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Critical Two-Point Function for Long-Range $O(n)$ Models Below the Upper Critical Dimension

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Abstract We consider the n-component $|\varphi|^4$ lattice spin model ($n \geq 1$) and the weakly self-avoiding walk ($n = 0$) on $\mathbb{Z}^d$, in dimensions $d = 1, 2, 3$. We study long-range models based on the fractional Laplacian, with spin-spin interactions or walk step probabilities decaying with distance r as $r^{-(d+\alpha)}$ with $\alpha \in (0, 2)$. The upper critical dimension is $d_c = 2\alpha$. For $\varepsilon > 0$, and $\alpha = \frac{1}{2}(d + \varepsilon)$, the dimension $d = d_c - \varepsilon$ is below the upper critical dimension. For small ε, weak coupling, and all integers $n \geq 0$, we prove that the two-point function at the critical point decays with distance as $r^{-(d-\alpha)}$. This "sticking" of the critical exponent at its mean-field value was first predicted in the physics literature in 1972. Our proof is based on a rigorous renormalisation group method. The treatment of observables differs from that used in recent work on the nearest-neighbour 4-dimensional case, via our use of a cluster expansion.

Keywords Renormalisation group · Critical phenomena · Two-point function · Spin systems · Self-avoiding walk

1 Introduction and Main Result

Broadly speaking, the mathematical understanding of critical phenomena for spin systems has progressed in dimension $d = 2$, where exact solutions and SLE are important tools; in dimensions $d > 4$, where infrared bounds and the lace expansion are useful; and in dimension $d = 4$, where renormalisation group (RG) methods have been applied. The physically most important case of $d = 3$ is more difficult, and mathematical methods are scarce.

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In the physics literature, the ε-expansion was introduced to study non-integer dimensions slightly below $d = 4$. An alternate approach is to consider long-range models, which change the upper critical dimension from $d_c = 4$ to a lower value $d_c = 2\alpha$ with $\alpha \in (0, 2)$. By choosing $d = 1, 2, 3$ and $\alpha = \frac{1}{2}(d + \varepsilon)$ with small ε, it is possible to study integer dimension d which is slightly below the upper critical dimension $2\alpha = d + \varepsilon$. In this paper, we consider n-component spins and the weakly self-avoiding walk in this long-range context, and prove that the critical two-point function has mean-field decay $r^{-(d-\alpha)}$ also *below* the upper critical dimension. Our method involves a RG analysis in the vicinity of a non-Gaussian fixed point.

1.1 Introduction

We consider long-range $O(n)$ models on $\mathbb{Z}^d$ for integers $n \geq 0$ and dimensions $d = 1, 2, 3$. The case $n = 0$ is the continuous-time weakly self-avoiding walk, and the case $n \geq 1$ is the n-component $|\varphi|^4$ lattice spin model. For $n = 0$ the underlying random walk model takes steps of length r with probabilities decaying as $r^{-(d+\alpha)}$ with $\alpha \in (0, 2)$, and for $n \geq 1$ the spin-spin interaction in the Hamiltonian has that same decay. More precisely, the models are based on the fractional Laplacian $(-\Delta)^{\alpha/2}$, whose kernel decays at large distance as $r^{-(d+\alpha)}$.

The upper critical dimension is predicted to be $d_c = 2\alpha$ for all $n \geq 0$. Thus, for $\alpha < \frac{d}{2}$, mean-field behaviour is predicted; this has been proved for self-avoiding walk, for the Ising model, for the 1-component φ^4 model, and for other models [3,17,22,23]. In the physics literature, it is observed that below the upper critical dimension the critical two-point function continues to exhibit the mean-field decay $r^{-(d-\alpha)}$ for $\alpha \in (\frac{d}{2}, 2 - \eta)$, and then crosses over to $r^{-(d-2+\eta)}$ decay for $\alpha \in (2 - \eta, 2)$. Here η is the exponent for the nearest-neighbour model; for $n = 1$ this is $\eta = \frac{1}{4}$ for $d = 2$ [35], and a recent estimate for $d = 3$ is $\eta = 0.03631(3)$ [18]. The earliest paper to elucidate the critical behaviour of long-range models is [20], with [33] roughly contemporaneous and [29] providing further development. A very recent paper which analyses the crossover for the two-point function in detail for $n = 1$ is [8]. At the crossover, when $\alpha = \alpha_* = 2 - \eta$, a logarithmic correction is predicted, with overall decay $\frac{1}{r^{d-\alpha_*}} \frac{1}{\log r}$ [8,11]. The relationship with conformal invariance is explained in [28].

Let $n = 0, 1, 2, \ldots$; $d = 1, 2, 3$; and $\alpha = \frac{1}{2}(d + \varepsilon)$. We use a rigorous RG argument to prove that for small $\varepsilon > 0$, the critical two-point function has decay $r^{-(d-\alpha)}$. This proves the "sticking" of the critical exponent at its mean-field value, for α slightly above $\frac{d}{2}$, or equivalently, for d slightly below the upper critical dimension $d_c = 2\alpha$. Our proof extends recent results and methods used to study the ε-expansion for the critical exponents for the susceptibility and specific heat of the long-range models [31]. It also relies on results and techniques developed to study related problems for the 4-dimensional nearest-neighbour models [5,16,32]. However, our treatment of observables differs from that used in the 4-dimensional case, via our application of a cluster expansion.

Earlier mathematical work which applies RG methods to long-range models includes the construction of global RG trajectories for $n = 0$ and $d = 3$ [27], and for a continuum version of the $n = 1$ model in [1,12]. These references do not study critical exponents. The exponents for critical correlations in a certain hierarchical version of the model, for $d = 3$ and $n = 1$, are computed in [2]. For a closely related continuum model with $n = 1$ in dimensions $d = 2, 3$, a proof of the "sticking" of the critical exponent for the critical two-point function was announced in a 2013 lecture [24].

1.2 Fractional Laplacian

The models we study are defined in terms of the fractional Laplacian. We now define the fractional Laplacian and list some of its properties. Further details can be found in [31, Sects. 2–3].

Let $d \geq 1$ and $\alpha \in (0, 2)$. We write $|x|$ for the Euclidean norm of $x \in \mathbb{Z}^d$. Let J be the $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix with $J_{xy} = 1$ if $|x-y| = 1$, and otherwise $J_{xy} = 0$. Let I denote the identity matrix. The lattice Laplacian on $\mathbb{Z}^d$ is $\Delta = J - 2dI$. For $k = (k_1, \dots, k_d) \in [-\pi, \pi]^d$, let

$$\lambda(k) = 4 \sum_{j=1}^{d} \sin^2(k_j/2) = 2 \sum_{j=1}^{d} \left(1 - \cos k_j\right). \tag{1.1}$$

The matrix element $-\Delta_{x,y}$ can be written as the Fourier integral

$$-\Delta_{x,y} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \lambda(k) e^{ik\cdot(x-y)} dk. \tag{1.2}$$

The fractional Laplacian is the matrix $(-\Delta)^{\alpha/2}$ defined by

$$(-\Delta)^{\alpha/2}_{x,y} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \lambda(k)^{\alpha/2} e^{ik\cdot(x-y)} dk. \tag{1.3}$$

For $|x-y| \to \infty$, the fractional Laplacian decays as

$$-(-\Delta)^{\alpha/2}_{x,y} \asymp |x-y|^{-(d+\alpha)} \tag{1.4}$$

(see [31, Lemma 2.1], or [10, Theorem 5.3] for a more precise and more general statement). Here, and in the following, we write $a \asymp b$ to denote the existence of $c > 0$ such that $c^{-1}b \leq a \leq cb$. For $d \geq 1$, $\alpha \in (0, 2 \wedge d)$, $\bar{m}^2 > 0$, $m^2 \in [0, \bar{m}^2]$, and $x \neq 0$, the resolvent obeys

$$\left((-\Delta)^{\alpha/2} + m^2\right)^{-1}_{0,x} \leq c \frac{1}{|x|^{d-\alpha}} \frac{1}{1 + m^4 |x|^{2\alpha}}, \tag{1.5}$$

with c depending on $d, \alpha, \bar{m}^2$ (see [31, Lemma 3.2]). For $m^2 = 0$, an asymptotic formula

$$\left((-\Delta)^{\alpha/2}\right)^{-1}_{0,x} \sim c_{d,\alpha} \frac{1}{|x|^{d-\alpha}} \tag{1.6}$$

is proven in [9, Theorem 2.4], with precise constant $c_{d,\alpha}$.

Given integers $L, N > 1$, let $\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ denote the d-dimensional discrete torus of side length L^N. The torus fractional Laplacian is defined by

$$\left(-\Delta_{\Lambda_N}\right)^{\alpha/2}_{x,y} = \sum_{z\in\mathbb{Z}^d} (-\Delta)^{\alpha/2}_{x,y+zL^N} \quad (x, y \in \Lambda_N). \tag{1.7}$$

The sum on the right-hand side of (1.7) converges, by (1.4).

1.3 The $|\varphi|^4$ Model

We first define the model on the torus $\Lambda = \Lambda_N$, as usual for spin systems. Let $d \geq 1$ and $\alpha \in (0, 2)$. Let $n \geq 1$. The *spin field* φ is a function $\varphi : \Lambda \to \mathbb{R}^n$, denoted $x \mapsto \varphi_x$, which we may regard as an element $\varphi \in (\mathbb{R}^n)^\Lambda$. The Euclidean norm of $v = (v^1, \dots, v^n) \in \mathbb{R}^n$ is $|v| = [\sum_{i=1}^n (v^i)^2]^{1/2}$, with inner product $v \cdot w = \sum_{i=1}^n v^i w^i$. We extend the action of the fractional Laplacian to act on the spin field component-wise, namely $((-\Delta_\Lambda)^{\alpha/2}\varphi)^i_x = \sum_{y\in\Lambda}(-\Delta_\Lambda)^{\alpha/2}_{x,y}\varphi^i_y$.

Given $g > 0$ and $\nu \in \mathbb{R}$, we define the interaction $V : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$ by

$$V(\varphi) = \sum_{x\in\Lambda}\left(\tfrac{1}{4}g|\varphi_x|^4 + \tfrac{1}{2}\nu|\varphi_x|^2 + \tfrac{1}{2}\varphi_x \cdot ((-\Delta_\Lambda)^{\alpha/2}\varphi)_x\right). \tag{1.8}$$

The *partition function* is defined by

$$Z_{g,\nu,N} = \int_{(\mathbb{R}^n)^\Lambda} e^{-V(\varphi)}d\varphi, \tag{1.9}$$

where $d\varphi$ is the Lebesgue measure on $(\mathbb{R}^n)^\Lambda$. The expectation of a random variable $F : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$ is

$$\langle F\rangle_{g,\nu,N} = \frac{1}{Z_{g,\nu,N}}\int_{(\mathbb{R}^n)^\Lambda} F(\varphi)e^{-V(\varphi)}d\varphi. \tag{1.10}$$

Given lattice points a, b, we define the finite- and infinite-volume *two-point function* by

$$G_{a,b,N}(g,\nu;n) = \langle\varphi^1_a\varphi^1_b\rangle_{g,\nu,N} = \frac{1}{n}\langle\varphi_a\cdot\varphi_b\rangle_{g,\nu,N}, \tag{1.11}$$

$$G_{a,b}(g,\nu;n) = \lim_{N\to\infty} G_{a,b,N}(g,\nu;n). \tag{1.12}$$

On the left-hand side of (1.12) we have $a, b \in \mathbb{Z}^d$, and on the right-hand side we identify these points with elements of Λ_N for large N, by regarding the vertices of Λ_N as a cube in $\mathbb{Z}^d$ (without boundaries identified) approximately centred at the origin. The *susceptibility* is defined by

$$\chi(g,\nu;n) = \lim_{N\to\infty}\sum_{b\in\Lambda_N} G_{a,b,N}(g,\nu;n) \tag{1.13}$$

and can be used to identify the critical point of the model. By translation invariance, χ is independent of a. Existence of the infinite volume limits in (1.12)–(1.13), in our context, is discussed below.

1.4 Weakly Self-Avoiding Walk

Let $d \geq 1$ and $\alpha \in (0, 2)$. Let X denote the continuous-time Markov chain with state space $\mathbb{Z}^d$ and infinitesimal generator $Q = -(-\Delta_{\mathbb{Z}^d})^{\alpha/2}$. Verification that Q has the attributes required of a generator is given in [31, Lemma 2.4]. Let P be the probability measure associated with X, and E the corresponding expectation; a subscript a specifies $X(0) = a$. The transition probabilities are given by

$$P_a(X(t) = b) = E_a(\mathbb{1}_{X(t)=b}) = (e^{tQ})_{a,b}. \tag{1.14}$$

The *local time* of X at x up to time T is the random variable $L^x_T = \int_0^T \mathbb{1}_{X(t)=x}\,dt$. The *self-intersection local time* up to time T is the random variable

$$I_T = \sum_{x\in\mathbb{Z}^d}\left(L^x_T\right)^2 = \int_0^T\int_0^T \mathbb{1}_{X(t_1)=X(t_2)}\,dt_1\,dt_2. \tag{1.15}$$

Given $g > 0$, $\nu \in \mathbb{R}$, and $a, b \in \mathbb{Z}^d$, the continuous-time weakly self-avoiding walk *two-point function* is defined by the integral

$$G_{a,b}(g, \nu; 0) = \int_0^\infty E_a\left(e^{-gI_T}\mathbb{1}_{X(T)=b}\right)e^{-\nu T}\,dT, \tag{1.16}$$

and the *susceptibility* is defined by

$$\chi(g, \nu; 0) = \sum_{b\in\mathbb{Z}^d} G_{a,b}(g, \nu; 0) = \int_0^\infty E_a(e^{-gI_T})e^{-\nu T}\,dT. \tag{1.17}$$

The labels 0 on the left-hand sides of (1.16)–(1.17) reflect the fact that the weakly self-avoiding walk corresponds to the formal $n = 0$ case of the n-component $|\varphi|^4$ model. As in earlier work on the 4-dimensional case, e.g., [31,32], we treat both cases $n \geq 1$ (spins) and $n = 0$ (self-avoiding walk) simultaneously and rigorously, via a supersymmetric spin representation for the weakly self-avoiding walk.

1.5 Susceptibility and Critical Point

Let $d = 1, 2, 3$; $n \geq 0$; L be sufficiently large; $\varepsilon > 0$ be sufficiently small; and $\alpha = \frac{1}{2}(d+\varepsilon)$. Let $\tau^{(\alpha)}$ denote the diagonal element of the Green function, i.e., $\tau^{(\alpha)} = ((-\Delta)^{\alpha/2})^{-1}_{0,0}$. One of the main results of [31] is that there exists $\bar{s} \asymp \varepsilon$ such that, for $g \in [\frac{63}{64}\bar{s}, \frac{65}{64}\bar{s}]$, there exist $\nu_c = \nu_c(g; n) = -(n+2)\tau^{(\alpha)}g(1 + O(g))$ and $C > 0$ such that for $\nu = \nu_c + t$ with $t \downarrow 0$,

$$C^{-1}t^{-(1+\frac{n+2}{n+8}\frac{\varepsilon}{\alpha}-C\varepsilon^2)} \leq \chi(g, \nu; n) \leq Ct^{-(1+\frac{n+2}{n+8}\frac{\varepsilon}{\alpha}+C\varepsilon^2)}. \tag{1.18}$$

This is a statement that there is a critical point at $\nu = \nu_c$, and that the critical exponent γ exists to order ε, with

$$\gamma = 1 + \frac{n+2}{n+8}\frac{\varepsilon}{\alpha} + O(\varepsilon^2) \qquad (n \geq 0). \tag{1.19}$$

It is part of the statement that for $n \geq 1$ the susceptibility is given by the infinite-volume limit (1.13), under the above hypotheses. The critical exponent for the specific heat is also computed to order ε in [31], for $n \geq 1$.

1.6 Main Result

Our main result is the following theorem, which shows that just below the upper critical dimension, the exponent for the critical two-point function "sticks" at its mean-field value (see (1.6)), as predicted by [20]. The theorem applies for all $n \geq 0$, including the case $n = 0$ of the weakly self-avoiding walk. The critical value $\nu_c = \nu_c(g; n)$ is the one mentioned in Sect. 1.5. As part of the proof of the theorem, it is shown that for $n \geq 1$ the infinite-volume limit (1.12) exists for $\nu = \nu_c$.

Theorem 1.1 *Let $d = 1, 2, 3$; $n \geq 0$; L be sufficiently large; $\varepsilon > 0$ be sufficiently small; and $\alpha = \frac{1}{2}(d+\varepsilon)$. For $g \in [\frac{63}{64}\bar{s}, \frac{65}{64}\bar{s}]$ the critical two-point function obeys, as $|a-b| \to \infty$,*

$$G_{a,b}(g, \nu_c; n) = (1 + O(\varepsilon))((-\Delta)^{\alpha/2})^{-1}_{a,b} \asymp \frac{1}{|a-b|^{d-\alpha}}. \tag{1.20}$$

Note that Theorem 1.1 identifies the constant in the decay of the interacting two-point function only up to an error of order ε. However, the error is uniformly bounded in a, b, so the power in the decay rate takes its mean-field value, and this is true to *all* orders in ε.

1.7 Strategy of Proof

The proof is based on a rigorous RG method developed in a series of papers by Bauerschmidt, Brydges and Slade, where the focus is on the nearest-neighbour models in dimension 4. The method is adapted to the long-range setting in [31].

Fix g as in the statement of Theorem 1.1. In [31], given small $m^2 > 0$, a critical value $\nu_0^c(m^2)$ is constructed, with the property that the critical point ν_c is given by $\nu_c = \lim_{m^2\downarrow 0} \nu_0^c(m^2)$. Let

$$U_0 = \sum_{x\in\Lambda} \left(\tfrac{1}{4}g|\varphi_x|^4 + \tfrac{1}{2}\nu_0^c|\varphi_x|^2\right) - \sigma_a\varphi_a^1 - \sigma_b\varphi_b^1. \tag{1.21}$$

For $x = a, b$, let $D_{\sigma_x} = \frac{\partial}{\partial\sigma_x}|_{\sigma_a=\sigma_b=0}$. For $n \geq 1$, the two-point function obeys

$$G_{a,b,N}(g, \nu_0^c(m^2) + m^2; n) = D_{\sigma_a} D_{\sigma_b} \log \mathbb{E}_C e^{-U_0}, \tag{1.22}$$

where $\mathbb{E}_C$ denotes Gaussian expectation with covariance $C = ((-\Delta_\Lambda)^{\frac{\alpha}{2}} + m^2)^{-1}$ ($m^2 > 0$ ensures existence of the inverse). Thus the two-point function is interpreted as a perturbation of a Gaussian expectation. A similar representation is valid for the weakly self-avoiding walk, using a Gaussian superexpectation.

Perturbation theory is performed inductively in a multi-scale fashion, using a finite-range decomposition $C = C_1 + \cdots + C_N$, with C_j of range $\sim L^j$. This is implemented via the Gaussian convolution identity $\mathbb{E}_C\theta = \mathbb{E}_{C_N}\theta \circ \cdots \circ \mathbb{E}_{C_1}\theta$, where $\mathbb{E}_C\theta$ denotes Gaussian convolution. At every step in the induction, we get a representation

$$\mathbb{E}_{C_j}\theta \circ \cdots \circ \mathbb{E}_{C_1}\theta e^{-U_0} \approx e^{-U_j}, \tag{1.23}$$

where the polynomial

$$U_j = u_j|\Lambda| + \sum_{x\in\Lambda} \left(\tfrac{1}{4}g_j|\varphi_x|^4 + \tfrac{1}{2}\nu_j|\varphi_x|^2\right) - \lambda_{a,j}\sigma_a\varphi_a^1 - \lambda_{b,j}\sigma_b\varphi_b^1 - \frac{1}{2}(q_{a,j} + q_{b,j})\sigma_a\sigma_b \tag{1.24}$$

includes all Euclidean- and $O(n)$-invariant monomials that are relevant and marginal according to the RG philosophy. The error in this approximation is irrelevant in the RG sense and is controlled uniformly in the volume by parametrising it as a polymer gas. According to (1.23), after the final step of the induction has been performed, we obtain

$$G_{a,b,N}(g, \nu_0^c + m^2; n) = D_{\sigma_a} D_{\sigma_b} \log \mathbb{E}_C e^{-U_0} \approx -D_{\sigma_a} D_{\sigma_b} U_N|_{\varphi=0} = \frac{1}{2}(q_{a,N} + q_{b,N}). \tag{1.25}$$

To control $q_{x,N}$ ($x = a, b$), we need to study the RG dynamical system

$$(g_j, \nu_j, u_j, \lambda_{x,j}, q_{x,j}) \to (g_{j+1}, \nu_{j+1}, u_{j+1}, \lambda_{x,j+1}, q_{x,j+1}), \tag{1.26}$$

and its non-perturbative corrections. The initial condition is $(g_0, \nu_0, u_0, \lambda_{x,0}, q_{x,0}) = (g, \nu_0^c, 0, 1, 1, 0, 0)$. (In fact, the coupling constant u_j does not play an important role for the two-point function.) For $d = 4$, the dynamical system has a Gaussian fixed point. We use the adaptation of the RG method, as developed in [31], to the long-range setting below the upper critical dimension, where the fixed point is instead non-Gaussian. In [31] only the flow of g_j, ν_j, u_j was studied and λ_j, q_j did not appear, but the flow of g_j, ν_j, u_j remains identical when these additional coupling constants do appear. For the nearest-neighbour model on

$\mathbb{Z}^4$, the RG method was applied in [5,32] to prove $|a-b|^{-2}$ decay of the critical two-point function for all $n \geq 0$. We mainly follow the approach of [5,32]. In particular, our treatment of the flow of $q_{x,j}$ remains the same and yields

$$q_{x,j} \approx \lambda_{a,j_{ab}} \lambda_{b,j_{ab}} w_{j;a,b}, \tag{1.27}$$

where $w_j = \sum_{k=1}^{j} C_k$, and where $j_{ab} = \lfloor \log_L(2|a-b|) \rfloor$ is the *coalescence scale* defined to ensure that $C_{k;a,b} = 0$ when $k \leq j_{ab}$. By definition of j_{ab}, the right-hand side of (1.27) is zero for j below the coalescence scale, and this remains true non-perturbatively as well: $q_{x,j} = 0$ for scales $j \leq j_{ab}$.

The flow of λ_j was analysed recursively for the Gaussian RG fixed point in [5,32], but for the non-Gaussian fixed point in our current setting the recursive analysis cannot be applied due to the non-summability of remainder terms, and a different approach is needed. Let $\bar{D} = \sum_{x \in \Lambda} \frac{\partial}{\partial \varphi_x^1}|_{\varphi=0}$ and $\bar{D}^2 = \sum_{x,y \in \Lambda} \frac{\partial^2}{\partial \varphi_x^1 \partial \varphi_y^1}|_{\varphi=0}$. According to (1.23),

$$\lambda_{a,j} = e^{u_j |\Lambda|} \bar{D} D_{\sigma_a} e^{-U_j} \approx e^{u_j |\Lambda|} \bar{D} D_{\sigma_a} \mathbb{E}_{w_j} \theta e^{-U_0}. \tag{1.28}$$

Let $w_j^{(1)} = \sum_{x \in \Lambda} w_{j;a,x}$, which is independent of a. Using Gaussian integration by parts and translation invariance, we show in (4.7) that

$$\bar{D} D_{\sigma_a} \mathbb{E}_{w_j} \theta e^{-U_0} = \mathbb{E}_{w_j} e^{-U_0|_{\sigma=0}} + \frac{1}{|\Lambda|} w_j^{(1)} \bar{D}^2 \mathbb{E}_{w_j} \theta e^{-U_0|_{\sigma=0}}. \tag{1.29}$$

By using (1.23) to evaluate the two terms in the above right-hand side approximately, we thus obtain

$$\lambda_{a,j} \approx 1 + w_j^{(1)} \nu_j. \tag{1.30}$$

This relates λ_j to the *bulk* coupling constants g_j, ν_j whose flow is known from [31]. In particular, it is shown in [31] that $w_j^{(1)} \nu_j = O(\varepsilon)$. All of the above is carried out uniformly in m^2, which permits the limit $m^2 \downarrow 0$ to be taken after the infinite-volume limit. Since $\lim_{m^2 \downarrow 0} \lim_{N \to \infty} w_{N:a,b} = ((-\Delta_{\mathbb{Z}^d})^{\alpha/2})^{-1}_{a,b} \asymp |a-b|^{-(d-\alpha)}$, all this, together with the rigorous versions of (1.27) and (1.25), implies our main result (1.20). The non-perturbative corrections to (1.30) due to the irrelevant error coordinate are controlled using a cluster expansion. This is the main innovation in the proof of Theorem 1.1.

The remainder of the paper is organised as follows. In Sect. 2, we provide some background and definitions needed for the RG method. In Sect. 3, we formulate the RG map and state the main theorem which provides estimates on the RG map; this is an adaptation of the main result of [16] as applied to the long-range model in [31]. The main difference, compared to [31], is the inclusion of observables in the RG map. The flow of the observable coupling constant λ_j is analysed in Sect. 4. The flow of the observable coupling constant q_j is then analysed in Sect. 5, where the proof of Theorem 1.1 is completed.

2 Set-Up for RG Method

In this section, we summarise some notation and background for the RG method, needed for the proof of Theorem 1.1. Additional details can be found in [31].

2.1 Formula for Two-Point Function

We begin with a formula for the two-point function that serves as our starting point.

2.1.1 The Case $n \geq 1$

For $n \geq 1$, we define

$$\tau_x = \frac{1}{2}|\varphi_x|^2 \qquad (n \geq 1). \tag{2.1}$$

Given $g > 0$, $\nu \in \mathbb{R}$, $m^2 > 0$, we set

$$g_0 = g, \quad \nu_0 = \nu - m^2. \tag{2.2}$$

Given $a, b \in \Lambda$, we introduce *observable fields* $\sigma_a, \sigma_b \in \mathbb{R}$, and define V_0 and Z_0 by

$$V_0(\varphi_x) = g_0\tau_x^2 + \nu_0\tau_x - \sigma_a\varphi_a^1 \mathbb{1}_{x=a} - \sigma_b\varphi_b^1 \mathbb{1}_{x=b}, \quad Z_0(\varphi) = e^{-V_0(\Lambda)}, \tag{2.3}$$

with $V_0(\Lambda) = \sum_{x\in\Lambda} V_0(\varphi_x)$.

Given a $\Lambda \times \Lambda$ covariance matrix w, let $\mathbb{E}_w$ denote the Gaussian expectation with covariance w. Let $C = ((-\Delta_{\Lambda_N})^{\alpha/2} + m^2)^{-1}$. By shifting part of the $|\varphi|^2$ term into the covariance, the expectation (1.10) can be rewritten as

$$\langle F\rangle_{g,\nu,N} = \frac{\mathbb{E}_C F e^{-V_0^{\varnothing}(\Lambda)}}{\mathbb{E}_C e^{-V_0^{\varnothing}(\Lambda)}}, \tag{2.4}$$

where $V_0^{\varnothing}(\Lambda)$ denotes the evaluation of $V_0(\Lambda)$ at $\sigma_a = \sigma_b = 0$. When F is a monomial, it is standard to write this ratio of expectations as a derivative of a logarithmic generating function. Let D_{σ_a} denote the operator $\frac{\partial}{\partial\sigma_a}|_{\sigma_a=\sigma_b=0}$, and similarly for higher derivatives. Then the two-point function is given, for $n \geq 1$, by

$$G_{a,b,N}(g,\nu;n) = \langle\varphi_a^1\varphi_b^1\rangle_{g,\nu,N} = D^2_{\sigma_a\sigma_b} \log \mathbb{E}_C e^{-V_0(\Lambda)}. \tag{2.5}$$

2.1.2 The Case $n = 0$

For $n = 0$, as in several previous papers (e.g., [5,6,32]) we formulate the weakly self-avoiding walk model as the infinite-volume limit of a *supersymmetric* version of the $|\varphi|^4$ model. The supersymmetric model involves a complex boson field $(\phi_x, \bar\phi_x)_{x\in\Lambda}$ and a fermion field given by the 1-forms $\psi_x = \frac{1}{\sqrt{2\pi i}}d\phi_x$, $\bar\psi_x = \frac{1}{\sqrt{2\pi i}}d\bar\phi_x$. For $n = 0$, in place of (2.1), we set

$$\tau_x = \phi_x\bar\phi_x + \psi_x \wedge \bar\psi_x \qquad (n = 0), \tag{2.6}$$

and we replace φ_a^1, φ_b^1 in (2.3) by $\bar\phi_a, \phi_b$.

For $n = 0$, a formula closely related to (2.5) is given, e.g., in [32, (6.5)], with $\mathbb{E}_C$ in (2.5) replaced by the Gaussian superexpectation. As in [32], our formalism applies to the supersymmetric model with only notational changes, with n interpreted as $n = 0$ in formulas such as (1.19), and with the Gaussian expectation replaced by a superexpectation. For notational simplicity, we concentrate throughout the paper on the case $n \geq 1$.

2.2 Progressive Integration

In our version of the RG method, the expectation $\mathbb{E}_C e^{-V_0(\Lambda)}$ of (2.5) is evaluated in a multi-scale fashion, via a finite-range decomposition of the covariance C. We use the same finite-range decomposition $C = C_1 + C_2 + \cdots + C_{N-1} + C_{N,N}$ of the covariance $C = ((-\Delta_{\Lambda_N})^{\alpha/2} + m^2)^{-1}$ that is described and analysed in [31, Sect. 3]. A closely related decomposition was first introduced in [25] and subsequently developed in [26]. The covariances C_j are translation invariant, and have the *finite-range* property

$$C_{j;x,y} = 0 \quad \text{if } |x-y| \geq \frac{1}{2}L^j. \tag{2.7}$$

Thus, we may regard C_j either as a covariance on $\mathbb{Z}^d$ or on Λ_N, as long as $N > j$. Viewing the C_j as covariances on $\mathbb{Z}^d$, we also have a decomposition of the infinite-volume covariance given by $((-\Delta)^{\alpha/2}+m^2)^{-1} = \sum_{j=1}^{\infty} C_j$. We leave implicit the dependence of the covariance C_j on m^2. According to [31, (3.11)], for m^2 bounded, the covariances C_j satisfy the estimates

$$|C_{j;x,y}| \leq cL^{-(d-\alpha)(j-1)}(1+m^4L^{2\alpha(j-1)})^{-1}. \tag{2.8}$$

For $n \geq 1$, and for an integrable $F : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$, we define the convolution $\mathbb{E}_C\theta F$ by

$$(\mathbb{E}_C\theta F)(\varphi) = \mathbb{E}_C F(\varphi+\zeta), \tag{2.9}$$

where the expectation $\mathbb{E}_C$ on the right-hand side acts on ζ and leaves φ fixed. A similar construction is used for $n=0$ (see, e.g., [6, Sect. 4.1]). By [13, Proposition 2.6], the Gaussian convolution can be evaluated as

$$\mathbb{E}_C\theta F = \big(\mathbb{E}_{C_N}\theta \circ \mathbb{E}_{C_{N-1}}\theta \circ \cdots \circ \mathbb{E}_{C_1}\theta\big)F, \tag{2.10}$$

with an abuse of notation where C_N means $C_{N,N}$. To compute the expectation $\mathbb{E}_C e^{-V_0(\Lambda)}$ in (2.5), we use (2.10) to evaluate $\mathbb{E}_C\theta e^{-V_0(\Lambda)}$ progressively, as follows. We write $\mathbb{E}_j = \mathbb{E}_{C_j}$ and let

$$Z_{j+1} = \mathbb{E}_{j+1}\theta Z_j \qquad (0 \leq j < N), \tag{2.11}$$

with $Z_0 = e^{-V_0(\Lambda)}$ as in (2.3). By (2.10), $\mathbb{E}_C Z_0$ is obtained by setting $\varphi = 0$ in

$$Z_N = \mathbb{E}_C\theta Z_0. \tag{2.12}$$

This leads us to study the recursion $Z_j \mapsto Z_{j+1}$.

2.3 Function Space

The observable fields σ_a, σ_b are needed only for the purpose of evaluating the second derivative in (2.5). Therefore, dependence on the observable fields which is higher order than quadratic plays no role. We make use of this by defining the function space $\mathcal{N}$ as explained below. We also define the T_φ seminorm on $\mathcal{N}$. These definitions are as in, e.g., [14,32]. We focus on the case $n \geq 1$; the modifications needed for $n=0$ are as in, e.g., [15].

2.3.1 The Space $\mathcal{N}$

Given $p_{\mathcal{N}} > 0$, let

$$\mathcal{N}^\varnothing = C^{p_{\mathcal{N}}}((\mathbb{R}^n)^\Lambda). \tag{2.13}$$

As in [31, Sect. 6.2.1], we fix any $p_{\mathcal{N}} \geq 10$. For $n=0$, $\mathcal{N}^\varnothing$ is instead a space of even differential forms with $p_{\mathcal{N}}$-times differentiable coefficients.

In order to treat functions of the observable fields σ_a, σ_b, we define an extension $\mathcal{N}$ of $\mathcal{N}^\varnothing$ exactly as in [32, Sect. 2.4.1]. Namely, let $\mathcal{N}'$ be the space of real-valued functions of $\varphi, \sigma_a, \sigma_b$ which are $C^{p_{\mathcal{N}}}$ in φ and C^∞ in σ_a, σ_b. An ideal $\mathcal{I}$ in $\mathcal{N}'$ is formed by those elements of $\mathcal{N}'$ whose formal power series expansion in the observable fields to order $1, \sigma_a, \sigma_b, \sigma_a\sigma_b$

is equal to zero. We define $\mathcal{N}$ as the quotient algebra $\mathcal{N} = \mathcal{N}'/\mathcal{I}$. Then $\mathcal{N}$ has a direct sum decomposition

$$\mathcal{N} = \mathcal{N}'/\mathcal{I} = \mathcal{N}^{\varnothing} \oplus \mathcal{N}^{a} \oplus \mathcal{N}^{b} \oplus \mathcal{N}^{ab}, \tag{2.14}$$

where elements of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$ are given by elements of $\mathcal{N}^{\varnothing}$ multiplied by σ_a, by σ_b, and by $\sigma_a\sigma_b$ respectively. Thus, elements of $\mathcal{N}$ can be identified with polynomials over $\mathcal{N}^{\varnothing}$ in the observable fields with terms only of order $1, \sigma_a, \sigma_b, \sigma_a\sigma_b$, i.e. $F \in \mathcal{N}$ can be written as

$$F = F_{\varnothing} + \sigma_a F_a + \sigma_b F_b + \sigma_a\sigma_b F_{ab} \tag{2.15}$$

with $F_c \in \mathcal{N}^{\varnothing}$ for each $c \in \{\varnothing, a, b, ab\}$. There are natural projections $\pi_c : \mathcal{N} \to \mathcal{N}^c$ defined for such F by $\pi_{\varnothing} F = F_{\varnothing}$, $\pi_a F = \sigma_a F_a$, $\pi_b F = \sigma_b F_b$, and $\pi_{ab} F = \sigma_a\sigma_b F_{ab}$. We set $\pi_* = 1 - \pi_{\varnothing}$. The expectation $\mathbb{E}_C$ acts term-by-term on $F \in \mathcal{N}$, namely $\mathbb{E}_C F = \mathbb{E}_C F_{\varnothing} + \sigma_a \mathbb{E}_C F_a + \sigma_b \mathbb{E}_C F_b + \sigma_a\sigma_b \mathbb{E}_C F_{ab}$ for F as in (2.15).

2.3.2 Seminorms

A family of seminorms is used to control the size of elements of $\mathcal{N}$. Let Λ^* denote the set of sequences of any finite length (including length 0), composed of elements of $\Lambda \times \{1, \dots, n\}$. Let $\varphi \in (\mathbb{R}^n)^{\Lambda}$ be a field, and let $F \in \mathcal{N}^{\varnothing}$. Given $\vec{x} = ((x_1, i_1), \dots, (x_p, i_p)) \in \Lambda^*$, we write $|\vec{x}| = p$ and let

$$F_{\vec{x}}(\varphi) = \frac{\partial^p F(\varphi)}{\partial \varphi_{x_1}^{i_1} \dots \partial \varphi_{x_p}^{i_p}}. \tag{2.16}$$

A *test function* g is a mapping $g : \Lambda^* \to \mathbb{R}$, written $\vec{x} \mapsto g_{\vec{x}}$. We define the *$\varphi$-pairing* of F with a test function g by

$$\langle F, g \rangle_{\varphi} = \sum_{|\vec{x}| \le p_{\mathcal{N}}} \frac{1}{|\vec{x}|!} F_{\vec{x}}(\varphi) g_{\vec{x}}. \tag{2.17}$$

Given a parameter $\mathfrak{h}_j > 0$, a scale-dependent norm $\|g\|_{\Phi} = \|g\|_{\Phi_j(\mathfrak{h}_j)}$ is defined on test functions in [31, (6.8)]. The $\Phi = \Phi_j(\mathfrak{h}_j)$ norm controls the size of a test function and its discrete gradients up to order $p_{\Phi} = 4$, but its precise definition is immaterial for the present discussion. With $B_{\Phi}(1)$ the unit ball in Φ, we define the $T_{\varphi} = T_{\varphi,j}(\mathfrak{h}_j)$ seminorm on $\mathcal{N}^{\varnothing}$ by

$$\|F\|_{T_{\varphi}} = \sup_{g \in B_{\Phi}(1)} |\langle F, g \rangle_{\varphi}|. \tag{2.18}$$

Given an additional parameter $\mathfrak{h}_{\sigma} = \mathfrak{h}_{\sigma,j}$, we extend this definition to all of $\mathcal{N}$ exactly as in [14], i.e., the seminorm of F of the form (2.15) is defined to be

$$\|F\|_{T_{\varphi}} = \|F_{\varnothing}\|_{T_{\varphi}} + \mathfrak{h}_{\sigma}(\|F_a\|_{T_{\varphi}} + \|F_b\|_{T_{\varphi}}) + \mathfrak{h}_{\sigma}^2 \|F_{ab}\|_{T_{\varphi}}. \tag{2.19}$$

2.4 Blocks, Polymers and Scales

2.4.1 Blocks and Polymers

The finite-range covariance decomposition is well-suited to a block decomposition of the torus Λ_N of period L^N into disjoint blocks of side L^j, for scales $0 \le j \le N$. This decomposition is an important ingredient in our choice of the coordinates in which we represent the RG map. We now describe it in detail, along with a number of useful related definitions, as in [16].

We partition the torus Λ_N, which has period L^N, into disjoint j*-blocks* of side L^j ($j \le N$). Each j-block is a translate of the block $\{x \in \Lambda : 0 \le x_i < L^j, i = 1, \dots, d\}$. We denote the collection of j-blocks by $\mathcal{B}_j$.

A j*-polymer* is any (possibly empty) union of j-blocks, and $\mathcal{P}_j$ denotes the set of j-polymers. Given $X \in \mathcal{P}_j$, we denote by $\mathcal{B}_j(X)$ the set of j-blocks in X, and denote by $\mathcal{P}_j(X)$ the set of j-polymers in X. A nonempty polymer X is *connected* if for any $x, x' \in X$, there is a sequence $x = x_0, \dots, x_n = x' \in X$ with $\|x_{i+1} - x_i\|_\infty = 1$ for $i = 0, \dots, n-1$. Let $\mathcal{C}_j$ denote the set of connected j-polymers and, for any $X \in \mathcal{P}_j$, let $\mathrm{Comp}_j(X) \subset \mathcal{C}_j(X)$ be the set of connected components of X. The empty set $\varnothing$ is not in $\mathcal{C}_j$.

We say that two polymers X, Y *do not touch* if $\min\{\|x - y\|_\infty : x \in X, y \in Y\} > 1$. We call a connected polymer $X \in \mathcal{C}_j$ a *small set* if it consists of at most 2^d j-blocks, and write $\mathcal{S}_j$ for the collection of small sets in $\mathcal{C}_j$. The *small-set neighbourhood* of a polymer $X \in \mathcal{P}_j$ is $X^\square = \cup_{Y \in \mathcal{S}_j(X) : X \cap Y \neq \varnothing} Y$.

For $F_1, F_2 : \mathcal{P}_j \to \mathcal{N}$, we define the scale-$j$ *circle product* $F_1 \circ F_2 : \mathcal{P}_j \to \mathcal{N}$ by

$$(F_1 \circ F_2)(Y) = \sum_{X \in \mathcal{P}_j(Y)} F_1(Y \setminus X) F_2(X) \qquad (Y \in \mathcal{P}_j). \tag{2.20}$$

We only consider maps $F : \mathcal{P}_j \to \mathcal{N}$ with the property that $F(\varnothing) = 1$. The identity element for the circle product is the map $\mathbb{1}_\varnothing : \mathcal{P}_j \to \mathcal{N}$ defined by $\mathbb{1}_\varnothing(\varnothing) = 1$, $\mathbb{1}_\varnothing(X) = 0$ if $X \neq \varnothing$.

2.4.2 Mass and Coalescence Scales

Two scales play an important role for the nature of the RG recursion (2.11). We define the *mass scale* j_m by

$$j_m = \lceil f_m \rceil, \qquad f_m = 1 + \frac{1}{\alpha} \log_L m^{-2}. \tag{2.21}$$

By definition, j_m is the smallest scale for which $m^2 L^{\alpha(j_m - 1)} \ge 1$. The mass scale is the scale beyond which the mass m^2 plays a significant helpful role in the decay of the covariance C_j. Indeed, by (2.8) and the elementary inequality (with notation $x_+ = \max\{x, 0\}$)

$$(1 + m^4 L^{2\alpha(j-1)})^{-1} \le L^{-2\alpha(j - j_m)_+}, \tag{2.22}$$

we have

$$|C_{j;x,y}| \le c L^{-(d-\alpha)(j-1) - 2\alpha(j - j_m)_+}. \tag{2.23}$$

We also define the *coalescence scale* j_{ab} by

$$j_{ab} = \big\lfloor \log_L(2|a - b|) \big\rfloor. \tag{2.24}$$

By definition, j_{ab} is the unique integer such that

$$\tfrac{1}{2} L^{j_{ab}} \le |a - b| < \tfrac{1}{2} L^{j_{ab}+1}. \tag{2.25}$$

By (2.7), $C_{j;a,b} = 0$ for all $j \le j_{ab}$, and hence

$$((-\Delta)^{\alpha/2} + m^2)^{-1}_{a,b} = \sum_{j=1}^{\infty} C_{j;a,b} = \sum_{j=j_{ab}+1}^{\infty} C_{j;a,b}. \tag{2.26}$$

Ultimately, we take the limit $m^2 \downarrow 0$ before considering large $|a - b|$, so we can and do assume that $j_m > j_{ab}$.

2.5 Localisation Operator Loc

We use the operator Loc defined and analysed in [14], to extract a local polynomial from an element $F \in \mathcal{N}$. For appropriate $X \subset \Lambda$, the local polynomial $\mathrm{Loc}_X F$ extracts the parts of F that are relevant and marginal in the RG sense.

2.5.1 Local Polynomials

The range of the operator Loc_X is a certain vector space $\mathcal{U}(X)$ of local polynomials in the field. We now define this vector space, taking into account that the elements $F \in \mathcal{N}$ to which Loc_X will be applied obey Euclidean covariance and $O(n)$ invariance on $\mathcal{N}^{\varnothing}$.

Given *bulk coupling constants* $g, \nu, u \in \mathbb{R}$; $a, b \in \Lambda$; *observable fields* $\sigma_a, \sigma_b \in \mathbb{R}$; and the *observable coupling constants* $\lambda_a, \lambda_b, q_a, q_b \in \mathbb{R}$; let

$$U^{\varnothing}(\varphi_x) = g\tau_x^2 + \nu\tau_x + u, \tag{2.27}$$

$$U(\varphi_x) = U^{\varnothing}(\varphi_x) - \sigma_a\lambda_a\varphi_a^1 \mathbb{1}_{x=a} - \sigma_b\lambda_b\varphi_b^1 \mathbb{1}_{x=b} - \tfrac{1}{2}(q_a\mathbb{1}_{x=a} + q_b\mathbb{1}_{x=b})\sigma_a\sigma_b. \tag{2.28}$$

The symbol $\varnothing$ denotes the bulk. (For $n = 0$, we can take $u = 0$ in $U^{\varnothing}$ due to supersymmetry; see [7].) For U as in (2.28) and $X \subset \Lambda$, we write

$$U(X) = \sum_{x\in X} U(\varphi_x). \tag{2.29}$$

Let $\mathcal{U}(X)$ denote the space of polynomials of the form (2.29). Let $\mathcal{V}(X) \subset \mathcal{U}(X)$ be the subspace for which $u = q_a = q_b = 0$. Note that V_0 of (2.3) obeys $V_0(X) \in \mathcal{V}(X)$ with $\lambda_a = \lambda_b = 1$.

2.5.2 Definition of Loc

To define Loc, we must first define a set of polynomial test functions, as in [14, Sect. 1.3]. Let $p > 0$, let $a = (a_1, \ldots, a_p)$ with each $a_r \in \mathbb{N}_0^d$, and let $k = (k_1, \ldots, k_p)$ with each $k_r \in \{1, \ldots, n\}$. Let $\Lambda' \subset \Lambda$ be a coordinate patch as defined in [14, Sect. 1.3] (e.g., Λ' can be any small set as defined in Sect. 2.4). Recall the set Λ^* of sequences, defined in Sect. 2.3.2. We define a test function $q^{a,k}$, supported on sequences $\vec{x} = ((x_1, i_1), \ldots, (x_p, i_p)) \in \Lambda^*$ with each $x_r \in \Lambda'$, by

$$q_{\vec{x}}^{a,k} = \prod_{r=1}^{p} \delta_{i_r,k_r} x_r^{a_r} = \prod_{r=1}^{p} \delta_{i_r,k_r} \prod_{l=1}^{d} x_{r,l}^{a_{r,l}}. \tag{2.30}$$

We include the case $p = 0$ by interpreting (2.30) as the constant number 1 in this case. The role of the coordinate patch, which cannot “wrap around” the torus, is to permit polynomial test functions such as (2.30) to be well-defined. We define the *field dimension* $[\varphi] = \frac{d-\alpha}{2}$. The *dimension* of the test function $q^{a,k}$ is defined to equal $p[\varphi] + \sum_{r=1}^{p}\sum_{l=1}^{d} a_{r,l}$. Given $d_+ \geq 0$, we let $\Pi = \Pi^{d_+}[\Lambda'] \subset \Phi$ denote the span of all test functions $q^{a,k}$ of dimension at most d_+.

Let $\mathcal{N}^{\varnothing}(\Lambda')$ denote the space of functionals of the field that only depend on field values at points in Λ'. By [14, Proposition 1.5], for $X \subset \Lambda'$, there is a unique operator $\mathrm{loc}_X : \mathcal{N}^{\varnothing}(\Lambda') \to \mathcal{V}(X)$ (independent of the choice of Λ') such that

$$\langle F, g\rangle_0 = \langle \mathrm{loc}_X\, F, g\rangle_0 \text{ for all } g \in \Pi, \tag{2.31}$$

with the pairing given by (2.17) with $\varphi = 0$. To extend loc from $\mathcal{N}^{\varnothing}$ to $\mathcal{N}$, suppose we are given d_+^c for $c = \varnothing, a, b, ab$. We let loc_X^c denote the operator loc_X with $d_+ = d_+^c$ and, for F as in (2.15), define

$$\mathrm{Loc}_Y F = \mathrm{loc}_X^{\varnothing} F_{\varnothing} + \sigma_a \mathrm{loc}_X^a F_a + \sigma_b \mathrm{loc}_X^b F_b + \sigma_a \sigma_b \mathrm{loc}_X^{ab} F_{ab}. \tag{2.32}$$

It remains to specify the d_+^c.

In [31], for scales below the mass scale the range of the restriction of Loc to $\mathcal{N}^{\varnothing}$ is specified as the span of $\{1, \tau, \tau^2\}$. This corresponds to the choice $d_+^{\varnothing} = d$ (using symmetry considerations to disregard non-symmetric monomials). Above the mass scale, a different range for Loc is used, namely $d_+^{\varnothing} = d - \alpha$, due to the enhanced decay of the covariance decomposition (see [31, Sect. 4.2]).

As in [7, Sect. 3.2], we set $d_+^a = d_+^b = [\varphi]$ when Loc acts at scales strictly less than j_{ab}, and set $d_+^a = d_+^b = 0$ for larger scales. We always take $d_+^{ab} = 0$.

The following elementary lemma will be useful. Let $\mathbb{1}^1$ denote the constant test function supported on sequences of length 1 and defined by $\mathbb{1}^1_{(x,i)} = \delta_{i,1}$. Likewise, let $\mathbb{1}^2$ denote the constant test function supported on sequences of length 2 and defined by $\mathbb{1}^2_{((x_1,i_1),(x_2,i_2))} = \delta_{i_1,1}\delta_{i_2,1}$. Note that $\mathbb{1}^1, \mathbb{1}^2$ are each of the form (2.30), with respective dimensions $[\varphi]$ and $2[\varphi]$.

Lemma 2.1 *Given $x \in \Lambda$ and a coordinate patch $\Lambda' \ni x$, suppose that $F \in \mathcal{N}^{\varnothing}(\Lambda')$. For $j < N$ and $m = 1, 2$,*

$$\langle (1 - \mathrm{Loc}_x) F, \mathbb{1}^m \rangle_0 = 0. \tag{2.33}$$

Moreover, if $j < j_{ab}$, then for $c = a, b$,

$$\langle (1 - \mathrm{loc}_x^c) F, \mathbb{1}^1 \rangle_0 = 0. \tag{2.34}$$

Proof The first statement is an immediate consequence of the definition of Loc together with the fact that the test function $\mathbb{1}^m$ has dimension $m[\varphi] \le d - \alpha \le d_+^{\varnothing}$ for $m = 1, 2$. The second statement follows similarly using the fact that the dimension $[\varphi]$ of $\mathbb{1}^1$ is equal to $d_+^a = d_+^b$ if $j < j_{ab}$. □

3 RG Map

In the absence of observables, i.e., with $\sigma_a = \sigma_b = 0$, the RG map for the long-range models is constructed and bounded in [31] using the main theorem of [16]. The result is given in [31, Theorem 6.4]. The extension of this construction to the case of nonzero observable fields σ_a, σ_b follows a similar route as in the 4-dimensional nearest neighbour case in [5,32], as we now explain. The coordinates for the RG map are discussed in Sect. 3.1, the domain of the RG map is discussed in Sect. 3.2, and the main estimates for the RG map are given in Theorem 3.3. These estimates, combined with a new estimate derived from a cluster expansion, are used in Sects. 4–5 to control the flow generated by the RG map.

3.1 RG Coordinates

The RG map will be defined so as to express the sequence Z_j defined by (2.11) as

$$Z_j = e^{\zeta_j} (I_j \circ K_j)(\Lambda), \tag{3.1}$$

for a real sequence ζ_j and sequences of maps $I_j : \mathcal{P}_j \to \mathcal{N}$ and $K_j : \mathcal{P}_j \to \mathcal{N}$. The *perturbative coordinate* I_j is an explicit function of $V_j \in \mathcal{V}$, and

$$\zeta_j = -u_j|\Lambda| + \tfrac{1}{2}(q_{a,j} + q_{b,j})\sigma_a\sigma_b. \tag{3.2}$$

The *nonperturbative coordinate* K_j is discussed in detail below. By (2.3), (3.1) holds at scale $j = 0$ with $u_0 = q_{x,0} = 0$, $I_0 = e^{-V_0}$, and $K_0 = \mathbb{1}_\varnothing$. We sometimes write an element of $\mathcal{U}$ as $U = (\zeta, V)$ with $V \in \mathcal{V}$, where ζ encodes u, q_a, q_b.

We express the map $Z_j \mapsto Z_{j+1}$ of (2.11) via a map $(V_j, K_j) \mapsto (\delta\zeta_{j+1}, V_{j+1}, K_{j+1})$, the *renormalisation group* (*RG*) *map*, in such a manner that

$$\mathbb{E}_{j+1}\theta(I_j \circ K_j)(\Lambda) = e^{\delta\zeta_{j+1}}(I_{j+1} \circ K_{j+1})(\Lambda) \tag{3.3}$$

with $I_{j+1} = I_{j+1}(V_{j+1})$ and $\delta\zeta_{j+1} = \zeta_{j+1} - \zeta_j$. This ensures that Z_{j+1} has the form (3.1) with $\zeta_{j+1} = \zeta_j + \delta\zeta_{j+1}$.

3.1.1 Perturbative Coordinate

The form of the perturbative coordinate I_j is as follows. Given a $\Lambda \times \Lambda$ matrix w, we define the operator $\mathcal{L}_w = \frac{1}{2}\sum_{u,v\in\Lambda} w_{u,v} \sum_{i=1}^n \frac{\partial}{\partial\varphi_u^i}\frac{\partial}{\partial\varphi_v^i}$. Recall the projections defined in Sect. 2.3.1. Given $V', V'' \in \mathcal{V}$, we also define

$$F_w(V', V'') = e^{\mathcal{L}_w}\big(e^{-\mathcal{L}_w}V'\big)\big(e^{-\mathcal{L}_w}V''\big) - V'V'', \tag{3.4}$$
$$F_{\pi,w}(V', V'') = F_w(V', \pi_\varnothing V'') + F_w(\pi_* V', V''). \tag{3.5}$$

For $j \geq 0$, we write the partial sums of the covariance decomposition as

$$w_j = \sum_{i=1}^{j} C_i, \qquad w_0 = 0. \tag{3.6}$$

As in [7, (3.21)], for $B \in \mathcal{B}_j$ we define

$$W_j(V, x) = \frac{1}{2}(1 - \mathrm{Loc}_x)F_{\pi,w_j}(V_x, V(\Lambda)), \qquad W_j(V, B) = \sum_{x\in B} W_j(V, x). \tag{3.7}$$

The polynomial $W_j(V, B)$ in the fields is thus an explicit quadratic function of V. In particular, $W_j^\varnothing$ is an even polynomial in the fields, and W_j is quadratic in the coupling constants and is irrelevant in the RG sense. Finally, for $V \in \mathcal{V}$, we define $I_j = I_j(V, \cdot) : \mathcal{P}_j \to \mathcal{N}$ by

$$I_j(V, X) = e^{-V(X)} \prod_{B\in\mathcal{B}_j(X)} (1 + W_j(V, B)). \tag{3.8}$$

As in (2.5), we write $D_{\sigma_a} = \frac{\partial}{\partial\sigma_a}|_{\sigma_a=\sigma_b=0}$. We also write $\bar{D} = \sum_{x\in\Lambda} \frac{\partial}{\partial\varphi_x^1}|_{\varphi=0}$ and $\bar{D}^2 = \sum_{x,y\in\Lambda} \frac{\partial^2}{\partial\varphi_x^1\partial\varphi_y^1}|_{\varphi=0}$. We will later make use of the following corollary of Lemma 2.1.

Corollary 3.1 *For $V \in \mathcal{V}$ and $x \in \Lambda$, and with $W_j = W_j(V, x)$,*

$$\bar{D}W_j^\varnothing = \bar{D}^2 W_j^\varnothing = 0. \tag{3.9}$$

Moreover, if $j < j_{ab}$, then

$$\bar{D}D_{\sigma_a}W_j = 0. \tag{3.10}$$

Proof The fact that $\bar{D}W_j^{\varnothing} = 0$ is immediate since $W_j^{\varnothing}$ is even in the fields. Also, since $\bar{D}^2 W_j^{\varnothing} = \langle W_j^{\varnothing}, \mathbb{1}^2\rangle_0$, it follows from Lemma 2.1 that $\bar{D}^2 W_j^{\varnothing} = 0$. To see that $\bar{D}D_{\sigma_a} W_j = 0$, note that $D_{\sigma_a} W_j = \frac{1}{2}(1 - \mathrm{loc}_x^a)F_a$, with $F_a \in \mathcal{N}^{\varnothing}$ the coefficient of σ_a in $F_{\pi,w_j}(V_x, V(\Lambda))$. Thus, by definition (2.17) of the pairing, $\bar{D}D_{\sigma_a} W_j = \frac{1}{2}\langle(1 - \mathrm{loc}_x^a)F_a, \mathbb{1}^1\rangle_0$, which vanishes by Lemma 2.1 when $j < j_{ab}$. This completes the proof. □

3.1.2 Nonperturbative Coordinate

We now define the space $\mathcal{K}_j$ of maps $K : \mathcal{P}_j \to \mathcal{N}$ which contains the nonperturbative RG coordinate. With $\mathcal{N}$ replaced by $\mathcal{N}^{\varnothing}$, such a space is defined in [31, Definition 6.2], and, as in [16], we extend it here to include observables. The symmetries (Euclidean covariance, gauge invariance, supersymmetry, and $O(n)$-invariance) used in Definition 3.2 are defined in [16, Sect. 1.6] and [4, Sect. 2.3]. For $n \geq 1$, as a replacement for the gauge invariance which holds for $n = 0$ we also introduce *sign invariance*, which is invariance under the map $(\sigma, \varphi) \mapsto (-\sigma, -\varphi)$. Note that V_0 of (2.3) is sign invariant. It can be verified that the property of sign invariance is preserved by the map $K \mapsto K_+$ of [16].

Definition 3.2 For $j < N$, let $\mathcal{CK}_j = \mathcal{CK}_j(\Lambda)$ denote the real vector space of functions $K : \mathcal{C}_j \to \mathcal{N}$ with the following properties:

- Field Locality: For all $X \in \mathcal{P}_j(\Lambda)$, $K(X) \in \mathcal{N}(X^{\square})$. Also, (i) $\pi_a K(X) = 0$ unless $a \in X$, (ii) $\pi_b K(X) = 0$ unless $b \in X$, and (iii) $\pi_{ab} K(X) = 0$ unless $a \in X$ and $b \in X^{\square}$ or vice versa, and $\pi_{ab} K(X) = 0$ if $X \in \mathcal{S}_j$ and $j < j_{ab}$.
- Symmetry: (i) $\pi_{\varnothing} K$ is Euclidean covariant, (ii) if $n = 0$, $\pi_{\varnothing} K$ is supersymmetric and K is gauge invariant and has no constant part; if $n \geq 1$, $\pi_{\varnothing} K$ is $O(n)$-invariant and K is sign invariant.

Let $\mathcal{K}_j = \mathcal{K}_j(\Lambda)$ be the real vector space of functions $K : \mathcal{P}_j \to \mathcal{N}$ which have the above field locality and symmetry properties, and, in addition:

- Component Factorisation: for all polymers X, $K(X) = \prod_{Y \in \mathrm{Comp}(X)} K(Y)$.

The nonperturbative coordinate K_j appearing in (3.1) is an element of $\mathcal{K}_j$. An element of $\mathcal{K}_j$ determines an element of $\mathcal{CK}_j$ by restriction to $X \in \mathcal{C}_j$. Also, an element of $\mathcal{CK}_j$ determines an element of $\mathcal{K}_j$ by the factorisation property. The same symbol is used for both elements related by this correspondence. Since the empty set is not a connected set, $\mathbb{1}_{\varnothing} \in \mathcal{K}_j$ becomes $0 \in \mathcal{CK}_j$ under this correspondence.

3.2 Norms and RG Domain

We now specify the domain of the RG map, which requires specification of norms on the spaces $\mathcal{V}$ and $\mathcal{CK}$. Without the observables fields, the norms are discussed in [31, Sect. 6.2]. For the nearest-neighbour 4-dimensional case, the adaptation of the norms to include observables is discussed, e.g., in [32, Sect. 5.1]. For our current long-range setting, we need only adjust some norm parameters, compared to [32, Sect. 5.1].

As in [31, (5.49)], the small number $\bar{s}$ in Theorem 1.1 is given by

$$\bar{s} = \frac{1}{a}(1 - L^{-\varepsilon}) = O(\varepsilon), \tag{3.11}$$

with the constant a specified in [31, Lemma 5.3] (not to be confused with the point $a \in \mathbb{Z}^d$ used for the two-point function). Recall the mass scale j_m defined in (2.21). Following [31, (3.19)–(3.20), (6.24)], we fix

$$\alpha' \in (0, \tfrac{1}{2}\alpha), \tag{3.12}$$

and define the bulk parameters

$$\ell_j = \ell_0 L^{-\frac{1}{2}(d-\alpha)j} L^{-\frac{1}{2}(\alpha+\alpha')(j-j_m)_+} = \begin{cases} \ell_0 L^{-\frac{1}{2}(d-\alpha)j} & (j \le j_m) \\ \ell_0 L^{-\frac{1}{2}(d-\alpha)j_m} L^{-\frac{1}{2}(d+\alpha')(j-j_m)} & (j > j_m), \end{cases} \tag{3.13}$$

$$h_j = \frac{1}{\bar{s}^{1/4}} k_0 L^{-\frac{1}{2}(d-\alpha)j} = \frac{1}{\bar{s}^{1/4}} \frac{k_0}{\ell_0} \ell_j \qquad (j \le j_m). \tag{3.14}$$

Here ℓ_0 can be chosen large (depending on L) and k_0 is a fixed (small) constant. We use $\mathfrak{h}_j$ to refer to either of the bulk parameters ℓ_j, h_j.

Now that observables are present, the pair of parameters $\mathfrak{h}_j$ is supplemented by the pair

$$\mathfrak{h}_{\sigma,j} = \ell_{j \wedge j_{ab}}^{-1} 2^{(j-j_{ab})_+} \times \begin{cases} \bar{s} & (\mathfrak{h} = \ell) \\ \bar{s}^{1/4} & (\mathfrak{h} = h). \end{cases} \tag{3.15}$$

We only use $h_{\sigma,j}$ for $j \le j_m$. Recall that we assume that the coalescence scale j_{ab} is smaller than the mass scale j_m, since the limit $m^2 \downarrow 0$ will be taken before considering arbitrarily large $|a - b|$.

For $U \in \mathcal{U} \simeq \mathbb{R}^7$, we define the scale-dependent norm

$$\|U\|_{\mathcal{U}} = \max \Big\{ |g| L^{\varepsilon(j \wedge j_m)}, |\nu| L^{\alpha(j\wedge j_m)}, |u| L^{jd}, \ell_j \ell_{\sigma,j}(|\lambda_a| \vee |\lambda_b|), \ell_{\sigma,j}^2(|q_a| \vee |q_b|) \Big\}. \tag{3.16}$$

We denote the restriction of $\| \cdot \|_{\mathcal{U}}$ to $\mathcal{V}$ by the same symbol. Given $C_{\mathcal{D}} > 0$, we define the domain

$$\mathcal{D}_j = \{ V \in \mathcal{V} : \|V\|_{\mathcal{U}} \le C_{\mathcal{D}} \bar{s}, \ g > C_{\mathcal{D}}^{-1} \bar{s} L^{-\varepsilon(j \wedge j_m)} \} \subset \mathcal{V}. \tag{3.17}$$

Note that $\mathcal{D}_j$ is a domain in $\mathcal{V}$, and as such, does not involve the coupling constants u or q.

A sequence $\mathcal{W}_j^{\varnothing}$ of Banach spaces is defined in terms of the $T_\varphi(\mathfrak{h}_j)$ seminorms in [31] (they are denoted $\mathcal{W}_j$ there). We extend $\mathcal{W}_j^{\varnothing}$ to a space $\mathcal{W}_j \subset \mathcal{CK}_j$ whose definition is the same with the exception that the T_φ seminorms are defined on the extended space $\mathcal{N}$. As in [31, Remark 6.3], we define a sequence

$$\vartheta_j = L^{-\frac{1}{4}\alpha(j-j_m)_+}. \tag{3.18}$$

Given a parameter $t > 0$, the domain of the RG map is defined by

$$\mathbb{D}_j = \mathcal{D}_j \times B_{\mathcal{W}_j}(t \vartheta_j^3 \bar{s}^3), \tag{3.19}$$

where $B_{\mathcal{W}_j}(r)$ is the open ball of radius r in the Banach space $\mathcal{W}_j$.

3.3 Estimates on RG Map

We now specify the RG map $(V_j, K_j) \mapsto (U_{j+1}, K_{j+1}) = (\delta\zeta_{j+1}, V_{j+1}, K_{j+1})$ and state our bounds on it. To shorten notation, we condense indices and write, e.g., (V, K) for (V_j, K_j) and (U_+, K_+) for (U_{j+1}, K_{j+1}). The definition of the maps U_+, K_+ is described in a general setup in [7,16], and is adapted to the long-range model with $\sigma_a = \sigma_b = 0$ in [31]. The same definitions extend to include observables.

In particular, the map $(V, K) \mapsto U_+ = (\delta\zeta_+, V_+) = \mathrm{PT}(V) + R_+(V, K)$ is explicit and consists of a perturbative part PT, incorporating second-order perturbation theory, and a nonperturbative, third-order error R_+. The explicit map PT is the one defined in [7] for $n = 0$, extended in [4] to $n \geq 1$, and used in [32] for general $n \geq 0$. Let λ denote λ_a or λ_b, and let q denote q_a or q_b. We denote the λ, q components of the map PT by $\lambda_{\mathrm{pt}}, \delta q_{\mathrm{pt}}$. For $j < N$, and with w_j given by (3.6), let

$$w_j^{(1)} = \sum_{y\in\mathbb{Z}^d} w_{j;x,y}. \tag{3.20}$$

By [31, (5.10) and Lemma 5.2],

$$w_j^{(1)} = O(L^{\alpha j}). \tag{3.21}$$

Let

$$\delta[\nu w^{(1)}] = (\nu + (n+2)gC_{+;0,0})w_+^{(1)} - \nu w^{(1)}. \tag{3.22}$$

Recall the definition of the coalescence scale j_{ab} in (2.24). Then, as in [32, Proposition 3.2], for general $n \geq 0$ the observable part of the map PT is the map $V \mapsto (\lambda_{\mathrm{pt}}, \delta q_{\mathrm{pt}})$ given by

$$\lambda_{\mathrm{pt}} = \begin{cases} (1 - \delta[\nu w^{(1)}])\lambda & (j+1 < j_{ab}) \\ \lambda & (j+1 \geq j_{ab}), \end{cases} \tag{3.23}$$

$$\delta q_{\mathrm{pt}} = \lambda_a \lambda_b \, C_{j+1;a,b}. \tag{3.24}$$

Note that $\lambda_{\mathrm{pt}} = \lambda$ for all scales $j \geq j_{ab} - 1$, i.e., the flow of λ stops evolving after scale $j_{ab} - 1$. Conversely, since $C_{j+1;a,b} = 0$ for $j + 1 \leq j_{ab}$, nonzero δq_{pt} can occur only at scales $j \geq j_{ab}$. The map $(V, K) \mapsto U_+$ is now defined by

$$U_+ = \mathrm{PT}(\hat{V}) \quad \text{with} \quad \hat{V} = V - \sum_{Y\in\mathcal{S}(\Lambda):Y\supset B} \mathrm{Loc}_{Y,B} I^{-Y} K(Y). \tag{3.25}$$

The localisation operator $\mathrm{Loc}_{Y,B}$ is defined in [14, Definition 1.17]. The higher-order correction $R_+ : \mathcal{V} \to \mathcal{U}$ to the perturbative calculation is then defined by

$$R_+(V, K) = \mathrm{PT}(\hat{V}) - \mathrm{PT}(V), \tag{3.26}$$

so that $U_+ = \mathrm{PT}(V) + R_+(V, K)$. We do not need the explicit form of R_+ and only use the bounds of Theorem 3.3 below.

The map $(V, K) \mapsto K_+$ is also given explicitly in [16], but it is complicated to write down. Like R_+, this nonperturbative part of the RG map is of order $O(\bar{s}^3)$. It is part of the statement of Theorem 3.3 below that the formula for K_+ constructed in [16] is well-defined on the domain specified in Theorem 3.3. We do not need to know more here about K_+ than the estimates provided by Theorem 3.3.

The RG map depends on the mass m^2 through its dependence on the covariance C_+. We require continuity in the mass in the limit $m^2 \downarrow 0$, which can only be taken after the infinite-volume limit $N \to \infty$. Given small $\delta > 0$, we define the mass domain for the RG map by

$$\mathbb{I}_j = \begin{cases} [0, \delta] & (j < N) \\ [\delta L^{-\alpha(N-1)}, \delta] & (j = N). \end{cases} \tag{3.27}$$

The special attention to $j = N$ is due to the fact that the final covariance $C_{N,N}$ is only defined for $m^2 > 0$, and it obeys good estimates for $m^2 \in \mathbb{I}_N$.

The following extends [31, Theorem 6.4] to allow for the presence of observables. Its estimates appear identical to [31, Theorem 6.4], but it is in fact an extension since the domain and range of the RG map now include observables in $(V, K) \in \mathbb{D}$, as well as in R_+ and K_+. Note that the map R_+, which acts on (V, K) with $V \in \mathcal{V}$, produces a polynomial in $\mathcal{U}$ which in particular contains the nonperturbative contributions to $\delta\zeta$. The bound (3.29) on R_+ controls these nonperturbative contributions to $\delta\zeta$. Note that the estimates (3.29) hold for $m^2 \in \mathbb{I}_+$, but the continuity is in the smaller interval $m^2 \in [0, L^{-\alpha j}]$. A restriction like this on the continuity interval is essential, because larger m^2 will put j above the mass scale, at which point the spaces themselves become dependent on m^2 through their dependence on ℓ_j and a continuity statement becomes meaningless.

Theorem 3.3 *Let $d = 1, 2, 3$; $n \geq 0$; $\alpha = \frac{1}{2}(d+\varepsilon)$ and $j < N$. Let $C_{\mathcal{D}}$ and L be sufficiently large, and let ε be sufficiently small. There exist $c > 0$, $C_{\mathrm{RG}} > 0$, $\delta > 0$, such that, with the domain $\mathbb{D}$ defined using $t = 4C_{\mathrm{RG}}$, the maps*

$$R_+ : \mathbb{D} \times \mathbb{I}_+ \to \mathcal{U}, \quad K_+ : \mathbb{D} \times \mathbb{I}_+ \to \mathcal{W}_+ \tag{3.28}$$

are analytic in (V, K), provide a solution to (3.3), and satisfy the estimates

$$\|R_+\|_{\mathcal{U}_+} \leq c\vartheta_+ \bar{s}^3, \qquad \|K_+\|_{\mathcal{W}_+} \leq C_{\mathrm{RG}} \vartheta_+^3 \bar{s}^3. \tag{3.29}$$

The coordinate in R_+ corresponding to $\delta q_{a,j}$, $\delta q_{b,j}$ is identically zero for $j \leq j_{ab}$, and the coordinate corresponding to $\lambda_{a,j}$, $\lambda_{b,j}$ is identically zero for $j \geq j_{ab}$. In addition, R_+, K_+ are jointly continuous in $m^2 \in [0, L^{-\alpha j}]$, V, K.

Proof The theorem is a consequence of the main result of [16], which focusses on the 4-dimensional nearest-neighbour case. For the long-range model, the appropriate modifications for the bulk part of the RG map are discussed in [31], and we assume familiarity with both the methodology and the modifications. In order to include observables, only minor further modifications are required, compared to [15,16].

One requirement is to verify that, for $V \in \mathcal{D}$, the basic small parameters ε_V and $\bar{\varepsilon}$ obey appropriate estimates when observables are present, as in [15, Sects. 3.2–3.3]. We verify this here; this verification validates our choice (3.15) for the norm parameters. (In fact, somewhat larger domains are used in [15, Sects. 3.2–3.3]; the main ideas are present for $V \in \mathcal{D}$, which we consider here, and the extension to the larger domains is a matter of bookkeeping.) A second requirement is to verify that the "crucial contraction" is maintained in the presence of observables, and we also verify this here.

Bound on ε_V. Let $V \in \mathcal{D}$. For ε_V, it suffices to observe that for $|\lambda_a| \leq C_{\mathcal{D}} \bar{s} \ell^{-1} \ell_\sigma^{-1}$,

$$\|\lambda_a \sigma_a \varphi_a^1\|_{T_0(\mathfrak{h})} = |\lambda_a| \mathfrak{h}_\sigma \mathfrak{h} \leq C_{\mathcal{D}} \bar{s} \frac{\mathfrak{h}_\sigma}{\ell_\sigma} \frac{\mathfrak{h}}{\ell} = \begin{cases} C_{\mathcal{D}} \bar{s} & (\mathfrak{h} = \ell) \\ C_{\mathcal{D}} k_0 \ell_0^{-1} & (\mathfrak{h} = h), \end{cases} \tag{3.30}$$

which implies stability on the domain $\mathcal{D}$ of (3.17), and complements the arguments of [16,31].

Bound on $\bar{\varepsilon}$. We must also verify the analogue of [15, Lemma 3.4]. To state the desired estimate, as in [31, (6.56)] we define the norm parameter

$$\hat{\ell}_j = \hat{\ell}_0 \ell_j L^{-\frac{1}{2}(\alpha - \alpha')(j - j_m)_+}, \tag{3.31}$$

and as in [31, (6.25)] we define the small parameter

$$\bar{\varepsilon} = \bar{\varepsilon}_j(\mathfrak{h}) = \begin{cases} \bar{s}\vartheta_j & (\mathfrak{h} = \ell) \\ \bar{s}^{1/4} & (\mathfrak{h} = h, \ j \leq j_m). \end{cases} \tag{3.32}$$

We write $U_{\rm pt} = {\rm PT}(V)$ and let $\delta V = \theta V - U_{\rm pt}$. Our goal then is to show that, for $V \in \mathcal{D}$,

$$\max_{B\in\mathcal{B}} \|\delta V(B)\|_{T_0(\mathfrak{h}\sqcup\hat{\ell})} \le C_{\delta V}\bar{\varepsilon} \tag{3.33}$$

where $C_{\delta V}$ is an L-dependent constant.

It is argued in [31, Sect .6.4.4] that (3.33) holds with δV replaced by $\delta V^{\varnothing} = \pi_{\varnothing}\delta V$. Thus, it suffices to establish (3.33) with δV replaced by $\delta V^* = \pi_*\delta V$. This can be done by writing

$$\|\delta V^*(B)\|_{T_0(\mathfrak{h}\sqcup\hat{\ell})} \le \|\theta V^*(B) - V^*(B)\|_{T_0(\mathfrak{h}\sqcup\hat{\ell})} + \|V^*(B) - U^*_{\rm pt}(B)\|_{T_0(\mathfrak{h}\sqcup\hat{\ell})} \tag{3.34}$$

and applying the triangle inequality to estimate each of the two terms on the right-hand side.

For instance, if $a \in B$, then the σ_a term of $\theta V^*(B) - V^*(B)$ is $\lambda_a\sigma_a\zeta_a^1$. By definition of the norm, by (3.31), (3.15), (3.32), (3.18), and by the fact that $\alpha' < \frac{1}{2}\alpha$,

$$\|\lambda_a\sigma_a\zeta_a^1\|_{T_0(\mathfrak{h}\sqcup\hat{\ell})} = |\lambda_a|\mathfrak{h}_\sigma\hat{\ell} \le C_{\mathcal{D}}\bar{s}\hat{\ell}_0 L^{-\frac{1}{2}(\alpha-\alpha')(j-j_m)_+}\frac{\mathfrak{h}_\sigma}{\ell_\sigma} \le C_{\mathcal{D}}\hat{\ell}_0\bar{\varepsilon}(\mathfrak{h}). \tag{3.35}$$

The σ_a term of $V^*(B) - U^*_{\rm pt}(B)$ is zero above the coalescence scale, whereas if $j+1 < j_{ab}$ then it is $\delta[\nu w^{(1)}]\lambda_a\sigma_a\varphi_a^1$, by (3.23). Thus, by (3.30), it is sufficient to show that

$$|\delta[\nu w^{(1)}]| \le \bar{s}^{1/4}\vartheta. \tag{3.36}$$

By its definition in (3.22),

$$\delta[\nu w^{(1)}] = \nu\sum_x C_{+:0,x} + (n+2)gC_{+;00}w_+^{(1)}. \tag{3.37}$$

By (3.17) and (2.23), and the finite-range property (2.7), the first term is bounded by

$$O(\bar{s})L^{-\alpha(j\wedge j_m)}L^{jd}L^{-(d-\alpha)j-2\alpha(j-j_m)_+} = O(\bar{s})L^{-\alpha(j-j_m)_+}, \tag{3.38}$$

and the second term is bounded by

$$O(\bar{s})L^{-\varepsilon(j\wedge j_m)}L^{-(d-\alpha)j-2\alpha(j-j_m)_+}L^{\alpha j} = O(\bar{s})L^{-d(j-j_m)_+}. \tag{3.39}$$

These bounds do better than what is required by (3.36).

For the $\sigma_a\sigma_b$ term, we can take $j \ge j_{ab}$. The $\sigma_a\sigma_b$ term of $\theta V^* - V^*$ is always 0 and the coefficient of $\sigma_a\sigma_b$ in $V^*(B) - U^*_{\rm pt}(B)$ is at most $|C_{+;a,b}\lambda_a\lambda_b|$. By (3.17), (2.23), and (3.13) and the fact that $\alpha' < \frac{1}{2}\alpha$,

$$\|C_{+;a,b}\lambda_a\lambda_b\sigma_a\sigma_b\|_{T_0(\mathfrak{h}\sqcup\hat{\ell})} \le O(\bar{s}^2)\frac{|C_{+;a,b}|}{\ell^2}\frac{\mathfrak{h}_\sigma^2}{\ell_\sigma^2} \le O(\bar{s}^2)\vartheta^2\frac{\mathfrak{h}_\sigma^2}{\ell_\sigma^2}. \tag{3.40}$$

When $\mathfrak{h} = h$ (hence $j < j_m$) this is $O(\bar{\varepsilon}(h)^2)$, and when $\mathfrak{h} = \ell$ it is $O(\bar{\varepsilon}(\ell)^2)$. This is better than what is required for (3.33).

Crucial contraction The adaptation of the crucial contraction to the long-range model is provided for the bulk in [31, Sect. 6.4.5–6.4.6]. We now extend the adaptation to include observables.

Below the mass scale, the least irrelevant of the sign invariant monomials involving the observable fields each have two additional spin fields compared to their marginal counterparts $\sigma_a\varphi_a^1$ and $\sigma_a\sigma_b$ (the latter occurs only above the coalescence scale), so have dimension which is larger by $2[\varphi] = d-\alpha$. Compared to [31, (6.64)], this gives rise to $\gamma = L^{-(d-\alpha)}$, and there is no factor L^d for observables, so the gain here is proportional to $L^{-(d-\alpha)}$. The worst

γ occurs for $d = 1$, where we have $\gamma = L^{-\frac{d}{2}+\frac{\varepsilon}{2}} = L^{-\frac{1}{2}+\frac{\varepsilon}{2}}$. This is consistent with the values of $L^d\gamma$ reported for the bulk in [31, (6.64)].

Above the mass scale, we extend the discussion in [31, Sect. 6.4.6], as follows. For the perturbative contribution to K, we have already verified that we can continue to use the $\bar{\varepsilon}$ given by (3.32) when observables are present, and there is therefore no change to [31] concerning this issue. It remains to consider the crucial contraction.

We recall and invoke our assumption that $j_{ab} < j_m$. Now $d_+^a = 0$, so the least irrelevant monomial in $\mathcal{N}^a$ is $\sigma_a\varphi$. This scales as

$$\begin{aligned}\ell_{\sigma,j}\ell_j &= \frac{\ell_j}{\ell_{j_{ab}}}2^{j-j_{ab}}\bar{s} = \frac{L^{-\frac{1}{2}(d-\alpha)j}L^{-(\alpha+\alpha')(j-j_m)}}{L^{-\frac{1}{2}(d-\alpha)j_{ab}}}2^{j-j_{ab}}\bar{s}\\ &\le L^{-\frac{1}{2}(d+\alpha')(j-j_m)}2^{j-j_m}\bar{s}.\end{aligned} \tag{3.41}$$

A change from scale j to scale $j+1$ in the above right-hand side gives rise to a factor $2L^{-\frac{1}{2}(d+\alpha')}$. As in [31, (6.68)–(6.69)], the essential condition here is that the product of this factor with $\vartheta^{-3} = L^{\frac{3}{4}\alpha}$ should be bounded above by an inverse power of L. This condition is indeed satisfied, since

$$\tfrac{1}{2}(d+\alpha') - \tfrac{3}{4}\alpha = \tfrac{1}{2}\left(d+\alpha' - \tfrac{3}{4}(d+\varepsilon)\right) = \tfrac{1}{2}\left(\tfrac{1}{4}d+\alpha'-\tfrac{3}{4}\varepsilon\right) > \tfrac{1}{8}d. \tag{3.42}$$

Similarly, the least irrelevant monomial in $\mathcal{N}^{ab}$ that is sign invariant is of the form $\sigma_a\sigma_b\varphi\varphi$, and has scaling dimension twice that considered in the previous paragraph, so twice as good. Thus the crucial contraction is not harmed by the presence of observables.

Estimate for R_+ above the mass scale Finally, we consider the extension of [31, Sect. 6.4.7] to include observables. The observable terms have the same T_0 and $\mathcal{U}$ norms: $\|\sigma_a\varphi_a^1\|_{T_0} = \ell_\sigma\ell = \|\sigma_a\varphi_a^1\|_{\mathcal{U}}$ and $\|\sigma_a\sigma_b\|_{T_0} = \ell_\sigma^2 = \|\sigma_a\sigma_b\|_{\mathcal{U}}$. This leads to an extension to [31, Lemma 6.6], as follows. Let

$$F_1 = \nu\tau + u + -\sigma_a\varphi_a^1\mathbb{1}_{x=a} - \sigma_b\varphi_b^1\mathbb{1}_{x=b} - \tfrac{1}{2}(q_a\mathbb{1}_{x=a}+q_b\mathbb{1}_{x=b})\sigma_a\sigma_b, \tag{3.43}$$

$$F_2 = g\tau^2 + \nu\tau + u + -\sigma_a\varphi_a^1\mathbb{1}_{x=a} - \sigma_b\varphi_b^1\mathbb{1}_{x=b}. \tag{3.44}$$

The estimates of [31, Lemma 6.6] now become

$$\|F_1\|_{\mathcal{U}} \le c_L L^{\alpha'(j-j_m)_+}\|F_1(B)\|_{T_0}, \qquad \|F_2(B)\|_{T_0} \le c\|F_2\|_{\mathcal{U}}, \tag{3.45}$$

i.e., the bound remains the same for F_1 but loses a helpful factor $L^{-\alpha'(j-j_m)_+}$ for F_2. The bound on F_1 then implies, as in [31, (6.74)], that

$$\|R_+\|_{\mathcal{U}} \le O(L^{\alpha'(j-j_m)_+})\|R_+(B)\|_{T_0}, \tag{3.46}$$

and the bound (3.29) follows from this as in [31, Sect. 6.4.7].

The introduction of observables does lead to a change in the bounds on F, W, P in [31, Lemma 6.7], due to the weakened estimate for F_2 in (3.45). The change is to replace the factors $L^{-(\alpha+\alpha')(j-j_m)_+}$ and $(c/L)^{-(\alpha+\alpha')(j-j_m)_+}$ in the three upper bounds of [31, Lemma 6.7] by the worse factor $\vartheta^2 = L^{-\frac{1}{2}\alpha(j-j_m)_+}$. Since we seek an upper bound which includes the factor ϑ^2 in $\bar{\varepsilon}^2$, the weakened bounds remain more than good enough.

For general reasons, $\pi_{ab}W = 0$ [15, Proposition 4.10], so there can be no such term in W. Thus, in the proof of [31, Lemma 6.7], only one factor $L^{-\alpha'(j-j_m)_+}$ can be lost by application of (3.45), not two. Also, by direct calculation, the relevant contribution to F is $F_C(\lambda_a\sigma_a\varphi_a^1, \lambda_b\sigma_b\varphi_b^1) = \frac{1}{2}\lambda_a\lambda_b C_{a,b}\sigma_a\sigma_b$, whose $T_0(\ell)$ norm is given as in the first inequality of (3.40) to be at most $L^{-(\alpha-\alpha')(j-j_m)}$, which is better than the required ϑ^2. The bound on P follows from the bounds on F, W as in [15, Proposition 4.1]. □

In the absence of observables, Theorem 3.3 is used in [31] to construct a global RG flow $(g_j, \nu_j, K_j^{\varnothing})$ that remains in the RG domain for all j. This requires tuning the initial ν to a mass-dependent critical value $\nu_0^c(m^2)$; this value converges to the critical point $\nu_c(g; n)$ as $m^2 \downarrow 0$ (see [31, (8.93)–(8.94)]). Throughout the remainder of the present paper, we always take (g_j, ν_j) to be this global flow of coupling constants. For general reasons this flow is the same in the presence of observables as in their absence: see [16, (1.68)–(1.69)]. The main task for the proof of Theorem 1.1 is to apply the estimates of Theorem 3.3 to control, in addition, the flow of the observable coupling constants λ and δq, and the observable part of the coordinate K. The flow of δq and K is analysed as in the 4-dimensional nearest-neighbour case [5,32].

The flow of λ is marginal, for the same reasons as in the 4-dimensional case. In [5,32], the perturbative approximation (3.23) to the recursion for λ is solved along the lines of the rough computation

$$\lambda_j = \prod_{k=1}^{j-1}(1 - \delta[\nu w^{(1)}]) = \exp\left[\sum_{k=1}^{j-1}\log(1 - \delta[\nu w^{(1)}])\right]$$
$$\approx \exp\left[-\sum_{k=1}^{j-1}\delta[\nu w^{(1)}]\right] = \exp\left[-\nu_j w_j^{(1)}\right] \approx 1 - \nu_j w_j^{(1)}. \tag{3.47}$$

In [5,32], the errors introduced by the map R_+ into (3.23) were summable over all scales because of the decay of the marginal coupling constant g_j with the scale (Gaussian fixed point), and the above computation survives the introduction of these errors.

For the long-range model, the fixed point is non-Gaussian, and the corrections due to R_+ are not summable. Instead of trying to follow the route laid out in [5,32], we derive an exact relation between $\lambda_{a,j}$ and the known bulk coupling constants, similar to (3.47), which gives better control of its flow than the recursion. This is done in Sect. 4.

4 Flow of λ

According to (3.23) and Theorem 3.3, the flow of $\lambda_{a,j}$ under the RG map is nontrivial only until scale $j_{ab}-1$, and stops beyond this scale. Conversely, $q_{a,j} = 0$ for $j < j_{ab}$, and the flow of $q_{a,j}$ is nontrivial only for scales $j \geq j_{ab}$. Our goal now is to determine the form of the flow until scale j_{ab}. Since we later take the limit $m^2 \downarrow 0$ before studying large $j_{ab} \sim \log_L |a-b|$, we can and do assume that $j_{ab} < j_m$. We will prove the following proposition.

Proposition 4.1 *Let $n \geq 0$, let L be sufficiently large, let Λ_N be the torus of period L^N, and let ε be sufficiently small. Let $g \in [\frac{63}{64}\bar{s}, \frac{65}{64}\bar{s}]$, and let $m^2 \in [L^{-\alpha(N-1)}, \delta]$ with $\delta > 0$ sufficiently small. Let $j_{ab} < j_m < N$. Let $g_0 = g$, let ν_0 be the critical value $\nu_0^c(m^2)$ constructed in [31], and let $\lambda_{a,0} = \lambda_{b,0} = 1$. Then the RG map can be iterated to scale j_{ab}, i.e., it produces a sequence $(V_j, K_j) \in \mathbb{D}_j$ with initial condition $(V_0, 0)$, such that (3.1) holds for all $j \leq j_{ab}$ with $I_j = I_j(V_j)$ and $\zeta_j = \sum_{k=1}^{j}\delta\zeta_j$. Moreover, $q_{x,j} = 0$, and for the component $\lambda_{x,j}$ of this flow we have the stronger statement*

$$\lambda_{x,j} = 1 - \nu_j w_j^{(1)} + O(\bar{s}^2) \qquad (j < j_{ab},\ x = a, b). \tag{4.1}$$

The proof of Proposition 4.1 is given in Sect. 4.1 below. Its statement holds trivially at $j = 0$, and will be established inductively for higher scales. The induction for the bulk quantities $U_j^\varnothing$, $K_j^\varnothing$ is the result of [31], and is unaffected by the presence of observables.

The main additional ingredient for the induction of the observable parts is to establish the flow (4.1) of $\lambda_{a,j}$. To achieve this, in Lemma 4.2 we use integration by parts to obtain a relation between $\lambda_{a,j}$, quantities of the bulk flow, and the observable parts of the coordinate K_j. This is achieved by taking suitable derivatives of the identity $Z_j = \mathbb{E}_{w_j}\theta Z_0$. The contribution due to K_j is bounded uniformly in the volume using a cluster expansion, in Sect. 4.2.

The formula (4.1) for $\lambda_{x,j}$ has a natural counterpart for the nearest-neighbour 4-dimensional case, with error term $O(g_j^2)$ instead of $O(\bar{s}^2)$. In that context, $-\nu_j w_j^{(1)} + O(g_j^2) \to 0$ as $j \to \infty$. This provides insight into the fact that $\lim_{j\to\infty} \lambda_{x,j} = 1$ in [5, Lemma 4.6] and [32, Corollary 6.4]. For the long-range model considered in Proposition 4.1, the non-Gaussian fixed point leads to a limit which is not equal to 1.

4.1 Integration by Parts

For notational convenience we restrict attention to $n \geq 1$; small modifications apply for $n = 0$. Recall that $\bar{D}$ and D_{σ_a} are defined above Corollary 3.1, and that $Z_j = \mathbb{E}_{w_j}\theta Z_0$. Let

$$z_j = z_j(\Lambda) = e^{-\zeta_j} \frac{Z_j(\Lambda)}{I_j(\Lambda)}, \qquad \mathcal{L}_j = \mathcal{L}_j(\Lambda) = \log z_j(\Lambda). \tag{4.2}$$

Then we have

$$Z_j = e^{\zeta_j} I_j(\Lambda) z_j(\Lambda) = e^{\zeta_j} I_j(\Lambda) e^{\mathcal{L}_j(\Lambda)}. \tag{4.3}$$

The existence of the logarithm $\mathcal{L}_j$ is discussed in Sect. 4.2, where it is constructed as an element of a Banach space $T_0(\ell_j)$ which only examines derivatives at zero field, using a cluster expansion. Bounds on $\mathcal{L}_j$ and its derivatives at zero field are proved in Proposition 4.4 below.

Lemma 4.2 *The functions I_j and $\mathcal{L}_j$ are related by the identity*

$$\bar{D} D_{\sigma_a} I_j(\Lambda) + \bar{D} D_{\sigma_a} \mathcal{L}_j(\Lambda) = 1 + \frac{1}{|\Lambda|} w_j^{(1)} \Big[\bar{D}^2 I_j^\varnothing(\Lambda) + \bar{D}^2 \mathcal{L}_j^\varnothing(\Lambda) \Big]. \tag{4.4}$$

Proof By definition, followed by Gaussian integration by parts,

$$\begin{aligned} D_{\sigma_a} \mathbb{E}_{w_j}\theta Z_0 &= \mathbb{E}_{w_j}(\varphi_a^1 + \zeta_a^1) Z_0^\varnothing(\varphi + \zeta) \\ &= \varphi_a^1 \mathbb{E}_{w_j} Z_0^\varnothing(\varphi + \zeta) + \sum_{y \in \Lambda} w_{j;a,y} \mathbb{E}_{w_j} \frac{\partial}{\partial \zeta_y^1} Z_0^\varnothing(\varphi + \zeta). \end{aligned} \tag{4.5}$$

On the right-hand side, $\frac{\partial}{\partial \zeta_y^1}$ can be replaced by $\frac{\partial}{\partial \varphi_y^1}$, and the latter commutes with the expectation. Then application of $\bar{D} = \sum_{x\in\Lambda} \frac{\partial}{\partial \varphi_x^1}|_{\varphi=0}$ gives

$$\bar{D} D_{\sigma_a} \mathbb{E}_{w_j}\theta Z_0 = \mathbb{E}_{w_j} Z_0^\varnothing + \sum_{y\in\Lambda} w_{j;a,y} \sum_{x\in\Lambda} \frac{\partial^2}{\partial \varphi_y^1 \partial \varphi_x^1}\Big|_{\varphi=0} \mathbb{E}_{w_j} Z_0^\varnothing(\varphi + \zeta), \tag{4.6}$$

which by translation invariance and by definition of Z_j is the same as

$$\bar{D} D_{\sigma_a} Z_j = Z_j^\varnothing|_{\varphi=0} + w_j^{(1)} \frac{1}{|\Lambda|} \bar{D}^2 Z_j^\varnothing. \tag{4.7}$$

Now we divide both sides of (4.7) by $Z_j^\varnothing|_{\varphi=0}$ and use (4.3). Since $I_j|_{\varphi=0} = 1$, and since $\bar{D}Z_j^\varnothing = \bar{D}I_j|_{\sigma_a=\sigma_b=0} = D_{\sigma_a}I_j|_{\varphi=0} = 0$ by symmetry, the result is (4.4). □

Note that the right-hand side of (4.4) involves only bulk quantities, while the left-hand side depends on $\lambda_{a,j}$ through $I_j(\Lambda)$ and $D_{\sigma_a}\mathcal{L}_j(\Lambda)$, and also on the observable part of the irrelevant coordinate K_j (through $D_{\sigma_a}\mathcal{L}_j(\Lambda)$). For the explicit terms, we have the following identities.

Lemma 4.3 *For $j \le N$ and $V \in \mathcal{V}$,*

$$\bar{D}^2 I_j^\varnothing(\Lambda) = -\nu_j|\Lambda|, \tag{4.8}$$

and if $j < j_{ab}$ then

$$\bar{D}D_{\sigma_a}I_j(\Lambda) = \lambda_{a,j}. \tag{4.9}$$

Proof We differentiate the formula $I_j(\Lambda) = e^{-V_j(\Lambda)}\prod_{B\in\mathcal{B}_j(\Lambda)}(1 + W_j(V_j, B))$, which is (3.8). We apply the product rule, Corollary 3.1, and the facts that $\bar{D}V_j^\varnothing = 0$ and $I_j|_{\varphi=0} = 1$, to obtain

$$\bar{D}^2 I_j^\varnothing(\Lambda) = -\nu_j|\Lambda| + \sum_{B\in\mathcal{B}_j(\Lambda)} \bar{D}^2 W_j^\varnothing(V_j, B) = -\nu_j|\Lambda|. \tag{4.10}$$

Similarly, for $j < j_{ab}$, we also use $D_{\sigma_a}V_j(\Lambda) = -\lambda_{a,j}\varphi_a^1$ to obtain

$$\bar{D}D_{\sigma_a}I_j(\Lambda) = \lambda_{a,j} + \sum_{B\in\mathcal{B}_j(\Lambda)} \bar{D}D_{\sigma_a}W_j(V_j, B) = \lambda_{a,j}, \tag{4.11}$$

and the proof is complete. □

We now state our bounds for the terms in (4.4) involving $\mathcal{L}_j$. The hypothesis $(V_j, K_j) \in \mathbb{D}_j$ of Proposition 4.4 will be established inductively.

Proposition 4.4 *Let $j \le j_{ab}$, let $\mathcal{L}_j$ be defined as in (4.2), and assume that $Z_j = e^{\zeta_j}(I_j \circ K_j)$ with $I_j = I_j(V_j)$ and $(V_j, K_j) \in \mathbb{D}_j$. Then there is a constant $c_1 > 0$ such that*

$$|\bar{D}D_{\sigma_a}\mathcal{L}_j(\Lambda)| \le c_1\bar{s}^2, \qquad |\bar{D}^2\mathcal{L}_j^\varnothing(\Lambda)| \le c_1|\Lambda|L^{-\alpha j}\bar{s}^3. \tag{4.12}$$

We defer the proof of Proposition 4.4 to Sect. 4.2.

Proof of Proposition 4.1 The proof is by induction on j. The statement of Proposition 4.1 for $j = 0$ is trivial. Without loss of generality, we consider the case $x = a$. We assume that we have (3.1) for Z_k with (V_k, K_k) constructed inductively using the RG map for $k \le j$, and we make the constant in the hypothesis (4.1) explicit by assuming that, with c_1 from (4.12),

$$|\lambda_{a,j} - 1 + \nu_j w_j^{(1)}| \le 2c_1\bar{s}^2. \tag{4.13}$$

Then we have (3.1) with a pair of RG coordinates $(V_j, K_j) \in \mathbb{D}_j$, satisfying in addition (4.13). Theorem 3.3 guarantees the existence of RG coordinates $(U_{j+1}, K_{j+1}) = (\delta\zeta_{j+1} = \delta u_{j+1}, V_{j+1}, K_{j+1})$ at scale $j+1$ such that Z_{j+1} obeys (3.1), with $U_{j+1} = \mathrm{PT}_j(V_j) + R_{j+1}(V_j, K_j)$, and bounds on $R_{j+1}(V_j, K_j)$ and K_{j+1} as in (3.29).

It has been proved in [31] that the bulk part of V_{j+1} lies in $\mathbb{D}_{j+1}$. The second bound in (3.29) is sufficient to guarantee that K_{j+1} also lies in $\mathbb{D}_{j+1}$. Therefore, to complete the proof that $(V_{j+1}, K_{j+1}) \in \mathbb{D}_{j+1}$, we only need to show that $|\lambda_{j+1}| < C_{\mathcal{D}}$, where $C_{\mathcal{D}} > 1$ is the constant in (3.17). By (3.23), and by the first bound of (3.29) together with the definition of the norm in (3.16), we have $\lambda_{j+1} = (1 + O(\bar{s}))\lambda_j + O(\bar{s}^2)$. It now follows immediately from (4.13) that $0 < \lambda_{j+1} = 1 + O(\bar{s}) < C_{\mathcal{D}}$, since $\bar{s}$ can be chosen small enough. This proves that $(V_{j+1}, K_{j+1}) \in \mathbb{D}_{j+1}$.

To complete the induction, we must prove (4.13) with j replaced by $j+1$. Since (4.1) is only required for scales below the coalescence scale, we may assume here that $j + 1 < j_{ab}$. The bounds of Proposition 4.4 at scale $j + 1$ can be applied, since the hypothesis $(V_{j+1}, K_{j+1}) \in \mathbb{D}_{j+1}$ has now been verified. Also, the hypothesis $j + 1 < j_{ab}$ of Lemma 4.3 is satisfied. We use Lemma 4.3 in conjunction with (4.4), and apply Proposition 4.4. This gives

$$\left|\lambda_{a,j+1} - 1 + \nu_{j+1} w_{j+1}^{(1)}\right| \le c_1 \bar{s}^2 + c_1 w_{j+1}^{(1)} L^{-\alpha(j+1)} \bar{s}^3 \le 2c_1 \bar{s}^2, \tag{4.14}$$

by (3.21) and by taking $\bar{s}$ sufficiently small. This advances (4.13) to scale $j+1$, and completes the proof. □

4.2 Cluster Expansion

In this section, we use a cluster expansion to construct a formula for $\mathcal{L}_j = \log z_j$ and prove Proposition 4.4. Let $p(X) = K_j(X)/I_j(X)$. By (3.1), (4.2), and by definition of the circle product,

$$z_j = I_j(\Lambda)^{-1}(I_j \circ K_j)(\Lambda) = \sum_{X \in \mathcal{P}_j(\Lambda)} p(X), \tag{4.15}$$

where the term in the sum with $X = \varnothing$ is interpreted as 1. In the sum, we decompose $X \in \mathcal{P}_j$ into its connected components $X_1, \ldots, X_n \in \mathcal{C}_j$, which may be labelled in $n!$ different ways. For $X, X' \in \mathcal{C}_j$, we set $g(X, X') = -1$ if X and X' touch, and otherwise set $g(X, X') = 0$. Using the component factorisation property of K_j, we obtain

$$z_j = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{X_1,\ldots,X_n \in \mathcal{C}_j} p(X_1) \cdots p(X_n) \prod_{1 \le i < j \le n} (1 + g(X_i, X_j)), \tag{4.16}$$

where the $n = 0$ term is again interpreted as 1. This has the form of the partition function of a polymer system, as defined, e.g., in [34, (1)]. It is a standard result, e.g., [19,34], that $\log z_j$ can be written as a cluster expansion and accurately bounded, provided the polymer activities $p(X)$ obey suitable estimates. In the following proof, we discuss this in detail and invoke a convergence criterion from [34]; see also [21,30] for pedagogical introductions to the cluster expansion. The verification of the criterion from [34] is an almost immediate consequence of the norm estimates in the definition of the domain $\mathbb{D}_j$.

Since we are interested only in the derivatives of $\mathcal{L}_j$ at zero external and observable fields, we do not construct $\mathcal{L}_j$ as a function of these fields (even though this would also be possible for suitably small fields), but rather as a Taylor polynomial (jet) of order $p_{\mathcal{N}}$ in the fields around zero. In other words, we work on the quotient of $\mathcal{N}$ by the ideal of elements of $F \in \mathcal{N}$ with $\|F\|_{T_0(\ell_j)} = 0$. On this quotient, the $T_0(\ell_j)$ seminorm becomes a norm, and the quotient becomes a finite-dimensional Banach algebra. This is discussed in detail in [16, Sect. 1.7.3, Appendix A]. We adopt the point of view in the following that we work in this normed space, and write simply $\|\cdot\|$ for $\|\cdot\|_{T_0(\ell_j)}$. Although the results of [34] are stated for complex-valued $p(X)$, the proofs hold verbatim for values in any Banach algebra. The completeness of the Banach algebra is important for the existence of $\mathcal{L}_j = \log z_j$, which is defined in terms of an infinite sum.

The estimates we use, for $(V_j, K_j) \in \mathbb{D}_j$ and $X \in \mathcal{C}_j$, are:

$$\|1/I_j(X)\| \le 2^{|X|}, \qquad \|K_j(X)\| \le M\bar{s}^{3+a(|X|-2^d)_+}, \tag{4.17}$$

where $a > 0$ is small; here, $|X| = |X|_j$ denotes the number of j-blocks in X. The bound on I_j^{-X} is a small adaptation of [15, Proposition 2.2] to our long-range setting, and the bound on K_j follows from the definition of the $\mathcal{W}_j$ norm and (3.19). Absorbing the factor $2^{|X|}$ by replacing M by $M' > M$, replacing a by $a' \in (0, a)$, and using the fact that $\bar{s}$ is sufficiently small, we conclude that the polymer activity obeys the bound

$$\|p(X)\| \leq M' \bar{s}^{3+a'(|X|-2^d)_+}. \tag{4.18}$$

The following lemma uses this bound and will be employed to verify the hypothesis of [34, Theorem 1].

Lemma 4.5 *If $B \in \mathcal{B}_j$, then for $\bar{s}$ sufficiently small (depending only on d)*

$$\sum_{Y \in \mathcal{C}_j} |g(B, Y)| \|p(Y)\| e^{|Y|} \leq O(\bar{s}^3), \tag{4.19}$$

where the constant depends on d.

Proof The number of connected polymers $Y \in \mathcal{C}_j$ that touch a block Y and have size $|Y| = n$ is at most A^n for some d-dependent constant A. Thus,

$$\sum_{Y \in \mathcal{C}_j} |g(B, Y)| \|p(Y)\| e^{|Y|} \leq M' \bar{s}^3 \sum_{n=1}^{|\Lambda|} (Ae)^n \bar{s}^{a'(n-2^d)_+}. \tag{4.20}$$

We split the sum on the right-hand side into sums with $n \leq 2^d$ and $n > 2^d$. The first of these is a constant that depends only on d. Taking $\bar{s}$ small so that $Ae\bar{s}^{a'} < 1$, the second sum is bounded as

$$\bar{s}^{-a'2^d} \sum_{n=2^d+1}^{|\Lambda|} (Ae\bar{s}^{a'})^n = O(\bar{s}^{a'}) \tag{4.21}$$

with a d-dependent constant, which suffices. □

Proof of Proposition 4.4 By Lemma 4.5, if $\bar{s}$ is sufficiently small, then

$$\sum_{Y \in \mathcal{C}_j} |g(X, Y)| \|p(Y)\| e^{|Y|} \leq \sum_{B \in \mathcal{B}_j(X)} \sum_{Y \in \mathcal{C}_j} |g(B, Y)| \|p(Y)\| e^{|Y|} \leq |X|, \tag{4.22}$$

which verifies the hypothesis [34, (3)] with $a(A) = |A|$. Also by Lemma 4.5,

$$\sum_{Y \in \mathcal{C}_j} \|p(Y)\| e^{|Y|} \leq \sum_{B \in \mathcal{B}_j} \sum_{Y \in \mathcal{C}_j} |g(B, Y)| \|p(Y)\| e^{|Y|} \leq O(L^{(N-j)d} \bar{s}^3) < \infty, \tag{4.23}$$

which verifies the other hypothesis of [34, Theorem 1].

Let $u(X_1, \ldots, X_n)$ denote the Ursell function, defined in [34, (2)] (with g written as ζ). We conclude from [34, Theorem 1] that $\mathcal{L}_j$ is given by the absolutely convergent sum

$$\mathcal{L}_j = \sum_{n=1}^{\infty} \sum_{X_1, \ldots, X_n \in \mathcal{C}_j(\Lambda)} p(X_1) \cdots p(X_n) u(X_1, \ldots, X_n), \tag{4.24}$$

and that for all $X_1 \in \mathcal{C}_j$ we have

$$\sum_{n=1}^{\infty} n \sum_{X_2, \ldots, X_n \in \mathcal{C}_j} \|p(X_2)\| \cdots \|p(X_n)\| |u(X_1, \ldots, X_n)| \leq e^{|X_1|}, \tag{4.25}$$

with the $n = 1$ term in (4.25) interpreted as 1.

By (4.24)–(4.25) (the factor n in (4.25) is not needed) and (4.23),

$$\|\mathcal{L}_j^{\varnothing}\| \leq \sum_{X_1 \in \mathcal{C}_j} \|p(X_1)\| e^{|X_1|} = O(L^{(N-j)d} \bar{s}^3). \tag{4.26}$$

Similarly, for $\|D_{\sigma_a} \mathcal{L}_j\|$, we use the product rule for differentiation, this time using the factor n (due to the product rule) in (4.25). With the definition of the T_0 seminorm in (2.19) and of $\ell_{\sigma,j}$ in (3.15) for $j \leq j_{ab}$, we obtain

$$\|D_{\sigma_a} \mathcal{L}_j\| \leq \sum_{X \in \mathcal{C}_j : X \ni a} \|D_{\sigma_a} p(X)\| e^{|X|} = \sum_{X \in \mathcal{C}_j : X \ni a} \ell_{\sigma,j}^{-1} \|p(X)\| e^{|X|} = O(\ell_j \bar{s}^2). \tag{4.27}$$

For $F \in \mathcal{N}$, $\bar{D}F = \langle F, \mathbb{1}^1 \rangle_0$ and $\bar{D}^2 F = \langle F, \mathbb{1}^2 \rangle_0$, with the test functions $\mathbb{1}^1, \mathbb{1}^2$ of Lemma 2.1. These two test functions have Φ_j-norms (as defined, e.g., in [31, (6.8)])

$$\|\mathbb{1}^1\|_{\Phi_j} = \ell_j^{-1}, \qquad \|\mathbb{1}^2\|_{\Phi_j} = \ell_j^{-2}. \tag{4.28}$$

Therefore, for $m = 1, 2$,

$$|\bar{D}^m F| \leq \|F\| \ell_j^{-m}. \tag{4.29}$$

In particular, since $L^{Nd} = |\Lambda|$,

$$|\bar{D} D_{\sigma_a} \mathcal{L}_j| \leq \|D_{\sigma_a} \mathcal{L}_j\| \ell_j^{-1} = O(\bar{s}^2), \tag{4.30}$$

$$|\bar{D}^2 \mathcal{L}_j^{\varnothing}| \leq \|\mathcal{L}_j^{\varnothing}\| \ell_j^{-2} = O(|\Lambda| L^{-\alpha j} \bar{s}^3), \tag{4.31}$$

and the proof is complete. □

5 Full RG Flow and Proof of Theorem 1.1

In Proposition 4.1, the RG flow (ζ_j, V_j, K_j) is constructed for scales $j \leq j_{ab}$. The sequence ζ_j of (3.2) contains in particular the coupling constants $q_{a,j}, q_{b,j}$; recall that $q_{x,j} = 0$ for $j \leq j_{ab}$. In Sect. 5.1, we apply Theorem 3.3 inductively to continue the RG flow (ζ_j, V_j, K_j) to scales $j_{ab} < j \leq N$. Using the extended flow, we prove Theorem 1.1 in Sect. 5.2. The analysis proceeds as in [5,32].

Once the RG flow has been extended to all scales, the combination of (2.12) and (3.1) gives, at the final scale $j = N$, the representation

$$\mathbb{E}_C e^{-V_0(\Lambda)} = Z_N\big|_{\varphi=0} = e^{\zeta_N}(I_N(\Lambda) + K_N(\Lambda))\big|_{\varphi=0}. \tag{5.1}$$

From this, we apply (2.5) to to calculate the two-point function as

$$G_{a,b,N}(g, \nu; n) = D^2_{\sigma_a \sigma_b} \log \mathbb{E}_C e^{-V_0(\Lambda)} = \frac{1}{2}(q_{a,N} + q_{b,N}) + A_N, \tag{5.2}$$

with

$$A_N = \frac{D^2_{\sigma_a \sigma_b} K_N}{1 + K_N^{\varnothing}}\bigg|_{\varphi=0} - \frac{\big(D_{\sigma_a} K_N\big)\big(D_{\sigma_b} K_N\big)}{(1 + K_N^{\varnothing})^2}\bigg|_{\varphi=0}. \tag{5.3}$$

5.1 Flow of q

The next proposition states that the RG flow exists for scales $j_{ab} \le j \le N$, and in particular analyses the flow of q and establishes control on the terms of the right-hand side of (5.2), which is needed to prove Theorem 1.1.

Proposition 5.1 *Let $n \ge 0$, let L be sufficiently large, let Λ_N be the torus of period L^N, and let ε be sufficiently small. Let $g \in [\frac{63}{64}\bar{s}, \frac{65}{64}\bar{s}]$, and let $m^2 \in [L^{-\alpha(N-1)}, \delta]$ with $\delta > 0$ sufficiently small. Suppose that $j_{ab} < j_m$. Starting with $(V_{j_{ab}}, K_{j_{ab}})$ produced by Proposition 4.1, the RG map can be iterated to scale N, i.e., it produces a sequence $(V_j, K_j) \in \mathbb{D}_j$ such that (3.1) holds for all $j \le N$ with $I_j = I_j(V_j)$ and $\zeta_j = \sum_{k=1}^{j} \delta\zeta_j$. The $q_{x,j}$ component of ζ_j is given by*

$$q_{x,j} = \lambda_{a,j_{ab}}\lambda_{b,j_{ab}} w_{j;a,b} + \sum_{i=j_{ab}}^{j-1} r_{x,i} \qquad (x = a, b) \tag{5.4}$$

with

$$|r_{x,i}| \le O(\bar{s}) \frac{1}{|a-b|^{d-\alpha}} 4^{-(i-j_{ab})_+}. \tag{5.5}$$

Moreover,

$$\lim_{N\to\infty} A_N = 0. \tag{5.6}$$

Proof For $j = j_{ab}$, we have $(V_{j_{ab}}, K_{j_{ab}}) \in \mathbb{D}_{j_{ab}}$ by Proposition 4.1. Also, (5.4)–(5.5) hold trivially, since $r_{x,j_{ab}} = 0$ by Theorem 3.3 and hence $q_{x,j_{ab}} = \lambda_{a,j_{ab}}\lambda_{b,j_{ab}} w_{j;a,b}$ by (3.24).

We fix $j \ge j_{ab}$ and assume inductively that (3.1) holds with a pair of RG coordinates $(V_j, K_j) \in \mathbb{D}_j$ and that (5.4)–(5.5) hold. As in the proof of Proposition 4.1, Theorem 3.3 guarantees the existence of RG coordinates (V_{j+1}, K_{j+1}) at scale $j+1$, with $V_{j+1} = \mathrm{PT}_{j+1}(V_j) + R_{j+1}(V_j, K_j)$, and bounds on $R_{j+1}(V_j, K_j)$ and K_{j+1} as in (3.29).

As before, it has been proved in [31] that the bulk part of V_{j+1} lies in $\mathbb{D}_{j+1}$. The coordinate $\lambda_{a,j} = \lambda_{a,j_{ab}}$ remains constant for $j > j_{ab}$, and thus still lies in $\mathbb{D}_{j+1}$. As before, the second bound in (3.29) is sufficient to guarantee that K_{j+1} also lies in $\mathbb{D}_{j+1}$. This shows that $(V_{j+1}, K_{j+1}) \in \mathbb{D}_{j+1}$.

We now show that $q_{a,j+1}$ satisfies (5.4) at scale $j+1$ and that (5.5) holds. Using (5.4) and (3.24), and denoting by $r_{a,j}$ the component of $R_{j+1}(U_j, K_j)$ corresponding to the component q_a, we see that

$$q_{a,j+1} = q_{a,j} + \lambda_{a,j_{ab}}\lambda_{b,j_{ab}} C_{j+1;a,b} + r_{a,j} = \lambda_{a,j_{ab}}\lambda_{b,j_{ab}} w_{j+1;a,b} + \sum_{i=j_{ab}}^{j} r_{x,i},$$

verifying (5.4) at scale $j+1$. By definition of the norm in (3.16) and by our assumption that $j_{ab} < j_m$, Theorem 3.3 gives the bound

$$r_{x,j} \le \ell_{\sigma,j+1}^{-2} \|R_{j+1}\| \le 1_{j \ge j_{ab}} \ell_{\sigma,j+1}^{-2} O(\bar{s}^3) = 1_{j \ge j_{ab}} L^{-j_{ab}(d-\alpha)} 4^{-(j-j_{ab})_+} O(\bar{s}), \tag{5.7}$$

which proves (5.5) since $L^{-j_{ab}(d-\alpha)} = O(|a-b|^{-(d-\alpha)})$ by (2.25).

Finally, we write D_σ^k to mean no derivative for $k = 0$, the derivative with respect to σ_a or σ_b for $k = 1$, and the second derivative with respect to σ_a, σ_b for $k = 2$. Since $(V_N, K_N) \in \mathbb{D}_N$, it follows from (3.29), with the fact that the $\mathcal{W}_N$ norm bounds the $T_0(\ell_N)$ norm, that

$$|D_\sigma^l K_N(\Lambda)|_{\varphi=0}| \le \ell_{\sigma,N}^{-l} C_{\mathrm{RG}} \vartheta_N^3 \bar{s}^3 \le O(\bar{s}^{3-l}) \vartheta_N^3 \left(2^{-(N-j_{ab})_+} L^{-\frac{1}{2} j_{ab}(d-\alpha)}\right)^l. \tag{5.8}$$

Since $\vartheta_N \to 0$ as $N \to \infty$, this implies (5.6) and completes the proof. □

5.2 Proof of Theorem 1.1

With (5.2) and Proposition 5.1, it is now straightforward to complete the proof of our main result. In the proof, we write $C_{a,b}$ for the massless free two-point function $((-\Delta)^{\alpha/2})^{-1}_{a,b}$ on $\mathbb{Z}^d$. According to (1.6), $C_{a,b} \asymp |a-b|^{-(d-\alpha)}$. The proof uses the following lemma.

Lemma 5.2 *Let $\nu_0 = \nu_0^c(m^2)$. Then for any $j < N$ the map $m^2 \mapsto (V_j, K_j)$ is continuous for $m^2 \in [0, L^{-\alpha j}]$. Moreover, the sequence V_j is independent of N. In particular, for any $j < \infty$, the maps $V_j, K_j, R_{+,j}$ depend continuously on m^2 at $m^2 = 0$.*

Proof We show by induction that (V_j, K_j) depends continuously on $m^2 \in [0, L^{-\alpha j}]$. The case $j = 0$ follows from [31, (7.14)] and [31, Corollary 7.5]. Now suppose the inductive hypothesis holds for some $j \geq 0$. Then the case $j+1$ follows from (3.23), [31, Lemma 5.2], and Theorem 3.3. The fact that V_j is independent of N is [16, Proposition 1.18]. □

Proof of Theorem 1.1 We first take the limit $N \to \infty$, then take the limit $m^2 \downarrow 0$, and finally consider large $|a-b|$. By (5.4) with $j = N$,

$$q_{x,N} = \lambda_{a,j_{ab}} \lambda_{b,j_{ab}} w_{N;a,b} + \sum_{i=j_{ab}}^{N-1} r_{x,i}. \tag{5.9}$$

By Proposition 5.1, the remainder term is bounded uniformly in N and in $m^2 \in [L^{-\alpha(N-1)}, \delta]$ by

$$\left| \sum_{i=j_{ab}}^{N-1} r_{x,i} \right| \leq O(\bar{s}) \frac{1}{|a-b|^{d-\alpha}} \leq O(\bar{s}) C_{a,b}. \tag{5.10}$$

By dominated convergence, and by the continuity of $r_{x,i}$ (a component of R_{i+1}) at $m^2 = 0$ guaranteed by Lemma 5.2, $\lim_{m^2 \downarrow 0} \lim_{N\to\infty} \sum_{i=j_{ab}}^{N-1} r_{x,i}$ exists and is bounded by $O(\bar{s}) C_{a,b}$. For the main term, since $\lambda_{j_{ab}} = 1 + O(\bar{s})$ by Proposition 4.1, it follows from the definition of w_N in (3.6) (together with the fact that that the covariance appearing in (3.24) is always the infinite-volume one), that

$$\lim_{m^2 \downarrow 0} \lim_{N\to\infty} \lambda_{j_{ab}}^2 w_{N,a,b} = (1 + O(\bar{s})) C_{a,b}. \tag{5.11}$$

The existence of the above limit as $m^2 \downarrow 0$ is a consequence of the fact that $w_{\infty,a,b} = ((-\Delta)^{\alpha/2} + m^2)^{-1}_{ab} \to C_{a,b}$, together with the mass continuity of $\lambda_{j_{ab}}$, which follows from Lemma 5.2. We apply (5.6) in (5.2), and find that the critical two-point function obeys

$$G_{a,b} = \lim_{N\to\infty} G_{a,b,N} = \frac{1}{2}(q_{a,\infty} + q_{b,\infty}) = (1 + O(\bar{s})) C_{a,b} + O(\bar{s}) C_{a,b} = (1 + O(\bar{s})) C_{a,b}. \tag{5.12}$$

This completes the proof. □

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