$

4.

$$E\Big[\sup_{u\in[0,T]}|\eta(u)|^2\Big]\to 0,$$

and there exists a constant $\varepsilon>0$ *such that*

$$E\Big[\sup_{u\in[s-m^{1/2}a_m,s]}|\eta(u)-\eta(s-m^{1/2}a_m)|^2\Big]\le C_6m^{\varepsilon} \tag{3.25}$$

for any $s\in[m^{1/2}a_m,T]$.

In particular, the distributions of $\{M(t)+\eta(t);t\in[0,t]\}$ *and* $\{I^k(t);t\in[0,T]\}$ *with* $k\in\{4,6,9\}$ *under* P_m *are tight in* $\wp(C([0,T];\mathbf{R}^d))$.

The tightness in Proposition 3.31 is a direct consequence of the other assertions of the same Proposition and Lemmas 3.29, 3.30. So it suffices to prove the first half of Proposition 3.31.

The quantity $\eta(t)$ of Proposition 3.31 is given by the sum of $I^1(t)$, $I^2(t)$, $I^5(t)$, $I^7(t)$, $I^8(t)$, $I^{10}(t)$, $I^{11}(t)$ and a part of $I^3(t)$. Lemma 3.32 below proves that there exists a constant $\varepsilon>0$ such that when $m\to 0$, all these terms except the last one are of order $o(m^{\varepsilon})$. Precisely, we prove the following:

Lemma 3.32 *There exists a constant* $\varepsilon>0$ *such that when* $m\to 0$,

$$E^{P_m}\Big[\sup_{0\le t\le T}|I(t)|^2\Big]^{1/2}=o(m^{\varepsilon}), \tag{3.26}$$

with I *given by* $I^1(t)$, $I^2(t)$, $I^5(t)$, $I^7(t)$, $I^8(t)$, $I^{10}(t)$, $I^{11}(t)$.

Lemma 3.32 certainly implies that the contribution from these terms in $\eta(t)$ satisfies the estimates in Proposition 3.31 (4).

We prove Lemma 3.32 in Sect. 3.4.3. (3.22) for $k=4$ is formulated as a part of Lemma 3.43. The proof of (3.22) for $k\in\{6,9\}$ is given in Sect. 3.4.4. The discussion with respect to $I^3(t)$ is presented in Sect. 3.4.5.

3.4.3 Proof of Lemma 3.32

We start our proof by noticing the following two facts, which hold by virtue of the translation property and the symmetry of $\varphi(t,\cdot,\cdot;X)$ and $\psi(t,\cdot,\cdot;X)$ with respect to X. See Appendix (Sect. A.2) for their proofs.

Lemma 3.33 1. *The following holds:*

$$\int_{\mathbf{R}\times E}\nabla U\left(X_0-\varphi^0\big(m^{-1/2}s,\Psi(m^{-1/2}r,x,v;X_0)\big)\right)\lambda(dr,dx,dv)=0. \tag{3.27}$$

2. *For any* $X\in\mathbf{R}^d$, *the following holds:*

$$\int_{\mathbf{R}\times E}\nabla U\big(X-\psi^0(u,x,v;X)\big)\overline{\lambda}(du,dx,dv)=0. \tag{3.28}$$

The proof of the following general result with respect to the Poisson point process follows easily from the definitions. We formulate it here without proof.

Lemma 3.34 *Let* $f : \mathbf{R} \times E \times \Omega \to \mathbf{R}$ *be a function such that* $f(r, x, v, \cdot)$ *is* $\mathcal{F}_{(-\infty,r]\times E}$*-measurable for any* $(r, x, v) \in \mathbf{R} \times E$*. Then we have that*

$$E\Big[\Big(\int f d(\mu_\omega - \lambda)\Big)^2\Big] = E\Big[\int f^2 d\lambda\Big].$$

□

As a consequence of Lemma 3.34, we get that

$$E\Big[\Big(\int f d\mu_\omega\Big)^2\Big] \le 2E\Big[\int f^2 d\lambda\Big] + 2E\Big[\Big(\int f d\lambda\Big)^2\Big], \tag{3.29}$$

$$E\Big[\Big(\int f d(\mu_\omega + \lambda)\Big)^2\Big] \le 2E\Big[\int f^2 d\lambda\Big] + 8E\Big[\Big(\int f d\lambda\Big)^2\Big]. \tag{3.30}$$

Lemmas 3.35 and 3.36 below imply that $I^1(t)$ and $I^2(t)$ satisfy (3.26), by the virtue of (2.10).

Lemma 3.35 *There exists a constant* $C_7 > 0$ *such that*

$$E^{P_m}\Big[\sup_{0\le t\le T} |I^1(t)|^2\Big] \le C_7 m a_m^6 e^{(1+C_1)a_m}, \qquad m \in (0, 1].$$

Proof Since $|X(\tilde{r})| \le |X_0| + nT$, we have that $1_{\{|x-\pi_v^\perp X(\tilde{r})|\le R_U+1\}} \le 1_{\{|x|\le R_0\}}$ and $1_{\{|x-\pi_v^\perp X_0|\le R_U+1\}} \le 1_{\{|x|\le R_0\}}$. So by (2.7), Proposition 2.1 (1) (3), Lemma 3.17 and (3.12), we get for any $s \in [0, T \wedge \sigma_n \wedge m^{1/2}a_m]$ that

$$\begin{aligned}
&\Big|\nabla U\big(X(s) - x(s, \Psi(r, x, m^{-1/2}v))\big) - \nabla U\big(X_0 - \varphi^0(m^{-1/2}s, \Psi(m^{-1/2}r, x, v; X_0))\big)\Big| \\
&\le 1_{[0,R_0]}(|x|)1_{[-m^{1/2}\tau, s+m^{1/2}\tau]}(r)C_1 \\
&\quad \cdot \Big(ns + \Big|x(s, \Psi\big(r, x, m^{-1/2}v)\big) - \varphi^0\big(m^{-1/2}s, \Psi(m^{-1/2}r, x, v; X_0)\big)\Big|\Big) \\
&\le 1_{[0,R_0]}(|x|)1_{\big[-m^{1/2}\tau, m^{1/2}a_m+m^{1/2}\tau\big]}(r)C_1 \cdot \frac{2C_1 n}{1+C_1} m^{1/2} a_m e^{\frac{1}{2}(1+C_1)a_m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|I^1(t)| \le\ & m^{1/2} a_m 2C_1 n m^{1/2} a_m e^{\frac{1}{2}(1+C_1)a_m} \\
&\cdot \int_{\mathbf{R}\times E} 1_{[0,R_0]}(|x|)1_{\big[-m^{1/2}\tau, m^{1/2}a_m+m^{1/2}\tau\big]}(r)\mu_\omega(dr, dx, dv).
\end{aligned}$$

Since $a_m \ge \tau \vee 1$ by definition, this combined with Lemma 4.4 (2) implies that

$$\begin{aligned}
&E^{P_m}\Big[\sup_{0\le t\le T} |I^1(t)|^2\Big] \\
&\le \Big(2C_1 n m a_m^2 e^{\frac{1}{2}(1+C_1)a_m}\Big)^2 C\Big(m^{-2}\Big(m^{1/2}a_m + m^{1/2}\tau\Big)^2 + 1\Big) \\
&\le 20 C_1^2 C n^2 m a_m^6 e^{(1+C_1)a_m}.
\end{aligned}$$

□

Lemma 3.36 *There exists a constant* $C_8 > 0$ *such that*

$$E^{P_m}\Big[\sup_{0\le t\le T} |I^2(t)|^2\Big] \le C_8 m^{1/2} a_m^3, \qquad m \in (0, 1].$$

Proof By Lemma 3.33 (1), we get that

$$I^2(t) = -\int_0^{t\wedge\sigma_n} 1_{[0,m^{1/2}a_m]}(s)ds \int_{\mathbf{R}\times E} \nabla U\big(X_0 - \varphi^0\big(m^{-1/2}s, \Psi(m^{-1/2}r, x, v; X_0)\big)\big)$$
$$(\mu_\omega - \lambda)(dr, dx, dv).$$

For any $s \in [0, m^{1/2}a_m]$, by Lemma 3.34, Proposition 2.1 (1) and Lemma 4.4 (1), we get that

$$\begin{aligned}
&E\Big[\Big|\int_{\mathbf{R}\times E} \nabla U\big(X_0 - \varphi^0\big(m^{-1/2}s, \Psi(m^{-1/2}r, x, v; X_0)\big)\big)(\mu_\omega - \lambda)(dr, dx, dv)\Big|^2\Big] \\
&= \int_{\mathbf{R}\times E} |\nabla U\big(X_0 - \varphi^0\big(m^{-1/2}s, \Psi(m^{-1/2}r, x, v; X_0)\big)\big)|^2 \lambda(dr, dx, dv) \\
&\le \|\nabla U\|_\infty^2 \int_{\mathbf{R}\times E} 1_{[0,R_0]}(|x|) 1_{[-m^{1/2}\tau, m^{1/2}a_m + m^{1/2}\tau]}(r)\lambda(dr, dx, dv) \\
&\le \|\nabla U\|_\infty^2 C m^{-1}\big(m^{1/2}a_m + m^{1/2}\tau\big) \le 2\|\nabla U\|_\infty^2 C m^{-1/2} a_m.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E^{P_m}\Big[\sup_{0\le t\le T} |I^2(t)|^2\Big] \\
&\le E^{P_m}\Big[m^{1/2}a_m \int_0^T 1_{[0,m^{1/2}a_m]}(s)ds \\
&\quad \Big|\int_{\mathbf{R}\times E} \nabla U(X_0 - \varphi^0(m^{-1/2}s, \Psi(m^{-1/2}r, x, v; X_0)))(\mu_\omega - \lambda)(dr, dx, dv)\Big|^2\Big] \\
&\le m a_m^2 2\|\nabla U\|_\infty^2 C m^{-1/2} a_m.
\end{aligned}$$

So we get our assertion with $C_8 := 2\|\nabla U\|_\infty^2 C$. □

Since $\alpha > \frac{1}{2(d-1)}$ by (2.9), Lemma 3.37 below implies that $I^5(t)$ also satisfies (3.26).

Lemma 3.37 *There exists a constant* $C_9 > 0$ *such that*

$$|I^5(t)| \le C_9 m^{\alpha(d-1)-\frac{1}{2}}$$

for any $t \in [0, T]$ *and* $m \in (0, 1]$.

Proof First, for any $s \in [0, T \wedge \sigma_n]$, we have by Lemma 3.33 (2) that

$$\int_{\mathbf{R}\times E} \nabla U\big(X(s) - \psi^0\big(m^{-1/2}(s-r), x, v; X(s)\big)\big)\overline{\lambda}(dr, dx, dv) = 0.$$

Assume $|x - \pi_v^\perp X(s)| \ge m^\alpha$ for a while. Then by Proposition 2.1 (2) and (2.14), for any $m \le (2\varepsilon_3)^{1/\alpha}$, we have that $\nabla U(X(s) - \psi^0(m^{-1/2}(s-r), x, v; X(s))) \ne 0$ implies $r \ge s - m^{1/2}t_1(v, |x - \pi_v^\perp X(s)|) \ge s - m^{1/2}(a_m - \tau)$. This combined with $s \ge m^{1/2}a_m$ implies that $r \ge m^{1/2}\tau$, so $|x - m^{-1/2}rv - X_0| \ge m^{-1/2}r|v| - |X_0| \ge R_1$, hence $\lambda(dr, dx, dv) = \overline{\lambda}(dr, dx, dv)$ by (A2).

Therefore,

$$
\begin{aligned}
I^5(t) &= I^5(t) + \int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \\
&\qquad \int_{\mathbf{R}\times E} \nabla U\big(X(s) - \psi^0\big(m^{-1/2}(s-r), x, v; X(s)\big)\big)\overline{\lambda}(dr, dx, dv) \\
&= \int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \\
&\qquad \int_{\mathbf{R}\times E, |x-\pi_v^{\perp}X(s)|<m^{\alpha}} \nabla U\big(X(s) - \psi^0\big(m^{-1/2}(s-r), x, v; X(s)\big)\big)\overline{\lambda}(dr, dx, dv) \\
&\qquad - \int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \\
&\qquad \int_{\mathbf{R}\times E, |x-\pi_v^{\perp}X(s)|<m^{\alpha}} \nabla U\big(X(s) - \psi^0\big(m^{-1/2}(s-r), x, v; X(s)\big)\big)\lambda(dr, dx, dv).
\end{aligned}
\tag{3.31}
$$

We notice that for any $(x, v) \in E$ satisfying $|x - \pi_v^{\perp}X(s)| \neq 0$, we have $|\psi^1(-\infty, x, v; X(s))| = |\psi^1(+\infty, x, v; X(s))| = |v|$, so

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}} \nabla U\big(X(s) - \psi^0(u, x, v; X(s))\big)du\Big| \\
&= |\psi^1(\infty, x, v; X(s)) - \psi^1(-\infty, x, v; X(s))| \leq 2|v|.
\end{aligned}
$$

Therefore, we have for any $s \in [0, T\wedge\sigma_n]$ that

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}\times E, |x-\pi_v^{\perp}X(s)|<m^{\alpha}} \nabla U\big(X(s) - \psi^0\big(m^{-1/2}(s-r), x, v; X(s)\big)\big)\lambda(dr, dx, dv)\Big| \\
&\leq \int_{E, |x-\pi_v^{\perp}X(s)|<m^{\alpha}} 2|v|m^{-1/2}\rho_{max}\Big(\frac{1}{2}|v|^2\Big)\nu(dxdv) \\
&\leq 2m^{-1/2}(2m^{\alpha})^{d-1}\int_{\mathbf{R}^d} |v|^2\rho_{max}\Big(\frac{1}{2}|v|^2\Big)dv.
\end{aligned}
$$

The same estimate holds for the assertion with $\lambda(dr, dx, dv)$ substituted by $\overline{\lambda}(dr, dx, dv)$, too. Therefore, we get our assertion with $C_9 := 2T \cdot 2^d \int_{\mathbf{R}^d} |v|^2\rho_{max}(\frac{1}{2}|v|^2)dv$. □

We next deal with the term $I^7(t)$. Lemma 3.39 below is more than enough to prove that $I^7(t)$ satisfies (3.26) under (2.8). We prove it in the present form since it is also used in the discussion with respect to $I^6(t)$ (see the proof of (3.22) for $k = 6$ in Sect. 3.4.4).

We first prepare the following estimate with respect to $z(\cdot; x, v, X, V, a)$.

Lemma 3.38 *For any $\bar{t} \geq \tau$ and $c_m \geq \tau$ $(m \in (0, 1])$, we have for any $|a| \leq c_m$, $|V| \leq n$, $X \in \mathbf{R}^d$ and $(x, v) \in E$ that*

$$
\begin{aligned}
&|z(t; x, v, X, V, a)| \vee \Big|\frac{d}{dt}z(t; x, v, X, V, a)\Big| \\
&\leq 2(\bar{t} + c_m)n\Big(e^{\frac{1}{2}(1+C_1)(t+\tau)} - 1\Big), \quad t \in [-\tau, \bar{t}].
\end{aligned}
$$

Proof We have by the definition of z, (2.6) and our assumption that

$$\left|\frac{d^2}{dt^2}z(t)\right| \le C_1|z(t)| + C_1(\bar{t}+c_m)n, \quad t \in [-\tau, \bar{t}].$$

Also, $z(-\tau) = \frac{d}{dt}z(-\tau) = 0$. Therefore, we get our assertion by Lemma 3.16. □

Lemma 3.39 *Under (2.8), there exists a constant $C_{10} > 0$ (which may depend on α) such that*

1. *for any $t \in [0, T]$, we have that*

$$E^{P_m}\left[1_{[m^{1/2}a_m, T\wedge\sigma_n]}(t)\left|\int_{\mathbf{R}\times E, |x-\pi_v^\perp X(\tilde{r})|\ge m^\alpha} f_2(t,r,x,v)(\mu_\omega - \lambda)(dr,dx,dv)\right|^2\right] < C_{10}m^{\frac{1}{2}}.$$

2. $E^{P_m}\left[\sup_{t\in[0,T]}|I^7(t)|^2\right] \le C_{10}m^{1/2}$.

Proof By re-choosing C_{10} if necessary, the second assertion is a direct consequence of the first assertion. We prove the first assertion in the following.

For any $t \in \mathbf{R}$, if $f_2(t,r,x,v) \ne 0$, then by Proposition 2.1 (2), we get that $m^{-1/2}(t-r) \in [-\tau, t_1(v, |x-\pi_v^\perp X(\tilde{r})|)]$. So if $t \in [m^{1/2}a_m, T\wedge\sigma_n]$ and $|x - \pi_v^\perp X(\tilde{r})| \ge m^\alpha$ in addition, then by (2.14), for any $m \le (2\varepsilon_3)^{1/\alpha}$, we have $r - m^{1/2}\tau \in [0, T\wedge\sigma_n]$. Also, we notice that $1_{\{r-m^{1/2}\tau\in[0,T\wedge\sigma_n]\}}1_{\{|x-\pi_v^\perp X(\tilde{r})|\ge m^\alpha\}}f_2(t,r,x,v)$ is $\mathcal{F}_{(-\infty,r]\times E}$-measurable for any r by Lemma 3.19. Therefore, with the help of Lemma 3.34, we get that

$$\begin{aligned}
&E^{P_m}\left[1_{[m^{1/2}a_m, T\wedge\sigma_n]}(t)\left|\int_{\mathbf{R}\times E, |x-\pi_v^\perp X(\tilde{r})|\ge m^\alpha} f_2(t,r,x,v)(\mu_\omega - \lambda)(dr,dx,dv)\right|^2\right]\\
&\le E^{P_m}\left[\left|\int_{\mathbf{R}\times E} 1_{\{r-m^{1/2}\tau\in[0,T\wedge\sigma_n]\}}1_{\{|x-\pi_v^\perp X(\tilde{r})|\ge m^\alpha\}}f_2(t,r,x,v)(\mu_\omega-\lambda)(dr,dx,dv)\right|^2\right]\\
&= \int_{\mathbf{R}\times E} E^{P_m}\left[1_{\{r-m^{1/2}\tau\in[0,T\wedge\sigma_n]\}}1_{\{|x-\pi_v^\perp X(\tilde{r})|\ge m^\alpha\}}f_2(t,r,x,v)^2\right]\lambda(dr,dx,dv).
\end{aligned}$$

So it suffices to prove that the right hand side above is dominated by $C_{10}m^{1/2}$ with some proper constant C_{10}.

On the other hand, if $f_2(t,r,x,v) \ne 0$, then by Proposition 2.1 (2), we get that $t - r \in [-m^{1/2}\tau, m^{1/2}t_1(v, |x-\pi_v^\perp X(\tilde{r})|)]$. This combined with $r - m^{1/2}\tau \in [0, T\wedge\sigma_n]$ implies that $t - \tilde{r} = t - r + m^{1/2}\tau \in [0, m^{1/2}t_1(v, |x-\pi_v^\perp X(\tilde{r})|) + m^{1/2}\tau]$, hence $|(t-\tilde{r})V(\tilde{r})| \le nm^{1/2}(t_1(v, |x-\pi_v^\perp X(\tilde{r})|)+\tau)$. So by Lemma 3.38 (with c_m and $\bar{t}$ given by $t_1(v, |x-\pi_v^\perp X(\tilde{r})|)$), we get that

$$\begin{aligned}
&\left|z\big(m^{-1/2}(t-r), x, v; X(\tilde{r}), V(\tilde{r}), -m^{-1/2}(\tilde{r}-r)\big) - m^{-1/2}(t-\tilde{r})V(\tilde{r})\right|\\
&\le 4nt_1\big(v, |x-\pi_v^\perp X(\tilde{r})|\big)\exp\left(\frac{1}{2}(1+C_1)\big(t_1\big(v,|x-\pi_v^\perp X(\tilde{r})|\big)+\tau\big)\right).
\end{aligned}$$

Therefore, by Proposition 2.1 (2),

$$\int_{\mathbf{R}\times E} E^{P_m}\Big[1_{\{r-m^{1/2}\tau\in[0,T\wedge\sigma_n]\}}1_{\{|x-\pi_v^\perp X(\tilde r)|\ge m^\alpha\}}f_2(t,r,x,v)^2\Big]\lambda(dr,dx,dv)$$
$$\le \int_{\mathbf{R}\times E}\lambda(dr,dx,dv)1_{\{|x|\le R_0\}}1_{\{t-r\in[-m^{1/2}\tau,m^{1/2}t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}}\|\nabla^2U\|_\infty^2$$
$$\times m\left(4nt_1(v,|x-\pi_v^\perp X(\tilde r)|)\exp\left(\frac{1}{2}(1+C_1)(t_1(v,|x-\pi_v^\perp X(\tilde r)|)+\tau)\right)\right)^2.$$

Since $(1+C_1)2\varepsilon_1^{-1/2}<d-1$ by (2.8), by Lemma 4.5 (1), we get that the right hand side above is dominated by $\|\nabla^2U\|_\infty^2 16n^2Cm^{1/2}$. This completes our proof. □

In order to prove that $I^8(t)$ also satisfies (3.26), we first prepare the following.

Lemma 3.40 *There exists a constant* $C_{11}>0$ *such that for any* $(x,v)\in\mathbf{R}^{2d}$ *and* $t\in[0,T\wedge\sigma_n]$, *we have*

$$|v(t,x,v)|\le|v|+C_{11}m^{-1/2}.$$

Proof For any $t\in[0,\sigma_n]$, we have that

$$\left|\frac{d}{dt}\left(\frac{m}{2}|v(t,x,v)|^2+U(x(t,x,v)-X(t))\right)\right|$$
$$=\Big|-\nabla U(x(t,x,v)-X(t))\cdot V(t)\Big|\le n\|\nabla U\|_\infty,$$

so

$$\left|\left(\frac{m}{2}|v(t,x,v)|^2+U(x(t,x,v)-X(t))\right)-\left(\frac{m}{2}|v|^2+U(x-X_0)\right)\right|\le n\|\nabla U\|_\infty t.$$

Therefore, for any $t\in[0,T\wedge\sigma_n]$, we have

$$|v(t,x,v)|^2\le\sqrt{|v|^2+2m^{-1}(nT\|\nabla U\|_\infty+\|U\|_\infty)}$$
$$\le|v|+m^{-1/2}\sqrt{2(nT\|\nabla U\|_\infty+\|U\|_\infty)}.$$

□

$I^8(t)$ satisfies (3.26) by Lemma 3.41 below, since $\alpha>\frac{1}{2(d-1)}$ by (2.9).

Lemma 3.41 *There exists a constant* $C_{12}>0$ *such that*

$$E^{P_m}\Big[\sup_{t\in[0,T]}|I^8(t)|^2\Big]\le C_{12}(m^{\alpha(d-1)}+m^{2\alpha(d-1)-1}),\qquad m\in(0,1].$$

Proof For any $(x,v)\in E$, $t\in[0,T]$ and $r\in\mathbf{R}$, by Lemmas 3.40 and 3.18, we have

$$\left|\int_0^{t\wedge\sigma_n}1_{(m^{1/2}a_m,\infty)}(s)ds\nabla U(X(s)-x(s,\Psi(r,x,m^{-1/2}v)))\right|$$
$$=\left|m\Big(v(t\wedge\sigma_n,\Psi(r,x,m^{-1/2}v))-v(t\wedge\sigma_n\wedge m^{1/2}a_m,\Psi(r,x,m^{-1/2}v))\Big)\right|$$
$$\le 2m^{1/2}(|v|+C_{11})1_{[-m^{1/2}\tau,T+m^{1/2}\tau]}(r). \tag{3.32}$$

This estimate is also used in the proof of Lemma 3.60. Similarly,

$$\left|\int_0^{t\wedge\sigma_n}1_{(m^{1/2}a_m,\infty)}(s)ds\nabla U(X(\widetilde r)-\psi^0(m^{-1/2}(s-r)x,v;X(\widetilde r))\right|$$
$$\le 2m^{1/2}|v|1_{[-m^{1/2}\tau,T+m^{1/2}\tau]}(r).$$

Therefore,

$$|I^8(t)| \le \int_{\mathbf{R}\times E, |x-\pi_v^\perp X(\tilde{r})|<m^\alpha} 4m^{1/2}(|v|+C_{11})1_{[-m^{1/2}\tau, T+m^{1/2}\tau]}(r)(\mu_\omega+\lambda)(dr,dx,dv).$$

Combining this with (3.30) and Lemma 4.5 (2), we get that

$$E^{P_m}\Big[\sup_{t\in[0,T]}|I^8(t)|^2\Big] \le 32m\cdot Cm^{\alpha(d-1)-1}(T+2\tau)+8\cdot 16m\cdot\Big(Cm^{\alpha(d-1)-1}(T+2\tau)\Big)^2,$$

which implies our assertion. □

Lemma 3.42 below implies that $I^{10}(t)$ also satisfies (3.26), since $\alpha > \frac{1}{2(d-1)}$ by (2.9).

Lemma 3.42 *There exists a constant* $C_{13} > 0$ *such that*

$$\sup_{t\in[0,T], m\in(0,1], \omega\in\Omega} |I^{10}(t)| \le C_{13} m^{\alpha(d-1)-\frac{1}{2}}.$$

Proof First we notice that if $|r| \ge m^{1/2}\tau$, then $|x - m^{-1/2}rv - X_0| \ge \tau|v| - |X_0| \ge R_1$, hence $\lambda(dr,dx,dv) = \overline{\lambda}(dr,dx,dv)$ by (A2).

Also, if $|x - \pi_v^\perp X(\tilde{r})| \ge m^\alpha$, $s \ge m^{1/2}a_m$, $s \in [0, T\wedge\sigma_n]$ and $\nabla U(X(s) - x(s, \Psi(r,x,m^{-1/2}v))) \ne 0$, then by Lemma 2.2, we get that $r \ge m^{1/2}\tau$. As discussed above, this implies in turn that $\lambda(dr,dx,dv) = \overline{\lambda}(dr,dx,dv)$. Therefore,

$$\begin{aligned}
&\Big|\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds\int_{\mathbf{R}\times E}\nabla U(X(s)-x(s,\Psi(r,x,m^{-1/2}v)))(\lambda-\overline{\lambda})(dr,dx,dv)\Big|\\
&\le \int_0^T ds\int_{\mathbf{R}\times E, |x-\pi_v^\perp X(\tilde{r})|<m^\alpha} 1_{\{|r|\le m^{1/2}\tau\}}\|\nabla U\|_\infty(\lambda+\overline{\lambda})(dr,dx,dv)\\
&\le \|\nabla U\|_\infty T m^{-1}(2m^\alpha)^{d-1}2m^{1/2}\tau\int_{\mathbf{R}^d} 2\rho_{max}\Big(\frac{1}{2}|v|^2\Big)|v|dv. \qquad (3.33)
\end{aligned}$$

Similarly, if $|x - \pi_v^\perp X(s)| \ge m^{1/2}\tau$, $s \ge m^{1/2}a_m$, $s \in [0, T\wedge\sigma_n]$ and $\nabla U(X(s) - \psi^0(m^{-1/2}(s-r), x, v, X(s)) \ne 0$, then by Proposition 2.1 (2) and (2.14), $m^{-1/2}(s-r) \le t_1(v, |x-\pi_v^\perp X(s)|) \le a_m - \tau$, so for any $m \le (2\varepsilon_3)^{1/\alpha}$, we have $r \ge s - m^{1/2}a_m + m^{1/2}\tau \ge m^{1/2}\tau$, hence $\lambda(dr,dx,dv) = \overline{\lambda}(dr,dx,dv)$. Therefore,

$$\begin{aligned}
&\Big|\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds\int_{\mathbf{R}\times E}\nabla U(X(s)-\psi^0(m^{-1/2}(s-r),x,v,X(s))(\lambda-\overline{\lambda})(dr,dx,dv)\Big|\\
&\le \int_0^T ds\int_{\mathbf{R}\times E, |x-\pi_v^\perp X(s)|<m^\alpha} 1_{\{|r|\le m^{1/2}\tau\}}\|\nabla U\|_\infty(\lambda+\overline{\lambda})(dr,dx,dv)\\
&\le \|\nabla U\|_\infty T m^{-1}(2m^\alpha)^{d-1}2m^{1/2}\tau\int_{\mathbf{R}^d} 2\rho_{max}\Big(\frac{1}{2}|v|^2\Big)|v|dv. \qquad (3.34)
\end{aligned}$$

Combining (3.33) and (3.34), we get our assertion. □

We prove in Lemma 3.43 below that $I^{11}(t)$ satisfies (3.26) and that $I^4(t)$ satisfies the estimate in (3.22). Lemma 3.43 is also used in the proof of (3.22) for $k = 9$ (see Sect. 3.4.4).

Lemma 3.43 *There exists a constant* $C_{14} > 0$ *such that*

1. $\sup_{m\in(0,1]}\sup_{t\in[0,T\wedge\sigma_n]}\Big|\int_{\mathbf{R}\times E} f_1(t,r,x,v)\overline{\lambda}(dr,dx,dv)\Big| \le C_{14}$.
2. $\sup_{m\in(0,1]}\sup_{t\in[0,T]} E^{P_m}\Big[\Big|\frac{d}{dt}I^4(t)\Big|^2\Big] \le C_{14}$.
3. $|I^{11}(t)| \le C_{14}m^{1/2}a_m$.

Proof By the definitions of $I^4(t)$ and $I^{11}(t)$, the second and the third assertions are trivial by the first assertion. We prove the first assertion in the following. By a simple change of variable, $\left|\int_{\mathbf{R}\times E} f_1(t,r,x,v)\overline{\lambda}(dr,dx,dv)\right|$ is equal to

$$\left|\int_{\mathbf{R}\times E} \nabla^2 U\big(X(t)-\psi^0(u,x,v;X(t)\big)z(u,x,v,X(t),V(t),-u)\rho_0\Big(\frac{1}{2}|v|^2\Big)du\nu(dx,dv)\right|.$$

So for any $t\in[0,T\wedge\sigma_n]$, we get by Proposition 2.1 (2) and Lemma 3.38 that

$$\begin{aligned}&\left|\int_{\mathbf{R}\times E} f_1(t,r,x,v)\overline{\lambda}(dr,dx,dv)\right|\\&\le\left|\int_{\mathbf{R}\times E}\|\nabla^2 U\|_\infty 1_{\{|x|\le R_0\}}1_{\{u\in[-\tau,t_1(v,|x-\pi_v^\perp X(t))]\}}\right.\\&\quad\times 4t_1(v,|x-\pi_v^\perp X(t)|)ne^{\frac{1}{2}(1+C_1)(u+\tau)}\rho_0\Big(\frac{1}{2}|v|^2\Big)du\nu(dx,dv),\end{aligned}$$

which, since $d>(1+C_1)\varepsilon_1^{-1/2}+1$ by (2.8), is bounded for $m\in(0,1]$ by Lemma 4.5 (1). This completes our proof. □

3.4.4 Smoothness for $I^6(t)$ and $I^9(t)$

In this section, we prove that $I^6(t)$ and $I^9(t)$ satisfy (3.22).
Proof of (3.22) *with* $k=6$

By Lemma 3.39 (1), it suffices to prove that

$$E^{P_m}\left[1_{\{m^{1/2}a_m\le t\le\sigma_n\}}\Big(\int_{\mathbf{R}\times E,|x-\pi_v^\perp X(\tilde r)|\ge m^\alpha}|f_3(t,r,x,v)|(\mu_\omega+\lambda)(dr,dx,dv)\Big)^2\right]$$

is bounded for $m\in(0,1]$ and $t\in[0,T]$. Since $|x-\pi_v^\perp X(\tilde r)|\ge m^\alpha$, we get by Proposition 2.1 (2), (4), (2.7), Lemma 3.20 and (3.12) that

$$\begin{aligned}&|f_3(t,r,x,v)|\\&\le 1_{\{|x-\pi_v^\perp X(\tilde r)|\le R_U+1\}}1_{\{m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}}\\&\quad\times C_1\Big(|X(t)-X(\tilde r)|+|x(t,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(t-r),x,v;X(\tilde r))|\Big)\\&\le 1_{\{|x-\pi_v^\perp X(\tilde r)|\le R_U+1\}}1_{\{m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}}\\&\quad\times C_1 2nm^{1/2}(t_1(v,|x-\pi_v^\perp X(\tilde r)|)+\tau)e^{\frac{1}{2}(1+C_1)(m^{-1/2}(t-r)+\tau)}.\end{aligned}$$

This combined with (3.30) and Lemma 4.5 (1) implies our assertion, since $d>2(1+C_1)\varepsilon_1^{-1/2}+1$ by (2.8). □

Before proving (3.22) for $k=9$, we first prepare several estimates.

Lemma 3.44 *There exists a constant $m_5\in(0,1]$ such that for any $m\in(0,m_5]$, if $t\in[0,T\wedge\sigma_n]$, $|x-\pi_v^\perp X(\tilde r)|\le m^\alpha$ and $m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(t)|)]$, then $|x-\pi_v^\perp X(t)|\le 2m^\alpha$.*

Proof First, we get by assumption that

$$t-\tilde r\in[0,m^{1/2}(t_1(v,|x-\pi_v^\perp X(t)|)+\tau)]. \tag{3.35}$$

Indeed, since $t \in [0, T \wedge \sigma_n]$ and $t - r \geq -m^{1/2}\tau$, we get that $r - m^{1/2}\tau \leq T \wedge \sigma_n$, hence $\tilde{r} = (r - m^{1/2}\tau) \vee 0 \geq r - m^{1/2}\tau$. So $t - \tilde{r} \leq t - r + m^{1/2}\tau \leq m^{1/2} t_1(v, |x - \pi_v^{\perp} X(t)|) + m^{1/2}\tau$, and $t - \tilde{r} = (t - (r - m^{1/2}\tau)) \wedge t \geq 0$.

By (3.35) and the assumption $|x - \pi_v^{\perp} X(\tilde{r})| \leq m^{\alpha}$, we get that

$$\begin{aligned} |x - \pi_v^{\perp} X(t)| &\leq |x - \pi_v^{\perp} X(\tilde{r})| + |X(t) - X(\tilde{r})| \\ &\leq m^{\alpha} + n m^{1/2} (t_1(v, |x - \pi_v^{\perp} X(t)|) + \tau). \end{aligned} \tag{3.36}$$

If $|v| \geq 2C_0 + 1$, then $t_1(v, |x - \pi_v^{\perp} X(t)|) = 2\tau$, so (3.36) becomes $|x - \pi_v^{\perp} X(t)| \leq m^{\alpha} + n m^{1/2} 3\tau$. Since $\alpha < \frac{1}{2}$ as a consequence of (2.10), as long as $m \leq (3n\tau)^{-1/(\frac{1}{2} - \alpha)}$, this implies that $|x - \pi_v^{\perp} X(t)| \leq 2m^{\alpha}$. We next consider the case $|v| \in [\overline{v}, 2C_0 + 1)$. To simplify notations, let $a := |x - \pi_v^{\perp} X(t)|$. So (3.36) becomes

$$a \leq m^{\alpha} + n m^{1/2} \left(C_2 + \tau + 2\varepsilon_1^{-1/2} \left(\log \frac{2\varepsilon_3}{a} \right) 1_{\left\{ a \leq \frac{R_U}{2} \wedge (2\varepsilon_3) \right\}} \right).$$

If $a > \frac{R_U}{2} \wedge (2\varepsilon_3)$, then we get $\frac{R_U}{2} \wedge (2\varepsilon_3) < a \leq m^{\alpha} + n m^{1/2}(C_2 + \tau)$, which is impossible for $m > 0$ small enough. If $a \leq \frac{R_U}{2} \wedge (2\varepsilon_3)$, then we get that $a \leq m^{\alpha} + n m^{1/2}\Big(C_2 + \tau + 2\varepsilon_1^{-1/2} \log \frac{2\varepsilon}{a} \Big)$. If $a \geq 2m^{\alpha}$ in addition, then this implies that $m^{\alpha} \leq n m^{1/2}\Big(C_2 + \tau + 2\varepsilon_1^{-1/2} \log \frac{2\varepsilon}{a} \Big) \leq n m^{1/2}\Big(C_2 + \tau + 2\varepsilon_1^{-1/2} \log \frac{2\varepsilon}{2m^{\alpha}} \Big)$, which, again, is impossible if $m > 0$ is small enough, since $\alpha < \frac{1}{2}$. □

For any $t \in [0, T \wedge \sigma_n]$, divide $\mathbf{R} \times E$ into $\mathbf{R} \times E = B_1 \cup B_2(t) \cup B_3(t)$, with

$$\begin{aligned} B_1 &:= \{(r, x, v) \in \mathbf{R} \times E;\, |x - \pi_v^{\perp} X(\tilde{r})| < m^{\alpha}\}, \\ B_2(t) &:= \{(r, x, v) \in \mathbf{R} \times E;\, |x - \pi_v^{\perp} X(\tilde{r})| \geq m^{\alpha},\, |x - \pi_v^{\perp} X(t)| < m^{\alpha}\}, \\ B_3(t) &:= \{(r, x, v) \in \mathbf{R} \times E;\, |x - \pi_v^{\perp} X(\tilde{r})| \geq m^{\alpha},\, |x - \pi_v^{\perp} X(t)| \geq m^{\alpha}\}. \end{aligned} \tag{3.37}$$

This decomposition is also used in the proof of Lemma 3.60.

By the virtue of (2.9), Lemma 3.45 implies that the integrals of f_4 on B_1 and B_2 converge to 0 as $m \to 0$.

Lemma 3.45 *There exists a constant $C_{15} > 0$ such that for any $t \in [0, T]$, we have*

1. $\left| \int_{B_1} f_4(t, r, x, v) \overline{\lambda}(dr, dx, dv) \right| \leq C_{15} \Big(m^{\alpha(d-1)-1} + m^{\alpha(d-1)-\frac{1}{2}} \log(m^{-1}) \Big),$
2. $\left| \int_{B_2(t)} f_4(t, r, x, v) \overline{\lambda}(dr, dx, dv) \right| \leq C_{15} \Big(m^{\alpha(d-1)-\frac{1}{2}} a_m + m^{\alpha(d-1)-\frac{1}{2}} \log(m^{-1}) \Big).$

Proof (1) By Proposition 2.1 (3), we have $\Big| \nabla U(X(t) - x(t, \Psi(r, x, m^{-1/2}v))) \Big| \leq \|\nabla U\|_{\infty} 1_{\{r \in [-m^{1/2}\tau, T + m^{1/2}\tau]\}}$; also, by Proposition 2.1 (2) and Lemma 3.44, we have on B_1 that $|\nabla U(X(t) - \psi^0(m^{-1/2}(t-r), x, v;\, X(t)))| \leq \|\nabla U\|_{\infty} 1_{\{m^{-1/2}(t-r) \in [-\tau, t_1(v, |x - \pi_v^{\perp} X(t)|)]\}}$ $1_{\{|x - \pi_v^{\perp} X(t)| \leq 2m^{\alpha}\}}$. Therefore, with the help of Lemma 4.5 (2) (3), we get for any $t \in [0, T]$ that

$$\begin{aligned} &\left| \int_{B_1} f_4(t, r, x, v) \overline{\lambda}(dr, dx, dv) \right| \\ &\quad \leq \int_{\mathbf{R} \times E} 1_{\{|x - \pi_v^{\perp} X(\tilde{r})| \leq m^{\alpha}\}} 1_{\{r \in [-m^{1/2}\tau, T + m^{1/2}\tau]\}} \|\nabla U\|_{\infty} \overline{\lambda}(dr, dx, dv) \end{aligned}$$

$$+ \int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(t)|\le 2m^\alpha\}} 1_{\{m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(t)|)]\}} \|\nabla U\|_\infty \overline{\lambda}(dr,dx,dv)$$
$$\le C\|\nabla U\|_\infty \Big((T+2\tau)m^{\alpha(d-1)-1} + m^{\alpha(d-1)-\frac{1}{2}}\log(m^{-1})\Big).$$

(2) It suffices to consider the case where $m \le (2\varepsilon_3)^{1/\alpha}$.

On $B_2(t)$, since $|x - \pi_v^\perp X(\tilde{r})| \ge m^\alpha$, we get by Proposition 2.1 (4) and (2.14) that $\Big|\nabla U(X(t) - x(t, \Psi(r, x, m^{-1/2}v)))\Big| \le \|\nabla U\|_\infty 1_{\{m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde{r})|)]\}} \le \|\nabla U\|_\infty 1_{\{m^{-1/2}(t-r)\in[-\tau,a_m-\tau]\}}$; also, by Proposition 2.1 (2), we have that $|\nabla U(X(t) - \psi^0(m^{-1/2}(t-r), x, v; X(t)))| \le \|\nabla U\|_\infty 1_{\{t-r\in[-m^{1/2}\tau, m^{1/2}t_1(v,|x-\pi_v^\perp X(t)|)]\}}$. So with the help of Lemma 4.5 (2) (3), we get that

$$\Big|\int_{B_2(t)} f_4(t,r,x,v)\overline{\lambda}(dr,dx,dv)\Big|$$
$$\le \int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(t)|\le m^\alpha\}} 1_{\{m^{-1/2}(t-r)\in[-\tau,a_m-\tau]\}} \|\nabla U\|_\infty \overline{\lambda}(dr,dx,dv)$$
$$+ \int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(t)|\le m^\alpha\}} 1_{\{m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(t)|)]\}} \|\nabla U\|_\infty \overline{\lambda}(dr,dx,dv)$$
$$\le C\|\nabla U\|_\infty \Big(m^{\alpha(d-1)-\frac{1}{2}}a_m + m^{\alpha(d-1)-\frac{1}{2}}\log(m^{-1})\Big).$$

□

Now we are ready to prove (3.22) for $k = 9$.

Proof of (3.22) *for* $k = 9$. By definition and Lemma 3.43 (1), it suffices to prove that

$$\sup_{m\in(0,1]}\sup_{t\in[0,T]} 1_{\{m^{1/2}a_m\le t\le\sigma_n\}}\Big|\int_{\mathbf{R}\times E} f_4(t,r,x,v)\overline{\lambda}(dr,dx,dv)\Big| < \infty.$$

The integrals on B_1 and $B_2(t)$ converge to 0 by Lemma 3.45. We prove in the following that the integral on $B_3(t)$ is bounded. Let $t_3(r,t,x,v) = t_1(v, |x - \pi_v^\perp X(t)|) \vee t_1(v, |x - \pi_v^\perp X(\tilde{r})|)$. Then on $B_3(t)$, for any $m \le (2\varepsilon_3)^{1/\alpha}$, we have by (2.14) that

$$t_3(r,t,x,v) \le a_m - \tau. \tag{3.38}$$

If $f_4(t,r,x,v) \ne 0$, then by Proposition 2.1 (2) (4), we have on $B_3(t)$ that $|x| \le R_0$ and $m^{-1/2}(t-r) \in [-\tau, t_3(r,t,x,v)]$, which combined with $t \ge m^{1/2}a_m$ and (3.38) implies that $r - m^{1/2}\tau \ge 0$, hence Lemma 3.20 (3) is applicable (with s, t, c_m and b_m given by r, $m^{-1/2}(t-r)$, $r-t$ and $t_3(r,t,x,v)$, respectively). Combining these, we get by Lemma 3.20 (3) that

$$\Big|f_4(t,r,x,v)\Big|$$
$$\le C_1\Big|x\big(t,\Psi(r,x,m^{-1/2}v)\big) - \psi^0\big(m^{-1/2}(t-r),x,v;X(t)\big)\Big|$$
$$\cdot 1_{\{|x|\le R_0\}} 1_{\{m^{-1/2}(t-r)\in[-\tau,t_3(r,t,x,v)]\}}$$
$$\le C_1 1_{\{|x|\le R_0\}} 1_{\{m^{-1/2}(t-r)\in[-\tau,t_3(r,t,x,v)]\}} 4nm^{1/2} t_3(r,t,x,v) e^{\frac{1}{2}(1+C_1)(m^{-1/2}(t-r)+\tau)}$$
$$\le C_1 1_{\{|x|\le R_0\}} 4nm^{1/2}\exp\left(\frac{1}{2}(1+C_1)(m^{-1/2}(t-r)+\tau)\right)$$
$$\times\Big(1_{\{m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(t)|)]\}} t_1(v,|x-\pi_v^\perp X(t)|)$$
$$+ 1_{\{m^{-1/2}(t-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde{r})|)]\}} t_1(v,|x-\pi_v^\perp X(\tilde{r})|)\Big).$$

Since $(1+C_1)\varepsilon_1^{-1/2} < d-1$ by (2.8), this combined with Lemma 4.5 (1) implies that the considered integral on $B_3(t)$ is bounded for $m \in (0,1]$, hence completes our proof. □

3.4.5 The Martingale Part

We deal with the term $I^3(t)$ in this section. Precisely, we prove that $I^3(t) - M(t)$ satisfies the same estimates as in Proposition 3.31 (4) with respect to $\eta(t)$, and that $M(t)$ satisfies Proposition 3.31 (3). This will complete the proof of Proposition 3.31.

The proof is similar to that of [12, Section 4.4] with slight modification, so we give only the sketch.

We notice that if $\nabla U(X(r - m^{1/2}\tau) - \psi^0(m^{-1/2}(s-r), x, v; X(r - m^{1/2}\tau))) \neq 0$, $s \geq m^{1/2}a_m$ and $|x - \pi_v^\perp X(\tilde{r})| \geq m^\alpha$, then by Proposition 2.1 (2) and (2.14), for any $m \leq (2\varepsilon_3)^{1/\alpha}$, we get that $m^{-1/2}(s-r) \in [-\tau, t_1(v, |x - \pi_v^\perp X(\tilde{r})|)]$, hence $r \in [s - m^{1/2}t_1(v, |x - \pi_v^\perp X(\tilde{r})|), s + m^{1/2}\tau] \subset [m^{1/2}\tau, s + m^{1/2}\tau]$, so if $s \in [0, T \wedge \sigma_n]$ in addition, then $\tilde{r} = r - m^{1/2}\tau$. So we have the following decomposition of $I^3(t)$ by definition:

$$I^3(t) = I^{3,1}(t) + I^{3,2}(t) + I^{3,3}(t) + I^{3,4}(t), \qquad t \in [0,T],$$

with

$$\begin{aligned}
I^{3,1}(t) &= -\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(\tilde{r})|<m^\alpha\}} \\
&\quad \nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)(\mu_\omega - \lambda)(dr, dx, dv), \\
I^{3,2}(t) &= -\int_0^{t\wedge\sigma_n} ds \int_{\mathbf{R}\times E} 1_{\{r\in[m^{1/2}\tau, s+m^{1/2}\tau]\}} \\
&\quad \nabla U\big(X(r - m^{1/2}\tau) - \psi^0(m^{-1/2}(s-r), x, v; X(r - m^{1/2}\tau))\big)(\mu_\omega - \lambda)(dr, dx, dv), \\
I^{3,3}(t) &= \int_0^{t\wedge\sigma_n} ds \int_{\mathbf{R}\times E} 1_{\{r\in[m^{1/2}\tau, s+m^{1/2}\tau]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})|<m^\alpha\}} \\
&\quad \nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)(\mu_\omega - \lambda)(dr, dx, dv), \\
I^{3,4}(t) &= \int_0^{t\wedge\sigma_n} 1_{(0,m^{1/2}a_m]}(s)ds \int_{\mathbf{R}\times E} 1_{\{r\in[m^{1/2}\tau, s+m^{1/2}\tau]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})|\geq m^\alpha\}} \\
&\quad \nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)(\mu_\omega - \lambda)(dr, dx, dv).
\end{aligned}$$

We prove in Lemmas 3.46 and 3.47 below that $I^{3,1}(t)$, $I^{3,3}(t)$ and $I^{3,4}(t)$ satisfy (3.26), which implies that they satisfy the same estimates as in Proposition 3.31 (4).

Lemma 3.46 *There exists a constant $C_{16} > 0$ such that*

$$E^{P_m}\Big[\sup_{t\in[0,T]} |I^{3,k}(t)|^2\Big] \leq C_{16} m^{\alpha(d-1)-\frac{1}{2}} \log(m^{-1}), \qquad k \in \{1,3\}, m \in (0,1].$$

Proof $X(\tilde{r})$ is $\mathcal{F}_{(-\infty,r]\times E}$-measurable by Lemma 3.19. So for $k \in \{1,3\}$, we have by Lemma 3.34 and Proposition 2.1 (2) that

$$\begin{aligned}
&E^{P_m}\Big[\sup_{t\in[0,T]} |I^{3,k}(t)|^2\Big] \\
&\leq T\int_0^T ds E\Big[\Big|\int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(\tilde{r})|<m^\alpha\}} \\
&\qquad \nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)(\mu_\omega - \lambda)(dr, dx, dv)\Big|^2\Big]
\end{aligned}$$

$$= T \int_0^T ds E\Big[\int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(\tilde{r})|<m^\alpha\}}$$
$$\Big|\nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)\Big|^2 \lambda(dr, dx, dv)\Big]$$
$$\leq T\|\nabla U\|_\infty^2 \int_0^T ds E\Big[\int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(\tilde{r})|<m^\alpha\}}$$
$$1_{\{m^{-1/2}(s-r)\in[-\tau, t_1(v, |x-\pi_v^\perp X(\tilde{r})|)]\}} \lambda(dr, dx, dv)\Big].$$

This combined with Lemma 4.5 (3) implies our assertion. □

Lemma 3.47 *There exists a constant* $C_{17} > 0$ *such that*

$$E^{P_m}\Big[\sup_{t\in[0,T]} |I^{3,4}(t)|^2\Big] \leq C_{17} m^{\frac{1}{2}} a_m^2, \qquad m \in (0, 1].$$

Proof We have by Lemma 3.34 that

$$E^{P_m}\Big[\sup_{t\in[0,T]} |I^{3,4}(t)|^2\Big]$$
$$\leq m^{\frac{1}{2}} a_m E^{P_m}\Big[\int_0^T 1_{(0,m^{1/2}a_m]}(s) ds \Big| \int_{\mathbf{R}\times E} 1_{\{r\in[m^{1/2}\tau, s+m^{1/2}\tau]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})|\geq m^\alpha\}}$$
$$\nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)(\mu_\omega - \lambda)(dr, dx, dv)\Big|^2\Big]$$
$$= m^{\frac{1}{2}} a_m \int_0^T 1_{(0,m^{1/2}a_m]}(s) ds E^{P_m}\Big[\int_{\mathbf{R}\times E} 1_{\{r\in[m^{1/2}\tau, s+m^{1/2}\tau]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})|\geq m^\alpha\}}$$
$$\nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)^2 \lambda(dr, dx, dv)\Big]. \tag{3.39}$$

By Proposition 2.1 (2) and Lemma 4.5 (1), we have that

$$E^{P_m}\Big[\int_{\mathbf{R}\times E} 1_{\{r\in[m^{1/2}\tau, s+m^{1/2}\tau]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})|\geq m^\alpha\}}$$
$$\nabla U\big(X(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; X(\tilde{r}))\big)^2 \lambda(dr, dx, dv)\Big]$$
$$\leq \|\nabla U\|_\infty^2 E^{P_m}\Big[\int_{\mathbf{R}\times E} 1_{\{m^{-1/2}(s-r)\in[-\tau, t_1(v, |x-\pi_v^\perp X(\tilde{r})|)]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})|\leq R_U+1\}} \lambda(dr, dx, dv)\Big]$$
$$\leq \|\nabla U\|_\infty^2 C m^{-1/2}.$$

This combined with (3.39) implies our assertion. □

Finally, we deal with the term $I^{3,2}$. Let

$$\widetilde{I^{3,2}}(t) := -\int_0^t ds \int_{(0,s]\times E} \nabla U(X(r\wedge\sigma_n)$$
$$-\psi^0(m^{-1/2}(s-r) - \tau, x, v; X(r\wedge\sigma_n)))\overline{N}(dr, dx, dv).$$

As discussed before, $r \geq m^{1/2}\tau$ implies $\lambda(dr, dx, dv) = \overline{\lambda}(dr, dx, dv)$. Therefore, by a trivial change of variable, we get that

$$I^{3,2}(t) = \widetilde{I^{3,2}}(t\wedge\sigma_n).$$

We first notice the following estimate with respect to $\frac{d}{dt}\widetilde{I^{3,2}}(t)$.

Lemma 3.48 *There exists a constant* $C_{18} > 0$ *such that*

$$E^{P_m}\Big[\Big|\frac{d}{dt}\widetilde{I^{3,2}}(t)\Big|^2\Big] \le C_{18} m^{-1/2}, \qquad \textit{for all } t \in [0,T], m \in (0,1].$$

Proof We have by definition, Lemma 3.34 and Proposition 2.1 (2) that

$$\begin{aligned} &E^{P_m}\Big[\Big|\frac{d}{dt}\widetilde{I^{3,2}}(t)\Big|^2\Big] \\ &= E^{P_m}\Big[\int_{(0,t]\times E}\Big|\nabla U(X(r\wedge\sigma_n) - \psi^0(m^{-1/2}(t-r)-\tau, x, v; X(r\wedge\sigma_n)))\Big|^2 \overline{\lambda}(dr,dx,dv)\Big] \\ &\le \|\nabla U\|_\infty^2 E^{P_m}\Big[\int_{(0,t]\times E} 1_{\{|x-\pi_v^\perp X(r\wedge\sigma_n)|\le R_U+1\}} \\ &\quad 1_{\{m^{-1/2}(t-r)-\tau\in[-\tau, t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)]\}}\overline{\lambda}(dr,dx,dv)\Big]. \end{aligned}$$

This combined with Lemma 4.5 (1) implies our assertion. □

Also, by a simple calculation, we have that

$$\begin{aligned} \widetilde{I^{3,2}}(t) = -&\int_{(0,t]\times E}\overline{N}(dr,dx,dv) \\ &\int_{[0,t-r]}\nabla U(X(r\wedge\sigma_n) - \psi^0(m^{-1/2}s-\tau, x, v; X(r\wedge\sigma_n)))ds. \end{aligned}$$

Since by Proposition 2.1 (2), $s \ge m^{1/2}\Big(t_1(v, |x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\Big)$ implies $\nabla U(X(r\wedge\sigma_n) - \psi^0(m^{-1/2}s-\tau, x, v; X(r\wedge\sigma_n))) = 0$, we get that $\widetilde{I^{3,2}}(t)$ can be rewritten as

$$\widetilde{I^{3,2}}(t) = M(t) + \eta_1(t), \tag{3.40}$$

with $M(t)$ as defined in (3.20), and

$$\begin{aligned} \eta_1(t) := &\int_{(0,t]\times E}\overline{N}(dr,dx,dv) \\ &\int_{[(t-r)\wedge\{m^{1/2}\left(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\right)\}, m^{1/2}\left(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\right)]} \\ &\nabla U\big(X(r\wedge\sigma_n) - \psi^0\big(m^{-1/2}s-\tau, x, v; X(r\wedge\sigma_n)\big)\big)ds. \end{aligned}$$

Summarizing our result up to now, we get that in order to complete our proof of Proposition 3.31, it suffices to prove that $M(t)$ satisfies Proposition 3.31 (3), and that $\eta_1(t)$ satisfies the same estimates as in Proposition 3.31 (4). Let $C_{19} := \Big(3\tau\|\nabla U\|_\infty\Big) \vee (4C_0+2)$. We first prove the following:

Lemma 3.49 $\{M(t)\}_{t\in[0,T]}$ *is a càdlàg martingale, and the jumps of* $\{M(t)\}_{t\in[0,T]}$ *and* $\{\eta_1(t)\}_{t\in[0,T]}$ *satisfy* $|\Delta M(t)| \le C_{19} m^{1/2}$, $|\Delta\eta_1(t)| \le C_{19} m^{1/2}$.

Proof The fact that $\{M(t)\}_{t\in[0,T]}$ is a càdlàg martingale is trivial.

We estimate $|\Delta M(t)|$ in the following. If $|v| \ge 2C_0+1$, then $t_1(v, |x-\pi_v^\perp X(r\wedge\sigma_n)|) = 2\tau$ by definition, so $\Big|\int_{[0,m^{1/2}\left(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\right)]}\nabla U(X(r\wedge\sigma_n) - \psi^0(m^{-1/2}s-\tau, x, v; X(r\wedge$

$\sigma_n)))ds\Big| \le m^{1/2}\big(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\big)\|\nabla U\|_\infty = m^{1/2}3\tau\|\nabla U\|_\infty$. If $|v| \le 2C_0+1$, then

$$\begin{aligned}&\Big|\int_{[0,m^{1/2}\left(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\right)]}\nabla U(X(r\wedge\sigma_n)-\psi^0(m^{-1/2}s-\tau,x,v;X(r\wedge\sigma_n)))ds\Big|\\&\quad = m^{1/2}\Big|\psi^1(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|),x,v;X(r\wedge\sigma_n))-\psi^1(-\tau,x,v;X(r\wedge\sigma_n))\Big|\\&\quad\le m^{1/2}2|v| \le m^{1/2}2(2C_0+1).\end{aligned}$$

The estimation with respect to $|\Delta\eta_1(t)|$ is gotten in exactly the same way, and we omit the proof here. □

Proof of Proposition 3.31 (3) By Lemma 3.49, if suffices to prove (3.23) and (3.24).

We prove (3.23) by using the following general result: for any $f:\mathbf{R}\times E\times\Omega\to\mathbf{R}$ such that $f(r,x,v,\cdot)$ is $\mathcal{F}_r$-measurable for any $(r,x,v)\in\mathbf{R}\times E$, we have for any $s,t\in\mathbf{R}$ satisfying $s<t$ that

$$\begin{aligned}&E\left[\Big(\int_{[s,t]\times E}f(r,x,v,\cdot)\overline{N}(dr,dx,dv)\Big)^2\Big|\mathcal{F}_s\right]\\&\quad = E\left[\int_{[s,t]\times E}f^2(r,x,v,\cdot)\overline{\lambda}(dr,dx,dv)\Big|\mathcal{F}_s\right].\end{aligned}\tag{3.41}$$

For any $s,t\in[0,T]$ satisfying $s<t$, we have by definition, (3.41) and Proposition 2.1 (2) that

$$\begin{aligned}&E^{P_m}\left[|M(t)-M(s)|^2\Big|\mathcal{F}_s\right]\\&\quad = E^{P_m}\Big[\int_{(s,t]\times E}\Big|\int_{[0,m^{1/2}\left(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\right)]}\\&\qquad\nabla U(X(r\wedge\sigma_n)-\psi^0(m^{-1/2}u-\tau,x,v;X(r\wedge\sigma_n)))du\Big|^2\overline{\lambda}(dr,dx,dv)\Big|\mathcal{F}_s\Big]\\&\quad\le m\|\nabla U\|_\infty^2 E^{P_m}\Big[\int_{(s,t]\times E}(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau)^2\\&\qquad\times 1_{\{|x-\pi_v^\perp X(r\wedge\sigma_n)|\le R_U+1\}}\overline{\lambda}(dr,dx,dv)\Big|\mathcal{F}_s\Big].\end{aligned}$$

Combining this with Lemma 4.5 (4), we get (3.23).

We next prove (3.24). Let $A_k(s,x,v) := m^{-\frac{1}{2}}\int_{\mathbf{R}}\nabla_k U(X(s)-\psi^0(m^{-\frac{1}{2}}u-\tau,x,v;X(s)))\,du$. Then $|A_k(s,x,v)|\le\|\nabla U\|_\infty(t_1(v,|x-\pi_v^\perp X(s)|)+\tau)1_{\{|x-\pi_v^\perp X(s)|\le R_U+1\}}$ by Proposition 2.1 (2). Also, for any $s\le\sigma_n$, we have by definition that

$$a_{kl}ds = m\int_E A_k(s,x,v)A_l(s,x,v)\overline{\lambda}(ds,dx,dv)$$

and

$$[M^k,M^l]_s = m\int_{(0,s]\times E}A_k(s,x,v)A_l(s,x,v)N(ds,dx,dv).$$

Therefore,

$$\int_{(0,t\wedge\sigma_n]}g(s)\Big(d[M^k,M^l]_s-a_{kl}ds\Big)=m\int_{(0,t\wedge\sigma_n]}g(s)A_k(s,x,v)A_l(s,x,v)\overline{N}(ds,dx,dv),$$

which is a martingale. So by Doob's inequality and (3.41), we get that

$$
\begin{aligned}
&E^{P_m}\Big[\sup_{t\in[0,T]}\Big|\int_{(0,t\wedge\sigma_n]} g(s)\Big(d[M^k, M^l]_s - a_{kl}ds\Big)\Big|^2\Big] \\
&\le 4E^{P_m}\Big[\Big|m\int_{(0,T\wedge\sigma_n]\times E} g(s)A_k(s,x,v)A_l(s,x,v)\overline{N}(ds,dx,dv)\Big|^2\Big] \\
&= 4m^2E^{P_m}\Big[\int_{(0,T\wedge\sigma_n]\times E} g(s)^2A_k(s,x,v)^2A_l(s,x,v)^2\overline{\lambda}(ds,dx,dv)\Big] \\
&\le 4mE^{P_m}\Big[\int_{(0,Ts\wedge\sigma_n]} ds\int_E \|g\|_\infty^2\|\nabla U\|_\infty^4(t_1(v,|x-\pi_v^\perp X(s)|)+\tau)^4 \\
&\quad\times 1_{\{|x-\pi_v^\perp X(s)|\le R_U+1\}}\rho_0\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv)\Big],
\end{aligned}
$$

which, by Lemma 4.5 (3), is bounded by $4\|g\|_\infty^2\|\nabla U\|_\infty^4 CTm$. So (3.24) holds. □

By Proposition 3.31 (3) and Lemma 3.30, we get easily the following.

Lemma 3.50 *{the distribution of $\{M(t)\}_{t\in[0,T]}$ under $P_m\}_{m\in(0,1]}$ is tight in $\wp(D([0,T];\mathbf{R}^d))$.* □

With Lemmas 3.49 and 3.50 in hand, we get Lemmas 3.51 and 3.52 below by exactly the same argument as in [12] (see the proof of [12, Lemmas 4.4.2–4.4.4]).

Lemma 3.51 *Any cluster point of $\{\{M(t)\}_{t\in[0,T]}$ under $P_m\}_{m\to0}$ in $\wp(D([0,T];\mathbf{R}^d))$ constitutes a continuous canonical process.* □

Lemma 3.52 *For any $\varepsilon>0$, we have that*

$$
\lim\sup_{\delta\to0}\lim\sup_{m\to0} P_m\Big(\sup_{0\le s\le t\le T,|s-t|\le\delta}|M(t)-M(s)|>\varepsilon\Big)=0. \tag{3.42}
$$

□

Also, since

$$
\begin{aligned}
\sup_{\omega\in\Omega,m\in(0,1]}\int_{(0,T]\times E}&\Big(m^{1/2}t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)\Big)^k \\
&1_{\{|x-\pi_v^\perp X(r\wedge\sigma_n)|\le R_U+1\}}\overline{\lambda}(dr,dx,dv)<\infty
\end{aligned}
$$

for $k\in\{2,4\}$, by exactly the same argument as in [12, Lemmas 4.4.5], we get the following.

Lemma 3.53 *There exists a constant $C_{20}>0$ such that*

$$
\sup_{m\in(0,1]} E^{P_m}\Big[\sup_{t\in[0,T]}|M(t)|^4\Big]\le C_{20}.
$$

□

Similarly, by the same argument as in the proof of [12, (4.18)], we get the following.

Lemma 3.54 *There exists a constant $C_{21}>0$ such that*

$$
E^{P_m}[|\eta_1(t)|^6]\le C_{21}m^{3/2}, \qquad t\in[0,T], m\in(0,1].
$$

Proof The quantity A in the proof of [12, (4.18)] is now given by $A = \left|\int_{(t-r)\wedge\{m^{1/2}\left(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\right)\}}^{m^{1/2}\left(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau\right)} \nabla U(X(r\wedge\sigma_n)-\psi^0(m^{-1/2}s-\tau,x,v;X(r\wedge\sigma_n)))ds\right|$. So

$$A \le 1_{\{r\ge t-m^{1/2}t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)-m^{1/2}\tau\}} \\ \times m^{1/2}(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau)\|\nabla U\|_\infty 1_{\{|x-\pi_v^\perp X(r\wedge\sigma_n)|\le R_U+1\}}.$$

So for any $k\in\mathbf{N}$, we get that

$$\int_{(0,t]\times E} A^k\overline{\lambda}(dr,dx,dv) \\ \le m^{k/2}\|\nabla U\|_\infty^k \int_{(0,t]\times E} 1_{\{r\ge t-m^{1/2}(t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau)\}} \\ (t_1(v,|x-\pi_v^\perp X(r\wedge\sigma_n)|)+\tau)^k 1_{\{|x-\pi_v^\perp X(r\wedge\sigma_n)|\le R_U+1\}}\overline{\lambda}(dr,dx,dv).$$

Hence by Lemma 4.5 (4), we get that

$$\int_{(0,t]\times E} A^k\overline{\lambda}(dr,dx,dv) \le \|\nabla U\|_\infty^k C m^{\frac{k-1}{2}}, \qquad k\in\{0,1,\cdots,6\}.$$

So by exactly the same argument as in the proof of [12, (4.18)], with a proper new C, we get that

$$E^{P_m}[|\eta_1(t)|^6] \le C\left(\left(m^{\frac{2-1}{2}}\right)^3+\left(m^{\frac{3-1}{2}}\right)^2+\left(m^{\frac{2-1}{2}}\right)\cdot\left(m^{\frac{4-1}{2}}\right)+m^{\frac{6-1}{2}}\right),$$

which implies our assertion. □

Recall that the jumps of $\eta_1(t)$ are dominated by $C_{19}m^{1/2}$ by Lemma 3.49. So by exactly the same argument as in the proof of [12, Lemma 4.4.6], we get the following result with the help of (3.40), Lemmas 3.48, 3.52 and 3.53:

Lemma 3.55 $\lim_{m\to 0} E^{P_m}\left[\sup_{0\le t\le T}\left|\eta_1(t)\right|^2\right]=0.$ □

Finally, we prove the following result with respect to the variations of $M(t)$ and $\eta_1(t)$:

Lemma 3.56 *There exists a constant $C_{22}>0$ such that for any $s\in[m^{1/2}a_m,T]$,*

1. $E\left[\sup_{u\in[s-m^{1/2}a_m,s]}|M(u)-M(s-m^{1/2}a_m)|^2\right]\le C_{22}m^{1/2}a_m.$
2. $E\left[\sup_{u\in[s-m^{1/2}a_m,s]}|\eta_1(u)-\eta_1(s-m^{1/2}a_m)|^2\right]\le C_{22}m^{1/2}a_m^2.$

Proof Since $M(\cdot)$ is a càdlàg martingale, we have by Doob's inequality and (3.23) that $E\left[\sup_{u\in[s-m^{1/2}a_m,s]}|M(u)-M(s-m^{1/2}a_m)|^2\right]\le 4E\left[|M(s)-M(s-m^{1/2}a_m)|^2\right]\le 4C_6m^{1/2}a_m$. So we get our first assertion.

For the second assertion, we have by definition that $|\eta_1(u)-\eta_1(s-m^{1/2}a_m)|^2\le 2|\widetilde{I^{3,2}}(u)-\widetilde{I^{3,2}}(s-m^{1/2}a_m)|^2+2|M(u)-M(s-m^{1/2}a_m)|^2$. By Lemma 3.48, we have that

$$
\begin{aligned}
&E\Big[\sup_{u\in[s-m^{1/2}a_m,s]} |\widetilde{I^{3,2}}(u)-\widetilde{I^{3,2}}(s-m^{1/2}a_m)|^2\Big]\\
&\le E\Big[\Big(\int_{s-m^{1/2}a_m}^{s}\Big|\frac{d}{du}\widetilde{I^{3,2}}(u)\Big|du\Big)^2\Big]\\
&\le (m^{1/2}a_m)^2\sup_{u\in[s-m^{1/2}a_m,s]} E\Big[\Big|\frac{d}{du}\widetilde{I^{3,2}}(u)\Big|^2\Big]\\
&\le (m^{1/2}a_m)^2 C_{18}m^{-1/2} = C_{18}m^{1/2}a_m^2.
\end{aligned}
$$

This combined with our first assertion implies our second assertion. □

Summarizing our result up to now, we have completed the proof of Proposition 3.31 with $\eta(t)$ given by $\eta(t) := I^1(t) + I^2(t) + I^5(t) + I^7(t) + I^8(t) + I^{10}(t) + I^{11}(t) + I^{3,1}(t) + I^{3,3}(t) + I^{3,4}(t) + \eta_1(t)$.

3.5 Decaying of $I^6(t)$ and $I^9(t)$

We completed the proof of Proposition 3.31 in Sect. 3.4.5. In this section, we prove that the terms $I^6(t)$ and $I^9(t)$ are actually negligible, by using Proposition 3.31. See Lemmas 3.59 and 3.60 below for the explicit expression. As explained in Sect. 2.1, by exactly the same argument of the martingale problem theory as was used in [12, Sect. 5], which was also briefly summarized in [13, page 317], this will complete the proof of Theorem 1.2.

We start our proof by the following observation.

Lemma 3.57 *Assume that* $(y, v) \in E$ *and* $s, r \in \mathbf{R}$ *satisfy* $|y| \ge m^\alpha$, $s \in [m^{\frac{1}{2}}a_m, T \wedge \sigma_n]$ *and* $-\tau \le m^{-1/2}(s-r) \le t_1(v,|y|)$. *Then we have for any* $m \le (2\varepsilon_3)^{1/\alpha}$ *that*

$$
\int_{r-m^{1/2}\tau}^{s} |V(u)-V(\tilde{r})|du \le 2m^{1/2}a_m \sup_{u\in[s-m^{1/2}a_m,s]} |V(s-m^{1/2}a_m)-V(u)|.
$$

Proof Since $|y| \ge m^\alpha$, by (2.14) we have for any $m \le (2\varepsilon_3)^{1/\alpha}$ that $t_1(v,|y|) \le a_m - \tau$. This combined with $m^{-1/2}(s-r) \le t_1(v,|y|)$ implies that $r - m^{1/2}\tau \ge s - m^{1/2}t_1(v,|y|) - m^{1/2}\tau \ge s - m^{1/2}a_m$. Our assertion is now trivial. □

Lemma 3.58 below is easy by Proposition 3.31.

Lemma 3.58 *There exist constants* $\varepsilon, C_{23} > 0$ *such that for any* $s \in [m^{1/2}a_m, T]$, *we have that*

$$
E^{P_m}\Big[1_{\{s\le\sigma_n\}}\sup_{u\in[s-m^{1/2}a_m,s]} |V(s-m^{1/2}a_m)-V(u)|^2\Big] \le C_{23}m^{\varepsilon}.
$$

Proof To simplify notation, let $P(t) := I^4(t) + I^6(t) + I^9(t)$. Then by Proposition 3.31 (1), we have that

$$
\begin{aligned}
&E^{P_m}\Big[1_{\{s\le\sigma_n\}}\sup_{u\in[s-m^{1/2}a_m,s]} |V(s-m^{1/2}a_m)-V(u)|^2\Big]\\
&\le 3E^{P_m}\Big[\sup_{u\in[s-m^{1/2}a_m,s]} |\eta(s-m^{1/2}a_m)-\eta(u)|^2\Big]\\
&\quad +3E^{P_m}\Big[\sup_{u\in[s-m^{1/2}a_m,s]} |M(s-m^{1/2}a_m)-M(u)|^2\Big]\\
&\quad +3E^{P_m}\Big[\sup_{u\in[s-m^{1/2}a_m,s]} |P(s-m^{1/2}a_m)-P(u)|^2\Big]. \qquad (3.43)
\end{aligned}
$$

The first expectation on the right hand side of (3.43) is dominated by $3C_6 m^{\varepsilon}$ by (3.25) of Proposition 3.31. Also, as proved in Lemma 3.56 (1), Proposition 3.31 (3) implies that the second expectation on the right hand side of (3.43) is dominated by $12C_6 m^{1/2} a_m$.

Finally, for the third term on the right hand side of (3.43), we have by Proposition 3.31 (2) that

$$E^{P_m}\Big[\sup_{u\in[s-m^{1/2}a_m,s]} |P(s-m^{1/2}a_m)-P(u)|^2\Big]$$
$$\le (m^{1/2}a_m)^2 \sup_{t\in[0,T]} E^{P_m}\Big[\Big|\frac{d}{dt}P(t)\Big|^2\Big] \le 3C_6 m a_m^2.$$

□

Lemma 3.59 *We have that*

$$\lim_{m\to 0} E\Big[\sup_{t\in[0,T]} |I^6(t)|\Big] = 0.$$

Proof Let

$$\begin{aligned}
F_1(s,r,x,v) &:= |X(s)-X(\tilde r)| + |x(s,\Psi(r,x,m^{-1/2}v)) \\
&\quad -\psi^0(m^{-1/2}(s-r),x,v;X(\tilde r))|, \\
F_2(s,r,x,v) &:= F_1(s,r,x,v)^2, \\
F_3(s,r,x,v) &:= |X(s)-X(\tilde r)-(s-\tilde r)V(\tilde r)| \\
&\quad +|x(s,\Psi(r,x,m^{-1/2}v)) - \psi^0(m^{-1/2}(s-r),x,v;X(\tilde r)) \\
&\quad -m^{1/2}z(m^{-1/2}(s-r),x,v;X(\tilde r),V(\tilde r),-m^{-1/2}(\tilde r - r))|.
\end{aligned}$$

Then by an easy calculation, we get the following:

$$|(f_3-f_2)(s,r,x,v)| \le \frac{1}{2}\|\nabla^3 U\|_\infty F_2(s,r,x,v) + \|\nabla^2 U\|_\infty F_3(s,r,x,v). \tag{3.44}$$

Indeed, by definition and a simple calculation, we have that

$$\begin{aligned}
&(f_3-f_2)(s,r,x,v) \\
&= \int_0^1 \Big\{\nabla^2 U\Big(X(\tilde r) - \psi^0(m^{-1/2}(s-r),x,v;X(\tilde r)) \\
&\quad + \theta[X(s)-X(\tilde r) - x(s,\Psi(r,x,m^{-1/2}v)) + \psi^0(m^{-1/2}(s-r),x,v;X(\tilde r))]\Big) \\
&\quad -\nabla^2 U\Big(X(\tilde r) - \psi^0(m^{-1/2}(s-r),x,v;X(\tilde r))\Big)\Big\} \\
&\quad \cdot\Big(X(s)-X(\tilde r) - x(s,\Psi(r,x,m^{-1/2}v)) + \psi^0(m^{-1/2}(s-r),x,v;X(\tilde r))\Big)d\theta \\
&\quad + \nabla^2 U\Big(X(\tilde r) - \psi^0(m^{-1/2}(s-r),x,v;X(\tilde r))\Big)\cdot\Big(X(s)-X(\tilde r)-(s-\tilde r)V(\tilde r) \\
&\quad - x(s,\Psi(r,x,m^{-1/2}v)) + \psi^0(m^{-1/2}(s-r),x,v;X(\tilde r)) \\
&\quad + m^{1/2}z(m^{-1/2}(s-r),x,v;X(\tilde r),V(\tilde r),-m^{-1/2}(\tilde r - r))\Big).
\end{aligned}$$

This gives us (3.44).

Also, by Proposition 2.1 (2) (4), we get that if $|x-\pi_v^{\perp}X(\tilde r)| \ge m^{\alpha}$, then $|(f_3 - f_2)(s,r,x,v)| \ne 0$ implies that $m^{-1/2}(s-r) \in [-\tau, t_1(v,|x-\pi_v^{\perp}X(\tilde r)|)]$ and $|x - \pi_v^{\perp}X(\tilde r)| \le R_U + 1$. So in order to prove Lemma 3.59, it suffices to prove the following:

$$E^{P_m}\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n}1_{(m^{1/2}a_m,\infty)}(s)\int_{\mathbf{R}\times E,|x-\pi_v^\perp X(\tilde r)|\ge m^\alpha}(\mu_\omega+\lambda)(dr,dx,dv)F_k(s,r,x,v)$$
$$\times 1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}}1_{\{|x-\pi_v^\perp X(\tilde r)|\le R_U+1\}}\Big|\Big]\to 0,\quad k=2,3. \tag{3.45}$$

We prove (3.45) in the following.

As same as in the proof of Lemma 2.2, if $m\le(2\varepsilon_3)^{1/\alpha}$, $m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]$, $s\ge m^{1/2}a_m$ and $|x-\pi_v^\perp X(\tilde r)|\ge m^\alpha$, then $\tilde r=r-m^{1/2}\tau$, hence $s-\tilde r=s-r+m^{1/2}\tau\in[0,m^{1/2}t_1(v,|x-\pi_v^\perp X(\tilde r)|)+m^{1/2}\tau]$. This combined with Lemma 3.20 (with s, t and b_m given by r, $m^{-1/2}(s-r)$ and $t_1(v,|x-\pi_v^\perp X(\tilde r)|)$, respectively) and (3.12) implies that

$$F_1(s,r,x,v)\le 4nm^{1/2}t_1(v,|x-\pi_v^\perp X(\tilde r)|)e^{\frac12(1+C_1)(m^{-1/2}(s-r)+\tau)}.$$

Therefore, since $(1+C_1)2\varepsilon_1^{-1/2}<d-1$, by Lemma 4.5 (1), we get that

$$\begin{aligned}
&E^{P_m}\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n}1_{(m^{1/2}a_m,\infty)}(s)\int_{\mathbf{R}\times E,|x-\pi_v^\perp X(\tilde r)|\ge m^\alpha}(\mu_\omega+\lambda)(dr,dx,dv)\\
&\quad\times F_2(s,r,x,v)1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}}1_{\{|x-\pi_v^\perp X(\tilde r)|\le R_U+1\}}\Big|\Big]\\
&\le 2\int_0^T dsE^{P_m}\Big[\int_{\mathbf{R}\times E}16n^2mt_1(v,|x-\pi_v^\perp X(\tilde r)|)^2e^{(1+C_1)(m^{-1/2}(s-r)+\tau)}\\
&\quad\times 1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}}1_{\{|x-\pi_v^\perp X(\tilde r)|\le R_U+1\}}\lambda(dr,dx,dv)\Big]\\
&\le Cm^{1/2}.
\end{aligned}$$

So (3.45) with $k=2$ holds.

We next prove (3.45) with $k=3$. In the considered domain, we have that $|X(s)-X(\tilde r)-(s-\tilde r)V(\tilde r)|\le\int_{r-m^{1/2}\tau}^s|V(u)-V(\tilde r)|du$. Combining this with Lemma 3.21 (with s, t, c_m and b_m given by r, $m^{-1/2}(s-r)$, $m^{1/2}\tau$ and $t_1(v,|x-\pi_v^\perp X(\tilde r)|)$, respectively) and (3.12), we get for $m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]$ that

$$\begin{aligned}
&F_3(s,r,x,v)\\
&\le 8n^2(\|\nabla^3U\|_\infty\vee 1)(2C_1\vee 1)e^{(1+\frac a2)(1+C_1)(t_1(v,|x-\pi_v^\perp X(\tilde r)|)+\tau)}\\
&\quad\times m^{\frac{1+a}{2}}(t_1(v,|x-\pi_v^\perp X(\tilde r)|)+\tau)^{1+a}\\
&\quad+2e^{\frac12(1+C_1)(t_1(v,|x-\pi_v^\perp X(\tilde r)|)+\tau)}\int_{r-m^{1/2}\tau}^s|V(u)-V(\tilde r)|du.
\end{aligned}$$

Choose $a\in(0,1)$ such that $(1+\frac a2)(1+C_1)2\varepsilon_1^{-1/2}<d-1$. Then by Lemma 4.5 (1), the integral with respect to the first term on the right hand side above has order $O(m^{a/2})$, hence converges to 0 as $m\to 0$. We deal with the second term on the right hand side above in the following.

By Lemma 3.57 and Schwarz inequality, we have that

$$\begin{aligned}
&E^{P_m}\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n}1_{(m^{1/2}a_m,\infty)}(s)\int_{\mathbf{R}\times E,|x-\pi_v^\perp X(\tilde r)|\ge m^\alpha}e^{\frac12(1+C_1)(t_1(v,|x-\pi_v^\perp X(\tilde r)|)+\tau)}\\
&\quad\times\int_{r-m^{1/2}\tau}^s|V(u)-V(\tilde r)|du1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}}\\
&\quad\times 1_{\{|x-\pi_v^\perp X(\tilde r)|\le R_U+1\}}(\mu_\omega+\lambda)(dr,dx,dv)\Big|\Big]
\end{aligned}$$

$$\leq 2m^{1/2}a_m \int_0^T ds E^{P_m}\Big[1_{\{m^{1/2}a_m \leq s \leq \sigma_n\}} \sup_{u \in [s-m^{1/2}a_m, s]} |V(s-m^{1/2}a_m) - V(u)|^2\Big]^{1/2}$$
$$\times E^{P_m}\Big[\Big(\int_{\mathbf{R}\times E} e^{\frac{1}{2}(1+C_1)(t_1(v,|x-\pi_v^\perp X(\tilde{r})|)+\tau)}$$
$$\times 1_{\{m^{-1/2}(s-r)\in[-\tau, t_1(v,|x-\pi_v^\perp X(\tilde{r})|)]\}} 1_{\{|x-\pi_v^\perp X(\tilde{r})|\leq R_U+1\}}(\mu_\omega+\lambda)(dr,dx,dv)\Big)^2\Big]^{1/2}.$$

Since $(1+C_1)2\varepsilon_1^{-1/2} < d-1$, by Lemma 3.58, (3.30) and Lemma 4.5 (1), the right hand side above is dominated by $2m^{1/2}a_m T(C_{23}m^{\varepsilon})^{1/2}\Big(2Cm^{-1/2}+8(Cm^{-1/2})^2\Big)^{1/2}$, which converges to 0 as $m \to 0$. This completes the proof of (3.45) with $k=3$, and in turn completes the proof of Lemma 3.59. □

In the second half of this section, we prove that I^9 is also negligible. This will complete our proof of Theorem 1.2.

Lemma 3.60 *We have that*

$$\lim_{m\to 0} E^{P_m}\Big[\sup_{t\in[0,T]} |I^9(t)|\Big] = 0.$$

Proof Let B_1, $B_2(\cdot)$ and $B_3(\cdot)$ be as defined in (3.37). By Lemma 3.45, it suffices to estimate the integrals of f_1 on B_1, $B_2(s)$, and the integral of $f_4 - f_1$ on $B_3(s)$. Precisely, we prove the following (3.46) and (3.47):

$$E^{P_m}\left[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{B_1\cup B_2(s)} f_1(s,r,x,v)\overline{\lambda}(dr,dx,dv)\Big|\right] \to 0, \tag{3.46}$$

$$E^{P_m}\left[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s)ds \int_{B_3(s)} (f_4-f_1)(s,r,x,v)\overline{\lambda}(dr,dx,dv)\Big|\right] \to 0. \tag{3.47}$$

We first prove (3.46). If $\nabla U^2(X(s)-\psi^0(m^{-1/2}(s-r),x,v;X(s)) \neq 0$, then by Proposition 2.1 (2), we have that $m^{-1/2}(s-r) \in [-\tau, t_1(v,|x-\pi_v^\perp X(s)|)]$. By Lemma 3.44, this combined with $|x-\pi_v^\perp X(\tilde{r})| \leq m^\alpha$ implies $|x-\pi_v^\perp X(s)| \leq 2m^\alpha$. Therefore, applying Lemma 3.38 (with $\bar{t}$ and c_m given by $t_1(v,|x-\pi_v^\perp X(s)|)$), with the help of (3.12), we get the following:

$$\Big|\int_{\mathbf{R}\times E} 1_{B_1\cup B_2(s)}(r,x,v) f_1(s,r,x,v)\overline{\lambda}(dr,dx,dv)\Big|$$
$$\leq m^{1/2}C_1 4n \int_{\mathbf{R}\times E} 1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(s)|)]\}} t_1(v,|x-\pi_v^\perp X(s)|)$$
$$e^{\frac{1}{2}(1+C_1)(t_1(v,|x-\pi_v^\perp X(s)|)+\tau)} 1_{\{|x-\pi_v^\perp X(s)|\leq 2m^\alpha\}} \overline{\lambda}(dr,dx,dv),$$

which, by Lemma 4.5 (3), is dominated by $C_1 4n \cdot Cm^{\alpha(d-1-(1+C_1)\varepsilon_1^{-1/2})}(\log(m^{-1}))^2$. Since $d-1 > (1+C_1)\varepsilon_1^{-1/2}$, this converges to 0 as $m \to 0$. This implies (3.46).

Finally, we prove (3.47). Let $t_3 := t_3(s,r,x,v) := t_1(v,|x-\pi_v^\perp X(\tilde{r})|) \vee t_1(v,|x-\pi_v^\perp X(s)|)$. Then on $B_3(s)$, by (2.14) we have for any $m \leq (2\varepsilon_3)^{1/\alpha}$ that $t_3 \leq a_m - \tau$. If $(f_4 - f_1)(s,r,x,v) \neq 0$, then by Proposition 2.1 (2) (4), we get that $|x| \leq R_0$ and

$m^{-1/2}(s-r) \le t_3$, which combined with $s \ge m^{1/2}a_m$ implies that $r - m^{1/2}\tau \ge 0$. Also, by exactly the same argument as that we used for (3.44), we get that

$$\begin{aligned}
&|(f_4 - f_1)(s,r,x,v)| \\
&\le \frac{1}{2}\|\nabla^3 U\|_\infty \Big| x\big(s, \Psi(r,x,m^{-1/2}v)\big) - \psi^0\big(m^{-1/2}(s-r), x, v; X(s)\big)\Big|^2 \\
&\quad + \|\nabla^2 U\|_\infty \Big| x\big(s, \Psi(r,x,m^{-1/2}v)\big) - \psi^0\big(m^{-1/2}(s-r), x, v; X(s)\big) \\
&\quad - m^{1/2} z\big(m^{-1/2}(s-r), x, v; X(s), V(s), -m^{-1/2}(s-r)\big)\Big|.
\end{aligned}$$

As same as in the proof of Lemma 3.59, choose $a \in (0,1)$ such that $(1+\frac{a}{2})(1+C_1)2\varepsilon_1^{-1/2} < d-1$. By Lemmas 3.20 and 3.21, we get that

$$\begin{aligned}
&|(f_4 - f_1)(s,r,x,v)| \\
&\le 1_{\{|x|\le R_0\}} 1_{\{m^{-1/2}(s-r)\in[-\tau,t_3]\}} \\
&\quad \times \Big(\frac{1}{2}\|\nabla^3 U\|_\infty \Big\{ \frac{2C_1 n}{1+C_1} m^{1/2} 2t_3 \Big(e^{\frac{1}{2}(1+C_1)(t_3+\tau)} - 1\Big)\Big\}^2 \\
&\quad + \|\nabla^2 U\|_\infty \Big\{ \frac{8n^2}{1+C_1}(2C_1 \vee 1)(\|\nabla^3 U\|_\infty \vee 1) e^{(1+\frac{a}{2})(1+C_1)(t_3+\tau)} \Big(m^{1/2} 2t_3\Big)^{1+a} \\
&\quad + \frac{2C_1}{1+C_1}\Big(e^{\frac{1}{2}(1+C_1)(t_3+\tau)} - 1\Big) \int_{r-m^{1/2}\tau}^{s} |V(u) - V(s)| du \Big\}\Big).
\end{aligned}$$

So with some proper constant C_{24}, we have that

$$\begin{aligned}
&|(f_4 - f_1)(s,r,x,v)| \le 1_{\{m^{-1/2}(s-r)\in[-\tau,t_3]\}} 1_{\{|x|\le R_0\}} C_{24} \\
&\quad \times \left(m t_3^2 e^{(1+C_1)t_3} + m^{\frac{1+a}{2}} t_3^2 e^{(1+\frac{a}{2})(1+C_1)t_3} + e^{\frac{1}{2}(1+C_1)t_3} \int_{r-m^{1/2}\tau}^{s} |V(u) - V(s)| du \right).
\end{aligned}$$

The integral corresponding to $mt_3^2 e^{(1+C_1)t_3}$ converges to 0 by Lemma 4.5 (1), by the virtue of (2.8). Therefore, in order to prove (3.47), it suffices to prove the following (3.48) $\sim$ (3.51).

$$\begin{aligned}
&m^{\frac{1+a}{2}} E^{P_m}\Big[\int_0^{T\wedge\sigma_n} ds \int_{\mathbf{R}\times E} 1_{\{m^{-1/2}(s-r)\in[-\tau, t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}} 1_{\{|x|\le R_0\}} \\
&\quad \times t_1(v, |x-\pi_v^\perp X(\tilde r)|)^2 e^{(1+\frac{a}{2})(1+C_1)t_1(v,|x-\pi_v^\perp X(\tilde r)|)} \overline{\lambda}(dr,dx,dv)\Big] \to 0, \quad (3.48)
\end{aligned}$$

$$\begin{aligned}
&m^{\frac{1+a}{2}} E^{P_m}\Big[\int_0^{T\wedge\sigma_n} ds \int_{\mathbf{R}\times E} 1_{\{m^{-1/2}(s-r)\in[-\tau, t_1(v,|x-\pi_v^\perp X(s)|)]\}} 1_{\{|x|\le R_0\}} \\
&\quad \times t_1(v, |x-\pi_v^\perp X(s)|)^2 e^{(1+\frac{a}{2})(1+C_1)t_1(v,|x-\pi_v^\perp X(s)|)} \overline{\lambda}(dr,dx,dv)\Big] \to 0, \quad (3.49)
\end{aligned}$$

$$\begin{aligned}
&E^{P_m}\Big[\int_0^{T\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s) ds \int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(\tilde r)|>m^\alpha\}} \\
&\quad \times 1_{\{m^{-1/2}(s-r)\in[-\tau, t_1(v,|x-\pi_v^\perp X(\tilde r)|)]\}} 1_{\{|x|\le R_0\}} e^{\frac{1}{2}(1+C_1)t_1(v,|x-\pi_v^\perp X(\tilde r)|)} \\
&\quad \times \int_{r-m^{1/2}\tau}^{s} |V(u)-V(s)| du \overline{\lambda}(dr,dx,dv)\Big] \to 0, \quad (3.50)
\end{aligned}$$

$$E^{P_m}\Big[\int_0^{T\wedge\sigma_n} 1_{(m^{1/2}a_m,\infty)}(s) ds \int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(s)|>m^\alpha\}}$$

$$\times 1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(s)|)]\}} 1_{\{|x|\le R_0\}} e^{\frac{1}{2}(1+C_1)t_1(v,|x-\pi_v^\perp X(s)|)}$$
$$\times \int_{r-m^{1/2}\tau}^{s} |V(u)-V(s)|du\overline{\lambda}(dr,dx,dv)\Big] \to 0. \quad (3.51)$$

Since $(1+\frac{a}{2})(1+C_1)2\varepsilon_1^{-1/2} < d-1$, (3.48) and (3.49) are direct consequences of Lemma 4.5 (1).

As for (3.50) and (3.51), by 2.14, we have for any $m \le (2\varepsilon_3)^{1/\alpha}$ that $|x-\pi_v^\perp X(\tilde{r})| > m^\alpha$ implies $t_1(v,|x-\pi_v^\perp X(s)| \le a_m - \tau$, so if $m^{-1/2}(s-r) \le t_1(v,|x-\pi_v^\perp X(s)|$ in addition, then $r - m^{1/2}\tau \ge s - m^{1/2}a_m$. Also, we have that $(1+C_1)\varepsilon_1^{-1/2} < d-1$. Therefore, by Lemmas 3.57 and 4.5 (1), both of the left hand sides of (3.50) and (3.51) are dominated by

$$CE^{P_m}\left[\int_0^T 1_{\{m^{1/2}a_m\le s\le\sigma_n\}} a_m \sup_{u\in[s-m^{1/2}a_m,s]} |V(u)-V(s-m^{1/2}\tau)|ds\right].$$

This combined with Lemma 3.58 implies (3.50) and (3.51). □

4 Concluding Remarks

For a Rayleigh gas model with a single massive particle and a repulsive potential, we presented a mathematical proof that the motion of the massive particle converges to a Brownian motion when the mass m of the light particles converges to 0, while their number density and velocity distribution scale like $m^{-\frac{1}{2}}$, under the assumption that the initial kinetic energies of the light particles are bounded below. We do not need any assumption requiring that the initial velocities of the environmental particles should be restricted to be "fast enough". As a result, the interaction time durations between the massive particle and the light particles are unbounded.

Here are several possible generalizations that are more physically relevant:

1. A model with no minimum constraint on the initial kinetic energies of the gas particles. In such a model, besides the possibly small initial skews between particles as discussed in this paper, the possibly slow initial velocities of the light particles is also a source of singularity. In other words, even for a freezing approximation particle, when the initial velocity v converges to 0, the effective interaction time duration diverges to infinity. Moreover, a careful calculation suggests that this divergence is much worse than a log order. So it is hopeless to remove this singularity by simply applying the method that we used in this paper (i.e., the introduction of α in this paper). Some new estimation is necessary. A generalization in this direction is now in progress by the author.
2. Lowering the dimensionality $d \le 5$. As explained in Remark 1.3, the method of this paper is not applicable to the low dimensional case. When $d \le 3$, we are even not able to prove that the generator L of the limiting process is well-defined. A more accurate estimate will be necessary.
3. A model with interaction potential that diverges to infinity as the inter-particle distance converges to 0. For example, the Weeks-Chandler-Andersen potential or the Lennard-Jones potential. In such a model, the second derivative of the potential function is not bounded, so estimations with respect to the effective interaction time durations do not imply directly estimations with respect to the distance between the light particles and our freezing approximations; and estimations with respect to the distances between the particles do not imply directly the estimations with respect to the differences between the

corresponding forces. It is expected that a method similar to the one used for low initial kinetic energies (i.e., the first problem formulated in this remark) might be effective. This is also planned as a forthcoming topic of our research.

Acknowledgements The author would like to thank Professor Sergio Albeverio who read carefully and made comments on the manuscript. The author is also grateful to the anonymous referees for their valuable comments and suggestions, which substantially improved the quality of this paper.

Appendix

We give the proof of (2.6) and (2.7) in Sect. A.1; give the proof of Lemma 3.33 in Sect. A.2; present several estimates that are used to prove that $I^1(t)$ and $I^2(t)$ converge to 0 fast enough in Sect. A.3; finally, in Sect. A.4, we present several necessary estimates with respect to integrals involving $t_1(v;*)$.

A.1 Proof of (2.6) and (2.7)

By a simple calculation, we have that $\nabla_i\nabla_j U(x) = \left(h''(|x|) - \frac{h'(|x|)}{|x|}\right)\frac{x_i}{|x|}\frac{x_j}{|x|} + \delta_{ij}\frac{h'(|x|)}{|x|}$ for any $i, j \in \{1, \cdots, d\}$, so $\nabla^2 U(x)y = \left(h''(|x|) - \frac{h'(|x|)}{|x|}\right)(y, \frac{x}{|x|})\frac{x}{|x|} + \frac{h'(|x|)}{|x|}y = h''(|x|)\pi_x y + \frac{h'(|x|)}{|x|}\pi_x^{\perp} y$. Since $|\frac{h'(|x|)}{|x|}| \leq \|h''\|_\infty$, this implies (2.6).

Also, (2.7) is an easy consequence of (2.6) since $|\nabla U(y_1) - \nabla U(y_2)| = \int_0^1 \nabla^2 U(y_2 + \theta(y_1 - y_2))(y_1 - y_2)d\theta$.

A.2 Proof of Lemma 3.33

We give the proof of Lemma 3.33 in this section. We first notice the following translation property (Lemma 4.1) and symmetry (Lemma 4.2) of φ, which are clear heuristically.

Lemma 4.1 *For any* $(x, v) \in \mathbf{R}^{2d}$, $t \geq 0$ *and* $X \in \mathbf{R}^d$, *we have that*

$$\varphi^0(t, x, v; X) - X = \varphi^0(t, x - X, v; \mathbf{0}).$$

Proof Just notice that both sides above satisfy the same ODE with the same initial conditions. □

Lemma 4.2 *For any* $X \in \mathbf{R}^d$ *and* $(x, v) \in E$, *let* $\iota(x, v; X) := \left(\iota^0(x, v; X), \iota^1(x, v; X)\right) := (2X - x, -v)$. *Then we have the following.*

$$\varphi\big(t, \iota(x, v; X); X\big) = \iota\big(\varphi(t, x, v; X); X\big), \qquad t \geq 0.$$

Proof First we notice that $\frac{d}{dt}\varphi^0(t, \iota(x, v; X); X) = \varphi^1(t, \iota(x, v; X); X)$ and $\frac{d}{dt}\iota^0(\varphi(t, x, v; X); X) = \frac{d}{dt}\left(2X - \varphi^0(t, x, v; X)\right) = -\varphi^1(t, x, v; X) = \iota^1(\varphi(t, x, v; X); X)$. We have by definition that $(\varphi^0(0, \iota(x, v; X); X), \varphi^1(0, \iota(x, v; X); X)) = (2X - x, -v) = \iota(\varphi(0, x, v; X); X)$. So $\varphi(t, \iota(x, v; X); X)$ and $\iota(\varphi(t, x, v; X); X)$ satisfy the same initial condition. Also,

$$\frac{d^2}{dt^2}\varphi^0(t, \iota(x, v; X); X) = -\nabla U(\varphi^0(t, \iota(x, v; X); X) - X),$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\iota^0(\varphi(t,x,v;X);X) &= \frac{d^2}{dt^2}\Big(2X-\varphi^0(t,x,v;X)\Big)\\ &= \nabla U\big(\varphi^0(t,x,v;X)-X\big) = -\nabla U\big((2X-\varphi^0(t,x,v;X))-X\big)\\ &= -\nabla U\big(\iota^0(\varphi(t,x,v;X);X)-X\big).\end{aligned}$$

So $\varphi^0(t,\iota(x,v;X);X)$ and $\iota^0(\varphi(t,x,v;X);X)$ satisfy the same ODE, too. Therefore, $\varphi^0(t,\iota(x,v;X);X)=\iota^0(\varphi(t,x,v;X);X)$, which implies our assertion. □

Now we are ready to prove the first assertion of Lemma 3.33.

Proof of Lemma 3.33 (1) We first notice that the left hand side of (3.27) does not depend on X_0. Indeed, by definition and Lemma 4.1, we have that

$$\begin{aligned}&\text{(LHS) of (3.27)}\\ &= \int_{\mathbf{R}^d}|v|dv\int_{\mathbf{R}}dr\int_{E_v}\nabla U\left(-\varphi^0(m^{-1/2}s,x-m^{-1/2}rv-X_0,v,\mathbf{0})\right)\\ &\quad\times m^{-1}\rho\left(\frac{1}{2}|v|^2,x-m^{-1/2}rv-X_0\right)\tilde{\nu}(dx;v).\end{aligned}$$

Writing $x-m^{-1/2}rv-X_0$ as $x-\pi_v^\perp X_0-m^{-1/2}(r+m^{1/2}\frac{(X_0,v)}{|v|^2})v$, by change of variable $x-\pi_v^\perp X_0\to x$, $r+m^{1/2}\frac{(X_0,v)}{|v|^2}\to r$ for any fix v, this gives us that

$$\begin{aligned}&\text{(LHS) of (3.27)}\\ &= \int_{\mathbf{R}^d}|v|dv\int_{\mathbf{R}}dr\int_{E_v}\nabla U\left(-\varphi^0(m^{-1/2}s,x-m^{-1/2}rv,v,\mathbf{0})\right)\\ &\quad\times m^{-1}\rho\left(\frac{1}{2}|v|^2,x-m^{-1/2}rv\right)\tilde{\nu}(dx;v),\end{aligned}\tag{5.1}$$

the right hand side of which does not depend on X_0.

Let J denote the right hand side of (5.1). So it suffices to prove that $J=0$. By Lemma 4.2 with $X=0$, we have that $-\varphi^0(m^{-1/2}s,x-m^{-1/2}r,v,\mathbf{0})=\varphi^0(m^{-1/2}s,-x+m^{-1/2}r,-v,\mathbf{0})$. Substituting this into the definition of J, since $\rho(u,-z)=\rho(u,z)$ for any $(u,z)\in[0,\infty)\times\mathbf{R}^d$ by (A2), we get by change of variable $x\to -x$ and $v\to -v$ that

$$\begin{aligned}J &= \int_{\mathbf{R}^d}|v|dv\int_{\mathbf{R}}dr\int_{E_v}\nabla U(\varphi^0(m^{-1/2}s,x-m^{-1/2}rv,v,\mathbf{0}))\\ &\qquad\times m^{-1}\rho\left(\frac{1}{2}|v|^2,x-m^{-1/2}rv\right)\tilde{\nu}(dx;v)\\ &= -J,\end{aligned}$$

hence $J=0$. □

The second assertion of Lemma 3.33 is proved similarly. We first notice the following property of ψ.

Lemma 4.3 *For any $(x,v)\in E$, $u\in\mathbf{R}$ and $X\in\mathbf{R}^d$,*

1. $\psi^0(u,x,v;X)-X=\psi^0(u-\frac{(X,v)}{|v|^2},x-\pi_v^\perp X,v;\mathbf{0})$.
2. $\psi^0(u,-x,-v;\mathbf{0})=-\psi^0(u,x,v;\mathbf{0})$.

Proof (1) By definition and Lemma 4.1, we have that

$$\begin{aligned}\psi^0(u,x,v;X)-X &= \lim_{s\to\infty}\varphi^0(u+s,x-sv,v;X)-X\\ &= \lim_{s\to\infty}\varphi^0(u+s,x-sv-X,v;\mathbf{0})\\ &= \lim_{s\to\infty}\varphi^0\left(\left(u-\frac{(X,v)}{|v|^2}\right)+\left(s+\frac{(X,v)}{|v|^2}\right),x-\pi_v^{\perp}X-\left(s+\frac{(X,v)}{|v|^2}\right)v,v;\mathbf{0}\right)\\ &= \psi^0\left(u-\frac{(X,v)}{|v|^2},x-\pi_v^{\perp}X,v;\mathbf{0}\right).\end{aligned}$$

(2) By definition and Lemma 4.2, we have that

$$\begin{aligned}\psi^0(u,-x,-v;\mathbf{0}) &= \lim_{s\to\infty}\varphi^0(u+s,-x+sv,-v;\mathbf{0})\\ &= -\lim_{s\to\infty}\varphi^0(u+s,x-sv,v;\mathbf{0}) = -\psi^0(u,x,v;\mathbf{0}).\end{aligned}$$

□

Proof of Lemma 3.33 (2) By Lemma 4.3 (1) and change of variable $u-\frac{(X,v)}{|v|^2}\to u$, $x-\pi_v^{\perp}X\to x$ for any fixed $v\in\mathbf{R}^d$, we have that

$$\begin{aligned}&\text{(LHS) of (3.28)}\\ &= \int_{\mathbf{R}\times E}\nabla U\left(-\psi^0\left(u-\frac{(X,v)}{|v|^2},x-\pi_v^{\perp}X,v;\mathbf{0}\right)\right)m^{-1}\rho_0\left(\frac{1}{2}|v|^2\right)du\nu(dx,dv)\\ &= \int_{\mathbf{R}\times E}\nabla U\left(-\psi^0(u,x,v;\mathbf{0})\right)m^{-1}\rho_0\left(\frac{1}{2}|v|^2\right)du\nu(dx,dv).\end{aligned}$$

Let J denote the right hand side above. Then by Lemma 4.3 (2) and change of variable $x\to -x$, $v\to -v$, we get that $J=-J$, hence $J=0$. □

A.3 Estimates for $I^1(t)$ and $I^2(t)$

Lemma 4.4 is used in the estimates with respect to $I^1(t)$ and $I^2(t)$ (see Lemmas 3.35 and 3.36).

Lemma 4.4 *For any $A>0$, there exists a constant $C>0$ such that for any $B_m\geq 0$ ($m\in(0,1]$), we have that*

1. $$\int_{\mathbf{R}\times E}1_{[0,A]}(|x|)1_{\{|r|\leq B_m\}}(1+|v|)^2\lambda_m(dr,dx,dv)\leq C_9m^{-1}B_m,$$

2. $$E^{P_m}\left[\left(\int_{\mathbf{R}\times E}1_{[0,A]}(|x|)1_{\{|r|\leq B_m\}}(1+|v|)\mu_\omega(dr,dx,dv)\right)^2\right]\leq C_9(m^{-2}B_m^2+1).$$

Proof Let $C:=2(2A)^{d-1}\int_{\mathbf{R}^d}(1+|v|^2)\rho_{max}\left(\frac{1}{2}|v|^2\right)|v|dv$. Then C is finite by assumption. We have that

$$\begin{aligned}
&\int_{\mathbf{R}\times E} 1_{[0,A]}(|x|)1_{\{|r|\le B_m\}}(1+|v|^2)\lambda_m(dr,dx,dv)\\
&\le \int_{\mathbf{R}\times E} 1_{[0,A]}(|x|)1_{\{|r|\le B_m\}}(1+|v|^2)m^{-1}\rho_{max}\left(\frac{1}{2}|v|^2\right)|v|dr\tilde{\nu}(dx;v)dv\\
&\le Cm^{-1}B_m,
\end{aligned}$$

which gives us our first assertion.

Therefore, by (3.29), we get that

$$\begin{aligned}
&E^{P_m}\left[\left(\int_{\mathbf{R}\times E} 1_{[0,A]}(|x|)1_{\{|r|\le B_m\}}(1+|v|)\mu_\omega(dr,dx,dv)\right)^2\right]\\
&\le 2\int_{\mathbf{R}\times E} 1_{[0,A]}(|x|)1_{\{|r|\le B_m\}}(1+|v|)^2\lambda_m(dr,dx,dv)\\
&\quad +2\left(\int_{\mathbf{R}\times E} 1_{[0,A]}(|x|)1_{\{|r|\le B_m\}}(1+|v|)\lambda_m(dr,dx,dv)\right)^2\\
&\le 2Cm^{-1}B_m+2(Cm^{-1}B_m)^2.
\end{aligned}$$

So we get our second assertion by re-choosing C in an appropriate way. □

A.4 Several Estimates with Respect to Integrals Involving $t_1(v,*)$

As seen, the valid interaction time of each light particle is $t_1(v,|x-\pi_v^\perp X(\tilde{r})|)+\tau$, which is not bounded with respect to (x,v). However, by Lemma 4.5, this does not cause any problem after taking integrals.

Lemma 4.5 *Fix any $T>0$ and $n\in\mathbf{N}$. Then there exists a constant C such that*

1. *for any $\beta>0$ such that $2\varepsilon_1^{-1/2}\beta<d-1$, $u_0\in\mathbf{R}$, $k\in\{0,1,2\}$, $s\in[0,T\wedge\sigma_n]$ and any $u(r,s)\in[0,T\wedge\sigma_n]$,*

$$\begin{aligned}
&\int_{\mathbf{R}\times E} 1_{\{|x|\le R_0\}}1_{\{m^{-1/2}(s-r)-u_0\in[-\tau,t_1(v,|x-\pi_v^\perp X(u(r,s))|)]\}}t_1(v,|x-\pi_v^\perp X(u(r,s))|)^k\\
&\qquad \exp\Big(\beta t_1(v,|x-\pi_v^\perp X(u(r,s))|)\Big)(\lambda+\overline{\lambda})(dr,dx,dv)\le Cm^{-1/2},
\end{aligned}$$

2. *for any $-\infty<c_1<c_2<\infty$ (which may depend on m),*

$$\int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(\tilde{r})|\le m^\alpha\}}1_{\{r\in[c_1,c_2]\}}(|v|+1)^2(\lambda+\overline{\lambda})(dr,dx,dv)\le Cm^{\alpha(d-1)-1}(c_2-c_1),$$

3. *for any $s\in[0,T\wedge\sigma_n]$ and $u(r,s)\in[0,T\wedge\sigma_n]$,*

$$\begin{aligned}
&\int_{\mathbf{R}\times E} 1_{\{|x-\pi_v^\perp X(u(r,s))|\le 2m^\alpha\}}1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|x-\pi_v^\perp X(u(r,s))|)]\}}(\lambda+\overline{\lambda})(dr,dx,dv)\\
&\quad\times t_1\big(v,|x-\pi_v^\perp X(u(r,s))|\big)^k\exp\left(\frac{1}{2}(1+C_1)t_1(v,|x-\pi_v^\perp X(u(r,s))|)\ell\right)\\
&\quad\le Cm^{\alpha\left((d-1-\ell(1+C_1)\epsilon_1^{-1/2}\right)-\frac{1}{2}}\left(\log m^{-1}\right)^{k+1},\quad k,\ell\in\{0,1\},
\end{aligned}$$

4.

$$\int_E t_1(v,|x-\pi_v^\perp X|)^k 1_{\{|x-\pi_v^\perp X|\le R_U+1\}}\rho\left(\frac{1}{2}|v|^2\right)\nu(dx,dv)\le C,\quad k\in\{0,1,\cdots,7\}.$$

Proof The proofs of these assertions are similar – first apply the change of variable $y = x - \pi_v^{\perp} X(\cdot)$ for any fixed v and r, then apply Fubini's Theorem to switch the order of integrations with respect to y and r if necessary, and finally estimate the integral with respect to x. We present the proofs of (1) and (3) in the following. The proofs of (2) and (4) are similar but simpler, since we do not need Fubini's Theorem in these proofs, and we omit them here.

(1) By definition and (A3), we get that the left hand side of (1) is dominated by

$$m^{-1} \int_{\mathbf{R}^d} 2\rho_{max}\Big(\frac{1}{2}|v|^2\Big)|v|dv \int_{\mathbf{R}} dr \int_{E_v} \tilde{\nu}(dx; v) \\ \times 1_{\{|x-\pi_v^{\perp} X(u(r,s))| \le 2R_0\}} 1_{\{m^{-1/2}(s-r)-u_0 \in [-\tau, t_1(v, |x-\pi_v^{\perp} X(u(r,s))|)\}} \\ \times t_1\left(v, |x - \pi_v^{\perp} X(u(r,s))|\right)^k \exp\left(\beta t_1\big(v, |x - \pi_v^{\perp} X(u(r,s))|\big)\right).$$

Applying the change of variable $y = x - \pi_v^{\perp} X(u(r, s))$ for any fixed v and r, this is equal to

$$2m^{-1} \int_{\mathbf{R}^d} \rho_{max}\Big(\frac{1}{2}|v|^2\Big)|v|dv \int_{\mathbf{R}} dr \int_{E_v} \tilde{\nu}(dy; v) \\ 1_{\{|y| \le 2R_0\}} 1_{\{m^{-1/2}(s-r)-u_0 \in [-\tau, t_1(v,|y|))\}} t_1(v, |y|)^k \exp\Big(\beta t_1(v, |y|)\Big).$$

Applying Fubini's Theorem to switch the order of integrations with respect to x and r, since $t_1(v, *) \ge \tau$, this is dominated by

$$4m^{-1/2} \int_{\mathbf{R}^d} \rho_{max}\Big(\frac{1}{2}|v|^2\Big)|v|dv \int_{E_v} 1_{\{|y| \le 2R_0\}} t_1(v, |y|)^{k+1} \exp\Big(\beta t_1(v, |y|)\Big)\tilde{\nu}(dy; v).$$

So in order to prove (1), it suffices to prove that

$$\sup_{v \in \mathbf{R}^d} \int_{E_v} 1_{\{|y| \le 2R_0\}} t_1(v, |y|)^{k+1} \exp\Big(\beta t_1(v, |y|)\Big)\tilde{\nu}(dy; v) < \infty. \tag{5.2}$$

By the definition of $t_1(v, |y|)$ and the help of polar coordinates, it in turn suffice to prove that

$$\int_{\left[0, (2R_0) \wedge \frac{R_U}{2} \wedge (2\varepsilon_3)\right]} r^{-\beta 2\varepsilon_1^{-1/2} + d - 2} \left(\log \frac{1}{r}\right)^{k+1} dr < \infty. \tag{5.3}$$

On the other hand, by (2.12), we have that (5.3) holds as long as $-\beta 2\varepsilon_1^{-1/2} + d - 2 > -1$. This completes our proof of (1).

(3) By definition and (A3), we get that the left hand side of (3) is dominated by

$$2m^{-1} \int_{\mathbf{R}^d} \rho_{max}\Big(\frac{1}{2}|v|^2\Big)|v|dv \int_{\mathbf{R}} dr \int_{E_v} \tilde{\nu}(dx; v) \\ \times 1_{\{|x-\pi_v^{\perp} X(u(r,s))| \le 2m^{\alpha}\}} 1_{\{m^{-1/2}(s-r) \in [-\tau, t_1(v, |x-\pi_v^{\perp} X(u(r,s))|)\}} \\ \times t_1(v, |x - \pi_v^{\perp} X(u(r,s))|)^k \exp\left(\frac{1}{2}(1 + C_1) t_1(v, |x - \pi_v^{\perp} X(u(r,s))|)\ell\right).$$

Applying the change of variable $y = x - \pi_v^{\perp} X(u(r,s))$ for any fixed v and r, this is equal to

$$\begin{aligned}
&2m^{-1}\int_{\mathbf{R}^d}\rho_{max}\left(\frac{1}{2}|v|^2\right)|v|dv\int_{\mathbf{R}}dr\int_{E_v}\tilde{\nu}(dy;v)\\
&\quad\times 1_{\{|y|\le 2m^{\alpha}\}}1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|y|)\}}t_1(v,|y|)^k\exp\left(\frac{1}{2}(1+C_1)t_1(v,|y|)\ell\right)\\
&= 2m^{-1}\int_{\mathbf{R}^d}\rho_{max}\left(\frac{1}{2}|v|^2\right)|v|dv\int_{E_v}\tilde{\nu}(dy;v)\int_{\mathbf{R}}dr\\
&\quad\times 1_{\{|y|\le 2m^{\alpha}\}}1_{\{m^{-1/2}(s-r)\in[-\tau,t_1(v,|y|)\}}t_1(v,|y|)^k\exp\left(\frac{1}{2}(1+C_1)t_1(v,|y|)\ell\right)\\
&\le 4m^{-\frac{1}{2}}\int_{\mathbf{R}^d}\rho_{max}\left(\frac{1}{2}|v|^2\right)|v|dv\int_{E_v}\tilde{\nu}(dy;v)\\
&\quad\times 1_{\{|y|\le 2m^{\alpha}\}}t_1(v,|y|)^{k+1}\exp\left(\frac{1}{2}(1+C_1)t_1(v,|y|)\ell\right).
\end{aligned}$$

It suffices to consider the case where $m > 0$ is small enough such that $2m^{\alpha} \le \frac{R_U}{2} \wedge (2\varepsilon_3)$. So by the definition of $t_1(v,|y|)$, it suffices to prove the following:

$$\begin{aligned}
&\sup_{v\in\mathbf{R}^d}\int_{E_v}\tilde{\nu}(dy;v)1_{\{|y|\le 2m^{\alpha}\}}(\log|y|^{-1})^{k+1}\exp\left(\frac{1}{2}\ell(1+C_1)2\epsilon_1^{-1/2}\log\frac{2\varepsilon_3}{|y|}\right)\\
&\le Cm^{\alpha\left((d-1-\ell(1+C_1)\epsilon_1^{-1/2}\right)}\left(\log m^{-1}\right)^{k+1}. \qquad (5.4)
\end{aligned}$$

On the other hand, by changing variables to polar coordinates, the left hand side of (5.4) is equal to

$$C\int_0^{2m^{\alpha}} r^{d-2}r^{-\ell(1+C_1)\epsilon_1^{-1/2}}\left(\log r^{-1}\right)^{k+1}dr.$$

For any fixed $\ell \in \{0,1\}$, we have that $d-1-\ell(1+C_1)\epsilon_1^{-1/2} > 0$. Now we get our assertion by the well-known result that for any $p \ne -1$,

$$\begin{aligned}
&\int r^p\log r\,dr = \frac{r^{p+1}}{p+1}\left(\log r - \frac{1}{p+1}\right),\\
&\int r^p\left(\log r\right)^2dr = \frac{r^{p+1}}{p+1}\left(\left(\log r\right)^2 - \frac{2}{p+1}\log r + \frac{2}{(p+1)^2}\right).
\end{aligned}$$

□

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