J Stat Phys (2017) 167:317–350
DOI 10.1007/s10955-017-1754-6

Four-Dimensional Weakly Self-avoiding Walk with Contact Self-attraction

Roland Bauerschmidt[1] · **Gordon Slade**[2] · **Benjamin C. Wallace**[2]

Received: 26 October 2016 / Accepted: 21 February 2017 / Published online: 1 March 2017

Abstract We consider the critical behaviour of the continuous-time weakly self-avoiding walk with contact self-attraction on $\mathbb{Z}^4$, for sufficiently small attraction. We prove that the susceptibility and correlation length of order p (for any $p > 0$) have logarithmic corrections to mean field scaling, and that the critical two-point function is asymptotic to a multiple of $|x|^{-2}$. This shows that small contact self-attraction results in the same critical behaviour as no contact self-attraction; a collapse transition is predicted for larger self-attraction. The proof uses a supersymmetric representation of the two-point function, and is based on a rigorous renormalisation group method that has been used to prove the same results for the weakly self-avoiding walk, without self-attraction.

Keywords Weakly self-avoiding walk · Collapse transition · Renormalisation group

1 The Model and Main Result

The self-avoiding walk is a basic model for a linear polymer chain in a good solution. The repulsive self-avoidance constraint models the excluded volume effect of the polymer. In a *poor* solution, the polymer tends to avoid contact with the solution by instead making contact with itself. This is modelled by a self-attraction favouring nearest-neighbour contacts. The

✉ Benjamin C. Wallace
bwallace@math.ubc.ca

Roland Bauerschmidt
rb812@cam.ac.uk

Gordon Slade
slade@math.ubc.ca

1 Statistical Laboratory, DPMMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK

2 Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

self-avoiding walk is already a notoriously difficult problem, and the combination of these two competing tendencies creates additional difficulties and an interesting phase diagram.

In this paper, we consider a continuous-time version of the weakly self-avoiding walk with nearest-neighbour contact self-attraction on $\mathbb{Z}^4$. When both the self-avoidance and self-attraction are sufficiently weak, we prove that the susceptibility and finite-order correlation length have logarithmic corrections to mean field scaling with exponents $\frac{1}{4}$ and $\frac{1}{8}$ for the logarithm, respectively, and that the critical two-point function is asymptotic to a multiple of $|x|^{-2}$ as $|x| \to \infty$.

1.1 Definition of the Model

For $d > 0$, let X denote the continuous-time simple random walk on $\mathbb{Z}^d$. That is, X is the stochastic process with right-continuous sample paths that takes its steps at the times of the events of a rate-$2d$ Poisson process. A step is independent both of the Poisson process and of all other steps, and is taken uniformly at random to one of the $2d$ nearest neighbours of the current position. The field of *local times* $L_T = (L_T^x)_{x\in\mathbb{Z}^d}$ of X, up to time $T \geq 0$, is defined by

$$L_T^x = \int_0^T \mathbb{1}_{X_t=x}\, dt. \tag{1.1}$$

The *self-intersection local time* and *self-contact local time* of X up to time T are the random variables defined, respectively, by

$$I_T = \sum_{x\in\mathbb{Z}^d} (L_T^x)^2 = \int_0^T ds \int_0^T dt\, \mathbb{1}_{X_s=X_t}, \tag{1.2}$$

$$C_T = \sum_{x\in\mathbb{Z}^d} \sum_{e\in\mathcal{U}} L_T^x L_T^{x+e} = \int_0^T ds \int_0^T dt\, \mathbb{1}_{X_s\sim X_t}, \tag{1.3}$$

where $\mathcal{U}$ is the set of unit vectors in $\mathbb{Z}^d$ and $y \sim x$ indicates that x and y are nearest neighbours.

Given $\beta > 0$ and $\gamma \in \mathbb{R}$, we define

$$U_{\beta,\gamma}(f) = \beta \sum_{x\in\mathbb{Z}^d} f_x^2 - \frac{\gamma}{2d} \sum_{x\in\mathbb{Z}^d} \sum_{e\in\mathcal{U}} f_x f_{x+e} \tag{1.4}$$

for $f : \mathbb{Z}^d \to \mathbb{R}$ with $f_x = 0$ for all but finitely many x. The potential that associates an energy to X in terms of its field of local times is given by

$$U_{\beta,\gamma,T} = U_{\beta,\gamma}(L_T) = \beta I_T - \frac{\gamma}{2d} C_T. \tag{1.5}$$

The energy $U_{\beta,\gamma,T}$ increases with the self-intersection local time, corresponding to weak self-avoidance. For $\gamma > 0$, the energy decreases when the self-contact local time increases, corresponding to a contact self-attraction. For $\gamma < 0$, the contact term is repulsive. We are primarily interested in the case of positive γ, but our results hold also for small negative γ.

Figure 1 shows a sample path X and indicates one self-intersection and four self-contacts. Although I_T also receives contributions from the time the walk spends at each vertex, and C_T receives a contribution from each step, these contributions have the same distribution for all walks taking the same number of steps. The depicted intersections and contacts are the meaningful ones.

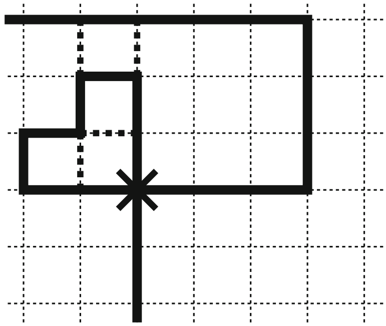

Fig. 1 Polymer with one self-intersection and four self-contacts shown

Let $a, b \in \mathbb{Z}^d$, and let E_a denote the expectation for the process X started at $X(0) = a$. We define

$$c_T = E_a\left(e^{-U_{\beta,\gamma,T}}\right), \quad c_T(a,b) = E_a\left(e^{-U_{\beta,\gamma,T}}\mathbb{1}_{X_T=b}\right). \tag{1.6}$$

By translation-invariance, c_T does not depend on a. For $\nu \in \mathbb{R}$, the *two-point function* is defined by

$$G_{\beta,\gamma,\nu}(a,b) = \int_0^\infty c_T(a,b)e^{-\nu T}\, dT, \tag{1.7}$$

and the *susceptibility* is defined by

$$\chi(\beta,\gamma,\nu) = \int_0^\infty c_T e^{-\nu T}\, dT = \sum_{x\in\mathbb{Z}^d} G_{\beta,\gamma,\nu}(0,x). \tag{1.8}$$

For $p > 0$, we define the *correlation length of order p* by

$$\xi_p(\beta,\gamma,\nu) = \left(\frac{1}{\chi(\beta,\gamma,\nu)}\sum_{x\in\mathbb{Z}^d}|x|^p G_{\beta,\gamma,\nu}(0,x)\right)^{1/p}. \tag{1.9}$$

In (1.7)–(1.9), self-intersections are suppressed by the factor $\exp[-\beta I_T]$, whereas nearest-neighbour contacts are encouraged by the factor $\exp[\frac{\gamma}{2d}C_T]$ when $\gamma > 0$.

1.2 The Critical Point

The right-hand sides of (1.7)–(1.8) are positive or $+\infty$, and monotone decreasing in ν by definition. We define the *critical point*

$$\nu_c(\beta,\gamma) = \inf\{\nu \in \mathbb{R} : \chi(\beta,\gamma,\nu) < \infty\}. \tag{1.10}$$

For $\gamma = 0$, an elementary argument shows that $\nu_c(\beta, 0) > -\infty$ for all dimensions, and furthermore that $\nu_c(\beta, 0) \in [-2\beta(-\Delta_{\mathbb{Z}^d}^{-1})_{0,0}, 0]$ for dimensions $d > 2$; see [3, Lemma A.1]. Here, $\Delta_{\mathbb{Z}^d}$ is the Laplacian on $\mathbb{Z}^d$, i.e., the $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix with entries

$$(\Delta_{\mathbb{Z}^d})_{x,y} = \mathbb{1}_{x\sim y} - 2d\mathbb{1}_{x=y}. \tag{1.11}$$

An equivalent definition is as follows: given a unit vector $e \in \mathbb{Z}^d$, the discrete gradient is defined by $\nabla^e f_x = f_{x+e} - f_x$, and the Laplacian is $\Delta_{\mathbb{Z}^d} f_x = \sum_{e\in\mathcal{U}} \nabla^e f_x = -\frac{1}{2}\sum_{e\in\mathcal{U}} \nabla^{-e}\nabla^e f_x$.

To estimate the critical point when $\gamma \neq 0$, we also define

$$|\nabla f_x|^2 = \sum_{e \in \mathcal{U}} |\nabla^e f_x|^2, \quad |\nabla f|^2 = \sum_{x \in \mathbb{Z}^d} |\nabla f_x|^2. \tag{1.12}$$

From the definition, we see that

$$\sum_{x \in \mathbb{Z}^d} f_x \Delta_{\mathbb{Z}^d} f_x = -\frac{1}{2} |\nabla f|^2. \tag{1.13}$$

It follows that

$$\sum_{x \in \mathbb{Z}^d} \sum_{e \in \mathcal{U}} f_x f_{x+e} = 2d \sum_{x \in \mathbb{Z}^d} f_x^2 + \sum_{x \in \mathbb{Z}^d} f_x \Delta_{\mathbb{Z}^d} f_x = 2d \sum_{x \in \mathbb{Z}^d} f_x^2 - \frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\nabla f_x|^2 \tag{1.14}$$

and so we get the useful representation:

$$U_{\beta,\gamma}(f) = (\beta - \gamma) \sum_{x \in \mathbb{Z}^d} f_x^2 + \frac{\gamma}{4d} \sum_{x \in \mathbb{Z}^d} \sum_{e \in \mathcal{U}} |\nabla^e f_x|^2. \tag{1.15}$$

In particular,

$$U_{\beta,\gamma,T} = (\beta - \gamma) I_T + \frac{\gamma}{4d} |\nabla L_T|^2. \tag{1.16}$$

A version of (1.16) can be found in [21].

Lemma 1.1 *Let $d > 0$. Let $\beta > 0$ and $|\gamma| < \beta$. If $\gamma \geq 0$ then $\nu_c(\beta, \gamma) \in [\nu_c(\beta, 0), \nu_c(\beta - \gamma, 0)]$. If $\gamma < 0$ then $\nu_c(\beta, \gamma) \in [\nu_c(\beta - \gamma, 0), \nu_c(\beta, 0)]$.*

Proof Suppose first that $\gamma \in [0, \beta)$. It follows from (1.5) and (1.16) that

$$U_{\beta-\gamma,0,T} \leq U_{\beta,\gamma,T} \leq U_{\beta,0,T}, \tag{1.17}$$

which implies the desired estimates for $\nu_c(\beta, \gamma)$.

On the other hand, if $\gamma \in (-\beta, 0)$ then the inequalities are reversed and now

$$U_{\beta,0,T} \leq U_{\beta,\gamma,T} \leq U_{\beta-\gamma,0,T}, \tag{1.18}$$

which again implies the desired result. □

1.3 The Main Result

Our main result is the following theorem. It shows that in dimension $d = 4$, for sufficiently small β and γ, the two-point function (1.7) has the same asymptotic decay, to leading order, as the simple random walk two-point function. It also shows that the susceptibility and correlation length of order p exhibit logarithmic corrections to mean-field behaviour. These results were all proved for $\gamma = 0$ in [2,3,6], and we extend them here to small nonzero γ.

We denote the Laplacian on $\mathbb{R}^d$ by $\Delta_{\mathbb{R}^d}$ and define a constant c_p by

$$\mathsf{c}_p^p = \int_{\mathbb{R}^4} |x|^p (-\Delta_{\mathbb{R}^4} + 1)^{-1}_{0x} \, dx. \tag{1.19}$$

Theorem 1.2 *Let $d = 4$. There exist $\beta_* > 0$ and a positive function $\gamma_* : (0, \beta_*) \to \mathbb{R}$ such that whenever $0 < \beta < \beta_*$ and $|\gamma| < \gamma_*(\beta)$, there are constants $A_{\beta,\gamma}$ and $B_{\beta,\gamma}$ such that the following hold:*

(i) The critical two-point function decays as

$$G_{\beta,\gamma,\nu_c}(0,x) = A_{\beta,\gamma}|x|^{-2}\left(1 + O\left(\frac{1}{\log|x|}\right)\right) \quad as |x| \to \infty, \tag{1.20}$$

with $A_{\beta,\gamma} = \frac{1}{4\pi^2}(1 + O(\beta))$ *as* $\beta \downarrow 0$.

(ii) The susceptibility diverges as

$$\chi(\beta,\gamma,\nu_c + \varepsilon) \sim B_{\beta,\gamma}\varepsilon^{-1}\left(\log\varepsilon^{-1}\right)^{1/4}, \quad \varepsilon \downarrow 0, \tag{1.21}$$

with $B_{\beta,\gamma} = \left(\frac{\beta}{2\pi^2}\right)^{1/4}(1 + O(\beta))$ *as* $\beta \downarrow 0$.

(iii) For any $p > 0$*, if* β_* *is chosen small depending on* p*, then the correlation length of order* p *diverges as*

$$\xi_p(\beta,\gamma,\nu_c + \varepsilon) \sim B_{\beta,\gamma}^{1/2}\mathsf{c}_p\varepsilon^{-1/2}\left(\log\varepsilon^{-1}\right)^{1/8}, \quad \varepsilon \downarrow 0. \tag{1.22}$$

Our method of proof extends the renormalisation group argument, used for $\gamma = 0$ in [2,3,6,27], to small nonzero γ. In Sect. 2, as a first step, we show that the two-point function can be approximated by a finite-volume one. The finite-volume two-point function has a supersymmetric integral representation [7,9,10], which we state in Sect. 3. These two sections do not involve the renormalisation group. The application of the renormalisation group method requires the following new ingredients: (i) in Sect. 4, we provide estimates on the contact attraction which show that it is compatible with the renormalisation group method developed in [13,14], and also with the dynamical systems theorem proved in [5], (ii) in Sect. 5, we use the implicit function theorem to extend the identification of the critical point from $\gamma = 0$ to $\gamma \neq 0$, and complete the proof of Theorem 1.2.

In fact, we demonstrate that after the introduction of γ, chosen sufficiently small depending on g, we may use the the same renormalisation group flow of the remaining coupling constants as in the case $\gamma = 0$, to second order in these coupling constants. Thus, since the critical exponents are determined by this second-order flow, they are independent of small γ, and take the same values as for $\gamma = 0$. The critical value $\nu_c(\beta,\gamma)$ does, however, depend on γ.

1.4 Critical Exponents and Polymer Collapse

It has been known for decades that self-avoiding walk obeys mean-field behaviour in dimensions $d \geq 5$. In particular, a version of Theorem 1.2 for the strictly self-avoiding walk (in discrete time with $\beta = \infty$ and $\gamma = 0$) in dimensions $d \geq 5$ was proved in [18,19] using the lace expansion [15]. In its original applications, the lace expansion relied on the purely repulsive nature of the self-avoidance interaction. Models incorporating attraction require new ideas. For a particular model with self-attraction and specially chosen exponentially decaying step weights, the lace expansion was used in [28] to prove that, for $d \geq 5$, the mean-square displacement grows diffusively for small attraction. More recently [20], the lace expansion has been applied in situations where repulsion occurs only in an average sense. In a further development [17], the lace expansion has been applied to a model of strictly self-avoiding walk with a self-attraction that rewards visits to adjacent parallel edges, to prove that sufficiently weak self-attraction does not affect the critical behaviour in dimensions $d \geq 5$. The results of [17,28] for $d \geq 5$ complement our results for $d = 4$, via entirely different methods.

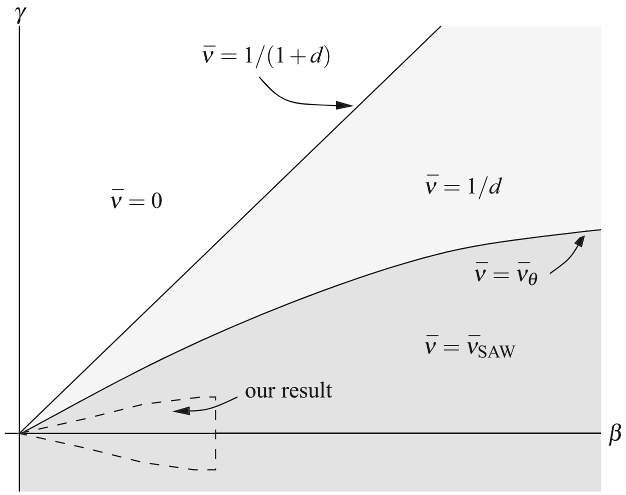

Fig. 2 The predicted phase diagram for $d \geq 2$

Assuming it exists, the critical exponent $\bar{\nu}$ for the mean-square displacement is defined by

$$\langle |X(T)|^2 \rangle = \frac{1}{c_T} E_0 \left(|X(T)|^2 e^{-U_{\beta,\gamma,T}} \right) \approx T^{2\bar{\nu}}, \tag{1.23}$$

possibly with logarithmic corrections. A general tenet of the theory of critical phenomena asserts that other natural length scales, such as the correlation length of order p, are also governed by the exponent $\bar{\nu}$. A typical argument for this, found in physics textbooks, goes as follows. It is predicted that $c_T \approx e^{\nu_c T} T^{\bar{\gamma}-1}$, where $\bar{\gamma}$ is the critical exponent for the susceptibility [for $d = 4$, $\bar{\gamma} = 1$ with a logarithmic correction, by (1.21)]. By definition,

$$\xi_2(\beta, \gamma, \nu)^2 = \frac{\int_0^\infty \langle |X(T)|^2 \rangle c_T e^{-\nu T} \, dT}{\int_0^\infty c_T e^{-\nu T} \, dT}. \tag{1.24}$$

In (1.24), we substitute the asymptotic formula for c_T, as well as (1.23), to obtain

$$\xi_2(\beta, \gamma, \nu) \approx (\nu - \nu_c)^{-\bar{\nu}} \quad \text{as } \nu \downarrow \nu_c, \tag{1.25}$$

with the same exponent $\bar{\nu}$ as in (1.23).

The weakly self-avoiding walk with contact self-attraction is a model for polymer collapse. Polymer collapse corresponds to a discontinuous reduction in the exponent $\bar{\nu}$ as γ increases. A summary of results, predictions, and references can be found in [23, Chapter 6]. See also [24,29]. The predicted phase diagram for dimensions $d \geq 2$ is shown in Figure 2. The predicted values of the exponent at the θ-transition are $\bar{\nu}_\theta = \frac{4}{7}$ for $d = 2$ and $\bar{\nu}_\theta = \frac{1}{2}$ for $d \geq 3$ [23]. The phase labelled $\bar{\nu}_{\rm SAW}$ takes its name from the fact that in this phase the model with attraction is predicted to be in the same universality class as the self-avoiding walk. The predicted values of the exponent $\bar{\nu}_{\rm SAW}$ for the self-avoiding walk are respectively $\frac{3}{4}$, 0.587597(7), $\frac{1}{2}$ for $d = 2, 3, 4$ (with a logarithmic correction for $d = 4$; see [16] for $d = 3$), and it has been proved that $\bar{\nu}_{\rm SAW} = \frac{1}{2}$ for $d \geq 5$ [15,19]. It remains a major challenge in the mathematical theory of polymers to prove the full validity of the phase diagram in all dimensions $d \geq 2$. Very recently, the existence of a collapse transition (a singularity of the free energy) has been proven for a two-dimensional *prudent* self-avoiding walk with contact self-attraction [26].

For $\gamma \geq 0$, the significance of the restriction $\gamma < \beta$ has been noted for a closely related discrete-time model, for which it is proved that for $\gamma > \beta$ the walk is in a compact phase in the sense that $\bar{\nu} = 0$, whereas for $\gamma < \beta$ it is the case that $\bar{\nu} \geq 1/d$ [21]. In the compact phase, the discrete-time model obeys the analogue of $c_T \approx e^{kT^2}$ with $k > 0$, so $\chi(\beta, \gamma, \nu) = \infty$ for all $\nu \in \mathbb{R}$ and $\nu_c = +\infty$. For the 1-dimensional case, the behaviour for the transition line $\gamma = \beta$ has been studied in [22].

The axis $\gamma = 0$ corresponds to the weakly self-avoiding walk which is well understood in dimensions $d \geq 5$ [15,19], and in dimension 4 [2,3,6]. Theorem 1.2 extends the results of [2,3,6] for dimension $d = 4$ to the region bounded by the dashed line. Our results show that for $d = 4$ there is no polymer collapse for small contact self-attraction, in the sense that the critical behaviour remains the same with small contact attraction as with no contact attraction. In particular, Theorem 1.2(iii) shows that, in the sense of (1.25), when γ is small, $\bar{\nu} = \frac{1}{2}$ holds with a logarithmic correction.

2 Finite-Volume Approximation

The first step in the proof of Theorem 1.2 is an approximation of $G_{\beta,\gamma,\nu}(a, b)$ and $\chi(\beta, \gamma, \nu)$ by finite-volume analogues of these quantities. This is the content of Proposition 2.2.

Before proving the proposition, we require some preliminaries. Let P^n be the projection of $\mathbb{Z}^d$ onto the discrete torus of side n, which we denote $\mathbb{Z}^d_n$. Then P^n has a natural action on the path space $(\mathbb{Z}^d)^{[0,\infty)}$. We let $X^n = P^n(X)$ be the projection of X and note that X^n is a simple random walk on $\mathbb{Z}^d_n$.

We call $h = (h_x)_{x\in\mathbb{Z}^d}$ a *field of path functionals* if $h_x : (\mathbb{Z}^d)^{[0,\infty)} \to \mathbb{R}$ is a function on continuous-time paths for each $x \in \mathbb{Z}^d$; a simple example is given by the local time functional. We assume that the *random* field $h(X) = (h_x(X))_{x\in\mathbb{Z}^d}$ has finite support almost surely, i.e., with probability 1, $h_x(X) = 0$ for all but finitely many x. Denote by $h(X^n)$ the corresponding random field for X^n, i.e., for $x \in \mathbb{Z}^d_n$,

$$h_x(X^n) = \sum_{y\in\mathbb{Z}^d} h_{x+ny}(X). \tag{2.1}$$

Given a positive integer k, we define $Q_k \subset \mathbb{Z}^d$ by $Q_k = \{y \in \mathbb{Z}^d : 0 \leq y_i < k,\ i = 1, \dots, d\}$. Then, for integers $n, k \geq 1$,

$$\sum_{y\in Q_k} h_{x+ny}(X^{kn}) = \sum_{y\in Q_k} \sum_{z\in\mathbb{Z}^d} h_{x+ny+knz}(X) = \sum_{y\in\mathbb{Z}^d} h_{x+ny}(X) = h_x(X^n), \tag{2.2}$$

and it follows by summation over $x \in \mathbb{Z}^d_n$ that

$$\sum_{x\in\mathbb{Z}^d_{kn}} h_x(X^{kn}) = \sum_{x\in\mathbb{Z}^d_n} h_x(X^n). \tag{2.3}$$

Lemma 2.1 *Let $n, k \geq 1$ and let f and g be nonnegative fields of path functionals with finite support almost surely. Then*

$$\sum_{x\in\mathbb{Z}^d_{kn}} f_x(X^{kn})g_x(X^{kn}) \leq \sum_{x\in\mathbb{Z}^d_n} f_x(X^n)g_x(X^n). \tag{2.4}$$

Proof By (2.3) and (2.2),

$$\sum_{x\in\mathbb{Z}^d_{kn}} f_x(X^{kn})g_x(X^{kn}) = \sum_{x\in\mathbb{Z}^d_n}\sum_{y\in Q_k} f_{x+ny}(X^{kn})g_{x+ny}(X^{kn}). \tag{2.5}$$

By nonnegativity and two more applications of (2.2),

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d_n}\sum_{y\in Q_k} f_{x+ny}(X^{kn})g_{x+ny}(X^{kn}) &\le \sum_{x\in\mathbb{Z}^d_n}\left(\sum_{y\in Q_k} f_{x+ny}(X^{kn})\right)\left(\sum_{y\in Q_k} g_{x+ny}(X^{kn})\right)\\ &= \sum_{x\in\mathbb{Z}^d_n} f_x(X^n)g_x(X^n).\end{aligned} \tag{2.6}$$

This completes the proof. □

Fix $L \ge 2$ and $N \ge 1$. We write Λ_N for the torus $\mathbb{Z}^d_n$ with $n = L^N$. Thus, X^{L^N} is the simple random walk on Λ_N. For $F_T = F_T(X)$ any one of the functions L^x_T, I_T, C_T of X defined in (1.1)–(1.3), we write $F_{N,T} = F_T(X^{L^N})$. For instance, with $n = L^N$,

$$L^x_{N,T} = \int_0^T \mathbb{1}_{X^n_t = x}\, dt, \quad I_{N,T} = \sum_{x\in\Lambda_N}(L^x_{N,T})^2. \tag{2.7}$$

We apply Lemma 2.1 with $k = L$ and $n = L^N$ for three choices of f, g:

$$I_{N+1,T} \le I_{N,T} \qquad \left(f_x = g_x = L^x_T\right), \tag{2.8}$$

$$C_{N+1,T} \le C_{N,T} \qquad \left(f_x = \textstyle\sum_{e\in\mathcal{U}} L^{x+e}_T,\ g_x = L^x_T\right), \tag{2.9}$$

$$\sum_{x\in\Lambda_{N+1}} |\nabla^e L^x_{N+1,T}|^2 \le \sum_{x\in\Lambda_N} |\nabla^e L^x_{N,T}|^2 \qquad \left(f_x = g_x = \left|\nabla^e L^x_T\right|\right). \tag{2.10}$$

Summation of (2.10) over $e \in \mathcal{U}$ also gives

$$\sum_{x\in\Lambda_{N+1}} |\nabla L^x_{N+1,T}|^2 \le \sum_{x\in\Lambda_N} |\nabla L^x_{N,T}|^2. \tag{2.11}$$

We identify the vertices of Λ_N with nested subsets of $\mathbb{Z}^d$, centred at the origin (approximately if L is even), with Λ_{N+1} paved by L^d translates of Λ_N. We can thus define $\partial\Lambda_N$ to be the inner vertex boundary of Λ_N. We denote the expectation of X^{L^N} started from $a \in \Lambda_N$ by $E^{\Lambda_N}_a$ and define

$$c_{N,T}(a,b) = E^{\Lambda_N}_a\left(e^{-U_{\beta,\gamma,T}}\mathbb{1}_{X(T)=b}\right) \quad (a,b\in\Lambda_N), \tag{2.12}$$

$$c_{N,T} = E^{\Lambda_N}_0\left(e^{-U_{\beta,\gamma,T}}\right). \tag{2.13}$$

The finite-volume two-point function and susceptibility are defined by

$$G_{N,\beta,\gamma,\nu}(a,b) = \int_0^\infty c_{N,T}(a,b)e^{-\nu T}\, dT, \tag{2.14}$$

$$\chi_N(\beta,\gamma,\nu) = \int_0^\infty c_{N,T}e^{-\nu T}\, dT. \tag{2.15}$$

Proposition 2.2 *Let $d > 0$, $\beta > 0$ and $\gamma < \beta$. For all $\nu \in \mathbb{R}$,*

$$\lim_{N\to\infty} G_{N,\beta,\gamma,\nu}(a,b) = G_{\beta,\gamma,\nu}(a,b) \tag{2.16}$$

and

$$\lim_{N\to\infty} \chi_N(\beta,\gamma,\nu) = \chi(\beta,\gamma,\nu). \tag{2.17}$$

Proof Fix $a, b \in \mathbb{Z}^d$, and consider N sufficiently large that a, b can be identified with points in Λ_N. By (1.16), (2.8) and (2.11) (if $0 \le \gamma < \beta$), or by (1.5), (2.8) and (2.9) (if $\gamma < 0$),

$$c_{N,T}(a,b) \le c_{N+1,T}(a,b). \tag{2.18}$$

Thus, (2.16) follows by monotone convergence, once we show that

$$\lim_{N\to\infty} c_{N,T}(a,b) = c_T(a,b). \tag{2.19}$$

This follows as in [2, (2.8)]. That is, first we define

$$c^*_{N,T}(a,b) = E_a^{\Lambda_N}\left(e^{-U_{\beta,\gamma,T}} \mathbb{1}_{X(T)=b} \mathbb{1}_{\{X([0,T])\cap\partial\Lambda_N \neq \varnothing\}}\right) \tag{2.20}$$

$$c^*_{T}(a,b) = E_a\left(e^{-U_{\beta,\gamma,T}} \mathbb{1}_{X(T)=b} \mathbb{1}_{\{X([0,T])\cap\partial\Lambda_N \neq \varnothing\}}\right). \tag{2.21}$$

Since walks which do not reach $\partial\Lambda_N$ make equal contributions to both $c_T(a,b)$ and $c_{N,T}(a,b)$, we have

$$c_T(a,b) - c^*_T(a,b) = c_{N,T}(a,b) - c^*_{N,T}(a,b). \tag{2.22}$$

Thus,

$$|c_T(a,b) - c_{N,T}(a,b)| = |c^*_T(a,b) - c^*_{N,T}(a,b)| \le c^*_T(a,b) + c^*_{N,T}(a,b). \tag{2.23}$$

Let $P_a^{\Lambda_N}$ and P_a be the measures associated with $E_a^{\Lambda_N}$ and E_a, respectively. With Y_t a rate-$2d$ Poisson process with measure P,

$$\begin{aligned} c^*_T(a,b) + c^*_{N,T}(a,b) &\le P_a(X([0,T]) \cap \partial\Lambda_N \neq \varnothing) + P_a^{\Lambda_N}(X([0,T]) \cap \partial\Lambda_N \neq \varnothing) \\ &\le 2\mathsf{P}(Y_T \ge \operatorname{diam}(\Lambda_N)) \to 0 \end{aligned} \tag{2.24}$$

as $N \to \infty$. This completes the proof of (2.16).

Finally, by monotone convergence of G_N to G, for $\nu \in \mathbb{R}$,

$$\lim_{N\to\infty} \chi_N(g,\gamma,\nu) = \sum_{b\in\mathbb{Z}^d} \lim_{N\to\infty} G_{N,g,\gamma,\nu}(a,b)\mathbb{1}_{b\in\Lambda_N} = \chi(g,\gamma,\nu), \tag{2.25}$$

which proves (2.17). $\square$

3 Integral Representation and Progressive Integration

In this section, we reformulate the model in terms of a perturbation of a supersymmetric Gaussian integral, in order to prepare for the application of the renormalisation group. The integral representation, which is a special case of a result from [9], makes use of the Grassmann integral. We begin by recalling the definition of the Grassmann integral in Sect. 3.1 and state the integral representation in Sect. 3.2. In Sect. 3.3, we split the integral into a Gaussian part and a perturbation. The basic idea underlying the renormalisation group is the progressive evaluation of this Gaussian integral via a multi-scale decomposition of its covariance, which we introduce in Sect. 3.4.

3.1 Boson and Fermion Fields

We fix N and write $\Lambda = \Lambda_N$. Given complex variables $\phi_x, \bar{\phi}_x$ (the boson field) for $x \in \Lambda$, we define the differentials (the fermion field)

$$\psi_x = \frac{1}{\sqrt{2\pi i}} d\phi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\phi}_x, \tag{3.1}$$

where we fix a choice of complex square root. The fermion fields are multiplied with each other via the anti-commutative wedge product, though we suppress this in our notation.

A differential form that is the product of a function of $(\phi, \bar{\phi})$ with p differentials is said to have degree p. A sum of forms of even degree is said to be *even*. We introduce a copy $\bar{\Lambda}$ of Λ and we denote the copy of $X \subset \Lambda$ by $\bar{X} \subset \bar{\Lambda}$. We also denote the copy of $x \in \Lambda$ by $\bar{x} \in \bar{\Lambda}$ and define $\phi_{\bar{x}} = \bar{\phi}_x$ and $\psi_{\bar{x}} = \bar{\psi}_x$. Then any differential form F can be written

$$F = \sum_{\vec{y}} F_{\vec{y}}(\phi, \bar{\phi}) \psi^{\vec{y}} \tag{3.2}$$

where the sum is over finite sequences $\vec{y}$ over $\Lambda \sqcup \bar{\Lambda}$, and $\psi^{\vec{y}} = \psi_{y_1} \ldots \psi_{y_p}$ when $\vec{y} = (y_1, \ldots, y_p)$. When $\vec{y} = \varnothing$ is the empty sequence, $F_\varnothing$ denotes the 0-degree (bosonic) part of F.

In order to apply the results of [2,3,6], we require smoothness of the coefficients $F_{\vec{y}}$ of F. For Theorem 1.2(i, ii), we need these coefficients to be C^{10}, and for Theorem 1.2(iii) we require a p-dependent number of derivatives for the analysis of ξ_p, as discussed in [6]. We let $\mathcal{N}$ be the algebra of even forms with sufficiently smooth coefficients and we let $\mathcal{N}(X) \subset \mathcal{N}$ be the sub-algebra of even forms only depending on fields in X. Thus, for $F \in \mathcal{N}(X)$, the sum in (3.2) runs over sequences $\vec{y}$ over $X \sqcup \bar{X}$. Note that $\mathcal{N} = \mathcal{N}(\Lambda)$.

Now let $F = (F_j)_{j \in J}$ be a finite collection of even forms indexed by a set J and write $F_\varnothing = (F_{\varnothing, j})_{j \in J}$. Given a C^∞ function $f : \mathbb{R}^J \to \mathbb{C}$, we define $f(F)$ by its Taylor expansion about $F_\varnothing$:

$$f(F) = \sum_\alpha \frac{1}{\alpha!} f^{(\alpha)}(F_\varnothing)(F - F_\varnothing)^\alpha. \tag{3.3}$$

The summation terminates as a finite sum, since $\psi_x^2 = \bar{\psi}_x^2 = 0$ due to the anti-commutative product.

We define the integral $\int F$ of a differential form F in the usual way as the Riemann integral of its top-degree part (which may be regarded as a function of the boson field). In particular, given a positive-definite $\Lambda \times \Lambda$ symmetric matrix C with inverse $A = C^{-1}$, we define the *Gaussian expectation* (or *super-expectation*) of F by

$$\mathbb{E}_C F = \int e^{-S_A} F, \tag{3.4}$$

where

$$S_A = \sum_{x \in \Lambda} \Big(\phi_x (A\bar{\phi})_x + \psi_x (A\bar{\psi})_x \Big). \tag{3.5}$$

Finally, for $F = f(\phi, \bar{\phi})\psi^{\vec{y}}$, we let

$$\theta F = f(\phi + \xi, \bar{\phi} + \bar{\xi})(\psi + \eta)^{\vec{y}}, \tag{3.6}$$

where ξ is a new boson field, $\eta = (2\pi i)^{-1/2} d\xi$ a new fermion field, and $\bar{\xi}, \bar{\eta}$ are the corresponding conjugate fields. We extend θ to all $F \in \mathcal{N}$ by linearity and define the convolution operator $\mathbb{E}_C\theta$ by letting $\mathbb{E}_C\theta F \in \mathcal{N}$ denote the Gaussian expectation of θF with respect to $(\xi, \bar{\xi}, \eta, \bar{\eta})$, with $\phi, \bar{\phi}, \psi, \bar{\psi}$ held fixed.

3.2 Integral Representation of the Two-Point Function

An integral representation formula applying to general local time functionals is given in [7,9]; see also [27, Appendix A]. We state the result we need in the proposition below.

Let Δ denote the Laplacian on Λ, i.e. Δ_{xy} is given by the right-hand side of (1.11) for $x, y \in \Lambda$. We define the differential forms:

$$\tau_x = \phi_x\bar{\phi}_x + \psi_x\bar{\psi}_x \tag{3.7}$$

$$\tau_{\Delta,x} = \frac{1}{2}\Big(\phi_x(-\Delta\bar{\phi})_x + (-\Delta\phi)_x\bar{\phi}_x + \psi_x(-\Delta\bar{\psi})_x + (-\Delta\psi)_x\bar{\psi}_x\Big) \tag{3.8}$$

$$|\nabla\tau_x|^2 = \sum_{e\in\mathcal{U}}(\nabla^e\tau)_x^2. \tag{3.9}$$

Proposition 3.1 *Let $d > 0$ and $\beta > 0$. For $\gamma < \beta$ and $\nu \in \mathbb{R}$,*

$$G_{N,\beta,\gamma,\nu}(a, b) = \int e^{-\sum_{x\in\Lambda}(U_{\beta,\gamma}(\tau)+\nu\tau_x+\tau_{\Delta,x})}\bar{\phi}_a\phi_b. \tag{3.10}$$

Proof The proof is identical to the proof of the $p = 1$ case of [27, Proposition 2.2] when, in the notation used in [27], we set

$$F(S) = e^{-U_{\beta,\gamma}(S)-(\nu-1)\sum_{x\in\Lambda}S_x} \tag{3.11}$$

in [27, (A.13)]. □

3.3 Gaussian Approximation

We divide the integral in (3.10) into a Gaussian part and a perturbation. Although the division is arbitrary here, a careful choice of the division must be made, and it is made in Theorem 5.1. We require several definitions. Let $z_0 > -1$ and $m^2 > 0$. We set

$$g_0 = (\beta - \gamma)(1 + z_0)^2, \quad \nu_0 = \nu(1 + z_0) - m^2, \quad \gamma_0 = \frac{1}{4d}\gamma(1 + z_0)^2, \tag{3.12}$$

and define

$$V_{0,x}^+ = g_0\tau_x^2 + \nu_0\tau_x + z_0\tau_{\Delta,x}, \quad U_x^+ = |\nabla\tau_x|^2. \tag{3.13}$$

The monomial U_x^+ should not be confused with the potential $U_{\beta,\gamma}$. We define

$$Z_0 = \prod_{x\in\Lambda} e^{-(V_{0,x}^+ + \gamma_0 U_x^+)}, \tag{3.14}$$

and, with $C = (-\Delta + m^2)^{-1}$ and with the expectation given by (3.4),

$$Z_N = \mathbb{E}_C\theta Z_0. \tag{3.15}$$

Recall that $Z_{N,\varnothing}$ denotes the 0-degree part of Z_N. We define a test function $\mathbb{1} : \Lambda_N \to \mathbb{R}$ by $\mathbb{1}_x = 1$ for all x, and write $D^2 Z_{N,\varnothing}(0, 0; \mathbb{1}, \mathbb{1})$ for the directional derivative of $Z_{N,\varnothing}$ at $(\phi, \bar{\phi}) = (0, 0)$, with both directions equal to $\mathbb{1}$. That is,

$$D^2 Z_{N,\varnothing}(0, 0; \mathbb{1}, \mathbb{1}) = \frac{\partial^2}{\partial s \partial t} Z_{N,\varnothing}(s\mathbb{1}, t\mathbb{1})\big|_{s=t=0}. \tag{3.16}$$

Proposition 3.2 *Let $d > 0$, $\gamma, \nu \in \mathbb{R}$, $\beta > 0$ and $\gamma < \beta$. If the relations (3.12) hold, then*

$$G_{N,\beta,\gamma,\nu}(a, b) = (1 + z_0)\mathbb{E}_C(Z_0 \bar{\phi}_a \phi_b), \tag{3.17}$$

and

$$\chi_N(\beta, \gamma, \nu) = (1 + z_0)\hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0, z_0), \tag{3.18}$$

with

$$\hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0, z_0) = \frac{1}{m^2} + \frac{1}{m^4}\frac{1}{|\Lambda|} D^2 Z_{N,\varnothing}(0, 0; \mathbb{1}, \mathbb{1}). \tag{3.19}$$

Proof We make the change of variables $\varphi_x \mapsto (1 + z_0)^{1/2}\varphi_x$ (with $\varphi = \phi, \bar{\phi}, \psi, \bar{\psi}$) in (3.10), and obtain

$$G_{N,\beta,\gamma,\nu}(a, b) = (1 + z_0) \int e^{-\sum_{x\in\Lambda}\left(g_0\tau_x^2 + \gamma_0|\nabla\tau_x|^2 + \nu(1+z_0)\tau_x + (1+z_0)\tau_{\Delta,x}\right)} \bar{\phi}_a \phi_b. \tag{3.20}$$

Then, for any $m^2 \in \mathbb{R}$, we have

$$G_{N,\beta,\gamma,\nu}(a, b) = (1 + z_0) \int e^{-\sum_{x\in\Lambda}(\tau_{\Delta,x} + m^2\tau_x)} Z_0 \bar{\phi}_a \phi_b \tag{3.21}$$

(m^2 simply cancels with ν_0 on the right-hand side). We use this with $m^2 > 0$, so that the inverse matrix $C = (-\Delta + m^2)^{-1}$ exists. By symmetry of the matrix Δ, (3.5) gives

$$S_{(-\Delta+m^2)} = \sum_{x\in\Lambda} \left(\tau_{\Delta,x} + m^2\tau_x\right). \tag{3.22}$$

Then (3.17) follows from (3.21)–(3.22) and (3.4). Summation over $b \in \Lambda_N$ gives the formula $\chi_N(\beta, \gamma, \nu) = (1 + z_0) \sum_{x\in\Lambda} \mathbb{E}_C(Z_0 \bar{\phi}_0 \phi_x)$. Then (3.18), with (3.19), follows by an elementary computation as in [3, Section 4.1]. □

3.4 Progressive Integration

The identity (3.17) splits the two-point function into a Gaussian part and a perturbation Z_0. The Gaussian part is parametrised by (m^2, z_0), although the dependence on z_0 has been shifted out of the integral. We analyse the integral (3.17) using the renormalisation group method developed in [4,11–14], which is itself inspired by [30]. This method is based on a decomposition

$$C = C_1 + \cdots + C_{N-1} + C_{N,N}, \tag{3.23}$$

of the covariance C used to define Z_N in (3.15), where $C_1, \ldots, C_{N-1}, C_{N,N}$ are covariances. For simplicity, we write $C_N = C_{N,N}$. A *finite-range* decomposition of this sort was constructed in [1,8]. Specifically, we use the decomposition of [1].

The covariance decomposition allows us to evaluate Z_N progressively by defining inductively

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j \quad (j < N). \tag{3.24}$$

It is a basic fact that a sum of two independent Gaussian random variables with covariances C' and C'' is itself Gaussian with covariance $C' + C''$. By [11, Proposition 2.6], this property extends to the Gaussian super-expectation in the sense that

$$\mathbb{E}_C\theta = \mathbb{E}_{C_N}\theta \circ \ldots \circ \mathbb{E}_{C_1}\theta. \tag{3.25}$$

Thus, the definition of Z_{j+1} in (3.24) agrees with (3.15) when $j+1 = N$.

From the perspective of the renormalisation group, we view the map $Z_j \mapsto Z_{j+1}$ as defining a dynamical system. The evaluation of Z_N can be accomplished by studying this system's dependence on its initial condition, as we discuss in the next section.

4 Initial Coordinates for the Renormalisation Group

Following the approach of [3], we represent Z_j by a pair of coordinates I_j and K_j that capture the *relevant* (expanding), *marginal*, and *irrelevant* (contracting) parts of Z_j. We begin in Sect. 4.1 by defining coordinates (I_0, K_0) for Z_0. Norms used to control the evolution of these coordinates are introduced in Sect. 4.2, and it is shown in Sects. 4.3, 4.4 that K_0 satisfies norm estimates that permit the results of [5,14] to be applied. The initial coordinate K_0 depends on the coupling constants $(g_0, \gamma_0, \nu_0, z_0)$ of (3.12), and regularity of K_0 as a function of these variables is established in Sect. 4.5.

4.1 Initial Coordinates for the Renormalisation Group

We now divide Z_0 into coordinates I_0 and K_0. The division depends on the sign of γ.

4.1.1 Coordinates for Positive γ

Assume that $\gamma \geq 0$. For $X \subset \Lambda$, we define

$$I_0^+(X) = \prod_{x\in X} e^{-V_{0,x}^+}, \qquad K_0^+(X) = \prod_{x\in X} I_{0,x}^+(e^{-\gamma_0 U_x^+} - 1). \tag{4.1}$$

Here, $I_{0,x}^+ = I_0^+(\{x\})$, and we usually denote evaluation at a singleton by a subscript. By definition and binomial expansion,

$$Z_0 = \prod_{x\in\Lambda}\left(I_{0,x}^+ + K_{0,x}^+\right) = \sum_{X\subset\Lambda} I_0^+(\Lambda\setminus X)K_0^+(X). \tag{4.2}$$

This *polymer gas representation* of Z_0 extends a much simpler representation used to study the weakly self-avoiding walk previously, e.g., in [2,3]. In particular, when $\gamma_0 = 0$,

$$K_0^+(X) = \mathbb{1}_\varnothing(X) = \begin{cases} 1 & X = \varnothing \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

and (4.2) agrees with [3, (5.27)]. Thus the effect of nonzero γ_0 is incorporated entirely into the non-trivial K_0^+ of (4.1), rather than (4.3).

Then (V_0^+, K_0^+) can be viewed as the initial condition of the dynamical system (3.24). This initial condition is *not* uniquely defined as a function of (β, γ, ν). Rather, the constraints (3.12) leave us with the freedom to choose ν_0 and z_0 as we please. The key to the success of the renormalisation group method is the identification of *critical* values ν_0^c, z_0^c that lie on a stable manifold for the *Gaussian fixed point* $(V_0, K_0) = 0$. The existence of the stable manifold, which is a highly non-trivial fact, is obtained using the main result of [5]. This result allows for the possibility that K_0^+ is non-zero as long as $\|K_0^+\| = O(g_0^3)$ in an appropriate norm. We take advantage of this additional generality in order to prove Theorem 1.2.

4.1.2 *Coordinates for Negative γ*

Assume that $\gamma < 0$. Define

$$V_{0,x}^- = V_{0,x}^+ + 4d\gamma_0\tau_x^2, \quad U_x^- = 2\sum_{e\in\mathcal{U}} \tau_x\tau_{x+e}. \tag{4.4}$$

By the identity

$$\sum_{x\in\Lambda}\Big(g_0\tau_x^2 + \gamma_0\sum_{e\in\mathcal{U}}(\nabla^e\tau_x)^2\Big) = \sum_{x\in\Lambda}\Big((g_0+4d\gamma_0)\tau_x^2 - 2\gamma_0\sum_{e\in\mathcal{U}}\tau_x\tau_{x+e}\Big), \tag{4.5}$$

we can write

$$Z_0 = \prod_{x\in\Lambda}(I_{0,x}^- + K_{0,x}^-) = \sum_{X\subset\Lambda} I_0^-(\Lambda\setminus X)K_0^-(X), \tag{4.6}$$

with

$$I_0^-(X) = \prod_{x\in X} e^{-V_{0,x}^-}, \qquad K_0^-(X) = \prod_{x\in X} I_{0,x}^-(e^{\gamma_0 U_x^-} - 1). \tag{4.7}$$

Thus, we can parametrise Z_0 via either pair $(I_0^\pm, K_0^\pm)$. We use (I_0^+, K_0^+) when $\gamma_0 \geq 0$ and (I_0^-, K_0^-) when $\gamma_0 < 0$. With this convention,

$$K_0^\pm(X) = \prod_{x\in X} I_{0,x}^\pm(e^{-|\gamma_0|U_x^\pm} - 1) \quad (\text{use} + \text{for } \gamma_0 \geq 0, \text{use} - \text{for } \gamma_0 < 0). \tag{4.8}$$

4.2 Norms

In this section, we recall some definitions and basic facts concerning norms, from [11]. For now, we only consider the case of scale $j = 0$.

Recall the notation introduced in Sect. 3.1. A *test function* g is defined to be a function $(\vec{x}, \vec{y}) \mapsto g_{\vec{x},\vec{y}}$, where $\vec{x}$ and $\vec{y}$ are finite sequences of elements in $\Lambda \sqcup \bar{\Lambda}$. When $\vec{x}$ or $\vec{y}$ is the empty sequence $\varnothing$, we drop it from the notation as long as this causes no confusion; e.g., we may write $g_{\vec{x}} = g_{\vec{x},\varnothing}$. The length of a sequence $\vec{x}$ is denoted $|\vec{x}|$. Gradients of test functions are defined component-wise. Thus, if $\vec{x} = (x_1, \ldots, x_m)$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with each $\alpha_i \in \mathbb{N}_0^{\mathcal{U}}$, and similarly for $\vec{y} = (y_1, \ldots, y_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, then

$$\nabla_{\vec{x},\vec{y}}^{\alpha,\beta} g_{\vec{x},\vec{y}} = \nabla_{x_1}^{\alpha_1}\ldots\nabla_{x_m}^{\alpha_m}\nabla_{y_1}^{\beta_1}\ldots\nabla_{y_n}^{\beta_n} g_{x_1,\ldots,x_m,y_1,\ldots,y_n}. \tag{4.9}$$

Let $\mathfrak{h}_0 > 0$ be a parameter, which we set below. We fix positive constants $p_\Phi \geq 4$ and $p_{\mathcal{N}}$ and assume that all test functions vanish when $|\vec{x}| + |\vec{y}| > p_{\mathcal{N}}$. For Theorem 1.2(i, ii), any choice of $p_{\mathcal{N}} \geq 10$ is sufficient, whereas for Theorem 1.2(iii) it is necessary to choose $p_{\mathcal{N}}$ large depending on p [6]. The $\Phi = \Phi(\mathfrak{h}_0)$ norm on such test functions is defined by

$$\|g\|_\Phi = \sup_{\vec{x},\vec{y}} \mathfrak{h}_0^{-(|\vec{x}|+|\vec{y}|)} \sup_{\alpha,\beta:|\alpha|_1+|\beta|_1\leq p_\Phi} |\nabla^{\alpha,\beta} g_{\vec{x},\vec{y}}|, \tag{4.10}$$

where $|\alpha|_1$ denotes the total order of the differential operator ∇^α. Thus, for any test function g and for sequences $\vec{x}$, $\vec{y}$ with $|\vec{x}|+|\vec{y}| \leq p_{\mathcal{N}}$ and corresponding α, β with $|\alpha|_1+|\beta|_1 \leq p_\Phi$,

$$|\nabla^{\alpha,\beta} g_{\vec{x},\vec{y}}| \leq \mathfrak{h}_0^{|\vec{x}|+|\vec{y}|} \|g\|_\Phi. \tag{4.11}$$

For any $F \in \mathcal{N}$, there exist *unique* functions $F_{\vec{y}}$ of $(\phi, \bar{\phi})$ that are anti-symmetric under permutations of $\vec{y}$, such that

$$F = \sum_{\vec{y}} \frac{1}{|\vec{y}|!} F_{\vec{y}}(\phi, \bar{\phi}) \psi^{\vec{y}}. \tag{4.12}$$

Given a sequence $\vec{x}$ with $|\vec{x}| = m$, we define

$$F_{\vec{x},\vec{y}} = \frac{\partial^m F_{\vec{y}}}{\partial \phi_{x_1} \dots \partial \phi_{x_m}}. \tag{4.13}$$

We define a ϕ-dependent pairing of elements of $\mathcal{N}$ with test functions, by

$$\langle F, g \rangle_\phi = \sum_{\vec{x},\vec{y}} \frac{1}{|\vec{x}|!|\vec{y}|!} F_{\vec{x},\vec{y}}(\phi, \bar{\phi}) g_{\vec{x},\vec{y}}. \tag{4.14}$$

Let $B(\Phi)$ denote the unit Φ-ball in the space of test functions. Then the $T_\phi = T_\phi(\mathfrak{h}_0)$ semi-norm on $\mathcal{N}$ is defined by

$$\|F\|_{T_\phi} = \sup_{g \in B(\Phi)} |\langle F, g \rangle_\phi|. \tag{4.15}$$

We need several properties of the T_ϕ semi-norm, whose proofs can be found in [11]. First, there is the important *product property* [11, Proposition 3.7]

$$\|FG\|_{T_\phi} \leq \|F\|_{T_\phi} \|G\|_{T_\phi}. \tag{4.16}$$

An immediate consequence is that $\|e^{-F}\|_{T_\phi} \leq e^{\|F\|_{T_\phi}}$. This is improved in [11, Proposition 3.8], which states that (recall that $F_\varnothing$ denotes the 0-degree part of F)

$$\|e^{-F}\|_{T_\phi} \leq e^{-2\mathrm{Re} F_\varnothing(\phi) + \|F\|_{T_\phi}}. \tag{4.17}$$

Each of the two choices $\varphi = \phi, \bar{\phi}$ can be viewed as a test function supported on sequences with $|\vec{x}| = 1$ and $|\vec{y}| = 0$ and satisfying $\varphi_{\vec{x}} = \bar{\varphi}_x$. In particular, $\|\phi\|_\Phi$ is defined as the norm of a test function. We use [11, Proposition 3.10], which states that if $F \in \mathcal{N}$ is a polynomial in $\phi, \bar{\phi}, \psi, \bar{\psi}$ of total degree $A \leq p_{\mathcal{N}}$, then

$$\|F\|_{T_\phi} \leq \|F\|_{T_0} (1 + \|\phi\|_\Phi)^A. \tag{4.18}$$

We write $x^\square = \{y : |y - x|_\infty \leq 2^d - 1\}$, where $|x|_\infty = \max\{|x_i| : 1 \leq i \leq d\}$ (this is the scale-0 version of [13, (1.37)] for a single point). The $\Phi_x \equiv \Phi(x^\square)$ norm of $\phi \in \mathbb{C}^\Lambda$ is defined by

$$\|\phi\|_{\Phi_x} = \inf \left\{ \|\phi - f\|_\Phi : f \in \mathbb{C}^\Lambda \text{ such that } f_y = 0 \ \forall y \in x^\square \right\}. \tag{4.19}$$

By taking the infimum in (4.18) over all possible re-definitions of ϕ_y for $y \notin x^\square$, we get

$$\|F\|_{T_\phi} \leq \|F\|_{T_0} (1 + \|\phi\|_{\Phi_x})^A \tag{4.20}$$

when $F \in \mathcal{N}(x^\square)$.

We need two choices of the parameter $\mathfrak{h}_0$ (for both choices, $\mathfrak{h}_0 \geq 1$): either $\mathfrak{h}_0 = \ell_0$, an L-dependent constant; or $\mathfrak{h}_0 = h_0 = k_0 \tilde{g}_0^{-1/4}$, where k_0 is a small constant and $\tilde{g}_0$ is a constant which must be chosen small depending on L. Some discussion of these constants occurs in the proof of Proposition 4.1. In [13], two *regulators* are defined. At scale 0, these are given by

$$G_0(x, \phi) = e^{\|\phi\|^2_{\Phi_x(\ell_0)}}, \qquad \tilde{G}_0(x, \phi) = e^{\frac{1}{2}\|\phi\|^2_{\tilde{\Phi}_x(\ell_0)}}. \tag{4.21}$$

The $\tilde{\Phi}_x$ norm in the definition of $\tilde{G}_0$, is defined in [13, (1.40)]; it is a modification of the Φ_x norm that is invariant under shifts by linear test functions. Its specific properties do not play a direct role in this paper. Two regulator norms are defined for $F \in \mathcal{N}(x^{\square})$ by

$$\|F\|_{G_0} = \sup_{\phi \in \mathbb{C}^\Lambda} \frac{\|F\|_{T_\phi(\ell_0)}}{G_0(x, \phi)}, \quad \|F\|_{\tilde{G}_0^{\mathfrak{t}}} = \sup_{\phi \in \mathbb{C}^\Lambda} \frac{\|F\|_{T_\phi(h_0)}}{\tilde{G}_0^{\mathfrak{t}}(x, \phi)}, \tag{4.22}$$

where $\mathfrak{t} \in (0, 1]$ is a constant power.

4.3 Bounds on K_0

The main estimate on $K_{0,x}^{\pm}$ is given by the following proposition. Consistent with [13, (1.83)], we fix a large constant $C_{\mathcal{D}}$ and define

$$\mathcal{D}_0 = \mathcal{D}_0(\tilde{g}_0) = \{(g, \nu, z) \in \mathbb{R}^3 : C_{\mathcal{D}}^{-1}\tilde{g}_0 < g < C_{\mathcal{D}}\tilde{g}_0,\ |\nu|, |z| < C_{\mathcal{D}}\tilde{g}_0\}. \tag{4.23}$$

Proposition 4.1 *Suppose that $V_0^{\pm} \in \mathcal{D}_0$, with $\tilde{g}_0$ sufficiently small. If $|\gamma_0| \leq \tilde{g}_0$, then (with constants that may depend on L)*

$$\|K_{0,x}^{\pm}\|_{G_0} = O(|\gamma_0|), \quad \|K_{0,x}^{\pm}\|_{\tilde{G}_0} = O(|\gamma_0|/g_0), \tag{4.24}$$

where the bounds on K^+ and K^- hold for $\gamma_0 \geq 0$ and $\gamma_0 < 0$, respectively.

The form of the estimates (4.24) can be anticipated from the definition of $K_0^{\pm}$ in (4.8). The upper bound arises from the small size of $e^{-|\gamma_0|U_x^{\pm}} - 1$. For small fields, hence small $U_x^{\pm}$, this is of order $|\gamma_0|$, as reflected by the G_0 norm estimate of (4.24). For large fields, namely fields of size $|\phi| \approx \tilde{g}_0^{-1/4}$, the difference $e^{-|\gamma_0|U_x^{\pm}} - 1$ is roughly of size $|\gamma_0|\,|\phi|^4 \approx |\gamma_0|/\tilde{g}_0$. This effect is measured by the $\tilde{G}_0$ norm.

Before proving the proposition, we write (4.8) for a singleton as

$$K_{0,x}^{\pm} = I_{0,x}^{\pm} J_x^{\pm}, \tag{4.25}$$

where, by the fundamental theorem of calculus,

$$I_{0,x}^{\pm} = e^{-V_{0,x}^{\pm}} \tag{4.26}$$

$$J_x^{\pm} = e^{-|\gamma_0|U_x^{\pm}} - 1 = -\int_0^1 |\gamma_0| U_x^{\pm} e^{-t|\gamma_0|U_x^{\pm}}\, dt. \tag{4.27}$$

As in (4.8), the $+$ versions of (4.25)–(4.27) hold only for $\gamma_0 \geq 0$ and the $-$ versions only for $\gamma_0 < 0$.

Let $F \in \mathcal{N}(x^{\square})$ be a polynomial of degree at most $p_{\mathcal{N}}$. Then the stability estimates [13, (2.1)–(2.2)] imply that there exists $c_3 > 0$ and, for any $c_1 \geq 0$, there exist positive constants

C, c_2 such that if $V_0^\pm \in \mathcal{D}_0$ then

$$\|I_{0,x}^\pm F\|_{T_\phi(\mathfrak{h}_0)} \le C\|F\|_{T_0(\mathfrak{h}_0)} \begin{cases} e^{c_3 g_0\left(1+\|\phi\|^2_{\Phi_x(\ell_0)}\right)} & \mathfrak{h}_0 = \ell_0 \\ e^{-c_1 k_0^4 \|\phi\|^2_{\Phi_x(h_0)}} e^{c_2 k_0^4 \|\phi\|^2_{\tilde{\Phi}_x(\ell_0)}} & \mathfrak{h}_0 = h_0. \end{cases} \tag{4.28}$$

This essentially reduces our task to estimating $J_x^\pm$. The next lemma is an ingredient for this.

Lemma 4.2 *There is a universal constant $\tilde{C}$ such that*

$$\|U_x^\pm\|_{T_\phi(\mathfrak{h}_0)} \le 2U_{\varnothing,x}^\pm + \tilde{C}\mathfrak{h}_0^4\left(1 + \|\phi\|^2_{\Phi_x(\mathfrak{h}_0)}\right), \tag{4.29}$$

where $U_\varnothing^\pm$ is the 0-degree part of $U^\pm$.

Proof Let

$$M^+ = M_e^+ = (\nabla^e \tau_x)^2, \quad M^- = M_e^- = 2\tau_x\tau_{x+e}, \tag{4.30}$$

so that $U_x^\pm = \sum_{e\in\mathcal{U}} M_e^\pm$. It suffices to prove (4.29) with $U_x^\pm$ replaced by $M^\pm$ (on both sides of the equation). In addition, we can replace the Φ_x norm by the Φ norm; the bound with the Φ_x norm then follows in the same way that (4.20) is a consequence of (4.18), since $M^\pm \in \mathcal{N}(x^\square)$.

By definition of τ_x,

$$M^\pm = M_\varnothing^\pm + R^\pm, \tag{4.31}$$

where

$$M_\varnothing^+ = \left(\nabla^e|\phi_x|^2\right)^2, \qquad R^+ = 2\left(\nabla^e|\phi_x|^2\right)\nabla^e(\psi_x\bar{\psi}_x), \tag{4.32}$$

$$M_\varnothing^- = 2|\phi_x|^2|\phi_{x+e}|^2, \qquad R^- = 2\left(|\phi_x|^2\psi_{x+e}\bar{\psi}_{x+e} + \psi_x\bar{\psi}_x|\phi_{x+e}|^2 + \psi_x\bar{\psi}_x\psi_{x+e}\bar{\psi}_{x+e}\right). \tag{4.33}$$

Thus, $\|M^\pm\|_{T_\phi} \le \|M_\varnothing^\pm\|_{T_\phi} + \|R^\pm\|_{T_\phi}$. A straightforward computation shows that

$$\|R^\pm\|_{T_\phi} = O\left(\mathfrak{h}_0^4(1 + \|\phi\|_\Phi)^2\right). \tag{4.34}$$

By definition of the T_ϕ semi-norm,

$$\|\nabla^e|\phi_x|^2\|_{T_\phi} \le \nabla^e|\phi_x|^2 + 2\mathfrak{h}_0(|\phi_x| + |\phi_{x+e}|) + 2\mathfrak{h}_0^2. \tag{4.35}$$

Together with (4.34), the product property, and (4.11), this implies that

$$\|M^+\|_{T_\phi} \le M_\varnothing^+ + 2|\nabla^e|\phi_x|^2|(2\mathfrak{h}_0(|\phi_x| + |\phi_{x+e}|)) + O\left(\mathfrak{h}_0^4\right)\left(1 + \|\phi\|_\Phi^2\right). \tag{4.36}$$

By the inequality

$$2|ab| \le |a|^2 + |b|^2 \tag{4.37}$$

and another application of (4.11),

$$2|\nabla^e|\phi_x|^2|(2\mathfrak{h}_0(|\phi_x| + |\phi_{x+e}|)) \le M_\varnothing^+ + O\left(\mathfrak{h}_0^2\|\phi\|_\Phi^2\right), \tag{4.38}$$

and the bound on M^+ follows.

For the bound on M^-, we use the identity

$$\|\tau_x\|_{T_\phi} = (|\phi_x| + \mathfrak{h}_0)^2 + \mathfrak{h}_0^2 \tag{4.39}$$

from [11, (3.27)]. By the product property and (4.11), this implies that

$$\begin{aligned}\|M^-\|_{T_\phi} &\le 2|\phi_x|^2|\phi_{x+e}|^2 + 2(|\phi_x||\phi_{x+e}|)(2\mathfrak{h}_0(|\phi_{x+e}| + |\phi_x|)) \\ &\quad + O(\mathfrak{h}_0^4)(1 + \|\phi\|_\Phi^2).\end{aligned} \tag{4.40}$$

Another application of (4.37) and (4.11) gives

$$2(|\phi_x||\phi_{x+e}|)(2\mathfrak{h}_0(|\phi_{x+e}| + |\phi_x|)) \le |\phi_x|^2|\phi_{x+e}|^2 + O(\mathfrak{h}_0^2\|\phi\|_\Phi^2), \tag{4.41}$$

and the proof is complete. □

An immediate consequence of Lemma 4.2, using (4.17), is that for any $s \ge 0$,

$$\|e^{-sU_x^\pm}\|_{T_\phi(\mathfrak{h}_0)} \le e^{\tilde{C}s\mathfrak{h}_0^4\left(1+\|\phi\|^2_{\Phi_x(\mathfrak{h}_0)}\right)}. \tag{4.42}$$

Proof of Proposition 4.1 According to the definition of the regulator norms in (4.21)–(4.22), it suffices to prove that, under the hypothesis on γ_0,

$$\|K_{0,x}^\pm\|_{T_\phi(\mathfrak{h}_0)} = O(|\gamma_0|\mathfrak{h}_0^4)\begin{cases} e^{\|\phi\|^2_{\Phi_x}} & (\mathfrak{h}_0 = \ell_0) \\ e^{\frac{\mathsf{t}}{2}\|\phi\|_{\tilde{\Phi}}} & (\mathfrak{h}_0 = h_0). \end{cases} \tag{4.43}$$

For $t \in [0, 1]$, let $\tilde{I}_x^\pm(t) = e^{-t|\gamma_0|U_x^\pm}$. By (4.25), (4.27), and the product property,

$$\|K_{0,x}^\pm\|_{T_\phi(\mathfrak{h}_0)} \le |\gamma_0|\|I_{0,x}^\pm U_x^\pm\|_{T_\phi(\mathfrak{h}_0)} \sup_{t\in[0,1]} \|\tilde{I}_x^\pm(t)\|_{T_\phi(\mathfrak{h}_0)}. \tag{4.44}$$

By (4.28) and Lemma 4.2, there exists $c_3 > 0$, and, for any $c_1 \ge 0$ there exists $c_2 > 0$, such that

$$\|I_{0,x}^\pm U_x^\pm\|_{T_\phi(\mathfrak{h}_0)} \le O(\mathfrak{h}_0^4)\begin{cases} e^{c_3 g_0\|\phi\|^2_{\Phi_x(\ell_0)}} & \mathfrak{h}_0 = \ell_0 \\ e^{-c_1 k_0^4\|\phi\|^2_{\Phi_x(h_0)}} e^{c_2 k_0^4\|\phi\|^2_{\tilde{\Phi}_x(\ell_0)}} & \mathfrak{h}_0 = h_0. \end{cases} \tag{4.45}$$

The constant in $O(|\gamma_0|\mathfrak{h}_0^4)$ may depend on c_1, but this is unimportant. Also, by (4.42),

$$\sup_{t\in[0,1]} \|\tilde{I}_x^\pm(t)\|_{T_\phi(\mathfrak{h}_0)} \le e^{\tilde{C}|\gamma_0|\mathfrak{h}_0^4\left(1+\|\phi\|^2_{\Phi_x(\mathfrak{h}_0)}\right)}. \tag{4.46}$$

Thus, for $\mathfrak{h}_0 = \ell_0$, the total exponent in our estimate for the right-hand side of (4.44) is

$$\tilde{C}|\gamma_0|\ell_0^4 + (c_3 g_0 + \tilde{C}|\gamma_0|\ell_0^4)\|\phi\|^2_{\Phi_x(\ell_0)}. \tag{4.47}$$

This gives the $\mathfrak{h}_0 = \ell_0$ version of (4.43) provided that g_0 is small and $|\gamma_0|$ is small depending on L.

For $\mathfrak{h}_0 = h_0$, the total exponent in our estimate for the right-hand side of (4.44) is

$$\tilde{C}|\gamma_0|k_0^4\tilde{g}_0^{-1} + (\tilde{C}|\gamma_0|k_0^4\tilde{g}_0^{-1} - c_1 k_0^4)\|\phi\|^2_{\Phi_x(h_0)} + c_2 k_0^4\|\phi\|^2_{\tilde{\Phi}_x(\ell_0)}. \tag{4.48}$$

This gives the $\mathfrak{h}_0 = h_0$ version of (4.43) provided that $|\gamma_0| \le \tilde{g}_0$, $c_1 \ge \tilde{C}$, and $c_2 k_0^4 \le \mathsf{t}/2$.

All the provisos are satisfied if we choose $c_1 \ge \tilde{C}$, k_0 small depending on c_1 and $\tilde{g}_0$ small. □

Remark 4.3 By a small modification to the proof of Proposition 4.1, it can be shown that if $M_x \in \mathcal{N}(x^\square)$ is a monomial of degree $r \le p_\mathcal{N} - 4$ (so that $M_x U_x^\pm$ has degree at most $p_\mathcal{N}$), then

$$\|M_x K_{0,x}^\pm\|_{\mathcal{G}_0} = O(|\gamma_0|\mathfrak{h}_0^{4+r}). \tag{4.49}$$

4.4 Unified Bound on K_0

The results of [5,14] are formulated in a sequence of spaces $\mathcal{W}_j$ that enable the combination of small-field and large-field estimates into a single norm estimate. In this section, we recast the result of Proposition 4.1 to see that $K_0^{\pm}$ fits into this formulation.

We restrict attention in this section to the $\mathcal{W}_0$ norm, whose definition is recalled below. This requires several preliminaries. Let $\mathcal{P}_0 = \mathcal{P}_0(\Lambda)$ denote the collection of subsets of vertices in Λ. We refer to the elements of $\mathcal{P}_0$ as *polymers*. We call a nonempty polymer $X \in \mathcal{P}_0$ *connected* if for any $x, x' \in X$, there is a sequence $x = x_0, \ldots, x_n = x' \in X$ such that $|x_{i+1} - x_i|_\infty = 1$ for $i = 0, \ldots, n-1$. Let $\mathcal{C}_0$ denote the set of connected polymers. The *small set neighbourhood* $X^\square$ of $X \in \mathcal{P}_0$ is defined by

$$X^\square = \{y \in \Lambda : \exists x \in \Lambda \text{ such that } |y - x|_\infty \le 2^d\}. \tag{4.50}$$

We extend the definitions of the regulators $\mathcal{G}_0 = G_0, \tilde{G}_0^{\mathrm{t}}$, defined in (4.21), by setting

$$\mathcal{G}_0(X, \phi) = \prod_{x \in X} \mathcal{G}_0(x, \phi), \tag{4.51}$$

and extend the definitions (4.22) to define norms, for $F \in \mathcal{N}(X^\square)$, by

$$\|F\|_{G_0} = \sup_{\phi \in \mathbb{C}^\Lambda} \frac{\|F\|_{T_\phi(\ell_0)}}{G_0(X, \phi)}, \quad \|F\|_{\tilde{G}_0^{\mathrm{t}}} = \sup_{\phi \in \mathbb{C}^\Lambda} \frac{\|F\|_{T_\phi(h_0)}}{\tilde{G}_0^{\mathrm{t}}(X, \phi)}. \tag{4.52}$$

It follows from the product property of the T_ϕ norm that these norms obey the product property

$$\|F_1 F_2\|_{\mathcal{G}_0} \le \|F_1\|_{\mathcal{G}_0} \|F_2\|_{\mathcal{G}_0} \quad \text{for } F_i \in \mathcal{N}(X_i^\square) \text{ with } X_1 \cap X_2 = \varnothing. \tag{4.53}$$

Given a map $K : \mathcal{P}_0 \to \mathcal{N}$ with the property that $K(X) \in \mathcal{N}(X^\square)$ for all $X \in \mathcal{P}_0$, we define the $\mathcal{F}_0(\mathcal{G})$ norms (for $\mathcal{G} = G, \tilde{G}$) by

$$\|K\|_{\mathcal{F}_0(G)} = \sup_{X \in \mathcal{C}_0} \tilde{g}_0^{-f_0(a,X)} \|K(X)\|_{G_0} \tag{4.54}$$

$$\|K\|_{\mathcal{F}_0(\tilde{G})} = \sup_{X \in \mathcal{C}_0} \tilde{g}_0^{-f_0(a,X)} \|K(X)\|_{\tilde{G}_0^{\mathrm{t}}}, \tag{4.55}$$

with

$$f_0(a, X) = a(|X| - 2^d)_+ = \begin{cases} a(|X| - 2^d) & \text{if } |X| > 2^d \\ 0 & \text{otherwise.} \end{cases} \tag{4.56}$$

Here a is a small constant; its value is discussed below [14, (1.46)]. The $\mathcal{W}_0$ norm is then defined by

$$\|K\|_{\mathcal{W}_0} = \max\left\{ \|K\|_{\mathcal{F}_0(G)}, \; \tilde{g}_0^{9/4} \|K\|_{\mathcal{F}_0(\tilde{G})} \right\}. \tag{4.57}$$

Since this definition depends on $\tilde{g}_0$ and the volume Λ, we sometimes write $\mathcal{W}_0 = \mathcal{W}_0(\tilde{g}_0, \Lambda)$. The following proposition uses Proposition 4.1 to obtain a bound on the $\mathcal{W}_0$ norm of the map $K_0^{\pm} : \mathcal{P}_0 \to \mathcal{N}$ defined by

$$K_0^{\pm}(X) = \prod_{x \in X} K_{0,x}^{\pm} \qquad (X \in \mathcal{P}_0). \tag{4.58}$$

Proposition 4.4 *If $V_0^\pm \in \mathcal{D}_0$ with $\tilde{g}_0$ sufficiently small (depending on L), and if $|\gamma_0| \le O(\tilde{g}_0^{1+a'})$ for some $a' > a$, then $\|K_0^\pm\|_{\mathcal{W}_0} \le O(|\gamma_0|)$, where all constants may depend on L.*

Proof Let X be a connected polymer in $\mathcal{P}_0$. By the product property and Proposition 4.1,

$$\|K_0^\pm(X)\|_{\mathcal{G}_0} \le (c|\gamma_0|\mathfrak{h}_0^4)^{|X|} = (c|\gamma_0|\mathfrak{h}_0^4)^{|X|\wedge 2^d}(c|\gamma_0|\mathfrak{h}_0^4)^{(|X|-2^d)_+}. \tag{4.59}$$

For $\mathcal{G}_0 = G_0$, we use $\mathfrak{h}_0 = \ell_0$, $(c|\gamma_0|\mathfrak{h}_0^4)^{|X|\wedge 2^d} \le O(|\gamma_0|)$, and

$$(c|\gamma_0|\mathfrak{h}_0^4)^{(|X|-2^d)_+} \le (c'\tilde{g}_0)^{(1+a')(|X|-2^d)_+} \le \tilde{g}_0^{f_0(a,X)}. \tag{4.60}$$

For $\mathcal{G}_0 = \tilde{G}_0$, we use $\mathfrak{h}_0 = h_0 = O(\tilde{g}_0^{-1/4})$ and, since $a' > a$,

$$(c|\gamma_0|\mathfrak{h}_0^4)^{(|X|-2^d)_+} \le (c'\tilde{g}_0)^{a'(|X|-2^d)_+} \le \tilde{g}_0^{f_0(a,X)}. \tag{4.61}$$

Since $|\gamma_0| \le \tilde{g}_0$, it follows from (4.59) that

$$\tilde{g}_0^{9/4}\|K_0^\pm\|_{\mathcal{F}_0(\tilde{G})} \le \tilde{g}_0^{9/4} O(|\gamma_0|\tilde{g}_0^{-1}) \le |\gamma_0|, \tag{4.62}$$

and the proof is complete. □

The above discussion is based on norms in the setting of the torus Λ. As in [14], a version on the infinite lattice $\mathbb{Z}^d$ is also required. This can be done in exactly the same manner, by defining the polymers $\mathcal{P}_0 = \mathcal{P}_0(\mathbb{Z}^d)$ to be the collection of subsets of $\mathbb{Z}^d$, with $K_0^\pm(X)$ defined for subsets of $\mathbb{Z}^d$ by $\prod_{x\in X} K_{0,x}^\pm$. The $\mathcal{W}_0 = \mathcal{W}_0(\tilde{g}_0, \mathbb{Z}^d)$ norm (in infinite volume) can be defined analogously to (4.57). The hypotheses and conclusion of Proposition 4.4 remain the same in the setting of $\mathbb{Z}^d$.

4.5 Smoothness of K_0

Let $\mathcal{C}_0(\mathbb{Z}^d) \subset \mathcal{P}_0(\mathbb{Z}^d)$ be the set of connected polymers. By definition, a connected polymer is nonempty. Given $\tilde{g}_0 > 0$, let $\mathcal{W}_0^*(\tilde{g}_0, \mathbb{Z}^d)$ denote the space of maps $F : \mathcal{C}_0(\mathbb{Z}^d) \to \mathcal{N}$, with $F(X) \in \mathcal{N}(X^\square)$ and $\|F\|_{\mathcal{W}_0(\tilde{g}_0,\mathbb{Z}^d)} < \infty$. Addition in this space is defined by $(F_1 + F_2)(X) = F_1(X) + F_2(X)$. We extend any $F : \mathcal{C}_0(\mathbb{Z}^d) \to \mathcal{N}$ to $F : \mathcal{P}_0(\mathbb{Z}^d) \to \mathcal{N}$ by taking $F(X) = \prod_Y F(Y)$ where the product is over the connected components Y of X.

Given any map $F : D \to \mathcal{W}_0^*(\tilde{g}_0, \mathbb{Z}^d)$ for $D \subset \mathbb{R}$ an open interval, write $F_X, F_X^\phi : D \to \mathcal{N}(X^\square)$ for the maps defined by partial evaluation of F at X and at (X, ϕ), respectively. We say F_X^ϕ is C^k if all of its coefficients in the decomposition (3.2) are C^k as functions $D \to \mathbb{R}$.

Lemma 4.5 *Let $D \subset \mathbb{R}$ be open and $F : D \to \mathcal{W}_0^*(\tilde{g}_0, \mathbb{Z}^d)$ be a map. Suppose that F_X^ϕ is C^2 for all $X \in \mathcal{C}_0$ and $\phi \in \mathbb{C}^\Lambda$, and define $F^{(i)} : D \to \mathcal{W}_0^*(\tilde{g}_0, \mathbb{Z}^d)$ by $(F^{(i)}(t))_X^\phi = (F_X^\phi)^{(i)}(t)$ for $i = 1, 2$, where the right-hand side denotes the (component-wise) i^{th} derivative of F_X^ϕ. If $\|F^{(i)}(t)\|_{\mathcal{W}_0} < \infty$ for $i = 1, 2$ and $t \in D$, then $F^{(1)}$ is the derivative of F.*

Proof For $t, t+s \in D$, define $R(t,s) \in \mathcal{W}_0$ by

$$R_X^\phi(t,s) = F_X^\phi(t+s) - F_X^\phi(t) - s(F_X^\phi)'(t). \tag{4.63}$$

By Taylor's theorem, for any ϕ and X,

$$R_X^\phi(t,s) = s^2 \int_0^1 (F_X^\phi)''(t+us)(1-u)\,du, \tag{4.64}$$

where the integral is taken component-wise. It follows that

$$\|R(t,s)\|_{\mathcal{W}_0} \le |s|^2 \sup_{u\in[0,1]} \|F''(t+us)\|_{\mathcal{W}_0} \le O(|s|^2), \tag{4.65}$$

so F is differentiable and its derivative satisfies $(F')^{\phi}_X = (F^{\phi}_X)'$. Continuity of F' follows similarly, since, by the fundamental theorem of calculus,

$$\|F'(t+s) - F'(t)\|_{\mathcal{W}_0} \le |s| \sup_{u\in[t,t+s]} \|F''(u)\|_{\mathcal{W}_0} \le O(|s|), \tag{4.66}$$

which suffices. □

Consider the map

$$(g_0, \gamma_0, \nu_0, z_0) \mapsto K_0 \in \mathcal{W}_0^*(\tilde{g}_0, \mathbb{Z}^d) \tag{4.67}$$

defined by

$$K_0(g_0,\gamma_0,\nu_0,z_0) = \begin{cases} K_0^+(g_0,\gamma_0,\nu_0,z_0) & (\gamma_0 \ge 0) \\ K_0^-(g_0,\gamma_0,\nu_0,z_0) & (\gamma_0 < 0), \end{cases} \tag{4.68}$$

for $(g_0, \gamma_0, \nu_0, z_0)$ satisfying the hypotheses of Proposition 4.4. The map K_0 is in fact analytic away from $\gamma_0 = 0$. However, we only prove the following, which is what we need later.

Proposition 4.6 *Suppose that $V_0^{\pm} \in \mathcal{D}_0$, with $\tilde{g}_0$ sufficiently small (depending on L) and $|\gamma_0| \le O(\tilde{g}_0^{1+a'})$ for some $a' > a$. The map $K_0(g_0,\gamma_0,\nu_0,z_0)$ is jointly continuous in its four variables, is C^1 in (g_0,ν_0,z_0), and (when $\gamma_0 \ne 0$) is C^1 in (g_0,γ_0,ν_0,z_0), with partial derivatives with respect to $t = g_0$, ν_0, and z_0 satisfying*

$$\|\partial K_0/\partial t\|_{\mathcal{W}_0} = O(|\gamma_0|\mathfrak{h}_0^8). \tag{4.69}$$

Moreover, K_0 is left- and right-differentiable in γ_0 at $\gamma_0 = 0$.

Proof Let t denote any one of the coupling constants g_0, γ_0, ν_0 or z_0. We drop the subscript 0, and let $K(t)$ denote K_0 viewed as a function of t, with the remaining coupling constants fixed. Then K_X^{ϕ} is smooth for any ϕ, X. If t is g_0, ν_0 or z_0, then

$$(K_x^{\phi})' = -M_x(\phi)K_x^{\phi}, \quad (K_x^{\phi})'' = M_x^2(\phi)K_x^{\phi}, \tag{4.70}$$

where M_x is τ_x^2, τ_x or $\tau_{\Delta,x}$, respectively. The maximal degree of M_x is 4, so (4.49) implies that

$$\|K_x'\|_{\mathcal{G}_0} \le O(|\gamma_0|\mathfrak{h}_0^8), \quad \|K_x''\|_{\mathcal{G}_0} \le O(|\gamma_0|\mathfrak{h}_0^{12}). \tag{4.71}$$

For t denoting γ_0, we restrict attention to $\gamma_0 > 0$, and write $U = U^+$ and $V_0 = V_0^+$ (the case $\gamma_0 < 0$ is similar). Then

$$(K_x^{\phi})' = -U_x(\phi)e^{-V_x(\phi)-\gamma_0 U_x(\phi)}, \quad (K_x^{\phi})'' = U_x^2(\phi)e^{-V_x(\phi)-\gamma_0 U_x(\phi)}, \tag{4.72}$$

and (4.28) and (4.42) imply that

$$\|K_x'\|_{\mathcal{G}_0} \le O(\mathfrak{h}_0^4), \quad \|K_x''\|_{\mathcal{G}_0} \le O(\mathfrak{h}_0^8). \tag{4.73}$$

By definition, $K_X = \prod_{x\in X} K_x$, so, for derivatives with respect to any one of the four variables (with $\gamma_0 \ne 0$ when differentiating with respect to γ_0),

$$(K_X^{\phi})' = \sum_{x\in X} (K_x^{\phi})' K_{X\setminus x}^{\phi}, \quad (K_X^{\phi})'' = \sum_{x\in X} ((K_x^{\phi})'' K_{X\setminus x}^{\phi} + (K_x^{\phi})'(K_{X\setminus x}^{\phi})'). \tag{4.74}$$

Thus, by the product property, (4.71), and Proposition 4.1,

$$\|K'_X\|_{\mathcal{G}_0} \le O(|X|)|\gamma_0|\mathfrak{h}_0^8(|\gamma_0|\mathfrak{h}_0^4)^{|X|-1}. \tag{4.75}$$

when differentiating with respect to g_0, ν_0, or z_0. The bound (4.69) then follows from the hypothesis on γ_0. Similarly, using (4.73),

$$\|K'_X\|_{\mathcal{G}_0} \le O(|X|)\mathfrak{h}_0^4(|\gamma_0|\mathfrak{h}_0^4)^{|X|-1} \tag{4.76}$$

when differentiating with respect to γ_0 away from $\gamma_0 = 0$. In both cases, we have

$$\|K''_X\|_{\mathcal{G}_0} \le O(|X|^2)\mathfrak{h}_0^8(|\gamma_0|\mathfrak{h}_0^4)^{(|X|-2)\wedge 0}. \tag{4.77}$$

Thus, by Lemma 4.5, K is C^1 in any of its variables. Therefore, K is C^1 in (g_0, ν_0, z_0) on the whole domain and in all the variables when $\gamma_0 \neq 0$.

To show right-continuity in γ_0 at $\gamma_0 = 0$, fix (g_0, ν_0, z_0) and define $F \in \mathcal{W}_0^*$ by

$$F(X) = \begin{cases} -U_x e^{-V_{0,x}} & X = \{x\} \\ 0 & |X| > 1, \end{cases} \tag{4.78}$$

where U_x, $V_{0,x}$ are defined above. Let $K'(\gamma_0)$ denote the γ_0 derivative of K evaluated at $\gamma_0 > 0$. Then (4.72) and (4.74) imply that

$$F(X) - K'_X(\gamma_0) = \begin{cases} U_x K_x(\gamma_0) & X = \{x\} \\ \sum_{x \in X} K'_x(\gamma_0) K_{X\setminus x}(\gamma_0) & |X| > 1. \end{cases} \tag{4.79}$$

Thus, by (4.49), (4.73), and Proposition 4.1,

$$\|F(X) - K'_X(\gamma_0)\|_{\mathcal{G}_0} \le \begin{cases} O(\gamma_0 \mathfrak{h}_0^8) & X = \{x\} \\ O(|X|)\mathfrak{h}_0^4(\gamma_0\mathfrak{h}_0^4)^{|X|-1} & |X| > 1. \end{cases} \tag{4.80}$$

It follows that

$$\lim_{\gamma_0 \downarrow 0} \|F - K'(\gamma_0)\|_{\mathcal{W}_0} = 0, \tag{4.81}$$

i.e., F is the right-derivative of K in γ_0 at $\gamma_0 = 0$. Left-continuity is handled similarly. □

5 Existence of Critical Parameters

In Sects. 5.1–5.2, we recall some facts about the renormalisation group map defined in [14]. In Sect. 5.3, we discuss the existence and properties of the finite-volume renormalisation group flow (a consequence of the main result of [5]), which is crucial to proving Theorem 1.2. Using the results of Sect. 5.3, we identify critical initial conditions for iteration of the renormalisation group in Sect. 5.4. In Sect. 5.5, we identify the critical point and discuss an important change of parameters. Then in Sect. 5.6 we obtain the asymptotic behaviour of the two-point function, susceptibility, and correlation length of order p, and thereby prove Theorem 1.2. Finally, Sect. 5.7 contains a version of the implicit function theorem that we apply in Sects. 5.4–5.5.

5.1 Renormalisation Group Coordinates

As discussed in Sect. 3.4, the evolution of Z_j defined in (3.24) is tracked via coordinates (I_j, K_j). In order to discuss these, we make the following definitions. We partition Λ into

L^{N-j} disjoint *scale-* j *blocks* of side L^j. We let $\mathcal{P}_j$ denote the set of *scale-* j *polymers*, which are unions of elements of $\mathcal{B}_j$. Given $X \in \mathcal{P}_j$, we denote the collection of scale-j blocks in X by $\mathcal{B}_j(X)$. Scale-0 blocks are simply elements of Λ, and scale-0 polymers are subsets of Λ, as in Section 4.4. Also, as in the scale-0 case, there is a version of blocks and polymers also on $\mathbb{Z}^d$ rather than Λ.

Given a polynomial V_j of the form

$$V_{j;x} = g_j \tau_x^2 + \nu_j \tau_x + z_j \tau_{\Delta,x}, \tag{5.1}$$

the *interaction* $I_j(X)$ is defined for $X \in \mathcal{P}_j(\Lambda)$ by

$$I_j(X) = e^{-\sum_{x \in X} V_{j;x}} \prod_{B \in \mathcal{B}_j(X)} (1 + W_j(B)), \tag{5.2}$$

where $W_j(B)$ is an explicit polynomial that is quadratic in V_j and is defined in [4, (3.21)]. In [14, Definition 1.7], a space $\mathcal{K}_j = \mathcal{K}_j(\Lambda)$ of maps $\mathcal{P}_j \to \mathcal{N}$ required to satisfy several properties is defined. The coordinate K_j is constructed in [14] as an element of $\mathcal{K}_j$. The renormalisation group is used to construct a sequence (V_j, K_j) from which Z_j can be recovered via the *circle product*

$$Z_j = (I_j \circ K_j)(\Lambda) = \sum_{X \in \mathcal{P}_j(\Lambda)} I_j(\Lambda \setminus X) K_j(X). \tag{5.3}$$

5.2 Renormalisation Group Map

We restrict the discussion in this section to a finite volume $\Lambda = \Lambda_N$ with $N > 1$.

For fixed $(\tilde{m}^2, \tilde{g}_0) \in [0, \delta) \times (0, \delta)$, we define a sequence $\tilde{g}_j = \tilde{g}_j(\tilde{m}^2, \tilde{g}_0)$ as in [3, (6.15)]; in particular, $\tilde{g}_0(\tilde{m}^2, \tilde{g}_0) = \tilde{g}_0$. In [14, Section 1.7.3], a sequence of norms $\| \cdot \|_{\mathcal{W}_j} = \| \cdot \|_{\mathcal{W}_j(\tilde{m}^2, \tilde{g}_j, \Lambda)}$ parametrised by $(\tilde{m}^2, \tilde{g}_j)$ is defined on maps $\mathcal{P}_j \to \mathcal{N}$. We let $\mathcal{W}_j$ denote the subspace of $\mathcal{K}_j$ consisting of all elements having finite $\mathcal{W}_j$ norm. Note that $\mathcal{W}_0 = \mathcal{K}_0 \cap \mathcal{W}_0^*$, where $\mathcal{W}_0^*$ is defined in Sect. 4.5.

In [3, (6.6)–(6.7)], a function $\vartheta_j = \vartheta_j(m^2)$ (denoted χ_j in [3]) is defined in such a way that ϑ_j decays exponentially when j is sufficiently large depending on m. We write $\tilde{\vartheta}_j = \vartheta_j(\tilde{m}^2)$. Given a constant $\alpha > 0$, we define the (finite-volume) renormalisation group domains $\mathbb{D}_j \subset \mathbb{R}^3 \oplus \mathcal{W}_j$ by

$$\mathbb{D}_j(\tilde{m}^2, \tilde{g}_j, \Lambda) = \mathcal{D}_j \times B_{\mathcal{W}_j(\tilde{m}^2, \tilde{g}_j, \Lambda)}(\alpha \tilde{\vartheta}_j \tilde{g}_j^3), \tag{5.4}$$

$$\mathcal{D}_j = \mathcal{D}_j(\tilde{g}_j) = \{(g, \nu, z) : C_{\mathcal{D}}^{-1} \tilde{g}_j < g < C_{\mathcal{D}} \tilde{g}_j ; |z|, L^{2j}|\nu| < C_{\mathcal{D}} \tilde{g}_j\}. \tag{5.5}$$

This definition of $\mathcal{D}_j$ is consistent with (4.23) when $j = 0$. We let $\tilde{\mathbb{I}}_j(\tilde{m}^2)$ be the neighbourhood of $\tilde{m}^2$ defined by

$$\tilde{\mathbb{I}}_j = \tilde{\mathbb{I}}_j(\tilde{m}^2) = \begin{cases} [\frac{1}{2}\tilde{m}^2, 2\tilde{m}^2] \cap \mathbb{I}_j & (\tilde{m}^2 \neq 0) \\ [0, L^{-2(j-1)}] \cap \mathbb{I}_j & (\tilde{m}^2 = 0), \end{cases} \tag{5.6}$$

where $\mathbb{I}_j = [0, \delta]$ if $j < N$ and $\mathbb{I}_N = [\delta L^{-2(N-1)}, \delta]$. The main result of [14] is the construction of the renormalisation group map on the domains $\mathbb{D}_j$. Although [14] constructs finite- and infinite-volume versions of this map, we only discuss the finite-volume map here.

For $m^2 \in \tilde{\mathbb{I}}_j(\tilde{m}^2)$, the finite-volume renormalisation group map at scale $j = 1, \dots, N-1$ is a map $\mathbb{D}_j(\tilde{m}^2, \tilde{g}_j, \Lambda) \to \mathbb{R}^3 \oplus \mathcal{W}_{j+1}(\tilde{m}^2, \tilde{g}_{j+1}, \Lambda)$, which we denote

$$(V_j, K_j) \mapsto (V_{j+1}, K_{j+1}). \tag{5.7}$$

The first component of this map takes the form

$$V_{j+1} = V_{\mathrm{pt}, j+1}(V_j) + R_{j+1}(V_j, K_j), \tag{5.8}$$

where the map $V_{\mathrm{pt}, j+1}$ defined in [4] captures the second-order evolution of V_j, and R_{j+1} is a third-order contribution. The main properties of the map (5.7) are listed in [3, Section 6.4]. Importantly, the renormalisation group map preserves the circle product in the sense that

$$(I_{j+1} \circ K_{j+1})(\Lambda) = \mathbb{E}_{C_{j+1}} \theta (I_j \circ K_j)(\Lambda). \tag{5.9}$$

Since $\mathcal{P}_N(\Lambda) = \{\varnothing, \Lambda_N\}$, this means that, if $(V_0, K_0) = (V_0^{\pm}, K_0^{\pm})$ and if the renormalisation group map can be iterated N times with this choice of initial condition, then

$$Z_N = I_N(\Lambda) + K_N(\Lambda) = e^{-\sum_{x \in \Lambda} V_{N;x}}(1 + W_N(\Lambda)) + K_N(\Lambda). \tag{5.10}$$

5.3 Renormalisation Group Flow

The following theorem is an extension of [3, Proposition 7.1] to non-trivial K_0. Such an extension is possible, with only minor modifications to the proof of the $K_0 = \mathbb{1}_\varnothing$ case, due to the generality allowed by the main result of [5].

The theorem provides, for any $N \geq 1$ and for initial error coordinate K_0 in a specified domain, a choice of initial condition (ν_0^c, z_0^c) for which there exists a finite-volume renormalisation group flow $(V_j, K_j) \in \mathbb{D}_j$ for $0 \leq j \leq N$. In order to ensure a degree of consistency amongst the sequences (V_j, K_j), which depend on the volume Λ_N, a notion of consistency must be imposed upon the collection of initial error coordinates $K_{0,\Lambda} \in \mathcal{K}_0(\Lambda)$ for varying Λ. Specifically, the family $K_{0,\Lambda}$ is required to satisfy the property $(\mathbb{Z}^d)$ of [14, Definition 1.15]. We refer to any such family as a Λ-family. As discussed in [14, Definition 1.15], any Λ-family induces an infinite-volume error coordinate $K_{0,\mathbb{Z}^d} \in \mathcal{K}_0(\mathbb{Z}^d)$ in a natural way.

Theorem 5.1 *Let $d = 4$. There exists a constant $a_* > 0$ and continuous functions ν_0^c, z_0^c of (m^2, g_0, K_0), defined for $(m^2, g_0) \in [0, \delta]^2$ (for some $\delta > 0$ sufficiently small) and for any $K_0 \in \mathcal{W}_0(m^2, g_0, \mathbb{Z}^d)$ with $\|K_0\|_{\mathcal{W}_0(m^2, g_0, \mathbb{Z}^d)} \leq a_* g_0^3$, such that the following holds for $g_0 > 0$: if $K_{0,\Lambda} \in \mathcal{K}_0(\Lambda)$ is a Λ-family that induces the infinite-volume coordinate K_0, and if*

$$V_0 = V_0^c(m^2, g_0, K_0) = (g_0, \nu_0^c(m^2, g_0, K_0), z_0^c(m^2, g_0, K_0)), \tag{5.11}$$

then for any $N \in \mathbb{N}$ and $m^2 \in [\delta L^{-2(N-1)}, \delta]$, there exists a sequence $(V_j, K_j) \in \mathbb{D}_j(m^2, g_0, \Lambda)$ such that

$$(V_{j+1}, K_{j+1}) = (V_{j+1}(V_j, K_j), K_{j+1}(V_j, K_j)) \text{ for all } j < N \tag{5.12}$$

and (5.3) is satisfied. Moreover, ν_0^c, z_0^c are continuously differentiable in $g_0 \in (0, \delta)$ and $K_0 \in B_{\mathcal{W}_0(m^2, g_0, \Lambda)}(a_* g_0^3)$, and

$$\nu_0^c(m^2, 0, 0) = z_0^c(m^2, 0, 0) = 0, \quad \frac{\partial \nu_0^c}{\partial g_0} = O(1), \quad \frac{\partial z_0^c}{\partial g_0} = O(1), \tag{5.13}$$

where the estimates above hold uniformly in $m^2 \in [0, \delta]$.

Proof The proof results from small modifications to the proofs of [3, Proposition 7.1] and then to [3, Proposition 8.1], where (in both cases) we relax the requirement that $K_0 = \mathbb{1}_\varnothing$, which was chosen in [3] due to the fact that $K_0 = \mathbb{1}_\varnothing$ when $\gamma = 0$. The more general condition that $K_0 \in B_{\mathcal{W}_0(m^2,g_0,\Lambda)}(a_* g_0^3)$ comes from the hypothesis of [5, Theorem 1.4] when $(m^2, g_0) = (\tilde{m}^2, \tilde{g}_0)$. By [5, Remark 1.5], no major changes to the proof result from this choice of K_0. The following paragraph outlines in more detail the modifications to the proof of [3, Proposition 7.1].

By [5, Theorem 1.4] and [5, Corollary 1.8], for any $(\tilde{m}^2, \tilde{g}_0) \in (0, \delta)^2$ and $\tilde{K}_0 \in B_{\mathcal{W}_0(\tilde{m}^2,\tilde{g}_0,\mathbb{Z}^d)}(a_* \tilde{g}_0^3)$, there is a neighbourhood $\mathsf{N}(\tilde{g}_0, \tilde{K}_0)$ of $(\tilde{g}_0, \tilde{K}_0)$ such that for all $(m^2, g_0, K_0) \in \tilde{\mathbb{I}}(\tilde{m}^2) \times \mathsf{N}(\tilde{g}_0, \tilde{K}_0)$, there is an infinite-volume renormalisation group flow

$$(\check{V}_j, K_j) = \check{x}_j^d(\tilde{m}^2, \tilde{g}_0, \tilde{K}_0; m^2, g_0, K_0) \tag{5.14}$$

in *transformed variables* $(\check{V}_j, K_j)$. The transformed variables are defined in [3, Section 6.6] and a flow in the original variables can be recovered from the transformed flow. The global solution is defined by $\check{x}_j^c(m^2, g_0, K_0) = \check{x}_j^d(m^2, g_0, K_0; m^2, g_0, K_0)$ (or $\check{x}^c \equiv 0$ if $g_0 = 0$). By [5, Remark 1.5], the proof of regularity of $\check{x}^c$ can proceed as in [3]. The functions (ν_0^c, z_0^c) are given by the (ν_0, z_0) components of $\check{x}_0^c = (\check{V}_0, K_0) = (V_0, K_0)$. □

Remark 5.2 The proof of [3, Proposition 7.1], hence of Theorem 5.1, makes important use of the parameter $\tilde{g}_0$ in order to prove regularity of the renormalisation group flow in g_0. However, once the flow has been constructed, we can and do set $\tilde{g}_0 = g_0$.

Suppose now that we are given some sufficiently small $\hat{g}_0 > 0$ and a Λ-family $K_{0,\Lambda} \in \mathcal{W}_0(m^2, \hat{g}_0, \Lambda)$ that satisfies the bounds $\|K_{0,\Lambda}\|_{\mathcal{W}_0(m^2,\hat{g}_0,\Lambda)} \le a_* \hat{g}_0^3$. Then in any fixed volume $\Lambda = \Lambda_N$, we can generalise (3.14) by defining $Z_0 = (I_0 \circ K_0)(\Lambda)$ [(3.14) is recovered when we set $K_0 = K_0^\pm$]. We also generalise (3.15) as $Z_N = \mathbb{E}_C \theta Z_0$, and let $\hat{\chi}_N(m^2, \hat{g}_0, K_0, \nu_0, z_0)$ be defined as in (3.19) from this Z_N [generalising (3.19)]. Then an analogue of [3, Theorem 4.1] (which corresponds to the case $K_0 = \mathbb{1}_\varnothing$) follows from Theorem 5.1. That is, if $(\nu_0^c, z_0^c) = (\nu_0^c(m^2, \hat{g}_0, K_0), z_0^c(m^2, \hat{g}_0, K_0))$, then the limit $\hat{\chi} = \lim_{N\to\infty} \hat{\chi}_N$ exists and

$$\hat{\chi}\left(m^2, \hat{g}_0, K_0, \nu_0^c, z_0^c\right) = \frac{1}{m^2}, \tag{5.15}$$

$$\frac{\partial \hat{\chi}}{\partial \nu_0}\left(m^2, \hat{g}_0, K_0, \nu_0^c, z_0^c\right) \sim -\frac{1}{m^4} \frac{c(\hat{g}_0^*, K_0)}{(\hat{g}_0^* \mathsf{B}_{m^2})^{1/4}} \quad \text{as} (m^2, \hat{g}_0) \to (0, \hat{g}_0^*), \tag{5.16}$$

where c is a continuous function and the *bubble diagram* B_{m^2} is is asymptotic to $(2\pi^2)^{-1} \log m^{-2}$, as $m^2 \downarrow 0$, when $d = 4$. For instance, (5.15) follows from (3.19), (5.10), the bound on K_N in Theorem 5.1, and the bound on W_N in [13, (4.57)]. See [3, Section 8.4] for details and for the proof of (5.16).

We wish to obtain a version of (5.15)–(5.16) with the initial conditions of Section 4.1, i.e. with $(\hat{g}_0, K_0) = (g_0, K_0^+)$ (if $\gamma_0 > 0$) or $(\hat{g}_0, K_0) = (g_0 + 4d\gamma_0, K_0^-)$ (if $\gamma_0 < 0$). It is straightforward to verify that $K_0^\pm \in \mathcal{K}_0$. For instance, the fact that $K_0^\pm$ is supersymmetric (which is required of all elements of $\mathcal{K}_0$) follows from the fact that $K_{0,x}^\pm$ is a function of τ_x (see [4, Section 5.2.1] for more on this topic). It also follows from the definition that the finite-volume coordinates $K_{0,\Lambda}^\pm$ form a Λ-family.

Moreover, by Proposition 4.4, if $|\gamma_0|$ is sufficiently small (depending on g_0; we now take $\tilde{g}_0 = g_0$) then $K_0 = K_0^\pm$ satisfies the bound required by Theorem 5.1. However, we cannot apply the theorem immediately with this choice of K_0, due to the fact that $K_0^\pm$ depends on (g_0, ν_0, z_0). We resolve this issue in the next section.

5.4 Critical Parameters

For convenience, let

$$\hat{g}_0 = \hat{g}_0(g_0, \gamma_0) = g_0 + 4d\gamma_0 \mathbb{1}_{\gamma_0 < 0}. \tag{5.17}$$

Thus, $\hat{g}_0$ is the coefficient of τ_x^2 in $V_{0,x}^+$ when $\gamma_0 \geq 0$, and in $V_{0,x}^-$ when $\gamma_0 < 0$. Recall the function $K_0(g_0, \gamma_0, \nu_0, z_0)$ defined in (4.68). We wish to initialise the renormalisation group with (ν_0, z_0) a solution to the system of equations

$$\nu_0 = \nu_0^c(m^2, \hat{g}_0(g_0, \gamma_0), K_0(g_0, \gamma_0, \nu_0, z_0)), \tag{5.18}$$

$$z_0 = z_0^c(m^2, \hat{g}_0(g_0, \gamma_0), K_0(g_0, \gamma_0, \nu_0, z_0)). \tag{5.19}$$

Such a choice of (ν_0, z_0) will be critical for K_0, where K_0 is itself evaluated at this same choice of (ν_0, z_0).

When $\gamma_0 = 0$, we get $K_0 = \mathbb{1}_\varnothing$, so K_0 no longer depends on (ν_0, z_0) and this system is solved by $(\nu_0^c(m^2, g_0, 0), z_0^c(m^2, g_0, 0))$ for any (small) $m^2, g_0 \geq 0$. Local solutions for $\gamma_0 \neq 0$ can then be constructed using a version of the implicit function theorem from [25] that allows for the continuous but non-smooth behaviour of K_0 in γ_0. In order to obtain global solutions with certain desired regularity properties (needed in the next section), we make use of Proposition 5.10, which is based on a version of the implicit function theorem from [25].

Suppose $\delta > 0$ and suppose $r : [0, \delta] \to [0, \infty)$ is a continuous positive-definite function; the latter means that $r(x) > 0$ if $x > 0$ and $r(0) = 0$. We define

$$D(\delta, r) = \{(w, x, y) \in [0, \delta]^2 \times (-\delta, \delta) : |y| \leq r(x)\} \tag{5.20}$$

and we let $C^{0,1,\pm}(D(\delta, r))$ denote the space of continuous functions on $D(\delta, r)$ that are C^1 in (x, y) away from $y = 0$, C^1 in x everywhere, and have left- and right-derivatives in y at $y = 0$. In our applications, we take $w = m^2$, $x = g_0$ or β, and $y = \gamma_0$ or γ.

Proposition 5.3 *There exists a continuous positive-definite function $\hat{r} : [0, \delta] \to [0, \infty)$ and continuous functions $\hat{\mu}_0^c, \hat{z}_0^c \in C^{0,1,\pm}(D(\delta, \hat{r}))$ such that the system (5.18)–(5.19) is solved by $(\nu_0, z_0) = (\hat{\mu}_0^c, \hat{z}_0^c)$ whenever $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$. Moreover, these functions satisfy the bounds*

$$\hat{\mu}_0^c = O(g_0), \quad \hat{z}_0^c = O(g_0) \tag{5.21}$$

uniformly in (m^2, γ_0).

Proof Recall the definition of $\hat{g}_0$ in (5.17), and let

$$F(m^2, g_0, \gamma_0, \nu_0, z_0) = (\nu_0, z_0) - (\nu_0^c(m^2, \hat{g}_0, K_0), z_0^c(m^2, \hat{g}_0, K_0)), \tag{5.22}$$

where $K_0 = K_0(g_0, \gamma_0, \nu_0, z_0)$. Then for $\delta > 0$ small and an appropriate constant $c > 0$ (depending on a_*), F is well-defined on

$$\{(m^2, g_0, \gamma_0, \nu_0, z_0) : (m^2, \hat{g}_0, \gamma_0) \in D(\delta, cg_0^3), |\nu_0|, |z_0| \leq C_{\mathcal{D}} g_0\}. \tag{5.23}$$

Indeed, for $(m^2, g_0, \gamma_0, \nu_0, z_0)$ in this domain, Proposition 4.4 (with $\tilde{g}_0 = g_0$) implies that $(m^2, \hat{g}_0, K_0)$ is in the domain of (ν_0^c, z_0^c). By Theorem 5.1 and Proposition 4.6, F is C^1 in (g_0, ν_0, z_0) and also in γ_0 away from $\gamma_0 = 0$, continuous in m^2, and has one-sided derivatives in γ_0 at $\gamma_0 = 0$.

For fixed $(\bar{m}^2, \bar{g}_0) \in [0, \delta]^2$, set $(\bar{\nu}_0, \bar{z}_0) = (\nu_0^c(\bar{m}^2, \bar{g}_0, 0), z_0^c(\bar{m}^2, \bar{g}_0, 0))$ so that

$$F(\bar{m}^2, \bar{g}_0, 0, \bar{\nu}_0, \bar{z}_0) = (0, 0). \tag{5.24}$$

By (4.69), at $(\bar{g}_0, 0, \bar{\nu}_0, \bar{z}_0)$,

$$\frac{\partial K_{0,x}}{\partial \nu_0} = \frac{\partial K_{0,x}}{\partial z_0} = 0. \tag{5.25}$$

It follows that $D_{\nu_0,z_0} F(\bar{m}^2, \bar{g}_0, 0, \bar{\nu}_0, \bar{z}_0)$ is the identity map on $\mathbb{R}^2$. The existence of δ, $\hat{r}$ and $\hat{\mu}_0^c$, $\hat{z}_0^c$ follows from Proposition 5.10 with $w = m^2$, $x = g_0$, $y = \gamma_0$, $z = (\nu_0, z_0)$, and with $r_1(g_0) = cg_0^3$, $r_2(g_0) = C_{\mathcal{D}} g_0$.

By the fundamental theorem of calculus, for any $0 < a < \gamma_0$,

$$\hat{\mu}_0^c(m^2, g_0, \gamma_0) = \hat{\mu}_0^c(m^2, g_0, a) + \int_a^{\gamma_0} \frac{\partial \hat{\mu}_0^c}{\partial \gamma_0}(m^2, g_0, t)\, dt. \tag{5.26}$$

Taking the limit $a \downarrow 0$ and using (5.13), we obtain

$$|\hat{\mu}_0^c(m^2, g_0, \gamma_0)| \le O(g_0) + \gamma_0 \sup_{t \in (0,\gamma_0]} \left| \frac{\partial \hat{\mu}_0^c}{\partial \gamma_0}(m^2, g_0, t) \right|. \tag{5.27}$$

The supremum above is bounded by a constant and so the first estimate of (5.21) for $\gamma_0 \ge 0$ follows from the fact that $|\gamma_0| \le \hat{r}(g_0)$ (since $\hat{r}(g_0)$ can be taken as small as desired). The case $\gamma_0 < 0$ and the second estimate follow similarly. □

Corollary 5.4 *Fix $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$ with $g_0 > 0$ and $m^2 \in [\delta L^{-2(N-1)}, \delta)$ and set $(V_0, K_0) = (V_0^{\pm}, K_0^{\pm})$ with $(\nu_0, z_0) = (\hat{\mu}_0^c, \hat{z}_0^c)$. Then for any $N \in \mathbb{N}$, there exists a sequence $(V_j, K_j) \in \mathbb{D}_j(m^2, g_0, \Lambda)$ such that*

$$(V_{j+1}, K_{j+1}) = (V_{j+1}(V_j, K_j), K_{j+1}(V_j, K_j)) \text{ for all } j < N \tag{5.28}$$

and (5.3) is satisfied. Moreover, the second-order evolution equation for V_j is independent of γ_0.

Proof By Proposition 4.4, and by taking $\hat{r}$ smaller if necessary, $K_0 = K_0^{\pm}$ satisfies the estimate required by Theorem 5.1 whenever $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$. The existence of the sequence (5.28) then follows from Theorem 5.1 and Proposition 5.3. Although the presence of γ_0 causes a shift in initial conditions, the second-order evolution of V_j is still given by the map $V_{\rm pt}$ [see (5.8)], which is independent of γ_0. □

By (3.19), $\hat{\chi}(m^2, g_0, \gamma_0, \nu_0, z_0) = \hat{\chi}(m^2, g_0, K_0, \nu_0, z_0)$, where $K_0 = K_0(g_0, \gamma_0, \nu_0, z_0)$ is defined in (4.68). Then by (5.15)–(5.16), Corollary 5.4, and (4.69), with $\hat{g}_0 = \hat{g}_0(g_0, \gamma_0)$, we have

$$\hat{\chi}\left(m^2, \hat{g}_0, \gamma_0, \hat{\mu}_0^c, \hat{z}_0^c\right) = \frac{1}{m^2}, \tag{5.29}$$

$$\frac{\partial \hat{\chi}}{\partial \nu_0}\left(m^2, \hat{g}_0, \gamma_0, \hat{\mu}_0^c, \hat{z}_0^c\right) \sim -\frac{1}{m^4} \frac{c(\hat{g}_0^*, \gamma_0)}{(\hat{g}_0^* \mathsf{B}_{m^2})^{1/4}} \quad \text{as} (m^2, g_0, \gamma_0) \to (0, g_0^*, \gamma_0^*), \tag{5.30}$$

where $\hat{g}_0^* = \hat{g}_0(g_0^*, \gamma_0^*)$ and we write $c(g_0, \gamma_0) = c(g_0, K_0)$. Although (5.30) depends on γ_0, this dependence ultimately only affects the computation of the critical point $\nu_c(\beta, \gamma)$ and the constants $A_{\beta,\gamma}$, $B_{\beta,\gamma}$ in the proof of Theorem 1.2. The asymptotic behaviour of the susceptibility in (1.21) results from the logarithmic divergence of the bubble diagram B_{m^2} and the exponent $\frac{1}{4}$ that appears in the denominator in (5.30).

Remark 5.5 We have invoked (4.69) above in order to satisfy the condition

$$\|\partial K_0 / \partial \nu_0\|_{\mathcal{W}_0} \le O(g_0^3) \tag{5.31}$$

required in the proof of [3, Lemma 8.6] (see [3, (8.34)]). This condition holds trivially when K_0 does not depend on ν_0, as in (5.15)–(5.16).

5.5 Change of Parameters

Recall from (3.18) that

$$\chi_N(\beta, \gamma, \nu) = (1 + z_0)\hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0, z_0), \tag{5.32}$$

whenever the variables on the left- and right-hand sides satisfy

$$g_0 = (\beta - \gamma)(1 + z_0)^2, \quad \nu_0 = \nu(1 + z_0) - m^2, \quad \gamma_0 = \frac{1}{4d}\gamma(1 + z_0)^2. \tag{5.33}$$

Given β, γ, ν, these relations leave free two of the variables $(m^2, g_0, \gamma_0, \nu_0, z_0)$. More generally, if any three of the variables $(\beta, \gamma, \nu, m^2, g_0, \gamma_0, \nu_0, z_0)$ are fixed, then two of the remaining variables are free. In the following two propositions, which together form an extension of [3, Proposition 4.2], we fix three variables and show that the addition of the constraints

$$\nu_0 = \hat{\nu}_0^c(m^2, g_0, \gamma_0), \quad z_0 = \hat{z}_0^c(m^2, g_0, \gamma_0) \tag{5.34}$$

allows us to uniquely specify the two remaining variables. First, in Proposition 5.6, the three fixed variables are (m^2, β, γ).

Proposition 5.6 *There exist $\delta_* > 0$, a continuous positive-definite function $r_* : [0, \delta_*] \to [0, \infty)$, and continuous functions $(\nu^*, g_0^*, \gamma_0^*, \nu_0^*, z_0^*)$ defined for $(m^2, \beta, \gamma) \in D(\delta_*, r_*)$, such that (5.33) and (5.34) hold with $\nu = \nu^*$ and $(g_0, \gamma_0, \nu_0, z_0) = (g_0^*, \gamma_0^*, \nu_0^*, z_0^*)$. Moreover,*

$$g_0^* = \beta + O(\beta^2), \quad \nu_0^* = O(\beta), \quad z_0^* = O(\beta). \tag{5.35}$$

Proof Suppose we have found the desired continuous functions (g_0^*, γ_0^*) and that g_0^* satisfies the first bound in (5.35). Then the functions defined by

$$\nu_0^* = \hat{\mu}_0^c(m^2, g_0^*, \gamma_0^*), \quad z_0^* = \hat{z}_0^c(m^2, g_0^*, \gamma_0^*), \quad \nu^* = \frac{\nu_0^* + m^2}{1 + z_0^*} \tag{5.36}$$

are continuous, (5.33) is satisfied, and the remaining bounds in (5.35) follow using (5.21).

We first solve the third equation of (5.33), and then solve the first equation of (5.33). To this end, we begin by defining

$$f_1(m^2, g_0, \gamma, \gamma_0) = \gamma_0 - (4d)^{-1}\gamma(1 + \hat{z}_0^c(m^2, g_0, \gamma_0))^2 \tag{5.37}$$

for $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$ and $|\gamma| \le \hat{r}(g_0)$ (recall that $\hat{r}$ is defined in Proposition 5.3); although f_1 is well-defined for any $\gamma \in \mathbb{R}$, we restrict the domain in preparation for our application of Proposition 5.10. Note that f_1 is C^1 in γ and $f_1(\cdot, \cdot, \gamma, \cdot) \in C^{0,1,\pm}(D(\delta, \hat{r}))$ for any γ. The equation $f_1(m^2, g_0, \gamma, \gamma_0) = 0$ has the solution $\gamma_0 = 0$ when $\gamma = 0$ and, for any $\gamma_0 \neq 0$,

$$\frac{\partial f_1}{\partial \gamma_0} = 1 - (2d)^{-1}\gamma(1 + \hat{z}_0^c(m^2, g_0, \gamma_0))\frac{\partial \hat{z}_0^c}{\partial \gamma_0}. \tag{5.38}$$

Since the one-sided γ_0 derivatives of $\hat{z}_0^c$ exist at $\gamma_0 = 0$, we can see that the γ_0 derivative of f_1 is well-defined and equal to 1 when $\gamma = 0$ for any small γ_0 (including $\gamma_0 = 0$). Thus, by Proposition 5.10 (with $w = m^2$, $x = g_0$, $y = \gamma$, $z = \gamma_0$ and $r_1 = r_2 = \hat{r}$), there exists a continuous function $\gamma_0^{(1)}(m^2, g_0, \gamma)$ on $D(\delta, r^{(1)})$ (for some continuous positive-definite function $r^{(1)}$ on $[0, \delta]$) such that $f_1(m^2, g_0, \gamma, \gamma_0^{(1)}) = 0$. Moreover, $\gamma_0^{(1)}$ is C^1 in (g_0, γ).

Next, we define

$$f_2(m^2, \beta, \gamma, g_0) = g_0 - (\beta - \gamma)(1 + \hat{z}_0^c(m^2, g_0, \gamma_0^{(1)}(m^2, g_0, \gamma)))^2 \tag{5.39}$$

for $(m^2, g_0, \gamma) \in D(\delta, r^{(1)})$ and $\beta \in [0, \delta_*]$, where $\delta_* > 0$ will be made sufficiently small below. Then $f_2(m^2, \beta, \gamma, g_0) = 0$ is solved by $(\gamma, g_0) = (0, g_0^*(m^2, \beta, 0))$, where $g_0^*(m^2, \beta, 0)$ was constructed in [3, (4.35)]. By [3, (4.37)], $g_0^* = \beta + O(\beta^2)$, so we may restrict the domain of f_2 so that $|g_0| \le 2\beta$. Moreover,

$$\frac{\partial f_2}{\partial g_0} = 1 - 2(\beta - \gamma)(1 + \hat{z}_0^c(m^2, g_0, \gamma_0^{(1)})) \left(\frac{\partial \hat{z}_0^c}{\partial g_0} + \frac{\partial \hat{z}_0^c}{\partial \gamma_0} \frac{\partial \gamma_0^{(1)}}{\partial g_0} \right). \tag{5.40}$$

Differentiating both sides of

$$\gamma_0^{(1)} = \frac{1}{4d} \gamma (1 + \hat{z}_0^c(m^2, g_0, \gamma_0^{(1)}))^2, \tag{5.41}$$

and solving for $\frac{\partial \gamma_0^{(1)}}{\partial g_0}$, gives

$$\frac{\partial \gamma_0^{(1)}}{\partial g_0} = \frac{\gamma (1 + \hat{z}_0^c) \frac{\partial \hat{z}_0^c}{\partial g_0}}{2d - \gamma (1 + \hat{z}_0^c) \frac{\partial \hat{z}_0^c}{\partial \gamma_0}}, \tag{5.42}$$

where $\hat{z}_0^c$ and its derivatives are evaluated at $(m^2, g_0, \gamma_0^{(1)})$. Thus, $\frac{\partial \gamma_0^{(1)}}{\partial g_0} = 0$ when $\gamma = 0$. It follows that $\partial f_2 / \partial g_0$ is well-defined when $(\gamma, g_0) = (0, g_0^*(m^2, \beta, 0))$ and equals

$$1 - 2\beta(1 + \hat{z}_0^c(m^2, g_0^*, 0)) \frac{\partial \hat{z}_0^c}{\partial g_0}(m^2, \beta, 0, g_0^*), \tag{5.43}$$

which is positive when δ_* is small, by (5.21). Thus, by Proposition 5.10 (with $w = m^2$, $x = \beta$, $y = \gamma$, $z = g_0$ and $r_1 = r^{(1)}$, $r_2(\beta) = 2\beta$), there exists a function $g_0^*(m^2, \beta, \gamma) \in C^{0,1,\pm}(D(\delta_*, r^{(2)}))$ (for some continuous positive-definite function $r^{(2)}$ on $[0, \delta_*]$) such that $f_2(m^2, \beta, \gamma, g_0^*) = 0$.

By the fact that g_0^* solves $f_2 = 0$,

$$g_0^* = (\beta - \gamma) + O((\beta - \gamma)^2). \tag{5.44}$$

Since $|\gamma| \le r^{(2)}(g_0)$ and $r^{(2)}(g_0)$ can be taken as small as desired, this implies the first estimate in (5.35). Thus, by taking r_* sufficiently small, if $|\gamma| \le r_*(\beta)$, then $|\gamma| \le r^{(2)}(g_0^*(m^2, \beta, \gamma))$. Thus, for $\beta < \delta_*$ and $|\gamma| \le r_*(\beta)$, we can define

$$\gamma_0^*(m^2, \beta, \gamma) = \gamma_0^{(1)}(m^2, g_0^*(m^2, \beta, \gamma), \gamma), \tag{5.45}$$

which completes the proof. □

Using Proposition 5.6, it is possible to identify the critical point ν_c, as follows. By (5.29), (5.32), Proposition 2.2, and Proposition 5.6,

$$\chi(\beta, \gamma, \nu^*) = \frac{1 + z_0^*}{m^2} = \frac{1 + O(\beta)}{m^2}. \tag{5.46}$$

Thus, with $\nu = \nu^*$, we see that $\chi < \infty$ when $m^2 > 0$, and $\chi = \infty$ when $m^2 = 0$. By (1.10), this implies that

$$\nu_c(\beta, \gamma) = \nu^*(0, \beta, \gamma) = O(\beta), \quad \nu_c(\beta, \gamma) < \nu^*(m^2, \beta, \gamma) \quad (m^2 > 0). \tag{5.47}$$

It follows that

$$\chi(\beta, \gamma, \nu_c) = \infty, \tag{5.48}$$

which is a fact that cannot be concluded immediately from the definition (1.10).

In (5.46), χ is evaluated at $\nu^* = \nu^*(m^2, \beta, \gamma)$. However, in the setting of Theorem 1.2, we need to evaluate χ at a *given* value of ν and then take $\nu \downarrow \nu_c$. To do so, we must determine a choice of m^2 in terms of ν such that (5.33) is satisfied and this choice must approach 0 (as it should by (5.47)) right-continuously as $\nu \downarrow \nu_c$. The following proposition carries out this construction. In the following, the functions $\tilde{m}^2, \tilde{g}_0$ should not be confused with the parameter $\tilde{m}^2, \tilde{g}_0$ that appeared previously in the $\mathcal{W}_j$ norms.

Proposition 5.7 *Write $\nu = \nu_c + \varepsilon$. There exist functions $\tilde{m}^2, \tilde{g}_0, \tilde{\gamma}_0, \tilde{\nu}_0, \tilde{z}_0$ of $(\varepsilon, \beta, \gamma) \in D(\delta_*, r_*)$ (all right-continuous as $\varepsilon \downarrow 0$) such that (5.33) and (5.34) hold with*

$$(m^2, g_0, \gamma_0, \nu_0, z_0) = (\tilde{m}^2, \tilde{g}_0, \tilde{\gamma}_0, \tilde{\nu}_0, \tilde{z}_0). \tag{5.49}$$

Moreover,

$$\tilde{m}^2(0, \beta, \gamma) = 0, \qquad \tilde{m}^2(\varepsilon, \beta, \gamma) > 0 \quad (\varepsilon > 0). \tag{5.50}$$

$$\tilde{g}_0 = \beta + O(\beta^2), \quad \tilde{\nu}_0 = O(\beta), \quad \tilde{z}_0 = O(\beta). \tag{5.51}$$

Proof The proof is a minor modification of the proof in [3], using Proposition 5.6. Define

$$\tilde{m}^2 = \tilde{m}^2(\varepsilon, \beta, \gamma) = \inf\{m^2 > 0 : \nu^*(m^2, \beta, \gamma) = \nu_c(\beta, \gamma) + \varepsilon\}, \tag{5.52}$$

on $D(\delta_*, r_*)$. By continuity of ν^*, the infimum is attained and

$$\nu_c(\beta, \gamma) + \varepsilon = \nu^*(\tilde{m}^2(\varepsilon, \beta, \gamma), \beta, \gamma). \tag{5.53}$$

From the above expression, continuity of ν^*, and (5.47), it follows that $\tilde{m}^2$ is right-continuous as $\varepsilon \downarrow 0$. It is immediate that (5.50) holds. Also, the functions of $(\varepsilon, \beta, \gamma)$ defined by

$$\tilde{\nu}_0 = \nu_0^*(\tilde{m}^2, \beta, \gamma), \quad \tilde{z}_0 = z_0^*(\tilde{m}^2, \beta, \gamma), \tag{5.54}$$

$$\tilde{g}_0 = (\beta - \gamma)(1 + \tilde{z}_0)^2, \quad \tilde{\gamma}_0 = \frac{1}{4d}\gamma(1 + \tilde{z}_0)^2 \tag{5.55}$$

are right-continuous as $\varepsilon \downarrow 0$ and satisfy (5.33). The bounds (5.51) follow from the definitions and (5.35), and the proof is complete. □

5.6 Conclusion of the Argument

By (5.29), (5.32), Propositions 2.2 and 5.7

$$\chi(\beta, \gamma, \nu) = \frac{1 + \tilde{z}_0}{\tilde{m}^2}. \tag{5.56}$$

Using this, (5.29), and (5.30), by exactly the same argument as in [3, Section 4.3], there is a differential relation between $\frac{\partial \chi}{\partial \nu}$ and χ, whose solution yields Theorem 1.2(ii).

The reason the susceptibility is handled first is that its leading-order critical behaviour can be computed from the second-order flow of the *bulk* coupling constants (g_j, ν_j, z_j). In contrast, in order to study the two-point function, we begin by writing

$$\bar{\phi}_a \phi_b = \frac{\partial^2}{\partial \sigma_a \partial \sigma_b} e^{\sigma_a \bar{\phi}_a + \sigma_b \phi_b} \Big|_{\sigma_a = \sigma_b = 0} \tag{5.57}$$

in (3.17). The incorporation of the exponential function $e^{\sigma_a \bar{\phi}_a + \sigma_b \phi_b}$ into Z_0 is equivalent to subtracting

$$\sigma_a \bar{\phi}_a \mathbb{1}_{x=a} + \sigma_b \bar{\phi}_b \mathbb{1}_{x=b} \tag{5.58}$$

from $V_0^{\pm}$. The renormalisation group map now acts on a polynomial of the form

$$\begin{aligned} g_j \tau^2 + \nu_j \tau + z_j \tau_\Delta & - \lambda_{a,j} \sigma_a \bar{\phi}_a \mathbb{1}_{x=a} - \lambda_{b,j} \sigma_b \phi_b \mathbb{1}_{x=b} \\ & - \tfrac{1}{2} \sigma_a \sigma_b (q_{a,j} \mathbb{1}_{x=a} + q_{b,j} \mathbb{1}_{x=b}). \end{aligned} \tag{5.59}$$

We have only included terms up to second order in (σ_a, σ_b) because, by (5.57), only these are needed to study the two-point function. The coefficients $(\lambda_{a,j}, \lambda_{b,j}, q_{a,j}, q_{b,j})$ are referred to as *observable* coupling constants and the behaviour of these coupling constants under the action of the renormalisation group is studied in detail in [2,27].

It was shown in [2] that the observable flow does not affect the bulk flow. Moreover, the second-order evolution of the observable flow remains identical to that of the case $\gamma_0 = 0$. This occurs for the same reason that the bulk flow is unaffected to second order by γ_0 (as in the statement of Corollary 5.4): namely, the second-order contributions to the observable flow are produced by an extension of the map V_{pt} [recall (5.8)], whose definition does not depend on γ_0. Thus, the analysis of the observable flow when γ_0 is small can proceed in the same way as when $\gamma_0 = 0$. That is, the same analysis that was carried out in [2] to study the two-point function applies directly here to prove Theorem 1.2(i).

The analysis of the correlation length of order p in [6] also applies directly here, and for the same reason: the second-order flow of coupling constants is independent of γ_0. This gives Theorem 1.2(iii).

5.7 A Version of the Implicit Function Theorem

We make use of [25, Chapter 4, Theorem 9.3], which is a version of the implicit function theorem that allows for a continuous, rather than differentiable, parameter. While the precise statement of [25, Chapter 4, Theorem 9.3] takes this parameter from an open subset of a Banach space, by [25, Chapter 4, Theorem 9.2], the parameter can in fact be taken from an arbitrary metric space. With this minor change, we restate [25, Chapter 4, Theorem 9.3] as the following proposition.

Proposition 5.8 *Let A be a metric space, let W, X be Banach spaces, and let $B \subset W$ be an open subset. Let $F : A \times B \to X$ be continuous, and suppose that F is C^1 in its second argument. Let $(\alpha, \beta) \in A \times B$ be a point such that $F(\alpha, \beta) = 0$ and $D_2 F(\alpha, \beta)^{-1}$ exists. Then there are open balls $M \ni \alpha$ and $N \ni \beta$ and a unique continuous mapping $f : M \to N$ such that $F(\xi, f(\xi)) = 0$ for all $\xi \in M$.*

We also use the following lemma, which is a small modification of [25, Chapter 3, Theorem 11.1]. In particular, it considers functions that may only be left- or right-differentiable.

Lemma 5.9 *Let F be a mapping as in the previous proposition with $A \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$. In addition, suppose that F is left-differentiable (respectively, right-differentiable) in α_2 at (α, β), with $\alpha = (\alpha_1, \alpha_2)$. If f is a continuous mapping defined in a neighbourhood of α, such that $F(\xi, f(\xi)) = 0$, then f is left-differentiable (respectively, right-differentiable) in α_2 at α.*

The above results lead to the following proposition, which we apply in the proofs of Propositions 5.3 and 5.6. Recall that $D(\delta, r)$ is defined in (5.20).

Proposition 5.10 *Let $\delta > 0$, and let r_1, r_2 be continuous positive-definite functions on $[0, \delta]$. Set*

$$D(\delta, r_1, r_2) = \{(w, x, y, z) \in D(\delta, r_1) \times \mathbb{R}^n : |z| \le r_2(x)\}, \tag{5.60}$$

and let F be a continuous function on $D(\delta, r_1, r_2)$ that is C^1 in (x, z). Suppose that for all $(\bar{w}, \bar{x}) \in [0, \delta]^2$ there exists $\bar{z}$ such that both $F(\bar{w}, \bar{x}, 0, \bar{z}) = 0$ and $D_Y F(\bar{w}, \bar{x}, 0, \bar{z})$ is invertible. Then there is a continuous positive-definite function r on $[0, \delta]$ and a continuous map $f : D(\delta, r) \to \mathbb{R}^n$ that is C^1 in x and such that $F(w, x, y, f(w, x, y)) = 0$ for all $(w, x, y) \in D(\delta, r)$. Moreover, if F is left-differentiable (respectively, right-differentiable) in y at some point (w, x, y, z), then f is left-differentiable (respectively, right-differentiable) at (w, x, y).

Proof Take any $(\bar{w}, \bar{x}) \in [0, \delta] \times (0, \delta]$ and let $R(\bar{w}, \bar{x})$ be the maximal radius s such that for all $(w, x, y) \in B(\bar{w}, \bar{x}, 0; s)$ there exists z such that both $F(w, x, y, z) = 0$ and $D_Z F(w, x, y, z)$ is invertible. By continuity of $(D_Z F(w, x, y, z))^{-1}$ near $(\bar{w}, \bar{x}, 0, \bar{z})$, and by Proposition 5.8 (applied to the restriction of F to $A \times B$, for some $A \ni (\bar{w}, \bar{x}, 0)$ and an open set $B \ni \bar{z}$), we have $R(\bar{w}, \bar{x}) > 0$ and there is a continuous function

$$f_{\bar{w},\bar{x}} : B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x})) \to \mathbb{R}^n \tag{5.61}$$

such that $F(w, x, y, f_{\bar{w},\bar{x}}(w, x, y)) = 0$ for all $(w, x, y) \in B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x}))$. Moreover, the unique solution to $F(w, x, y, z) = 0$ is given by $z = f_{\bar{w},\bar{x}}(w, x, y)$ for all such (w, x, y). By an application of Lemma 5.9 (with $\alpha_1 = (w, x), \alpha_2 = y$), we see that $f_{\bar{w},\bar{x}}$ is left- or right-differentiable in y wherever F is. By another application of Lemma 5.9 (with $\alpha_1 = (w, y), \alpha_2 = x$), we see that $f_{\bar{w},\bar{x}}$ is C^1 in x.

Set $R(\bar{w}, 0) = 0$ for all $\bar{w} \in [0, \delta]$, and let

$$D_f = \bigcup_{(\bar{w},\bar{x}) \in [0,\delta]^2} B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x})). \tag{5.62}$$

We define $f(w, 0, 0) = 0$ and, for $x > 0$,

$$f(w, x, y) = f_{\bar{w},\bar{x}}(w, x, y) \quad \text{for} \quad (w, x, y) \in B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x})). \tag{5.63}$$

By uniqueness, this function is well-defined. Continuity of f at $(w, 0, 0)$ follows from the fact that $|f(w, x, y)| \le r_2(x)$. The remaining desired regularity properties of f follow from those of the $f_{\bar{w},\bar{x}}$. It remains to show that $D(\delta, r) \subset D_f$ for some continuous positive-definite function r on $[0, \delta]$.

First, let us show that R is continuous on $[0, \delta]^2$. Let $\bar{x} > 0$ and fix $0 < \epsilon < R(\bar{w}, \bar{x})$. Then for any $(\bar{w}', \bar{x}') \in [0, \delta] \times (0, \delta]$ such that $|(\bar{w}, \bar{x}) - (\bar{w}', \bar{x}')| < \epsilon$, we have $B(\bar{w}', \bar{x}', 0; R(\bar{w}, \bar{x}) - \epsilon) \subset B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x}))$ by maximality of R. It follows that $R(\bar{w}', \bar{x}') \ge R(\bar{w}, \bar{x}) - \epsilon$. By a similar argument, $R(\bar{w}', \bar{x}') \le R(\bar{w}, \bar{x}) + \epsilon$, so $|R(\bar{w}, \bar{x}) - R(\bar{w}', \bar{x}')| \le \epsilon$. Thus, R is continuous on $[0, \delta] \times (0, \delta]$. Continuity at $\bar{x} = 0$ follows from the fact that $R(\bar{w}, \bar{x}) \le r_1(\bar{x})$ uniformly in $\bar{w}$.

For $\bar{x} \in [0, \delta]$, let

$$r(\bar{x}) = \inf(R(\bar{w}, \bar{x}) : \bar{w} \in [0, \delta]). \tag{5.64}$$

Since $R(\cdot, \bar{x})$ is continuous, $r(\bar{x}) > 0$ for $\bar{x} > 0$. Moreover, $0 \le r(0) \le r_1(0) = 0$, so r is positive-definite. Continuity of r follows from joint continuity of R. For any $(w, x, y) \in D(\delta, r)$ (with this choice of r),

$$|(w, x, y) - (w, x, 0)| = |y| < r(x) \le R(w, x), \tag{5.65}$$

so $(w, x, y) \in B(w, x, 0; R(w, x))$. We conclude that $D(\delta, r) \subset D_f$. □

Acknowledgements The work of RB was supported in part by the Simons Foundation. The work of GS and BCW was supported in part by NSERC of Canada. We thank the referees for useful suggestions.

References

1. Bauerschmidt, R.: A simple method for finite range decomposition of quadratic forms and Gaussian fields. Probab. Theory Relat. Fields **157**, 817–845 (2013)
2. Bauerschmidt, R., Brydges, D.C., Slade, G.: Critical two-point function of the 4-dimensional weakly self-avoiding walk. Commun. Math. Phys. **338**, 169–193 (2015)
3. Bauerschmidt, R., Brydges, D.C., Slade, G.: Logarithmic correction for the susceptibility of the 4-dimensional weakly self-avoiding walk: a renormalisation group analysis. Commun. Math. Phys. **337**, 817–877 (2015)
4. Bauerschmidt, R., Brydges, D.C., Slade, G.: A renormalisation group method. III. Perturbative analysis. J. Stat. Phys. **159**, 492–529 (2015)
5. Bauerschmidt, R., Brydges, D.C., Slade, G.: Structural stability of a dynamical system near a non-hyperbolic fixed point. Ann. Henri Poincaré **16**, 1033–1065 (2015)
6. Bauerschmidt, R., Slade, G., Tomberg, A., Wallace, B.C.: Finite-order correlation length for 4-dimensional weakly self-avoiding walk and $|\varphi|^4$ spins. Ann. Henri Poincaré **18**, 375–402 (2016)
7. Brydges, D., Evans, S.N., Imbrie, J.Z.: Self-avoiding walk on a hierarchical lattice in four dimensions. Ann. Probab. **20**, 82–124 (1992)
8. Brydges, D.C., Guadagni, G., Mitter, P.K.: Finite range decomposition of Gaussian processes. J. Stat. Phys. **115**, 415–449 (2004)
9. Brydges, D.C., Imbrie, J.Z., Slade, G.: Functional integral representations for self-avoiding walk. Probab. Surv. **6**, 34–61 (2009)
10. Brydges, D.C., Maya, I.M.: An application of Berezin integration to large deviations. J. Theor. Probab. **4**, 371–389 (1991)
11. Brydges, D.C., Slade, G.: A renormalisation group method. I. Gaussian integration and normed algebras. J. Stat. Phys. **159**, 421–460 (2015)
12. Brydges, D.C., Slade, G.: A renormalisation group method. II. Approximation by local polynomials. J. Stat. Phys. **159**, 461–491 (2015)
13. Brydges, D.C., Slade, G.: A renormalisation group method. IV. Stability analysis. J. Stat. Phys. **159**, 530–588 (2015)
14. Brydges, D.C., Slade, G.: A renormalisation group method. V. A single renormalisation group step. J. Stat. Phys. **159**, 589–667 (2015)
15. Brydges, D.C., Spencer, T.: Self-avoiding walk in 5 or more dimensions. Commun. Math. Phys. **97**, 125–148 (1985)
16. Clisby, N.: Accurate estimate of the critical exponent ν for self-avoiding walks via a fast implementation of the pivot algorithm. Phys. Rev. Lett. **104**, 055702 (2010)
17. Hammond, A., Helmuth, T.: Self-avoiding walk with a self-attraction. In preparation
18. Hara, T.: Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. Ann. Probab. **36**, 530–593 (2008)
19. Hara, T., Slade, G.: Self-avoiding walk in five or more dimensions. I. The critical behaviour. Commun. Math. Phys. **147**, 101–136 (1992)
20. Helmuth, T.: Loop-weighted walk. Ann. Inst. Henri Poincaré Comb. Phys. Interact. **3**, 55–119 (2016)
21. van der Hofstad, R., Klenke, A.: Self-attractive random polymers. Ann. Appl. Probab. **11**, 1079–1115 (2001)

22. van der Hofstad, R., Klenke, A., König, W.: The critical attractive random polymer in dimension one. J. Stat. Phys. **106**, 477–520 (2002)
23. den Hollander, F.: Random Polymers. In: Ecole d'Eté de Probabilités de Saint-Flour XXXVII-2007. Lecture Notes in Mathematics, Vol. 1974. Springer, Berlin (2009)
24. Janse van Rensburg, E.J.: The Statistical Mechanics of Interacting Walks, Polygons. Animals and Vesicles, 2nd edn. Oxford University Press, Oxford (2015)
25. Loomis, L.H., Sternberg, S.: Advanced Calculus. World Scientific, Singapore (2014)
26. Pétrélis, N., Torri, N.: Collapse transition of the interacting prudent walk. arXiv:1610.07542 (2016)
27. Slade, G., Tomberg, A.: Critical correlation functions for the 4-dimensional weakly self-avoiding walk and n-component $|\varphi|^4$ model. Commun. Math. Phys. **342**, 675–737 (2016)
28. Ueltschi, D.: A self-avoiding walk with attractive interactions. Probab. Theory Relat. Fields **124**, 189–203 (2002)
29. Vanderzande, C.: Lattice Models of Polymers. Cambridge University Press, Cambridge (1998)
30. Wilson, K.G., Kogut, J.: The renormalization group and the ϵ expansion. Phys. Rep. **12**, 75–200 (1974)