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On Unique Ergodicity in Nonlinear Stochastic Partial Differential Equations

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Abstract We illustrate how the notion of asymptotic coupling provides a flexible and intuitive framework for proving the uniqueness of invariant measures for a variety of stochastic partial differential equations whose deterministic counterpart possesses a finite number of determining modes. Examples exhibiting parabolic and hyperbolic structure are studied in detail. In the later situation we also present a simple framework for establishing the existence of invariant measures when the usual approach relying on the Krylov–Bogolyubov procedure and compactness fails.

Keywords Ergodicity · Coupling · SPDEs · Stochastic fluid equations

1 Introduction

The goal of this work is to give a simple exposition, distillation and refinement of methods developed over the last decade and a half to analyze ergodicity in nonlinear stochastic PDEs with an additive forcing. To this end, we detail a number of examples which highlight different difficulties and help clarify the domain of applicability and flexibility of the core ideas. For

Dedicated to David Ruelle and Yakov G. Sinai on the occasion of their 80th birthdays with thanks for all they have and will teach us.

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each example, we provide a simple proof of unique ergodicity with a presentation which should be adaptable to other settings. Though our calculations often lay the ground work for stronger results such as convergence of transition measures, exponential mixing or spectral gaps, we resist the urge to expand the discussions here, and opt to make the uniqueness arguments as simple as possible. Although some of our examples are close to those in the existing literature, many are not and require an involved analysis to develop the required PDE estimates. For all the situations considered we present a relatively succinct proof of unique ergodicity, particularly when compared with existing expositions.

The feature common to all of our examples is the existence of a finite number of determining modes in the spirit of [24] and a sufficiently rich stochastic forcing structure to ensure that all determining modes are directly excited. There has been a larger body of work in these directions in recent years beginning with [3,39,64] and continuing with [17,21,31–35,40,41,52,53,65] to name a few. Very roughly speaking, the presence of noise terms allows for the 'coupling' of all the relevant large scales of motion, which contain any unstable directions. The small scales, which are then provably stable, contract asymptotically in time.

The heart of the calculations presented below are very much in the spirit of [6,53,64] although the approach here does not pass through a reduction to an equation with memory. In that sense our presentation is closer to [52,53] which decomposes the future starting from an initial condition and proves a coupling along a subset of futures of positive probability. There however, the analysis was complicated by an attempt at generality and the desire to prove exponential convergence. In [31], which followed [52], the control used to produce the coupling drove all of the modes together only asymptotically.[1] In particular, [31] did not force the large scales ("low modes") to match exactly as was the case in previous works. While this leads to slightly weaker results, it can be conceptually simpler in some settings. In parallel to these works, two other groups developed their own takes on these same questions. One vein of work is contained in [3,4] and the other beginning in [39,40] is nicely summarized in [41].

We proceed through the lens of a variation on the 'asymptotic coupling' framework from [35] (and equally in the spirit of [31,53,64]). This formalism allows us to highlight the underlying flexibility and the wide range applicability of the above mentioned body of work by treating a number of interesting systems simply and without extraneous complications dictated by previous abstract frameworks. Indeed, the examples selected below were chosen to underline a variety of commonly encountered difficulties which can be surmounted, including the lack of exponential moments of critical norms, the lack of well-posedness in the space where the convergence analysis is performed, or, as in the case of weakly damped hyperbolic systems, situations in which the dissipative mechanism is uniform across (spatial) scales.

Our first example is the most classical: the 2-D Navier–Stokes equations (NDEs) posed on a domain. Here the presence of boundaries prevents a closed vorticity-formulation and hence interferes with higher order constraints of motion resulting in a 'critical' problem. This criticality makes attaining the gradient bounds on the Markov-semigroup difficult. Such bounds are central to the infinitesimal approach of Asymptotic Strong Feller developed in [32,34] to address the hypoelliptic setting. As currently presented, the Asymptotic Strong Feller approach does not localize easily. However the analysis presented in [52,53,64] was localized from the start, through the estimates used there do not apply to this setting directly due to the lack of a vorticity formulation. Our presentation is simplified in comparison to previous works, producing an immediate and transparent proof of unique ergodicity.

We then turn to address two other interesting dissipative equations arising from fluid systems which have received much less attention in the SPDE literature. The first example

[1] [31] also coined the term 'asymptotic coupling' which was later defined more generally in [34].

is provided by the so-called hydrostatic Navier–Stokes (or simplified Primitive Equations) arising for fluids spanning geophysical scales and therefore of interest in climate and weather applications. See [59] and Sect. 3.2 for extensive further references. Our second example provides a streamlined analysis of the so-called fractionally dissipative stochastic Euler equation, introduced recently in [8]. The theme shared by these examples is that the non-linearity is relatively stronger than the dissipative structure in comparison to the 2D NSEs. This leads to a situation in which the continuous dependence of solutions, and hence the Foias-Prodi estimate, is tractable only when carried out in a weaker topology. As made explicit in Corollary 2.1 (see below), performing this convergence analysis in a weaker topology is sufficient to provide a suitable asymptotic coupling.

The last two examples are illustrative of the difficulties encountered in studying weakly damped hyperbolic systems. The first equation is a variation of the damped Euler–Voigt equation, a hyperbolic regularization of the Euler equations. As a second example we address the weakly damped stochastic Sine–Gordon equations. Here rather than using the parabolic structure to produce an effective large damping at small scales, higher-order regularity constraints restrict the strength of transfer to high frequencies and allow us to obtain stronger control on the non-linear terms.

Regarding the Euler–Voigt equation, it is notable that the existence of solutions also requires special consideration. For this stage in the analysis, the lack of any obvious finite time smoothing mechanism or alternatively of any invariant quantities in spaces more regular than those for which the equations are well posed suggest that the usual approach relying on the Krylov–Bogolyubov procedure and compactness may fail. To address this difficulty we show how a limiting procedure involving a parabolic regularization can be used to guarantee the existence of stationary states. This stage of the analysis makes use of another abstract criteria which we think will prove useful in other future applications in hyperbolic SPDEs.

Note that in all of these examples we will focus our attention exclusively on the "effectively elliptic" setting where all of the "determining modes", or "presumptively unstable directions", are directly forced stochastically. The hypoelliptic setting, where most of the determining modes are not directly forced and the drift is used to spread the noise to all of the determining modes, remains unexplored for all of the examples we discuss and will almost certainly require significantly more machinery, though much is provided by [32,34]. We emphasize that while we may extend our analysis in many of the examples to obtain convergence rates or possibly spectral gaps, we only explicitly address the question of unique ergodicity here in order to keep the exposition minimal and to maintain the broadest range of applicability. That being said, the analysis here lays the framework for obtaining convergence rates using the ideas outlined in [31,52,53].

The manuscript is organized as follows. In Sect. 2, we recall the asymptotic coupling framework and introduce several refinements which were not made explicit in previous work. We conclude this section by recalling a specific form the Girsanov theorem which will be crucially used to apply the abstract asymptotic coupling in each of the examples and give a general recipe for building asymptotic couplings. In Sect. 3, we successively treat each of the SPDE examples described, proving the existence and uniqueness of an ergodic invariant measure in each case. The appendix is devoted to proving some abstract lemmata which we use to prove the existence of an invariant measure for Euler–Voigt system considered in Sect. 3.4.

2 An Abstract Framework for Unique Ergodicity

We now introduce the simple abstract framework which we use for proving unique ergodicity. After briefly recalling some generalities about Markov processes and ergodic theory we present a refinement of the asymptotic coupling arguments developed in [35]. The approach is very much in the spirit of the general results given in this previous work but the packaging here is a little different. We emphasize the exact formulation we will use and make explicit the ability to establish convergence in a different topology than the one associated to the space on which the Markov dynamics are defined (see Corollary 2.1 and Remark 2.2 below). Furthermore, to better illuminate the connection of these ideas to those in [38], we weaken the condition for convergence at infinity to one only involving time averages (even if we will not make direct use of this generalization here). The proof given is essentially the same one given in [35] which in turn closely mirrors the proofs in [64] and [53].

In the following section, we also recall a form of the Girsanov theorem which is crucially used to apply our abstract results. In particular this formulation is convenient for establishing the absolute continuity on path space required by the abstract framework. In the final section, we giving a general recipe for building an asymptotic coupling leveraging the discussions of the preceding two sections.

2.1 Generalities

Let P be a family of Markov transition kernels on a Polish space H with a metric ρ. Given a bounded, measurable function $\phi\colon H \to \mathbb{R}$, we define a new function $P\phi$ on H by $P\phi(u) = \int_H \phi(v)P(u, dv)$. For any probability measure ν on H, we take νP to be the probability measure on H defined according to $\nu P(A) = \int_H P(u, A)\nu(du)$ for any measurable $A \subset H$.[2] Then $\nu P\phi$ is simply the expected value of ϕ evaluated on one step of the Markov chain generated by P when the initial condition is distributed as ν.

An invariant measure for P is a probability measure μ which is a fixed point for P in that $\mu = \mu P$. Since starting the Markov chain with an initial condition distributed as μ produces a stationary sequence of random variables, μ is also called a stationary measure for the Markov Chain generated by P. An invariant measure μ is ergodic if any set A which is invariant for P relative to μ has $\mu(A) \in \{0, 1\}$.[3] Notice that the set $\mathcal{I}$ of invariant measures for P is a convex set. It is classical that an invariant measure is ergodic if and only if it is an extremal point of $\mathcal{I}$. Furthermore, different ergodic measures are mutually singular. In this sense, the ergodic invariant measures form the "atoms" for the set of invariant measures as each invariant measure can be written as a convex combination of ergodic invariant measures and the ergodic invariant measures cannot be decomposed as a convex combination of other invariant measures.

We now lift any probability measure μ on H to a canonical probability measure on the pathspace representing trajectories of the Markov process generated by P. We denote the pathspace over H by

$$H^{\mathbf{N}} = \{u : \mathbf{N} \to H\} = \{u = (u_1, u_2, \dots) : u_i \in H\}$$

[2] These left and right actions of P are consistent with the case when H is the finite set $\{1, \dots, n\}$ and P is a $n \times n$ matrix given by $P_{ik} = \mathbb{P}(\text{ transition } i \to j)$. Then $\phi \in \mathbb{R}^n$ is a column vector and $\nu \in \mathbb{R}^n$ is a row vector whose nonnegative entries sum to one. As such, $P\phi$ and νP have the standard meaning given my matrix multiplication.

[3] Recall that a measurable set A is invariant for P relative to μ if $P(u, A) = 1$ for μ-a.e. $u \in A$ and if $P(u, A) = 0$ for μ-a.e. $u \notin A$. More compactly this say that A is invariant for P relative μ if $P\mathbb{1}_A = \mathbb{1}_A$, μ-a.e.

where $\mathbf{N} = \{1, 2, \dots\}$. The suspension of any initial measure ν on H to $H^{\mathbf{N}}$ is denoted by $\nu P^{\mathbf{N}}$. This pathspace measure $\nu P^{\mathbf{N}}$ is defined in the standard way from the Kolmogorov extension theorem by defining the probability of the cylinder sets $\{U_1 \in A_1, U_2 \in A_2, \dots, U_n \in A_n\}$ where n is arbitrary. Here A_n are any measurable subsets of H whereas U_0 is distributed as ν and given U_{k-1}, the U_k's are distributed as $P(U_{k-1}, \cdot)$. The measure $\nu P^{\mathbf{N}}$ should be understood as the measure on the present state and the entire future trajectory of the Markov chain P starting from the initial distribution ν. The canonical action of the left shift map $\theta(u)_j = u_{j+1}$ on $H^{\mathbf{N}}$ corresponds to taking one step under P. Hence under θ, the state tomorrow becomes the state today and all other states shift one day closer to the present.

In general a measure M on $H^{\mathbf{N}}$ is invariant under the shift θ if $M\theta^{-1} = M$, where $M\theta^{-1}$ is defined by $M\theta^{-1}(A) = M(\theta^{-1}(A))$ for all measurable $A \subset H^{\mathbf{N}}$. [4] As in the previous setting, an invariant measure is ergodic if and only if it is an extremal point of the set of invariant measures for θ on $H^{\mathbf{N}}$.

The notions of ergodic and extremal measures on H and $H^{\mathbf{N}}$ are self consistent in that the following statements are equivalent:

(i) a measure μ on H is an ergodic invariant measure for P
(ii) $\mu P^{\mathbf{N}}$ is an ergodic invariant measure on $H^{\mathbf{N}}$ relative to the shift map θ
(iii) $\mu P^{\mathbf{N}}$ is an extremal point for the set of θ-invariant measures on $H^{\mathbf{N}}$
(iv) μ is an extremal point for the set of P-invariant measures on H.

For further discussion of all these ergodic theory generalities see [18,23,36]

2.2 Equivalent Asymptotic Couplings and Unique Ergodicity

We say that a probability measure Γ on $H^{\mathbf{N}} \times H^{\mathbf{N}}$ is an *asymptotically equivalent coupling* of two measures M_1 and M_2 on $H^{\mathbf{N}}$ if $\Gamma\Pi_i^{-1} \ll M_i$, for $i = 1, 2$, where $\Pi_1(u, v) = u$ and $\Pi_2(u, v) = v$. We will write $\tilde{\mathcal{C}}(M_1, M_2)$ for the set of all such asymptotically equivalent couplings.

Given any bounded (measurable) function $\phi\colon H \to \mathbb{R}$, we define $\bar{D}_\phi \subset H^{\mathbf{N}} \times H^{\mathbf{N}}$ by

$$\bar{D}_\phi := \left\{(u, v) \in H^{\mathbf{N}} \times H^{\mathbf{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\phi(u_k) - \phi(v_k)) = 0\right\}. \tag{2.1}$$

On the other hand a set $\mathcal{G}$ of bounded, real-valued, measurable functions on H is said to *determine measures* if, whenever $\mu_1, \mu_2 \in Pr(H)$ are such that $\int \phi d\mu_1 = \int \phi d\mu_2$ for all $\phi \in \mathcal{G}$, then $\mu_1 = \mu_2$.

Theorem 2.1 *Let $\mathcal{G}\colon H \to \mathbb{R}$ be collection of functions which determines measures. Assume that there exists a measurable $H_0 \subset H$ such that for any $(u_0, v_0) \in H_0 \times H_0$ and any $\phi \in \mathcal{G}$ there exists a $\Gamma = \Gamma(u_0, v_0, \phi) \in \tilde{\mathcal{C}}(\delta_{u_0} P^{\mathbf{N}}, \delta_{v_0} P^{\mathbf{N}})$ such that $\Gamma(\bar{D}_\phi) > 0$. Then there exist at most one ergodic invariant measure μ for P with $\mu(H_0) > 0$. In particular if $H_0 = H$, then there exists at most one, and hence ergodic, invariant measure.*

Remark 2.1 At first glance it might be surprising that equivalence is sufficient to determine the long time statistics. However since the Birkhoff Ergodic theorem implies that time averages along typical trajectories converge to the integral against an ergodic invariant measure, one only needs to draw typical infinite trajectories. From this it is clear that absolutely continuity on the entire future trajectory indexed by $\mathbf{N}$ is critical and can not be replaced with

[4] Here a measurable set A is invariant for θ relative to a probability measure M on $H^{\mathbf{N}}$ if $\theta^{-1}(A) = A \bmod M$, which is to say the symmetric difference $\theta^{-1}(A)\Delta A$ is measure zero for M.

absolutely continuity on $\{1, 2, \ldots, N\}$ for all $N \in \mathbf{N}$. Since absolutely continuous measures have the same paths, only with different weights, we see that absolutely continuous measures are sufficient to ensure one is drawing typical trajectories.

Proof of Theorem 2.1 Assume that there are two ergodic invariant probability measures μ and ν on H such that both $\mu(H_0) > 0$ and $\nu(H_0) > 0$. Fix any $\phi \in \mathcal{G}$. By Birkhoff's ergodic theorem there exists sets $A^\mu_\phi, A^\nu_\phi \subset H^\mathbf{N}$ such that $\mu P^\mathbf{N}(A^\mu_\phi) = \nu P^\mathbf{N}(A^\nu_\phi) = 1$ and such that if $u \in A^\mu_\phi$ and $v \in A^\nu_\phi$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(u_k) = \int_H \phi d\mu \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(v_k) = \int_H \phi d\nu\,. \tag{2.2}$$

Define $A^\mu_\phi(u_0) = \{\tilde{u} = (\tilde{u}_0, \tilde{u}_1, \ldots) \in A^\mu_\phi : \tilde{u}_0 = u_0\}$ and $A^\nu_\phi(u_0)$ analogously. Notice that $\delta_{u_0} P^\mathbf{N}(A^\mu_\phi(u_0)) = \delta_{u_0} P^\mathbf{N}(A^\mu_\phi)$ for any $u_0 \in H$. Hence, by Fubini's theorem we have, for μ-a.e. $u_0 \in H$, that $\delta_{u_0} P^\mathbf{N}(A^\mu_\phi(u_0)) = \delta_{u_0} P^\mathbf{N}(A^\mu_\phi) = 1$, and that $\delta_{v_0} P^\mathbf{N}(A^\nu_\phi(v_0)) = 1$ for ν-a.e. $v_0 \in H$.

Since we have presumed that H_0 is non-trivial relative to both μ, ν, we may now select a pair of initial conditions $u_0, v_0 \in H_0$ such that $\delta_{u_0} P^\mathbf{N}(A^\mu_\phi(u_0)) = \delta_{v_0} P^\mathbf{N}(A^\nu_\phi(v_0)) = 1$. Let Γ be the measure in $\tilde{\mathcal{C}}(\delta_{u_0} P^\mathbf{N}, \delta_{v_0} P^\mathbf{N})$ given in the assumptions of the Theorem corresponding to these initial points u_0, v_0 and the test function ϕ. Since $\Gamma\Pi_1^{-1} \ll \delta_{u_0} P^\mathbf{N}$ and $\Gamma\Pi_2^{-1} \ll \delta_{v_0} P^\mathbf{N}$, where again $\Pi_1(u, v) = u$ and $\Pi_2(u, v) = v$, we have that $\Gamma(A^\mu_\phi(u_0) \times A^\nu_\phi(v_0)) = 1$. Defining $\bar{D}'_\phi = \bar{D}_\phi \cap \big(A^\mu_\phi(u_0) \times A^\nu_\phi(v_0)\big)$, one has $\Gamma(\bar{D}'_\phi) > 0$ and thus we infer that $\bar{D}'_\phi$ is nonempty. Observe that for any $(u, v) \in \bar{D}'_\phi$, in view of (2.1), (2.2) and the definition of $\bar{D}'_\phi$, we have

$$\int_H \phi\, d\mu - \int_H \phi\, d\nu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\phi(u_k) - \phi(v_n)) = 0.$$

Since ϕ was an arbitrary function in $\mathcal{G}$, which was assumed to be sufficiently rich to determine measures, we conclude that $\mu_1 = \mu_2$ and the proof is complete. □

We next provide a simple corollary of Theorem 2.1 to be used directly in the examples provided below. To this end, we consider a possibly different distance $\tilde{\rho}$ on H and define

$$D_{\tilde{\rho}} := \left\{(u, v) \in H^\mathbf{N} \times H^\mathbf{N} : \lim_{n\to\infty} \tilde{\rho}(u_n, v_n) = 0\right\}.$$

We also consider the class of test functions

$$\mathcal{G}_{\tilde{\rho}} = \left\{\phi \in C_b(H) : \sup_{u\neq v} \frac{|\phi(u) - \phi(v)|}{\tilde{\rho}(u, v)} < \infty\right\}.$$

The corollary is as follows:

Corollary 2.1 *Suppose that $\mathcal{G}_{\tilde{\rho}}$ determines measures on (H, ρ) and assume that $D_{\tilde{\rho}}$ is a measurable subset of $H^\mathbf{N} \times H^\mathbf{N}$. If $H_0 \subset H$ is a measurable set such that for each pair $u_0, v_0 \in H_0$ there exists an element $\Gamma \in \tilde{\mathcal{C}}(\delta_{u_0} P^\mathbf{N}, \delta_{v_0} P^\mathbf{N})$ with $\Gamma(D_{\tilde{\rho}}) > 0$, then there exists at most one ergodic invariant measure μ with $\mu(H_0) > 0$.*

Proof With the observation that

$$D_{\tilde{\rho}} \subset \bar{D}_\phi \text{ for every } \phi \in \mathcal{G}_{\tilde{\rho}},$$

the desired result follows immediately from Theorem 2.1. □

Remark 2.2 The conditions imposed on $\tilde{\rho}$, $\mathcal{G}_{\tilde{\rho}}$ and $D_{\tilde{\rho}}$ in Corollary 2.1 are easily verified in practice. For instance, consider $H = H^1(\mathbb{T}^d)$ with the usual topology and take $\tilde{\rho}(u_0, v_0) = \|u_0 - v_0\|_{L^2}$, the $L^2(\mathbb{T}^d)$ distance. Since $\tilde{\rho}$ is continuous on H^1 we see that D_ρ is measurable. Furthermore, a simple mollification argument allows one to show that $\mathcal{G}_{\tilde{\rho}}$ is determining. Similar considerations will allow us to directly apply Corollary 2.1 in each of the examples below.

2.3 Girsanov's Theorem Through a Particular Lens

Let $\{W_k(t) : k = 1, \ldots, d\}$ be a collection of independent one-dimensional Brownian motions and define $W(t) = (W_1(t), \ldots, W_d(t))$. Take $h(t)$ to be an $\mathbb{R}^d$-valued stochastic process adapted to the filtration generated by $W(t)$ such that

$$\int_0^\infty |h(s)|^2 ds \leq C \quad \text{almost surely,} \tag{2.3}$$

for some finite (deterministic) constant C. Now define $\widetilde{W}$ by

$$\widetilde{W}(t) = W(t) + \int_0^t h(s)ds$$

for all $t \geq 0$. The following result is a restating of the Girsanov theorem (see e.g. [60] for further details):

Theorem 2.2 *In the above setting, the law of $\widetilde{W}$ is equivalent to that of W as measures on $C([0,\infty), \mathbb{R}^d)$. Furthermore if Φ is a measurable map from $C([0,\infty), \mathbb{R}^d)$ into $H^{\mathbf{N}}$ for some Polish space H, then the law of $\Phi(\widetilde{W})$ is equivalent to that of $\Phi(W)$ as measures on $H^{\mathbf{N}}$.*

Remark 2.3 The assumption given in (2.3) is an overkill, as Girsanov's Theorem holds under much less restrictive assumptions. However, in all of our applications (2.3) will hold.

2.4 A Recipe for Asymptotic Coupling

We now outline the basic logic used to in all of our examples. To avoid technicalities arising in the infinite dimensional setting such as the domain of various operators, we begin with an example in finite dimensions. However, as we will see, these arguments directly apply to infinite dimensional SPDEs when all of the objects involved are well defined.

Consider the stochastic differential equation given by

$$dx = F(x)dt + \sigma dW \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^d, \tag{2.4}$$

where $F: \mathbb{R}^d \to \mathbb{R}^d$, W is an n-dimensional Brownian Motion, and σ is a $d \times n$-dimensional matrix chosen so that (2.4) has global solutions. Fundamentally, the question of unique ergodicity turns on showing that (2.4) and a second copy

$$dy = F(y)dt + \sigma d\widetilde{W} \quad \text{with} \quad y(0) = y_0 \in \mathbb{R}^d, \tag{2.5}$$

have identical long time statistics even when $x_0 \neq y_0$. Since we are only interested in showing that the marginals of (x, y) have the same statistics, we are free to couple the two Brownian motions $(W, \widetilde{W})$ in any way we wish, building in correlations which are useful in the analysis. The essence of Corollary 2.1 is that we can even replace $\widetilde{W}$ with another process as long as it is absolutely continuous with respect to a Brownian Motion on the infinite time horizon, namely $C([0,\infty); \mathbb{R}^n)$. To this end, we use $\tilde{y}$ to represent a solution to (2.5) driven by this modified process $\widetilde{W}$, and take $\widetilde{W}$ to be a Brownian Motion shifted with a feedback control G which depends on the current state of $(x, \widetilde{y})$ and is designed to send $|x(t) - \widetilde{y}(t)| \to 0$ as $t \to \infty$. One could consider more general adapted controls (or even non-adapted as in [32,34][5]) but this class has proven sufficient for all of the problems we present here. In addition to the control G, we will introduce a stopping time τ which will turn off the control should $(x(t), \widetilde{y}(t))$ separate too much. This stopping time ensures that (2.3) holds and hence guarantees that our shifted process $\widetilde{y}$ is absolutely continuous with respect to y.

In light of this discussion, consider the system

$$\begin{aligned} dx &= F(x)dt + \sigma dW \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^d\,, \\ d\widetilde{y} &= F(\widetilde{y})dt + G(x, \widetilde{y})1\!\!1_{t\le\tau}dt + \sigma dW \quad \text{with} \quad y(0) = \widetilde{y}_0 \in \mathbb{R}^d\,, \end{aligned}$$

where $G\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is our feedback control and τ is a stopping time adapted to the filtration generated by $\{(x_s, \widetilde{y}_s) : s \le t\}$. We assume that G and τ are such that the system $(x, \widetilde{y})$ has global solutions. Furthermore, we assume that everything is constructed so that $\mathbb{P}(\tau = \infty) > 0$ and $|x(t) - \widetilde{y}(t)| \to 0$ on the event $\{\tau = \infty\}$. If in addition,

$$\int_0^\infty |\sigma^{-1}G(x(t), \widetilde{y}(t))|^2 1\!\!1_{t\le\tau}\, dt < C \quad \text{a.s.,} \tag{2.6}$$

for some deterministic (finite) $C > 0$, then by Theorem 2.2

$$\widetilde{W}(t) = W(t) + \int_0^t \sigma^{-1}G(x(s), \widetilde{y}(s))1\!\!1_{s\le\tau}ds$$

is equivalent to a Brownian motion on $C([0,\infty), \mathbb{R}^n)$. Implicit in equation (2.6) is the assumption that the range of G is contained in the range of σ.[6] Moreover, by Theorem 2.2 and (2.6), we also see that $\widetilde{y}$ is equivalent to the solution of (2.5) on $C([0,\infty), \mathbb{R}^d)$. In summary, if $P\phi(x_0) = \mathbb{E}\phi(x(t, x_0))$ for some $t > 0$ where $x(t, x_0)$ solves (2.4) (starting from the initial condition $x_0 \in \mathbb{R}^d$), $H_0 = H = \mathbb{R}^d$, and $\delta_{x_0}P^{\mathbf{N}}$ is defined as in Sect. 2.1, then the law induced by $\{(x(nt, x_0), \widetilde{y}(nt, y_0)) : n = 1, 2\ldots\}$ is an element of $\widetilde{\mathcal{C}}(\delta_{x_0}P^{\mathbf{N}}, \delta_{\widetilde{y}_0}P^{\mathbf{N}})$ which charges $D_{|\cdot|}$ with positive probability. And hence by Corollary 2.1, we know that (2.4) has at most one invariant measure.

The question remains, which G and τ do we choose? There is no unique choice. One only needs to ensure that the range of G is contained in the range of σ, that (2.6) holds, and that $|x(t) - \widetilde{y}(t)| \to 0$ on the event $\tau = \infty$. Typically, we will take $\tau = \inf\{t > 0 : \int_0^t |\sigma^{-1}G(x_s, \widetilde{y}_s)|^2 ds \ge R\}$ for some $R > 0$. To explore this question informally, we define $\rho(t) = x(t) - \widetilde{y}(t)$ and observe that

$$\frac{d}{dt}\rho(t) = F(x) - F(\widetilde{y}) - G(x, \widetilde{y})1\!\!1_{t\le\tau},$$

provided $\sigma^{-1}G$ is well defined on the interval $[0, \tau]$. If σ is invertible, one choice is to take $G(x, \widetilde{y}) = F(x) - F(\widetilde{y}) + \lambda(x - \widetilde{y})$ which results in

$$\frac{d}{dt}\rho(t) = -\lambda\rho$$

5 Of course, non-adapted controls make things significantly more technical. In particular, the classical Girsanov Theorem can not be used.

6 σ need not be invertible. As long as the range of G is contained in the range of σ then σ^{-1} can be taken to be the pseudo-inverse.

for $t < \tau$, so that $|x(t) - \widetilde{y}(t)|$ clearly decays towards zero on $[0, \tau)$ with a rate independent of R. Furthermore, (2.6) holds provided F does not grow too fast and has some Hölder regularity, and $\tau = \infty$ almost surely provided $R > 0$ is chosen sufficiently large. Often taking $G(x, \widetilde{y}) = \lambda(x - \widetilde{y})$ is sufficient for a λ large enough (see [31,35] for example). If only part of F leads to instability, say ΠF for some projection Π, then we can often take $G(x, \widetilde{y}) = \Pi(F(x) - F(\widetilde{y})) + \lambda\Pi(x - \widetilde{y})$ or even simply $G(x, \widetilde{y}) = \lambda\Pi(x - \widetilde{y})$, and only assume that the range of σ contains the range of Π. This loosening of the assumptions on the range of σ is one of the principle advantages of this point of view for SPDEs.[7]

3 Examples

3.1 Navier–Stokes on a Domain

Our first example is the 2D stochastic Navier–Stokes equation

$$d\mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} dt = (\nu \Delta \mathbf{u} - \nabla \pi + \mathbf{f})dt + \sum_{k=1}^{d} \sigma_k dW^k, \quad \nabla \cdot \mathbf{u} = 0, \tag{3.1}$$

for an unknown velocity field $\mathbf{u} = (u_1, u_2)$ and pressure π evolving on a bounded domain $\mathcal{D} \subset \mathbb{R}^2$ where we assume that $\partial\mathcal{D}$ is smooth and $\mathbf{u}$ satisfies the no-slip (Dirichlet) boundary condition

$$\mathbf{u}_{|\partial\mathcal{D}} = 0. \tag{3.2}$$

Here, in addition to the given vector fields $\sigma_j \in L^2(\mathcal{D})$ and a corresponding collection of $W = (W_1, \ldots, W_d)$ independent standard Brownian motions, the dynamics of (3.1)–(3.2) are also driven by a fixed, deterministic $\mathbf{f} \in L^2(\mathcal{D})$. We refer to e.g. [7,62] and to [1] for further details on the mathematical setting of the Navier–Stokes equations in the deterministic and stochastic frameworks respectively.

Remark 3.1 If either $\mathbf{f}$ or any of the σ_k are not divergence free, they can be replaced with their projection onto the divergence free vector fields without changing the dynamics as this only changes the pressure which acts as a Lagrange multiplier in this setting, keeping solutions on the space of divergence free vector fields.

3.1.1 Mathematical Preliminaries

We consider (3.1) on the phase space

$$H := \{\mathbf{u} \in L^2(\mathcal{D})^2 : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0\},$$

where $\mathbf{n}$ is the outward normal to $\partial\mathcal{D}$. Denote P_L as the orthogonal projection of $L^2(\mathcal{D})^2$ onto H. The space of vector fields whose gradients are integrable in $L^2(\mathcal{D})^2$ are also relevant and we define $V := \{\mathbf{u} \in H^1(\mathcal{D})^2 : \nabla \cdot \mathbf{u} = 0, \mathbf{u}_{|\partial\mathcal{D}} = 0\}$. We denote the norms associated to H and V respectively as $|\cdot|$ and $\|\cdot\|$.

[7] Taking $G(x, \widetilde{y}) = F(x) - F(\widetilde{y}) + \lambda \frac{x - \widetilde{y}}{|x - \widetilde{y}|}$ leads to ρ dynamics which converge to zero in finite time. This can be used to prove convergence in total variation norm. However it is less useful when one takes $G(x, \widetilde{y}) = \Pi(F(x) - F(\widetilde{y}) + \lambda \frac{x - \widetilde{y}}{|x - \widetilde{y}|})$ as the remaining degrees of freedom only contract asymptotically at $t \to \infty$. Nonetheless, such a control can simplify the convergence analysis in some cases.

The Stokes operator is defined as $A\mathbf{u} = -P_L \Delta \mathbf{u}$, for any vector field $\mathbf{u} \in V \cap H^2(\mathcal{D})^2$. Since A is self-adjoint with a compact inverse we infer that A admits an increasing sequence of eigenvalues $\lambda_k \sim k$ diverging to infinity with the corresponding eigenvectors e_k forming a complete orthonormal basis for H. We denote by P_N and Q_N the projection onto $H_N = \text{span}\{e_k : k = 1, \dots N\}$ and its orthogonal complement, respectively. Recall the generalized Poincaré inequalities

$$\|P_N \mathbf{u}\|^2 \leq \lambda_N |P_N \mathbf{u}|^2 \quad |Q_N \mathbf{u}|^2 \leq \lambda_N^{-1} \|Q_N \mathbf{u}\|^2 \tag{3.3}$$

hold for all sufficiently smooth $\mathbf{u}$ and any $N \geq 1$.

Recall that for all $\mathbf{u}_0 \in H$ and any fixed, finite d, (3.1) admits a unique solution

$$\mathbf{u}(\cdot) = \mathbf{u}(\cdot, \mathbf{u}_0) \in L^2(\Omega; C([0, \infty); H) \cap L^2_{loc}([0, \infty); V)),$$

which depends continuously in H on $\mathbf{u}_0$ for each $t \geq 0$. As such the transition functions $P_t(A, \mathbf{u}_0) = \mathbb{P}(\mathbf{u}(t, \mathbf{u}_0) \in A)$ are well defined for any $\mathbf{u}_0 \in H$, $t \geq 0$ and any Borel subset A of H, and define an associated Feller Markov semigroup $\{P_t\}_{t \geq 0}$ on $C_b(H)$.

We next recall some basic energy estimates for (3.1). Applying the Itō lemma to (3.1) we find that

$$d|\mathbf{u}|^2 + 2\nu \|\mathbf{u}\|^2 dt = 2\langle \mathbf{f}, \mathbf{u} \rangle dt + |\sigma|^2 dt + 2\langle \sigma, \mathbf{u} \rangle dW$$

where, for any $\mathbf{v} \in H$, $\langle \sigma, \mathbf{v} \rangle : \mathbb{R}^d \to \mathbb{R}$ is the linear operator defined by $\langle \sigma, \mathbf{v} \rangle w := \sum_{k=1}^d \langle \sigma_k, \mathbf{v} \rangle w_k$, $w \in \mathbb{R}^d$ and $|\sigma|^2 := \sum_{k=1}^n |\sigma_k|^2_{L^2}$ is the mean instantaneous energy injected into the system per unit time. Thus, for $R \geq 0$, using exponential martingale estimates[8] we infer that, for $\alpha = \alpha(|\sigma|, \nu) = \frac{\nu}{|\sigma|^2}$, independent of R

$$\mathbb{P}\left(\sup_{t \geq 0} |\mathbf{u}(t)|^2 + \nu \int_0^t \|\mathbf{u}\|^2 ds - (|\sigma|^2 + \tfrac{|A^{-\frac{1}{2}} f|^2}{2\nu}) t - |\mathbf{u}_0|^2 \geq R \right) \leq \exp(-\alpha R). \tag{3.5}$$

Note that (3.5) implies that time averaged measures $\nu_T(A) = \frac{1}{T} \int_0^T \mathbb{P}(\mathbf{u}(s) \in A) ds$ are a tight sequence since V is compactly embedding into H. Since $\mathbf{u}_0 \mapsto \mathbb{E}\phi(\mathbf{u}(t, \mathbf{u}_0))$ is continuous and bounded in L^2 whenever ϕ is (namely the Markov semigroup is Feller), the Krylov–Bogolyubov theorem implies the collection of invariant measures corresponding to (3.1) is non-empty.

3.1.2 Asymptotic Coupling Arguments

Having now reviewed the basic mathematical setting of (3.1), the uniqueness of invariant measures corresponding to (3.1) is established using the asymptotic coupling framework introduced above. Fix any $\mathbf{u}_0, \tilde{\mathbf{u}}_0 \in H$ and consider $\mathbf{u}(\cdot) = \mathbf{u}(\cdot, \mathbf{u}_0)$ solving (3.1) with initial data $\mathbf{u}_0$, and $\tilde{\mathbf{u}}$ solving

$$d\tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} dt = (\nu \Delta \tilde{\mathbf{u}} + \mathbb{1}_{\{\tau_K > t\}} \lambda P_N (\mathbf{u} - \tilde{\mathbf{u}})$$

[8] Recall that for any continuous martingale $\{M(t)\}_{t \geq 0}$,

$$\mathbb{P}\left(\sup_{t \geq 0} M(t) - \gamma \langle M \rangle (t) \geq R \right) \leq e^{-\gamma R} \tag{3.4}$$

for any $R, \gamma > 0$ where $\langle M \rangle (t)$ is the quadratic variation of $M(t)$.

$$+\nabla\tilde{\pi} + f)dt + \sum_{k=1}^{d} \sigma_k dW^k, \quad \nabla\cdot\tilde{\mathbf{u}} = 0, \quad \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0, \tag{3.6}$$

where

$$\tau_K := \inf_{t\geq 0}\left\{\int_0^t |P_N(\mathbf{u}-\tilde{\mathbf{u}})|^2 ds \geq K\right\}$$

and $K, \lambda > 0$ are fixed positive parameters which we will specify below as a function of $\mathbf{u}_0, \tilde{\mathbf{u}}_0$. In the context of the framework presented in Sect. 2.4, $G(\mathbf{u},\tilde{\mathbf{u}}) = \lambda P_N(\mathbf{u}-\tilde{\mathbf{u}})$ and the stopping time is τ_K.

We now make the connection with Corollary 2.1 explicit. Fix $T > 0$ and take $t_n = nT$. Define the measures $\mathfrak{m}$ and $\mathfrak{n}$ on $H^{\mathbf{N}}$ to be, respectively, the laws of the random vectors

$$\big(\mathbf{u}(t_1,\tilde{\mathbf{u}}_0), \mathbf{u}(t_2,\tilde{\mathbf{u}}_0),\dots\big) \qquad \text{and} \qquad \big(\tilde{\mathbf{u}}(t_1,\tilde{\mathbf{u}}_0), \tilde{\mathbf{u}}(t_2,\tilde{\mathbf{u}}_0),\dots\big).$$

The Girsanov theorem as presented in Theorem 2.2, implies that $\mathfrak{n}$ is mutually absolutely continuous with respect to $\mathfrak{m}$. Indeed, let $h(t) = 1\!\!1_{\{\tau_K>t\}}\lambda\sigma^{-1}P_N(\mathbf{u}-\tilde{\mathbf{u}})$, where σ^{-1} is the psuedo-inverse of σ. Thanks to the definition of the stopping times τ_K, we have, for any choice of $\lambda > 0$ and $K > 0$, that h satisfies the condition (2.3). We again emphasize that the equivalence of the measures $\mathfrak{m}$ and $\mathfrak{n}$ holds on the entire infinite trajectory sampled at the times $\{T, 2T, \dots\}$ which is significantly stronger than absolute continuity for the trajectories sampled at finite number of times $\{T, 2T, \dots, nT\}$ for all $n > 0$.

We now define the measure Γ on the space $H^{\mathbf{N}} \times H^{\mathbf{N}}$ as the law of the random vector

$$\big(\mathbf{u}(t_n,\mathbf{u}_0), \tilde{\mathbf{u}}(t_n,\tilde{\mathbf{u}}_0)\big)_{n\in\mathbf{N}}.$$

In view of the discussions in the previous paragraph, for any λ, $K > 0$, Γ is an element of $\tilde{C}(\delta_{u_0}P^{\mathbf{N}}, \delta_{v_0}P^{\mathbf{N}})$. The uniqueness of invariant measures corresponding to (3.1) therefore follows immediately from Corollary 2.1 if, for each $\mathbf{u}_0, \tilde{\mathbf{u}}_0 \in H$, we can find a corresponding $\lambda, K > 0$ (which may well depend on $\mathbf{u}_0, \tilde{\mathbf{u}}_0 \in H$) such that $\mathbf{u}(t) - \tilde{\mathbf{u}}(t) \to 0$ in H as $t \to \infty$ on a set of nontrivial probability.

Take $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$ and $q = \pi - \tilde{\pi}$. We have that

$$\partial_t\mathbf{v} - \nu\Delta\mathbf{v} + 1\!\!1_{\{\tau_K>t\}}\lambda P_N\mathbf{v} = -\nabla q - \mathbf{v}\cdot\nabla\mathbf{u} - \tilde{\mathbf{u}}\cdot\nabla\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0. \tag{3.7}$$

Taking $\lambda = \nu\lambda_N$, the Poincaré inequality, (3.3) implies that

$$1\!\!1_{\{\tau_K>t\}}\lambda_N\nu|P_N\mathbf{v}|^2 + \nu\|\mathbf{v}\|^2 \geq 1\!\!1_{\{\tau_K>t\}}(\nu\lambda_N|P_N\mathbf{v}|^2 + \nu\|Q_N\mathbf{v}\|^2) \geq 1\!\!1_{\{\tau_K>t\}}\nu\lambda_N|\mathbf{v}|^2. \tag{3.8}$$

Multiplying (3.7) with $\mathbf{v}$, integrating over $\mathcal{D}$, using that $\mathbf{v}, \mathbf{u}, \tilde{\mathbf{u}}$ are all divergence free and (3.8) we obtain

$$\frac{d}{dt}|\mathbf{v}|^2 + \nu\|\mathbf{v}\|^2 + \lambda_N\nu 1\!\!1_{\{\tau_K>t\}}|\mathbf{v}|^2 \leq \left|\int_{\mathcal{D}} \mathbf{v}\cdot\nabla\mathbf{u}\cdot\mathbf{v}dx\right| \leq C|\mathbf{v}|\|\mathbf{v}\|\|\mathbf{u}\| \leq \nu\|\mathbf{v}\|^2 + C_1|\mathbf{v}|^2\|\mathbf{u}\|^2$$

where C_1 depends only on ν and universal quantities from Sobolev embedding. Rearranging and using the Grönwall lemma we obtain

$$|\mathbf{v}(t)|^2 \leq |\mathbf{u}_0 - \tilde{\mathbf{u}}_0|^2 \exp\Big(-\lambda_N\nu t + C_1\int_0^t \|\mathbf{u}\|^2 ds\Big), \tag{3.9}$$

for any $t \in [0, \tau_K]$.

Now, for any $R > 0$, consider the sets

$$E_R := \left\{ \sup_{t\geq 0} \left(|\mathbf{u}(t)|^2 + \nu \int_0^t \|\mathbf{u}\|^2 ds - (|\sigma|^2 + \tfrac{|A^{-\frac{1}{2}} f|^2}{2\nu})t - |\mathbf{u}_0|^2 \right) < R \right\}.$$

Notice that, in view of (3.5), these sets have nonzero probability for every $R = R(\nu, |\sigma|) > 0$ sufficiently large. On the other hand, on E_R, (3.9) implies

$$|\mathbf{v}(t)|^2 \leq |\mathbf{u}_0 - \tilde{\mathbf{u}}_0|^2 \exp\left(\frac{C_1}{\nu}(R + |\mathbf{u}_0|^2)\right) \exp\left(-\lambda_N \nu t + \frac{C_1}{\nu}(|\sigma|^2 + \tfrac{|A^{-\frac{1}{2}} f|^2}{4\nu})t\right).$$

for each $t \in [0, \tau_K]$. Note carefully that the constant C_1 is independent of $N, \lambda, K > 0$ and $\mathbf{u}_0, \tilde{\mathbf{u}}_0$. By picking N such that

$$\frac{\nu\lambda_N}{2} - \frac{C_1}{\nu}\big(|\sigma|^2 + \frac{|A^{-\frac{1}{2}} f|^2}{2\nu}\big) > 0,$$

we infer, for $\lambda = \nu\lambda_N$,

$$|\mathbf{v}(t)|^2 \leq |\mathbf{u}_0 - \tilde{\mathbf{u}}_0|^2 \exp\left(\frac{C_1}{\nu}(R + |\mathbf{u}_0|^2)\right) \exp\left(-\frac{\lambda}{2}t\right), \tag{3.10}$$

on E_R for every $t \in [0, \tau_K]$ and where we again emphasize that C_1 does not depend on K in (3.6). By now choosing $K = K(\nu, |\sigma|^2)$ sufficiently large we are forced to conclude from (3.10) that $\{\tau_K = \infty\} \supset E_R$ and hence on the non-trivial set E_R we infer that

$$\mathbf{u}(t) - \tilde{\mathbf{u}}(t) \longrightarrow 0 \quad \text{in} \quad H.$$

as $t \to \infty$.

In conclusion we have proven that

Proposition 3.1 *For every $\nu > 0$ there exists $N = N(\nu, |\sigma|^2, |A^{-\frac{1}{2}} f|)$ such that if* Range$(\sigma) \supset H_N = P_N H$ *then* (3.1) *has a unique ergodic invariant measure.*

Remark 3.2 It is worth emphasizing here that the above analysis shows that unique ergodicity results can be easily obtained in the presence of a deterministic forcing. This observation applies to each of the examples considered below, but we omit its explicit inclusion for brevity and clarity of presentation. Of course, the addition of a body forcing $\mathbf{f}$ in the hypo-elliptic setting can bring extra complications, primarily in proving topological irreducibility which is often required to prove unique ergodicity.

3.2 2D Hydrostatic Navier–Stokes Equations

We next consider a stochastic version of the 2D Hydrostatic Naver-Stokes equations

$$du + (u\partial_x u + w\partial_z u + \partial_x p - \nu\Delta u)dt = \sum_{k=1}^{d} \sigma_k dW^k \tag{3.11}$$

$$\partial_z p = 0 \tag{3.12}$$

$$\partial_x u + \partial_z w = 0, \tag{3.13}$$

for an unknown velocity field (u, w) and pressure p evolving on the domain $\mathcal{D} = (0, L) \times (-h, 0)$. The boundary $\partial\mathcal{D}$ is decomposed into its vertical sides $\Gamma_v = [0, L] \times \{0, -h\}$ and lateral sides $\Gamma_l = \{0, L\} \times [-h, 0]$, and we impose the boundary conditions

$$u = 0 \text{ on } \Gamma_l, \quad \partial_z u = w = 0 \text{ on } \Gamma_v. \tag{3.14}$$

The system is driven by a collection of independent Brownian motions $(W^1, \ldots, W^d)$ acting in directions $\sigma_k \in L^2(\mathcal{D})$ to be specified below.

The hydrostatic Navier–Stokes equations serve as a simple mathematical model which maintains some of the crucial anisotropic structure present in the more involved Primitive equations of the oceans and atmosphere. This latter system forms the numerical core of sophisticated general circulation models used in climate and weather prediction [58,63]. The Primitive equations have been studied extensively in the mathematics literature in both deterministic [12,37,44,49–51,59] and stochastic [13,14,19,25–28] settings.

Note that, in contrast to the Navier–Stokes equations, global existence of strong solutions to the Primitive equations has been proven in 3D [12,37,44], but the uniqueness of weak solutions in 2-D remains an outstanding open problem. Indeed, for the hydrostatic Navier–Stokes equations, we rely on H^1 well-posedness results [27,29] to provide suitable Markovian dynamics associated to (3.11)–(3.13). The existence of invariant measures follows from H^2-moment bounds (see [25] and (3.21) below) and the Krylov-Bogoliubov Theorem. For uniqueness of the invariant measure we invoke another asymptotic coupling argument. In comparison to the previous example, this argument will invoke the flexibility of Corollary 2.1 by proving convergence in the (weaker) L^2 topology. The more involved cases of other boundary conditions, couplings with proxies for density (temperature and salinity), and three space dimensions, will be pursued in future work.

3.2.1 The Mathematical Setting

We begin with some observations about the structure of (3.11)–(3.13) when subject to (3.14). Firstly notice that, in view of (3.13) and (3.14): $\int_{-h}^0 \partial_x u(x,z)dz = -\int_{-h}^0 \partial_z w(x,z)dz = 0$ for $x \in [0, L]$. It follows that

$$\int_{-h}^0 u(x,z)dz \equiv 0.$$

The divergence free condition (3.13) coupled with (3.14) also allows us to write w as a functional of u, namely $w(x,z) = -\int_{-h}^z \partial_x u(x,\bar{z})d\bar{z}$. In the geophysical literature w is referred to as a 'diagnostic variable' and we define

$$w(u)(x,z) = \int_{-h}^z \partial_x u(x,\bar{z})d\bar{z}$$

Consider the spaces

$$H = \left\{ u \in L^2(\mathcal{D}) : \int_{-h}^0 u dz \equiv 0 \right\} \quad \text{and} \quad V = \left\{ u \in H^1(\mathcal{D}) : \int_{-h}^0 u dz \equiv 0, u|_{\Gamma_l} = 0 \right\}$$

and the projection operator $P_H : L^2(\mathcal{D}) \to H$,

$$P_H(v) = v - \frac{1}{h}\int_{-h}^0 v dz.$$

As in the previous example we use $|\cdot|$ and $\|\cdot\|$ to denote the L^2 and H^1 norms, respectively. We define $A = -P_H \Delta$, and identify its domain as

$$D(A) = \left\{ u \in H^2(\mathcal{D}) : \int_{-h}^0 u dz \equiv 0, u|_{\Gamma_l} = 0, \partial_z u|_{\Gamma_v} = 0 \right\}.$$

Then for fixed $u_0 \in V$, and $\sigma_k \in D(A)$ with finite d (or sufficently fast decay in $\|\sigma_k\|_{H^2}$), the system (3.11)–(3.14) possesses a unique solution

$$u(\cdot) = u(\cdot, u_0) \in L^2(\Omega; C([0,\infty); V) \cap L^2_{loc}([0,\infty); D(A))),$$

which depends continuously on $u_0 \in V$; see [27,29]. Moreover, the dynamics of (3.11)–(3.14) generate a Feller Markov semigroup $\{P_t\}_{t\geq 0}$ on $C_b(V)$.

3.2.2 A Priori Estimates

We proceed to establish some estimates on solutions needed for existence and uniqueness of invariant measures. The energy estimate is standard. In view of (3.13) the pressure and nonlinear terms drop and Itō's lemma gives

$$d|u|^2 + 2\nu\|u\|^2 dt = |\sigma|^2 dt + \langle u, \sigma\rangle dW. \tag{3.15}$$

From the exponential martingale bound, cf. (3.4), we infer that

$$\mathbb{P}\left(\sup_{t\geq 0} |u(t)|^2 + \nu \int_0^t \|u\|^2 ds - |\sigma|^2 t - |u_0|^2 > R \right) \leq e^{-\gamma R}, \tag{3.16}$$

for every $R > 0$ where $\gamma = \gamma(\nu, |\sigma|^2)$ does not depend on R or the number of forced modes.

We will also require bounds on $\|\partial_z u\|$, and appeal to the 'vorticity form' of (3.11)–(3.14) defined by taking ∂_z of (3.11). In view of (3.12) the pressure p is independent of z, and we obtain

$$d\partial_z u + \partial_z(u\partial_x u + w\partial_z u)dt - \nu\Delta\partial_z u dt = \sum_{k=1}^N \partial_z \sigma_k dW^k.$$

Notice that in view of the boundary conditions (3.14), $-\int \Delta\partial_z u \partial_z u dx dz = \|\partial_z u\|^2$, and moreover we have the cancelation

$$\begin{aligned}\int \partial_z(u\partial_x u + w\partial_z u)\partial_z u dx dz &= \int (\partial_z u \partial_x u + u\partial_{xz} u + \partial_z w \partial_z u + w\partial_{zz} u)\partial_z u dx dz \\ &= \frac{1}{2}\int (\partial_x u(\partial_z u)^2 + \partial_z w(\partial_z u)^2) dx dz = 0.\end{aligned}$$

Combining these observations we obtain

$$d|\partial_z u|^2 + 2\nu\|\partial_z u\|^2 dt = |\partial_z\sigma|^2 + 2\langle\partial_z\sigma, \partial_z u\rangle dW, \tag{3.17}$$

and hence, for every $R > 0$,

$$\mathbb{P}\left(\sup_{t\geq 0} |\partial_z u(t)|^2 + \nu \int_0^t \|\partial_z u\|^2 ds - |\partial_z\sigma|^2 t - |\partial_z u_0|^2 > R \right) \leq e^{-\gamma R}, \tag{3.18}$$

for some $\gamma = \gamma(\nu, |\partial\sigma|) > 0$ independent of R.

3.2.3 Existence of Invariant States

We can now prove existence of invariant measures for (3.11)–(3.13) by considering the evolution equation for $\|u\|^2$. Here we follow an approach similar to [25]. By applying the Itō lemma to a Galerkin truncation of (3.11)–(3.14), and passing to a limit, we obtain

$$d\|u\|^2 + \nu\|u\|^2_{H^2} dt \leq C(|\partial_x u|^4 + |\partial_x u|^2|\partial_z u|\|\partial_z u\|) + \|\sigma\|^2 dt + 2\langle\sigma, u\rangle_{H^1} dW. \tag{3.19}$$

Here we have used the anisotropic estimate

$$\begin{aligned}\left|\int_0^L\int_{-h}^0 w(v)\partial_z u_1 u_2 dz dx\right| &\leq h^{1/2}\int_0^L |\partial_x v|_{L^2_z}|\partial_z u_1|_{L^2_z}|u_2|_{L^2_z}dx\\ &\leq h^{1/2}|u_2||\partial_x v|\left(\sup_{x\in[0,L]}\int_{-h}^0(\partial_z u_1)^2 dz\right)^{1/2}\\ &\leq h^{1/2}|u_2||\partial_x v|\left(\sup_{x\in[0,L]}\int_0^x \partial_{\bar{x}}\int_{-h}^0(\partial_z u_1)^2 dz d\bar{x}\right)^{1/2}\\ &\leq 2h^{1/2}|u_2||\partial_x v||\partial_z u_1|^{1/2}\|\partial_z u_1\|^{1/2} \end{aligned} \tag{3.20}$$

for all suitably regular v, u_1, u_2. With (3.20) and the Sobolev embedding of $H^{1/3}$ into L^3 in dimension 2 we infer that

$$\begin{aligned}\left|\int (u\partial_x u + w\partial_z u)\partial_{xx} u dx dz\right| &\leq C(|\partial_x u|_{L^3}^3 + |\partial_x u|\|\partial_x u\||\partial_z u|^{1/2}\|\partial_z u\|^{1/2})\\ &\leq C(|\partial_x u|^2\|\partial_x u\| + |\partial_x u|\|\partial_x u\||\partial_z u|^{1/2}\|\partial_z u\|^{1/2})\\ &\leq \nu\|u\|_{H^2}^2 + C(|\partial_x u|^4 + |\partial_x u|^2|\partial_z u|\|\partial_z u\|),\end{aligned}$$

and from here the derivation of (3.19) is straightforward. From (3.19) we now compute $\log(1+\|u\|^2)$ and observe

$$d\log(1+\|u\|^2) + \nu\frac{\|u\|_{H^2}^2}{1+\|u\|^2}dt \leq C(\|u\|^2 + |\partial_z u|\|\partial_z u\|)dt + \|\sigma\|^2 dt + 2\frac{\langle\sigma, u\rangle_{H^1}}{1+\|u\|^2}dW.$$

Hence from this bound and (3.15), (3.17) we infer

$$\int_0^T \mathbb{E}\|u\|_{H^2} ds \leq \frac{1}{2}\mathbb{E}\int_0^T\left(1+\|u\|^2 + \frac{\|u\|_{H^2}^2}{1+\|u\|^2}\right)ds \leq C(T+1), \tag{3.21}$$

for a constant $C = C(\|u_0\|^2, \|\sigma\|^2, \nu)$ independent of $T > 0$. The existence of invariant measures associated to (3.11)–(3.13) now follows by applying the Krylov–Bogolyubov Theorem.

3.2.4 Asymptotic Coupling Arguments

Similar to the previous example and again following Sect. 2.4, we fix any $u_0, \tilde{u}_0 \in V$ and consider u a solution of (3.11)–(3.13) starting from u_0 and $\tilde{u}$ solving the same system with an additional control G given as

$$G(u,\tilde{u}) = \lambda 1\!\!1_{\{\tau_K > t\}}P_N(u-\tilde{u})dt, \quad \text{with} \quad \tau_K := \inf_{t\geq 0}\left\{\int_0^t \|P_N(u-\tilde{u})\|^2 ds \geq K\right\},$$

and starting from $\tilde{u}_0$; cf. (3.6). Once again, the parameters λ, $K > 0$ and N will be specified below. As above $\tilde{u}$ is subject to a Girsonov shift of the form $\sigma^{-1}G$ and Theorem 2.2 applies.

Subtracting $\tilde{u}$ from u and taking $v = u - \tilde{u}$, $q = p - \tilde{p}$ we obtain

$$\partial_t v - \nu\Delta v + \lambda 1\!\!1_{\{\tau_K > t\}}P_N v = -\tilde{u}\partial_x v - w(\tilde{u})\partial_z v - v\partial_x u - w(v)\partial_z u - \partial_x q.$$

It follows, as in (3.8) for suitably large N, that on $[0, \tau_K]$

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \frac{\nu}{2}\|v\|^2 + \lambda|v|^2 \leq \left|\int (v\partial_x u + w(v)\partial_z u)v dx dz\right|$$

$$\leq C|v|\|v\|(\|u\| + |\partial_z u|^{1/2}\|\partial_z u\|^{1/2}),$$

where we have applied the anisotropic estimate (3.20) in the last line. On the interval $[0, \tau_K]$ this gives

$$|v(t)|^2 \leq \exp\left(-2\lambda t + C\int_0^t (\|u\|^2 + |\partial_z u|\|\partial_z u\|)ds\right)|v(0)|^2, \tag{3.22}$$

where, to emphasize, the constant C is independent of K. Thus, following the strategy of Sect. 3.1 above, we can combine (3.16), (3.18) and (3.22) to apply Corollary 2.1 with $H_0 = H^1(\mathcal{D})$ and $\tilde{\rho}$ corresponding to the $L^2(\mathcal{D})$-topology. We have proven the following result:

Proposition 3.2 *For every* $\nu > 0$ *there exists* $N = N(\nu, |\sigma|^2, |\partial_z\sigma|^2)$ *such that if* $\text{Range}(\sigma) \supset V_N = P_N V$ *then* (3.11)–(3.13) *has a unique ergodic invariant measure.*

3.3 The Fractionally Dissipative Euler Model

Our next example considers the fractionally dissipative Euler equations introduced in [8]. This system takes the form

$$d\xi + (\Lambda^\gamma \xi + \mathbf{u}\cdot\nabla\xi)dt = \sum_{k=1}^d \sigma_k dW^k, \quad \mathbf{u} = \mathcal{K} * \xi, \tag{3.23}$$

for any unknown vorticity field ξ. Here $\Lambda^\gamma = (-\Delta)^{\gamma/2}$ is the fractional Laplacian which we consider for *any* $\gamma \in (0, 2]$, $\mathcal{K}$ is the Biot-Savart kernel, so that $\nabla^\perp \cdot \mathbf{u} = \xi$ and $\nabla \cdot \mathbf{u} = 0$, and we suppose that (3.23) is posed on the periodic box $\mathbb{T}^2 = [-\pi, \pi]^2$. Conditions on the forced directions σ_k will be specified below.

In [8] it was demonstrated that with "effectively elliptic" forcing, the system (3.23) possesses a unique ergodic invariant measure, and in the course of the proof, significant effort was made to establish arbitrary order polynomial moment bounds in high order Sobolev spaces (H^r for any $r > 2$). These bounds are interesting and hold significance for questions regarding the rate of convergence to the invariant measure. However, we show here that much less effort is required if one simply wishes to prove existence and uniqueness of the invariant measure. As in the last example, the argument is significantly simplified by invoking Corollary 2.1 and proving convergence in the L^2-topology.

3.3.1 Mathematical Preliminaries

We consider (3.23) in its velocity formulation

$$d\mathbf{u} + (\Lambda^\gamma \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi)dt = \sum_{k=1}^d \rho_k dW^k, \quad \nabla\cdot\mathbf{u} = 0. \tag{3.24}$$

Here the unknowns are the velocity field $\mathbf{u} = (u_1, u_2)$ and pressure π both posed on the periodic box $\mathbb{T}^2$. One may show using a Galerkin regularization argument, that for any $r > 2$ and $\mathbf{u}_0 \in H^r$, there exists a unique $\mathbf{u} = \mathbf{u}(\cdot, \mathbf{u}_0)$ solving (3.24), with $\mathbf{u} \in C([0, \infty), H^r)$. As in [8] we may infer that for any such $r > 2$ and any $t > 0$, $\mathbf{u}(t, \mathbf{u}_0^n) \to \mathbf{u}(t, \mathbf{u}_0)$ almost surely in H^r whenever $\mathbf{u}_0^n \to \mathbf{u}_0$ in H^r. It follows that the transition function $P_t(\mathbf{u}_0, A) = \mathbb{P}(\mathbf{u}(t, \mathbf{u}_0) \in A)$, $\mathbf{u}_0 \in H^r$, $A \in \mathcal{B}(H^r)$ defines a Feller Markovian semigroup.

To prove the existence of an invariant measure for (3.23) we argue as follows. Applying Λ^r for any $r > 2$ to (3.24) and then integrating we find that

$$d\|\mathbf{u}\|^2_{H^r} + 2\|\mathbf{u}\|^2_{H^{r+\frac{\gamma}{2}}} dt = -2\int_{\mathbb{T}^2} (\Lambda^r(\mathbf{u}\cdot\nabla\mathbf{u}) - \mathbf{u}\cdot\nabla\Lambda^r\mathbf{u})\Lambda^r\mathbf{u}dx + \|\rho\|^2_{H^r}dt + 2\langle\rho,\mathbf{u}\rangle_{H^r}dW,$$

where we have used that $\mathbf{u}$ is divergence free to eliminate the pressure and rewrite the nonlinear terms. In order to estimate these nonlinear terms we recall the Kenig-Ponce-Vega commutator estimate

$$\|\Lambda^s(f\cdot\nabla g) - f\cdot\nabla\Lambda^s g\|_{L^p} \le C(\|\Lambda^s f\|_{L^{q_1}}\|\nabla g\|_{L^{r_1}} + \|\nabla f\|_{L^{q_2}}\|\Lambda^s g\|_{L^{r_2}}),$$

valid for any suitably regular $f, g, s > 1$ and any trios p, q_i, r_i with $1 < p, q_i, r_i < \infty$ and $p^{-1} = q_j^{-1} + r_j^{-1}$, $j = 1, 2$; see [56]. With this bound we obtain

$$\left|\int_{\mathbb{T}^2} (\Lambda^r(\mathbf{u}\cdot\nabla\mathbf{u}) - \mathbf{u}\cdot\nabla\Lambda^r\mathbf{u})\Lambda^r\mathbf{u}dx\right| \le C\|\nabla\mathbf{u}\|_{L^{4/\delta}}\|\Lambda^r\mathbf{u}\|_{L^{4/(2-\delta)}}|\Lambda^r\mathbf{u}|, \tag{3.25}$$

valid for any $\delta \in [0, 1)$. Next recall the Gagliardo-Nirenberg interpolation inequality

$$\|\Lambda^\alpha f\|_{L^p} \le C\|f\|^\theta_{L^q}\|\Lambda^\beta f\|^{1-\theta}_{L^m},$$

which holds for any $1 < p, q, m \le \infty$, $0 < \alpha < \beta < \infty$ such that

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{m}, \quad \text{where} \quad \theta = 1 - \frac{\alpha}{\beta}.$$

Taking $p = 4/(2-\delta)$, $m = 2$, $\alpha = r - 1$ and $\beta = r - 1 + \frac{\gamma}{2}$ in this inequality we infer that for every $\delta < \frac{2\gamma}{\gamma+2r-2}$,

$$\|\Lambda^r\mathbf{u}\|_{L^{4/(2-\delta)}} \le C\|\nabla\mathbf{u}\|^{\frac{\gamma}{2r-2+\gamma}}_{L^q}|\Lambda^{r+\frac{\gamma}{2}}\mathbf{u}|^{\frac{2r-2}{2r-2+\gamma}} \quad \text{with} \quad q = \frac{4\gamma}{2\gamma - \delta(\gamma+2r-2)}.$$

Hence by choosing $\delta = \frac{\gamma}{\gamma+2r-2}$ and combining this bound with (3.25) we find

$$\begin{aligned}\left|\int_{\mathbb{T}^2} (\Lambda^r(\mathbf{u}\cdot\nabla\mathbf{u}) - \mathbf{u}\cdot\nabla\Lambda^r\mathbf{u})\Lambda^r\mathbf{u}dx\right| &\le C\|\xi\|_{L^{4/\delta}}\|\xi\|^{\frac{\gamma}{2r-2+\gamma}}_{L^4}\|\mathbf{u}\|^{\frac{2r-2}{2r-2+\gamma}}_{H^{r+\frac{\gamma}{2}}}\|\mathbf{u}\|_{H^r}\\ &\le C\|\xi\|^2_{L^{4/\delta}}\|\mathbf{u}\|^{2\left(\frac{2r-2+\gamma}{2r-2+2\gamma}\right)}_{H^r} + \|\mathbf{u}\|^2_{H^{r+\frac{\gamma}{2}}}.\end{aligned}$$

With this estimate in mind we now compute a differential for $(1 + \|\mathbf{u}\|^2_{H^r})^\kappa$ with $\kappa \in (0, 1)$. This yields

$$\begin{aligned}d(1 + \|\mathbf{u}\|^2_{H^r})^\kappa + \kappa\|\mathbf{u}\|^2_{H^{r+\frac{\gamma}{2}}}(1+\|\mathbf{u}\|^2_{H^r})^{\kappa-1}dt &\le C\|\xi\|^2_{L^{4/\delta}}\|\mathbf{u}\|^{2\left(\frac{2r-2+\gamma}{2r-2+2\gamma}\right)}_{H^r}(1 + \|\mathbf{u}\|^2_{H^r})^{\kappa-1}dt\\ &+ \kappa\|\rho\|^2_{H^r}(1 + \|\mathbf{u}\|^2_{H^r})^{\kappa-1}dt + 2\kappa\langle\rho,\mathbf{u}\rangle_{H^r}(1 + \|\mathbf{u}\|^2_{H^r})^{\kappa-1}dW.\end{aligned}$$

By taking $\kappa = \frac{\gamma}{2r-2+2\gamma}$ we infer

$$\int_0^t \mathbb{E}\|\mathbf{u}\|^{\kappa+1}_{H^{r+\frac{\gamma}{2}}}ds \le C\left((1 + \|\mathbf{u}_0\|^2_{H^r})^\kappa + \int_0^t (\|\rho\|^2_{H^r} + \mathbb{E}\|\xi\|^2_{L^{4/\delta}} + 1)ds\right), \tag{3.26}$$

for each $t \geq 0$ where the constant $C = C(\gamma, r)$ is independent of $\mathbf{u}_0$ and t. The existence of an invariant measure now follows once we establish a suitable bound on ξ in $L^p(\mathbb{T}^2)$ for any $p \geq 2$.

To this end, we next observe that from (3.23) we have for any $p \geq 2$,

$$d\|\xi\|_{L^p}^p + p\int_{\mathbb{T}^2} \Lambda^\gamma \xi \xi^{p-1} dxdt = \frac{p(p-1)}{2} \sum_{k=1}^d \int \sigma_k^2 \xi^{p-2} dxdt + p\sum_{k=1}^d \int \sigma_k \xi^{p-1} dxdW^k.$$

Recalling the nonlinear Poincaré inequality from [8]

$$p\int_{\mathbb{T}^2} \Lambda^\gamma \xi \xi^{p-1} dx \geq \frac{1}{C_\gamma} \|\xi\|_{L^p}^p,$$

where the constant C_γ depends only on $\gamma > 0$, we infer

$$d\|\xi\|_{L^p}^p + \frac{1}{C_\gamma}\|\xi\|_{L^p}^p dt \leq \frac{p(p-1)}{2}\|\sigma\|_{L^p}^2 \|\xi\|_{L^p}^{p-2} dt + p\sum_{k=1}^d \int \sigma_k \xi^{p-1} dxdW^k. \quad (3.27)$$

The existence of an (ergodic) invariant measure follows immediately by combining (3.27) with (3.26).

3.3.2 Asymptotic Coupling Arguments

Fix any $\mathbf{u}_0, \tilde{\mathbf{u}}_0 \in H^r$ and let $\mathbf{u} = \mathbf{u}(\cdot, \mathbf{u}_0)$ be the corresponding solution of (3.23) while we suppose that $\tilde{\mathbf{u}}$ solves

$$d\tilde{\mathbf{u}} + (\Lambda^\gamma \tilde{\mathbf{u}} - 1\!\!1_{\tau_K > t}\lambda P_N(\mathbf{u} - \tilde{\mathbf{u}}) + \tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}} + \nabla\tilde{\pi})dt = \sum_{k=1}^d \sigma_k dW, \quad \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0,$$

where

$$\tau_K := \inf_{t\geq 0}\left\{\int_0^t |P_N(\mathbf{u} - \tilde{\mathbf{u}})|^2 ds \geq K\right\}.$$

The parameters $K, \lambda > 0$ are to be determined presently. It is easy to see from Theorem 2.2 that, for any choice of λ, $K > 0$, the law of $\tilde{\mathbf{u}}$ is absolutely continuous with respect to the solution $\mathbf{u}(\cdot, \tilde{\mathbf{u}}_0)$ of (3.23) corresponding to $\tilde{\mathbf{u}}_0$. As previous examples, unique ergodicity follows from Corollary 2.1 once we can find some λ, $K > 0$ (where K may depend on $\mathbf{u}_0, \tilde{\mathbf{u}}_0$) such that $\mathbf{u}(t) - \tilde{\mathbf{u}}(t) \to 0$ in $L^2(\mathbb{T}^2)$ on a set of non-trivial measure. Take $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$ and $q = \pi - \tilde{\pi}$. We find

$$\partial_t \mathbf{v} + \Lambda^\gamma \mathbf{v} + 1\!\!1_{\tau_K > t}\lambda P_N \mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{u} + \tilde{\mathbf{u}} \cdot \nabla\mathbf{v} + \nabla q = 0.$$

Hence, using that $\tilde{\mathbf{u}}$ is divergence free and the generalized Poincaré inequality (similarly to (3.8) above) we have

$$\frac{d}{dt}|\mathbf{v}|^2 + 2\lambda|\mathbf{v}|^2 + \|\mathbf{v}\|_{H^{\gamma/2}}^2 \leq 2\left|\int \mathbf{v} \cdot \nabla\mathbf{u} \cdot \mathbf{v} dx\right|$$

for every $t \in [0, \tau_K]$. Here $\lambda = \lambda(N, \gamma)$ can be chosen as large as desired by decreeing the space H_N spanned by the forced modes to be commensurately big. By choosing $p = p(\gamma) > 0$ sufficiently large we infer

$$2\left|\int \mathbf{v} \cdot \nabla\mathbf{u} \cdot \mathbf{v} dx\right| \leq C\|\xi\|_{L^p}|\mathbf{v}|\|\mathbf{v}\|_{H^{\gamma/2}}$$

and hence the bound

$$\frac{d}{dt}|\mathbf{v}|^2 + (2\lambda - C\|\xi\|_{L^p}^2)|\mathbf{v}|^2 \leq 0$$

holds on $[0, \tau_K]$. Grönwall's inequality implies that, on this same interval $[0, \tau_K]$,

$$|\mathbf{v}(t)|^2 \leq |\mathbf{v}(0)|^2 \exp\big(-2\lambda t + C\int_0^t \|\xi\|_{L^p}^2 ds\big). \tag{3.28}$$

Here we again emphasize that the constant C appearing in the exponential depends only on universal quantities and in particular is independent of K in the definition of τ_K.

To finally infer the desired contraction from (3.28) we thus need a further bound on the L^p norms of ξ. For this we compute $d(1 + \|\xi\|_{L^p}^p)^{2/p}$ which with (3.27) yields

$$\begin{aligned} &d(1 + \|\xi\|_{L^p}^p)^{2/p} + \frac{2}{pC_\gamma}\|\xi\|_{L^p}^2 dt \\ &\quad \leq (p-1)\|\sigma\|_{L^p}^2 dt + 2(1 + \|\xi\|_{L^p}^p)^{\frac{2}{p}-1}\sum_{k=1}^d \int \sigma_k \xi^{p-1} dx dW^k. \end{aligned} \tag{3.29}$$

Observe that the martingale term in the inequality above has a quadratic variation which can be estimated as

$$\begin{aligned} &4\int_0^t (1 + \|\xi\|_{L^p}^p)^{\frac{4}{p}-2}\sum_{k=1}^d \left(\int \sigma_k \xi^{p-1} dx\right)^2 ds \\ &\quad \leq 4\int_0^t (1 + \|\xi\|_{L^p}^p)^{\frac{4}{p}-2}\left(\int \left(\sum_{k=1}^d \sigma_k^2\right)^{1/2} \xi^{p-1} dx\right)^2 ds \\ &\quad \leq 4\|\sigma\|_{L^p}^2 \int_0^t (1 + \|\xi\|_{L^p}^p)^{2/p} ds. \end{aligned} \tag{3.30}$$

By now combining (3.29), (3.30) we infer from exponential martingale bounds, (3.4), that

$$\mathbb{P}\left(\sup_{t\geq 0} \frac{1}{pC_\gamma}\int_0^t \|\xi\|_{L^p}^2 dt - (p + 2^{2/p+2})\|\sigma\|_{L^p}^2 t \ \geq R\right) \leq e^{-\alpha R}, \tag{3.31}$$

for every $R \geq 0$ where $\alpha = \alpha(\|\sigma\|_{L^p}, p, \gamma)$ is independent of R and does not depend on the number of forced modes but only on the norm of $\|\sigma\|_{L^p}$.

Combining (3.28) and (3.31) and arguing as in the previous examples we infer that, for an appropriate choice of $K > 0$, $|v(t)| \to 0$ on a set of non-trivial measure. In summary we have proven the following:

Proposition 3.3 *The system (3.23) possesses an ergodic invariant measure. When $N = N(\|\sigma\|_{H^r}, \gamma)$ is sufficiently large and* Range$(\sigma) \supset P_N H^r$ *this invariant measure is unique.*

3.4 The Damped Stochastically Forced Euler–Voigt Model

The next system that we will consider is an inviscid 'Voigt-type' regularization (see e.g. [57] and further references below) of the damped stochastic Euler equations. This example is significant as, in contrast to the previous equations, it illustrates a case for which the existence and uniqueness of invariant measures can be demonstrated in the absence of a parabolic regularization mechanism. In fact both the questions of the existence and the uniqueness of the invariant measure leads to interesting new twists in the analysis in comparison to the previous examples. For the question of existence we make use of an inviscid limit procedure along with an abstract result presented in Corollary 4.2 in Appendix below.

The governing equations read

$$d\mathbf{u} + (\gamma \mathbf{u} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla p)dt = \sum_{k=1}^{d} \sigma_k dW^k, \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{3.32}$$

for some $\gamma > 0$ with the unknown vector field $\mathbf{u}$ subject to the divergence-free condition $\nabla \cdot \mathbf{u} = 0$ and where the non-linear terms are subject to an α degree regularization

$$(-\Delta)^{\alpha/2}\mathbf{u}_\alpha = \Lambda^\alpha \mathbf{u}_\alpha = \mathbf{u}. \tag{3.33}$$

We suppose that (3.32) evolves on the periodic box $\mathbb{T}^n$ where $n = 2, 3$. To streamline our presentation and in view of the fact that damping terms are more natural for two dimensional flows, our main focus will be on the case $n = 2$. Here the assumed degree of regularization α in (3.33) is greater $2/3$. This lower bound is a strict inequality for the question of uniqueness. Note however that the case $n = 3$ can be addressed by a similar approach when we suppose that $\alpha \geq 2$. See Remark 3.3 at the conclusion of this section for further details.

There is a vast literature around regularizations (or mollifications) of the nonlinear terms in the Navier–Stokes and Euler equations. In fact, it is notable that such a regularization procedure was the basis for the first existence results for weak solutions dating back to the seminal work of Leray, [45]. In the more recent literature a variety of related systems explore this theme in the context of turbulence closure models, viscoelastic and non-newtonian fluids and a variety of other applications. See, for example, [9,11,16,22,42,47,48,54,55,57] and numerous containing references.

3.4.1 A Priori Estimates

We begin by illustrating some a-priori energy estimate for (3.32)–(3.33) which will guide us in the sequel. Notice that if we apply $\Lambda^{-\alpha/2}$ to (3.32) we obtain from the Itō lemma that

$$d\|\Lambda^{-\alpha/2}\mathbf{u}\|^2 + 2\gamma\|\Lambda^{-\alpha/2}\mathbf{u}\|^2 dt = \|\Lambda^{-\alpha/2}\sigma\|^2 + 2\langle\Lambda^{-\alpha/2}\sigma, \Lambda^{-\alpha/2}\mathbf{u}\rangle dW, \tag{3.34}$$

and hence exponential martingale bounds imply

$$\mathbb{P}\left(\sup_{t\geq 0}\left(\|\Lambda^{-\alpha/2}\mathbf{u}(t)\|^2 + \gamma\int_0^t \|\Lambda^{-\alpha/2}\mathbf{u}\|^2 ds - \|\Lambda^{-\alpha/2}\sigma\|^2 t + \|\Lambda^{-\alpha/2}\mathbf{u}_0\|^2\right) \geq K\right) \leq \exp(-cK),$$

for each $K > 0$ and some $c = c(\|\Lambda^{-\alpha/2}\sigma\|^2, \gamma)$ independent of K.

Next observe that, by taking $\xi = \operatorname{curl}\mathbf{u}$, $\rho = \operatorname{curl}\sigma$, we obtain the vorticity formulation of (3.32)

$$d\xi + (\gamma\xi + \mathbf{u}_\alpha \cdot \nabla\xi_\alpha - \xi_\alpha \cdot \nabla\mathbf{u}_\alpha)dt = \sum_{k=1}^{N} \rho_k dW^k,$$

In $n = 2$, our main concern here, the 'vortex stretching term' $\xi_\alpha \cdot \nabla\mathbf{u}_\alpha$ is absent and we obtain

$$d\xi + (\gamma\xi + \mathbf{u}_\alpha \cdot \nabla\xi_\alpha)dt = \sum_{k=1}^{N} \rho_k dW^k,$$

which, in this two dimensional case, implies

$$d\|\Lambda^{-\alpha/2}\xi\|^2 + 2\gamma\|\Lambda^{-\alpha/2}\xi\|^2 dt = \|\Lambda^{-\alpha/2}\rho\|^2 + 2\langle\Lambda^{-\alpha/2}\rho, \Lambda^{-\alpha/2}\xi\rangle dW, \tag{3.35}$$

and hence yields

$$\mathbb{P}\left(\sup_{t\geq 0}\left(\|\Lambda^{-\alpha/2}\xi(t)\|^2 + \gamma\int_0^t \|\Lambda^{-\alpha/2}\xi\|^2 ds - \|\Lambda^{-\alpha/2}\rho\|^2 t + \|\Lambda^{-\alpha/2}\xi_0\|^2\right) \geq K\right) \leq \exp(-cK),$$

for each $K > 0$ and some $c = c(\|\Lambda^{-\alpha/2}\rho\|^2, \gamma)$ independent of K. From (3.35) we can further prove that for $\eta = \eta(\|\Lambda^{-\alpha/2}\rho\|^2, \gamma)$

$$\mathbb{E}\exp(\eta\|\Lambda^{-\alpha/2}\xi(t)\|^2) \leq \exp(\eta(\gamma^{-1}\|\Lambda^{-\alpha/2}\rho\|^2 + e^{-\gamma t/2}\|\Lambda^{-\alpha/2}\xi_0\|^2)) \tag{3.36}$$

and we also have that

$$\mathbb{E}\exp\left(\eta\gamma\int_0^t \|\Lambda^{-\alpha/2}\xi(t)\|^2 ds\right) \leq \exp(\|\Lambda^{-\alpha/2}\rho\|^2 t + \|\Lambda^{-\alpha/2}\xi_0\|^2). \tag{3.37}$$

Note that the constant η appearing in (3.36), (3.37) may be taken to be less than 1.

Suppose that $\mathbf{u}$, $\tilde{\mathbf{u}}$ solve both (3.32)–(3.33) and take $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$ which satisfies

$$\partial_t\mathbf{v} + \gamma\mathbf{v} + \mathbf{v}_\alpha\cdot\nabla\mathbf{u}_\alpha + \tilde{\mathbf{u}}_\alpha\cdot\nabla\mathbf{v}_\alpha + \nabla q = 0, \quad \mathbf{v}(0) = \mathbf{u}(0) - \tilde{\mathbf{u}}(0)$$

with q the difference of the pressures. We immediately infer that

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\alpha/2}\mathbf{v}\|^2 + \gamma\|\Lambda^{-\alpha/2}\mathbf{v}\|^2 = -\int \mathbf{v}_\alpha\cdot\nabla\mathbf{u}_\alpha\cdot\mathbf{v}_\alpha dx. \tag{3.38}$$

When $\alpha \geq 2/3$ we have $\frac{1}{3} \geq \frac{1}{2} - \frac{\alpha}{4}$ and hence (in $n = 2$) with the Sobolev imbedding of $H^{\alpha/2} \subset L^3$ and basic properties of the Biot-Savart kernel we infer

$$\begin{aligned}\left|\int \mathbf{v}_\alpha\cdot\nabla\mathbf{u}_\alpha\cdot\mathbf{v}_\alpha dx\right| &\leq \|\mathbf{v}_\alpha\|_{L^3}^2\|\nabla\mathbf{u}_\alpha\|_{L^3} \leq C\|\Lambda^{-\alpha}\mathbf{v}\|_{H^{\alpha/2}}^2\|\Lambda^{-\alpha}\nabla\mathbf{u}\|_{H^{\alpha/2}}\\ &\leq C\|\Lambda^{-\alpha}\mathbf{v}\|_{H^{\alpha/2}}^2\|\Lambda^{-\alpha}\xi\|_{H^{\alpha/2}} \leq C\|\Lambda^{-\alpha/2}\mathbf{v}\|^2\|\Lambda^{-\alpha/2}\xi\|.\end{aligned} \tag{3.39}$$

3.4.2 Existence and Uniqueness of Solutions and Markov Semigroup

With these observation in hand we turn now to address the well-posedness for (3.32)–(3.33). The a priori bounds (3.34), (3.35) in combination with (3.38)–(3.39) are the basis of:

Proposition 3.4 *Assume that $\alpha \geq 2/3$ and consider (3.32) in the case $n = 2$. Then, for all $\mathbf{u}_0 \in H^{1-\alpha/2}$, there exists a unique*

$$\mathbf{u} \in L^2(\Omega; L^\infty_{loc}([0,\infty); H^{1-\alpha/2}))$$

evolving continuously in L^2 which is an adapted, pathwise solution of (3.32). Taking $\mathbf{u}(t, \mathbf{u}_0)$ as the unique solution associated to a given $\mathbf{u}_0 \in H^{1-\alpha/2}$ we have that

$$\mathbf{u}(t, \mathbf{u}_0^n) \to \mathbf{u}(t, \mathbf{u}_0) \textit{ almost surely in the} H^{-\alpha/2} \text{ topology}$$

for any sequence $\{\mathbf{u}_0^n\}_{n\geq 1} \subset H^{1-\alpha/2}$ converging in $H^{-\alpha/2}$.

The existence of solutions in this class may be established with a standard Faedo-Galerkin procedure. We omit further details.

Given Proposition 3.4 we may thus define the Markov transition kernels $\{P_t\}_{t\geq 0}$ associated to (3.32)–(3.33) as

$$P_t(\mathbf{u}_0, A) = \mathbb{P}(\mathbf{u}(t, \mathbf{u}_0) \in A).$$

These kernels are Feller in $H^{-\alpha/2}$ namely, given any $\phi \in C_b(H^{-\alpha/2})$, $t \geq 0$, $P_t\phi \in C_b(H^{-\alpha/2})$.

3.4.3 The Existence of an Invariant Measure ($n = 2$)

To prove the existence of an invariant measure we make use of the abstract results in Appendix. In the present concrete setting we take $V = H^{1-\alpha/2}$ and $H = H^{-\alpha/2}$. It is easy to see that (by for example taking ρ_n to be the projection onto H_n, the span of the first n elements of a sinusoidal basis) these spaces satisfy the conditions imposed on V, H in the Appendix. Notice moreover that, as we identified in Proposition 3.4 and the surrounding commentary, the Markov transition kernel associated to (3.32)–(3.33) is defined on V and is readily seen to be H-Feller.

In order to apply Corollary 4.2 and hence infer the existence of invariant states we now consider, for each $\epsilon > 0$, the viscous regularizations of (3.32) given as

$$d\mathbf{u}^\epsilon + (\gamma \mathbf{u}^\epsilon - \epsilon \Delta \mathbf{u}^\epsilon + \mathbf{u}^\epsilon_\alpha \cdot \nabla \mathbf{u}^\epsilon_\alpha + \nabla p)dt = \sum_{k=1}^{N} \sigma_k dW^k, \quad \nabla \cdot \mathbf{u}^\epsilon = 0, \quad \mathbf{u}^\epsilon(0) = \mathbf{u}_0. \tag{3.40}$$

As above (3.40) has an associated vorticity form

$$d\xi^\epsilon + (\gamma \xi^\epsilon - \epsilon \Delta \xi^\epsilon + \mathbf{u}^\epsilon_\alpha \cdot \nabla \xi^\epsilon_\alpha)dt = \sum_{k=1}^{N} \rho_k dW^k.$$

For the same reasons as (3.32)–(3.33) these equations define a collections of Markov kernels $\{P^\epsilon_t\}_{t\geq 0}$ for each $\epsilon > 0$ on $V = H^{1-\alpha/2}$.

From the Itō lemma we obtain an evolution like (3.35) for $\|\Lambda^{-\alpha/2}\xi^\epsilon\|^2$ but which has the additional viscous term $2\epsilon\|\nabla\Lambda^{-\alpha/2}\xi^\epsilon\|^2 dt$. We thus obtain, for any $t > 0$

$$\epsilon \mathbb{E} \int_0^t \|\nabla \Lambda^{-\alpha/2}\xi^\epsilon\|^2 ds = \epsilon \mathbb{E} \int_0^t \|\Lambda^{2-\alpha/2}\mathbf{u}^\epsilon\|^2 ds \leq \|\Lambda^{1-\alpha/2}\mathbf{u}^\epsilon_0\|^2 + \|\Lambda^{1-\alpha/2}\sigma\|^2 t. \tag{3.41}$$

Hence, by applying the Krylov-Bogoliubov averaging procedure we immediately infer, for all ϵ strictly positive, that there existence of an invariant μ^ϵ for the Markov semigroup P^ϵ associated with (3.40). Noting that the bound (3.36) also holds for ξ^ϵ with all of the constants independent of $\epsilon > 0$ and we infer

$$\sup_{\epsilon > 0} \int \exp(\eta \|\Lambda^{1-\alpha/2}\mathbf{u}\|^2) d\mu^\epsilon(\mathbf{u}) \leq C < \infty. \tag{3.42}$$

We have thus established the condition (4.2) for the collection of invariant measure for P^ϵ. The existence now follows once we establish (4.1) in our setting.

For this purpose fix any initial condition $\mathbf{u}_0 \in H^{1-\alpha/2}$. Observe that $\mathbf{v}^\epsilon = \mathbf{u}(t, \mathbf{u}_0) - \mathbf{u}^\epsilon(t, \mathbf{u}_0)$ satisfies

$$\partial_t \mathbf{v}^\epsilon + \gamma \mathbf{v}^\epsilon + \mathbf{v}^\epsilon_\alpha \cdot \nabla \mathbf{u}_\alpha + \mathbf{u}^\epsilon_\alpha \cdot \nabla \mathbf{v}^\epsilon_\alpha + \nabla p + \epsilon \Delta \mathbf{u}^\epsilon = 0, \quad \mathbf{v}^\epsilon(0) = 0. \tag{3.43}$$

Similarly to above in (3.39),

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\|^2 + \gamma\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\|^2 \le C\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\|^2\|\Lambda^{-\alpha/2}\xi\| + \epsilon\|\Delta\Lambda^{-\alpha/2}\mathbf{u}^\epsilon\|\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\|$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\| \le C\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\|\|\Lambda^{-\alpha/2}\xi\| + \epsilon\|\Delta\Lambda^{-\alpha/2}\mathbf{u}^\epsilon\|$$

which implies

$$\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\| \le \sqrt{\epsilon}\exp\left(Ct + \frac{\eta\gamma}{2}\int_0^t \|\Lambda^{-\alpha/2}\xi\|^2 ds\right)\int_0^t \sqrt{\epsilon}\|\nabla\Lambda^{-\alpha/2}\xi^\epsilon\| ds.$$

Taking expected values we find

$$\mathbb{E}\|\Lambda^{-\alpha/2}\mathbf{v}^\epsilon\| \le \sqrt{\epsilon}\sqrt{t}\left(\mathbb{E}\exp\left(Ct + \eta\gamma\int_0^t \|\Lambda^{-\alpha/2}\xi\|^2 ds\right)\right)^{1/2}\left(\epsilon\mathbb{E}\int_0^t \|\nabla\Lambda^{-\alpha/2}\xi^\epsilon\|^2 ds\right)^{1/2}. \tag{3.44}$$

Combining this bound with (3.37) (which holds for solution of (3.40) with constant independent of $\epsilon > 0$) and (3.41) we conclude that

$$\mathbb{E}\|\Lambda^{-\alpha/2}(\mathbf{u}(t, \mathbf{u}_0) - \mathbf{u}^\epsilon(t, \mathbf{u}_0))\| \le \sqrt{\epsilon}\exp(C(t + \|\Lambda^{1-\alpha/2}\mathbf{u}_0\|^2))$$

for a constant C independent of t, ϵ and $\mathbf{u}_0$. The condition 4.1 now follows and in conclusion we have that

Proposition 3.5 *Assume that $\alpha \ge 2/3$ and consider (3.32) in the case $n = 2$. Then for any $\gamma > 0$ there exists at least one invariant measure μ of (3.32) such that*

$$\int \exp(\eta\|\Lambda^{1-\alpha/2}\mathbf{u}\|^2) d\mu(\mathbf{u}) < \infty.$$

3.4.4 Uniqueness of the Invariant Measure (n = 2)

In order to establish the uniqueness of the invariant measure identified in (3.5) fix any $\mathbf{u}_0, \tilde{\mathbf{u}}_0$. We take $\mathbf{u} = \mathbf{u}(t, \mathbf{u}_0)$ as the associated solution of (3.32) and consider $\tilde{\mathbf{u}}$ solving

$$d\tilde{\mathbf{u}} + (\gamma\tilde{\mathbf{u}} + \tilde{\mathbf{u}}_\alpha \cdot \nabla\tilde{\mathbf{u}}_\alpha + \nabla p)dt = \lambda P_N(\mathbf{u} - \tilde{\mathbf{u}})dt\,\mathbb{1}_{t\le\tau_R}dt + \sum_{k=1}^N \sigma_k dW^k, \quad \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0,$$

where τ_R is the stopping time

$$\tau_R := \inf_{t\ge 0}\left\{\int_0^t \lambda^2\|P_N(\mathbf{u} - \tilde{\mathbf{u}})\|^2 dt > R\right\}.$$

Here λ, R are parameters to be determined presently. Let $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$ and observe that

$$\partial_t \mathbf{v} + \gamma\mathbf{v} + \mathbf{u}_\alpha \cdot \nabla\mathbf{v}_\alpha + \mathbf{v}_\alpha \cdot \nabla\mathbf{u}_\alpha + \nabla p = -\lambda P_N \mathbf{v}\,\mathbb{1}_{t\le\tau_R},$$

so that on the interval $[0, \tau_R]$

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\alpha/2}\mathbf{v}\|^2 + \gamma\|\Lambda^{-\alpha/2}\mathbf{v}\|^2 + \lambda\|P_N\Lambda^{-\alpha/2}\mathbf{v}\|^2 = -\int \mathbf{v}_\alpha \cdot \nabla\mathbf{u}_\alpha \cdot \mathbf{v}_\alpha dx. \qquad (3.45)$$

We suppose that $\alpha > 2/3$ so that for some $\delta = \delta(\alpha) > 0$ we have that $H^{\alpha/2-\delta} \subset L^3$. As such, cf. (3.39), we have from the inverse poincare inequality that

$$\begin{aligned}&\left|\int \mathbf{v}_\alpha \cdot \nabla\mathbf{u}_\alpha \cdot \mathbf{v}_\alpha dx\right| \\ &\qquad \le C(\|P_N\Lambda^{-\alpha}\mathbf{v}\|^2_{H^{\alpha/2}} + \|Q_N\Lambda^{-\alpha}\mathbf{v}\|^2_{H^{\alpha/2-\delta}})\|\Lambda^{-\alpha}\xi\|_{H^{\alpha/2}} \\ &\qquad \le \lambda\|P_N\Lambda^{-\alpha}\mathbf{v}\|^2_{H^{\alpha/2}} + \frac{C}{\lambda}\|\Lambda^{-\alpha}\mathbf{v}\|^2_{H^{\alpha/2}}\|\Lambda^{-\alpha}\xi\|^2_{H^{\alpha/2}} + \frac{C}{N^\delta}\|\Lambda^{-\alpha}\mathbf{v}\|^2_{H^{\alpha/2}}\|\Lambda^{-\alpha}\xi\|_{H^{\alpha/2}}.\end{aligned} \qquad (3.46)$$

Combining this bound with (3.45) and rearranging we find that $[0, \tau_R]$

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\alpha/2}\mathbf{v}\|^2 + \left(\gamma - C(\lambda^{-1} + N^{-\delta})(1 + \|\Lambda^{-\alpha}q\|^2_{H^{\alpha/2}})\right)\|\Lambda^{-\alpha/2}\mathbf{v}\|^2 \le 0$$

We emphasize that C depends only on quantities coming from Sobolev embedding and that δ only depends on α. Both quantities are independent of our choice of $R > 0$. Thus, by choosing λ and N sufficiently large (depending again only on $\alpha, \gamma, \|\Lambda^{-\alpha/2}\rho\|^2$ and universal quantities), we obtain the bound

$$\begin{aligned}&\|\Lambda^{-\alpha/2}\mathbf{v}(t \wedge \tau_R)\|^2 \\ &\quad \le \exp\left(-\frac{\gamma}{2}t \wedge \tau_R + \frac{\gamma \min\{\gamma, 1\}}{4\max\{\|\Lambda^{-\alpha/2}\rho\|^2, 1\}}\int_0^{t\wedge\tau_R}\|\Lambda^{-\alpha}\xi\|^2_{H^{\alpha/2}}ds\right)\|\Lambda^{-\alpha/2}(\mathbf{u}_0 - \tilde{\mathbf{u}}_0)\|^2\end{aligned}$$

This implies that on the set

$$E_K := \left\{\sup_{t\ge 0}\left(\|\Lambda^{-\alpha/2}\xi(t)\|^2 + \gamma\int_0^t \|\Lambda^{-\alpha/2}\xi\|^2 ds - (\|\Lambda^{-\alpha/2}\rho\|^2 t + \|\Lambda^{-\alpha/2}\xi_0\|^2)\right) \le K\right\}$$

we have

$$\|\Lambda^{-\alpha/2}\mathbf{v}(t \wedge \tau_R)\|^2 \le \exp\left(-\frac{\gamma}{4}t \wedge \tau_R + K + \|\Lambda^{-\alpha/2}\xi_0\|^2\right)\|\Lambda^{-\alpha/2}(\mathbf{u}_0 - \tilde{\mathbf{u}}_0)\|^2.$$

By now choosing K large enough that $\mathbb{P}(E_K) > 1/2$ and then taking R sufficiently large we now obtain

Proposition 3.6 *Consider* (3.32) *in the case* $n = 2$. *Then for any* $\gamma > 0$ *and any* $\alpha > 2/3$ *there exists an* $N = N(\alpha, \gamma, \|\Lambda^{-\alpha/2}\rho\|^2)$ *such that if* $H_N \subset Range(\sigma)$ *then* (3.32) *has at most one invariant measure.*

Remark 3.3 (The Three Dimensional Case) As already mentioned the approach taken here also yields the existence and uniqueness of invariant measures for (3.32)–(3.33) in dimensional three whenever $\alpha \ge 2$. The following modifications of the proof are required primarily as a consequence of the fact that we are not able to make use of the vorticity formulation in 3D as above in (3.35). Firstly we note that we consider solutions $u \in L^2(\Omega; L^\infty([0,\infty); H^{-\alpha/2})$. Taking $\mathbf{v}$ to be the difference of two solutions, uniqueness and continuous dependence on data in $H^{-\alpha/2}$ follows from the estimate

$$\left|\int \mathbf{v}_\alpha \cdot \nabla \mathbf{u}_\alpha \cdot \mathbf{v}_\alpha dx\right| \leq \|\nabla \mathbf{u}_\alpha\| \|\mathbf{v}_\alpha\|_{L^4}^2 \leq \|\Lambda^{-\alpha/2}\mathbf{u}\| \|\Lambda^{3/4-\alpha}\mathbf{v}\|^2 \leq \|\Lambda^{-\alpha/2}\mathbf{u}\| \|\Lambda^{-\alpha/2}\mathbf{v}\|^2 \tag{3.47}$$

which we may combine with (3.34) to close (3.38). The estimates leading to the existence of an invariant measure are also a little different. Here we take $V = H^{-\alpha/2}$ and $H = H^{-\alpha}$. The bounds (3.41) and (3.42) are replaced with

$$\sup_{\epsilon>0, t\geq 1} \epsilon \mathbb{E}\frac{1}{t}\int_0^t \|\Lambda^{1-\alpha/2}\mathbf{u}^\epsilon\|^2 ds + \sup_{\epsilon>0}\int \exp(\eta\|\Lambda^{-\alpha/2}\mathbf{u}\|^2) d\mu^\epsilon(\mathbf{u}) \leq C < \infty.$$

and the convergence $\mathbf{u}^\epsilon \to \mathbf{u}$ is now carried out in the $H^{-\alpha}$ topology. For the convergence, taking $\mathbf{v}^\epsilon = \mathbf{u} - \mathbf{u}^\epsilon$, (3.43) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\alpha}\mathbf{v}^\epsilon\|^2 + \gamma\|\Lambda^{-\alpha}\mathbf{v}^\epsilon\|^2 \\
&\leq \left|\int (\mathbf{v}^\epsilon_\alpha \cdot \nabla \mathbf{u}_\alpha + \mathbf{u}^\epsilon_\alpha \cdot \nabla \mathbf{v}^\epsilon_\alpha)\Lambda^{-2\alpha}\mathbf{v}^\epsilon dx\right| + \epsilon\|\Delta\Lambda^{-\alpha}\mathbf{u}^\epsilon\|\|\Lambda^{-\alpha}\mathbf{v}^\epsilon\| \\
&\leq C\|\Lambda^{-\alpha}\mathbf{v}^\epsilon\|^2(\|\Lambda^{-\alpha/2}\mathbf{u}\| + \|\Lambda^{-\alpha/2}\mathbf{u}^\epsilon\|) + \epsilon\|\Lambda^{1-\alpha/2}\mathbf{u}^\epsilon\|\|\Lambda^{-\alpha}\mathbf{v}^\epsilon\|.
\end{aligned}$$

so that the convergence required by the abstract condition (4.1) now follows in a similar fashion to (3.44) above. Finally regarding the uniqueness, the strategy is essentially the same once we notice that (3.47) provides the sub-criticality necessary to replace the estimate (3.46).

3.5 A Damped Nonlinear Wave Equation

Our final example is the damped Sine–Gordon equation which we write formally as

$$\partial_{tt}u + \alpha\partial_t u - \Delta u + \beta\sin(u) = \sum_{k=1}^d \sigma_k \dot{W}^k. \tag{3.48}$$

Here the unknown u evolves on a bounded domain $\mathcal{D} \subset \mathbb{R}^n$ with smooth boundary and satisfies the Dirichlet boundary condition $u_{\partial\mathcal{D}} \equiv 0$. The parameter α is strictly positive and β is a given real number. The functions σ_k on $\mathcal{D}$ will be specified below, and $\dot{W}^k$ represent a sequence of independent white noise processes. This is written more rigorously as the system of stochastic partial differential equations

$$dv + (\alpha v - \Delta u + \beta\sin(u))dt = \sum_{k=1}^d \sigma_k dW^k, \quad \frac{du}{dt} = v, \tag{3.49}$$

which we supplement with the initial condition $u(0) = u_0$, $v(0) = v_0$.

The deterministic Sine–Gordon equation appears in the description of continuous Josephson junctions [46], and has been studied extensively in a variety of contexts [2,5,10,15,20,30,43,66]. For example, analysis of the existence and finite dimensionality of the attractor for the deterministic counterpart of (3.48) can be found in [61].

3.5.1 Mathematical Preliminaries

For any given $(u_0, v_0) \in X := H^1_0(\mathcal{D}) \times L^2(\mathcal{D})$ there exists a unique $U = (u, v) \in L^2(\Omega; C([0,\infty), X)$ which is a (weak) solution of (3.49). These solutions $U(t) = U(t, U_0)$ depend continuously on $U_0 = (u_0, v_0) \in X$ and hence $P_t\phi(U_0) := \mathbb{E}\phi(U(t, U_0))$ is a Feller

Markov semigroup acting on $C_b(X)$. Moreover when $(u_0, v_0) \in Y := (H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})) \times H_0^1(\mathcal{D})$ the corresponding solution satisfies $U \in L^2(\Omega; C([0,\infty), Y)$. In what follows we will maintain the standing convention that $|\cdot| = \|\cdot\|_{L^2}$ and $\|\cdot\| = \|\cdot\|_{H^1}$ with all other norms given explicitly.

The existence of solutions may be established via standard compactness methods starting from a Galerkin truncation of (3.49) and making use of the following a priori estimates. Take $r = v + \epsilon u$ with $\epsilon > 0$ to be specified presently. Evidently

$$dr + (\alpha - \epsilon) r dt = (\epsilon(\alpha - \epsilon)u + \Delta u - \beta \sin(u))\, dt + \sum_{k=1}^{d} \sigma_k dW^k. \tag{3.50}$$

From the Itō lemma we infer

$$d|r|^2 + 2(\alpha - \epsilon)|r|^2 dt = \left(2\epsilon(\alpha - \epsilon)\langle u, r\rangle + 2\langle \Delta u, r\rangle - 2\beta\langle \sin(u), r\rangle + |\sigma|^2\right) dt + 2\langle \sigma, r\rangle dW.$$

Now since

$$2\langle \Delta u, r\rangle = -\frac{d}{dt}\|u\|^2 - 2\epsilon\|u\|^2,$$

we infer that when $\epsilon \leq \alpha/2$

$$\begin{aligned} &d(|r|^2 + \|u\|^2) + (\alpha|r|^2 + 2\epsilon\|u\|^2)dt \\ &\quad\leq \left(\frac{\epsilon\alpha}{\sqrt{\lambda}}\|u\||r| + 2|\beta||\mathcal{D}|^{1/2}|r| + |\sigma|^2\right) dt + 2\langle \sigma, r\rangle dW, \end{aligned}$$

where $\lambda = \lambda(\mathcal{D})$ is the Poincaré constant. By now choosing

$$\epsilon := \min\left\{\frac{\lambda}{\alpha}, \frac{\alpha}{2}, \sqrt{\frac{\lambda}{2}}\right\}, \tag{3.51}$$

we have that

$$d(|r|^2 + \|u\|^2) + \epsilon\left(|r|^2 + \|u\|^2\right) dt \leq \left(\frac{4|\beta|^2|\mathcal{D}|}{\alpha} + |\sigma|^2\right) dt + 2\langle \sigma, r\rangle dW. \tag{3.52}$$

and that

$$\frac{1}{2}(|v|^2 + \|u\|^2) \leq |r|^2 + \|u\|^2 \leq 2(|v|^2 + \|u\|^2). \tag{3.53}$$

Combining the previous two inequalities and using the exponential Martingale bound, (3.4), we conclude

$$\begin{aligned} &\mathbb{P}\Bigg(\sup_{t\geq 0}\Bigg[\frac{1}{2}|v(t)|^2 + \|u(t)\|^2 + \frac{\epsilon}{4}\int_0^t \left(|v(s)|^2 + \|u(s)\|^2\right) ds \\ &\quad - \left(\frac{4|\beta|^2|\mathcal{D}|}{\alpha} + |\sigma|^2\right)t - 2\left(|v_0|^2 + \|u_0\|^2\right)\Bigg] \geq K\Bigg) \leq e^{-\gamma K}, \end{aligned}$$

for every $K > 0$ where $\gamma = \gamma(|\sigma|, \alpha) > 0$ is independent of K and of the solution $U = (u, v)$.

In order to prove the existence of an invariant measure for $\{P_t\}_{t\geq 0}$ we next establish suitable bounds for $U = (u, v)$ in $Y = (H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})) \times H_0^1(\mathcal{D})$. Denote $-\Delta$ with

Dirchlet boundary conditions as A. Applying $A^{1/2}$ to (3.50) and then invoking the Itō lemma we obtain

$$
\begin{aligned}
&d\|r\|^2 + 2(\alpha - \epsilon)\|r\|^2 dt \\
&\quad = \big(2\epsilon(\alpha - \epsilon)\langle A^{1/2}u, A^{1/2}r\rangle - 2\langle \Delta u, \Delta r\rangle - 2\beta\langle A^{1/2}\sin(u), A^{1/2}r\rangle + \|\sigma\|^2\big)\, dt \\
&\qquad + 2\langle A^{1/2}\sigma, A^{1/2}r\rangle dW,
\end{aligned}
$$

and hence estimating as above and imposing the same condition on ϵ we find

$$
d(\|r\|^2 + |Au|^2) + \epsilon(\|r\|^2 + |Au|^2)dt \leq \left(\|\sigma\|^2 + \frac{4|\beta|^2}{\alpha}\|u\|^2\right)dt + 2\langle A^{1/2}\sigma, A^{1/2}r\rangle dW. \tag{3.54}
$$

Combining (3.54) and (3.52) and noting that, similarly to (3.53), $\|r\|^2 + \|u\|^2_{H^2} \leq 2(\|v\|^2 + \|u\|^2_{H^2})$ we now infer

$$
\int_0^T \mathbb{E}(\|v(t)\|^2 + \|u(t)\|^2_{H^2})dt \leq C\left(\int_0^T (\mathbb{E}\|u(t)\|^2 + 1)dt\right) \leq CT,
$$

for any $T > 0$ when $u_0 = v_0 \equiv 0$. Here the constant $C = C(\sigma, \beta, \alpha, \mathcal{D})$ but is independent of T. The existence of an ergodic invariant measure $\mu \in Pr(X)$ for (3.49) now follows from the Krylov–Bogolyubov theorem.

3.5.2 Asymptotic Coupling Arguments

To establish the uniqueness of invariant measures for (3.49) we fix arbitrary $U_0, \tilde{U}_0 \in X = H^1_0(\mathcal{D}) \times L^2(\mathcal{D})$. Take $U = (u, v)$ to be the solution of (3.49) corresponding to U_0 and let $\tilde{U} = (\tilde{u}, \tilde{v})$ be the solution of

$$
d\tilde{v} + (\alpha\tilde{v} - \Delta\tilde{u} + \beta\sin(\tilde{u}) - \beta 1\!\!1_{\tau_K > t} P_N(\sin(u) - \sin(\tilde{u})))dt = \sum_k \sigma_k dW^k, \quad \frac{d}{dt}\tilde{u} = \tilde{v} \tag{3.55}
$$

where $\tilde{u}(0) = \tilde{u}_0$, $\tilde{v}(0) = \tilde{v}_0$, and

$$
\tau_K := \inf_{t \geq 0}\left\{\int_0^t |u - \tilde{u}|^2 ds \geq K\right\}.
$$

In the framework of Sect. 2.4, we have taken $G(u, \tilde{u}) = P_N(\sin(u) - \sin(\tilde{u}))$ rather than $\lambda P_N(u - \tilde{u})$ as in the preceding sections. It follows that $h(t) = 1\!\!1_{\tau_K > t}\sigma^{-1}\beta P_N(\sin(u) - \sin(\tilde{u}))$ is a continuous adapted process in $\mathbb{R}^N$ which satisfies the Novikov condition (2.3). Taking $w = u - \tilde{u}$ and subtracting (3.55) from (3.49) we obtain

$$
\partial_{tt}w + \alpha\partial_t w - \Delta w = \beta(\sin(\tilde{u}) - \sin(u)) - 1\!\!1_{\tau_K > t}\beta P_N(\sin(\tilde{u}) - \sin(u)).
$$

Modifying slightly the method of previous examples, uniqueness of the invariant measure will follow from showing that for $N, K > 0$ sufficiently large, $\tau_K = \infty$ almost surely, and moreover

$$
|\partial_t w(t)|^2 + \|w(t)\|^2 \to 0 \text{ as } t \to \infty.
$$

To this end, we again pursue the strategy leading to (3.52), (3.54) and introduce $y = \partial_t w + \epsilon w$ with ϵ as in (3.51). Similarly to (3.50) above y satisfies

$$\partial_t y + (\alpha - \epsilon)y - \Delta w = \epsilon(\alpha - \epsilon)w + \beta(\sin(\tilde{u}) - \sin(u)).$$

This equation can be projected to low and high frequencies, giving

$$\partial_t P_N y + (\alpha - \epsilon)P_N y - \Delta P_N w = \epsilon(\alpha - \epsilon)P_N w + 1\!\!1_{\tau_K \le t}\beta P_N(\sin(\tilde{u}) - \sin(u)).$$
$$\partial_t Q_N y + (\alpha - \epsilon)Q_N y - \Delta Q_N w = \epsilon(\alpha - \epsilon)Q_N w + \beta Q_N(\sin(\tilde{u}) - \sin(u)).$$

Multiplying these expressions by y, and integrating over $\mathcal{D}$, when $t < \tau_K$ this gives

$$\frac{d}{dt}(|P_N y|^2 + \|P_N w\|^2) + \epsilon(|P_N y|^2 + \|P_N w\|^2) \le 0,$$
$$\frac{d}{dt}(|Q_N y|^2 + \|Q_N w\|^2) + \epsilon(|Q_N y|^2 + \|Q_N w\|^2) \le \beta\langle Q_N(\sin(\tilde{u}) - \sin(u)), Q_N y\rangle.$$

By Grönwall's inequality

$$(|P_N y|^2 + \|P_N w\|^2)(t \wedge \tau_K) \le e^{-\epsilon t \wedge \tau_K}(|P_N y_0|^2 + \|P_N w_0\|^2), \tag{3.56}$$

and using the inverse Poincaré inequality, taking $N = N(\beta, \epsilon)$ sufficiently large, we find

$$\begin{aligned}
&\frac{d}{dt}(|Q_N y|^2 + \|Q_N w\|^2) + \epsilon(|Q_N y|^2 + \|Q_N w\|^2) \\
&\quad \le |\beta||w||Q_N y| \le |\beta||P_N w||Q_N y| + |\beta||Q_N w||Q_N y| \\
&\quad \le \frac{\epsilon}{4}|Q_N y|^2 + C_\epsilon|\beta|^2|P_N w|^2 + \frac{|\beta|}{\lambda_N}\|Q_N w\||Q_N y| \\
&\quad \le \frac{\epsilon}{2}(|Q_N y|^2 + \|Q_N w\|^2) + C_\epsilon|\beta|^2|P_N w|^2.
\end{aligned}$$

Applying Grönwall once more and then making use of (3.56) we find that for for $t < \tau_K$,

$$\begin{aligned}
(|Q_N y|^2 + \|Q_N w\|^2)(t) &\le e^{-\frac{\epsilon}{2}t}(|Q_N y_0|^2 + \|Q_N w_0\|^2) + C_\epsilon|\beta|^2\int_0^t e^{-\frac{\epsilon}{2}(t-s)}|P_N w(s)|^2 ds \\
&\le e^{-\frac{\epsilon}{2}t}(|Q_N y_0|^2 + \|Q_N w_0\|^2) + \tilde{C}_\epsilon|\beta|^2 e^{-\frac{\epsilon}{2}t}(|P_N y_0|^2 + \|P_N w_0\|^2).
\end{aligned}$$

Combining the estimates on the high and low modes,

$$\begin{aligned}
&(|y|^2 + \|w\|^2)(t \wedge \tau_K) \\
&\quad \le e^{-\frac{\epsilon}{2}t\wedge\tau_K}\left((e^{-\frac{\epsilon}{2}t\wedge\tau_K} + \tilde{C}_\epsilon|\beta|^2)(|P_N y_0|^2 + \|P_N w_0\|^2) + |Q_N y_0|^2 + \|Q_N w_0\|^2\right),
\end{aligned}$$

and we conclude that $\tau_K = \infty$ almost surely for K sufficiently large. Moreover, due to (3.53) the convergence $|\partial_t w|^2 + \|w\|2 \le 2(|y|^2 + \|w\|^2) \to 0$ is obtained, almost surely.

In summary we have proven the following result

Proposition 3.7 *For every $\alpha > 0$, $\beta \in \mathbb{R}$ and $N \ge 0$ (3.49) possesses an ergodic invariant measure μ. Moreover for each $\alpha > 0$ and $\beta \in \mathbb{R}$ there exists an $N = N(\alpha, |\beta|)$ such that if* $\text{Range}(\sigma) \supset P_N L^2(\mathcal{D})$*, then μ is unique.*

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Appendix: Existence of Invariant Measures by a Limiting Procedure

We now present some absotract results which are used above to infer the existence of an invariant measure via an approximation procedure relying on invariant measures for a collection of regularized systems. It was used in Sect. 3.4.3 to prove the existence of an invariant measure.

Let $(H, \|\cdot\|_H)$, $(V, \|\cdot\|_V)$ be two separable Banach spaces. The associated Borel σ-algebras are denoted as $\mathcal{B}(H)$ and $\mathcal{B}(V)$ respectively. We suppose that V is continuously and compactly embedded in H. Moreover we assume that there exists continuous functions $\rho_n : H \to V$ for $n \geq 1$ such that

$$\lim_{n\to\infty} \|\rho_n(u)\|_V = \begin{cases} \|u\|_V & \text{for } u \in V \\ \infty & \text{for } u \in H \setminus V. \end{cases}$$

Notice that, under these circumstances, $\mathcal{B}(V) \subset \mathcal{B}(H)$ and moreover that $A \cap V \in \mathcal{B}(V)$ for any $A \in \mathcal{B}(H)$. We can therefore extend any Borel measure μ on V to a measure μ_E on H by setting $\mu_E(A) = \mu(A \cap V)$ and hence we identify $Pr(V) \subset Pr(H)$. This natural extension will be made without further comment in what follows.

By appropriately restricting the domain of elements $\phi \in C_b(H)$ to V we have that $C_b(H) \subset C_b(V)$. Similarly $\text{Lip}(H) \subset \text{Lip}(V)$, etc. Furthermore, under the given conditions on H and V, $C_b(H) \cap \text{Lip}(H)$ determines measures in $Pr(V)$ namely if $\int_V \phi\, d\mu = \int_V \phi\, d\nu$ for all $\phi \in C_b(H) \cap \text{Lip}(H)$ then $\mu = \nu$.

On V we consider a Markov transition kernel P, which is assumed to be Feller in H, that is to say P maps $C_b(H)$ to itself. We also suppose that $\{P^\epsilon\}_{\epsilon>0}$ is a sequence of Markov transition kernels (again defined on V) such that, for any $\phi \in C_b(H) \cap \text{Lip}(H)$, and $R > 0$,

$$\lim_{\epsilon\to 0} \sup_{u\in B_R(V)} |P^\epsilon \phi(u) - P\phi(u)| = 0, \tag{4.1}$$

where $B_R(V)$ is the ball of radius R in V.

Lemma 4.1 *In the above setting, let $\{\mu^\epsilon\}_{\epsilon>0}$ be a sequence of probability measures on V. Assume that there is an increasing continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(r) \to \infty$ as $r \to \infty$ and a finite constant $C_0 > 0$ so that*

$$\sup_{\epsilon>0} \int \psi(\|u\|_V) d\mu^\epsilon \leq C_0. \tag{4.2}$$

Then there exists a probability measure μ, supported on V, with $\int \psi(\|u\|_V) d\mu(u) \leq C_0$ such that (up to a subsequence) $\mu^\epsilon P^\epsilon$ converges weakly in H to μP that is, for all $\phi \in C_b(H)$,

$$\lim_{\epsilon\to 0} \mu^\epsilon P^\epsilon \phi = \mu P \phi \tag{4.3}$$

Proof of Lemma 4.1 From our assumption we know that

$$\mu^\epsilon(\psi(\|u\|_V) \geq R) \leq C_0/\psi(R) \tag{4.4}$$

for all $\epsilon > 0$. We infer that the family of measures $\{\mu^\epsilon\}_{\epsilon>0}$ is tight on H and thus that there exists a measure μ on H such that μ^{ϵ_n} converges weakly in H to μ for some decreasing subsequence $\epsilon_n \to 0$. For $k, m \geq 1$ define $f_{k,m} \in C_b(H)$ as $f_{k,m}(u) := \psi(\|\rho_m(u)\|_V) \wedge k$. Weak convergence in H implies that $\int f_{k,m} d\mu^{\epsilon_n} \to \int f_{k,m} d\mu \leq C_0$ as $n \to \infty$ for each fixed k, m. Fatou's lemma then implies that

$$\int \psi(\|u\|_V) d\mu(u) \leq \lim_{k,m\to\infty} \int f_{k,m}(u) d\mu(u) \leq C_0$$

and in particular that $\mu(V) = 1$.

We now turn to demonstrate (4.3). Observe that, for any $\phi \in C_b(H)$ and any $\epsilon > 0$,

$$\left|\mu^\epsilon P^\epsilon \phi - \mu P\phi\right| \leq \left|\mu^\epsilon P^\epsilon \phi - \mu^\epsilon P\phi\right| + \left|\mu^\epsilon P\phi - \mu P\phi\right| \quad (4.5)$$

Taking $\epsilon = \epsilon_n$, the first term is bounded as

$$\left|\mu^{\epsilon_n} P^{\epsilon_n}\phi - \mu^{\epsilon_n} P\phi\right| \leq \sup_{u\in B_S(V)} |P^{\epsilon_n}\phi(u) - P\phi(u)| + 2\sup_u |\phi(u)|\, \mu^{\epsilon_n}(B_S(V)^c) \quad (4.6)$$

for any $S > 0$. Combining (4.5), (4.6) with (4.1), (4.4), using that μ^{ϵ_n} converges weakly in H and that $P^\epsilon\phi \in C_b(H)$ we infer (4.3), completing the proof. □

This produces the following corollary.

Corollary 4.2 *In the above setting, if in addition we assume that, for every $\epsilon > 0$, μ^ϵ is an invariant measure for P^ϵ then the limiting measure μ is an invariant measure of P.*

Proof By the above result we may pick $\epsilon_n \to 0$ such that μ^{ϵ_n} and $\mu^{\epsilon_n} P^{\epsilon_n}$ converge weakly in H to μ and μP respectively. However since $\mu^{\epsilon_n} P^{\epsilon_n} = \mu^{\epsilon_n}$ we also have that $\mu^{\epsilon_n} P^{\epsilon_n}$ converges weakly in H to μ. Hence we conclude that $\mu P = \mu$ which is means the μ is an invariant measure for P. □

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