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Overlap Functions for Measures in Conformal Iterated Function Systems

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Abstract We employ thermodynamic formalism for the study of conformal iterated function systems (IFS) $\mathcal{S} = \{\phi_i\}_{i \in I}$ with arbitrary overlaps, and of measures μ on limit sets Λ, which are projections of equilibrium measures $\hat{\mu}$ with respect to a certain lift map Φ on $\Sigma_I^+ \times \Lambda$. No type of Open Set Condition is assumed. We introduce a notion of overlap function and overlap number for such a measure $\hat{\mu}$ with respect to $\mathcal{S}$; and, in particular a notion of (topological) overlap number $o(\mathcal{S})$. These notions take in consideration the n-chains between points in the limit set. We prove that $o(\mathcal{S}, \hat{\mu})$ is related to a conditional entropy of $\hat{\mu}$ with respect to the lift Φ. Various types of projections to Λ of invariant measures are studied. We obtain upper estimates for the Hausdorff dimension $HD(\mu)$ of μ on Λ, by using pressure functions and $o(\mathcal{S}, \hat{\mu})$. In particular, this applies to projections of Bernoulli measures on Σ_I^+. Next, we apply the results to Bernoulli convolutions ν_λ for $\lambda \in (\frac{1}{2}, 1)$, which correspond to self-similar measures determined by composing, with equal probabilities, the contractions of an IFS with overlaps $\mathcal{S}_\lambda$. We prove that for *all* $\lambda \in (\frac{1}{2}, 1)$, there exists a relation between $HD(\nu_\lambda)$ and the overlap number $o(\mathcal{S}_\lambda)$. We also estimate $o(\mathcal{S}_\lambda)$ for certain values of λ.

Keywords Thermodynamic formalism for iterated function systems with overlaps · Equilibrium measures · One-sided symbolic spaces · Overlap numbers for fractals · Dimensions of measures · Pressure functions · Entropies of measures

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1 Introduction and Outline

Iterated function systems (IFS) have been studied by many authors, and a lot about their theory is known. In many instances, systems which satisfy the Open Set Condition were studied. When arbitrary overlaps of the images of the contractions are allowed, the theory is different and the results from the case of Open Set Condition do not work anymore.

Let us consider a finite set I and an iterated function system $\mathcal{S} = \{\phi_i, i \in I\}$ consisting of injective conformal contractions ϕ_i defined on the closure of an open set $V \subset \mathbb{R}^q, q \geq 1$. Denote by Σ_I^+ the one-sided space $\{\omega = (\omega_1, \omega_2, \ldots), \omega_j \in I, j \geq 1\}$, with its shift endomorphism $\sigma : \Sigma_I^+ \to \Sigma_I^+, \sigma(\omega) = (\omega_2, \omega_3, \ldots)$. For an arbitrary sequence ω and for an integer $n \geq 1$, let the n-truncation $\omega|_n$ be the finite sequence $(\omega_1, \ldots, \omega_n)$. Also by $[i_1 \ldots i_n]$ we denote the n-cylinder $\{\omega \in \Sigma_I^+, \omega_1 = i_1, \ldots, \omega_n = i_n\}, n \geq 1, i_1, \ldots, i_n \in I$.

Let denote now by Λ the fractal *limit set* of the iterated function system $\mathcal{S}$, where:

$$\Lambda := \bigcup_{\omega \in \Sigma_I^+} \bigcap_{n \geq 1} \phi_{\omega|_n}(V)$$

Since all the maps ϕ_i are contractions, we can define the canonical coding map $\pi : \Sigma_I^+ \to \Lambda$, $\pi(\omega) = \lim_{n \to \infty} \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}(V)$, for all $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma_I^+$. The singleton $\pi(\omega)$ will also be denoted by $\phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots$, as this infinite composition is in fact a point. We will denote the composition $\phi_{i_1} \circ \cdots \circ \phi_{i_m}$ also by $\phi_{i_1 \ldots i_m}$, for $m \geq 1, i_j \in I, 1 \leq j \leq m$. The map π is called the canonical projection onto the limit set Λ of the system $\mathcal{S}$. Various properties of conformal IFS's with overlaps were studied by several authors, for e.g. in [4,12,15,16,21], etc. Let us fix now some more terminology and notation.

Definition 1 By *overlaps* we mean intersections of type $\phi_i(\Lambda) \cap \phi_j(\Lambda) \neq \emptyset$, $i \neq j$. If for a point $x \in \Lambda$ and an integer $m \geq 1$, there exists a point $\zeta \in \Lambda$ and a finite sequence $i_1, \ldots i_m \in I$ such that $\phi_{i_1} \circ \cdots \circ \phi_{i_m}(\zeta) = x$, then ζ is called an *m-root* of x, and $(i_1, \ldots, i_m)$ is called an *m-chain* from ζ to x.

In general, the number of roots/overlaps depends on the point $x \in \Lambda$, so it is not constant. Notice also that the m-chain from a certain root ζ to x is not uniquely defined, i.e. there may exist two different m-chains $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ so that $\phi_{i_1 \ldots i_m}(\zeta) = \phi_{j_1 \ldots j_m}(\zeta) = x$. Considering the above, how can we define a good notion of average number of overlaps of the IFS $\mathcal{S}$, and how is such a notion dependent on a probability measure μ on Λ; also, how does such a number of overlaps affect the Hausdorff dimension of μ? It is clear that we have to look at n-roots of points, since the limit set Λ is invariant under the system $\mathcal{S}$, i.e. $\Lambda = \bigcup_{i \in I} \phi_i(\Lambda)$, thus for k-iterations of $\mathcal{S}$ we have $\Lambda = \bigcup_{i_1, \ldots, i_k \in I} \phi_{i_1 \ldots i_k}(\Lambda)$, for any $k \geq 2$. In [12] we studied the effect, of the bounds for the number of overlaps, on the Hausdorff dimension of the limit set Λ. This hints to the fact that the overlap number should be given by an average rate of growth of the number of n-chains between points in Λ. Another question is, what probabilities μ on Λ should be considered, and what roots in Λ should we use. Some n-roots and n-chains which are non-generic with respect to μ and to a lift map $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$ will thus be ignored when defining the overlap number relative to μ.

Besides the canonical coding projection $\pi : \Sigma_I^+ \to \Lambda$, one can consider also the projection $\pi_2 : \Sigma_I^+ \times \Lambda \to \Lambda$, $\pi_2(\omega, x) = x$, and the projection $\tilde{\pi} : \Sigma_I^+ \times \Sigma_I^+ \to \Sigma_I^+ \times \Lambda$, $\tilde{\pi}(\omega, \eta) = (\omega, \pi\eta)$; so we obtain projections of σ-invariant measures on Σ_I^+, Φ-invariant measures on $\Sigma_I^+ \times \Lambda$ or $\tilde{\Phi}$-invariant measures on $\Sigma_I^+ \times \Sigma_I^+$ (where $\tilde{\Phi}$ is a lift of Φ to $\Sigma_I^+ \times \Sigma_I^+$). In

Theorem 1 we will prove that, for Bernoulli measures, the corresponding projection measures on Λ are in fact the same.

We introduce a notion of *overlap number* $o(\mathcal{S}, \hat{\mu}_\psi)$ associated to a Φ-invariant Gibbs state $\hat{\mu}_\psi$ on $\Sigma_I^+ \times \Lambda$ (and to its π_2-projection μ_ψ on Λ), and we use thermodynamic formalism to relate it to the dimension of μ_ψ. In Theorem 2 and Corollary 1 we show that the overlap number $o(\mathcal{S}, \hat{\mu}_\psi)$ is related to the folding entropy of $\hat{\mu}_\psi$ with respect to the lift map Φ. In particular, this applies to Bernoulli measures on Σ_I^+ and their lifts on $\Sigma_I^+ \times \Lambda$. When $\mu = \mu_0$ is the projection of the measure of maximal entropy $\hat{\mu}_0$ from $\Sigma_I^+ \times \Lambda$, one obtains a topological overlap number $o(\mathcal{S})$ of $\mathcal{S}$, which quantifies the average level of overlapping in $\mathcal{S}$, and indicates how far is $\mathcal{S}$ from satisfying the Open Set Condition. By using Theorem 1, we compute in Corollary 2 the overlap number $o(\mathcal{S})$ as a limit of integrals over Σ_I^+ w.r.t the uniform Bernoulli measure $\nu_{(\frac{1}{|I|}, ..., \frac{1}{|I|})}$. And in general for Bernoulli measures $\nu_\mathbf{p}$, Corollary 2 gives a simpler formula for $o(\mathcal{S}, \hat{\mu}_\mathbf{p})$.

Next, in Theorem 3 we use the overlap number of $\hat{\mu}_\psi$ to obtain estimates for the Hausdorff dimension of a set of full μ_ψ-measure in Λ, which set is constructed explicitly. This gives upper bounds for $HD(\mu_\psi)$, by using zeros of pressure functions associated to $o(\mathcal{S}, \hat{\mu}_\psi)$, which are computable in certain cases of interest.

In Sect. 3 we apply the results to the case of Bernoulli convolutions ν_λ for $\lambda \in (\frac{1}{2}, 1)$, where ν_λ gives the distribution of the random series $\sum_{n \geq 0} \pm \lambda^n$ with the $+, -$ signs taken independently and with equal probabilities. In this case, one has an iterated function system with overlaps $\mathcal{S}_\lambda$, whose limit set is an interval I_λ, and ν_λ appears as the projection of the measure of maximal entropy $\nu_{(\frac{1}{2}, \frac{1}{2})}$ from Σ_2^+ to I_λ. Bernoulli convolutions have attracted a lot of attention (see [15]), starting with Erdös [3] who showed that ν_λ is singular for λ^{-1} Pisot; then, continuing with the result of Solomyak [21] about the absolute continuity of ν_λ for Lebesgue-a.e $\lambda \in (\frac{1}{2}, 1)$, and the result of Przytycki and Urbański [17] that $HD(\nu_\lambda) < 1$ for λ^{-1} Pisot, and other more recent results. In Theorem 4 we find a relation between $HD(\nu_\lambda)$ and the overlap number $o(\mathcal{S}_\lambda)$, for all $\lambda \in (\frac{1}{2}, 1)$. We show how to approximate $o(\mathcal{S}_\lambda)$ with integrals on Σ_2^+ with respect to the uniform Bernoulli measure $\nu_{(\frac{1}{2}, \frac{1}{2})}$. By using known results on $HD(\nu_\lambda)$, one obtains then upper estimates for $o(\mathcal{S}_\lambda)$; in particular, one can estimate $o(\mathcal{S}_\lambda)$ more precisely for specific values of λ, like $\lambda = 2^{-\frac{1}{m}}$, $m \geq 2$ (i.e. $\frac{1}{\lambda}$ non-Pisot), or $\lambda = \frac{\sqrt{5}-1}{2}$ (i.e. $\frac{1}{\lambda}$ Pisot). In Corollary 3 we prove that $o(\mathcal{S}_\lambda)$ is strictly less than 2, for all $\lambda \in (\frac{1}{2}, 1)$. In the end, we obtain dimension estimates for biased Bernoulli convolutions $\nu_{\lambda, p}$, for $\lambda \in (\frac{1}{2}, 1)$ and $p \in (0, 1)$. The results about overlap numbers can be applied also to other conformal iterated function systems with overlaps.

2 Overlap Numbers of Measures and Dimension Estimates

First, let us define an *overlap lift function* which allows to associate the dynamics of a map to our IFS $\mathcal{S}$. With regard to this function, the contractions ϕ_i appear as restrictions to cylinders $[i], i \in I$.

Definition 2 In the above setting, for the finite IFS $\mathcal{S} = \{\phi_i\}_{i \in I}$, define the *overlap lift map*

$$\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda, \ \Phi(\omega, x) = (\sigma\omega, \phi_{\omega_1}(x)), \ (\omega, x) \in \Sigma_I^+ \times \Lambda$$

Let us now consider a Hölder continuous function $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$. Since the lift map Φ is distance-expanding in the first coordinate and contracting in the second coordinate, it

follows that it is expansive and we can apply the theory of equilibrium states (for e.g. [7,22]). As ψ is Hölder, there exists a unique equilibrium measure for ψ with respect to Φ on $\Sigma_I^+ \times \Lambda$, denoted by $\hat{\mu}_\psi$.

In particular, if we take a Hölder continuous function $g : \Lambda \to \mathbb{R}$ and the associated function $\psi_g : \Sigma_I^+ \times \Lambda \to \mathbb{R}$, $\psi_g(\omega, x) = g(x)$, then we have the equilibrium measure $\hat{\mu}_{\psi_g}$ on $\Sigma_I^+ \times \Lambda$ (relative to Φ) and its projection $(\pi_2)_*(\hat{\mu}_{\psi_g})$ on Λ, where π_2 is the projection on the second coordinate. In general this measure is different from the projection $\pi_*(\bar{\mu}_{g \circ \pi})$, where $\pi : \Sigma_I^+ \to \Lambda, \pi(\omega) = \phi_{\omega_1} \circ \cdots$, and where in general $\bar{\mu}_\chi$ denotes the equilibrium measure of a Hölder continuous χ on Σ_I^+ (relative to the shift σ).

For any $n \geq 1$ and any $(\omega, x) \in \Sigma_I^+ \times \Lambda$, we have $\Phi^n(\omega, x) = (\sigma^n \omega, \phi_{\omega_n} \circ \phi_{\omega_{n-1}} \circ \cdots \circ \phi_{\omega_1}(x))$. Notice that, if $\eta_1, \cdots, \eta_n$ are given and if $\phi_{\omega_n} \circ \cdots \circ \phi_{\omega_1}(x) = \phi_{\eta_n} \circ \cdots \circ \phi_{\eta_1}(y)$, then from the injectivity of the contractions $\phi_i, i \in I$, there exists *exactly one* point y with this property. By Definition 1, this means that, given the n-chain $(\eta_n, \ldots, \eta_1)$ as above, the corresponding n-root y is uniquely defined such that $(\eta_n, \ldots, \eta_1)$ is an n-chain from y to $\phi_{\omega_n \ldots \omega_1}(x)$.

Given now a measure $\hat{\mu}_\psi$ as above, an arbitrary point $(\omega, x) \in \Sigma_I^+ \times \Lambda$, and $\tau > 0$, define the set of n-chains from points in Λ to $\phi_{\omega_n \ldots \omega_1}(x)$, which are τ-generic relative to $\hat{\mu}_\psi$:

$$\Delta_n\big((\omega, x), \tau, \hat{\mu}_\psi\big) = \{(\eta_1, \ldots, \eta_n) \in I^n, \ \exists y \in \Lambda, \ \phi_{\eta_n \ldots \eta_1}(y) \tag{1}$$
$$= \phi_{\omega_n \ldots \omega_1}(x) \text{ and } |\frac{S_n \psi(\eta, y)}{n} - \int_{\Sigma_I^+ \times \Lambda} \psi \, d\hat{\mu}_\psi| < \tau\},$$

where $\eta = (\eta_1, \ldots, \eta_n, \omega_{n+1}, \omega_{n+2}, \ldots) \in \Sigma_I^+$, and where $S_n \psi(\eta, y) = \psi(\eta, y) + \psi(\Phi(\eta, y)) + \cdots + \psi(\Phi^n(\eta, y))$. We denote the cardinality of the set Δ_n by b_n, so

$$b_n((\omega, x), \tau, \hat{\mu}_\psi) := \text{Card} \, \Delta_n\big((\omega, x), \tau, \hat{\mu}_\psi\big), \ \forall (\omega, x) \in \Sigma_I^+ \times \Lambda$$

Remark that, if $(i_1, \ldots, i_n) \in \Delta_n\big((\omega, x), \tau, \hat{\mu}_\psi\big)$ with corresponding n-root y of $\phi_{\omega_n \ldots \omega_1}(x)$, then $\Delta_n\Big(((i_1, \ldots, i_n, \omega_{n+1}, \omega_{n+2}, \ldots), y), \tau, \hat{\mu}_\psi\Big) = \Delta_n\Big((\omega, x), \tau, \hat{\mu}_\psi\Big)$.

Definition 3 Given a Hölder continuous potential ψ on $\Sigma_I^+ \times \Lambda$ and $\tau > 0$, we call $b_n(\cdot, \tau, \hat{\mu}_\psi) : \Sigma_I^+ \times \Lambda \to \mathbb{N}$ the n-**overlap function** associated to the measure $\hat{\mu}_\psi$ and τ.

The function $b_n(\cdot, \tau, \hat{\mu}_\psi)$ is measurable and bounded, but in general discontinuous on $\Sigma_I^+ \times \Lambda$. In the sequel, we will use the folding entropy of a Φ-invariant measure $\hat{\mu}$ on $\Sigma_I^+ \times \Lambda$; the notion of folding entropy of a measure was introduced by Ruelle [19]. The folding entropy of a Φ-invariant probability μ with respect to $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$, is defined as the conditional entropy $F_\Phi(\mu) := H_\mu(\epsilon | \Phi^{-1}\epsilon)$, where ϵ is the point partition of the Lebesgue space $\Sigma_I^+ \times \Lambda$. For entropy production see also [13,19,20]. And for measures invariant under endomorphisms, for e.g. [9–11,14,20]. In [14] Parry introduced a notion of Jacobian of an invariant measure for an endomorphism, and studied its properties; in particular, the Jacobian satisfies the Chain Rule. Given a map $f : X \to X$ on a Lebesgue space X and an f-invariant probability μ, such that f is essentially countable-to-one, we denote the Jacobian of μ by $J_f(\mu)$. From [14,19] it follows that, in general, the folding entropy of a measure μ is equal to the integral of the logarithm of the Jacobian of μ. So in our case, the folding entropy of $\hat{\mu}_\psi$ with respect to Φ is given by:

$$F_\Phi(\hat{\mu}_\psi) = \int_{\Sigma_I^+ \times \Lambda} \log J_\Phi(\hat{\mu}_\psi) \, d\hat{\mu}_\psi$$

We investigate now the structure of the Φ-invariant probabilities on the product space $\Sigma_I^+ \times \Lambda$. Let define also the lift homeomorphism $\tilde{\Phi}$ on $\Sigma_I^+ \times \Sigma_I^+$, namely:

$$\tilde{\Phi} : \Sigma_I^+ \times \Sigma_I^+ \to \Sigma_I^+ \times \Sigma_I^+, \quad \tilde{\Phi}(\omega, \eta) = (\sigma\omega, \omega_1\eta)$$

If $\tilde{\pi}(\omega, \eta) := (\omega, \pi(\eta))$, for $(\omega, \eta) \in \Sigma_I^+ \times \Sigma_I^+$, then we obtain the following diagram of maps on $\Sigma_I^+ \times \Sigma_I^+$, respectively $\Sigma_I^+ \times \Lambda$, where both vertical maps below are equal to $\tilde{\pi} : \Sigma_I^+ \times \Sigma_I^+ \to \Sigma_I^+ \times \Lambda$:

$$
\begin{array}{ccc}
\Sigma_I^+ \times \Sigma_I^+ & \xrightarrow{\tilde{\Phi}} & \Sigma_I^+ \times \Sigma_I^+ \\
\downarrow & & \downarrow \\
\Sigma_I^+ \times \Lambda & \xrightarrow{\Phi} & \Sigma_I^+ \times \Lambda
\end{array}
\tag{2}
$$

This diagram is commutative. Indeed, $\tilde{\pi} \circ \tilde{\Phi}(\omega, \eta) = (\sigma\omega, \pi(\omega_1\eta)) = (\sigma\omega, \phi_{\omega_1} \circ \phi_{\eta_1} \circ \phi_{\eta_2} \circ \cdots)$; on the other hand, $\Phi \circ \tilde{\pi}(\omega, \eta) = \Phi(\omega, \phi_{\eta_1} \circ \phi_{\eta_2} \circ \cdots) = (\sigma\omega, \phi_{\omega_1} \circ \phi_{\eta_1} \circ \cdots)$. Hence $\tilde{\pi} \circ \tilde{\Phi} = \Phi \circ \tilde{\pi}$.

Also $\tilde{\Phi}$ is a homeomorphism. Then as in [18], by using Hahn–Banach theorem and Markov–Kakutani theorem and by approximating integrals of functions from $\mathcal{C}(\Sigma_I^+ \times \Sigma_I^+, \mathbb{R})$ with integrals of functions $g \circ \tilde{\pi} \circ \tilde{\Phi}^n$, $n \in \mathbb{Z}$, for $g \in \mathcal{C}(\Sigma_I^+ \times \Lambda, \mathbb{R})$, it follows that for any Φ-invariant probability ν on $\Sigma_I^+ \times \Lambda$, there exists a unique $\tilde{\Phi}$-invariant probability $\tilde{\nu}$ on $\Sigma_I^+ \times \Sigma_I^+$ such that $\tilde{\pi}_*(\tilde{\nu}) = \nu$. In particular, the equilibrium measure $\hat{\mu}_\psi$ of the Hölder continuous ψ on $\Sigma_I^+ \times \Lambda$, is the $\tilde{\pi}$-projection of the equilibrium measure $\tilde{\mu}_{\tilde{\psi}}$ of $\tilde{\psi} := \psi \circ \tilde{\pi}$ on $\Sigma_I^+ \times \Sigma_I^+$. Hence, the measure of maximal entropy $\hat{\mu}_0$ on $\Sigma_I^+ \times \Lambda$ is the $\tilde{\pi}$-projection of the measure of maximal entropy $\tilde{\mu}_0$ for $\tilde{\Phi}$ on $\Sigma_I^+ \times \Sigma_I^+$, i.e.

$$\hat{\mu}_0 = \tilde{\pi}_*(\tilde{\mu}_0)$$

Moreover, the topological entropy of the map Φ is equal to the topological entropy of the shift $\sigma : \Sigma_I^+ \to \Sigma_I^+$, i.e. $\log |I|$, because in the second coordinate we have contractions, so the separated sets are determined only by the expansion σ in the first coordinate. With the canonical distance on Σ_I^+, $d(\omega, \eta) = \sum_{i \geq 1} \frac{|\omega_i - \eta_i|}{2^i}$, the ball of center ω and radius $\frac{1}{2^n}$ is the cylinder $[\omega_1, \ldots, \omega_n]$, so $B((\omega, x), \frac{1}{2^n}) = [\omega_1, \ldots, \omega_n] \times B(x, \frac{1}{2^n})$. If we consider n-roots of x and the measure of maximal entropy $\hat{\mu}_0$ w.r.t Φ, then all these n-roots are generic. Since in this case the overlap function b_n does not depend on τ, we denote it simply by $b_n(\omega, x)$, for $(\omega, x) \in \Sigma_I^+ \times \Lambda$.

In general, there are several ways to define **projections of invariant measures** on the fractal limit set Λ, depending whether we project σ-invariant measures on Σ_I^+, or Φ-invariant measures on $\Sigma_I^+ \times \Lambda$, or $\tilde{\Phi}$-invariant measures on $\Sigma_I^+ \times \Sigma_I^+$. In many cases, for example for Bernoulli measures, these projections will be shown to coincide. Let us first consider a Hölder continuous potential ψ on $\Sigma_I^+ \times \Lambda$, and as above let $\hat{\mu}_\psi$ its (unique) equilibrium state on $\Sigma_I^+ \times \Lambda$; if $\pi_2 : \Sigma_I^+ \times \Lambda \to \Lambda$ is the projection on the second coordinate $\pi_2(\omega, x) = x$, denote the projection measure on Λ by:

$$\mu_\psi := (\pi_2)_*(\hat{\mu}_\psi) \tag{3}$$

Consider next g a Hölder continuous potential on Σ_I^+, and let $\bar{\mu}_g$ be its unique equilibrium measure on Σ_I^+. Then we can define two kinds of projection measures on Λ. The first type is μ_ψ defined above in (3), where $\psi = g \circ \pi_1$; so $\mu_\psi = (\pi_2)_*(\hat{\mu}_\psi)$. The second type is the self-conformal measure:

$$\pi_*(\bar{\mu}_g), \tag{4}$$

where $\pi : \Sigma_I^+ \to \Lambda$, $\pi(\omega_1\omega_2\ldots) = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots$ is the canonical coding map for Λ.

We now prove that, for **Bernoulli measures** on Σ_I^+, the two types of projection measures defined above, are in fact equal. This will make our results about overlap numbers apply to π-projections of Bernoulli measures onto Λ. Consider then a Bernoulli measure $\nu_{\mathbf{p}}$ on Σ_I^+ determined by an arbitrary probabilistic vector $\mathbf{p} = (p_1, \ldots, p_{|I|})$. Thus the $\nu_{\mathbf{p}}$-measure of the cylinder $[\omega_1, \ldots, \omega_n] = \{\eta \in \Sigma_I^+, \eta_1 = \omega_1, \ldots, \eta_n = \omega_n\}$, is equal to $p_{\omega_1} \ldots p_{\omega_n}$ for any $n \geq 1$ and $\omega_i \in I, 1 \leq i \leq n$. Consider the potential $\phi : \Sigma_I^+ \to \mathbb{R}$, $\phi(\omega_1\omega_2\ldots) = \log p_{\omega_1}$, for $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma_I^+$. Then $S_n\phi(\omega) = \phi(\omega) + \phi(\sigma(\omega)) + \cdots + \phi(\sigma^{n-1}(\omega)) = \log p_{\omega_1} \ldots p_{\omega_n}$. By taking Bowen balls for the shift σ (which are cylinders in our case), we see immediately that

$$P_\sigma(\phi) = 0$$

Clearly, ϕ is Hölder continuous on Σ_I^+ and its unique equilibrium measure $\bar{\mu}_\phi$ is equal to the Bernoulli measure $\nu_{\mathbf{p}}$; this is due to the expression of $\bar{\mu}_\phi$ on cylinders $[\omega_1 \ldots \omega_n]$ (see [2,7]), i.e.

$$\frac{1}{C} e^{S_n\phi(\omega) - nP_\sigma(\phi)} \leq \bar{\mu}_\phi(B_n(\omega, \varepsilon)) \leq C e^{S_n\phi(\omega) - nP_\sigma(\phi)},$$

so we conclude that

$$\bar{\mu}_\phi = \nu_{\mathbf{p}}$$

In case of Bernoulli measures, we can now prove that the various projection measures are **equal** on Λ:

Theorem 1 *In the above setting, let* $\mathbf{p} = (p_1, \ldots, p_{|I|})$ *an arbitrary probabilistic vector, and* $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$, $\psi((\omega_1\ldots), x) := \log p_{\omega_1}$, *with* $\hat{\mu}_\psi$ *denoting the unique equilibrium measure of* ψ *with respect to* $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$. *Then the following measures are equal on* Λ:

$$\pi_*\nu_{\mathbf{p}} = \pi_{2*}\hat{\mu}_\psi = (\pi_2 \circ \tilde{\pi})_*(\nu_{\mathbf{p}} \times \nu_{\mathbf{p}}),$$

where $\pi_2 : \Sigma_I^+ \times \Lambda \to \Lambda$, $\pi_2(\omega, x) = x$, *and* $\pi : \Sigma_I^+ \to \Lambda$ *is the canonical coding map, and where* $\tilde{\pi} : \Sigma_I^+ \times \Sigma_I^+ \to \Sigma_I^+ \times \Lambda$, $\tilde{\pi}(\omega, \eta) = (\omega, \pi(\eta))$.

Proof In order to prove the first equality, let us define $\tilde{\psi} = \psi \circ \tilde{\pi}$, where $\tilde{\pi}(\omega, \eta) = (\omega, \pi\eta)$. So $\tilde{\psi}$ is a Hölder continuous potential on $\Sigma_I^+ \times \Sigma_I^+$. Then recalling that $\tilde{\Phi}(\omega, \eta) = (\sigma\omega, \omega_1\eta)$ is an expansive homeomorphism with specification property, it follows [7] that there exists a unique equilibrium measure $\tilde{\mu}_{\tilde{\psi}}$ on $\Sigma_I^+ \times \Sigma_I^+$. Also we have the projection $\tilde{\pi}(\omega, \eta) = (\omega, \pi\eta)$ from $\Sigma_I^+ \times \Sigma_I^+$ to $\Sigma_I^+ \times \Lambda$. Moreover, from definitions it can be seen that

$$\tilde{\pi}\tilde{\Phi}(\omega, \eta) = (\sigma\omega, \phi_{\omega_1}(\pi\eta)) = \Phi \circ \tilde{\pi}(\omega, \eta),$$

so $\tilde{\pi} \circ \tilde{\Phi} = \Phi \circ \tilde{\pi}$. This implies that $\tilde{\pi}_*(\tilde{\mu}_{\tilde{\psi}}) = \hat{\mu}_\psi$, i.e. the projection to $\Sigma_I^+ \times \Lambda$ of the equilibrium measure of $\tilde{\psi}$ on $\Sigma_I^+ \times \Sigma_I^+$, is equal to the equilibrium measure of ψ. Hence from above,

$$\pi_{2*}(\hat{\mu}_\psi)(A) = \hat{\mu}_\psi(\pi_2^{-1}(A)) = \tilde{\mu}_{\tilde{\psi}}(\Sigma_I^+ \times \pi^{-1}(A)) \tag{5}$$

On the other hand, notice that the Bowen ball for $\tilde{\Phi}$ is given by $B_n((\omega, \eta), \varepsilon) = [\omega_1 \ldots \omega_n] \times \Sigma_I^+$, and for any $1 \leq i \leq n$, we have $\tilde{\Phi}^i(B_n((\omega, \eta), \varepsilon)) = [\omega_{i+1} \ldots \omega_n] \times [\omega_i \ldots \omega_1]$. From the $\tilde{\Phi}$-invariance of the equilibrium measure $\tilde{\mu}_{\tilde{\psi}}$, it follows that for any $1 \leq i \leq n$,

$$\tilde{\mu}_{\tilde{\psi}}(\tilde{\Phi}^i(B_n((\omega, \eta), \varepsilon))) = \tilde{\mu}_{\tilde{\psi}}([\omega_1 \ldots \omega_n] \times \Sigma_I^+) = \tilde{\mu}_{\tilde{\psi}}([\omega_{i+1} \ldots \omega_n] \times [\omega_i \ldots \omega_1]) \tag{6}$$

However recall that $\pi_{1*}\hat{\mu}_\psi = \bar{\mu}_\phi = \nu_\mathbf{p}$, and thus $(\pi_1 \circ \tilde{\pi})_* \tilde{\mu}_{\tilde{\psi}} = \nu_\mathbf{p}$. Therefore using also (6) we obtain that, for any $j \geq 1$ and any $\omega, \eta \in \Sigma_I^+$,

$$\tilde{\mu}_{\tilde{\psi}}([\omega_1] \times [\eta_1 \ldots \eta_j]) = \nu_\mathbf{p}([\eta_j \ldots \eta_1 \omega_1]) = p_{\eta_j} \cdot \ldots \cdot p_{\eta_1} p_{\omega_1} \tag{7}$$

By adding over $\omega_1 \in \Sigma_I^+$ we obtain that, for any $j \geq 1$ and for any $\eta = (\eta_1 \eta_2 \ldots) \in \Sigma_I^+$,

$$\tilde{\mu}_{\tilde{\psi}}(\Sigma_I^+ \times [\eta_1 \ldots \eta_j]) = p_{\eta_1} \ldots p_{\eta_j} = \nu_\mathbf{p}([\eta_1 \ldots \eta_j])$$

But this works for any cylinder in Σ_I^+. Also, for any Borel set $A \subset \Lambda$, we have $\pi_* \nu_\mathbf{p}(A) = \nu_\mathbf{p}(\pi^{-1}(A))$. Hence from the above, and by using also (5), we can infer that $\pi_{2*}\hat{\mu}_\psi$ is in fact a self-conformal measure on Λ, namely,

$$\pi_{2*}\hat{\mu}_\psi = \pi_* \nu_\mathbf{p}$$

We now prove the second equality. From before, $\tilde{\Phi} : \Sigma_I^+ \times \Sigma_I^+ \to \Sigma_I^+ \times \Sigma_I^+$ is a homeomorphism which preserves $\tilde{\mu}_{\tilde{\psi}}$. Also notice that for any $\omega_1, \omega_2, \eta_1, \ldots, \eta_m \in I$, one has $\tilde{\Phi}([\omega_1 \omega_2] \times [\eta_1 \ldots \eta_m]) = [\omega_2] \times [\omega_1 \eta_1 \eta_2 \ldots \eta_m]$. But, from (7), $\tilde{\mu}_{\tilde{\psi}}([\omega_2] \times [\omega_1 \eta_1 \ldots \eta_m]) = p_{\omega_2} p_{\omega_1} p_{\eta_1} \ldots p_{\eta_m}$, and from the $\tilde{\Phi}$-invariance of $\tilde{\mu}_{\tilde{\psi}}$, it follows that $\tilde{\mu}_{\tilde{\psi}}([\omega_1 \omega_2] \times [\eta_1 \ldots \eta_m]) = \tilde{\mu}_{\tilde{\psi}}(\tilde{\Phi}([\omega_1 \omega_2] \times [\eta_1 \ldots \eta_m])) = p_{\omega_1} p_{\omega_2} p_{\eta_1} \ldots p_{\eta_m}$. Hence by induction it follows similarly that, for any $k, m \geq 1$,

$$\tilde{\mu}_{\tilde{\psi}}([\omega_1 \ldots \omega_k] \times [\eta_1 \ldots \eta_m]) = p_{\omega_1} \ldots p_{\omega_k} \cdot p_{\eta_1} \ldots p_{\eta_m}$$

This means that $\tilde{\mu}_{\tilde{\psi}} = \nu_\mathbf{p} \times \nu_\mathbf{p}$, and that $\pi_* \nu_\mathbf{p} = (\pi_2 \circ \tilde{\pi})_*(\nu_\mathbf{p} \times \nu_\mathbf{p})$. □

The equality of the projection measures for Bernoulli probabilities has useful consequences when computing the associated overlap numbers, see Corollary 2.

For any conformal iterated function system $\mathcal{S}$, we want to prove now that the exponential rate of growth in n, of the number of generic n-chains/roots from Δ_n, is approaching the folding entropy of the measure $\hat{\mu}_\psi$. In particular it follows that, on average, the number of n-chains associated to the n-overlaps of Λ grows exponentially like $e^{nF_\Phi(\hat{\mu}_0)}$.

Theorem 2 *Let a finite conformal IFS $\mathcal{S} = \{\phi_i, i \in I\}$ with limit set Λ, and a Hölder continuous potential ψ on the lift space $\Sigma_I^+ \times \Lambda$; denote the equilibrium measure of ψ on $\Sigma_I^+ \times \Lambda$ by $\hat{\mu}_\psi$. Then,*

$$\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \log b_n((\omega, x), \tau, \hat{\mu}_\psi) \, d\hat{\mu}_\psi(\omega, x) = F_\Phi(\hat{\mu}_\psi)$$

Proof In our case the map $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$ is distance-expanding in the first coordinate, and distance contracting in the second coordinate. Let $B_m(z, \varepsilon)$ denote the (m, ε)-Bowen ball around z in the canonical product metric on the compact metric space $\Sigma_I^+ \times \Lambda$ with respect to the endomorphism Φ; hence in particular it is expansive. Since $\hat{\mu}_\psi$ is the equilibrium measure of a Hölder continuous potential on $\Sigma_I^+ \times \Lambda$, we can apply the properties of equilibrium measures with respect to expansive maps on compact metric spaces (see [7]). We will use first the ideas of Theorem 1 from [9], giving the comparison between the (equilibrium) measure of various parts of the preimage set. So, from [9] there exists a constant $C > 0$ such that, for any positive integer m and for any sets A_1, A_2 satisfying $A_1 \subset B_m(z_1, \varepsilon), A_2 \subset B_m(z_2, \varepsilon)$ and $\Phi^m(A_1) = \Phi^m(A_2)$, we have:

$$\frac{1}{C} \frac{\hat{\mu}_\psi(A_2)}{e^{S_m \psi(z_2)}} \leq \frac{\hat{\mu}_\psi(A_1)}{e^{S_m \psi(z_1)}} \leq C \frac{\hat{\mu}_\psi(A_2)}{e^{S_m \psi(z_2)}} \tag{8}$$

Now the Jacobian of the measure $\hat{\mu}_\psi$ with respect to Φ^n gives the change in the measure of a set by applying the map Φ^n (see [14]); hence for any integer $n \geq 1$, $\hat{\mu}_\psi(\Phi^n(\mathcal{A})) = \int_{\mathcal{A}} J_{\Phi^n}(\hat{\mu}_\psi) d\hat{\mu}_\psi$, for any measurable set $\mathcal{A} \subset \Sigma_I^+ \times \Lambda$, on which Φ^n is injective. But in fact, $J_{\Phi^n}(\hat{\mu}_\psi)(\omega, x) = \lim_{r \to 0} \frac{\hat{\mu}_\psi(\Phi^n(B((\omega,x),r))}{\hat{\mu}_\psi(B((\omega,x),r))}$, for $\hat{\mu}_\psi$-a.e $(\omega, x) \in \Sigma_I^+ \times \Lambda$. However from the Φ-invariance of the measure $\hat{\mu}_\psi$ it follows that $\hat{\mu}_\psi(\Phi^n(\mathcal{A})) = \hat{\mu}_\psi(\Phi^{-n}(\Phi^n(\mathcal{A})))$, for any Borel set $\mathcal{A}$. Hence we can apply the above comparison between the various parts of the preimage set $\Phi^{-n}(\Phi^n(\mathcal{A}))$ for n arbitrary (i.e. in fact the comparison between various sets taken by different compositions $\phi_{j_1} \circ \cdots \circ \phi_{j_n}$ to the same image), in order to obtain that there exists a constant $C > 0$ independent of n such that:

$$\frac{\sum_{(\eta,y),\Phi^n(\eta,y)=\Phi^n(\omega,x)} \exp(S_n\psi(\eta, y))}{C \cdot \exp(S_n\psi(\omega, x))} \leq J_{\Phi^n}(\hat{\mu}_\psi)(\omega, x)$$

$$\leq C \cdot \frac{\sum_{(\eta,y),\Phi^n(\eta,y)=\Phi^n(\omega,x)} \exp(S_n\psi(\eta, y))}{\exp(S_n\psi(\omega, x))}, \tag{9}$$

for $\hat{\mu}_\psi$-a.e pair $(\omega, x) \in \Sigma_I^+ \times \Lambda$. Now, as the probability $\hat{\mu}_\psi$ is Φ-invariant on the product space $\Sigma_I^+ \times \Lambda$, it follows from (9) and from the properties of the folding entropy that

$$F_\Phi(\hat{\mu}_\psi) = \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \log J_{\Phi^n}(\hat{\mu}_\psi)(\omega, x) d\hat{\mu}_\psi(\omega, x)$$

$$= \lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \log \frac{\sum_{\Phi^n(\eta,y)=\Phi^n(\omega,x)} \exp(S_n\psi(\eta, y))}{\exp(S_n\psi(\omega, x))} d\hat{\mu}_\psi(\omega, x) \tag{10}$$

From Birkhoff Ergodic Theorem we know that, $\hat{\mu}_\psi\left((\omega, x) \in \Sigma_I^+ \times \Lambda, |\frac{S_n\psi(\omega,x)}{n} - \int_{\Sigma_I^+ \times \Lambda} \psi d\hat{\mu}_\psi| > \tau/2\right) \underset{n \to \infty}{\to} 0$. Then, for any positive small number ξ, there exists an integer $n = n(\xi) \geq 1$ so that for all integers $n \geq n(\xi)$, we have

$$\mu_\psi\left((\omega, x) \in \Sigma_I^+ \times \Lambda, \left|\frac{S_n\psi(\omega, x)}{n} - \int_{\Sigma_I^+ \times \Lambda} \psi d\hat{\mu}_\psi\right| > \tau/2\right) < \xi \tag{11}$$

Recall that, if $(\eta_1, \ldots, \eta_n) \in \Delta_n((\omega, x), \tau, \hat{\mu}_\psi)$, then the n-chain $(\eta_n, \ldots, \eta_1)$ uniquely determines an n-root y of $\phi_{\omega_n \ldots \omega_1}(x)$. Hence with $\eta_{n+i} = \omega_{n+i}, i \geq 1$, we can consider also the finite set

$$\Delta_n'((\omega, x), \tau, \hat{\mu}_\psi) = \left\{(\eta, y) \in \Sigma_I^+ \times \Lambda, \ \Phi^n(\eta, y)\right.$$

$$= \Phi^n(\omega, x), \left|\frac{S_n\psi(\eta, y)}{n} - \int \psi d\hat{\mu}_\psi\right| < \tau\right\},$$

and there exists a bijection between $\Delta_n((\omega, x), \tau, \hat{\mu}_\psi)$ and $\Delta_n'((\omega, x), \tau, \hat{\mu}_\psi)$, taking $(\eta_1, \ldots, \eta_n)$ to $((\eta_1, \ldots, \eta_n, \omega_{n+1}, \omega_{n+2}, \ldots), y)$. Thus $b_n((\omega, x), \tau, \hat{\mu}_\psi) = \text{Card}\Delta_n'((\omega, x), \tau, \hat{\mu}_\psi)$. We now define the following set of n-roots,

$$\Gamma_n((\omega, x), \tau, \hat{\mu}_\psi) := \{(\eta, y) \in \Sigma_I^+ \times \Lambda, \Phi^n(\eta, y)$$

$$= \Phi^n(\omega, x), (\eta_1, \ldots, \eta_n) \notin \Delta_n((\omega, x), \tau, \hat{\mu}_\psi)\}$$

Denote the sum corresponding to the roots from $\Gamma_n((\omega, x), \tau, \hat{\mu}_\psi)$ by

$$\vartheta_n((\omega, x), \tau, \hat{\mu}_\psi) := \sum_{(\eta, y) \in \Gamma_n((\omega, x), \tau, \hat{\mu}_\psi)} \exp(S_n \psi(\eta, y))$$

Let us now see what a typical Bowen ball for the map $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$ looks like. If $d(\cdot, \cdot)$ denotes the product metric, and if $d(\Phi^i(\omega, x), \Phi^i(\eta, y)) < \varepsilon, 0 \le i \le n-1$, then there exists an integer $N(\varepsilon)$ so that $\omega_i = \eta_i, i = 1, \ldots, n + N(\varepsilon)$, and $d(x, y) < \varepsilon$, since the maps ϕ_j are all contractions. For an arbitrary $n \ge 2$, we now consider a measurable partition of $\Sigma_I^+ \times \Lambda$ modulo $\hat{\mu}_\psi$, into sets $L_i^n, 1 \le i \le p_n$, such that for any $1 \le i \le p_n$ there exists a point $\zeta_i \in L_i^n$ so that for any point $\zeta_{ij} \in \Phi^{-n}(\zeta_i), 1 \le j \le p_{i,n}$, we have $L_i^n \subset \Phi^n(B_n(\zeta_{ij}, \varepsilon))$. The integer $p_{i,n} \ge 1$ depends on i for $1 \le i \le p_n$, and it is given by the number of n-roots of ζ_i in Λ, with respect to $\mathcal{S}$. This is possible to do if we take the sets L_i^n small enough. Then, let us denote by $L_{ij}^n := \Phi^{-n}(L_i^n) \cap B_n(\zeta_{ij}, \varepsilon)$, for $1 \le i \le p_n, 1 \le j \le p_{i,n}$. Notice that if $\Phi(\eta, y) = \Phi(\eta', y') = (\omega, x) \in \Sigma_I^+ \times \Lambda$, then $\sigma\eta = \sigma\eta' = \omega$, i.e. $\eta_2 = \omega_2, \ldots$, and $\phi_{\eta_1}(y) = \phi_{\eta_1'}(y') = x$. If $\eta_1 \ne \eta_1'$, then $d((\eta, y), (\eta', y')) \ge d(\eta_1, \eta_1') > \varepsilon_0 > \varepsilon$, for some $\varepsilon_0 > 0$. If $\eta_1 = \eta_1'$, then $\phi_{\eta_1}(y) = \phi_{\eta_1'}(y')$; but $\phi_\eta, \eta \in I$ are injective and thus $y = y'$. This implies that the sets L_{ij}^n are mutually disjoint in i, j. We now decompose the integral of the logarithm of the Jacobian of $\hat{\mu}_\psi$ with respect to Φ^n, along this partition with the sets $L_{ij}^n, 1 \le i \le p_n, 1 \le j \le p_{i,n}$. Therefore, for an arbitrary $n \ge 2$, we have:

$$\int_{\Sigma_I^+ \times \Lambda} \log \frac{\displaystyle\sum_{\Phi^n(\eta, y) = \Phi^n(\omega, x)} \exp(S_n \psi(\eta, y))}{\exp(S_n \psi(\omega, x))} d\hat{\mu}_\psi(\omega, x) \tag{12}$$

$$= \sum_{\substack{1 \le i \le p_n \\ 1 \le j \le p_{i,n}}} \int_{L_{ij}^n} \log \frac{\displaystyle\sum_{\Phi^n(\eta, y) = \Phi^n(\omega, x)} \exp(S_n \psi(\eta, y))}{\exp(S_n \psi(\omega, x))} d\hat{\mu}_\psi(\omega, x)$$

Now, in regards to formula (9), we can write in general

$$\sum_{(\eta, y) \in \Phi^{-n}\Phi^n(\omega, x)} e^{S_n \psi(\eta, y)} = \sum_{(\eta_1, \ldots, \eta_n) \in \Delta_n((\omega, x), \tau, \hat{\mu}_\psi)} e^{S_n \psi(\eta, y)} + \vartheta_n((\omega, x), \tau, \hat{\mu}_\psi)$$

Denote also $\rho_n(i, \tau, \hat{\mu}_\psi) := \sum_{j, \zeta_{ij} \notin \Delta_n'(\zeta_{i1}, \tau, \hat{\mu}_\psi)} \hat{\mu}_\psi(L_{ij}^n)$. Thus by using (8), the definition of $\Delta_n'((\omega, x), \tau, \hat{\mu}_\psi)$ and the fact that $b_n((\omega, x), \tau, \hat{\mu}_\psi) = \text{Card}(\Delta_n'((\omega, x), \tau, \hat{\mu}_\psi))$, we obtain that the above sum in (12) is comparable to the sum:

$$\sum_{i, j} \hat{\mu}_\psi(L_{ij}^n) \log \frac{b_n(\zeta_{ij}, \tau, \hat{\mu}_\psi)\hat{\mu}_\psi(L_{ij}^n) + \rho_n(i, \tau, \hat{\mu}_\psi)}{\hat{\mu}_\psi(L_{ij}^n)},$$

where we recall that the comparability constant $C > 0$ does not depend on n, nor on L_{ij}^n. Now in general, if $(\eta, y) \in \Delta_n'((\omega, x), \tau, \hat{\mu}_\psi)$, and if $0 < \varepsilon < \tau$ and $(\eta, y) \in B_n(\zeta_{ij}, \varepsilon)$, then since the potential ψ is Hölder continuous, it follows that

$$\left| \frac{S_n \psi(\eta, y)}{n} - \frac{S_n \psi(\zeta_{ij})}{n} \right| \le \upsilon(\tau),$$

for some small $\upsilon(\tau) > 0$ where $\lim_{\tau \to 0} \upsilon(\tau) = 0$. Also, if $K := \sup_{\Sigma_I^+ \times \Lambda} |\psi|$, then $e^{S_n \psi(\eta, y)} \le e^{nK}$. Notice in addition, that the set $\Phi^{-n}\Phi^n(\omega, x)$ has at most $|I|^n$ elements in $\Sigma_I^+ \times \Lambda$. Denote

the set of indices j corresponding to nongeneric roots by $Q(n, i, \tau, \hat{\mu}_\psi) := \{j, 1 \le j \le p_{i,n}, \zeta_{ij} \in \Gamma_n(\zeta_{i1}, \tau, \hat{\mu}_\psi)\}$. Then if $j \in Q(n, i, \tau, \hat{\mu}_\psi)$, then $\frac{1}{n}|S_n\psi(\zeta_{ij}) - \int_{\Sigma_I^+ \times \Lambda} \psi \, d\hat{\mu}_\psi| > \tau$. Hence we can use the measure estimate in (11) to obtain that:

$$\sum_{1 \le i \le p_n, \ j \in Q(n,i,\tau,\hat{\mu}_\psi)} \frac{1}{n} \int_{L_{ij}^n} \log \frac{\sum\limits_{(\eta,y) \in \Phi^{-n}\Phi^n(\omega,x)} \exp(S_n\psi(\eta, y))}{\exp(S_n\psi(\omega, x))} d\hat{\mu}_\psi(\omega, x)$$
$$\le \frac{1}{n}\xi \log(2K|I|^n)$$

Therefore, from the comparison in (8) and from the above discussion, it follows that there exists a positive constant C, independent of n, of the partition $\{L_i^n\}_{1 \le i \le p_n}$ and of the points $\zeta_i \in L_i^n$, such that:

$$\frac{1}{n} \sum_{\substack{1 \le i \le p_n \\ j \notin Q(n,i,\tau,\hat{\mu}_\psi)}} \hat{\mu}_\psi(L_{ij}^n) \log b_n(\zeta_{i1}, \tau, \hat{\mu}_\psi)$$

$$+ \frac{1}{n} \sum_{i,j \notin Q(n,i,\tau,\hat{\mu}_\psi)} \hat{\mu}_\psi(L_{ij}^n) \log\left(1 + \frac{\rho_n(i, \tau, \hat{\mu}_\psi)}{b_n(\zeta_{i1}, \tau, \hat{\mu}_\psi)\hat{\mu}_\psi(L_{ij}^n)}\right) - v(\tau) - C\xi$$

$$\le \int_{\Sigma_I^+ \times \Lambda} \frac{1}{n} \log \frac{\sum\limits_{(\eta,y) \in \Phi^{-n}\Phi^n(\omega,x)} \exp(S_n\psi(\eta, y))}{\exp(S_n\psi(\omega, x))} d\hat{\mu}_\psi(\omega, x)$$

$$\le \frac{1}{n} \sum_{\substack{1 \le i \le p_n \\ j \notin Q(n,i,\tau,\hat{\mu}_\psi)}} \hat{\mu}_\psi(L_{ij}^n) \log b_n(\zeta_{i1}, \tau, \hat{\mu}_\psi)$$

$$+ \frac{1}{n} \sum_{i,j \notin Q(n,i,\tau,\hat{\mu}_\psi)} \hat{\mu}_\psi(L_{ij}^n) \log\left(1 + \frac{\rho_n(i, \tau, \hat{\mu}_\psi)}{b_n(\zeta_{i1}, \tau, \hat{\mu}_\psi)\hat{\mu}_\psi(L_{ij}^n)}\right) + v(\tau) + C\xi, \quad (13)$$

where we recall that ξ is the bound on the measure of non-generic points in (11). But in general, $\log(1 + x) \le x$ for any $x > 0$, hence $\log(1 + \frac{\rho_n(i,\tau,\hat{\mu}_\psi)}{b_n(\zeta_{i1},\tau,\hat{\mu}_\psi)\hat{\mu}_\psi(L_{ij}^n)}) \le \frac{\rho_n(i,\tau,\hat{\mu}_\psi)}{b_n(\zeta_{i1},\tau,\hat{\mu}_\psi)\hat{\mu}_\psi(L_{ij}^n)}$. Therefore from (11), the second sum in the right-hand term of (13) is less than ξ, which implies that:

$$\left| \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \frac{1}{n} \log \frac{\sum\limits_{(\eta,y) \in \Phi^{-n}\Phi^n(\omega,x)} \exp(S_n\psi(\eta, y))}{\exp(S_n\psi(\omega, x))} d\hat{\mu}_\psi(\omega, x) \right.$$

$$\left. - \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \log b_n((\omega, x), \tau, \hat{\mu}_\psi) d\hat{\mu}_\psi(\omega, x) \right| \le v(\tau) + C\xi$$

Therefore, using the expression for the folding entropy $F_\Phi(\hat{\mu}_\psi)$ from (10), and the fact that ξ converges to 0 when τ converge to 0 (and also that $v(\tau)$ converges to 0 at the same time), we obtain the conclusion of the Theorem. $\qquad\square$

We now want to define a notion of overlap number of $\mathcal{S}$ associated to an equilibrium state $\hat{\mu}_\psi$. This notion will take into consideration the $\hat{\mu}_\psi$-generic n-roots in Λ and all the corresponding n-chains starting from them, for n large. In particular, we obtain a (topological)

overlap number of the system $\mathcal{S}$, which gives the average rate of growth of the number of n-chains from n-roots to points in Λ.

Corollary 1 *If $\mathcal{S} = \{\phi_i, i \in I\}$ is an arbitrary finite conformal iterated function system with overlaps and Λ is its limit set, and if ψ is a Hölder continuous potential on $\Sigma_I^+ \times \Lambda$ with equilibrium measure $\hat{\mu}_\psi$, we call the **overlap number of** $\mathcal{S}$ **with respect to** $\hat{\mu}_\psi$,*

$$o(\mathcal{S}, \hat{\mu}_\psi) := \exp \left(\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \log b_n((\omega, x), \tau, \hat{\mu}_\psi) \, d\hat{\mu}_\psi(\omega, x) \right) \qquad (14)$$

*If $\hat{\mu}_0$ is the measure of maximal entropy for Φ on $\Sigma_I^+ \times \Lambda$, then the (topological) **overlap number of** $\mathcal{S}$ is given by:*

$$o(\mathcal{S}) := o(\mathcal{S}, \hat{\mu}_0) = \exp \left(\lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \log b_n(\omega, x) \, d\hat{\mu}_0(\omega, x) \right) = \exp \left(F_\Phi(\hat{\mu}_0) \right)$$

$$= \exp \left(\int_{\Sigma_I^+ \times \Lambda} \log \lim_{n \to \infty} \frac{\hat{\mu}_0([\omega_2, \ldots, \omega_n] \times \phi_{\omega_1} \left(B \left(x, \frac{1}{2^n} \right) \right)}{\hat{\mu}_0 \left([\omega_1, \ldots, \omega_n] \times B \left(x, \frac{1}{2^n} \right) \right)} \, d\hat{\mu}_0(\omega, x) \right)$$

The Φ-invariant measure $\hat{\mu}_0$ on $\Sigma_I^+ \times \Lambda$ is the projection of the measure of maximal entropy $\tilde{\mu}_0$ of $\tilde{\Phi}$ on $\Sigma_I^+ \times \Sigma_I^+$, i.e. $\tilde{\pi}_*(\tilde{\mu}_0) = \hat{\mu}_0$. But in general, for $n \geq 2$, $\tilde{\Phi}^n(\omega, \eta) = (\sigma^n(\omega), \omega_n \ldots \omega_1 \eta)$. This means that for the product metric on $\Sigma_I^+ \times \Sigma_I^+$ the Bowen ball $B_n((\omega, \eta), \varepsilon)$ is equal to $[\omega_1 \ldots \omega_n] \times \Sigma_I^+$ (where $[\omega_1 \ldots \omega_n]$ is the cylinder of sequences in Σ_I^+ with starting coordinates $\omega_1, \ldots, \omega_n$), i.e. the second coordinate does not matter in Bowen balls for $\tilde{\Phi}$. Moreover, we have for any $1 \leq i \leq n$, $\tilde{\Phi}^i(B_n((\omega, \eta), \varepsilon)) = [\omega_{i+1} \ldots \omega_n] \times [\omega_i \ldots \omega_1]$, and we know that the measure $\tilde{\mu}_0$ is $\tilde{\Phi}$-invariant, hence

$$\tilde{\mu}_0([\omega_1 \ldots \omega_n] \times \Sigma_I^+) = \tilde{\mu}_0([\omega_{i+1} \ldots \omega_n] \times [\omega_i \ldots \omega_1])$$

On the other hand, we know that $\pi_1 \circ \tilde{\Phi}(\omega, \eta) = \sigma \omega = \sigma \circ \pi_1(\omega, \eta)$, so $\pi_{1*}(\tilde{\mu}_0) = \bar{\mu}_0$, where $\bar{\mu}_0$ is the measure of maximal entropy for the shift on Σ_I^+. Therefore we have $\tilde{\mu}_0([\omega_1 \ldots \omega_n] \times \Sigma_I^+) = \bar{\mu}_0([\omega_1 \ldots \omega_n])$. This implies, from above, that we can say exactly what is $\tilde{\mu}_0$ on a neighbourhood basis in the product space $\Sigma_I^+ \times \Sigma_I^+$; namely for any $i, j \geq 1$, the measure $\tilde{\mu}_0$ on the product of any two cylinders $[\omega_1 \ldots \omega_i], [\eta_1 \ldots \eta_j]$ in $\Sigma_I^+ \times \Sigma_I^+$ is given by:

$$\tilde{\mu}_0([\omega_1 \ldots \omega_i] \times [\eta_1 \ldots \eta_j]) = \bar{\mu}_0([\omega_1 \ldots \omega_i]) \cdot \bar{\mu}_0([\eta_1 \ldots \eta_j]) = \frac{1}{|I|^{i+j}}.$$

In the case of **projections of Bernoulli measures**, we can use now Theorem 1 to compute more easily the overlap numbers. Let us take an arbitrary probability vector $\mathbf{p} = (p_1, \ldots, p_{|I|})$, which gives a Bernoulli measure $\nu_\mathbf{p}$ on Σ_I^+. According to the discussion before Theorem 1, there exists an equilibrium measure denoted $\hat{\mu}_\mathbf{p}$ of the potential $\psi((\omega_1, \ldots), x) = \log p_{\omega_1}, (\omega, x) \in \Sigma_I^+ \times \Lambda$, with respect to Φ on $\Sigma_I^+ \times \Lambda$, so that $\pi_* \nu_\mathbf{p} = \pi_{2*} \hat{\mu}_\mathbf{p}$. The measure $\hat{\mu}_\mathbf{p}$ is called the equilibrium measure (with respect to Φ) associated to $\mathbf{p}$. Denote also by $h(\mathbf{p}) := \sum_{1 \leq j \leq |I|} p_j \log p_j$, and notice that $h(\mathbf{p}) = \int \psi \, d\hat{\mu}_\mathbf{p}$. Let us denote now by

$$\beta_n(x) := \text{Card}\{(\eta_1, \ldots, \eta_n) \in I^n, \ x \in \phi_{\eta_1} \circ \cdots \circ \phi_{\eta_n}(\Lambda)\}, \ \forall x \in \Lambda$$

More generally, we define for $\tau > 0$,

$$\beta_n(x, \tau, \mathbf{p})$$
$$:= \mathrm{Card}\left\{(\eta_1, \ldots, \eta_n) \in I^n, \ x \in \phi_{\eta_1} \circ \cdots \circ \phi_{\eta_n}(\Lambda), \ \left|\frac{\log(p_{\eta_1} \cdots p_{\eta_n})}{n} - h(\mathbf{p})\right| < \tau\right\}$$
(15)

As before if $x \in \phi_{\eta_1} \circ \cdots \circ \phi_{\eta_n}(\Lambda)$, then there exists a unique point $y \in \Lambda$ with $x = \phi_{\eta_1} \circ \cdots \circ \phi_{\eta_n}(y)$. When the system $\mathcal{S}$ satisfies Open Set Condition, then the overlap number $o(\mathcal{S}, \hat{\mu}_{\mathbf{p}})$ is equal to 1.

We prove now the following simpler expression for the overlap number in the case of Bernoulli projections for conformal IFS's with overlaps $\mathcal{S}$, by employing the function $\beta_n(\cdot)$, that counts the number of n-chains from n-roots in the limit set Λ:

Corollary 2 *Let a conformal iterated function system with overlaps $\mathcal{S} = \{\phi_i, i \in I\}$ with limit set Λ, and consider $\mathbf{p}$ an arbitrary probabilistic vector, with $\hat{\mu}_{\mathbf{p}}$ being the equilibrium measure on $\Sigma_I^+ \times \Lambda$ associated to $\mathbf{p}$. Then, the overlap number $o(\mathcal{S}, \hat{\mu}_{\mathbf{p}})$ can be computed as:*

$$o(\mathcal{S}, \hat{\mu}_{\mathbf{p}}) = \exp\left(\lim_{\tau \to 0} \lim_{n} \frac{1}{n} \int_{\Sigma_I^+} \log \beta_n(\pi\omega, \tau, \mathbf{p}) \, d\nu_{\mathbf{p}}(\omega)\right)$$

In particular, we obtain the (topological) overlap number of $\mathcal{S}$, by integrating with respect to the uniform Bernoulli measure $\nu_{(\frac{1}{|I|}, \ldots, \frac{1}{|I|})}$,

$$o(\mathcal{S}) = \exp\left(\lim_{n} \frac{1}{n} \int_{\Sigma_I^+} \log \beta_n(\pi\omega) \, d\nu_{(\frac{1}{|I|}, \ldots, \frac{1}{|I|})}(\omega)\right)$$

Proof We prove here the second part of the statement, about the topological overlap number; the first part follows similarly. Let us denote by $\mathbf{p} = (\frac{1}{|I|}, \ldots, \frac{1}{|I|})$, and consider $\mu_{\mathbf{p}} = \pi_* \nu_{\mathbf{p}}$. As in Theorem 1 there exists a corresponding Φ-invariant measure $\hat{\mu}_{\mathbf{p}}$ on $\Sigma_I^+ \times \Lambda$. We have from Theorem 1 that $\pi_* \nu_{\mathbf{p}} = \pi_{2*} \hat{\mu}_{\mathbf{p}}$, hence

$$\int_\Lambda \log \beta_n(x) \, d\mu_{\mathbf{p}}(x) = \int_{\Sigma_I^+ \times \Lambda} \log \beta_n \circ \pi_2(\omega, x) \, d\hat{\mu}_{\mathbf{p}}(\omega, x)$$
$$= \int_{\Sigma_I^+ \times \Lambda} \log \beta_n \circ \pi_2 \circ \Phi^n(\omega, x) \, d\hat{\mu}_{\mathbf{p}}(\omega, x)$$

But notice that $\beta_n \circ \pi_2 \circ \Phi^n(\omega, x) = \beta_n(\phi_{\omega_n} \circ \cdots \circ \phi_{\omega_1}(x)) = \mathrm{Card}\{(\eta_1, \ldots, \eta_n) \in I^n, \ \phi_{\omega_n} \circ \cdots \circ \phi_{\omega_1}(x) \in \phi_{\eta_1} \circ \cdots \circ \phi_{\eta_n}(\Lambda)\} = b_n(\omega, x)$, for any (ω, x). Therefore, from the last displayed equality, it follows that:

$$\int_{\Sigma_I^+} \log \beta_n(\pi\omega) \, d\nu_{(\frac{1}{|I|}, \ldots, \frac{1}{|I|})}(\omega) = \int_\Lambda \log \beta_n(x) \, d\mu_{\mathbf{p}}(x) = \int_{\Sigma_I^+ \times \Lambda} \log b_n(\omega, x) \, d\hat{\mu}_{\mathbf{p}}(\omega, x)$$
$\square$

We now show that overlap numbers of conformal IFS and of equilibrium measures on $\Sigma_I^+ \times \Lambda$, can be used to estimate the dimensions of the associated projection measures on Λ. Denote the Hausdorff dimension (for sets or measures) by HD. Recall that, in general for a measure μ on a metric space X, its Hausdorff dimension is defined by:

$$HD(\mu) := \inf\{HD(Z), Z \subset X \text{ with } \mu(X \setminus Z) = 0\}$$

In the following Theorem, we give an upper estimate for $HD(\mu_\psi)$, by estimating $HD(\Lambda \backslash Z(\psi))$ for some set $Z(\psi) \subset \Lambda$ of μ_ψ-measure zero with the help of the overlap number $o(\mathcal{S}, \hat{\mu}_\psi)$. Moreover, we will construct explicitly this set of μ_ψ-measure zero $Z(\psi)$ below.

Theorem 3 *Consider a finite conformal iterated function system $\mathcal{S} = \{\phi_i\}_{i \in I}$ with limit set Λ, $\pi : \Sigma_I^+ \to \Lambda$ be the canonical projection, and let a Hölder continuous potential $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$, with its (unique) equilibrium measure $\hat{\mu}_\psi$; let $\mu_\psi := \pi_{2*}\hat{\mu}_\psi$ be the projection as in (3). Then,*

$$HD(\mu_\psi) \leq t(\mathcal{S}, \psi),$$

where $t(\mathcal{S}, \psi)$ is the unique zero of the pressure function with respect to the shift $\sigma : \Sigma_I^+ \to \Sigma_I^+$,

$$t \to P_\sigma(t \log |\phi'_{\omega_1}(\pi(\sigma\omega))| - \log o(\mathcal{S}, \hat{\mu}_\psi))$$

Proof Let denote by $R_n(\hat{\mu}_\psi, \delta)$ the set of points $(\omega, x) \in \Sigma_I^+ \times \Lambda$ for which the number of generic roots satisfies $b_n((\omega, x), \tau, \hat{\mu}_\psi) < \frac{1}{2} \cdot e^{n(F_\Phi(\hat{\mu}_\psi) - \delta)}$. We want to show that the $\hat{\mu}_\psi$-measure of these sets converges to 0, when $n \to \infty$. If this does not happen, then there exist an infinite sequence $\{k_n\}_n$ and a number $\beta > 0$, such that $\hat{\mu}_\psi(R_{k_n}(\hat{\mu}_\psi, \delta)) > \beta > 0, \forall n \geq 1$. Then, for all pairs $(\omega, x) \in R_{k_n}(\hat{\mu}_\psi, \delta)$,

$$\frac{\log b_{k_n}((\omega, x), \tau, \hat{\mu}_\psi)}{k_n} < \frac{-\log 2}{k_n} + F_\Phi(\hat{\mu}_\psi) - \delta$$

Therefore, after integrating with respect to $\hat{\mu}_\psi$,

$$\int_{R_{k_n}(\hat{\mu}_\psi, \delta)} \frac{\log b_{k_n}((\omega, x), \tau, \hat{\mu}_\psi)}{k_n} d\hat{\mu}_\psi(\omega, x) < \hat{\mu}_\psi(R_{k_n}(\hat{\mu}_\psi, \delta)) \cdot \left(F_\Phi(\hat{\mu}_\psi) - \delta - \frac{\log 2}{k_n}\right)$$

We now use the last displayed inequality, and the properties of $J_{\Phi^n}(\hat{\mu}_\psi)$ from the proof of Theorem 2 (namely relation (9)); thus by adding the integral of $\frac{\log b_{k_n}((\omega, x), \tau, \hat{\mu}_\psi)}{k_n}$ over R_{k_n} and the integral of $\frac{\log b_{k_n}((\omega, x), \tau, \hat{\mu}_\psi)}{k_n}$ over the complement of R_{k_n}, we obtain that:

$$\int_{\Sigma_I^+ \times \Lambda} \frac{\log b_{k_n}((\omega, x), \tau, \hat{\mu}_\psi)}{k_n} d\hat{\mu}_\psi(\omega, x) < \hat{\mu}_\psi(R_{k_n}(\hat{\mu}_\psi, \delta)) \cdot \left(F_\Phi(\hat{\mu}_\psi) - \delta - \frac{\log 2}{k_n}\right)$$
$$+ \int_{\Sigma_I^+ \times \Lambda \backslash R_{k_n}(\hat{\mu}_\psi, \delta)} \frac{\log J_{\Phi^{k_n}}(\hat{\mu}_\psi)}{k_n} d\hat{\mu}_\psi(\omega, x) \tag{16}$$

On the other hand, from the Chain rule we know that $\log J_{\Phi^n}(\hat{\mu}_\psi)(\omega, x) = \log J_\Phi(\omega, x) + \cdots + \log J_\Phi(\hat{\mu}_\psi)(\Phi^{n-1}(\omega, x))$, for all $n \geq 1$. Therefore from the Birkhoff Ergodic Theorem,

$$\frac{\log J_{\Phi^n}(\hat{\mu}_\psi)(\omega, x)}{n} \underset{n \to \infty}{\to} F_\Phi(\hat{\mu}_\psi),$$

for $\hat{\mu}_\psi$-almost all $(\omega, x) \in \Sigma_I^+ \times \Lambda$. Moreover, from (9) we have that

$$J_{\Phi^n}(\hat{\mu}_\psi)(\omega, x) \leq C \cdot \frac{\sum_{\Phi^n(\eta, y) = \Phi^n(\omega, x)} e^{S_n \psi(\eta, y)}}{e^{S_n \psi(\omega, x)}} \leq C|I|^n \cdot e^{n(C_1 - C_2)}, \tag{17}$$

for all $n \geq 1$, where $C_2 \leq \psi \leq C_1$ on $\Sigma_I^+ \times \Lambda$ (as the potential ψ is continuous). This implies that the sequence $\{\frac{1}{n} \log J_{\Phi^n}(\hat{\mu}_\psi)(\omega, x)\}_n$ is bounded by $\log C + \log |I| + C_1 - C_1$,

independently of (ω, x). Since $\log J_\Phi(\hat\mu_\psi)$ is integrable, we obtain then from the Birkhoff Ergodic Theorem, that $\int_{\Sigma_I^+ \times \Lambda} \frac{\log J_{\Phi^n}(\hat\mu_\psi)(\omega, x)}{n} d\hat\mu_\psi(\omega, x) \underset{n\to\infty}{\to} F_\Phi(\hat\mu_\psi)$, and similarly,

$$\gamma_n(\hat\mu_\psi, \delta) := \int_{\Sigma_I^+ \times \Lambda \setminus R_n(\hat\mu_\psi, \delta)} \left(\frac{\log J_{\Phi^n}(\hat\mu_\psi)}{n} - F_\Phi(\hat\mu_\psi) \right) d\hat\mu_\psi(\omega, x) =$$

$$= \int_{\Sigma_I^+ \times \Lambda} \left(\frac{\log J_{\Phi^n}(\hat\mu_\psi)}{n} - F_\Phi(\hat\mu_\psi) \right) \cdot \chi_{\Sigma_I^+ \times \Lambda \setminus R_n(\hat\mu_\psi, \delta)} d\hat\mu_\psi(\omega, x) \underset{n\to\infty}{\to} 0$$

Hence for any integer $n \geq 1$,

$$\int_{\Sigma_I^+ \times \Lambda \setminus R_n(\hat\mu_\psi, \delta)} \frac{\log J_{\Phi^n}(\hat\mu_\psi)}{n} d\hat\mu_\psi = \gamma_n(\hat\mu_\psi, \delta) + F_\Phi(\hat\mu_\psi) \cdot \hat\mu_\psi(\Sigma_I^+ \times \Lambda \setminus R_n(\hat\mu_\psi, \delta))$$

Therefore, we obtain from (16) that:

$$\int_{\Sigma_I^+ \times \Lambda} \frac{\log b_{k_n}((\omega, x), \tau, \hat\mu_\psi)}{k_n} d\hat\mu_\psi(\omega, x)$$

$$< \hat\mu_\psi(R_{k_n}(\hat\mu_\psi, \delta)) \left(F_\Phi(\hat\mu_\psi) - \delta - \frac{\log 2}{k_n} \right) + \gamma_{k_n}(\hat\mu_\psi, \delta)$$

$$+ F_\Phi(\hat\mu_\psi) \cdot \hat\mu_\psi(\Sigma_I^+ \times \Lambda \setminus R_{k_n}(\hat\mu_\psi, \delta))$$

$$= \gamma_{k_n}(\hat\mu_\psi, \delta) + F_\Phi(\hat\mu_\psi) - \hat\mu_\psi(R_{k_n}(\hat\mu_\psi, \delta)) \left(\delta + \frac{\log 2}{k_n} \right)$$

However if $\hat\mu_\psi(R_{k_n}(\hat\mu_\psi, \delta)) > \beta$ for $n > n(\delta)$ (for some integer $n(\delta) \geq 1$), then it follows from the above and from the fact that: $\gamma_n(\hat\mu_\psi, \delta) \to 0$, that

$$\int_{\Sigma_I^+ \times \Lambda} \frac{\log b_{k_n}((\omega, x), \tau, \hat\mu_\psi)}{k_n} d\hat\mu_\psi(\omega, x) < F_\Phi(\hat\mu_\psi) - \beta \left(\delta + \frac{\log 2}{k_n} \right)$$

$$+ \gamma_{k_n}(\hat\mu_\psi, \delta) < F_\Phi(\hat\mu_\psi)$$

But then, this would give contradiction with Theorem 2. Hence, for $\delta > 0$ fixed there exists a sequence of positive numbers $\alpha_n \underset{n\to\infty}{\to} 0$, such that the set $R_n(\hat\mu_\psi, \delta)$ of points $(\omega, x) \in \Sigma_I^+ \times \Lambda$ for which $b_n((\omega, x), \tau, \hat\mu_\psi) < \frac{1}{2} e^{n(F_\Phi(\hat\mu_\psi) - \delta)}$, has $\hat\mu_\psi$-measure that satisfies:

$$\hat\mu_\psi(R_n(\hat\mu_\psi, \delta)) < \alpha_n, \text{ for } n > n(\delta)$$

Let denote now the complement of the set $R_n(\hat\mu_\psi, \delta)$ in $\Sigma_I^+ \times \Lambda$ by:

$$Q_n(\hat\mu_\psi, \delta) := \Sigma_I^+ \times \Lambda \setminus R_n(\hat\mu_\psi, \delta)$$

From the Φ-invariance of $\hat\mu_\psi$ on $\Sigma_I^+ \times \Lambda$, and from the definition of $Q_n(\hat\mu_\psi, \delta)$, we obtain that

$$\hat\mu_\psi(\Phi^n(Q_n(\hat\mu_\psi, \delta))) > 1 - \alpha_n, \ n \geq n(\delta)$$

And from the definition of the set $\Phi^n(Q_n(\hat\mu_\psi, \delta))$, it follows that for any for point $(\eta', y') \in \Phi^n(Q_n(\hat\mu_\psi, \delta))$, there exist at least $\frac{1}{2} e^{n(F_\Phi(\hat\mu_\psi) - \delta)}$ indices $\underline{i} = (i_1, \dots, i_n) \in I^n$, such that $y' \in \phi_{\underline{i}}(\Lambda) = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(\Lambda)$. From above, the sequence $\hat\mu_\psi(R_n(\hat\mu_\psi, \delta))$ converges to 0, so there exists an increasing sequence of integers $m_n \to \infty$ such that: $\hat\mu_\psi(R_{m_1}(\hat\mu_\psi, \delta)) <$

$\frac{1}{2}$, $\hat{\mu}_\psi(R_{m_2}(\hat{\mu}_\psi)) < \frac{1}{2^2}, \ldots, \hat{\mu}_\psi(R_{m_n}(\hat{\mu}_\psi, \delta)) < \frac{1}{2^n}, \ldots$. Employing the sequence $\{m_n\}_n$, define now the following measurable subsets of Λ,

$$\Lambda_n(\hat{\mu}_\psi, \delta) := \pi_2 \left(\underset{s \geq n}{\cap} \Phi^{m_s}(Q_{m_s}(\hat{\mu}_\psi, \delta)) \right),$$

where $\pi_2 : \Sigma_I^+ \times \Lambda \to \Lambda$ is the canonical projection to the second coordinate. Moreover, denote the union of the Borel subsets in Λ introduced above by,

$$\Lambda(\hat{\mu}_\psi, \delta) := \underset{n \geq 1}{\cup} \Lambda_n(\hat{\mu}_\psi, \delta) = \pi_2 \left(\underset{n \geq 1}{\cup} \underset{s \geq n}{\cap} \Phi^{m_s}(Q_{m_s}(\hat{\mu}_\psi, \delta)) \right)$$

Firstly, notice that from the definition of the sequence of integers $\{m_n\}_{n \geq 1}$, we have

$$\hat{\mu}_\psi \left(\underset{s \geq n}{\cap} \Phi^{m_s}(Q_{m_s}(\hat{\mu}_\psi, \delta)) \right) \geq 1 - \sum_{s \geq n} \hat{\mu}_\psi \left(\Sigma_I^+ \times \Lambda \backslash \Phi^{m_s}(Q_{m_s}(\hat{\mu}_\psi, \delta)) \right)$$

$$\geq 1 - \sum_{s \geq n} \frac{1}{2^s} = 1 - \frac{1}{2^{n-1}}$$

Therefore by taking the union of these sets over all $n \geq 1$, recalling that $\mu_\psi = \pi_{2*}(\hat{\mu}_\psi)$, and observing that $\mu_\psi(\Lambda(\hat{\mu}_\psi, \delta)) = \hat{\mu}_\psi \left(\pi_2^{-1}(\Lambda(\hat{\mu}_\psi, \delta)) \right) \geq \hat{\mu}_\psi \left(\underset{n \geq 1}{\cup} \underset{s \geq n}{\cap} \Phi^{m_s}(Q_{m_s}(\hat{\mu}_\psi, \delta)) \right)$, we obtain that

$$\hat{\mu}_\psi \left(\underset{n \geq 1}{\cup} \underset{s \geq n}{\cap} \Phi^{m_s}(Q_{m_s}(\hat{\mu}_\psi, \delta)) \right) = 1, \text{ hence } \mu_\psi(\Lambda(\hat{\mu}_\psi, \delta)) = 1 \qquad (18)$$

We now investigate the influence of the number of roots on the Hausdorff dimension of the set $\Lambda(\hat{\mu}_\psi, \delta)$. Recall from above that, for any $(\eta', y') \in \Phi^n(Q_n(\hat{\mu}_\psi, \delta))$, there exist at least $\frac{1}{2} e^{n(F_\Phi(\hat{\mu}_\psi) - \delta)}$ indices $\underline{i} = (i_1, \ldots, i_n) \in I^n$, such that $y' = \phi_{\underline{i}}(\Lambda) = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(\Lambda)$. Hence the points in the projection $\pi_2(\Phi^n(Q_n(\hat{\mu}_\psi, \delta)))$ are covered at least $\frac{1}{2} e^{n(F_\Phi(\hat{\mu}_\psi) - \delta)}$ times by images of Λ, through compositions of n maps of type ϕ_i. Now, $\tilde{S}$ satisfies the condition that there exists $\kappa \in (0, 1)$ such that $|\phi_i'| < \kappa$ on Λ. It follows that, for any indices $i_1, \ldots, i_n \in I$, $\mathrm{diam}(\phi_{i_1} \circ \cdots \circ \phi_{i_n}(\Lambda)) \leq \kappa^n$. Thus, every point in $\pi_2(\Phi^n(Q_n(\hat{\mu}_\psi, \delta)))$ can be covered at least $\frac{1}{2} e^{n(F_\Phi(\hat{\mu}_\psi) - \delta)}$ times with sets of diameter less than κ^n. For $\alpha \geq 0$, let us denote now by $t(\alpha)$ the unique zero of the following pressure function with respect to the shift map $\sigma : \Sigma_I^+ \to \Sigma_I^+$,

$$t \to P_\sigma(t|\phi_{\omega_1}'(\sigma\omega)| - \alpha) \qquad (19)$$

Take an arbitrary number $t' > t(F_\Phi(\hat{\mu}_\psi) - \delta)$; we assume without loss of generality that $F_\Phi(\hat{\mu}_\psi) > 0$ and that δ is small enough, so that $\delta < F_\Phi(\hat{\mu}_\psi)$. Let define the pressure function

$$p_\delta(s) := P(s|\phi_{\omega_1}'(\sigma\omega)| - F_\Phi(\hat{\mu}_\psi) + \delta), \ s \in \mathbb{R}$$

From assumption above, $p_\delta(t') < 0$. So from the conformality of the contractions ϕ_i, and by denoting in general $\phi_\eta := \phi_{\eta_1} \circ \cdots \circ \phi_{\eta_m}$ for $\eta = (\eta_1, \ldots, \eta_m) \in I^m$, $m \geq 1$, it follows that for n large:

$$\sum_{|\omega|=n} |\phi_\omega'|^{t'} e^{-n(F_\Phi(\hat{\mu}_\psi) - \delta)} \leq e^{\frac{n \cdot p_\delta(t')}{4}} \qquad (20)$$

Now for any $s \geq n$, from the above definition of $Q_{m_s}(\hat{\mu}_\psi, \delta)$, it follows that any point in $\Lambda_n(\hat{\mu}_\psi, \delta)$ can be covered with at least $M_s := \frac{1}{2} e^{m_s(F_\Phi(\hat{\mu}_\psi) - \delta)}$ sets $\phi_\eta(V)$ for $|\eta| = m_s$, and

every one of these sets $\phi_\eta(V)$ has diameter less than κ^{m_s}. Denote the collection of the above sets $\phi_\eta(V)$ by $\mathcal{U}_s$, so $\mathcal{U}_s$ is a cover of $\Lambda_n(\hat{\mu}_\psi, \delta)$. We want now to perform extractions from this cover $\mathcal{U}_s$ of $\Lambda_n(\hat{\mu}_\psi, \delta)$ (by using its large multiplicity), in such a way that in the end we obtain a subcover which is minimal, from the point of view of the sum of diameters raised to power t. This will be the subcover which we shall use to estimate the Hausdorff dimension of the set $\Lambda_n(\hat{\mu}_\psi, \delta)$. We have that the maps ϕ_η are conformal, so we can apply the $5r$-Covering Theorem (see [8]), where we consider $5U$ to denote the ball with the same center as U and 5 times the radius of U. One can then extract a subfamily $\mathcal{U}_s(1) \subset \mathcal{U}_s$, such that the sets $5U$, $U \in \mathcal{U}_s(1)$, cover $\Lambda_n(\hat{\mu}_\psi, \delta)$, and so that the sets in $\mathcal{U}_s(1)$ are mutually disjoint. From conformality we have that there exists x, r and a fixed constant C independent of U, such that $B(x, r) \subset U \subset B(x, Cr)$. We then eliminate this subfamily $\mathcal{U}_s(1)$. Since it was disjointed, the multiplicity of the cover $\mathcal{U}_s$ of $\Lambda_n(\hat{\mu}_\psi, \delta)$ is still at least $M_s - 1$. Therefore we can repeat this procedure and will extract a second subfamily $\mathcal{U}_s(2)$ in $\mathcal{U}_s \backslash \mathcal{U}_s(1)$, which is disjointed and such that $5U$, $U \in \mathcal{U}_s(2)$ cover the set $\Lambda_n(\hat{\mu}_\psi, \delta)$. After eliminating both $\mathcal{U}_s(1)$ and $\mathcal{U}_s(2)$ from $\mathcal{U}_s$, the multiplicity of the cover is at least $M_s - 2$. By induction, we obtain thus M_s subfamilies $\mathcal{U}_s(j)$, which are disjointed and such that $5U$, $U \in \mathcal{U}_s(j)$, cover $\Lambda_n(\hat{\mu}_\psi, \delta)$. We then take, out of these subfamilies constructed above, the subfamily $\mathcal{U}_s(j_0)$ for which the expression $\sum_{U \in \mathcal{U}_s(j_0)} (\text{diam} U)^{t'}$ is minimal. Then from (20), we obtain:

$$\sum_{U \in \mathcal{U}_s(j_0)} (\text{diam} U)^{t'} \leq \frac{1}{M_s} \sum_{U \in \mathcal{U}_s} (\text{diam} U)^{t'} \leq C e^{m_s p_\delta(t')/4} < 1, \tag{21}$$

for some constant $C > 0$, independent of s, n large. Since for any $s \geq n$, we can obtain such minimal covers $\mathcal{U}_s(j_0)$ for the set $\Lambda_n(\hat{\mu}_\psi, \delta)$, and since t' was chosen arbitrarily larger than $t(F_\Phi(\hat{\mu}_\psi) - \delta)$, it follows from (21) that:

$$HD(\Lambda_n(\hat{\mu}_\psi, \delta)) \leq t(F_\Phi(\hat{\mu}_\psi) - \delta)$$

Now recall the definition of $\Lambda(\hat{\mu}_\psi, \delta) = \bigcup_{n \geq 1} \Lambda_n(\hat{\mu}_\psi, \delta)$. From the last estimate, we infer that

$$HD(\Lambda(\hat{\mu}_\psi, \delta)) \leq t(F_\Phi(\hat{\mu}_\psi) - \delta)$$

Also from (18), $\mu_\psi(\Lambda(\hat{\mu}_\psi, \delta)) = 1$. Define now the set $\Lambda(\psi) := \bigcap_{\delta > 0} \Lambda(\hat{\mu}_\psi, \delta) = \bigcap_{n \geq 1} \Lambda(\hat{\mu}_\psi, \frac{1}{n})$. We have then that $\mu_\psi(\Lambda(\psi)) = 1$. Let us now remark that from definition (19) of the zero $t(\alpha)$, and from the continuity of the pressure function, we obtain that $t(F_\Phi(\hat{\mu}_\psi) - \delta) \to t(F_\Phi(\hat{\mu}_\psi))$ when $\delta \to 0$. But from Theorem 2, we know that $\log o(\mathcal{S}, \psi) = F_\Phi(\hat{\mu}_\psi)$. Hence, by taking the set $Z(\psi) := \Lambda \backslash \Lambda(\psi)$, we have $\mu_\psi(Z(\psi)) = 0$; thus from the definition of $HD(\mu_\psi)$, $HD(\mu_\psi) \leq HD(\Lambda \backslash Z(\psi)) \leq t(\mathcal{S}, \psi)$. □

3 Applications to Bernoulli Convolutions

Consider the random series $\sum_{n \geq 0} \pm \lambda^n$ for $\lambda \in (0, 1)$ where the $+, -$ signs are taken independently and with equal probability, and let us denote its distribution by ν_λ. This is called a Bernoulli convolution, since it is in fact the infinite convolution of the atomic measures $\frac{1}{2}(\delta_{-\lambda^n} + \delta_{\lambda^n})$, for $n \geq 0$ (for e.g. [3,21]). The probability measure ν_λ can be written also as the self-similar measure associated to the probability vector $(\frac{1}{2}, \frac{1}{2})$ and to the iterated

function system

$$\mathcal{S}_\lambda = \{S_1, S_2\},$$

where $S_1(x) = \lambda x - 1$, $S_2(x) = \lambda x + 1$, $x \in \mathbb{R}$. Hence, ν_λ satisfies the self-similarity relation:

$$\nu_\lambda = \frac{1}{2}\nu_\lambda \circ S_1^{-1} + \frac{1}{2}\nu_\lambda \circ S_2^{-1}$$

The case $\lambda \in (0, \frac{1}{2})$ corresponds to $\mathcal{S}_\lambda$ having no overlaps, while the case when $\lambda \in [\frac{1}{2}, 1)$ corresponds to the more difficult situation of the iterated function system $\mathcal{S}_\lambda$ having overlaps. We assume in the sequel that $\lambda \in (\frac{1}{2}, 1)$, thus we are in the case when $\mathcal{S}_\lambda$ has overlaps. The limit set Λ_λ is in this case the whole interval $I_\lambda = [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$. The measure ν_λ can be viewed as the projection $\pi_{\lambda*}\nu_{(\frac{1}{2}, \frac{1}{2})}$, where $\nu_{(\frac{1}{2}, \frac{1}{2})}$ is the Bernoulli measure on Σ_2^+ generated by the vector $(\frac{1}{2}, \frac{1}{2})$, and $\pi_\lambda : \Sigma_2^+ \to I_\lambda$ is the canonical coding map. It is well-known that ν_λ can be either singular or absolutely continuous. Several results on Bernoulli convolutions are in the paper by Peres, Schlag and Solomyak [15]. The case $\lambda > \frac{1}{2}$ attracted a lot of interest, starting with Erdös who proved in [3] that, when $\frac{1}{\lambda}$ is a Pisot number, then ν_λ is singular. Later Solomyak showed in [21] that the measure ν_λ is absolutely continuous for Lebesgue-a.e $\lambda \in [\frac{1}{2}, 1)$. If ν_λ is absolutely continuous, then $HD(\nu_\lambda) = 1$. From the point of view of actual values of λ, Garsia proved in [5] that ν_λ is absolutely continuous when λ^{-1} is an algebraic integer in $(1, 2)$, whose monic polynomial has other roots outside the unit circle and constant coefficient ± 2; for example if $\lambda^{-1} = 2^{\frac{1}{m}}$, $m \geq 2$, ν_λ is absolutely continuous. Przytycki and Urbański proved in [17] that, if λ is the inverse of a Pisot number in $(1, 2)$, then $HD(\nu_\lambda) < 1$. In the special case when $\lambda = \frac{\sqrt{5}-1}{2}$ (the reciprocal of the Pisot number $\frac{\sqrt{5}+1}{2}$), Alexander and Zagier [1] found precise estimates for $HD(\nu_\lambda)$, showing that $0.99557 < HD(\nu_\lambda) < 0.99574$. Hochman showed recently in [6] that $HD(\nu_\lambda) = 1$ for λ outside a set of dimension zero.

For arbitrary $\lambda \in (\frac{1}{2}, 1)$, Theorem 4 below gives an upper estimate for $HD(\nu_\lambda)$, by using an expression involving $o(\mathcal{S}_\lambda)$; this allows to obtain bounds also for the overlap numbers $o(\mathcal{S}_\lambda)$. In particular, if $HD(\nu_\lambda) = 1$ for some value $\lambda \in (\frac{1}{2}, 1)$, then $o(\mathcal{S}_\lambda) \leq 2\lambda$. In general, $1 \leq o(\mathcal{S}_\lambda) \leq 2$, for any $\lambda \in (\frac{1}{2}, 1)$; we show that in fact, the overlap number $o(\mathcal{S}_\lambda)$ is *never* equal to 2 (even if, for $\lambda \to 1$ the overlaps become larger). For specific values of λ (for e.g. $\lambda = 2^{-\frac{1}{m}}, m \geq 2$, or $\lambda = \frac{\sqrt{5}-1}{2}$), we obtain then more precise bounds for $o(\mathcal{S}_\lambda)$. First, for arbitrary $\lambda \in (\frac{1}{2}, 1)$, the measure ν_λ is supported on the limit set of $\mathcal{S}_\lambda$, which is the interval $I_\lambda = [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$; the coding map is $\pi_\lambda : \Sigma_2^+ \to I_\lambda$. Recall that for $x \in I_\lambda$ and $n \geq 2$, $\beta_n(x)$ denotes the number of n-chains $(\zeta_1, \ldots, \zeta_n) \in \{1, 2\}^n$ from points in I_λ to x, i.e. $x \in \phi_{\zeta_1 \ldots \zeta_n}\left([-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]\right)$. From Corollary 2, in the formula for $o(\mathcal{S}_\lambda)$ we integrate $\log \beta_n$ with respect to the uniform Bernoulli measure $\nu_{(\frac{1}{2}, \frac{1}{2})}$.

Theorem 4 *For all $\lambda \in (\frac{1}{2}, 1)$, the following relation is satisfied for the Bernoulli convolution ν_λ:*

$$HD(\nu_\lambda) \leq \frac{\log \frac{2}{o(\mathcal{S}_\lambda)}}{|\log \lambda|},$$

where $o(\mathcal{S}_\lambda)$ denotes the overlap number of $\mathcal{S}_\lambda$, which can be computed as:

$$o(\mathcal{S}_\lambda) = \exp\left(\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_2^+} \log \beta_n(\pi_\lambda \omega) \, dv_{(\frac{1}{2}, \frac{1}{2})}(\omega)\right)$$

And from the above, $o(\mathcal{S}_\lambda) \leq 2\lambda^{HD(v_\lambda)}$.

Proof From Theorem 1, in our case the measure v_λ can be written as $\pi_{\lambda*} v_{(\frac{1}{2}, \frac{1}{2})}$ and it is equal to the π_2-projection of an equilibrium state $\hat{\mu}_\psi$ on $\Sigma_2^+ \times I_\lambda$. Therefore, from Corollary 2,

$$o(\mathcal{S}_\lambda) = \exp\left(\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_2^+} \log \beta_n(\pi_\lambda \omega) \, dv_{(\frac{1}{2}, \frac{1}{2})}(\omega)\right)$$

$\mathcal{S}_\lambda$ is a system of similarities, thus from Theorem 3, $HD(v_\lambda)$ is bounded above by the unique zero of the pressure function with respect to $\sigma : \Sigma_2^+ \to \Sigma_2^+$:

$$t \to P_\sigma(t \log \lambda - o(\mathcal{S}_\lambda)) = t \log \lambda + \log 2 - \log o(\mathcal{S}_\lambda)$$

Hence it follows that $HD(v_\lambda) \leq \frac{\log \frac{2}{o(\mathcal{S}_\lambda)}}{|\log \lambda|}$, and the corresponding bound for $o(\mathcal{S}_\lambda)$.   □

For any $\lambda \in (\frac{1}{2}, 1)$, the number of overlaps between images $S_{i_1 \ldots i_n}(I_\lambda)$ is less than 2^n, so $1 \leq o(\mathcal{S}_\lambda) \leq 2$. In fact, it turns out that the overlap number of $\mathcal{S}_\lambda$ is always strictly less than 2:

Corollary 3 *In the above setting, it follows that for all parameters $\lambda \in (\frac{1}{2}, 1)$,*

$$o(\mathcal{S}_\lambda) < 2$$

Proof If $o(\mathcal{S}_\lambda) = 2$, then from Theorem 4, it would follow that $\lambda = 1$. Hence contradiction.

For a large set of values of λ, by using Theorem 4 and the above mentioned results of [1,5,21], we can obtain more precise estimates for the overlap number:

Corollary 4 (a) *For Lebesgue-almost all parameters λ in $(\frac{1}{2}, 1)$, we have*

$$o(\mathcal{S}_\lambda) \leq 2\lambda$$

This happens for example when λ^{-1} is an algebraic number whose monic polynomial has other roots outside the unit circle and constant coefficient ± 2. In particular, if $\lambda = 2^{-\frac{1}{m}}$ for $m \geq 2$, then

$$o(\mathcal{S}_\lambda) \leq 2^{\frac{m-1}{m}}$$

(b) *In case $\lambda = \frac{\sqrt{5}-1}{2}$, then $o(\mathcal{S}_\lambda) \leq 2\lambda^{0.99557} < 1.25$.*

Let now p arbitrary in $(0, 1)$ and denote by $v_{(p, 1-p)}$ the Bernoulli measure on Σ_2^+ determined by the vector $(p, 1 - p)$. For $\lambda \in (\frac{1}{2}, 1)$, one defines the *biased* Bernoulli convolution $v_{\lambda, p}$ (see for e.g. [16]), where $v_{\lambda, p}$ is the π_λ-projection of $v_{(p, 1-p)}$ onto the limit set $I_\lambda = [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$. We have as above the associated lift map $\Phi_\lambda : \Sigma_2^+ \times I_\lambda \to \Sigma_2^+ \times I_\lambda$. From the discussion before Theorem 1, there exists a Φ_λ-invariant equilibrium measure $\hat{v}_{\lambda, p}$ on $\Sigma_2^+ \times I_\lambda$, such that $\pi_{2*} \hat{v}_{\lambda, p} = v_{\lambda, p}$. For integers $0 < k < n$, denote by $W(x, n, k)$ the set

of n-chains $(i_1, \ldots, i_n) \in \{1, 2\}^n$ from points in I_λ to x, having exactly k indices i_j equal to 1. From (15), for any $x \in I_\lambda$, $\tau > 0$ and $n \geq 2$, we have

$$\beta_n \left(x, \tau \left| \log \frac{p}{1-p} \right|, (p, 1-p) \right) = \sum_{k, \, |\frac{k}{n} - p| < \tau} \text{Card } W(x, n, k)$$

Thus, for any parameter $\lambda \in (\frac{1}{2}, 1)$, it follows from Theorem 3 and Corollary 2 that:

Corollary 5 *For all* $\lambda \in (\frac{1}{2}, 1)$ *and* $p \in (0, 1)$, *the biased Bernoulli convolution* $\nu_{\lambda, p}$ *satisfies:*

$$HD(\nu_{\lambda, p}) \leq \frac{\log \frac{2}{o(\mathcal{S}_\lambda, \hat{\nu}_{\lambda, p})}}{|\log \lambda|},$$

where $o(\mathcal{S}_\lambda, \hat{\nu}_{\lambda, p})$ *denotes the overlap number of* $\mathcal{S}_\lambda$ *with respect to* $\hat{\nu}_{\lambda, p}$, *which can be computed by:*

$$o(\mathcal{S}_\lambda, \hat{\nu}_{\lambda, p}) = \exp \left(\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_2^+} \log \sum_{|\frac{k}{n} - p| < \tau} \text{Card } W(\pi_\lambda \omega, n, k) \, d\nu_{(p, 1-p)}(\omega) \right).$$

References

1. Alexander, J.C., Zagier, D.: The entropy of a certain infinitely convolved Bernoulli measure. J. Lond. Math. Soc. (2) **44**, 121–134 (1991)
2. Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Mathematics, vol. 470. Springer, Berlin (1975)
3. Erdös, P.: On a family of symmetric Bernoulli convolutions. Am. J. Math. **61**, 974–976 (1939)
4. Falconer, K.: The Hausdorff dimension of some fractals and attractors of overlapping construction. J. Stat. Phys. **47**, 1–2 (1987)
5. Garsia, A.: Arithmetic properties of Bernoulli convolutions. Trans. Am. Math. Soc. **102**, 409–432 (1962)
6. Hochman, M.: On self-similar sets with overlaps and inverse theorems for entropy. Ann. Math. (2) **180**(2), 773–822 (2014)
7. Katok, A., Hasselblatt, B.: Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press, London (1995)
8. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1995)
9. Mihailescu, E.: On a class of stable conditional measures. Ergod. Theory Dyn. Syst. **31**, 1499–1515 (2011)
10. Mihailescu, E.: On some coding and mixing properties for a class of chaotic systems. Monatsh. Math. **167**, 241–255 (2012)
11. Mihailescu, E., Stratmann, B.: Upper estimates for stable dimensions on fractal sets with variable numbers of foldings. Int. Math. Res. Not. **23**, 6474–6496 (2014)
12. Mihailescu, E., Urbański, M.: Hausdorff dimension of the limit set of conformal iterated function systems with overlaps. Proc. Am. Math Soc. **139**(8), 2767–2775 (2011)
13. Mihailescu, E., Urbański, M.: Entropy production for a class of inverse SRB measures. J. Stat. Phys. **150**, 881–888 (2013)
14. Parry, W.: Entropy and Generators in Ergodic Theory. W. A. Benjamin, New York (1969)
15. Peres, Y., Schlag, W., Solomyak, B.: Sixty Years of Bernoulli Convolutions in Fractal Geometry and Stochastics II (Greifswald/Koserow, 1998). Progress in Probability, vol. 46, pp. 39–65. Birkhäuser, Basel (2000)
16. Peres, Y., Solomyak, B.: Self-similar measures and intersections of Cantor sets. Trans. Am. Math Soc **350**, 4065–4087 (1998)

17. Przytycki, F., Urbański, M.: On Hausdorff dimension of some fractal sets. Studia Math. **93**, 155–186 (1989)
18. Ruelle, D.: Thermodynamic Formalism. Addison-Wesley, Reading (1978)
19. Ruelle, D.: Positivity of entropy production in nonequilibrium statistical mechanics. J. Stat. Phys. **85**(1/2), 1–23 (1996)
20. Ruelle, D.: Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics. J. Stat. Phys. **95**, 393–468 (1999)
21. Solomyak, B.: On the random series $\sum \pm \lambda^n$ (an Erdös problem). Ann. Math. **142**, 611–625 (1995)
22. Walters, P.: An Introduction to Ergodic Theory, 2nd edn. Springer, New York (2000)