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Non-equilibrium Dynamics for a Widom–Rowlinson Type Model with Mutations

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Abstract A dynamical version of the Widom–Rowlinson model in the continuum is considered. The dynamics is modelled by a spatial two-component birth-and-death Glauber process where particles, in addition, are allowed to change their type with density dependent rates. An evolution of states is constructed in terms of correlation function evolution in a certain Ruelle space. It is shown that such evolution provides the unique weak solution to the associated Fokker–Planck equation. Existence of a unique invariant measure and ergodicity with exponential rate is established. Vlasov scaling is performed and the chaos preservation property is shown.

Keywords Widom–Rowlinson model · Mutations · Fokker–Planck equation · Ergodicity · Vlasov scaling

1 Introduction

The study of critical behaviour of complex systems and related invariant states is one of the central problems for statistical and mathematical physics. Particular classes of complex systems can be modelled either as lattice or as continuous models. The Ising model is probably one of the most famous examples on the lattice, where each particle is allowed to have only two possible states, the so-called spins $\pm$. Its generalization to particles with any fixed number of spins is known as the Potts model. It was introduced in [29] and has been intensively studied on various lattices, see, e.g., [4,34] and the review paper [18]. In contrast to lattice models, much less is known for their continuous counterparts, i.e., for continuous interacting particle systems. Below we consider the case of interacting particle systems in the continuum.

We suppose that all particles are located in $\mathbb{R}^d$, are identical by properties and, indistinguishable. A particle at position $x \in \mathbb{R}^d$ is assumed to have two different spins $\pm$. The

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extension to dynamics with N distinct spins is then a straightforward modification of the results presented here. The phase space for the dynamics is chosen as the configuration space $\Gamma^2 = \Gamma^+ \times \Gamma^-$, where

$$\Gamma^{\pm} = \left\{ \gamma^{\pm} \subset \mathbb{R}^d \,|\, \left|\gamma^{\pm} \cap \Lambda\right| < \infty \text{ for all compacts } \Lambda \subset \mathbb{R}^d \right\},$$

is the configuration space of all locally finite subsets of $\mathbb{R}^d$. Here, $|A|$ stands for the number of elements in the set $A \subset \mathbb{R}^d$. For simplicity of notation we write $\gamma = (\gamma^+, \gamma^-) \in \Gamma^2$ and let $\gamma^{\pm}\backslash x$ and $\gamma^{\pm} \cup x$ stand for $\gamma^{\pm}\backslash\{x\}$ and $\gamma^{\pm} \cup \{x\}$, respectively. The one-component configuration space $\Gamma^{\pm}$ is a Polish space w.r.t. the smallest topology such that all mappings $\Gamma^{\pm} \ni \gamma^{\pm} \longmapsto \sum_{x \in \gamma^{\pm}} f(x)$ are continuous, where $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ is any continuous function having compact support (see [23]). Hence Γ^2 equipped with the product topology is a Polish space. Measurable functions $F\colon \Gamma^2 \longrightarrow \mathbb{R}$ are called observables and probability measures μ on Γ^2 are called states.

Interactions between particles of different (or the same type) are assumed to be given by a symmetric pair potential $\phi\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying the usual conditions such as stability, lower regularity and, integrability. Associated to a particle at position $x \in \mathbb{R}^d$ is the relative energy

$$E_{\phi}\left(x,\, \gamma^{\pm}\right) := \begin{cases} \sum\limits_{y \in \gamma^{\pm}} \phi(x-y), & \sum\limits_{y \in \gamma^{\pm}} |\phi(x-y)| < \infty, \\ \infty, & \text{otherwise}, \end{cases}$$

w.r.t. the configuration $\gamma^{\pm}$.

In this work we consider birth-and-death Markov dynamics with Markov (pre-)generator $L = L_0 + V$. The first operator describes two-interacting Glauber-type dynamics, whereas the second one describes mutations of particles. We assume that L_0 is for a suitable class of observables $F\colon \Gamma^2 \longrightarrow \mathbb{R}$ given by

$$\begin{aligned}(L_0 F)(\gamma) = &\sum_{x \in \gamma^+} \left(F\left(\gamma^+\backslash x,\, \gamma^-\right) - F(\gamma)\right) + \sum_{x \in \gamma^-} \left(F\left(\gamma^+,\, \gamma^-\backslash x\right) - F(\gamma)\right) \\ &+ z^+ \int_{\mathbb{R}^d} e^{-E_{\phi^-}(x,\gamma^-)} e^{-E_{\psi^+}(x,\gamma^+)} \left(F\left(\gamma^+ \cup x,\, \gamma^-\right) - F(\gamma)\right) \mathrm{d}x \\ &+ z^- \int_{\mathbb{R}^d} e^{-E_{\phi^+}(x,\gamma^+)} e^{-E_{\psi^-}(x,\gamma^-)} \left(F\left(\gamma^+,\, \gamma^- \cup x\right) - F(\gamma)\right) \mathrm{d}x.\end{aligned}$$

Here, $z^{\pm} > 0$ are the activities of $\pm$ particles and $\phi^{\pm},\ \psi^{\pm}$ are symmetric, non-negative and satisfy some reasonable integrability condition (see Sect. 3). The pair potentials $\phi^{\pm}$ describe the interaction of a new particle $x \in \mathbb{R}^d$ added to the configuration $\gamma^{\mp}$ with particles of different type, i.e., $E_{\phi^{\pm}}(x,\, \gamma^{\pm})$ is the relative energy of the configuration $\gamma^{\pm}$ w.r.t. $x \in \mathbb{R}^d$. Likewise, $\psi^{\pm}$ describe the interaction of a particle $x \in \mathbb{R}^d$ added to $\gamma^{\pm}$ and interacting with particles of the same type, i.e., $E_{\psi^{\pm}}(x,\, \gamma^{\pm})$ is the relative energy of x w.r.t. $\gamma^{\pm}$. Such interactions imply that the birth rate for a particle at position $x \in \mathbb{R}^d$ is small, provided there are many particles of types $\pm$ close to x.

Birth-and-death models have also various applications in biology, in particular in the modelling of tumour evolution (see, e.g., [8]). In such a case particles are considered as tumour cells and it is natural to admit cells to change their type. Such events are called mutations, they are modelled by the elementary Markov events $(\gamma^+,\, \gamma^-) \longmapsto (\gamma^+\backslash x,\, \gamma^- \cup x)$ and $(\gamma^+,\, \gamma^-) \longmapsto (\gamma^+ \cup x,\, \gamma^-\backslash x)$. In this work we consider mutations with Markov generator given by the heuristic formula

$$(VF)(\gamma) = q^+ \sum_{x\in\gamma^+} e^{-E_{\kappa^+}(x,\gamma^+\backslash x)} e^{-E_{\tau^+}(x,\gamma^-)} \left(F\left(\gamma^+\backslash x,\ \gamma^-\cup x\right) - F(\gamma)\right)$$
$$+ q^- \sum_{x\in\gamma^-} e^{-E_{\kappa^-}(x,\gamma^-\backslash x)} e^{-E_{\tau^-}(x,\gamma^+)} \left(F\left(\gamma^+\cup x,\ \gamma^-\backslash x\right) - F(\gamma)\right).$$

The constants $q^\pm > 0$ are the so-called mutation activities and play a similar role as the usual activities $z^\pm$. The pair potentials $\kappa^\pm \geq 0$ take the interactions with particles of same type into account, whereas $\tau^\pm$ take the interactions with particles of different type into account.

In this work we construct an non-equilibrium evolution of states and study its time-behaviour as $t \to \infty$. An evolution of states corresponding to L is, by definition, a family of states $(\mu_t)_{t\geq 0}$ which satisfies the Fokker–Planck equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma^2} F(\gamma)\mathrm{d}\mu_t(\gamma) = \int_{\Gamma^2} (LF)(\gamma)\mathrm{d}\mu_t(\gamma), \quad \mu_t|_{t=0} = \mu_0, \quad t \geq 0, \tag{1}$$

for a suitable large class of observables F. It is worth to mention that, without additional assumptions, $(LF)(\gamma)$ makes no sense for any $\gamma \in \Gamma^2$. A rigorous meaning for LF and the definition of solutions to (1) will be given in Sect. 3. At this point we have to restrict the class of admissible states μ. Namely, we consider an evolution of states $(\mu_t)_{t\geq 0}$ for which there exists an associated sequence of correlation functions $k_{\mu_t} = (k_{\mu_t}^{(n,m)})_{n,m=0}^{\infty}$ satisfying a certain (space-inhomogeneous) Ruelle bound (see Sect. 2). Such bound is used to define a certain Banach space of correlation functions with some norm $\|\cdot\|_{\mathcal{K}_\alpha}$. We realize the following scheme

(a) For each initial state μ_0 there exists a unique solution $(\mu_t)_{t\geq 0}$ to (1). Moreover, such solution is constructed in the class of states for which the associated sequence of correlation functions exists and satisfies a certain (space-inhomogeneous) Ruelle bound. As a consequence of the construction, it will be shown that the corresponding evolution of correlation functions $t \longmapsto k_{\mu_t}$ is continuous w.r.t. $\|\cdot\|_{\mathcal{K}_\alpha}$
(b) There exists a unique invariant measure μ_{inv} with correlation function k_{inv} which is associated with the evolution of states. Moreover, the evolution of states is ergodic with exponential rate, i.e., there exist constants $a,\ b > 0$ such that

$$\left\|k_{\mu_t} - k_{\mathrm{inv}}\right\|_{\mathcal{K}_\alpha} \leq ae^{-bt}\left\|k_{\mu_0} - k_{\mathrm{inv}}\right\|_{\mathcal{K}_\alpha}, \quad t \geq 0.$$

(c) Using the notion of Vlasov scaling (see [9]), we derive the kinetic equation for the approximate density of the particle system. We show convergence of the scaled evolution to solutions corresponding to a certain (limiting) hierarchical system of equations. Solutions to the latter system of equations satisfy the chaos preservation property.

Let us briefly comment on the results. Many results known for continuous systems are related with the analysis of equilibrium states which are the so-called Gibbs-type measures (see, e.g., [5,20] and the references therein). The analysis of non-equilibrium evolution of states is a non-trivial problem on its own which has to be realized for each model separately.

In the first step we use a (two-component) modification of the techniques developed in [10] to provide an evolution of correlation functions. Since this construction is already well known (see [24]) and has been realized for several models, we keep all computations and arguments short and simply point out the main differences for this model. It is worth to mention that in contrast to the latter works (see also [10,11]) we do not suppose that the correlation functions are bounded w.r.t. the spatial variables. In the second step we show that the evolution of correlation functions is, in fact, associated to an evolution of states which provides a solution to the Fokker–Planck equation (1). Known techniques for the construction of an evolution of states are related with some kind of approximation by finite volumes (see, e.g., [3,12,21]). In our approach we use an approximation by finite systems (the number of particles remains finite at any moment of time) simultaneously with an approximation of the operator L. Solutions to such approximation enjoy the additional property that the corresponding correlation functions are integrable. Hence we can identify them with an evolution of densities on the space of finite configurations. Details concerning the evolution of densities for Markov evolutions on the space of finite configurations can be found in [16]. The ergodicity statement is known for the Sourgailis model (see [7]) and for the so-called G^--dynamics (see [25]). We extend and improve the techniques from the latter work for this particular model. The Vlasov scaling is, on the formal level, easy to derive (see [9] for one-component models and [8] for two-component models). It is, however, an important task to show that such formal convergence of equations implies convergence of solutions to these equations.

This work is organized as follows. Preliminaries and notations are introduced in Sect. 2. Section 3 is devoted to the construction of an evolution of states and correlation function evolution. Ergodicity is proved in Sect. 4, whereas Vlasov scaling is studied in Sect. 5. Two particular models which are included in the general form of the Markov operator L are considered in the last section.

2 Preliminaries

2.1 Space of Finite Configurations

Let $\Gamma_0 = \{\eta \subset \mathbb{R}^d \,|\, |\eta| < \infty\}$ be the space of all finite configurations over $\mathbb{R}^d$. It has the natural decomposition $\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}$, where $\Gamma_0^{(0)} = \{\emptyset\}$ and $\Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d \,|\, |\eta| = n\}$. Let $\widetilde{(\mathbb{R}^d)^n}$ be the collection of all ordered $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ such that $x_j \neq x_k$ whenever $j \neq k$. Then,

$$\mathrm{sym}_n \colon \widetilde{\left(\mathbb{R}^d\right)^n} \longrightarrow \Gamma_0^{(n)}, \quad (x_1, \dots, x_n) \longmapsto \{x_1, \dots, x_n\}, \quad n \geq 1,$$

is a bijection. A set $O \subset \Gamma_0$ is said to be open, if $\mathrm{sym}_n^{-1}(O) \subset \widetilde{(\mathbb{R}^d)^n}$ is open for any $n \in \mathbb{N}$. The latter space is endowed with the subspace topology. In the case $n = 0$ we require $\{\emptyset\}$ to be open. This defines a topology on Γ_0.

Let $m^{\otimes n}$ be the Lebesgue measure on $(\mathbb{R}^d)^n$. Then $m^{\otimes n}((\mathbb{R}^d)^n \backslash \widetilde{(\mathbb{R}^d)^n}) = 0$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda = \delta_\emptyset + \sum_{n=1}^{\infty} \frac{1}{n!} m^{(n)},$$

where $m^{(n)} = m^{\otimes n} \circ \mathrm{sym}_n^{-1}$ is a measure on $\Gamma_0^{(n)}$, $n \geq 1$. We will need the following well-known identity.

Lemma 1 *Let $G \colon \Gamma_0 \times \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$ be measurable. Then*

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} G(\xi, \eta \backslash \xi, \eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\xi, \eta, \eta \cup \xi) \mathrm{d}\lambda(\xi) \mathrm{d}\lambda(\eta), \tag{2}$$

whenever one side of the equality is finite for $|G|$.

For convenience of the reader, a proof is given in the Appendix.

The Lebesgue exponential is for a measurable function $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $e_\lambda(f;\,\eta) = \prod_{x\in\eta} f(x)$ and satisfies, provided $f \in L^1(\mathbb{R}^d)$,

$$\int\limits_{\Gamma_0} e_\lambda(f;\,\eta)\mathrm{d}\lambda(\eta) = \exp(\langle f\rangle),$$

where $\langle f\rangle = \int_{\mathbb{R}^d} f(x)\mathrm{d}x$ denotes the mean of f. Below we give a brief extension to the two-component case.

Let $\Gamma_0^2 = \Gamma_0^+ \times \Gamma_0^-$, where $\Gamma_0^\pm$ are two identical copies of Γ_0. It is equipped with the product topology. For simplicity of notation, we extend all set-operations component-wise. Namely, $\eta \cup \xi$, $\eta\backslash\xi$, $\xi \subset \eta$ stand for $(\eta^+ \cup \xi^+,\ \eta^- \cup \xi^-)$, $(\eta^+\backslash\xi^+,\ \eta^-\backslash\xi^-)$ and, $\xi^+ \subset \eta^+$, $\xi^- \subset \eta^-$, where $\eta,\ \xi \in \Gamma_0^2$. The two-component Lebesgue–Poisson measure $\lambda^+ \otimes \lambda^-$ satisfies

$$\lambda^+ \otimes \lambda^- \left(\left\{\left(\eta^+,\ \eta^-\right) \in \Gamma_0^2 \middle| \eta^+ \cap \eta^- \neq \emptyset\right\}\right) = 0,$$

see [14]. Since no confusion can arise we use the notation λ instead of $\lambda^+ \otimes \lambda^-$. Thus, for any measurable function G

$$\int\limits_{\Gamma_0^2} G(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0^+}\int\limits_{\Gamma_0^-} G\left(\eta^+,\ \eta^-\right)\mathrm{d}\lambda^+\left(\eta^+\right)\mathrm{d}\lambda^-\left(\eta^-\right),$$

provided one side is finite for $|G|$. A function $G\colon \Gamma_0^2 \longrightarrow \mathbb{R}$ is said to have bounded support if there exists $N \in \mathbb{N}$ and a compact $\Lambda \subset \mathbb{R}^d$ such that

$$G(\eta) = 0, \quad \text{whenever} \quad \eta \cap \Lambda^c \neq \emptyset \quad \text{or} \quad |\eta| > N.$$

Here, we let $|\eta| := |\eta^+| + |\eta^-|$ and $\eta \cap \Lambda^c := (\eta^+ \cap \Lambda^c,\ \eta^- \cap \Lambda^c)$. Denote by $B_{bs}(\Gamma_0^2)$ the space of all bounded functions having bounded support.

2.2 Harmonic Analysis on Configuration Spaces

Let X be a locally compact Hausdorff space and

$$\Gamma(X) = \{\gamma \subset X \,|\, |\gamma \cap \Lambda| < \infty \text{ for all compacts } \Lambda \subset X\}.$$

General results concerning harmonic analysis on $\Gamma(X)$ and $\Gamma_0(X)$ (defined analogously with $\mathbb{R}^d$ replaced by X) can be found in [22]. Here we consider the two-component case which corresponds to $X = \mathbb{R}^d \times \{0,\ 1\}$, where $(x,\ 0)$ is identified with a particle of type $+$ and $(x,\ 1)$ with a particle of type $-$. Recall that $\Gamma^2 = \Gamma^+ \times \Gamma^-$, where

$$\Gamma^\pm = \left\{\gamma^\pm \subset \mathbb{R}^d \,\middle|\, \left|\gamma^\pm \cap \Lambda\right| < \infty \text{ for any compact } \Lambda \subset \mathbb{R}^d\right\}.$$

Then

$$\Gamma^2 \longrightarrow \Gamma\left(\mathbb{R}^d \times \{0,\ 1\}\right), \quad \left(\gamma^+,\ \gamma^-\right) \longmapsto \left\{(x,\ 0) \,|\, x \in \gamma^+\right\} \cup \left\{(x,\ 1) | x \in \gamma^-\right\}, \qquad (3)$$

is a bijection. Hence any function $F\colon \Gamma(\mathbb{R}^d \times \{0,\ 1\}) \longrightarrow \mathbb{R}$ can be identified with a function $\widetilde{F}\colon \Gamma^2 \longrightarrow \mathbb{R}$. Similarly, any function $G\colon \Gamma_0(\mathbb{R}^d \times \{0,\ 1\}) \longrightarrow \mathbb{R}$ can be identified with a function $\widetilde{G}\colon \Gamma_0^2 \longrightarrow \mathbb{R}$. Most of the results given below are due to [22] obtained by above identification.

The Poisson measure $\pi_{e^{\alpha^\pm}}$ on $\Gamma^\pm$, where $\alpha^\pm \in \mathbb{R}$, is the unique probability measure on $\Gamma^\pm$ such that its Laplace transform satisfies

$$\int_{\Gamma^\pm} e^{\sum\limits_{x\in\gamma^\pm} f(x)} \mathrm{d}\pi_{e^{\alpha^\pm}}\left(\gamma^\pm\right) = \exp\left(e^{\alpha^\pm} \int_{\mathbb{R}^d} \left(e^{f(x)} - 1 \right) \mathrm{d}x \right),$$

for any continuous function f on $\mathbb{R}^d$ having compact support. The two-component Poisson measure is defined by $\pi_{e^\alpha} := \pi_{e^{\alpha^+}} \otimes \pi_{e^{\alpha^-}}$. A state μ on Γ^2 is said to have finite local moments, if for all compacts $\Lambda \subset \mathbb{R}^d$

$$\int_{\Gamma^2} \left|\gamma^+ \cap \Lambda\right|^n \left|\gamma^- \cap \Lambda\right|^n \mathrm{d}\mu(\gamma) < \infty, \quad \forall n \in \mathbb{N}.$$

Given two compacts $\Lambda^+, \ \Lambda^- \subset \mathbb{R}^d$, consider the projection

$$p_{\Lambda^+,\Lambda^-}\colon \Gamma^2 \longrightarrow \Gamma^2_{\Lambda^+,\Lambda^-}, \quad p_{\Lambda^+,\Lambda^-}\left(\gamma^+, \gamma^-\right) := \left(\gamma^+ \cap \Lambda^+, \ \gamma^- \cap \Lambda^-\right),$$

where $\Gamma^2_{\Lambda^+,\Lambda^-} = \{\gamma \in \Gamma^2 | \gamma^\pm \subset \Lambda^\pm\}$. A state μ is said to be locally absolutely continuous w.r.t. the Poisson measure, if $\mu^{\Lambda^+,\Lambda^-} := \mu \circ p^{-1}_{\Lambda^+,\Lambda^-}$ is absolutely continuous w.r.t. $\pi_{e^\alpha} \circ p^{-1}_{\Lambda^+,\Lambda^-}$ for some $\alpha = (\alpha^+, \alpha^-) \in \mathbb{R}$ and all compacts $\Lambda^+, \ \Lambda^- \subset \mathbb{R}^d$. It can be shown that this definition is independent of the particular choice of $\alpha^\pm$. Denote by $\mathcal{P}$ the space of all states which have finite local moments and are locally absolutely continuous w.r.t. the Poisson measure. For any $\mu \in \mathcal{P}$ it holds that

$$\mu\left(\left\{\left(\gamma^+, \gamma^-\right) \in \Gamma^+ \times \Gamma^- \middle| \gamma^+ \cap \gamma^- = \emptyset\right\}\right) = 1,$$

i.e., events where particles of different types are located at the same position are negligible (see [14]).

Given $G \in B_{bs}(\Gamma_0^2)$, the K-transform is defined by

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma^2, \tag{4}$$

where $\Subset$ means that the sum runs over all finite subsets of γ. Let $\mathcal{FP}(\Gamma^2) := K(B_{bs}(\Gamma_0^2))$. For each $F \in \mathcal{FP}(\Gamma^2)$ there exists $A > 0$, $N \in \mathbb{N}$ and a compact $\Lambda \subset \mathbb{R}^d$ such that $F(\gamma) = F(\gamma \cap \Lambda)$ and

$$|F(\gamma)| \le A(1 + |\gamma \cap \Lambda|)^N, \quad \gamma \in \Gamma^2,$$

i.e., F is a polynomially bounded cylinder function. Here, $\gamma \cap \Lambda := (\gamma^+ \cap \Lambda, \ \gamma^- \cap \Lambda)$. The map $K\colon B_{bs}(\Gamma_0^2) \longrightarrow \mathcal{FP}(\Gamma^2)$ is a positivity preserving isomorphism with inverse

$$\left(K^{-1}F\right)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi).$$

Denote by K_0 the restriction $KF|_{\Gamma_0^2}$ and by K_0^{-1} its inverse given as above.

Let $\mu \in \mathcal{P}$. It follows by [15] that the correlation function k_μ exists and satisfies

$$\int_{\Gamma^2} KG(\gamma) \mathrm{d}\mu(\gamma) = \int_{\Gamma_0^2} G(\eta) k_\mu(\eta) \mathrm{d}\lambda(\eta), \quad G \in B_{bs}\left(\Gamma_0^2\right). \tag{5}$$

The correlation function is uniquely determined by such property. Note that $K|G|$ is integrable w.r.t. μ. The K-transform satisfies $\|KG\|_{L^1(\Gamma^2,\mathrm{d}\mu)} \leq \|G\|_{L^1(\Gamma_0^2,k_\mu \mathrm{d}\lambda)}$ for any $G \in B_{bs}(\Gamma_0^2)$. It can be extended to a bounded linear operator $K\colon L^1(\Gamma^2,\ \mathrm{d}\mu) \longrightarrow L^1(\Gamma_0^2,\ k_\mu \mathrm{d}\lambda)$ in such a way that (4) holds for μ-a.a. $\gamma \in \Gamma^2$ (see [22] together with the identification (3)).

2.3 Ruelle Space of Correlation Functions

Let $\rho\colon \mathbb{R}^d \longrightarrow [1,\ \infty)$ be a measurable, locally bounded function and take $\alpha = (\alpha^+,\ \alpha^-) \in \mathbb{R}^2$. For simplicity of notation, we let

$$e_\lambda(\rho;\ \eta)e^{\alpha|\eta|} := e_\lambda\left(\rho;\ \eta^+\right) e_\lambda\left(\rho;\ \eta^-\right) e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|}.$$

Let $\mathcal{L}_\alpha := L^1(\Gamma_0^2,\ e^{\alpha|\cdot|}e_\lambda(\rho)\mathrm{d}\lambda)$ with the norm

$$\|G\|_{\mathcal{L}_\alpha} = \int_{\Gamma_0^2} |G(\eta)| e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|} e_\lambda\left(\rho;\ \eta^+\right) e_\lambda\left(\rho;\ \eta^-\right) \mathrm{d}\lambda(\eta).$$

Since ρ is fixed, we omit here and in the following the additional dependence of $\mathcal{L}_\alpha$ on ρ. Denote by $(\mathcal{L}_\alpha)^*$ the dual Banach space to $\mathcal{L}_\alpha$. We use the duality

$$\langle G,\ k\rangle := \int_{\Gamma_0^2} G(\eta)k(\eta)\mathrm{d}\lambda(\eta), \quad G \in \mathcal{L}_\alpha,$$

to identify $(\mathcal{L}_\alpha)^*$ with the space of equivalence classes of functions k with the norm

$$\|k\|_{\mathcal{K}_\alpha} = \operatorname*{ess\ sup}_{\eta \in \Gamma_0^2} \frac{|k(\eta)|}{e_\lambda(\rho;\ \eta^+)e_\lambda(\rho;\ \eta^-)} e^{-\alpha^+|\eta^+|} e^{-\alpha^-|\eta^-|}.$$

Let $\mathcal{K}_\alpha$ the Banach space of all such equivalence classes of functions k. Then, each $k \in \mathcal{K}_\alpha$ satisfies the (space-inhomogeneous) Ruelle bound

$$|k(\eta)| \leq \|k\|_{\mathcal{K}_\alpha} e_\lambda\left(\rho;\ \eta^+\right) e_\lambda\left(\rho;\ \eta^-\right) e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|}, \quad \eta \in \Gamma_0^2.$$

In general, not any non-negative function $k \in \mathcal{K}_\alpha$ is the correlation function of a state μ. Such property can be characterized by an additional positivity property. A function $G \in \mathcal{L}_\alpha$ is called positive definite if $KG \geq 0$. Let $B_{bs}^+(\Gamma_0^2)$ be the cone of all positive definite functions in $B_{bs}(\Gamma_0^2)$. A function $k \in \mathcal{K}_\alpha$ is called positive definite (in the sense of Lenard), if

$$\langle G,\ k\rangle = \int_{\Gamma_0^2} G(\eta)k(\eta)\mathrm{d}\lambda(\eta) \geq 0, \quad G \in B_{bs}^+\left(\Gamma_0^2\right).$$

Note that any positive definite function k is non-negative. It follows from [27] that any positive definite function k such that $k(\emptyset,\ \emptyset) = 1$ is the correlation function of some $\mu \in \mathcal{P}$. There may exist, in general, many different $\mu \in \mathcal{P}$ such that k is the correlation function for μ, i.e., (5) holds. A growth condition is sufficient to show that μ is unique with such property (see [26]). The following is a particular case of [26,27] together with the identification (3).

Theorem 1 *Let $k \in \mathcal{K}_\alpha$. The following are equivalent.*

(1) *There exists a unique $\mu \in \mathcal{P}_\alpha$ such that k is its correlation function, i.e., $k_\mu = k$.*
(2) *$k(\emptyset,\ \emptyset) = 1$ and k is positive definite (in the sense of Lenard).*

Let $\mathcal{P}_\alpha$ be the collection of all $\mu \in \mathcal{P}$ such that $k_\mu \in \mathcal{K}_\alpha$. It is a complete, non-separable metric space w.r.t. $d_\alpha(\mu, \nu) := \|k_\mu - k_\nu\|_{\mathcal{K}_\alpha}$. Note that for each $F \in \mathcal{FP}(\Gamma^2)$ there exists a constant $C_\alpha(F) > 0$ such that

$$\left| \int_{\Gamma^2} F(\gamma)\mathrm{d}\mu(\gamma) - \int_{\Gamma^2} F(\gamma)\mathrm{d}\nu(\gamma) \right| \le d_\alpha(\mu, \nu) C_\alpha(F).$$

3 Evolution of States

3.1 Scheme of Construction

Let L be the Markov operator given by the heuristic formulas in the introduction. A general approach to the construction of an evolution of states can be found in [13], we consider a modification of such scheme. Details, assumptions and the rigorous statements are given later on. The aim is to solve the Fokker–Planck equation (1) in the class $\mathcal{P}_\alpha$ for some $\alpha = (\alpha^+, \alpha^-) \in \mathbb{R}^2$ and a measurable, locally bounded function $\rho : \mathbb{R}^d \longrightarrow [1, \infty)$. This parameters will be specified later on.

Definition 1 A family $(\mu_t)_{t \ge 0} \subset \mathcal{P}_\alpha$ is a weak solution to (1), if for all $F \in \mathcal{FP}(\Gamma^2)$ the following conditions are satisfied:

(a) $LF \in L^1(\Gamma^2, \mathrm{d}\mu_t)$ for all $t \ge 0$.
(b) $t \longmapsto \int_{\Gamma^2} (LF)(\gamma)\mathrm{d}\mu_t(\gamma)$ is locally integrable.
(c) We have

$$\int_{\Gamma^2} F(\gamma)\mathrm{d}\mu_t(\gamma) = \int_{\Gamma^2} F(\gamma)\,\mathrm{d}\mu_0(\gamma) + \int_0^t \int_{\Gamma^2} (LF)(\gamma)\mathrm{d}\mu_s(\gamma)\mathrm{d}s, \quad t \ge 0. \tag{6}$$

For any $\mu \in \mathcal{P}_\alpha$ and $F \in L^1(\Gamma^2, \mathrm{d}\mu)$ let

$$\langle\langle F, \mu \rangle\rangle := \int_{\Gamma^2} F(\gamma)\mathrm{d}\mu(\gamma).$$

Remark 1 Let $(\mu_t)_{t \ge 0} \subset \mathcal{P}_\alpha$ be a weak solution to (1). Then, $t \longmapsto \langle\langle F, \mu_t \rangle\rangle$ is absolutely continuous (see (6)) and hence (1) holds for each $F \in \mathcal{FP}(\Gamma^2)$ and a.a. $t \ge 0$.

Let $(\mu_t)_{t \ge 0} \subset \mathcal{P}_\alpha$ be a solution to (1) and denote by $(k_t)_{t \ge 0} \subset \mathcal{K}_\alpha$ its associated evolution of correlation functions (see (5)). Define a linear mapping $\widehat{L} := K_0^{-1} L K_0$ acting on functions $B_{bs}(\Gamma_0^2)$. Then $(k_t)_{t \ge 0}$ is (at least formally) a solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_0^2} G(\eta) k_t(\eta)\mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} (\widehat{L}G)(\eta) k_t(\eta)\mathrm{d}\lambda(\eta), \quad k_t|_{t=0} = k_0, \quad G \in B_{bs}\left(\Gamma_0^2\right). \tag{7}$$

Uniqueness for weak solutions to (1) holds, provided that (7) has at most one solution. For existence of solutions to (1) one first constructs a solution to (7). Afterwards it is necessary to show that such solution corresponds to an evolution of states $(\mu_t)_{t \ge 0} \subset \mathcal{P}_\alpha$ and this evolution of states is a weak solution to (1).

In order to find a solution to (7) we consider the pre-dual Cauchy problem

$$\frac{\partial G_t}{\partial t} = \widehat{L} G_t, \quad G_t|_{t=0} = G_0, \tag{8}$$

on $\mathcal{L}_\alpha$. Here $(G_t)_{t\geq 0}$ is the so-called evolution of quasi-observables. We will prove that $(\widehat{L},\ B_{bs}(\Gamma_0^2))$ can be realized as a linear operator on $\mathcal{L}_\alpha$, it is closable, and its closure is the generator of an analytic semigroup of contractions on $\mathcal{L}_\alpha$. Solutions to (7) are then obtained by duality. The main part of this section consists of the proof that an evolution $(k_t)_{t\geq 0}$, obtained by such procedure, is, in fact, positive definite (see Theorem 1).

Following [10,17], we apply general perturbation theory for sub-stochastic, analytic semigroups (see [1,33]). In order that $\widehat{L}$ is a well-defined linear mapping on $B_{bs}(\Gamma_0^2)$, the potentials should satisfy the following assumption.

There exists a measurable, locally bounded function $\rho\colon \mathbb{R}^d \longrightarrow [1,\ \infty)$ such that $(1 - e^{-g(x-\cdot)})\rho$ is integrable for any $x \in \mathbb{R}^d$ and any $g \in \{\phi^\pm,\ \psi^\pm,\ \kappa^\pm,\ \tau^\pm\}$.

The linear mapping $\widehat{L}$ is given by $\widehat{L} = A + B$, where $(AG)(\eta) = -M(\eta)G(\eta)$ and

$$\begin{aligned}
M(\eta) &= \left|\eta^+\right| + \left|\eta^-\right| + q^+ \sum_{x\in\eta^+} e^{-E_{\kappa^+}(x,\eta^+\backslash x)} e^{-E_{\tau^+}(x,\eta^-)} \\
&\quad + q^- \sum_{x\in\eta^-} e^{-E_{\kappa^-}(x,\eta^-\backslash x)} e^{-E_{\tau^-}(x,\eta^+)},
\end{aligned}$$

$$\begin{aligned}
(BG)(\eta) &= z^+ \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} e^{-E_{\phi^-}(x,\xi^-)} e^{-E_{\psi^+}(x,\xi^+)} f_x\left(\phi^-;\ \eta^-\backslash\xi^-\right) f_x\left(\psi^+;\ \eta^+\backslash\xi^+\right) \\
&\quad \times G\left(\xi^+ \cup x,\ \xi^-\right) \mathrm{d}x \\
&\quad + z^- \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} e^{-E_{\phi^+}(x,\xi^+)} e^{-E_{\psi^-}(x,\xi^-)} f_x\left(\phi^+;\ \eta^+\backslash\xi^+\right) f_x\left(\psi^-;\ \eta^-\backslash\xi^-\right) \\
&\quad \times G\left(\xi^+,\ \xi^- \cup x\right) \mathrm{d}x \\
&\quad + q^+ \sum_{\xi\subset\eta} \sum_{x\in\xi^+} e^{-E_{\kappa^+}(x,\xi^+\backslash x)} e^{-E_{\tau^+}(x,\xi^-)} f_x\left(\kappa^+;\ \eta^+\backslash\xi^+\right) f_x\left(\tau^+;\ \eta^-\backslash\xi^-\right) \\
&\quad \times G\left(\xi^+\backslash x,\ \xi^- \cup x\right) \\
&\quad + q^- \sum_{\xi\subset\eta} \sum_{x\in\xi^-} e^{-E_{\kappa^-}(x,\xi^-\backslash x)} e^{-E_{\tau^-}(x,\xi^+)} f_x\left(\kappa^-;\ \eta^-\backslash\xi^-\right) f_x\left(\tau^-;\ \eta^+\backslash\xi^+\right) \\
&\quad \times G\left(\xi^+ \cup x,\ \xi^-\backslash x\right) \\
&\quad - q^+ \sum_{\substack{\xi\subset\eta \\ \xi\neq\eta}} \sum_{x\in\xi^+} e^{-E_{\kappa^+}(x,\xi^+\backslash x)} e^{-E_{\tau^+}(x,\xi^-)} f_x\left(\kappa^+;\ \eta^+\backslash\xi^+\right) f_x\left(\tau^+;\ \eta^-\backslash\xi^-\right) G(\xi),
\end{aligned} \tag{9}$$

$$- q^- \sum_{\substack{\xi\subset\eta \\ \xi\neq\eta}} \sum_{x\in\xi^-} e^{-E_{\kappa^-}(x,\xi^-\backslash x)} e^{-E_{\tau^-}(x,\xi^+)} f_x\left(\kappa^-;\ \eta^-\backslash\xi^-\right) f_x\left(\tau^-;\ \eta^+\backslash\xi^+\right) G(\xi). \tag{10}$$

Here $f_x(g;\ \eta) := e_\lambda(e^{-g(x-\cdot)} - 1;\ \eta) = \prod_{y\in\eta}(e^{-g(x-y)} - 1)$. Fix any $\alpha = (\alpha^+,\ \alpha^-) \in \mathbb{R}$. The multiplication operator A is well-defined on the domain

$$D(A) = \{G \in \mathcal{L}_\alpha | M \cdot G \in \mathcal{L}_\alpha\}.$$

We show that, given some mild additional conditions, also B is well-defined on $B_{bs}(\Gamma_0^2)$ but also on $D(A)$. Let B' be the linear mapping B, where $f_x(g;\, \eta)$ is replaced by $\prod_{y\in\eta} |e^{-g(x-y)} - 1|$ and in the last two terms, see (9) and (10), the $-$ is replaced by $+$. Moreover, define

$$C_g\left(x,\, \alpha^{\pm}\right) := \exp\left(e^{\alpha^{\pm}} \int\limits_{\mathbb{R}^d} \left(1 - e^{-g(x-y)}\right) \rho(y)\mathrm{d}y \right), \quad x \in \mathbb{R}^d, \tag{11}$$

where $g \in \{\phi^{\pm},\, \psi^{\pm},\, \kappa^{\pm},\, \tau^{\pm}\}$. Then, for any $0 \le G \in D(A)$, see $\rho \ge 1$, we get by (2)

$$\int\limits_{\Gamma_0^2} B'G(\eta)e_\lambda(\rho;\, \eta)e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) \le \int\limits_{\Gamma_0^2} \beta(\alpha;\, \eta)G(\eta)e_\lambda(\rho;\, \eta)e^{\alpha|\eta|}\mathrm{d}\lambda(\eta),$$

where

$$\begin{aligned}
\beta(\alpha;\, \eta) = {} & z^+ e^{-\alpha^+} \sum_{x\in\eta^+} e^{-E_{\phi^-}(x,\eta^-)} e^{-E_{\psi^+}(x,\eta^+\setminus x)} C_{\phi^-}\left(x,\, \alpha^-\right) C_{\psi^+}\left(x,\, \alpha^+\right) \\
& + z^- e^{-\alpha^-} \sum_{x\in\eta^-} e^{-E_{\phi^+}(x,\eta^+)} e^{-E_{\psi^-}(x,\eta^-\setminus x)} C_{\phi^+}\left(x,\, \alpha^+\right) C_{\psi^-}\left(x,\, \alpha^-\right) \\
& + q^+ e^{\alpha^+-\alpha^-} \sum_{x\in\eta^-} e^{-E_{\kappa^+}(x,\eta^+)} e^{-E_{\tau^+}(x,\eta^-\setminus x)} C_{\kappa^+}\left(x,\, \alpha^+\right) C_{\tau^+}\left(x,\, \alpha^-\right) \\
& + q^- e^{\alpha^--\alpha^+} \sum_{x\in\eta^+} e^{-E_{\kappa^-}(x,\eta^-)} e^{-E_{\tau^-}(x,\eta^+\setminus x)} C_{\kappa^-}\left(x,\, \alpha^-\right) C_{\tau^-}\left(x,\, \alpha^+\right) \\
& + q^+ \sum_{x\in\eta^+} e^{-E_{\kappa^+}(x,\eta^+\setminus x)} e^{-E_{\tau^+}(x,\eta^-)} \left(C_{\kappa^+}\left(x,\, \alpha^+\right) C_{\tau^+}\left(x,\, \alpha^-\right) - 1\right) \\
& + q^- \sum_{x\in\eta^-} e^{-E_{\kappa^-}(x,\eta^-\setminus x)} e^{-E_{\tau^-}(x,\eta^+)} \left(C_{\kappa^-}\left(x,\, \alpha^-\right) C_{\tau^-}\left(x,\, \alpha^+\right) - 1\right).
\end{aligned}$$

Remark 2 Suppose that $C_g(x,\, \alpha^{\pm})$ are bounded for any $g \in \{\phi^{\pm},\, \psi^{\pm},\, \kappa^{\pm},\, \tau^{\pm}\}$. Then there exists a constant $a = a(\alpha) > 0$ such that

$$\beta(\alpha;\, \eta) \le a(\alpha)M(\eta), \quad \eta \in \Gamma_0^2.$$

Consequently, $(B',\, D(A))$ and since $|BG| \le B'|G|$ also $(\widehat{L},\, D(A)) = (A + B,\, D(A))$ is a well-defined operator on $\mathcal{L}_\alpha$.

3.2 Assumptions

Here and below we suppose that the following conditions are satisfied.

(A) Suppose that $\phi^{\pm},\, \psi^{\pm},\, \kappa^{\pm},\, \tau^{\pm}: \mathbb{R}^d \longrightarrow [0,\, \infty)$ are symmetric. Moreover, there exists a measurable, locally bounded function $\rho: \mathbb{R}^d \longrightarrow [1,\, \infty)$ such that $(1 - e^{-g(x-\cdot)})\rho$ is integrable for any $x \in \mathbb{R}^d$ and any $g \in \{\phi^{\pm},\, \psi^{\pm},\, \kappa^{\pm},\, \tau^{\pm}\}$.

(B) There exists $\alpha = (\alpha^+, \alpha^-)$ and a constant $a(\alpha) \in (0,\, 1)$ such that

$$\beta(\alpha;\, \eta) \le a(\alpha)M(\eta), \quad \eta \in \Gamma_0^2. \tag{12}$$

Below we give some comments and sufficient conditions on assumptions (A) and (B). Note that $C_g(x,\, \alpha^{\pm})$ given as in (11) is (for $\rho = 1$) related to the construction of Gibbs measures (see, e.g., [31]).

Remark 3 Let $\phi^{\pm}, \psi^{\pm}, \tau^{\pm}, \kappa^{\pm} \geq 0$ be symmetric, measurable functions.

(a) Suppose that $\phi^{\pm}, \psi^{\pm}, \tau^{\pm}, \kappa^{\pm}$ are continuous and have compact support. Then condition (A) holds for any measurable, locally bounded function $\rho \geq 1$,
(b) Take $\rho = 1$. Then condition (A) holds, provided all potentials $\phi^{\pm}, \psi^{\pm}, \tau^{\pm}, \kappa^{\pm}$ are integrable.

The following lemma shows that condition (B) is satisfied in the low activity regime.

Lemma 2 *Suppose that condition* (*A*) *is satisfied and assume that for any* $g \in \{\phi^{\pm}, \psi^{\pm}, \tau^{\pm}, \kappa^{\pm}\}$

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 - e^{-g(x-y)}\right) \rho(y) \mathrm{d}y < \infty. \tag{13}$$

Then for any $\alpha = (\alpha^+, \alpha^-) \in \mathbb{R}$ *there exit* $q_0^{\pm}(\alpha), z_0^{\pm}(\alpha) > 0$ *such that condition* (*B*) *holds for all* $q^{\pm} < q_0^{\pm}(\alpha)$ *and* $z^{\pm} < z_0^{\pm}(\alpha)$.

Proof By (13) it follows that $C_g(x, \alpha^{\pm})$ is bounded in x by a constant depending on $\alpha^{\pm}$ for any $g \in \{\phi^{\pm}, \psi^{\pm}, \tau^{\pm}, \kappa^{\pm}\}$. This implies that there exists a constant $C(\alpha) > 0$ such that

$$\beta(\alpha; \eta) \leq \left(z^+ + q^- + q^+\right) C(\alpha) \left|\eta^+\right| + \left(z^- + q^+ + q^-\right) C(\alpha) \left|\eta^-\right|,$$

which yields the assertion. □

We consider two examples for which this lemma is applicable, i.e., (13) is satisfied.

3.2.1 Bounded Correlation Functions

Let $\rho(x) = 1$ for all $x \in \mathbb{R}^d$. This case corresponds to bounded correlation functions, since for each $\mu \in \mathcal{P}_\alpha$ its correlation function k_μ satisfies

$$k_\mu(\eta) \leq \|k\|_{\mathcal{K}_\alpha} e^{\alpha^+|\eta^+|} e^{\alpha^-|\eta^-|}, \quad \eta \in \Gamma_0^2.$$

In particular, k_μ is bounded on $\{(\eta^+, \eta^-) | |\eta^+| = n, |\eta^-| = m\}$ for all $n, m \in \mathbb{N}_0$. Suppose that all potentials are integrable. Then condition (A) holds and $C_g(x, \alpha^{\pm})$ given by (11) is independent of x for any $g \in \{\phi^{\pm}, \psi^{\pm}, \tau^{\pm}, \kappa^{\pm}\}$. Hence (13) holds and condition (B) is satisfied, provided $z^{\pm}$ and $q^{\pm}$ are small enough. Such condition reflects a balance condition of the interactions between particles of the same and of different type.

3.2.2 Unbounded Correlation Functions

Suppose that all potentials $\phi^{\pm}, \psi^{\pm}, \tau^{\pm}, \kappa^{\pm}$ are given by $g(x) := e^{-\delta|x|^2}$ for some $\delta > 0$. Take any $\delta' \in (0, \delta)$ and let

$$\rho(x) = e^{\delta'|x|^2}, \quad x \in \mathbb{R}^d.$$

Then, for each $\mu \in \mathcal{P}_\alpha$ its correlation function k_μ satisfies for all $n, m \in \mathbb{N}_0$

$$\begin{aligned} k_\mu^{(n,m)}(x_1, \dots, x_n; y_1, \dots, y_m) &\leq \|k\|_{\mathcal{K}_\alpha} e^{\alpha^+ n} e^{\alpha^- m} \prod_{k=1}^{n} e^{\delta'|x_k|^2} \prod_{k=1}^{m} e^{\delta'|y_k|^2} \\ &= \|k\|_{\mathcal{K}_\alpha} e^{\delta' \sum_{k=1}^{n}(\alpha^+ + |x_k|^2)} e^{\delta' \sum_{k=1}^{m}(\alpha^- + |y_k|^2)}, \end{aligned}$$

and hence is not necessarily bounded. Let us show that (13) holds.

Denote by ω_d the surface volume of the sphere $\{y \in \mathbb{R}^d | |y| = 1\}$. For any $x \in \mathbb{R}^d$ it follows by $g \geq 0$

$$\int_{\mathbb{R}^d} \left(1 - e^{-g(x-y)}\right) e^{\delta'|y|^2} \mathrm{d}y \leq \int_{\mathbb{R}^d} e^{-\delta|x-y|^2} e^{\delta'|y|^2} \mathrm{d}y$$

$$\leq \omega_d e^{-\delta|x|^2} \int_0^\infty r^{d-1} e^{-(\delta-\delta')r^2 + 2|x|\delta r} \mathrm{d}r = \omega_d e^{-\frac{\delta\delta'}{\delta-\delta'}|x|^2} \int_0^\infty r^{d-1} e^{-(\delta-\delta')\left(r - \frac{|x|\delta}{\delta-\delta'}\right)^2} \mathrm{d}r$$

$$= \omega_d e^{-\frac{\delta\delta'}{\delta-\delta'}|x|^2} \int_{-\frac{|x|\delta}{\delta-\delta'}}^\infty \left(r + \frac{|x|\delta}{\delta - \delta'}\right)^{d-1} e^{-(\delta-\delta')r^2} \mathrm{d}r$$

$$\leq \omega_d e^{-\frac{\delta\delta'}{\delta-\delta'}|x|^2} \int_{\mathbb{R}} \left(|r| + \frac{|x|\delta}{\delta - \delta'}\right)^{d-1} e^{-(\delta-\delta')r^2} \mathrm{d}r.$$

The integral on the right-hand side is due to

$$\int_{\mathbb{R}} \left(r + \frac{|x|\delta}{\delta - \delta'}\right)^{d-1} e^{-(\delta-\delta')r^2} \mathrm{d}r = \sum_{k=0}^{d-1} \binom{d-1}{k} \left(\frac{\delta}{\delta - \delta'}\right)^k |x|^k \int_{\mathbb{R}} |r|^{d-1-k} e^{-(\delta-\delta')r^2} \mathrm{d}r,$$

a polynomial of order $d - 1$ in $|x|$. Hence there exists a constant $c(d, \delta, \delta') > 0$ such that

$$\int_{\mathbb{R}^d} \left(1 - e^{-g(x-y)}\right) e^{\delta'|y|^2} \mathrm{d}y \leq c(d, \delta, \delta'), \quad x \in \mathbb{R}^d.$$

3.3 Evolution of Quasi-observables and Correlation Functions

Suppose that conditions (A) and (B) are satisfied. Let $\mathbb{1}^*(\eta) = \begin{cases} 1, & |\eta| = 0 \\ 0, & \text{otherwise} \end{cases}$. The next proposition shows that the Cauchy problem (8) is well-posed in $\mathcal{L}_\alpha$.

Proposition 1 *The operator $(\widehat{L}, D(A))$ is the generator of an analytic semigroup $(\widehat{T}(t))_{t\geq 0}$ of contractions on $\mathcal{L}_\alpha$ such that $\widehat{T}(t)\mathbb{1}^* = \mathbb{1}^*$. Moreover, $B_{bs}(\Gamma_0^2)$ is a core for $(\widehat{L}, D(A))$.*

Proof Observe that $(A, D(A))$ is the generator of a positive, analytic semigroup of contractions on $\mathcal{L}_\alpha$. By condition (B) it follows that

$$\int_{\Gamma_0^2} B'G(\eta)e^{\alpha|\eta|}e_\lambda(\rho; \eta)\mathrm{d}\lambda(\eta) \leq a(\alpha) \int_{\Gamma_0^2} M(\eta)G(\eta)\mathrm{d}\lambda(\eta). \tag{14}$$

Hence B' is a well-defined positive linear operator on $D(A)$ which satisfies $|BG| \leq B'|G|$ for any $G \in D(A)$. Take $r \in (0, 1)$ such that $\frac{a(\alpha)}{r} < 1$, then

$$\int_{\Gamma_0^2} \left(A + \frac{1}{r}B'\right) G(\eta)e_\lambda(\rho; \eta)e^{\alpha|\eta|}\mathrm{d}\lambda(\eta) \leq 0, \quad 0 \leq G \in D(A).$$

Hence, $(A + B', D(A))$ is the generator of a positive, strongly continuous semigroup of contractions, cf. [33, Theorem 2.2]. By [1, Theorem 1.1], this semigroup is analytic. Moreover, [1, Theorem 1.2] implies that $(A + B,\ D(A))$ is the generator of an analytic semigroup of contractions on $\mathcal{L}_\alpha$. Since $\mathbb{1}^* \in D(A)$ and $\widehat{L}\mathbb{1}^* = 0$, it follows that $\widehat{T}(t)\mathbb{1}^* = \mathbb{1}^*$. In order to see that $B_{bs}(\Gamma_0^2)$ is a core, let $(\Lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence of compacts in $\mathbb{R}^d$ and $G \in D(A)$. Define

$$G_n(\eta) := \begin{cases} G(\eta) \wedge n, & |\eta| \le n \text{ and } \eta \subset \Lambda_n, \\ 0, & \text{otherwise}, \end{cases},$$

then $G_n \in B_{bs}(\Gamma_0^2)$, $G_n \longrightarrow G$ a.e. as $n \to \infty$ and $|G_n| \le |G|$, for all $n \in \mathbb{N}$. Dominated convergence yields $G_n \longrightarrow G$ in $\mathcal{L}_\alpha$. Moreover, dominated convergence also implies $\widehat{L}G_n \longrightarrow \widehat{L}G$ a.e., as $n \to \infty$. Since $|MG_n| \le M|G|$ and $|BG_n| \le B'|G_n| \le B'|G|$ applying again dominated convergence shows that $\widehat{L}G_n \longrightarrow \widehat{L}G$ in $\mathcal{L}_\alpha$. □

For convenience of notation, we let $D(A) = D(\widehat{L})$, so that $(\widehat{L},\ D(\widehat{L}))$ is the generator of an analytic semigroup on $\mathcal{L}_\alpha$. Let $\widehat{T}(t)^*$ be the adjoint semigroup on $\mathcal{K}_\alpha$, solutions to (7) are given by $\widehat{T}(t)^* k_0$. Denote by $(\widehat{L}^*,\ D(\widehat{L}^*))$ the adjoint operator to $\widehat{L}$. It is given by

$$D(\widehat{L}^*) = \left\{ k \in \mathcal{K}_\alpha \,|\, \exists k_1 \in \mathcal{K}_\alpha \text{ such that } \langle \widehat{L}G,\ k\rangle = \langle G,\ k_1\rangle\ \forall G \in D(A) \right\},$$

with $\widehat{L}^* k := k_1$. Using condition (B) together with (2) it is possible to give an explicit formula for $\widehat{L}^*$ and characterize $D(\widehat{L}^*)$ as the maximal domain for the action of $\widehat{L}^*$ given by this formula.

Let $g_0,\ g_1 \colon \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be given with $(1 - e^{-g_j(x-\cdot)})\rho \in L^1(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$ and $j = 0,\ 1$. Let $\mathcal{Q}_x(g_0,\ g_1)$ be a linear operator on $\mathcal{K}_\alpha$ given by

$$\mathcal{Q}_x\left(g_0,\ g_1\right) k(\eta) = \int\limits_{\Gamma_0^2} f_x\left(g_0;\ \xi^+\right) f_x\left(g_1;\ \xi^-\right) k(\eta \cup \xi) \mathrm{d}\lambda(\xi), \quad x \in \mathbb{R}^d. \tag{15}$$

This operator satisfies

$$\left|\mathcal{Q}_x\left(g_0,\ g_1\right) k(\eta)\right| \le e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) C_{g_0}\left(x,\ \alpha^+\right) C_{g_1}\left(x,\ \alpha^-\right) \|k\|_{\mathcal{K}_\alpha}. \tag{16}$$

Let L^Δ be a linear mapping given by

$$\begin{aligned}
\left(L^\Delta k\right)(\eta) = &-|\eta| k(\eta) &(17)\\
&- q^+ \sum_{x\in\eta^+} e^{-E_{\kappa^+}(x,\eta^+\backslash x)} e^{-E_{\tau^+}(x,\eta^-)} \mathcal{Q}_x\left(\kappa^+,\ \tau^+\right) k(\eta) &(18)\\
&- q^- \sum_{x\in\eta^-} e^{-E_{\kappa^-}(x,\eta^-\backslash x)} e^{-E_{\tau^-}(x,\eta^+)} \mathcal{Q}_x\left(\tau^-,\ \kappa^-\right) k(\eta) &(19)\\
&+ z^+ \sum_{x\in\eta^+} e^{-E_{\phi^-}(x,\eta^-)} e^{-E_{\psi^+}(x,\eta^+\backslash x)} \mathcal{Q}_x\left(\psi^+,\ \phi^-\right) k\left(\eta^+\backslash x,\ \eta^-\right)\\
&+ z^- \sum_{x\in\eta^-} e^{-E_{\phi^+}(x,\eta^+)} e^{-E_{\psi^-}(x,\eta^-\backslash x)} \mathcal{Q}_x\left(\phi^+,\ \psi^-\right) k\left(\eta^+,\ \eta^-\backslash x\right)\\
&+ q^+ \sum_{x\in\eta^-} e^{-E_{\kappa^+}(x,\eta^+)} e^{-E_{\tau^+}(x,\eta^-\backslash x)} \mathcal{Q}_x\left(\kappa^+,\ \tau^+\right) k\left(\eta^+ \cup x,\ \eta^-\backslash x\right)\\
&+ q^- \sum_{x\in\eta^+} e^{-E_{\kappa^-}(x,\eta^-)} e^{-E_{\tau^-}(x,\eta^-\backslash x)} \mathcal{Q}_x\left(\tau^-,\ \kappa^-\right) k\left(\eta^+\backslash x,\ \eta^- \cup x\right).
\end{aligned}$$

Lemma 3 *There exists $M(\alpha^+),\ N(\alpha^-) > 0$ such that for any $\alpha_0^+ < \alpha^+$ and $\alpha_0^- < \alpha^-$*

$$\left\| L^{\Delta} k \right\|_{\mathcal{K}_\alpha} \leq \left(\frac{M(\alpha^+)}{\alpha^+ - \alpha_0^+} + \frac{N(\alpha^-)}{\alpha^- - \alpha_0^-} \right) \|k\|_{\mathcal{K}_{\alpha_0}}. \tag{20}$$

Consequently, L^{Δ} is a bounded linear operator from $\mathcal{K}_{\alpha_0}$ to $\mathcal{K}_\alpha$. Moreover, for each $G \in D(\widehat{L})$ and each $k \in \mathcal{K}_\alpha$ we have $L^{\Delta} k \in \bigcap_{\alpha_0^\pm > \alpha^\pm} \mathcal{K}_{\alpha_0}$, $G \cdot L^{\Delta} k$ is integrable and

$$\int_{\Gamma_0^2} (\widehat{L} G)(\eta) k(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} G\left(\eta\right) (L^{\Delta} k\right) (\eta) \mathrm{d}\lambda(\eta). \tag{21}$$

Proof The first assertion follows from (16), $M(\eta) \leq (1 + q^+)|\eta^+| + (1 + q^-)|\eta^-|$ and

$$\left|\eta^\pm\right| e^{-(\alpha^\pm - \alpha_0^\pm)|\eta^\pm|} \leq \frac{1}{e(\alpha^\pm - \alpha_0^\pm)}, \quad \eta \in \Gamma_0^2.$$

Let us show the second assertion. By (20) it follows that $L^{\Delta} k \in \bigcap_{\alpha_0^\pm > \alpha^\pm} \mathcal{K}_{\alpha_0}$ holds for any $k \in \mathcal{K}_\alpha$. Property (21) follows from (2) provided we can show that (2) is applicable. Denote by L'^{Δ} the linear mapping L^{Δ} where the $-$ in (17)–(19) is replaced by $+$ and $f_x(g;\ \eta)$ is replaced by $\prod_{y \in \eta} |1 - e^{-g(x-y)}|$ in the definition of (15). Then (2) is applicable, provided that $|G| \cdot L'^{\Delta}|k|$ is integrable. But this follows from

$$\left(L'^{\Delta}|k|\right)(\eta) \leq \beta(\alpha;\ \eta) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) \|k\|_{\mathcal{K}_\alpha} \leq a(\alpha) M(\eta) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) \|k\|_{\mathcal{K}_\alpha}.$$

Since $|G \cdot L^{\Delta} k| \leq |G| \cdot L'^{\Delta}|k|$ we see that also $G \cdot L^{\Delta} k$ is integrable, which completes the proof. □

The next statement shows that L^{Δ} can be identified with $\widehat{L}^*$.

Proposition 2 *Consider L^{Δ} on its maximal domain*

$$D\left(L^{\Delta}\right) = \left\{k \in \mathcal{K}_\alpha | L^{\Delta} k \in \mathcal{K}_\alpha\right\},$$

then $(L^{\Delta},\ D(L^{\Delta})) = (\widehat{L}^*,\ D(\widehat{L}^*))$.

Proof Let $G \in D(\widehat{L})$ and $k \in D(\widehat{L}^*)$. By (21) it follows that

$$\int_{\Gamma_0^2} G(\eta)(\widehat{L}^* k)(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} (\widehat{L} G)(\eta) k(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} G(\eta) \left(L^{\Delta} k\right)(\eta) \mathrm{d}\lambda(\eta).$$

Since G was arbitrary, we get $L^{\Delta} k = \widehat{L}^* k$ and $D(\widehat{L}^*) \subset D(L^{\Delta})$. Conversely, let $k \in D(L^{\Delta})$. Then (21) implies $k \in D(\widehat{L}^*)$ and $\widehat{L}^* k = L^{\Delta} k$. □

Since $\mathcal{K}_\alpha$ is not reflexive, $\widehat{T}(t)^*$ is, in general, not strongly continuous. However, it is continuous w.r.t. the weak topology $\sigma(\mathcal{K}_\alpha,\ \mathcal{L}_\alpha)$. Here, $\sigma(\mathcal{K}_\alpha,\ \mathcal{L}_\alpha)$ is the smallest topology such that all functionals $G \longmapsto \langle G,\ k\rangle$ are continuous for any $k \in \mathcal{K}_\alpha$. It is well-known, see [6, Chap. 2, pp. 77–79], that $\widehat{T}(t)^*$ leaves the proper subspace $\mathcal{K}_\alpha^{\odot} := \overline{D(L^{\Delta})}$ invariant. Moreover, the restriction $\widehat{T}(t)^{\odot} := \widehat{T}(t)^*|_{\mathcal{K}_\alpha^{\odot}}$ is a strongly continuous semigroup with generator $\widehat{L}^{\odot} k = L^{\Delta} k$,

$$D\left(\widehat{L}^{\odot}\right) = \left\{k \in D\left(L^{\Delta}\right) | L^{\Delta} k \in \mathcal{K}_\alpha^{\odot}\right\}.$$

Thus, for any $k_0 \in D(\widehat{L}^{\odot})$, $k_t := \widehat{T}(t)^* k_0$ is the unique classical solution to

$$\frac{\partial k_t}{\partial t} = L^{\Delta} k_t, \quad k_t|_{t=0} = k_0. \tag{22}$$

in $\mathcal{K}_\alpha$. Such system of equations is an Markov analogue of the BBGKY-hierarchy known in the physical literature (see [32]). Our aim is to get uniqueness for (7), which is simply a weak formulation of (22). The precise definition of a solution to (7) is given below. We use the topology of uniform convergence on compact subsets of $\mathcal{L}_\alpha$ on $\mathcal{K}_\alpha$. A basis of neighbourhoods around 0 is given by sets of the form

$$\left\{ k \in \mathcal{K}_\alpha \middle| \sup_{G \in K} |\langle G, k\rangle| < \varepsilon \right\}, \tag{23}$$

where $\varepsilon > 0$ and $K \subset \mathcal{L}_\alpha$ is a compact, cf. [35]. Denote by $\mathcal{C}$ the topology generated by the basis of neighbourhoods (23). Note that $\mathcal{C}$ coincides with $\sigma(\mathcal{K}_\alpha, \mathcal{L}_\alpha)$ on norm-bounded sets, cf. [35, Lemma 1.10].

Definition 2 Given $k_0 \in \mathcal{K}_\alpha$, a weak solution to (7) is a family $(k_t)_{t\geq 0} \subset \mathcal{K}_\alpha$ being continuous w.r.t. $\mathcal{C}$ and

$$\langle G, k_t\rangle = \langle G, k_0\rangle + \int_0^t \left\langle \widehat{L}G, k_s \right\rangle \mathrm{d}s, \quad G \in B_{bs}\left(\Gamma_0^2\right), \tag{24}$$

holds for all $t \geq 0$.

Theorem 2 *For any $k \in \mathcal{K}_\alpha$ there exists a unique weak solution to (7), given by $k_t = \widehat{T}(t)^* k_0$. Moreover, the following holds:*

(1) *For any $G \in D(\widehat{L})$, $t \longmapsto \langle G, k_t\rangle$ is continuously differentiable and satisfies (7) for each $t \geq 0$.*
(2) *If $k_0 \in \mathcal{K}_{\alpha_0}$ for some $\alpha_0^+ < \alpha^+$ and $\alpha_0^- < \alpha^-$, then k_t is infinitely often differentiable w.r.t. to the norm in $\mathcal{K}_\alpha$ and satisfies (22).*

Proof Since $\widehat{L}$ is the generator of a strongly continuous semigroup, the first assertion follows by [35, Theorem 2.1].

(1) The contraction property implies $\|k_t\|_{\mathcal{K}_\alpha} \leq \|k_0\|_{\mathcal{K}_\alpha}$ and hence, by $\widehat{L}G \in \mathcal{L}_\alpha$, we see that $s \longmapsto \langle \widehat{L}G, k_s\rangle$ is continuous. By (24), we see that $t \longmapsto \langle G, k_t\rangle$ is continuously differentiable and satisfies (7) for any $t \geq 0$.
(2) Let $n \in \mathbb{N}$ and take $\alpha_0^{\pm} < \alpha_1^{\pm} < \cdots < \alpha_n^{\pm} < \alpha^{\pm}$. Then $(L^{\Delta})^j k_0 \in \mathcal{K}_{\alpha_n} \subset D(L^{\Delta})$ for all $j = 0, \ldots, n$ and hence $(L^{\Delta})^j k_0 \in D(\widehat{L}^{\odot})$ for all $j = 0, \ldots, n$.

□

3.4 Positive Definiteness

In this section we show existence and uniqueness of weak solutions to (1).

Lemma 4 *Fix any $\mu \in \mathcal{P}_\alpha$. Then, $F, LF \in L^1(\Gamma^2, \mathrm{d}\mu)$ holds for each $F \in \mathcal{FP}(\Gamma^2)$.*

Proof Fix $\mu \in \mathcal{P}_\alpha$. Let $G \in B_{bs}(\Gamma_0^2)$ be such that $F = KG \in \mathcal{FP}(\Gamma^2)$. By $k_\mu(\eta) \leq \|k_\mu\|_{\mathcal{K}_\alpha} e^{\alpha|\eta|} e_\lambda(\rho; \eta)$ we have $G, \widehat{L}G \in \mathcal{L}_\alpha \subset L^1(\Gamma_0^2, k_\mu \mathrm{d}\lambda)$. Since $K\colon L^1(\Gamma_0^2, k_\mu \mathrm{d}\lambda) \longrightarrow L^1(\Gamma^2, \mathrm{d}\mu)$ is continuous, it follows that $KG, K\widehat{L}G \in L^1(\Gamma^2, \mathrm{d}\mu)$. The assertion follows from $K\widehat{L}G = LKG$. □

The next theorem establishes uniqueness for weak solutions to (1).

Theorem 3 *The Fokker–Planck equation* (1) *has at most one weak solution* $(\mu_t)_{t\geq 0} \subset \mathcal{P}_\alpha$ *such that its correlation functions* $(k_{\mu_t})_{t\geq 0}$ *satisfy*

$$\sup_{t\in[0,T]} \left\| k_{\mu_t} \right\|_{\mathcal{K}_\alpha} < \infty, \quad \forall T > 0. \tag{25}$$

Proof Let $(\mu_t)_{t\geq 0} \subset \mathcal{P}_\alpha$ be a weak solution to (1). Denote by $(k_{\mu_t})_{t\geq 0}$ the associated family of correlation functions. Let $G \in B_{bs}(\Gamma_0^2) \subset D(\widehat{L})$ and $F = KG \in \mathcal{FP}(\Gamma^2)$. Then $G, \widehat{L}G \in \mathcal{L}_\alpha \subset L^1(\Gamma_0^2, k_{\mu_t}\mathrm{d}\lambda)$, $t \geq 0$ and hence F, LF belong to $L^1(\Gamma^2, \mathrm{d}\mu_t)$ with $LF = K\widehat{L}G$ (see Lemma 4). By $\langle\langle F, \mu_t\rangle\rangle = \langle G, k_{\mu_t}\rangle$, $\langle\langle LF, \mu_t\rangle\rangle = \langle \widehat{L}G, k_{\mu_t}\rangle$ and, (1) it follows that $t \longmapsto \langle \widehat{L}G, k_{\mu_t}\rangle$ is locally integrable and (24) holds. In particular, k_{μ_t} is continuous w.r.t. $\sigma(\mathcal{K}_\alpha, \mathcal{L}_\alpha)$. By (25) and [35, Lemma 1.10] it is also continuous w.r.t. $\mathcal{C}$. Hence $(k_{\mu_t})_{t\geq 0}$ is a weak solution to (7). The assertion follows from weak uniqueness of solutions to (7). □

Theorem 4 *For each* $\mu_0 \in \mathcal{P}_\alpha$ *there exists exactly one weak solution* $(\mu_t)_{t\geq 0} \subset \mathcal{P}_\alpha$ *to* (1) *such that its correlation functions satisfy* (25)*. This solution is uniquely determined by the associated family of correlation functions* $k_{\mu_t} = \widehat{T}(t)^* k_{\mu_0}$.

Since uniqueness was already shown, it remains to prove existence of a weak solution to (1). To this end, it suffices to show that $k_t := \widehat{T}(t)^* k_{\mu_0} \in \mathcal{K}_\alpha$ is positive definite for each $t \geq 0$.

3.4.1 Step 1: Evolution of Local Densities

Let $R_\delta(x) := \frac{e^{-\delta|x|^2}}{1+\delta\rho(x)}$ and $z_\delta^\pm(x) := R_\delta(x)z^\pm$, $\delta > 0$. Then

(1) $R_\delta(x) \longrightarrow 1$ as $\delta \to 0$ for any $x \in \mathbb{R}^d$.
(2) $R_\delta(x) \leq e^{-\delta|x|^2} \leq 1$ for any $x \in \mathbb{R}^d$, $\delta > 0$.
(3) $\rho \cdot R_\delta$ is integrable for any $\delta > 0$.

Denote by L_δ the associated Markov operator given by L where $z^\pm$ are replaced by $z_\delta^\pm$. It defines a linear mapping, when restricted to measurable functions $F\colon \Gamma_0^2 \longrightarrow \mathbb{R}$. Note that $L_\delta F$ is, in general, not bounded on Γ_0^2 even if F is bounded on Γ_0^2. Below we consider its (formal) adjoint operator. Namely, let

$$\begin{aligned} D_\delta(\eta) = |\eta| &+ \int_{\mathbb{R}^d} z_\delta^+(x) e^{-E_{\phi^-}(x,\eta^-)} e^{-E_{\psi^+}(x,\eta^+)} \mathrm{d}x + \int_{\mathbb{R}^d} z_\delta^-(x) e^{-E_{\phi^+}(x,\eta^+)} e^{-E_{\psi^-}(x,\eta^-)} \mathrm{d}x \\ &+ q^+ \sum_{x\in\eta^+} e^{-E_{\kappa^+}(x,\eta^+\setminus x)} e^{-E_{\tau^+}(x,\eta^-)} + q^- \sum_{x\in\eta^-} e^{-E_{\kappa^-}(x,\eta^-\setminus x)} e^{-E_{\tau^-}(x,\eta^+)}, \end{aligned}$$

and $\mathcal{D}_\delta = \{R \in L^1(\Gamma_0^2, \mathrm{d}\lambda) | D_\delta \cdot R \in L^1(\Gamma_0^2, \mathrm{d}\lambda)\}$. Then $(-D_\delta, \mathcal{D}_\delta)$ is the generator of a positive analytic semigroup of contractions on $L^1(\Gamma_0^2, \mathrm{d}\lambda)$. Let $\mathcal{Q}_\delta$ be another (positive) operator on $\mathcal{D}_\delta$ given by

$$
\begin{aligned}
(\mathcal{Q}_\delta R)(\eta) = &\int_{\mathbb{R}^d} R\left(\eta^+ \cup x,\ \eta^-\right) \mathrm{d}x + \sum_{x\in\eta^+} z_\delta^+(x) e^{-E_{\phi^-}(x,\eta^-)} e^{-E_{\psi^+}(x,\eta^+\backslash x)} R\left(\eta^+\backslash x,\ \eta^-\right)\\
&+ \int_{\mathbb{R}^d} R\left(\eta^+,\ \eta^- \cup x\right) \mathrm{d}x + \sum_{x\in\eta^-} z_\delta^-(x) e^{-E_{\phi^+}(x,\eta^+)} e^{-E_{\psi^-}(x,\eta^-\backslash x)} R\left(\eta^+,\ \eta^-\backslash x\right)\\
&+ \sum_{x\in\eta^-} e^{-E_{\kappa^+}(x,\eta^+)} e^{-E_{\tau^+}(x,\eta^-\backslash x)} R\left(\eta^+ \cup x,\ \eta^-\backslash x\right)\\
&+ \sum_{x\in\eta^+} e^{-E_{\kappa^-}(x,\eta^-)} e^{-E_{\tau^-}(x,\eta^+\backslash x)} R\left(\eta^+\backslash x,\ \eta^- \cup x\right).
\end{aligned}
$$

Then

$$
\int_{\Gamma_0^2} (\mathcal{Q}_\delta R)(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} D_\delta(\eta) R(\eta) \mathrm{d}\lambda(\eta), \quad 0 \le R \in \mathcal{D}_\delta.
$$

Consequently, there exists an extension $(\mathcal{J}_\delta,\ D(\mathcal{J}_\delta))$ of $(-D_\delta + \mathcal{Q}_\delta,\ \mathcal{D}_\delta)$ such that $\mathcal{J}_\delta$ is the generator of a sub-stochastic semigroup $(S_\delta(t))_{t\ge 0}$ on $L^1(\Gamma_0^2,\ \mathrm{d}\lambda)$, cf. [33, Theorem 2.2].

Lemma 5 *$\mathcal{D}_\delta$ is a core for $\mathcal{J}_\delta$. Moreover, $S_\delta(t)$ leaves $L^1(\Gamma_0^2,\ (1 + |\cdot|)\mathrm{d}\lambda)$ invariant.*

Proof Let $V(\eta) = |\eta|$, we want to find a constant $c = c(\delta) > 0$ such that

$$
L_\delta V(\eta) \le c(\delta)(1 + V(\eta)) - \frac{1}{2} D_\delta(\eta), \quad \eta \in \Gamma_0^2. \tag{26}
$$

In such a case the assertion follows from [33, Proposition 5.1]. Observe that

$$
(L_\delta V)(\eta) \le -|\eta| + \left\langle z_\delta^+ \right\rangle + \left\langle z_\delta^- \right\rangle.
$$

Then (26) holds, provided

$$
\left\langle z_\delta^+ \right\rangle + \left\langle z_\delta^- \right\rangle + \frac{1}{2} D_\delta(\eta) \le (1 + c)|\eta| + c.
$$

By $D_\delta(\eta) \le 2|\eta| + \langle z_\delta^+ \rangle + \langle z_\delta^- \rangle$ this holds true, provided

$$
\frac{3}{2}\left(\left\langle z_\delta^+ \right\rangle + \left\langle z_\delta^- \right\rangle\right) + |\eta| \le (1 + c)|\eta| + c.
$$

Above inequality is satisfied if $c > 0$ is such that $c > \frac{3}{2}(\langle z_\delta^+ \rangle + \langle z_\delta^- \rangle)$. □

Let $(\mathcal{I}_\delta,\ D(\mathcal{I}_\delta))$ be the adjoint operator to $(\mathcal{J}_\delta,\ D(\mathcal{J}_\delta))$. This operator is defined on $L^\infty(\Gamma_0^2,\ \mathrm{d}\lambda)$. The next lemma follows immediately by (2).

Lemma 6 *For each $F \in D(\mathcal{I}_\delta)$ the action of $\mathcal{I}_\delta$ is given by $L_\delta F$, i.e., $\mathcal{I}_\delta F = L_\delta F$.*

This shows that for each $R_0 \in L^1(\Gamma_0^2,\ \mathrm{d}\lambda)$ there exists exactly one weak solution $(R_t^\delta)_{t\ge 0} \subset L^1(\Gamma_0^2,\ \mathrm{d}\lambda)$ to

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_0^2} F(\eta) R_t^\delta(\eta) \mathrm{d}\lambda(\eta) = \int_{\Gamma_0^2} L_\delta F(\eta) R_t^\delta(\eta) \mathrm{d}\lambda(\eta), \quad R_t^\delta|_{t=0} = R_0, \tag{27}
$$

where $F \in D(\mathcal{I}_\delta)$, cf. [2]. This solution is given by $R_t^\delta = S_\delta(t) R_0$, $t \ge 0$.

3.4.2 Step 2: Evolution of Localized Correlation Functions

Let $\widehat{L}_\delta := K_0^{-1} L_\delta K_0$ be the operator given by $\widehat{L}$ where $z^\pm$ are replaced by $z_\delta^\pm$. It is considered on the same domain $D(\widehat{L})$ as $\widehat{L}$.

Lemma 7 *For any $\delta > 0$, the assertions of Proposition 1 and Theorem 2 hold with $z_\delta^\pm$ instead of $z^\pm$, i.e., $(\widehat{L}_\delta,\ D(\widehat{L}))$ is the generator of an analytic semigroup of contractions. Let $\widehat{T}_\delta(t)$ and $\widehat{T}_\delta(t)^*$ be the semigroups on $\mathcal{L}_\alpha$ and $\mathcal{K}_\alpha$, respectively. Then, for any $G \in \mathcal{L}_\alpha$,*

$$\widehat{T}_\delta(t)G \longrightarrow \widehat{T}(t)G, \quad \delta \to 0,$$

holds in $\mathcal{L}_\alpha$.

Proof Note that condition (A) is independent of δ and hence is still satisfied. Concerning condition (B), let $\beta_\delta(\alpha;\ \eta)$ be defined similarly to $\beta(\alpha;\ \eta)$, where $z^\pm$ are replaced by $z_\delta^\pm$. From $R_\delta \le 1$ it follows that $\beta_\delta(\alpha;\ \eta) \le \beta(\alpha;\ \eta)$ which shows that condition (B) holds. This shows the first assertion. Let us prove the second assertion. To this end it suffices to show that $\widehat{L}_\delta G \longrightarrow \widehat{L}G$ holds in $\mathcal{L}_\alpha$ as $\delta \to 0$ for any $G \in B_{bs}(\Gamma_0^2)$ (see [6, Chap. 3, Theorem 4.8]). Since $|z_\delta^\pm - z^\pm| = z^\pm(1 - R_\delta) \longrightarrow 0$ and $|z_\delta^\pm - z^\pm| \le z^\pm$ this follows by dominated convergence. □

Let $\mathcal{B}_\alpha^\delta$ be the Banach space of all equivalence classes of functions G with norm

$$\|G\|_{\mathcal{B}_\alpha^\delta} = \int_{\Gamma_0^2} |G(\eta)| e_\lambda\left(R_\delta;\ \eta^+\right) e_\lambda\left(R_\delta;\ \eta^-\right) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta)\mathrm{d}\lambda(\eta).$$

Its dual Banach space is identified with the Banach space $\mathcal{R}_\alpha^\delta$ of all equivalence classes of functions u equipped with the norm

$$\|u\|_{\mathcal{R}_\alpha^\delta} = \operatorname*{ess\ sup}_{\eta \in \Gamma_0^2} \frac{|u(\eta)|}{e_\lambda(R_\delta;\ \eta^+) e_\lambda(R_\delta;\ \eta^-) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta)}.$$

The same arguments as in the proof of Proposition 1 and Theorem 2 show that we can replace $\mathcal{L}_\alpha$, $\mathcal{K}_\alpha$ by $\mathcal{B}_\alpha^\delta$ and $\mathcal{R}_\alpha^\delta$. Let $U_\delta(t)$ and $U_\delta(t)^*$ be the corresponding semigroups and let $(\widehat{L}_\delta,\ D^{\mathcal{B}}(\widehat{L}_\delta))$ be the generator of $U_\delta(t)$. It is easily seen that

$$D^{\mathcal{B}}\left(\widehat{L}_\delta\right) = \left\{G \in \mathcal{B}_\alpha^\delta | M \cdot G \in \mathcal{B}_\alpha^\delta\right\},$$

and, in particular, $B_{bs}(\Gamma_0^2) \subset D^{\mathcal{B}}(\widehat{L}_\delta)$ is a core. Thus, the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle G,\ u_t^\delta\right\rangle = \left\langle \widehat{L}_\delta G,\ u_t^\delta\right\rangle, \quad u_t^\delta|_{t=0} = u_0, \quad G \in B_{bs}\left(\Gamma_0^2\right), \tag{28}$$

has for every $u_0 \in \mathcal{R}_\alpha^\delta$ a unique weak solution in $\mathcal{R}_\alpha^\delta$ given by $u_t^\delta = U_\delta(t)^* u_0$. Here the notion of weak solutions to (28) is defined analogously to (7).

Lemma 8 *Let $k_0 \in \mathcal{R}_\alpha^\delta$, then $\widehat{T}_\delta(t)^* k_0 = U_\delta(t)^* k_0$ holds.*

Proof Observe that $\mathcal{R}_\alpha^\delta \subset \mathcal{K}_\alpha$ is embedded continuously. Consequently, $u_t^\delta = U_\delta(t)^* k_0$ and $k_t^\delta = \widehat{T}_\delta(t)^* k_0$ are well-defined. Since also $\mathcal{L}_\alpha \subset \mathcal{B}_\alpha^\delta$ is continuously embedded, we obtain $D(\widehat{L}) \subset D^{\mathcal{B}}(\widehat{L}_\delta)$, i.e., $(\widehat{L}_\delta,\ D^{\mathcal{B}}(\widehat{L}_\delta))$ is an extension of $(\widehat{L}_\delta,\ D(\widehat{L}))$. Therefore, $(u_t^\delta)_{t \ge 0}$ is also a weak solution to (7) and hence uniqueness implies $k_t^\delta = u_t^\delta$, $t \ge 0$. □

Lemma 9 *Let* $\alpha_0^+ < \alpha^+,\ \alpha_0^- < \alpha^-,\ k_0 \in \mathcal{R}_{\alpha_0}^\delta$ *and assume that* k_0 *is positive definite. Then,* $u_t^\delta := U_\delta(t)^* k_0$ *is positive definite, for any* $t \geq 0$.

Proof Define a bounded linear operator $\mathcal{H}: \mathcal{R}_\alpha^\delta \longrightarrow \mathcal{L}_c$, for any $c = (c^+,\ c^-) \in \mathbb{R}^2$, by

$$\mathcal{H}u(\eta) := \int_{\Gamma_0^2} (-1)^{|\xi|} u(\eta \cup \xi) \mathrm{d}\lambda(\xi).$$

Let $G \in \mathcal{B}_\alpha^\delta$ be arbitrary. Then, for any $u \in \mathcal{R}_\alpha^\delta$, we get by Fubini's theorem and (2)

$$\langle K_0 G,\ \mathcal{H}u \rangle = \langle G,\ u \rangle. \tag{29}$$

We can apply Fubini's theorem and (2) since

$$\begin{aligned}
&\int_{\Gamma_0^2}\int_{\Gamma_0^2}\int_{\Gamma_0^2} |G(\xi)||u(\eta \cup \xi \cup \zeta)| \mathrm{d}\lambda(\zeta)\mathrm{d}\lambda(\xi)\mathrm{d}\lambda(\eta) \\
&\quad \leq \|u\|_{\mathcal{R}_\alpha^\delta} e^{2e^{\alpha^+}\langle R_\delta \rangle_\rho} e^{2e^{\alpha^-}\langle R_\delta \rangle_\rho} \int_{\Gamma_0^2} |G(\xi)| e^{\alpha|\xi|} e_\lambda(\rho;\ \xi) e_\lambda\left(R_\delta;\ \xi^+\right) e_\lambda\left(R_\delta;\ \xi^-\right) \mathrm{d}\lambda(\xi),
\end{aligned}$$

is satisfied, where $\langle R_\delta \rangle_\rho := \int_{\mathbb{R}^d} R_\delta(x)\rho(x)\mathrm{d}x$. For the same u and $G \in D^{\mathcal{B}}(\widehat{L})$ we obtain by (29) and $K_0 \widehat{L}_\delta G = L_\delta K_0 G$

$$\left\langle \widehat{L}_\delta G,\ u \right\rangle = \left\langle K_0 \widehat{L}_\delta G,\ \mathcal{H}u \right\rangle = \left\langle L_\delta K_0 G,\ \mathcal{H}u \right\rangle. \tag{30}$$

Observe that

$$\left\langle G,\ u_t^\delta \right\rangle = \langle G,\ u_0 \rangle + \int_0^t \left\langle \widehat{L}_\delta G,\ u_s^\delta \right\rangle \mathrm{d}s, \qquad G \in D^{\mathcal{B}}(\widehat{L}). \tag{31}$$

Let $G \in \mathcal{K}_c$, where $c := (\log(2),\ \log(2))$. By $|G(\eta)| \leq 2^{|\eta|}\|k\|_{\mathcal{K}_c}$ and by $M(\eta) \leq 2|\eta|$ we get

$$\begin{aligned}
&\int_{\Gamma_0^2} M(\eta)|G(\eta)| e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) e_\lambda\left(R_\delta;\ \eta^+\right) e_\lambda\left(R_\delta;\ \eta^-\right) \mathrm{d}\lambda(\eta) \\
&\quad \leq 2\|G\|_{\mathcal{K}_c} \int_{\Gamma_0^2} |\eta| 2^{|\eta|} e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) e_\lambda\left(R_\delta;\ \eta^+\right) e_\lambda\left(R_\delta;\ \eta^-\right) \mathrm{d}\lambda(\eta) \\
&\quad = 2\|G\|_{\mathcal{K}_c} \sum_{n,m=0}^{\infty} \frac{2^{n+m}}{n!m!}(n+m) e^{\alpha^+ n} e^{\alpha^- m} \langle R_\delta \rangle_\rho^{n+m} < \infty.
\end{aligned}$$

This implies $\mathcal{K}_c \subset D^{\mathcal{B}}(\widehat{L})$. By (29)–(31) it follows for $R_t^\delta := \mathcal{H}u_t^\delta \in L^1(\Gamma_0^2,\ \mathrm{d}\lambda),\ t \geq 0$,

$$\left\langle K_0 G,\ R_t^\delta \right\rangle = \langle K_0 G,\ R_0 \rangle + \int_0^t \left\langle L_\delta K_0 G,\ R_s^\delta \right\rangle \mathrm{d}s, \quad G \in \mathcal{K}_c. \tag{32}$$

For any $F \in D(\mathcal{J}_\delta) \subset L^\infty(\Gamma_0^2, \mathrm{d}\lambda)$ we get $|K_0^{-1}F(\eta)| \le \|F\|_{L^\infty} 2^{|\eta|}$ and hence $D(\mathcal{J}_\delta) \subset K_0\mathcal{K}_c$. Thus, we can find $G \in \mathcal{K}_c$ such that $K_0 G = F$. By (32), it follows that

$$\left\langle F,\ R_t^\delta \right\rangle = \langle F,\ R_0\rangle + \int_0^t \left\langle \mathcal{J}_\delta F,\ R_s^\delta \right\rangle \mathrm{d}s, \quad F \in D\left(\mathcal{J}_\delta\right).$$

Recall that $k_0 \in \mathcal{R}_{\alpha_0}^\delta$. By Theorem 2 we see that $u_t^\delta = U_\delta(t)^* k_0$ is continuous in $t \ge 0$ w.r.t. the norm in $\mathcal{R}_\alpha^\delta$. Since $\mathcal{H}\colon \mathcal{R}_\alpha^\delta \longrightarrow L^1(\Gamma_0^2, \mathrm{d}\lambda)$ is continuous, $R_t^\delta = \mathcal{H}u_t^\delta$ is continuous in $t \ge 0$ w.r.t. the norm in $L^1(\Gamma_0^2, \mathrm{d}\lambda)$. Hence, $(R_t^\delta)_{t\ge 0}$ is a weak solution to (27). Uniqueness implies that $R_t^\delta = S_\delta(t)R_0 \ge 0$. Finally, for any $G \in B_{bs}^+(\Gamma_0^2)$ we get

$$\left\langle G,\ u_t^\delta \right\rangle = \left\langle K_0 G,\ R_t^\delta \right\rangle \ge 0, \quad t \ge 0.$$

□

3.4.3 Step 3: Proof of Theorem 4

Let $\alpha_0^+ < \alpha^+$, $\alpha_0^- < \alpha^-$. First, we consider the special case $\mu_0 \in \mathcal{P}_{\alpha_0}$. Let $k_0 \in \mathcal{K}_{\alpha_0}$ be the associated correlation function. Define

$$k_{0,\delta}(\eta) := k_0(\eta) e_\lambda\left(R_\delta;\ \eta^+\right) e_\lambda\left(R_\delta;\ \eta^-\right), \quad \delta > 0, \quad \eta \in \Gamma_0^2,$$

then $k_{0,\delta} \in \mathcal{R}_{\alpha_0}^\delta$. The following lemma shows that $k_{0,\delta}$ is positive definite. It is a two-component generalization of [7, Lemma 3.9]. A computationally similar, but technically different proof is given in the Appendix.

Lemma 10 *Let $k\colon \Gamma_0^2 \longrightarrow \mathbb{R}$ be positive definite such that for any $C > 0$ and any compact $\Lambda \subset \mathbb{R}^d$*

$$\int_{\Gamma_0^2} C^{|\eta|} e_\lambda\left(\mathbb{1}_\Lambda;\ \eta^+\right) e_\lambda\left(\mathbb{1}_\Lambda;\ \eta^-\right) k(\eta)\mathrm{d}\lambda(\eta) < \infty.$$

Let $0 \le f^\pm \le 1$ be integrable and define

$$\widetilde{k}(\eta) := k(\eta) \prod_{x\in\eta^+} f^+(x) \prod_{x\in\eta^-} f^-(x), \quad \eta \in \Gamma_0^2.$$

Then $\widetilde{k}$ is positive definite.

Lemma 8 implies $\widehat{T}_\delta(t)^* k_{0,\delta} = U_\delta(t)^* k_{0,\delta} \in \mathcal{R}_\alpha^\delta$ and Lemma 9 shows that $U_\delta(t)^* k_{0,\delta}$ is positive definite. Let $G \in B_{bs}^+(\Gamma_0^2)$, then $\langle G,\ \widehat{T}_\delta(t)^* k_{0,\delta}\rangle \ge 0$. Observe that

$$\left\langle G,\ \widehat{T}_\delta(t)^* k_{0,\delta} \right\rangle = \langle \widehat{T}_\delta(t)G - \widehat{T}(t)G, k_{0,\delta}\rangle + \left\langle \widehat{T}(t)G,\ k_{0,\delta} \right\rangle. \tag{33}$$

For the first term we obtain, by $\|k_{0,\delta}\|_{\mathcal{K}_\alpha} \le \|k_0\|_{\mathcal{K}_\alpha}$,

$$\left|\left\langle \widehat{T}_\delta(t)^* G - \widehat{T}(t)G,\ k_{0,\delta} \right\rangle\right| \le \left\| \widehat{T}_\delta(t)G - \widehat{T}(t)G \right\|_{\mathcal{L}_\alpha} \|k_0\|_{\mathcal{K}_\alpha}.$$

The latter tends to zero, see Lemma 7. The second term in (33) tends, by dominated convergence, to $\langle \widehat{T}(t)G,\ k_0\rangle = \langle G,\ \widehat{T}(t)^* k_0\rangle$. Thus

$$\left\langle G,\ \widehat{T}_\delta(t)^* k_{0,\delta} \right\rangle \longrightarrow \left\langle G,\ \widehat{T}(t)^* k_0 \right\rangle, \quad \delta \to 0,$$

and hence $\langle G,\ \widehat{T}(t)^* k_0\rangle \ge 0$, i.e., $\widehat{T}(t)^* k_0$ is positive definite.

For the general case, let $\mu_0 \in \mathcal{P}_\alpha$ with correlation function $k_0 \in \mathcal{K}_\alpha$. Then $k_{0,\delta}(\eta) := e^{-\delta|\eta|}k_0(\eta)$ belongs to $\mathcal{K}_{\alpha-\delta}$ for any $\delta > 0$. By previous case, we see that $\widehat{T}(t)^* k_{0,\delta} \in \mathcal{K}_\alpha$ is positive definite. Taking the limit $\delta \to 0$ yields the assertion.

4 Ergodicity

Suppose the conditions (A) and (B) are satisfied. The following is the main statement for this section.

Theorem 5 *There exists a unique invariant measure $\mu_{\mathrm{inv}} \in \mathcal{P}_\alpha$ associated to L, i.e.,*

$$\int_{\Gamma^2} LF(\gamma)\mathrm{d}\mu_{\mathrm{inv}}(\gamma) = 0, \quad F \in \mathcal{FP}\left(\Gamma^2\right). \tag{34}$$

Let k_{inv} be the associated correlation function.

(1) *The semigroup $\widehat{T}(t)$ is uniformly ergodic with exponential rate and projection operator*

$$\widehat{P}G(\eta) = \int_{\Gamma_0^2} G(\xi)k_{\mathrm{inv}}(\xi)\mathrm{d}\lambda(\xi)\mathbb{1}^*(\eta). \tag{35}$$

(2) *The adjoint semigroup $\widehat{T}(t)^*$ is uniformly ergodic with exponential rate and projection operator*

$$\widehat{P}^* k(\eta) = k_{\mathrm{inv}}(\eta)k(\emptyset). \tag{36}$$

(3) *There exists constants $a,\ b > 0$ such that for all $\mu_0 \in \mathcal{P}_\alpha$*

$$\left\| k_{\mu_t} - k_{\mu_{\mathrm{inv}}} \right\|_{\mathcal{K}_\alpha} \le a e^{-bt} \left\| k_{\mu_0} - k_{\mu_{\mathrm{inv}}} \right\|_{\mathcal{K}_\alpha}, \quad t \ge 0,$$

holds, where $(\mu_t)_{t\ge 0}$ is the unique weak solution to (1).

The rest of this section is devoted to the proof of this theorem. Multiplication by $\mathbb{1}^*$ and $1-\mathbb{1}^*$ defines projection operators $\mathbb{1}^*\colon \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha^0$ and $(1-\mathbb{1}^*)\colon \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha^{\ge 1}$, respectively. Here, $\mathcal{K}_\alpha^{\ge 1} = \{k \in \mathcal{K}_\alpha | k^{(0,0)} = 0\}$ and $\mathcal{K}_\alpha^0 = \{k \in \mathcal{K}_\alpha | k^{(n,m)} = 0,\ \ n+m \ge 1\}$. By $\mathbb{1}^*(1-\mathbb{1}^*) = (1-\mathbb{1}^*)\mathbb{1}^* = 0$ we obtain $\mathcal{K}_\alpha = \mathcal{K}_\alpha^0 \oplus \mathcal{K}_\alpha^{\ge 1}$. Define a linear operator S by $Sk(\emptyset) = 0$ and

$$Sk(\eta) = \frac{1}{M(\eta)}(Bk)(\eta), \quad \eta \ne \emptyset.$$

It is not difficult to see that S leaves $\mathcal{K}_\alpha^{\ge 1}$ invariant and $\|S\|_{L(\mathcal{K}_\alpha)} < 1$. The next lemma provides existence and uniqueness of solutions to $L^\Delta k = 0$. Its proof is an easy modification of the arguments in [10].

Lemma 11 *The equation*

$$L^\Delta k_{\mathrm{inv}} = 0, \quad k_{\mathrm{inv}}(\emptyset, \emptyset) = 1, \tag{37}$$

has a unique solution $k_{\mathrm{inv}} \in \mathcal{K}_\alpha$. This solution is given by $k_{\mathrm{inv}} = \mathbb{1}^* + (1-S)^{-1}S\mathbb{1}^*$, where

$$S\mathbb{1}^*(\eta) = \mathbb{1}_{\Gamma_0^{(1)}}\left(\eta^+\right)0^{|\eta^-|}z^+ + \mathbb{1}_{\Gamma_0^{(1)}}\left(\eta^-\right)0^{|\eta^+|}z^-.$$

In particular, (36) *is a projection operator on* $\mathcal{K}_\alpha$ *with range*

$$\mathrm{Ran}(\widehat{P}^*) = \left\{k \in D\left(L^{\Delta}\right) | L^{\Delta} k = 0\right\},$$

and it is given by $\widehat{P}^* = \mathbb{1}^* + (1-S)^{-1} S\mathbb{1}^*$, *where* $\mathbb{1}^*$ *acts as a multiplication operator.*

First we establish ergodicity for $\widehat{T}(t)$. Let $\mathcal{L}_\alpha^0 := \{G \in \mathcal{L}_\alpha | G = \kappa \mathbb{1}^*, \ \kappa \in \mathbb{R}\}$ and $\mathcal{L}_\alpha^{\geq 1} := \{G \in \mathcal{L}_\alpha | G(\emptyset) = 0\}$. Then $\mathcal{L}_\alpha = \mathcal{L}_\alpha^0 \oplus \mathcal{L}_\alpha^{\geq 1}$ and the projection onto $\mathcal{L}_\alpha^0$ is given by multiplication with $\mathbb{1}^*$. Likewise, $1 - \mathbb{1}^*$ projects onto $\mathcal{L}_\alpha^{\geq 1}$. Define $B_{01} \colon \mathcal{L}_\alpha^{\geq 1} \longrightarrow \mathcal{L}_\alpha^0, \ B_{01} G = \mathbb{1}^* BG$ and $L_{11} \colon \mathcal{L}_\alpha^{\geq 1} \longrightarrow \mathcal{L}_\alpha^{\geq 1}, \ L_{11} G = AG + (1-\mathbb{1}^*) BG$. Taking into account $\widehat{L} = \widehat{L}(1-\mathbb{1}^*)$ yields

$$\widehat{L} G = B_{01}(1-\mathbb{1}^*) G + L_{11}(1-\mathbb{1}^*) G, \quad G \in \mathcal{L}_\alpha. \tag{38}$$

Moreover, since $D(\widehat{L}) = \{G \in \mathcal{L}_\alpha | M \cdot G \in \mathcal{L}_\alpha\}$ and $\mathcal{L}_\alpha^0 \subset D(\widehat{L})$ it follows that $D(L_{11}) = D(\widehat{L}) \cap \mathcal{L}_\alpha^{\geq 1}$. Note that B_{01} is given by

$$\mathbb{1}^* BG(\eta) = \mathbb{1}^*(\eta) z^- \int_{\mathbb{R}^d} G(\emptyset, x) \mathrm{d}x + \mathbb{1}^*(\eta) z^+ \int_{\mathbb{R}^d} G(x, \emptyset) \mathrm{d}x,$$

and hence is a positive operator. The next statement was shown for the one-component G^--dynamics in [25]. Based on their techniques we present an extension to the two-component case. Such extension includes a better estimate on the spectral gap of $\widehat{L}$ and admits a larger constant in condition (B).

Proposition 3 *Let* $a(\alpha) \in (0, 1)$ *be given as in condition* (*B*) *and*

$$\omega_0 := \sup\left\{\omega \in \left[0, \frac{\pi}{4}\right] \middle| a(\alpha) < \cos(\omega)\right\}. \tag{39}$$

Then the following statements hold:

(1) *The point 0 is an eigenvalue for* $(\widehat{L}, \ D(\widehat{L}))$ *with eigenspace* $\mathcal{L}_\alpha^0$ *and eigenvector* $\mathbb{1}^*$.
(2) *Let* $\lambda_0 := (1 - a(\alpha)) > 0$. *Then*

$$I_1 := \{\lambda \in \mathbb{C} | \mathrm{Re}(\lambda) > -\lambda_0\} \backslash \{0\},$$

and

$$I_2 := \left\{\lambda \in \mathbb{C} \middle| |\arg(\lambda)| < \frac{\pi}{2} + \omega_0\right\} \backslash \{0\},$$

belong to the resolvent set $\rho(\widehat{L})$ *of* $\widehat{L}$ *on* $\mathcal{L}_\alpha$.

Proof Let $(A_1, \ D(L_{11}))$ be the restriction of $(A, \ D(\widehat{L}))$ to $\mathcal{L}_\alpha^{\geq 1}$ and denote by $\| \cdot \|_{\mathcal{L}_\alpha^{\geq 1}}$ the norm on $\mathcal{L}_\alpha^{\geq 1}$. Observe that $M(\eta) \geq 1$ for all $|\eta| \geq 1$. Then, for any $\lambda = u + iw, \ u \geq 0, \ w \in \mathbb{R}$, by $M(\eta) \geq 1$ for all $|\eta| \geq 1$,

$$\left| \frac{G}{\lambda + M(\eta)} \right| \leq \frac{|G|}{\sqrt{(u+1)^2 + w^2}} \leq |G| \min\left(\frac{1}{|\lambda|}, \ \frac{1}{\sqrt{1 + w^2}} \right).$$

This implies $\lambda \in \rho(A_1)$ and

$$\| R(\lambda; \ A_1) G \|_{\mathcal{L}_\alpha^{\geq 1}} \leq \min\left(\frac{1}{|\lambda|}, \ \frac{1}{\sqrt{1+w^2}} \right) \| G \|_{\mathcal{L}_\alpha^{\geq 1}}. \tag{40}$$

Consider the decomposition

$$(\lambda - L_{11}) = \left(1 - (1-\mathbb{1}^*)BR(\lambda;\ A_1)\right)(\lambda - A_1). \tag{41}$$

Then, by (14),

$$\|(1-\mathbb{1}^*)BG\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \int\limits_{\Gamma_0^2} |BG(\eta)|e^{\alpha|\eta|}e_\lambda(\rho;\ \eta)\mathrm{d}\lambda(\eta) \leq a(\alpha)\|M\cdot G\|_{\mathcal{L}_\alpha^{\geq 1}},$$

for any $G \in \mathcal{L}_\alpha^{\geq 1}$. This implies that $(1-(1-\mathbb{1}^*)BR(\lambda;\ A_1))$ is invertible on $\mathcal{L}_\alpha^{\geq 1}$, i.e., $\lambda \in \rho(L_{11})$, and

$$R(\lambda;\ L_{11}) = R(\lambda;\ A_1)\left(1-(1-\mathbb{1}^*)BR(\lambda;\ A_1)\right)^{-1}. \tag{42}$$

In particular, we obtain for $\lambda = u+iw,\ u\geq 0,\ w\in\mathbb{R}$ by (42) and (40)

$$\|R(\lambda;\ L_{11})G\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \frac{\min\left(\frac{1}{|\lambda|},\ \frac{1}{\sqrt{1+w^2}}\right)}{1-a(\alpha)}\|G\|_{\mathcal{L}_\alpha^{\geq 1}},$$

and for $\lambda = iw,\ w\in\mathbb{R}$

$$\|R(iw,\ L_{11})G\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \frac{\sqrt{1+w^2}^{-1}}{1-a(\alpha)}\|G\|_{\mathcal{L}_\alpha^{\geq 1}}.$$

For $\lambda = u+iw,\ 0>u>-\lambda_0$ and $w\in\mathbb{R}$ write

$$(u+iw-L_{11}) = (1+uR(iw;\ L_{11}))(iw-L_{11}).$$

Then, by $|u|<\lambda_0$ and $\frac{|u|}{\sqrt{1+w^2}}\frac{1}{1-a} \leq \frac{|u|}{\lambda_0} < 1$ we obtain $\lambda\in\rho(L_{11})$ and

$$\|R(\lambda;\ L_{11})G\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \frac{\sqrt{1+w^2}^{-1}}{1-a(\alpha)}\left(1-\frac{|u|}{\lambda_0}\right)^{-1}\|G\|_{\mathcal{L}_\alpha^{\geq 1}}.$$

Therefore, I_1 belongs to the resolvent set of L_{11}. For I_2 let $\lambda = u+iw\in I_2$ and $u<0$. Then, there exists $\omega\in(0,\ \omega_0)$ such that $|\arg(\lambda)| < \frac{\pi}{2}+\omega$ and hence

$$|w| = |\tan(\arg(\lambda))||u| \geq \cot(\omega)|u|.$$

This implies for $\eta\neq\emptyset$

$$|\lambda+M(\eta)|^2 = (u+M(\eta))^2+w^2 \geq (u+M(\eta))^2+\cot(\omega)^2u^2.$$

The right-hand side is minimal for the choice $u = -\frac{M(\eta)}{1+\cot(\omega)^2}$ which yields

$$\begin{aligned}|\lambda+M(\eta)|^2 &\geq M(\eta)^2\left(\left(\frac{\cot(\omega)^2}{1+\cot(\omega)^2}\right)^2+\frac{\cot(\omega)^2}{(1+\cot(\omega)^2)^2}\right)\\ &= M(\eta)^2\frac{\cot(\omega)^2}{1+\cot(\omega)^2} = M(\eta)^2\cos(\omega)^2.\end{aligned}$$

Then, by

$$\left\|(1-\mathbb{1}^*)BR(\lambda;\ A_1)G\right\|_{\mathcal{L}_\alpha^{\geq 1}} \leq a(\alpha)\|A_1R(\lambda;\ A_1)G\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \frac{a(\alpha)}{\cos(\omega)}\|G\|_{\mathcal{L}_\alpha^{\geq 1}},$$

and (39) we have $a(\alpha) < \cos(\omega)$. By (41) we obtain $I_2 \subset \rho(L_{11})$. Moreover, for each $\lambda = u + iw$ such that $\frac{\pi}{2} < |\arg(\lambda)| < \frac{\pi}{2} + \omega$ and, for some $\omega \in (0, \omega_0)$,

$$\|R(\lambda;\ L_{11})\, G\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \frac{\sqrt{(u^2+1)^2 + w^2}^{-1}}{1 - \frac{a(\alpha)}{\cos(\omega)}} \|G\|_{\mathcal{L}_\alpha^{\geq 1}}$$
$$\leq \frac{\left(1 - \frac{a(\alpha)}{\cos(\omega)}\right)^{-1}}{|w|} \|G\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \sqrt{2} \frac{\left(1 - \frac{a(\alpha)}{\cos(\omega)}\right)^{-1}}{|\lambda|} \|G\|_{\mathcal{L}_\alpha^{\geq 1}},$$

where we have used $|w| \geq \frac{|\lambda|}{\sqrt{2}}$. For the first claim let $\psi \in D(\widehat{L})$ be an eigenvector to the eigenvalue 0. The decomposition $\psi = \mathbb{1}^*\psi + (1 - \mathbb{1}^*)\psi = \psi_0 + \psi_1$ with $\psi_0 \in \mathcal{L}_\alpha^0$ and $\psi_1 \in \mathcal{L}_\alpha^{\geq 1} \cap D(\widehat{L}) = D(L_{11})$ yields, by (38),

$$0 = \widehat{L}\psi = \mathbb{1}^* B\psi_1 + L_{11}\psi_1 \in \mathcal{L}_\alpha^0 \oplus \mathcal{L}_\alpha^{\geq 1}.$$

Hence $L_{11}\psi_1 = 0$ and since $0 \in \rho(L_{11})$ also $\psi_1 = 0$. For the second statement let $\lambda \in I_1 \cup I_2$ and $H = H_0 + H_1 \in \mathcal{L}_\alpha^0 \oplus \mathcal{L}_\alpha^{\geq 1}$. Then, we have to find $G \in D(\widehat{L})$ such that

$$(\lambda - \widehat{L})G = H.$$

Using again the decomposition of $\widehat{L}$, above equation is equivalent to the system of equations

$$\lambda G_0 - \mathbb{1}^* B G_1 = H_0,$$
$$(\lambda - L_{11})\, G_1 = H_1.$$

Since $\lambda \in I_1 \cup I_2 \subset \rho(L_{11})$ the second equation has a unique solution on $\mathcal{L}_\alpha^{\geq 1}$ given by $G_1 = R(\lambda;\ L_{11})H_1$. Therefore, G_0 is given by

$$G_0 = \frac{1}{\lambda}\left(H_0 + \mathbb{1}^* B R(\lambda;\ L_{11})\, H_1\right).$$

□

Remark 4 The proof shows that for any $\varepsilon > 0$ there exists $\omega = \omega(\varepsilon) \in \left(0, \frac{\pi}{2}\right)$ such that

$$\Sigma(\varepsilon) := \left\{\lambda \in \mathbb{C} \middle| |\arg(\lambda + \lambda_0 - \varepsilon)| \leq \frac{\pi}{2} + \omega\right\} \subset I_1 \cup I_2 \cup \{0\},$$

and there exists $M(\varepsilon) > 0$ with

$$\|R(\lambda;\ L_{11})\, G\|_{\mathcal{L}_\alpha^{\geq 1}} \leq \frac{M(\varepsilon)}{|\lambda|} \|G\|_{\mathcal{L}_\alpha^{\geq 1}},$$

for all $\lambda \in \Sigma(\varepsilon)\backslash\{0\}$. Moreover, $(L_{11},\ D(L_{11}))$ is a sectorial operator of angle ω_0 on $\mathcal{L}_\alpha^{\geq 1}$. Denote by $\widetilde{T}(t)$ the bounded analytic semigroup on $\mathcal{L}_\alpha^{\geq 1}$ given by

$$\widetilde{T}(t) = \frac{1}{2\pi i} \int_\sigma e^{\zeta t} R(\zeta;\ L_{11})\, \mathrm{d}\zeta, \quad t > 0, \tag{43}$$

where the integral converges in the uniform operator topology, see [28]. Here, σ denotes any piecewise smooth curve in

$$\left\{\lambda \in \mathbb{C} \middle| |\arg(\lambda)| < \frac{\pi}{2} + \omega_0\right\} \backslash\{0\},$$

running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\theta \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \omega_0\right)$.

The $\mathcal{L}_\alpha^{\geq 1}$ part of $\widehat{T}(t)$ is given by $(1-\mathbb{1}^*)\widehat{T}(t)|_{\mathcal{L}_\alpha^{\geq 1}}$ and hence has the generator $(1-\mathbb{1}^*)\widehat{L}G = L_{11}G$. As a consequence, we obtain $\widetilde{T}(t) = (1-\mathbb{1}^*)\widehat{T}(t)|_{\mathcal{L}_\alpha^{\geq 1}}$. This yields the decomposition

$$\widehat{T}(t) = \mathbb{1}^* + \mathbb{1}^*\widehat{T}(t)(1-\mathbb{1}^*) + \widetilde{T}(t)(1-\mathbb{1}^*), \quad t \geq 0. \tag{44}$$

By duality we see that the adjoint semigroup $(\widehat{T}(t)^*)_{t\geq 0}$ on $\mathcal{K}_\alpha$ admits the decomposition

$$\widehat{T}(t)^* = \mathbb{1}^* + (1-\mathbb{1}^*)\widehat{T}(t)^*\mathbb{1}^* + \widetilde{T}(t)^*(1-\mathbb{1}^*), \quad t \geq 0, \tag{45}$$

where $\widetilde{T}(t)^* \in L(\mathcal{K}_\alpha^{\geq 1})$ is the adjoint semigroup to $(\widetilde{T}(t))_{t\geq 0}$.

Lemma 12 *The projection operator $\widehat{P}\colon \mathcal{L}_\alpha \longrightarrow \mathcal{L}_\alpha^0$, given by (35), satisfies*

$$\langle \widehat{P}G,\ k\rangle = \langle G,\ \widehat{P}^*k\rangle.$$

Moreover, we have $\widehat{P} = \widehat{T}(t)\widehat{P} = \widehat{P}\widehat{T}(t)$ and

$$\widehat{T}(t)^*\widehat{P}^* = \widehat{P}^*\widehat{T}(t)^* = \widehat{P}^*. \tag{46}$$

Now we are prepared to prove Theorem 5.

Proof (Theorem 5) The spectral properties stated in Remark 4, formulas (43)–(45) imply that for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|(1-\mathbb{1}^*)\widehat{T}(t)G\|_{\mathcal{L}_\alpha} \leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|G\|_{\mathcal{L}_\alpha}, \quad G \in \mathcal{L}_\alpha^{\geq 1}, \quad t \geq 0.$$

Repeat, e.g., the arguments in [25]. This yields

$$\|\widehat{T}(t)^*k\|_{\mathcal{K}_\alpha} \leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|k\|_{\mathcal{K}_\alpha}, \quad k \in \mathcal{K}_\alpha^{\geq 1}.$$

Let $k \in \mathcal{K}_\alpha$, we obtain, by (36),

$$k - \widehat{P}^*k = (1-\mathbb{1}^*)k \cdot k_{\text{inv}} \in \mathcal{K}_\alpha^{\geq 1}.$$

Using (46), we see that

$$\|\widehat{T}(t)^*k - \widehat{P}^*k\|_{\mathcal{K}_\alpha} = \|\widehat{T}(t)^*(k-\widehat{P}^*k)\|_{\mathcal{K}_\alpha} \leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\|k-\widehat{P}^*k\|_{\mathcal{K}_\alpha}, \tag{47}$$

holds. This shows that $\widehat{T}(t)^*$ is uniformly ergodic with exponential rate. By duality also $\widehat{T}(t)$ is uniformly ergodic with exponential rate. Let $\mu_0 \in \mathcal{P}_\alpha$ and $\mu_t \in \mathcal{P}_\alpha$ be the weak solution to (1). Denote by $(k_{\mu_t})_{t\geq 0} \subset \mathcal{K}_\alpha$ its associated family of correlation functions. Then, for any $t \geq 0$,

$$\left\|k_{\mu_t} - k_{\text{inv}}\right\|_{\mathcal{K}_\alpha} \leq C(\varepsilon)e^{-(\lambda_0-\varepsilon)t}\left\|k_{\mu_0} - k_{\text{inv}}\right\|_{\mathcal{K}_\alpha},$$

shows that k_{inv} is a limit of positive definite functions. Hence, it is positive definite. Thus, there exists a unique measure $\mu_{\text{inv}} \in \mathcal{P}_\alpha$ having k_{inv} as its correlation function. Since $\widehat{T}(t)^*k_{\text{inv}} = k_{\text{inv}}$, it follows that μ_{inv} is invariant for L. Property (34) follows immediately from $L^\Delta k_{\text{inv}} = 0$. □

5 Vlasov Scaling

The general scheme of Vlasov scaling for (one-component) interacting particle systems in the continuum can be found in [9]. Particular examples for two-component models have been considered in [8]. For convenience of the reader, we give a brief description.

The aim is to produce a certain scaling $L \longmapsto L_n$, $n \in \mathbb{N}$ such that the following scheme holds. Let L_n^{Δ} be the scaled operator on correlation functions and $e^{tL_n^{\Delta}}$ the (heuristic) representation of the scaled evolution of correlation functions. The particular choice of $L \longrightarrow L_n$ should preserve the order of singularity. Namely, for $\beta > 0$ let $R_\beta G(\eta) := \beta^{|\eta|}G(\eta)$, then

$$R_{n^{-1}} e^{tL_n^{\Delta}} R_n k \longrightarrow T_V^{\Delta}(t)k, \quad n \to \infty, \tag{48}$$

should exist. The evolution $T_V^{\Delta}(t)$ should preserve Lebesgue–Poisson exponentials, i.e., if $r_0(\eta) = e_\lambda(\rho_0^-, \eta^-)e_\lambda(\rho_0^+; \eta^+)$, then $T_V^{\Delta}(t)r_0(\eta) = e_\lambda(\rho_t^-, \eta^-)e_\lambda(\rho_t^+; \eta^+)$. We will show that ρ_t^-, ρ_t^+ satisfy a certain system of non-linear integro-differential equations.

Instead of investigating the limit (48), observe that formally

$$R_{n^{-1}} e^{tL_n^{\Delta}} R_n = e^{tR_{n^{-1}} L_n^{\Delta} R_n}.$$

Thus, it is the same to consider renormalized operators $L_{n,\text{ren}}^{\Delta} := R_{n^{-1}} L_n^{\Delta} R_n$ and study the behaviour of the renormalized semigroups $T_{n,\text{ren}}^{\Delta}(t) := e^{tL_{n,\text{ren}}^{\Delta}}$, as $n \to \infty$. We will prove that the limit

$$L_{n,\text{ren}}^{\Delta} \longrightarrow L_V^{\Delta}, \tag{49}$$

exists and L_V^{Δ} is associated with a semigroup $T_V^{\Delta}(t) = e^{tL_V^{\Delta}}$. The limit (48) is then obtained by showing the convergence

$$T_{n,\text{ren}}^{\Delta}(t) \longrightarrow T_V^{\Delta}(t), \tag{50}$$

in a proper sense.

Note that $L_{n,\text{ren}}^{\Delta}$ and L_V^{Δ} are operators on $\mathcal{K}_\alpha$ and therefore cannot be generators of strongly continuous semigroups. Hence, we consider first the scaled evolution on quasi-observables $\widehat{L}_n := K^{-1}L_n K$ and the renormalized operators $\widehat{L}_{n,\text{ren}} = R_n \widehat{L}_n R_{n^{-1}}$. We show that $\widehat{L}_{n,\text{ren}}$ is the generator of an analytic semigroup $\widehat{T}_{n,\text{ren}}(t)$ of contractions and prove that $\widehat{L}_{n,\text{ren}} \longrightarrow \widehat{L}_V$, as $n \to \infty$. Here, $\widehat{L}_V$ is again the generator of an analytic semigroup $\widehat{T}_V(t)$ of contractions. By Trotter–Kato approximation (see [6, Chap. 3, Theorem 4.8], it follows that $\widehat{T}_{n,\text{ren}}(t) \longrightarrow \widehat{T}_V(t)$ strongly in $\mathcal{L}_\alpha$. By duality we obtain (49) and (50).

5.1 Assumptions and Scaling

Put $z^{\pm} \longmapsto nz^{\pm}$ and scale the potentials by $\frac{1}{n}$, i.e., $g \longmapsto \frac{1}{n}g$ where $g \in \{\phi^{\pm}, \psi^{\pm}, \kappa^{\pm}, \tau^{\pm}\}$. Denote by L_n the corresponding (heuristic) Markov operator obtained by this scaling. Similarly to the case $n = 1$, we consider the following assumptions.

(V1) Suppose that there exists $\alpha = (\alpha^+, \alpha^-) \in \mathbb{R}^2$ and a locally bounded measurable function $\rho: \mathbb{R}^d \longrightarrow [1, \infty)$ such that $(g * \rho)(x)$ exists for all $x \in \mathbb{R}^d$ and $g \in \{\phi^{\pm}, \psi^{\pm}, \kappa^{\pm}, \tau^{\pm}\}$.

(V2) We have

$$\sup_{x\in\mathbb{R}^d}\left(z^+e^{-\alpha^+}e^{e^{\alpha^-}(\phi^-*\rho)(x)}e^{e^{\alpha^+}(\psi^+*\rho)(x)}+q^-e^{\alpha^--\alpha^+}e^{e^{\alpha^-}(\kappa^-*\rho)(x)}e^{e^{\alpha^+}(\tau^-*\rho)(x)}\right)<1,$$

$$\sup_{x\in\mathbb{R}^d}\left(z^-e^{-\alpha^-}e^{e^{\alpha^+}(\phi^+*\rho)(x)}e^{e^{\alpha^-}(\psi^-*\rho)(x)}+q^+e^{\alpha^+-\alpha^-}e^{e^{\alpha^+}(\kappa^+*\rho)(x)}e^{e^{\alpha^-}(\tau^+*\rho)(x)}\right)<1,$$

$$\sup_{x\in\mathbb{R}^d} e^{e^{\alpha^+}(\kappa^+*\rho)(x)}e^{e^{\alpha^-}(\tau^+*\rho)(x)}<\frac{1+q^+}{q^+},$$

$$\sup_{x\in\mathbb{R}^d} e^{e^{\alpha^-}(\kappa^-*\rho)(x)}e^{e^{\alpha^+}(\tau^-*\rho)(x)}<\frac{1+q^-}{q^-}.$$

Let us comment of the assumptions. It can be shown that (V1) and (V2) are stronger then (A) and (B). Similarly to Lemma 2, one can show that

$$\sup_{x\in\mathbb{R}^d}(g*\rho)(x)<\infty,\quad \forall g\in\left\{\phi^{\pm},\ \psi^{\pm},\ \tau^{\pm},\ \kappa^{\pm}\right\},$$

implies that for each $\alpha=(\alpha^+,\ \alpha^-)$ there exists $z^{\pm}(\alpha),\ q^{\pm}(\alpha)>0$ such that (V1) and (V2) are satisfied for all $z^{\pm}<z^{\pm}(\alpha)$ and $q^{\pm}<q^{\pm}(\alpha)$.

5.2 Statements

Let $\widehat{L}_n:=K^{-1}L_nK$ be a linear mapping defined on $B_{bs}(\Gamma_0^2)$ and put $\widehat{L}_{n,\mathrm{ren}}:=R_n\widehat{L}_nR_{n^{-1}}$. Then, $\widehat{L}_{n,\mathrm{ren}}=A_n+B_n$ where $(A_nG)(\eta)=-M_n(\eta)G(\eta)$ with cumulative death rate

$$M_n(\eta)=\left|\eta^+\right|+\left|\eta^-\right|+\sum_{x\in\eta^+}e^{-\frac{1}{n}E_{\kappa^+}(x,\eta^+\setminus x)}e^{-\frac{1}{n}E_{\tau^+}(x,\eta^-)}$$
$$+\sum_{x\in\eta^-}e^{-\frac{1}{n}E_{\kappa^-}(x,\eta^-\setminus x)}e^{-\frac{1}{n}E_{\tau^-}(x,\eta^+)}.$$

Let $f_x^n(g;\ \eta):=\prod_{y\in\eta}n\left(e^{-\frac{1}{n}g(x-y)}-1\right)$, the second linear mapping is given by

$$\begin{aligned}
&(B_nG)(\eta)\\
&=z^+\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}e^{-\frac{1}{n}E_{\phi^-}(x,\xi^-)}e^{-\frac{1}{n}E_{\psi^+}(x,\xi^+)}f_x^n\left(\phi^-;\ \eta^-\setminus\xi^-\right)f_x^n\left(\psi^+;\ \eta^+\setminus\xi^+\right)\\
&\quad\times G\left(\xi^+\cup x,\ \xi^-\right)\mathrm{d}x\\
&\quad+z^-\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}e^{-\frac{1}{n}E_{\phi^+}(x,\xi^+)}e^{-\frac{1}{n}E_{\psi^-}(x,\xi^-)}f_x^n\left(\phi^+;\ \eta^+\setminus\xi^+\right)f_x^n\left(\psi^-;\ \eta^-\setminus\xi^-\right)\\
&\quad\times G\left(\xi^+,\ \xi^-\cup x\right)\mathrm{d}x\\
&\quad+q^+\sum_{\xi\subset\eta}\sum_{x\in\xi^+}e^{-\frac{1}{n}E_{\kappa^+}(x,\xi^+\setminus x)}e^{-\frac{1}{n}E_{\tau^+}(x,\xi^-)}f_x^n\left(\kappa^+;\ \eta^+\setminus\xi^+\right)f_x^n\left(\tau^+;\ \eta^-\setminus\xi^-\right)\\
&\quad\times G\left(\xi^+\setminus x,\ \xi^-\cup x\right)\\
&\quad+q^+\sum_{\xi\subset\eta}\sum_{x\in\xi^-}e^{-\frac{1}{n}E_{\kappa^-}(x,\xi^-\setminus x)}e^{-\frac{1}{n}E_{\tau^-}(x,\xi^+)}f_x^n\left(\kappa^-;\ \eta^-\setminus\xi^-\right)f_x^n\left(\tau^-;\ \eta^+\setminus\xi^+\right)\\
&\quad\times G\left(\xi^+\cup x,\ \xi^-\setminus x\right)
\end{aligned}$$

$$- q^- \sum_{\substack{\xi \subset \eta \\ \xi \neq \eta}} \sum_{x \in \xi^+} e^{-\frac{1}{n} E_{\kappa^+}(x, \xi^+ \backslash x)} e^{-\frac{1}{n} E_{\tau^+}(x, \xi^-)} f_x^n \left(\kappa^+;\ \eta^+ \backslash \xi^+\right) f_x^n \left(\tau^+;\ \eta^- \backslash \xi^-\right) G(\xi)$$
$$- q^- \sum_{\substack{\xi \subset \eta \\ \xi \neq \eta}} \sum_{x \in \xi^-} e^{-\frac{1}{n} E_{\kappa^-}(x, \xi^- \backslash x)} e^{-\frac{1}{n} E_{\tau^-}(x, \xi^+)} f_x^n \left(\kappa^-;\ \eta^- \backslash \xi^-\right) f_x^n \left(\tau^-;\ \eta^+ \backslash \xi^+\right) G(\xi).$$

As before, we consider this linear map on the domain

$$D\left(A_n\right) := \{G \in \mathcal{L}_\alpha | M_n \cdot G \in \mathcal{L}_\alpha\}.$$

One can show that there exists a constant $a_n(\alpha) \in (0,\ 1)$ such that

$$\int_{\Gamma_0^2} B_n' G(\eta) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) \mathrm{d}\lambda(\eta) \le a_n(\alpha) \int_{\Gamma_0^2} M_n(\eta) G(\eta) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) \mathrm{d}\lambda(\eta), \quad 0 \le G \in D\left(A_n\right),$$

where B_n' is defined analogously to B'. Hence $(\widehat{L}_{n,\mathrm{ren}},\ D(A_n))$ is a well-defined operator on $\mathcal{L}_\alpha$. The next statement follows by the same arguments as Proposition 1 and Theorem 2.

Proposition 4 *Let $n \in \mathbb{N}$ be arbitrary and fixed. The following assertions are satisfied.*

(1) *$(\widehat{L}_{n,\mathrm{ren}},\ D(A_n))$ is the generator of an analytic semigroup $(\widehat{T}_{n,\mathrm{ren}}(t))_{t\ge 0}$ of contractions on $\mathcal{L}_\alpha$. Moreover, $B_{bs}(\Gamma_0^2)$ is a core.*
(2) *Let $(\widehat{T}_{n,\mathrm{ren}}(t)^*)_{t\ge 0}$ be the adjoint semigroup. For any $k_0 \in \mathcal{K}_\alpha$ there exists a unique weak solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle G,\ k_{t,n} \right\rangle = \left\langle \widehat{L}_{n,\mathrm{ren}} G,\ k_{t,n} \right\rangle, \quad k_{t,n}|_{t=0} = k_0, \quad G \in B_{bs}\left(\Gamma_0^2\right).$$

This solution is given by $k_{t,n} = \widehat{T}_{n,\mathrm{ren}}(t)^* k_0$.

In the next step we consider the limiting operators, as $n \to \infty$. These operators are formally given by $\widehat{L}_V = A_V + B_V$, where $A_V G(\eta) = -M_V(\eta) G(\eta)$ and

$$M_V(\eta) = 2\left|\eta^+\right| + 2\left|\eta^-\right|$$
$$\left(B_V G\right)(\eta) = z^+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda\left(-\phi^-(x - \cdot);\ \eta^- \backslash \xi^-\right) e_\lambda\left(-\psi^+(x - \cdot);\ \eta^+ \backslash \xi^+\right)$$
$$\times G\left(\xi^+ \cup x,\ \xi^-\right) \mathrm{d}x$$
$$+ z^- \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda\left(-\phi^+(x - \cdot);\ \eta^+ \backslash \xi^+\right) e_\lambda\left(-\psi^-(x - \cdot);\ \eta^- \backslash \xi^-\right) G\left(\xi^+,\ \xi^- \cup x\right) \mathrm{d}x$$
$$+ q^+ \sum_{\xi \subset \eta} \sum_{x \in \xi^+} e_\lambda\left(-\kappa^+(x - \cdot);\ \eta^+ \backslash \xi^+\right) e_\lambda\left(-\tau^+(x - \cdot);\ \eta^- \backslash \xi^-\right) G\left(\xi^+ \backslash x,\ \xi^- \cup x\right)$$
$$+ q^- \sum_{\xi \subset \eta} \sum_{x \in \xi^-} e_\lambda\left(-\kappa^-(x - \cdot);\ \eta^- \backslash \xi^-\right) e_\lambda\left(-\tau^-(x - \cdot);\ \eta^+ \backslash \xi^+\right) G\left(\xi^+ \cup x,\ \xi^- \backslash x\right)$$
$$- q^+ \sum_{\substack{\xi \subset \eta \\ \xi \neq \eta}} \sum_{x \in \xi^+} e_\lambda\left(-\kappa^+(x - \cdot);\ \eta^+ \backslash \xi^+\right) e_\lambda\left(-\tau^+(x - \cdot);\ \eta^- \backslash \xi^-\right) G(\xi)$$
$$- q^- \sum_{\substack{\xi \subset \eta \\ \xi \neq \eta}} \sum_{x \in \xi^-} e_\lambda\left(-\kappa^-(x - \cdot);\ \eta^- \backslash \xi^-\right) e_\lambda\left(-\tau^-(x - \cdot);\ \eta^+ \backslash \xi^+\right) G(\xi).$$

As before, it can be shown that for any $0 \leq G \in D(\widehat{L}_V)$ with

$$D\left(\widehat{L}_V\right) := \{G \in \mathcal{L}_\alpha | M_V \cdot G \in \mathcal{L}_\alpha\},$$

we have

$$\int_{\Gamma_0^2} B_V' G(\eta) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) \mathrm{d}\lambda(\eta) \leq a(\alpha) \int_{\Gamma_0^2} M_V(\eta) G(\eta) e^{\alpha|\eta|} e_\lambda(\rho;\ \eta) \mathrm{d}\lambda(\eta),$$

for some constant $a_V(\alpha) \in (0,\ 1)$. Here B_V' is defined analogously to B'. Hence $\widehat{L}_V$ is a well-defined operator on $\mathcal{L}_\alpha$ with domain $D(\widehat{L}_V)$.

Theorem 6 *The following assertions are satisfied*:

(1) *The operator $(\widehat{L}_V,\ D(\widehat{L}_V))$ is the generator of an analytic semigroup $(\widehat{T}^V(t))_{t\geq 0}$ of contractions on $\mathcal{L}_\alpha$. Moreover, $B_{bs}(\Gamma_0^2)$ is a core for the generator.*
(2) *Let $(\widehat{T}^V(t)^*)_{t\geq 0}$ be the adjoint semigroup on $\mathcal{K}_\alpha$. Then, for any $r_0 \in \mathcal{K}_\alpha$ there exists a unique solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle G,\ r_t\rangle = \left\langle \widehat{L}_V G,\ r_t\right\rangle, \quad r_t|_{t=0} = r_0, \quad G \in B_{bs}\left(\Gamma_0^2\right). \tag{51}$$

The solution is given by $r_t = \widehat{T}^V(t)^* r_0$.
(3) *Let $r_0(\eta) = \prod_{x\in\eta^+} \rho_0^+(x) \prod_{x\in\eta^-} \rho_0^-(x)$ with*

$$\rho_0^\pm(x) \leq e^{\alpha^\pm} \rho(x), \quad x \in \mathbb{R}^d.$$

Assume that $(\rho_t^+,\ \rho_t^-)$ is a classical solution to

$$\begin{aligned}\frac{\partial \rho_t^+(x)}{\partial t} &= -\rho_t^+(x) + z^+ e^{-(\psi^+ * \rho_t^+)(x)} e^{-(\phi^- * \rho_t^-)(x)} \\ &\quad - q^+ e^{-(\kappa^+ * \rho_t^+)(x)} e^{-(\tau^+ * \rho_t^-)(x)} \rho_t^+(x) + q^- e^{-(\tau^- * \rho_t^+)(x)} e^{-(\kappa^- * \rho_t^-)(x)} \rho_t^-(x),\end{aligned} \tag{52}$$

$$\begin{aligned}\frac{\partial \rho_t^-(x)}{\partial t} &= -\rho_t^-(x) + z^- e^{-(\phi^+ * \rho_t^+)(x)} e^{-(\psi^- * \rho_t^-)(x)} \\ &\quad - q^- e^{-(\kappa^- * \rho_t^-)(x)} e^{-(\tau^- * \rho_t^+)(x)} \rho_t^-(x) + q^+ e^{-(\kappa^+ * \rho_t^+)(x)} e^{-(\tau^+ * \rho_t^-)(x)} \rho_t^+(x),\end{aligned} \tag{53}$$

such that

$$\rho_t^\pm(x) \leq e^{\alpha^\pm} \rho(x), \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

Then $r_t(\eta) := \prod_{x\in\eta^+} \rho_t^+(x) \prod_{x\in\eta^-} \rho_t^-(x)$ is a weak solution to (51).

Proof The first two assertions follow by a modification of the arguments given in the proof of Proposition 1 and Theorem 2. For the last assertion, observe that $r_t(\eta)$ is continuous w.r.t. $\sigma(\mathcal{K}_\alpha,\ \mathcal{L}_\alpha)$. Since $\|r_t\|_{\mathcal{K}_\alpha} \leq 1$ for all $t \geq 0$, it follows that r_t is continuous w.r.t. $\mathcal{C}$. The adjoint operator to $\widehat{L}_V$ is given by

$$\begin{aligned}\left(L_V^{\Delta}k\right)(\eta) = &-|\eta|k(\eta) - q^+ \sum_{x\in\eta^+} \mathcal{Q}_x^V\left(\kappa^+,\, \tau^+\right)k(\eta) - q^- \sum_{x\in\eta^-} \mathcal{Q}_x^V\left(\tau^-,\, \kappa^-\right)k(\eta)\\ &+ z^+ \sum_{x\in\eta^+} \mathcal{Q}_x^V\left(\psi^+,\, \phi^-\right)k\left(\eta^+\backslash x,\, \eta^-\right) + z^- \sum_{x\in\eta^-} \mathcal{Q}_x^V\left(\phi^+,\, \psi^-\right)k\left(\eta^+,\, \eta^-\backslash x\right)\\ &+ q^+ \sum_{x\in\eta^-} \mathcal{Q}_x^V\left(\kappa^+,\, \tau^+\right)k\left(\eta^+\cup x,\, \eta^-\backslash x\right)\\ &+ q^- \sum_{x\in\eta^+} \mathcal{Q}_x^V\left(\tau^-,\, \kappa^-\right)k\left(\eta^+\backslash x,\, \eta^-\cup x\right),\end{aligned}$$

defined on its maximal domain $D(L_V^{\Delta}) = \{k \in \mathcal{K}_\alpha \,|\, L_V^{\Delta}k \in \mathcal{K}_\alpha\}$ and

$$\mathcal{Q}_x^V\left(g_0,\, g_1\right)k(\eta) = \int\limits_{\Gamma_0^2} e_\lambda\left(-g_0(x-\cdot);\, \xi^+\right)e_\lambda\left(-g_1(x-\cdot);\, \xi^-\right)k(\eta\cup\xi)\mathrm{d}\lambda(\xi).$$

We have

$$\frac{\partial r_t(\eta)}{\partial t} = \sum_{x\in\eta^+} r_t\left(\eta^+\backslash x,\, \eta^-\right)\frac{\partial \rho_t^+(x)}{\partial t} + \sum_{x\in\eta^-} r_t\left(\eta^+,\, \eta^-\backslash x\right)\frac{\partial \rho_t^-(x)}{\partial t}.$$

An easy computation (see, e.g., [8]) shows that r_t is (formally) a solution to

$$\frac{\partial r_t(\eta)}{\partial t} = L_V^{\Delta}r_t(\eta), \quad r_t|_{t=0} = r_0,$$

provided $(\rho_t^+,\, \rho_t^-)$ solve (52) and (53). By (2) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle G,\, r_t\rangle = \left\langle G,\, L_V^{\Delta}r_t\right\rangle = \left\langle \widehat{L}_V G,\, r_t\right\rangle,$$

which implies (51). □

The next statement establishes convergence of the scaled evolution to the limiting solutions.

Theorem 7 *Let $G \in \mathcal{L}_\alpha$, then, $\widehat{T}_{n,\mathrm{ren}}(t)G \longrightarrow \widehat{T}^V(t)G$ as $n \to \infty$. In particular, for any r_0 we have*

$$\left\langle G,\, \widehat{T}_{n,\mathrm{ren}}(t)^*r_0\right\rangle \longrightarrow \left\langle G,\, \widehat{T}_V(t)^*r_0\right\rangle, \quad n \to 0, \quad G \in \mathcal{L}_\alpha.$$

Proof Since $B_{bs}(\Gamma_0^2)$ is a core for $\widehat{L}_{n,\mathrm{ren}}$ and $\widehat{L}_V$, it suffices to show that

$$\widehat{L}_{n,\mathrm{ren}}G \longrightarrow \widehat{L}_V G, \quad n \to \infty$$

holds in $\mathcal{L}_\alpha$, for any $G \in B_{bs}(\Gamma_0^2)$. But this follows, by dominated convergence, similarly to [8]. □

6 Examples

6.1 Dynamical Widom–Rowlinson Model

The model with $\psi^\pm = 0$ and $q^\pm = 0$ is known as the dynamical Widom–Rowlinson model. It was introduced in [30] and an extension is provided in [19,20]. It has been recently studied in [11] where a local evolution of correlation functions was constructed. Moreover, its kinetic limit was derived and for the solution (one particle density) of the kinetic equation the dynamical phase transition was shown. In contrast to [11] we obtain a global evolution of states and show that in a certain regime there exists only one invariant state and the non-equilibrium evolution of states is ergodic. Below we consider an extension of this model with additional mutations.

Assume that $\psi^+ = \psi^- = 0$ and $\kappa^+ = \kappa^- = 0$. The (formal) Markov generator is for $F \in \mathcal{FP}(\Gamma^2)$ given by

$$\begin{aligned}
(LF)(\gamma) &= \sum_{x\in\gamma^+} \left(F\left(\gamma^+\backslash x,\, \gamma^-\right) - F(\gamma)\right) + \sum_{x\in\gamma^-} \left(F\left(\gamma^+,\, \gamma^-\backslash x\right) - F(\gamma)\right)\\
&\quad + z^+ \int\limits_{\mathbb{R}^d} e^{-E_{\phi^-}(x,\gamma^-)} \left(F\left(\gamma^+ \cup x,\, \gamma^-\right) - F(\gamma)\right) \mathrm{d}x\\
&\quad + z^- \int\limits_{\mathbb{R}^d} e^{-E_{\phi^+}(x,\gamma^+)} \left(F\left(\gamma^+,\, \gamma^- \cup x\right) - F(\gamma)\right) \mathrm{d}x\\
&\quad + q^+ \sum_{x\in\gamma^+} e^{-E_{\tau^+}(x,\gamma^-)} \left(F\left(\gamma^+\backslash x,\, \gamma^- \cup x\right) - F(\gamma)\right)\\
&\quad + q^- \sum_{x\in\gamma^-} e^{-E_{\tau^-}(x,\gamma^+)} \left(F\left(\gamma^+ \cup x,\, \gamma^-\backslash x\right) - F(\gamma)\right).
\end{aligned}$$

Let $\rho \geq 1$ be a locally bounded function, $\alpha = (\alpha^+,\, \alpha^-) \in \mathbb{R}^2$ and assume that the potentials are such that condition (A) is fulfilled. Concerning condition (B) we obtain

$$\begin{aligned}
\beta(\alpha;\, \eta) &= e^{-\alpha^+} z^+ \sum_{x\in\eta^+} e^{-E_{\phi^-}(x,\eta^+\backslash x)} C_{\phi^-}\left(x,\, \alpha^+\right)\\
&\quad + e^{-\alpha^-} z^- \sum_{x\in\eta^-} e^{-E_{\phi^+}(x,\eta^-\backslash x)} C_{\phi^+}\left(x,\, \alpha^-\right)\\
&\quad + q^+ e^{\alpha^+-\alpha^-} \sum_{x\in\eta^-} e^{-E_{\tau^+}(x,\eta^-\backslash x)} C_{\tau^+}\left(x,\, \alpha^-\right)\\
&\quad + q^- e^{\alpha^--\alpha^+} \sum_{x\in\eta^+} e^{-E_{\tau^-}(x,\eta^+\backslash x)} C_{\tau^-}\left(x,\, \alpha^+\right)\\
&\quad + q^+ \sum_{x\in\eta^+} e^{-E_{\tau^+}(x,\eta^-)} \left(C_{\tau^+}\left(x,\, \alpha^-\right) - 1\right)\\
&\quad + q^- \sum_{x\in\eta^-} e^{-E_{\tau^-}(x,\eta^+)} \left(C_{\tau^-}\left(x,\, \alpha^+\right) - 1\right)\\
&\leq \sum_{x\in\eta^+} \left(e^{-\alpha^+} z^+ C_{\phi^-}\left(x,\, \alpha^+\right) + q^- e^{\alpha^--\alpha^+} C_{\tau^-}\left(x,\, \alpha^+\right)\right)\\
&\quad + \sum_{x\in\eta^-} \left(e^{-\alpha^-} z^- C_{\phi^+}\left(x,\, \alpha^-\right) + q^+ e^{\alpha^+-\alpha^-} C_{\tau^+}\left(x,\, \alpha^-\right)\right)\\
&\quad + q^+ \sum_{x\in\eta^+} e^{-E_{\tau^+}(x,\eta^-)} \left(C_{\tau^+}\left(x,\, \alpha^-\right) - 1\right)\\
&\quad + q^- \sum_{x\in\eta^-} e^{-E_{\tau^-}(x,\eta^+)} \left(C_{\tau^-}\left(x,\, \alpha^+\right) - 1\right).
\end{aligned}$$

Note that we have

$$M(\eta) = |\eta^+| + |\eta^-| + q^+ \sum_{x\in\eta^+} e^{-E_{\tau^+}(x,\eta^-)} + q^- \sum_{x\in\eta^-} e^{-E_{\tau^-}(x,\eta^+)}.$$

Hence there exists $a(\alpha) \in (0,\ 1)$ such that

$$\beta(\alpha;\ \eta) \le a(\alpha)M(\eta), \quad \eta \in \Gamma_0^2,$$

provided the following conditions are fulfilled

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d} \left(e^{-\alpha^+} z^+ C_{\phi^-}\left(x,\ \alpha^-\right) + q^- e^{\alpha^- - \alpha^+} C_{\tau^-}\left(x,\ \alpha^+\right)\right) < 1,\\
&\sup_{x\in\mathbb{R}^d} \left(e^{-\alpha^-} z^- C_{\phi^+}\left(x,\ \alpha^+\right) + q^+ e^{\alpha^+ - \alpha^-} C_{\tau^+}\left(x,\ \alpha^-\right)\right) < 1,\\
&\sup_{x\in\mathbb{R}^d} C_{\tau^+}\left(x,\ \alpha^-\right) < \frac{1+q^+}{q^+},\\
&\sup_{x\in\mathbb{R}^d} C_{\tau^-}\left(x,\ \alpha^+\right) < \frac{1+q^-}{q^-}.
\end{aligned}$$

The second pair of conditions is satisfied, provided

$$\sup_{x\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \left(1 - e^{-\tau^\pm(x-y)}\right)\rho(y)\mathrm{d}y < e^{-\alpha^\mp} \ln\left(\frac{1+q^\pm}{q^\pm}\right),$$

i.e., $\alpha^\pm$ are large enough. The first two conditions are satisfied, if the activities $z^\pm$, $q^\pm$ are small enough. In such a case conditions (V1) and (V2) are fulfilled. The kinetic equations are given by

$$\begin{aligned}
\frac{\partial \rho_t^+(x)}{\partial t} &= -\rho_t^+(x) + z^+ e^{-(\phi^- * \rho_t^-)(x)} - q^+ e^{-(\tau^+ * \rho_t^-)(x)} \rho_t^+(x) + q^- e^{-(\tau^- * \rho_t^+)(x)} \rho_t^-(x),\\
\frac{\partial \rho_t^-(x)}{\partial t} &= -\rho_t^-(x) + z^- e^{-(\phi^+ * \rho_t^+)(x)} - q^- e^{-(\tau^- * \rho_t^+)(x)} \rho_t^-(x) + q^+ e^{-(\tau^+ * \rho_t^-)(x)} \rho_t^+(x).
\end{aligned}$$

Let us consider, for simplicity, the case where the dynamics for the $\pm$ particles is determined by the same parameters, i.e., $z^\pm = z$, $q^\pm = q$, $\tau^\pm = \tau$ and $\phi^\pm = \phi$. In such a case let $\alpha^\pm = \alpha$. Above conditions reduce to the pair of conditions

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d} \left(z e^{-\alpha} C_\phi(x,\ \alpha) + q C_\tau(x,\ \alpha)\right) < 1,\\
&\sup_{x\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \left(1 - e^{-\tau(x-y)}\right)\rho(y)\mathrm{d}y < e^{-\alpha} \ln\left(\frac{1+q}{q}\right).
\end{aligned}$$

Then again conditions (V1) and (V2) are fulfilled and the kinetic equations take the simple form

$$\begin{aligned}
\frac{\partial \rho_t^+(x)}{\partial t} &= -\rho_t^+(x) + z e^{-(\phi * \rho_t^-)(x)} - q e^{-(\tau * \rho_t^-)(x)} \rho_t^+(x) + q e^{-(\tau * \rho_t^+)(x)} \rho_t^-(x),\\
\frac{\partial \rho_t^-(x)}{\partial t} &= -\rho_t^-(x) + z^- e^{-(\phi^+ * \rho_t^+)(x)} - q e^{-(\tau * \rho_t^+)(x)} \rho_t^-(x) + q e^{-(\tau * \rho_t^-)(x)} \rho_t^+(x).
\end{aligned}$$

6.2 Pure Mutation Dynamics

Suppose that particles are only allowed to change their type, i.e., consider the dynamics for the operator V from the introduction. Moreover, assume that $\kappa^+ = \tau^-$ and $\kappa^- = \tau^+$. The Markov (pre-)generator is therefore given by

$$(VF)(\gamma) = q^+ \sum_{x\in\gamma^+} e^{-E_{\kappa^+}(x,\gamma^+\backslash x)} e^{-E_{\kappa^-}(x,\gamma^-)} \left(F\left(\gamma^+\backslash x,\ \gamma^- \cup x\right) - F(\gamma)\right)$$
$$+ q^- \sum_{x\in\gamma^-} e^{-E_{\kappa^-}(x,\gamma^-\backslash x)} e^{-E_{\kappa^+}(x,\gamma^+)} \left(F\left(\gamma^+ \cup x,\ \gamma^-\backslash x\right) - F(\gamma)\right).$$

The corresponding dynamics describes the time evolution of spins associated with randomly distributed particles. We suppose that the potentials $\kappa^\pm \geq 0$ are measurable and symmetric. Moreover, assume that there exists a measurable, locally bounded function $\rho\colon \mathbb{R}^d \longrightarrow [1,\ \infty)$ such that $(1 - e^{-\kappa^\pm(x-\cdot)}) \cdot \rho$ are integrable for any $x \in \mathbb{R}^d$, i.e., condition (A) holds. Suppose that there exists $\alpha \in \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}^d} C_{\kappa^+}(x,\ \alpha) C_{\kappa^-}(x,\ \alpha) < \frac{1+q^\pm}{q^+ + q^-}.$$

Then assumption (B) holds for $\alpha^+ = \alpha^- = \alpha$. Moreover, conditions (V1) and (V2) are satisfied, provided

$$\sup_{x\in\mathbb{R}^d} \left(\kappa^+ * \rho\right)(x) \left(\kappa^- * \rho\right)(x) < \frac{1+q^\pm}{q^+ + q^-}.$$

Let μ_0 be an initial state and μ_t be its time evolution on Γ^2. It is not difficult to see that

$$\int_{\Gamma^2} H\left(\gamma^+ \cup \gamma^-\right) \mu_0\left(\mathrm{d}\gamma^+,\ \mathrm{d}\gamma^-\right) = \int_{\Gamma^2} H\left(\gamma^+ \cup \gamma^-\right) \mu_t\left(\mathrm{d}\gamma^+,\ \mathrm{d}\gamma^-\right), \quad t \geq 0,$$

holds for any H polynomially bounded cylinder function on the one-component configuration space $\Gamma := \Gamma^+$. This relation shows that the distribution of the particles in the space $\mathbb{R}^d$ is conserved in the time evolution. The kinetic equation is given by

$$\frac{\partial \rho_t^+(x)}{\partial t} = -q^+ e^{-(\kappa^+ * \widehat{\rho}_t)(x)} \rho_t^+(x) + q^- e^{-(\kappa^- * \widehat{\rho}_t)(x)} \rho_t^-(x),$$
$$\frac{\partial \rho_t^-(x)}{\partial t} = -q^- e^{-(\kappa^- * \widehat{\rho}_t)(x)} \rho_t^-(x) + q^+ e^{-(\kappa^+ * \widehat{\rho}_t)(x)} \rho_t^+(x),$$

where $\widehat{\rho} := \rho_t^+ + \rho_t^-$ is the total particle density. Note that $\widehat{\rho}$ is preserved in the time evolution.

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Appendix

Proof of Lemma 1

Let $G \geq 0$, then (for the notation see [22])

$$\begin{aligned}
&\int\limits_{\Gamma_0}\int\limits_{\Gamma_0} G(\xi,\ \eta,\ \eta \cup \xi)\mathrm{d}\lambda(\eta)\mathrm{d}\lambda(\xi) \\
&\quad = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n!}\frac{1}{m!}\int\limits_{(\mathbb{R}^d)^n}\int\limits_{(\mathbb{R}^d)^m} G\left(\{x\}_1^n,\ \{x\}_{n+1}^{n+m},\ \{x\}_1^{n+m}\right)\mathrm{d}x_1^{n+m} \\
&\quad = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{n}\binom{n}{m}\int\limits_{(\mathbb{R}^d)^l} G\left(\{x\}_1^m,\ \{x\}_{m+1}^n,\ \{x\}_1^n\right)\mathrm{d}x_1^n \\
&\quad = \int\limits_{\Gamma_0}\sum_{\xi\subset\eta} G(\xi,\ \eta\backslash\xi,\ \eta)\mathrm{d}\lambda(\eta).
\end{aligned}$$

This shows the assertion in the case $G \geq 0$. For the general case, let $G = G^+ - G^-$ with $G^{\pm} \geq 0$. Then

$$\int\limits_{\Gamma_0}\int\limits_{\Gamma_0} G^{\pm}(\xi,\ \eta,\ \eta \cup \xi)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_0}\sum_{\xi\subset\eta} G^{\pm}(\xi,\ \eta\backslash\xi,\ \eta)\mathrm{d}\lambda(\eta).$$

In particular the left-hand side is finite if and only if the right-hand side is finite. (2) can be checked by using $G = G^+ - G^-$ and above equality.

Proof of Lemma 10

It suffices to show that

$$\int\limits_{\Gamma_0^2} G(\eta)\widetilde{k}(\eta)\mathrm{d}\lambda(\eta) \geq 0,$$

for any $G \in B_{bs}(\Gamma_0^2)$ such that $KG \geq 0$. By approximation it suffices to consider G of the form

$$G(\eta) = \sum_{i=1}^{N} b_i e_\lambda\left(g_i^+;\ \eta^+\right)e_\lambda\left(g_i^-;\ \eta^-\right),\quad N \in \mathbb{N},\quad b_i \in \mathbb{C},\quad g_i^{\pm} \in C_c\left(\mathbb{R}^d;\ \mathbb{C}\right),$$

with $KG \geq 0$. Here $C_c(\mathbb{R}^d;\ \mathbb{C})$ denotes the space of continuous functions, having compact support in $\mathbb{R}^d$ with values in $\mathbb{C}$. For such functions G we obtain

$$G(\eta)e_\lambda\left(f^+;\ \eta^+\right)e_\lambda\left(f^-;\ \eta^-\right) = \sum_{i=1}^{N} b_i e_\lambda\left(g_i^+ f^+;\ \eta^+\right)e_\lambda\left(g_i^- f^-;\ \eta^-\right).$$

Let $\Lambda \subset \mathbb{R}^d$ be the unions of the supports of $g_i^{\pm}$, $i = 1, \dots, N$ and $A_{\pm} := \max\{\|g_i^{\pm}\|_\infty | i = 1, \dots, N\}$. Then

$$\int_{\Gamma_0^2} \left| e_\lambda \left(g_i^+ f^+;\ \eta^+ \right) e_\lambda \left(g_i^- f^-;\ \eta^- \right) \right| k(\eta) \mathrm{d}\lambda(\eta)$$
$$\le \int_{\Gamma_0^2} A_+^{|\eta^+|} A_-^{|\eta^-|} e_\lambda \left(\mathbb{1}_\Lambda;\ \eta^+ \right) e_\lambda \left(\mathbb{1}_\Lambda;\ \eta^- \right) k(\eta) \mathrm{d}\lambda(\eta) < \infty.$$

Let $\mu \in \mathcal{P}$ be such that k is its correlation function. Then $\prod_{x \in \gamma^+} (1 + f^+(x) g_i^+(x)) \prod_{x \in \gamma^-} (1 + f^-(x) g_i^-(x))$ is integrable w.r.t. μ. Moreover, since $g_i^{\pm}$ have compact support, it follows that

$$\prod_{x \in \gamma^+} \left(1 + f^+(x) g_i^+(x) \right) \prod_{x \in \gamma^-} \left(1 + f^-(x) g_i^-(x) \right)$$
$$= \prod_{x \in \gamma^+ \cap \Lambda} \left(1 + f^+(x) g_i^+(x) \right) \prod_{x \in \gamma^- \cap \Lambda} \left(1 + f^-(x) g_i^-(x) \right),$$

and hence

$$\int_{\Gamma^2} \prod_{x \in \gamma^+} \left(1 + f^+(x) g_i^+(x) \right) \prod_{x \in \gamma^-} \left(1 + f^-(x) g_i^-(x) \right) \mathrm{d}\mu(\gamma)$$
$$= \int_{\Gamma^2_{\Lambda,\Lambda}} \prod_{x \in \gamma^+} \left(1 + f^+(x) g_i^+(x) \right) \prod_{x \in \gamma^-} \left(1 + f^-(x) g_i^-(x) \right) \mathrm{d}\mu^{\Lambda,\Lambda}(\gamma).$$

We obtain

$$\begin{aligned}
\int_{\Gamma_0^2} G(\eta) \widetilde{k}(\eta) \mathrm{d}\lambda(\eta) &= \sum_{i=1}^N b_i \int_{\Gamma_0^2} e_\lambda \left(f^+ g_i^+;\ \eta^+ \right) e_\lambda \left(f^- g_i^-;\ \eta^- \right) k(\eta) \mathrm{d}\lambda(\eta) \\
&= \sum_{i=1}^N b_i \int_{\Gamma^2} \prod_{x \in \gamma^+} \left(1 + f^+(x) g_i^+(x) \right) \prod_{x \in \gamma^-} \left(1 + f^-(x) g_i^-(x) \right) \mathrm{d}\mu(\gamma) \\
&= \sum_{i=1}^N b_i \int_{\Gamma^2_{\Lambda,\Lambda}} \prod_{x \in \gamma^+} \left(1 + f^+(x) g_i^+(x) \right) \prod_{x \in \gamma^-} \left(1 + f^-(x) g_i^-(x) \right) \mathrm{d}\mu^{\Lambda,\Lambda}(\gamma).
\end{aligned}$$

Introduce the notation $e_\lambda(h;\ \beta^{\pm}) := e_\lambda(h^+;\ \beta^+) e_\lambda(h^-;\ \beta^-)$ for $h^{\pm}\colon \mathbb{R}^d \longrightarrow \mathbb{C}$ and $\beta = (\beta^+,\ \beta^-) \in \Gamma_0^2$. By $1 + f^{\pm} g_i^{\pm} = (1 - f^{\pm}) + f^{\pm}(1 + g_i^{\pm})$ we get for the integrand

$$\prod_{x \in \gamma^+} \left(1 + f^+(x) g_i^+(x) \right) \prod_{x \in \gamma^-} \left(1 + f^-(x) g_i^-(x) \right)$$
$$= \sum_{\xi^{\pm} \subset \gamma^{\pm}} e_\lambda \left(1 - f^{\pm};\ \xi^{\pm} \right) e_\lambda \left(f^{\pm} \left(1 + g_i^{\pm} \right);\ \gamma^{\pm} \backslash \xi^{\pm} \right).$$

This implies

$$\int\limits_{\Gamma_0^2} G(\eta)\widetilde{k}(\eta)\mathrm{d}\lambda(\eta) = \int\limits_{\Gamma_{\Lambda,\Lambda}^2} \sum_{\xi^\pm\subset\gamma^\pm} e_\lambda\left(1-f^\pm;\,\xi^\pm\right) e_\lambda\left(f^\pm;\,\gamma^\pm\backslash\xi^\pm\right)$$

$$\times \sum_{i=1}^{N} b_i e_\lambda\left(1+g_i^\pm;\,\gamma^\pm\backslash\xi^\pm\right) \mathrm{d}\mu^{\Lambda,\Lambda}(\gamma)$$

$$= \int\limits_{\Gamma_{\Lambda,\Lambda}^2} \sum_{\xi^\pm\subset\gamma^\pm} e_\lambda\left(1-f^\pm;\,\xi^\pm\right) e_\lambda\left(f^\pm;\,\gamma^\pm\backslash\xi^\pm\right)(KG)(\gamma\backslash\xi)\mathrm{d}\mu^{\Lambda,\Lambda}(\gamma)$$

$$\geq 0.$$

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