

Global Solutions and Time Decay of the Non-cutoff Vlasov–Maxwell–Boltzmann System in the Whole Space

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Abstract The global in time classical solutions to the Cauchy problem of the Vlasov–Maxwell–Boltzmann system near Maxwellians are obtained under the lower regularity index assumption and the weaker smallness condition on the initial perturbation in comparison with the work (Duan et al. in *Kinet Relat Models* 6(1):159–204, 2013). In particular, we show the relation between time decay rates and spatial derivatives of solutions. Our analysis relies on a refined energy estimates and the interpolation techniques between negative Sobolev norms and positive Sobolev norms without linear decay analysis.

Keywords Global in time classical solutions · Time decay rates · The Vlasov–Maxwell–Boltzmann system · Global Maxwellians

1 Introduction and Main Results

This paper concerns the existence and time decay rates of global in time classical solutions to the Cauchy problem of the Vlasov–Maxwell–Boltzmann system (called VMB for simplicity) near Maxwellians, which takes the form:

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$$\begin{cases} \partial_t F_+ + v \cdot \nabla_x F_+ + (E + v \times B) \cdot \nabla_v F_+ = Q(F_+, F_+) + Q(F_+, F_-), \\ \partial_t F_- + v \cdot \nabla_x F_- - (E + v \times B) \cdot \nabla_v F_- = Q(F_-, F_+) + Q(F_-, F_-), \\ \partial_t E - \nabla_x \times B = - \int_{\mathbb{R}^3} v(F_+ - F_-)dv, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \int_{\mathbb{R}^3} (F_+ - F_-)dv, \quad \nabla_x \cdot B = 0. \end{cases} \tag{1.1}$$

with prescribed initial data

$$F_{\pm}(0, x, v) = F_{0,\pm}(v, x), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x) \tag{1.2}$$

which satisfy the compatibility conditions

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{0,+} - F_{0,-})dv, \quad \nabla_x \cdot B_0 = 0.$$

Here the unknown functions $F_{\pm} = F_{\pm}(t, x, v) \geq 0$ are the number density functions for the ions (+) and electrons (-) with position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geq 0$, respectively. $E(t, x)$ and $B(t, x)$ denote the electro and magnetic fields, respectively. The Boltzmann collision operator Q is given by

$$Q(F, G)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} \mathbf{B}(v - u, \sigma) \{F(v')G(u') - F(v)G(u)\}d\sigma du$$

where in terms of velocities u and v before the collision, velocities v' and u' after the collision are defined by

$$v' = \frac{v + u}{2} + \frac{|v - u|}{2}\sigma, \quad u' = \frac{v + u}{2} - \frac{|v - u|}{2}\sigma.$$

The Boltzmann collision kernel $\mathbf{B}(v - u, \sigma)$ depends only on the relative velocity $|v - u|$ and on the deviation angle θ given by $\cos \theta = \langle \sigma, (v - u)/|v - u| \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual dot product in \mathbb{R}^3 . As in [1–3, 7], we suppose that $\mathbf{B}(v - u, \sigma)$ is supported on $\cos \theta \geq 0$. Notice also that all the physical parameters have been chosen to be unit for simplicity of presentation.

Throughout the paper, the collision kernel is further supposed to satisfy the following assumptions:

(A1). $\mathbf{B}(v - u, \sigma)$ takes the product form in its argument as

$$\mathbf{B}(v - u, \sigma) = \Phi(|v - u|)\mathbf{b}(\cos \theta)$$

with Φ and \mathbf{b} being non-negative functions;

(A2). The angular function $\sigma \rightarrow \mathbf{b}(\langle \sigma, (v - u)/|v - u| \rangle)$ is not integrable on \mathbb{S}^2 , i.e.

$$\int_{\mathbb{S}^2} \mathbf{b}(\cos \theta)d\sigma = 2\pi \int_0^{\pi/2} \sin \theta \mathbf{b}(\cos \theta)d\theta = \infty.$$

Moreover, there are two positive constants $c_b > 0, 0 < s < 1$ such that

$$\frac{c_b}{\theta^{1+2s}} \leq \sin \theta \mathbf{b}(\cos \theta) \leq \frac{1}{c_b \theta^{1+2s}};$$

(A3). The kinetic function $z \rightarrow \Phi(|z|)$ satisfies

$$\Phi(|z|) = C_{\Phi}|z|^{\gamma}$$

for some positive constant $C_{\Phi} > 0$. Here we should notice that the exponent $\gamma > -3$ is determined by the intermolecular interactive mechanism.

It is convenient to call hard potentials when $\gamma + 2s \geq 0$ and soft potentials when $-3 < \gamma < -2s$ with $0 < s < 1$. Interested readers may refer to the textbooks [12, 14, 20] for more details. The current work is restricted to the case of

$$\max \left\{ -3, -\frac{3}{2} - 2s \right\} < \gamma < -2s, \quad \frac{1}{2} \leq s < 1. \tag{1.3}$$

Our goal of the paper is to study the Cauchy problem (1.1) around the following normalized global Maxwellian $\mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$. For this purpose, we define the perturbation $f_{\pm} = f_{\pm}(t, x, v)$ by $F_{\pm}(t, x, v) = \mu + \mu^{1/2} f_{\pm}(t, x, v)$. Then, the VMB system (1.1) is reformulated as

$$\begin{cases} \partial_t f_{\pm} + v \cdot \nabla_x f_{\pm} \pm (E + v \times B) \cdot \nabla_v f_{\pm} \mp E \cdot v \mu^{1/2} \mp \frac{1}{2} E \cdot v f_{\pm} + L_{\pm} f = \Gamma_{\pm}(f, f), \\ \partial_t E - \nabla_x \times B = - \int_{\mathbb{R}^3} v \mu^{1/2} (f_+ - f_-) dv, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = \int_{\mathbb{R}^3} \mu^{1/2} (f_+ - f_-) dv, \quad \nabla_x \cdot B = 0, \end{cases} \tag{1.4}$$

with initial data

$$f_{\pm}(0, x, v) = f_{0,\pm}(x, v), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x), \tag{1.5}$$

which satisfy the compatibility conditions

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} \mu^{1/2} (f_{0,+} - f_{0,-}) dv, \quad \nabla_x \cdot B_0 = 0. \tag{1.6}$$

If we denote $f = [f_+, f_-]$, then (1.4)₁ can be written as

$$\partial_t f + v \cdot \nabla_x f + q_0(E + v \times B) \cdot \nabla_v f - E \cdot v \mu^{1/2} q_1 + Lf = \frac{q_0}{2} E \cdot v f + \Gamma(f, f) \tag{1.7}$$

where $q_0 = \text{diag}(1, -1)$, $q_1 = [1, -1]$, the linearized collision operator L and the nonlinear collision term $\Gamma(f, f)$ are respectively defined by

$$Lf = [L_+ f, L_- f], \quad \Gamma(f, g) = [\Gamma_+(f, g), \Gamma_-(f, g)]$$

with

$$\begin{aligned} L_{\pm} f &= -\mu^{-1/2} \{ Q(\mu, \mu^{1/2} (f_{\pm} + f_{\mp})) + 2Q(\mu^{1/2} f_{\pm}, \mu) \}, \\ \Gamma_{\pm}(f, g) &= \mu^{-1/2} Q(\mu^{1/2} f_{\pm}, \mu^{1/2} g_{\pm}) + \mu^{-1/2} Q(\mu^{1/2} f_{\pm}, \mu^{1/2} g_{\mp}). \end{aligned}$$

For the linearized Boltzmann collision operator L , it is well known [9] that it is non-negative and the null space of L is given by

$$\mathcal{N} = \text{span} \{ [1, 0] \mu^{1/2}, [0, 1] \mu^{1/2}, [v_i, v_i] \mu^{1/2} (1 \leq i \leq 3), [|v|^2, |v|^2] \mu^{1/2} \}.$$

If we define \mathbf{P} as the orthogonal projection from $L^2(\mathbb{R}_v^3) \times L^2(\mathbb{R}_v^3)$ to \mathcal{N} , then for any given function $f(t, x, v) \in L^2(\mathbb{R}_v^3)$, one has

$$\begin{aligned} \mathbf{P} f &= a_+(t, x) [1, 0] \mu^{1/2} + a_-(t, x) [0, 1] \mu^{1/2} + \sum_{i=1}^3 b_i(t, x) [1, 1] v_i \mu^{1/2} \\ &\quad + c(t, x) [1, 1] (|v|^2 - 3) \mu^{1/2} \end{aligned}$$

with

$$a_{\pm} = \int_{\mathbb{R}^3} \boldsymbol{\mu}^{1/2} f_{\pm} dv, \quad b_i = \frac{1}{2} \int_{\mathbb{R}^3} v_i \boldsymbol{\mu}^{1/2} (f_+ + f_-) dv,$$

$$c = \frac{1}{6} \int_{\mathbb{R}^3} (|v|^2 - 3) \boldsymbol{\mu}^{1/2} (f_+ + f_-) dv.$$

Therefore, we have the following macro-micro decomposition with respect to a given global Maxwellian which was introduced in [8, 10, 16]

$$f(t, x, v) = \mathbf{P}f(t, x, v) + \{\mathbf{I} - \mathbf{P}\}f(t, x, v) \tag{1.8}$$

where \mathbf{I} denotes the identity operator.

Before stating our main results, we first introduce some notations used throughout the paper. C denotes some positive constant (generally large) and κ, λ denotes some positive constant (generally small), where C, κ and λ may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leq CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. The multi-indices $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$ will be used to record spatial and velocity derivatives, respectively. And $\partial_{\beta}^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}$. Similarly, the notation ∂^{α} will be used when $\beta = 0$ and likewise for ∂_{β} . The length of α is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. $\alpha' \leq \alpha$ means that no component of α' is greater than the corresponding component of α , and $\alpha' < \alpha$ means that $\alpha' \leq \alpha$ and $|\alpha'| < |\alpha|$. And it is convenient to write $\nabla_x^k = \partial^{\alpha}$ with $|\alpha| = k$. We also use $\langle \cdot, \cdot \rangle$ to denotes the L^2_v inner product in \mathbb{R}^3_v , with the L^2 norm $|\cdot|_{L^2}$. For notational simplicity, (\cdot, \cdot) denotes the L^2 inner product either in $\mathbb{R}^3_x \times \mathbb{R}^3_v$ or in \mathbb{R}^3_x with the L^2_v norm $\|\cdot\|$. $B_C \subset \mathbb{R}^3$ denotes the ball of radius C centered at the origin, and $L^2(B_C)$ stands for the space L^2 over B_C and likewise for other spaces.

As in [11], we introduce the operator Λ^s with $s \in \mathbb{R}$ by

$$(\Lambda^s g)(t, x, v) = \int_{\mathbb{R}^3} |\xi|^s \hat{g}(t, \xi, v) e^{2\pi i x \cdot \xi} d\xi = \int_{\mathbb{R}^3} |\xi|^s \mathcal{F}[g](t, \xi, v) e^{2\pi i x \cdot \xi} d\xi$$

with $\hat{g}(t, \xi, v) \equiv \mathcal{F}[g](t, \xi, v)$ being the Fourier transform of $g(t, x, v)$ with respect to x . The homogeneous Sobolev space H^s is the Banach space consisting of all g satisfying $\|g\|_{\dot{H}^s} < +\infty$, where

$$\|g(t)\|_{\dot{H}^s} \equiv \|(\Lambda^s g)(t, x, v)\|_{L^2_{x,v}} = \| |\xi|^s \hat{g}(t, \xi, v) \|_{L^2_{\xi,v}}.$$

We now list series of notations introduced in [3]. Introduce

$$|f|_D^2 \equiv \int_{\mathbb{R}^6 \times \mathbb{S}^2} \mathbf{B}(v - u, \sigma) \mu(u) (f' - f)^2 dudv d\sigma$$

$$+ \int_{\mathbb{R}^6 \times \mathbb{S}^2} f(u)^2 (\mu(u')^{1/2} - \mu(u)^{1/2})^2 dudv d\sigma.$$

For $l \in \mathbb{R}$, $\langle v \rangle = \sqrt{1 + |v|^2}$, L^2_l denotes the weighted space with norm $|f|_{L^2_l}^2 \equiv \int_{\mathbb{R}^3} |f(v)|^2 \langle v \rangle^{2l} dv$. The weighted frictional Sobolev norm $|f(v)|_{H^s_l}^2 = |\langle v \rangle^l f(v)|_{H^s}^2$ is given by

$$|f(v)|_{H^s_l}^2 = |f|_{L^2_l}^2 + \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv' \frac{[\langle v \rangle^l f(v) - \langle v' \rangle^l f(v')]^2}{|v - v'|^{3+2s}} \chi_{|v-v'| \leq 1},$$

where χ_{Ω} is the standard indicator function of the set Ω . Moreover, in $\mathbb{R}^3_x \times \mathbb{R}^3_v$, $\|\cdot\|_{H^s_l} \equiv \|\cdot\|_{H^s_l} \| \cdot \|_{L^2_x}$ is used.

As in [5], we introduce the time-velocity weight function corresponding to the Boltzmann operator:

$$w_\ell(\alpha, \beta) = \langle v \rangle^{-\gamma(\ell-|\alpha|-\beta)} e^{\frac{q(v)}{(1+t)^\vartheta}}, \quad \max \left\{ -3, -\frac{3}{2} - 2s \right\} < \gamma < -2s, \\ \frac{1}{2} \leq s < 1, \quad 0 < q \ll 1. \tag{1.9}$$

with the parameter ϑ being taken as

$$\begin{cases} 0 < \vartheta \leq \min \left\{ \frac{\varrho}{2} - \frac{1}{4}, \frac{1}{3} \right\}, & \text{when } \varrho \in \left(\frac{1}{2}, \frac{3}{2} \right) \text{ and } N_0 \geq 4, \\ 0 < \vartheta \leq \frac{\varrho}{2} - \frac{1}{2}, & \text{when } \varrho \in \left(1, \frac{3}{2} \right) \text{ and } N_0 = 3. \end{cases} \tag{1.10}$$

Moreover, for an integer $N \geq 0$ and $\ell \in \mathbb{R}$, we define the energy functional $\bar{\mathcal{E}}_{N,\ell}(t)$ and the corresponding dissipation rate functional $\bar{\mathcal{D}}_{N,\ell}(t)$ by

$$\bar{\mathcal{E}}_{N,\ell}(t) \sim \mathcal{E}_{N,\ell}(t) + \|\Lambda^{-\ell}(f, E, B)\|^2 \tag{1.11}$$

and

$$\bar{\mathcal{D}}_{N,\ell}(t) \sim \mathcal{D}_{N,\ell}(t) + \|\Lambda^{1-\ell}(a, b, c, E, B)\|^2 + \|\Lambda^{-\ell}(a_+ - a_-, E)\|^2 + \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \tag{1.12}$$

respectively. Here

$$\mathcal{E}_{N,\ell}(t) \sim \sum_{|\alpha|+|\beta|\leq N} \|w_\ell(\alpha, \beta)\partial_\beta^\alpha f\|^2 + \|(E, B)\|_{H^N}^2, \tag{1.13}$$

$$\begin{aligned} \mathcal{D}_{N,\ell}(t) \sim & \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha(a_\pm, b, c)\|^2 + \sum_{|\alpha|+|\beta|\leq N} \|w_\ell(\alpha, \beta)\partial_\beta^\alpha\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|a_+ - a_-\|^2 \\ & + \|E\|_{H^{N-1}}^2 + \|\nabla_x B\|_{H^{N-2}}^2 + (1+t)^{-1-\vartheta} \\ & \times \sum_{|\alpha|+|\beta|\leq N} \|\langle v \rangle^{1/2} w_\ell(\alpha, \beta)\partial_\beta^\alpha\{\mathbf{I} - \mathbf{P}\}f\|^2, \end{aligned} \tag{1.14}$$

In our analysis, we also need to define the energy functional without weight $\mathcal{E}_N(t)$, the higher order energy functional without weight $\mathcal{E}_{N_0}^k(t)$, and the higher order energy functional with weight $\mathcal{E}_{N_0,\ell}^k(t)$ as follows

$$\mathcal{E}_N(t) \sim \sum_{k=0}^N \|\nabla^k(f, E, B)\|^2, \tag{1.15}$$

$$\mathcal{E}_{N_0}^k(t) \sim \sum_{|\alpha|=k}^{N_0} \|\partial^\alpha(f, E, B)\|^2, \tag{1.16}$$

$$\mathcal{E}_{N_0,\ell}^k(t) \sim \sum_{\substack{|\alpha|+|\beta|\leq N, \\ |\alpha|\geq k}} \|w_\ell(\alpha, \beta)\partial_\beta^\alpha f\|^2 + \sum_{|\alpha|=k}^{N_0} \|\partial^\alpha(E, B)\|^2, \tag{1.17}$$

and the corresponding energy dissipation rate functionals are given by

$$\begin{aligned} \mathcal{D}_N(t) \sim & \|(E, a_+ - a_-)\|^2 + \sum_{1\leq|\alpha|\leq N-1} \|\partial^\alpha(\mathbf{P}f, E, B)\|^2 \\ & + \sum_{|\alpha|=N} \|\partial^\alpha \mathbf{P}f\|^2 + \sum_{|\alpha|\leq N} \|\partial^\alpha\{\mathbf{I} - \mathbf{P}\}f\|_D^2, \end{aligned} \tag{1.18}$$

$$\begin{aligned} \mathcal{D}_{N_0}^k(t) &\sim \|\nabla^k(E, a_+ - a_-)\|^2 + \sum_{k+1 \leq |\alpha| \leq N_0-1} \|\partial^\alpha(\mathbf{P}f, E, B)\|^2 \\ &+ \sum_{|\alpha|=N_0} \|\partial^\alpha \mathbf{P}f\|^2 + \sum_{k \leq |\alpha| \leq N_0} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_D^2, \end{aligned} \tag{1.19}$$

$$\begin{aligned} \mathcal{D}_{N_0, \ell}^k(t) &\sim \|\nabla^k(E, a_+ - a_-)\|^2 + \sum_{k+1 \leq |\alpha| \leq N_0-1} \|\partial^\alpha(\mathbf{P}f, E, B)\|^2 \\ &+ \sum_{\substack{|\alpha|+|\beta| \leq N_0, \\ |\alpha| \geq k}} \|w_\ell(\alpha, \beta) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \sum_{|\alpha|=N_0} \|\partial^\alpha \mathbf{P}f\|^2 + (1+t)^{-1-\vartheta} \\ &\times \sum_{\substack{|\alpha|+|\beta| \leq N_0, \\ |\alpha| \geq k}} \|\langle v \rangle^{1/2} w_\ell(\alpha, \beta) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}f\|^2. \end{aligned} \tag{1.20}$$

respectively.

With the above preparation in hand, our main result concerning the Cauchy problem (1.4), (1.5) is

Theorem 1.1 *Suppose that $F_0(x, v) = \mu + \sqrt{\mu} f_0(x, v) \geq 0$, $\frac{1}{2} < \varrho < \frac{3}{2}$, and $\max\{-3, -\frac{3}{2} - 2s\} < \gamma < -2s$ with $\frac{1}{2} \leq s < 1$. Let*

$$\begin{cases} N_0 \geq 4, N = 2N_0 - 1, & \text{when } \varrho \in (\frac{1}{2}, 1], \\ N_0 \geq 3, N = 2N_0, & \text{when } \varrho \in (1, \frac{3}{2}), \end{cases} \tag{1.21}$$

and take $l \geq N$ and $l_0 \geq \max\{l, l + \frac{1}{2} - \frac{1-s}{\gamma} - N_0\}$. If

$$\begin{aligned} Y_0 &= \sum_{|\alpha|+|\beta| \leq N_0} \|w_{l_0+l^*}(\alpha, \beta) \partial_\beta^\alpha f_0\| \\ &+ \sum_{|\alpha|+|\beta| \leq N} \|w_l(\alpha, \beta) \partial_\beta^\alpha f_0\| + \|(E_0, B_0)\|_{H^N} \cap \dot{H}^{-e} + \|f_0\|_{\dot{H}^{-e}} \end{aligned} \tag{1.22}$$

is sufficiently small where $l^* = l' + \frac{3(\gamma+2s)}{2\gamma}$ and l' will be specified in the proof for detail, the Cauchy problem (1.4), (1.5) admits a unique global solution $(f(t, x, v), E(t, x), B(t, x))$ satisfying $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geq 0$.

Remark 1.2 Several remarks concerning Theorem 1.1 are listed below:

- For brevity, the precise value of the parameter l' can be specified in the proofs of Theorem 1.1, cf. the proof of Lemma 4.1.

Our second result is concerned with the temporal decay estimates on the global solution $(f(t, x, v), E(t, x), B(t, x))$ obtained in Theorem 1.1. For results in this direction, we have from Theorem 1.1 that

Theorem 1.3 *Under the assumptions of Theorem 1.1, we have the following results:*

(1) Taking $k = 0, 1, 2, \dots, N_0 - 2$, it follows that

$$\mathcal{E}_{N_0}^k(t) \lesssim Y_0^2(1+t)^{-(e+k)}. \tag{1.23}$$

(2) Let $0 \leq i \leq k \leq N_0 - 3$ be an integer, if we take l^* in Theorem 1.1 as $l^* = l' + \frac{3(\gamma+2s)}{2\gamma} + \chi_{N_0 \geq 4} \left(1 - \frac{2s-1}{\gamma}\right) (N_0 - 4)$ and if we set $l_{0,0} = l_0, l_{0,k} + 1 + \frac{2s-1}{\gamma} = l_{0,k-1}$ and

$l_{0,k} + \frac{(\gamma+2s)(k+2)}{2} \geq N_0$ for $2 \leq k \leq N_0 - 3$, one has

$$\mathcal{E}_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}^k(t) \lesssim Y_0^2(1+t)^{-k-\varrho+i}, \quad i = 0, 1, \dots, k. \tag{1.24}$$

furthermore, when $\varrho \in [1, \frac{3}{2})$,

$$\mathcal{E}_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}^k(t) \lesssim Y_0^2(1+t)^{1-\varrho}. \tag{1.25}$$

(3) When $N_0 + 1 \leq |\alpha| \leq N - 1$, we have

$$\|\partial^\alpha f\|^2 \lesssim Y_0^2(1+t)^{-\frac{(N-|\alpha|)(N_0-2+\varrho)}{N-N_0}}, \tag{1.26}$$

Remark 1.4 In fact, compared with the results in [5], the main results obtained in this manuscript show that:

- The smallness of $\|w_\ell f_0\|_{Z^1}$ and $\|(E_0, B_0)\|_{L^1}$ can be replaced by the weaker assumption that $\|(E_0, B_0)\|_{\dot{H}^{-s}}^2 + \|f_0\|_{\dot{H}^{-s}}^2$ ($\frac{1}{2} < s < \frac{3}{2}$) is small.
- The minimal regularity index 14 is reduced into $N = 7$ for $s \in (\frac{1}{2}, 1]$ and $N = 6$ for $s \in (1, \frac{3}{2})$.
- The restriction on $\vartheta = \frac{1}{4}$ is relaxed to $0 < \vartheta \leq \frac{\varrho}{2} - \frac{1}{4}$, $\varrho \in (\frac{1}{2}, \frac{3}{2})$ for $N_0 \geq 4$ and $0 < \vartheta \leq \frac{\varrho}{2} - \frac{1}{2}$, $\varrho \in (1, \frac{3}{2})$ for $N_0 = 3$.
- The time decay rates of the higher-order spatial derivatives of solutions are obtained.

Let's review some former results on the construction of global smooth solutions to the Vlasov–Maxwell–Boltzmann system (1.1) near Maxwellians. Guo in [9] firstly constructed periodic classical solutions near Maxwellian for the (non-relativistic) two-species Vlasov–Maxwell–Boltzmann system for hard sphere, which was extended to the whole space \mathbb{R}_x^3 by Strain in [18]. The large-time behavior of classical solutions to the Vlasov–Maxwell–Boltzmann system for hard sphere in the whole space was studied by Duan–Strain [6]. Recently, Duan–Liu–Yang–Zhao in [5] deal with the Vlasov–Maxwell–Boltzmann system for non-cutoff soft potentials in the whole space \mathbb{R}_x^3 .

As pointed out in [5], to overcome the mathematical difficulties, which are produced by the velocity-growth of the nonlinear term with the velocity-growth rate $|v|$ and the regularity-loss of the electromagnetic field, the main arguments used in [5] are as follows:

- They introduce the exponential time-velocity weight $w_\ell(t, v)$ to generate the extra dissipation corresponding to the last term in the energy dissipation rate functional $\bar{D}_{N,l}(t)$;
- Motivated by the argument developed in [13] to deduce the decay property of solutions to nonlinear equations of regularity-loss type, a time-weighted energy estimate is designed to close the analysis, which implies that although the L^2 -norm of terms with the highest order derivative with respect to x of the solutions of the Vlasov–Maxwell–Boltzmann system may can only be bounded by some function of t which increases as time evolves, the L^2 -norm of terms with lower order derivatives with respect to x still enjoy some decay rates.

Based on the above arguments and combining the decay of solutions to the corresponding linearized system with the Duhamel principle, Duan–Liu–Yang–Zhao [5] can indeed close the analysis provided that the regularity index imposed on the initial perturbation is 14, i.e. $N \geq 14$ and certain norms of the initial perturbation, especially $\|w_\ell f_0\|_{Z^1}$ and $\|(E_0, B_0)\|_{L^1}$, are assumed to be sufficiently small, meanwhile the choice of $\vartheta = \frac{1}{4}$ is critical in their proof.

Inspired by the work [11], the main purpose of our present manuscript is trying to study such a problem by a different method, which does not rely on the decay analysis of the corresponding linearized system and the Duhamel principle.

Now we sketch the main ideas to deduce our main results:

- As in [5], we apply the exponential time-velocity weight $w_\ell(t, v)$, which can deduce the extra dissipation $(1 + t)^{-1-\vartheta} \|\langle v \rangle^{\frac{1}{2}} w_\ell(\alpha, \beta) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\} f\|^2$, to control the term $\|E\|_{L^\infty} \|\langle v \rangle^{\frac{1}{2}} w_\ell(\alpha, \beta) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\} f\|^2$. The key point of the above argument rests with the fact that the time decay rates of $\|E\|_{L^\infty}$ is greater than $1 + \vartheta$. Unlike the techniques to obtain the time decay of $\|E\|_{L^\infty}$ in [5], which heavily rely on linear analysis and Duhamel principle, to deduce such a result, we hope that the following estimates

$$\frac{d}{dt} \mathcal{E}_{N_0}^k(t) + \mathcal{D}_{N_0}^k(t) \leq 0, \quad 0 \leq k \leq N_0 - 2, \tag{1.27}$$

and

$$\frac{d}{dt} \mathcal{E}_{N_0, \ell}^k(t) + \mathcal{D}_{N_0, \ell}^k(t) \lesssim \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2, \quad k = 0, 1, \tag{1.28}$$

hold for $0 \leq t < T$.

- In the proof of (1.27), the terms like $(\nabla^k(v \cdot \mathbf{E}\mathbf{P}f), \nabla^k f)$ and $(\nabla^k(v \times \mathbf{B} \cdot \nabla_v \mathbf{P}f), \nabla^k f)$ ask us to use the interpolation techniques between negative Sobolev norms i.e. $\|\Lambda^{-\varrho}(f, E, B)\|$ and positive Sobolev norms i.e. $\|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|$ or $\|\nabla^{k+1}(\mathbf{P}f, E, B)\|$, and when we deal with the terms such as $(\nabla^k(v \times \mathbf{B} \nabla_v \{\mathbf{I} - \mathbf{P}\}f), \nabla^k \{\mathbf{I} - \mathbf{P}\}f)$, even we have to apply the interpolation techniques with respect to velocity derivatives. In a word, the terms including electric–magnetic field (E,B) directly cause us to use the above techniques, which is mainly different from what Guo in [11] used to get the estimates like (1.27) with respect to Boltzmann equation (1.28) can be obtained in a similar way.
- With the help of the interpolation techniques between negative or the higher order Sobolev norms and corresponding dissipation functions introduced by [11], we deduce from (1.27) and (1.28) the time decay of $\mathcal{E}_{N_0}^k(t)$ and $\mathcal{E}_{N_0, \ell}^k(t)$. We notice that $\|E\|_{L^\infty}$ can be dominated by $\mathcal{E}_{N_0}^k(t)$ for $k = 0, 1, 2$. Therefore, combing the time decay rates of $\|E\|_{L^\infty}$ and $\mathcal{E}_{N_0, \ell}^1(t)$ with other energy estimates, we can then close the a priori assumption given in (3.1) and then the global solvability result follows immediately from the continuation argument. It is worth pointing out that (1.27) and (1.28) play an essential role in the proof of Theorem 1.1.

The rest of this paper is organized as follows. In Sect. 2, we list some basic lemmas for the later proof. Section 3 is devoted to deducing the desired energy estimates for the energy functionals and the proofs of Theorems 1.1 and 1.3 will be given in the end of Sect. 3. For brevity, the detail proofs of some lemmas in Sect. 3 will be given in the Appendix.

2 Preliminary

In this section, we will cite some fundamental results for later use. The first lemma is concerned with the estimates of the linear operator L .

Lemma 2.1 (cf.[3,5]) (i) *It holds that*

$$(Lg, g) \gtrsim \|\{\mathbf{I} - \mathbf{P}\}g\|_{\mathcal{D}}^2. \tag{2.1}$$

(ii) Let $\ell \in \mathbb{R}$, $\lambda \geq 0$, $0 < s < 1$ and $-3 < \gamma < -2s$, it holds that

$$\sum_{|\beta| \leq N} (w_\ell^2 \partial_\beta Lg, \partial_\beta g) \gtrsim \|w_\ell \partial_\beta g\|_D^2 - C \|g\|_{L^2_{BC}}^2. \tag{2.2}$$

The second and third lemma concern the estimates on the nonlinear collision operator Γ .

Lemma 2.2 (cf. [3,5]) For all $0 < s < 1$, $\max\{-3, -\frac{3}{2} - 2s\} < \gamma < -2s$, $\lambda_0 > 0$, $\ell \geq 0$ and for some $\bar{\lambda} > 0$, then one has

$$\begin{aligned} & |(\partial_\beta^\alpha \Gamma(f, g), w_\ell^2(\alpha, \beta) \partial_\beta^\alpha h)| \\ & \lesssim \sum \left\{ \left| w_{\ell, \lambda_0} \partial_{\beta_1}^{\alpha_1} f \right|_{L^2_{s+\gamma/2}} \left| \partial_{\beta_2}^{\alpha_2} g \right|_D + \left| \partial_{\beta_2}^{\alpha_2} g \right|_{L^2_{s+\gamma/2}} \left| w_\ell(\alpha, \beta) \partial_{\beta_1}^{\alpha_1} f \right|_D \right\} \left| w_\ell(\alpha, \beta) \partial_\beta^\alpha h \right|_D \\ & + \min \left\{ \left| w_\ell(\alpha, \beta) \partial_{\beta_1}^{\alpha_1} f \right|_{L^2} \left| \partial_{\beta_2}^{\alpha_2} g \right|_{L^2_{s+\gamma/2}}, \left| \partial_{\beta_2}^{\alpha_2} g \right|_{L^2} \left| w_\ell(\alpha, \beta) \partial_{\beta_1}^{\alpha_1} f \right|_{L^2_{s+\gamma/2}} \right\} \left| w_\ell(\alpha, \beta) \partial_\beta^\alpha h \right|_D \\ & + \sum \left| e^{\frac{\lambda_0(v)}{(1+t)^\vartheta}} \partial_{\beta_2}^{\alpha_2} g \right|_{L^2} \left| w_\ell(\alpha, \beta) \partial_{\beta_1}^{\alpha_1} f \right|_{L^2_{\gamma/2}} \left| w_\ell(\alpha, \beta) \partial_\beta^\alpha h \right|_D \\ & + \sum \left| \mu^{\frac{\bar{\lambda}}{128}} \partial_{\beta_2}^{\alpha_2} g \right|_{L^2} \left| w_\ell(\alpha, \beta) \partial_{\beta_1}^{\alpha_1} f \right|_{L^2_{\gamma/2+1/2}} \left| w_\ell(\alpha, \beta) \partial_\beta^\alpha h \right|_{L^2_{\gamma/2+1/2}} \\ & + \sum (1+t)^{-\vartheta} \left| \mu^{\frac{\bar{\lambda}}{128}} \partial_{\beta_2}^{\alpha_2} g \right|_{L^2} \left| w_\ell(\alpha, \beta) \partial_{\beta_1}^{\alpha_1} f \right|_{L^2_{\gamma/2+1}} \left| w_\ell(\alpha, \beta) \partial_\beta^\alpha h \right|_{L^2_{\gamma/2+s}} \\ & + \sum (1+t)^{-2\vartheta} \left| \mu^{\frac{\bar{\lambda}}{128}} \partial_{\beta_2}^{\alpha_2} g \right|_{L^2} \left| w_\ell(\alpha, \beta) \partial_{\beta_1}^{\alpha_1} f \right|_{L^2_{\gamma/2+1}} \left| w_\ell(\alpha, \beta) \partial_\beta^\alpha h \right|_{L^2_{\gamma/2+1}}, \end{aligned} \tag{2.3}$$

where the summation \sum is taken over $\alpha_1 + \alpha_2 \leq \alpha$ and $\beta_1 + \beta_2 \leq \beta$. Furthermore, from [3],

$$|(\Gamma(f, g), h)| \lesssim \left\{ |f|_{L^2_{\gamma/2+s}} |g|_D + |g|_{L^2_{\gamma/2+s}} |f|_D + \min \left\{ |f|_{L^2} |g|_{L^2_{\gamma/2+s}}, |g|_{L^2} |f|_{L^2_{\gamma/2+s}} \right\} \right\} |h|_D. \tag{2.4}$$

Lemma 2.3 (cf. [19]) Let $\ell > 0$, $\gamma > -3$ with $\gamma + 2s > -\frac{3}{2}$ and $s \in [\frac{1}{2}, 1)$, it holds that

$$|w_\ell \Gamma(f, f)|_{L^2} \lesssim |w_\ell f|_{L^2_{\gamma/2+s}}^2 |w_\ell f|_{H^2_{\gamma/2+s}}^2. \tag{2.5}$$

The following lemma concerns the trillion estimates on the nonlinear term $\Gamma(f, f)$.

Lemma 2.4 For all $\frac{1}{2} \leq s < 1$, $\max\{-3, -\frac{3}{2} - 2s\} < \gamma < -2s$, and assume $\ell \geq N$, $0 < \vartheta \leq \frac{1}{3}$, one has the following estimates:

$$\sum_{|\alpha| \leq N} |(\partial^\alpha \Gamma(f, f), \partial^\alpha \{\mathbf{I} - \mathbf{P}\}f)| \lesssim \mathcal{E}_{N_0, N_0}^{1/2}(t) \mathcal{D}_N(t), \tag{2.6}$$

$$\begin{aligned} |(w_\ell^2(0, 0) \Gamma(f, f), \{\mathbf{I} - \mathbf{P}\}f)| & \lesssim (1+t)^{\frac{1-3\vartheta}{2}} \left\{ \mathcal{E}_{N_0, \ell-N_0+2}^1(t) \right\}^{1/2} \mathcal{D}_{N, \ell}(t) \\ & + \mathcal{E}_{N_0, N_0}^{1/2}(t) \mathcal{D}_{N, \ell}(t), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \sum_{0 < |\alpha| \leq N_0} |(w_\ell^2(\alpha, 0) \partial^\alpha \Gamma(f, f), \partial^\alpha f)| & \lesssim (1+t)^{\frac{1-3\vartheta}{2}} \left\{ \mathcal{E}_{N_0, \ell-N_0+2}^1(t) \right\}^{1/2} \mathcal{D}_{N, \ell}(t) \\ & + \mathcal{E}_{N_0, N_0}^{1/2}(t) \mathcal{D}_{N, \ell}(t), \end{aligned} \tag{2.8}$$

and

$$\sum_{\substack{|\alpha|+|\beta|\leq N_0, \\ |\beta|\geq 1}} \left| \left(w_\ell^2(\alpha, \beta) \partial_\beta^\alpha \Gamma(f, f), \partial_\beta^\alpha (\mathbf{I} - \mathbf{P})f \right) \right| \lesssim (1+t)^{\frac{1-3\vartheta}{2}} \{ \mathcal{E}_{N_0, \ell-N_0+2}^1(t) \}^{1/2} \mathcal{D}_{N, \ell}(t) + \mathcal{E}_{N_0, N_0}^{1/2}(t) \mathcal{D}_{N, \ell}(t). \tag{2.9}$$

Proof For brevity, we only prove (4.3),

$$\begin{aligned} & |(\partial^\alpha \Gamma(f, f), w_\ell(\alpha, 0)^2 \partial^\alpha f)| \\ & \lesssim \int_{\mathbb{R}^3} \sum \left\{ |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2_{s+\gamma/2}} |\partial^{\alpha_2} f|_D + |\partial^{\alpha_2} f|_{L^2_{s+\gamma/2}} |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_D \right\} \\ & \quad \times |w_\ell(\alpha, 0) \partial^\alpha f|_D dx \\ & + \int_{\mathbb{R}^3} \min \left\{ |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2} |\partial^{\alpha_2} f|_{L^2_{s+\gamma/2}}, |\partial^{\alpha_2} f|_{L^2} |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2_{s+\gamma/2}} \right\} \\ & \quad \times |w_\ell(\alpha, 0) \partial^\alpha f|_D dx \\ & + \int_{\mathbb{R}^3} \sum \left| e^{\frac{\lambda_0(v)}{(1+t)^\vartheta}} \partial^{\alpha_2} f \right|_{L^2} |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2_{\gamma/2}} |w_\ell(\alpha, 0) \partial^\alpha f|_D dx \\ & + \int_{\mathbb{R}^3} \sum \left| \mu^{\frac{\tilde{\lambda}}{128}} \partial^{\alpha_2} f \right|_{L^2} |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2_{\gamma/2+1/2}} |w_\ell(\alpha, 0) \partial^\alpha f|_{L^2_{\gamma/2+1/2}} dx \\ & + \int_{\mathbb{R}^3} \sum (1+t)^{-\vartheta} \left| \mu^{\frac{\tilde{\lambda}}{128}} \partial^{\alpha_2} f \right|_{L^2} |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2_{\gamma/2+1}} |w_\ell(\alpha, 0) \partial^\alpha f|_{L^2_{\gamma/2+s}} dx \\ & + \int_{\mathbb{R}^3} \sum (1+t)^{-2\vartheta} \left| \mu^{\frac{\tilde{\lambda}}{128}} \partial^{\alpha_2} f \right|_{L^2} |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2_{\gamma/2+1}} |w_\ell(\alpha, 0) \partial^\alpha f|_{L^2_{\gamma/2+1}} dx \\ & \equiv \sum_{i=1}^6 R_i, \end{aligned} \tag{2.10}$$

where we should notice the summation \sum is taken over $\alpha_1 + \alpha_2 \leq \alpha$. We only prove the last term on right hand of (2.10) since the estimates for the other terms on the right hand of (2.10) are more simple than the last term. Since $\gamma \leq -2s$ and $1/2 \leq s < 1$, we see that $\gamma + 2 \leq 1$. Therefore,

$$(1+t)^{-(1+\vartheta)} \|(v)^{\gamma/2+1} w_\ell(\alpha, \beta) \partial_\beta^\alpha (\mathbf{I} - \mathbf{P})f\|^2 \lesssim \mathcal{D}_{N, \ell}(t). \tag{2.11}$$

When $\alpha_2 = \alpha$ with $|\alpha| \geq N_0 + 1$, we use $L^2 - L^\infty - L^2$,

$$\begin{aligned} & \int_{\mathbb{R}^3} (1+t)^{-2\vartheta} \left| \mu^{\frac{\tilde{\lambda}}{128}} \partial^{\alpha_2} f \right|_{L^2} |w_\ell(\alpha, 0) f|_{L^2_{\gamma/2+1}} |w_\ell(\alpha, 0) \partial^\alpha f|_{L^2_{\gamma/2+1}} dx \\ & \lesssim (1+t)^{-2\vartheta} \|\mu^\delta \partial^{\alpha_2} f\| \|w_\ell(\alpha, 0) \nabla f\|_{H^1_x L^2_{\gamma/2+1}} \|w_\ell(\alpha, 0) \partial^\alpha f\|_{L^2_{\gamma/2+1}} \\ & \lesssim (1+t)^{\frac{1-3\vartheta}{2}} \{ \mathcal{E}_{N_0, \ell-N_0+2}(t) \}^{1/2} \mathcal{D}_{N, \ell}(t). \end{aligned} \tag{2.12}$$

When $N_0 + 1 \leq |\alpha_2| \leq |\alpha| - 1$ with $|\alpha| \geq N_0 + 2$, we use $L^3 - L^6 - L^2$,

$$\begin{aligned} & \int_{\mathbb{R}^3} (1+t)^{-2\vartheta} \left| \mu^{\frac{\tilde{\lambda}}{128}} \partial^{\alpha_2} f \right|_{L^2} |w_\ell(\alpha, 0) \partial^{\alpha_1} f|_{L^2_{\gamma/2+1}} |w_\ell(\alpha, 0) \partial^\alpha f|_{L^2_{\gamma/2+1}} dx \\ & \lesssim (1+t)^{-2\vartheta} \|\mu^\delta \partial^{\alpha_2} f\| \|w_\ell(\alpha, 0) \nabla \partial^{\alpha_1} f\|_{L^2_{\gamma/2+1}} \|w_\ell(\alpha, 0) \partial^\alpha f\|_{L^2_{\gamma/2+1}} \\ & \lesssim (1+t)^{\frac{1-3\vartheta}{2}} \{ \mathcal{E}_{N_0, \ell-N_0+2}(t) \}^{1/2} \mathcal{D}_{N, \ell}(t). \end{aligned} \tag{2.13}$$

Collecting the above estimates yields $R_6 \lesssim (1 + t)^{\frac{1-3\vartheta}{2}} \{\mathcal{E}_{N_0, \ell-N_0+2}(t)\}^{1/2} \mathcal{D}_{N, \ell}(t) + \{\mathcal{E}_{N_0, N_0}(t)\}^{1/2} \mathcal{D}_{N, \ell}(t)$.

For other terms $R_1 \sim R_5$, noticing that $0 < \vartheta \leq \frac{1}{3}$, we can obtain easily by the similar way as R_6

$$\sum_{i=1}^5 R_i \lesssim (1 + t)^{\frac{1-3\vartheta}{2}} \{\mathcal{E}_{N_0, \ell-N_0+2}^1(t)\}^{1/2} \mathcal{D}_{N, \ell}(t) + \mathcal{E}_{N_0, N_0}^{1/2}(t) \mathcal{D}_{N, \ell}(t).$$

Collecting the above estimates gives (4.3) for the case $N_0 + 1 \leq |\alpha| \leq N$. While for the case $|\alpha| \leq N_0$, (4.3) can be obtained in a similar way. Thus we have completed the proof of this lemma. \square

In what follows, we will collect the analytic tools which will be used in this paper. The Sobolev interpolation among the spatial regularity is

Lemma 2.5 (cf. [21]) *Let $2 \leq p < \infty$ and $k, \ell, m \in \mathbb{R}$; then we have*

$$\|\nabla^k f\|_{L^p} \lesssim \|\nabla^\ell f\|_{L^2}^\theta \|\nabla^m f\|_{L^2}^{1-\theta}. \tag{2.14}$$

where $0 \leq \theta \leq 1$ and ℓ satisfy

$$\frac{1}{p} - \frac{k}{3} = \left(\frac{1}{2} - \frac{\ell}{3}\right)\theta + \left(\frac{1}{2} - \frac{m}{3}\right)(1 - \theta). \tag{2.15}$$

Also we have that

$$\|\nabla^k f\|_{L^\infty} \lesssim \|\nabla^\ell f\|_{L^2}^\theta \|\nabla^m f\|_{L^2}^{1-\theta}. \tag{2.16}$$

where $0 \leq \theta \leq 1$ and ℓ satisfy

$$-\frac{k}{3} = \left(\frac{1}{2} - \frac{\ell}{3}\right)\theta + \left(\frac{1}{2} - \frac{m}{3}\right)(1 - \theta), \tag{2.17}$$

here we require $\ell \leq k + 1$ and $m \geq k + 2$.

In this paper, we should estimate $\|\Lambda^{-\ell} f\|$, we need the following L^p inequality for $\Lambda^{-\ell}$.

Lemma 2.6 (cf. [11, 17]) *Let $0 < \varrho < 3$, $1 < p < q < \infty$, $\frac{1}{q} + \frac{\varrho}{3} = \frac{1}{p}$, then*

$$\|\Lambda^{-\ell} f\|_{L^q} \lesssim \|f\|_{L^p}. \tag{2.18}$$

In many places, we will use Minkowski’s integral inequality to interchange the orders of integration over x and v .

Lemma 2.7 (cf. [21]) *For $1 \leq p \leq q \leq \infty$, we have*

$$\|f\|_{L_x^q L_y^p} \leq \|f\|_{L_y^p L_x^q}. \tag{2.19}$$

3 The Proofs of Our Main Results

This section is devoted to proving our main results based on the continuation argument. For this purpose, suppose that the Cauchy problem (1.4) and (1.5) admits a unique local solution $f(t, x, v)$ defined on the time interval $0 \leq t \leq T$ for some $0 < T < \infty$ and the solution $f(t, x, v)$ satisfies the a priori assumption

$$X(t) = \sup_{0 \leq \tau \leq t} \left\{ \bar{\mathcal{E}}_{N_0, l_0+l^*}(\tau) + \mathcal{E}_N(\tau) + \mathcal{E}_{N-1, l}(\tau) + (1 + \tau)^{-\frac{1+\epsilon_0}{2}} \mathcal{E}_{N, l}(\tau) \right\} \leq M, \tag{3.1}$$

where the parameters $N_0, N, l, l_0,$ and l^* are given in Theorem 1.1 and M is a sufficiently small positive constant. Then use the continuation argument to extend such a solution step by step to a global one, one only need to deduce certain uniform-in-time energy type estimates on $f(t, x, v)$ such that the a priori assumption (3.1) can be closed.

For this purpose, we first deduce the temporal decay of the energy functional $\mathcal{E}_{N_0}^k(t)$ in the following lemma.

Lemma 3.1 *Let N_0 and N satisfy (1.21), $n \geq \frac{2}{3}N_0 - \frac{5}{3}$, and take $k = 0, 1, 2, \dots, N_0 - 2$, then one has*

$$\frac{d}{dt} \mathcal{E}_{N_0}^k(t) + \mathcal{D}_{N_0}^k(t) \leq 0, \quad 0 \leq t \leq T \tag{3.2}$$

provided that

$$(H_1) \quad \max \left\{ \sup_{0 \leq \tau \leq T} \mathcal{E}_{N_0+n}(\tau), \sup_{0 \leq \tau \leq T} \mathcal{E}_{N-1, N-1}(\tau), \sup_{0 \leq \tau \leq T} \bar{\mathcal{E}}_{N_0, N_0+l'}(\tau) \right\}$$

is sufficiently small.

Furthermore, as a consequence of (3.2), we can get that

$$\mathcal{E}_{N_0}^k(t) \lesssim \max \left\{ \sup_{0 \leq \tau \leq t} \bar{\mathcal{E}}_{N_0, N_0+\frac{\gamma+2s}{2\gamma}}(\tau), \sup_{0 \leq \tau \leq t} \mathcal{E}_{N_0+k+\varrho}(\tau) \right\} (1+t)^{-(k+\varrho)} \tag{3.3}$$

holds for $0 \leq t \leq T$.

Proof Under the smallness assumption (H_1) , one can deduce that

$$\frac{d}{dt} \mathcal{E}_{N_0}^k(t) + \mathcal{D}_{N_0}^k(t) \leq 0,$$

which is a immediately consequence of Lemma 4.1 and Lemma 4.2 whose proofs are postponed to the next section for simplicity, where we used the fact that \bar{m} of Lemma 4.1 is less or equal to $N - 1$.

To get (3.3), for the macroscopic component $\mathbf{P}f(t, x, v)$ and the electromagnetic field $[E(t, x), B(t, x)]$ one has by Lemma 2.5 that

$$\left\| \nabla^k(\mathbf{P}f, B) \right\| \leq \left\| \nabla^{k+1}(\mathbf{P}f, B) \right\|^{\frac{k+\varrho}{k+\varrho+1}} \left\| \Lambda^{-\varrho}(\mathbf{P}f, B) \right\|^{\frac{1}{k+\varrho+1}}$$

and

$$\left\| \nabla^{N_0}(E, B) \right\| \lesssim \left\| \nabla^{N_0-1}(E, B) \right\|^{\frac{k+\varrho}{k+\varrho+1}} \left\| \nabla^{N_0+k+\varrho}(E, B) \right\|^{\frac{1}{k+\varrho+1}},$$

while for the microscopic component $\{\mathbf{I} - \mathbf{P}\}f(t, x, v)$, employing the Hölder inequality gives

$$\begin{aligned} \sum_{k \leq |\alpha| \leq N_0} \left\| \partial^\alpha \{\mathbf{I} - \mathbf{P}\}f \right\| &\leq \left\| \partial^\alpha \{\mathbf{I} - \mathbf{P}\}f \langle v \rangle^{\frac{\gamma+2s}{2}} \right\|^{\frac{k+\varrho}{k+\varrho+1}} \left\| \partial^\alpha \{\mathbf{I} - \mathbf{P}\}f \langle v \rangle^{-\frac{(\gamma+2s)(k+\varrho)}{2}} \right\|^{\frac{1}{k+\varrho+1}} \\ &\leq \left\| \partial^\alpha \{\mathbf{I} - \mathbf{P}\}f \right\|_v^{\frac{k+\varrho}{k+\varrho+1}} \left\| w_{|\alpha|+\frac{\gamma+2s}{2\gamma}} \partial^\alpha \{\mathbf{I} - \mathbf{P}\}f \right\|^{\frac{1}{k+\varrho+1}}. \end{aligned}$$

Thus, we have

$$\mathcal{E}_{N_0}^k(t) \leq \left\{ \mathcal{D}_{N_0}^k(t) \right\}^{\frac{k+\varrho}{k+\varrho+1}} \left\{ \max \left\{ \sup_{0 \leq \tau \leq t} \bar{\mathcal{E}}_{N_0, N_0+\frac{\gamma+2s}{2\gamma}}(\tau), \sup_{0 \leq \tau \leq t} \mathcal{E}_{N_0+k+\varrho}(\tau) \right\} \right\}^{\frac{1}{k+\varrho+1}},$$

which combining with (3.2) yields that

$$\frac{d}{dt} \mathcal{E}_{N_0}^k(t) + \left\{ \max \left\{ \sup_{0 \leq \tau \leq t} \bar{\mathcal{E}}_{N_0, N_0 + \frac{\gamma+2s}{2\gamma}}(\tau), \sup_{0 \leq \tau \leq t} \mathcal{E}_{N_0+k+\varrho}(\tau) \right\} \right\}^{-\frac{1}{k+\varrho}} \left\{ \mathcal{E}_{N_0}^k(t) \right\}^{1+\frac{1}{k+\varrho}} \leq 0.$$

Solving the above inequality directly gives

$$\mathcal{E}_{N_0}^k(t) \lesssim \max \left\{ \sup_{0 \leq \tau \leq t} \bar{\mathcal{E}}_{N_0, N_0 + \frac{\gamma+2s}{2\gamma}}(\tau), \sup_{0 \leq \tau \leq t} \mathcal{E}_{N_0+k+\varrho}(\tau) \right\} (1+t)^{-(k+\varrho)}.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2 *Let $\ell \geq N_0$, $n \geq \frac{2}{3}N_0 - \frac{5}{3}$ and suppose that*

$$(H_2) \quad \max \left\{ \sup_{0 \leq \tau \leq t} \mathcal{E}_{N_0+n}(\tau), \sup_{0 \leq \tau \leq t} \mathcal{E}_{N_0, \ell}(\tau) \right\} \text{ is sufficiently small}$$

with $\tilde{\ell}$ being given in Lemma 3.1,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_{N_0, \ell}^k(t) + \mathcal{D}_{N_0, \ell}^k(t) &\lesssim \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2 + \chi_{k \geq 2} \sum_{\substack{1 \leq |\alpha'| \leq k-1, \\ |\alpha|+|\beta|=N_0}} \left\| \partial^{\alpha'}(E, B) \right\|_{L^\infty}^2 \\ &\times \left\| \langle v \rangle^{1-s-\frac{\gamma}{2}} w_\ell(\alpha - \alpha', \beta + e_i) \partial_{\beta+e_i}^{\alpha-\alpha'} \{\mathbf{I} - \mathbf{P}\} f \right\|^2 \end{aligned} \tag{3.4}$$

for any $0 \leq t \leq T$, where $k = 0, 1, \dots, N_0 - 3$. Furthermore, based on (3.4), if we set $l_{0,0} = l_{0,1} = \ell$, $l_{0,k} + 1 + \frac{2s-1}{\gamma} = l_{0,k-1}$ and $l_{0,k} + \frac{(\gamma+2s)(k+2)}{2} \geq N_0$ for $2 \leq k \leq N_0 - 3$, we can deduce

$$\mathcal{E}_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}^k(t) \lesssim X(t)(1+t)^{-k-\varrho+i}, \quad i = 0, 1, \dots, k, \tag{3.5}$$

while for $\varrho \in [1, 3/2)$,

$$\mathcal{E}_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}^k(t) \lesssim X(t)(1+t)^{1-\varrho}, \tag{3.6}$$

Proof Since the proof of (3.4) is nearly same to (3.2), we only point out the main difference between the proof of (3.4) and (3.2). For instance, when $|\alpha| = N_0$, one has

$$\begin{aligned} &(v \times \partial^{e_i} B \partial^{\alpha-e_i} \nabla_v \{\mathbf{I} - \mathbf{P}\} f, w_\ell(\alpha, 0)^2 \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f) \\ &\lesssim \|\nabla B\|_{L^\infty}^2 \|w_\ell(\alpha, 0) \partial^{\alpha-e_i} \nabla_v \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{1-\frac{\gamma}{2}-s}\|^2 + \varepsilon \|w_\ell(\alpha, \beta) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_D^2 \end{aligned} \tag{3.7}$$

which will be included on the right hand side of (3.4) for $k \geq 2$.

For $k = 0, 1$, multiplying (3.4) with $\ell = l_0 + \frac{\gamma+2s}{2\gamma}i$ by $(1+t)^{k+\varrho-i+\epsilon}$ gives

$$\begin{aligned} &\frac{d}{dt} \left\{ (1+t)^{k+\varrho-i+\epsilon} \mathcal{E}_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}^k(t) \right\} + (1+t)^{k+\varrho-i+\epsilon} \mathcal{D}_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}^k(t) \\ &\lesssim \sum_{|\alpha|=N_0} (1+t)^{k+\varrho-i+\epsilon} \|\partial^\alpha E\|^2 + (1+t)^{k+\varrho-i-1+\epsilon} \mathcal{E}_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}^k(t), \end{aligned} \tag{3.8}$$

where ϵ is taken as a sufficiently small positive constant. When $\varrho \in (1/2, 1)$, we take $\ell = l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}$ in (3.4), it holds that

$$\frac{d}{dt} \mathcal{E}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t) + \mathcal{D}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t) \lesssim \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2. \tag{3.9}$$

Using the relation between the energy functional $\mathcal{E}^k_{N_0, \ell}(t)$ and its dissipation rate $\mathcal{D}^k_{N_0, \ell}(t)$, the proper linear combination of (3.8) and (3.9) yields

$$\begin{aligned} & \frac{d}{dt} \sum_{i=0}^k \left\{ C_i (1+t)^{k+\varrho-i+\epsilon} \mathcal{E}^k_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}(t) + C_{k+1} \mathcal{E}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t) \right\} \\ & + \left(\sum_{i=0}^k (1+t)^{k+\varrho-i+\epsilon} \mathcal{D}^k_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}(t) + \mathcal{D}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t) \right) \\ & \lesssim \sum_{|\alpha|=N_0} (1+t)^{k+\varrho+\epsilon} \|\partial^\alpha E\|^2 + (1+t)^{k+\varrho-1+\epsilon} \|\nabla^k(\mathbf{P}f, E, B)\|^2. \end{aligned} \tag{3.10}$$

Lemma 3.1 tells us that

$$\sum_{|\alpha|=N_0} (1+t)^{k+\varrho+\epsilon} \|\partial^\alpha E\|^2 + (1+t)^{k+\varrho-1+\epsilon} \|\nabla^k(\mathbf{P}f, E, B)\|^2 \lesssim X(t)(1+t)^{-1+\epsilon}. \tag{3.11}$$

Plugging (3.11) into (3.10) and taking the time integration, it follows that

$$\begin{aligned} & \sum_{i=0}^k (1+t)^{k+\varrho-i+\epsilon} \mathcal{E}^k_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}(t) + \mathcal{E}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t) \\ & + \int_0^t \left(\sum_{i=0}^k (1+\tau)^{k+\varrho-i+\epsilon} \mathcal{D}^k_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}(\tau) + \mathcal{D}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(\tau) \right) d\tau \lesssim X(t)(1+t)^\epsilon. \end{aligned} \tag{3.12}$$

Using this, it follows that

$$\mathcal{E}^k_{N_0, l_{0,k} + \frac{\gamma+2s}{2\gamma}i}(t) \lesssim X(t)(1+t)^{-k-\varrho+i}, \quad i = 0, 1, \dots, k. \tag{3.13}$$

When $\varrho \in [1, 3/2)$, multiplying (3.4) with $\ell = l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}$ by $(1+t)^{\varrho-1+\epsilon}$ gives

$$\begin{aligned} & \frac{d}{dt} \left\{ (1+t)^{\varrho-1+\epsilon} \mathcal{E}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t) \right\} + (1+t)^{\varrho-1+\epsilon} \mathcal{D}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t) \\ & \lesssim \sum_{|\alpha|=N_0} (1+t)^{\varrho-1+\epsilon} \|\partial^\alpha E\|^2 + (1+t)^{\varrho-2+\epsilon} \mathcal{E}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+1)}{2\gamma}}(t). \end{aligned} \tag{3.14}$$

we take $\ell = l_{0,k} + \frac{(\gamma+2s)(k+2)}{2\gamma}$ in (3.4), it holds that

$$\frac{d}{dt} \mathcal{E}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+2)}{2\gamma}}(t) + \mathcal{D}^k_{N_0, l_{0,k} + \frac{(\gamma+2s)(k+2)}{2\gamma}}(t) \lesssim \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2. \tag{3.15}$$

As in the case $\varrho \in (\frac{1}{2}, 1)$, the proper linear combination of (3.8), (3.14) and (3.14) yields

$$\begin{aligned} & \frac{d}{dt} \sum_{i=0}^{k+1} \left\{ C_i (1+t)^{k+\varrho-i+\epsilon} \mathcal{E}_{N_0, l_0, k+\frac{\gamma+2s}{2\gamma}i}^k(t) + C_{k+2} \mathcal{E}_{N_0, l_0, k+\frac{(\gamma+2s)(k+2)}{2\gamma}}^k(t) \right\} \\ & + \left(\sum_{i=0}^{k+1} (1+t)^{k+\varrho-i+\epsilon} \mathcal{D}_{N_0, l_0, k+\frac{\gamma+2s}{2\gamma}i}^k(t) + \mathcal{D}_{N_0, l_0, k+\frac{(\gamma+2s)(k+2)}{2\gamma}}^k(t) \right) \tag{3.16} \\ & \lesssim \sum_{|\alpha|=N_0} (1+t)^{k+\varrho+\epsilon} \|\partial^\alpha E\|^2 + (1+t)^{k+\varrho-1+\epsilon} \|\nabla^k(\mathbf{P}f, E, B)\|^2. \end{aligned}$$

By the same way as (3.5), it follows that

$$\begin{aligned} & \sum_{i=0}^{k+1} (1+t)^{k+\varrho-i+\epsilon} \mathcal{E}_{N_0, l_0, k+\frac{\gamma+2s}{2\gamma}i}^k(t) + \mathcal{E}_{N_0, l_0, k+\frac{(\gamma+2s)(k+2)}{2\gamma}}^k(t) \\ & + \int_0^t \left(\sum_{i=0}^{k+1} (1+\tau)^{k+\varrho-i+\epsilon} \mathcal{D}_{N_0, l_0, k+\frac{\gamma+2s}{2\gamma}i}^k(\tau) + \mathcal{D}_{N_0, l_0, k+\frac{(\gamma+2s)(k+2)}{2\gamma}}^k(\tau) \right) d\tau \lesssim X(t)(1+t)^\epsilon. \end{aligned} \tag{3.17}$$

Combining (3.12) and (3.17) yields (3.5) and (3.6).

When $k \geq 2$, we let $l_{0,0} = l_{0,1} = \ell$, by using the principle of mathematical induction, we can get that

$$\mathcal{E}_{N_0, l_0, k+\frac{\gamma+2s}{2\gamma}i}^k(t) \lesssim X(t)(1+t)^{-k-\varrho+i}, \quad \varrho \in (1/2, 3/2) \quad i = 0, 1, \dots, k. \tag{3.18}$$

especially, when $\varrho \in [1, 3/2)$,

$$\mathcal{E}_{N_0, l_0, k+\frac{(\gamma+2s)(k+1)}{2\gamma}}^k(t) \lesssim X(t)(1+t)^{1-\varrho}, \tag{3.19}$$

where $l_{0,k} + 1 + \frac{2s-1}{\gamma} = l_{0,k-1}$. This completes the proof of Lemma 3.2. □

Bases on the time decay estimates on $\mathcal{E}_{N_0}^k(t)$ and $\mathcal{E}_{N_0, \ell}^k(t)$, and suppose $\max \{ \sup_{0 \leq t \leq T} \mathcal{E}_{N_0, N_0}(t), \sup_{0 \leq t \leq T} \mathcal{E}_N(t) \}$ is sufficiently small, we will have the following three lemmas for $\mathcal{E}_N(t)$, $\mathcal{E}_{N,l}(t)$, $\mathcal{E}_{N-1,l}(t)$ and $\bar{\mathcal{E}}_{N_0, l_0+l^*}(t)$ respectively:

Lemma 3.3 *Let N_0, N and ϑ satisfy (1.21) and (1.10) respectively, under the assumptions of Lemma 3.1 and $l \geq N, l_0 \geq N - N_0 + \frac{1}{2} - \frac{1-s}{\gamma}$, we can deduce that*

$$\frac{d}{dt} \mathcal{E}_N(t) + \mathcal{D}_N(t) \lesssim \frac{\delta}{(1+t)^{1+\vartheta}} \mathcal{D}_{N,l}(t) + \mathcal{E}_N(t) \mathcal{E}_{N_0, l_0}^1(t) \tag{3.20}$$

holds for all $0 \leq t \leq T$.

Proof First of all, it is straightforward to establish the energy identities

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \sum_{N_0+1 \leq |\alpha| \leq N} \left(\|\partial^\alpha f\|^2 + \|\partial^\alpha(E, B)\|^2 \right) + \sum_{N_0+1 \leq |\alpha| \leq N} (L\partial^\alpha f, \partial^\alpha f) \\
 &= \underbrace{\sum_{N_0+1 \leq |\alpha| \leq N} \left(\partial^\alpha \left(\frac{q_0}{2} E \cdot v f \right), \partial^\alpha f \right)}_{J_1} - \underbrace{\sum_{N_0+1 \leq |\alpha| \leq N} \left(\partial^\alpha (q_0(E + v \times B) \cdot \nabla_v f), \partial^\alpha f \right)}_{J_2} \\
 &+ \underbrace{\sum_{N_0+1 \leq |\alpha| \leq N} \left(\partial^\alpha \Gamma(f, f), \partial^\alpha f \right)}_{J_3}. \tag{3.21}
 \end{aligned}$$

For the ∂^α derivative term related to (E, B) with $N_0 + 1 \leq |\alpha| \leq N$, i.e., the estimates on J_1 and J_2 , one has

$$\begin{aligned}
 J_1 &\lesssim \|E\|_{L^\infty} \|\langle v \rangle^{1/2} \partial^\alpha f\| \|\langle v \rangle^{1/2} \partial^\alpha f\| \\
 &+ \sum_{1 \leq |\alpha_1| \leq N_0-1} \|\partial^{\alpha_1} E\|_{L^6} \|\langle v \rangle^{1/2} \partial^{\alpha-\alpha_1} f\|_{L^3} \|\langle v \rangle^{1/2} \partial^\alpha f\| \\
 &+ \sum_{|\alpha_1|=N_0} \|\partial^{\alpha_1} E\| \|\langle v \rangle^{1/2} \partial^{\alpha-\alpha_1} f\|_{L^\infty} \|\langle v \rangle^{1/2} \partial^\alpha f\| \\
 &+ \sum_{|\alpha_1| \geq N_0+1, \alpha_1 \neq \alpha} \|\partial^{\alpha_1} E\|_{L^6} \|\langle v \rangle^{1-s-\frac{\gamma}{2}} \partial^{\alpha-\alpha_1} f\|_{L^3} \|\partial^\alpha f\|_D \\
 &+ \|\partial^\alpha E\| \|\langle v \rangle^{1-s-\frac{\gamma}{2}} f\|_{L^\infty} \|\partial^\alpha f\|_D \\
 &\lesssim (\|E\|_{L^\infty} + \|\nabla^2 E\|_{H^{N_0-2}}) \mathcal{D}_{N,l}(t) + \mathcal{E}_N(t) \mathcal{E}_{N_0,l_0}^1(t) + \varepsilon \mathcal{D}_N(t). \tag{3.22}
 \end{aligned}$$

For J_2 , due to $((E + v \times B) \cdot \partial^\alpha \nabla_v f, \partial^\alpha f) = 0$, we can deduce by employing the same argument to deal with J_1 that

$$J_2 \lesssim \|\nabla^2(E, B)\|_{H^{N_0-2}} \mathcal{D}_{N,l}(t) + \mathcal{E}_N(t) \mathcal{E}_{N_0,l_0}^1(t) + \varepsilon \mathcal{D}_N(t).$$

Here we choose $l_0 \geq N - N_0 + \frac{1}{2} - \frac{1-s}{\gamma}$ and $N \leq 2N_0$. It follows from Lemma 2.2 that $J_3 \lesssim (\mathcal{E}_N(t) + \varepsilon) \mathcal{D}_N(t)$.

Plugging the estimates of J_1, J_2, J_3 into (3.21) yields

$$\begin{aligned}
 & \frac{d}{dt} \sum_{N_0+1 \leq |\alpha| \leq N} \left(\|\partial^\alpha f\|^2 + \|\partial^\alpha(E, B)\|^2 \right) + \sum_{N_0+1 \leq |\alpha| \leq N} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim (\|E\|_{L^\infty} + \|\nabla^2(E, B)\|_{H^{N_0-2}}) \mathcal{D}_{N,l}(t) + \mathcal{E}_N(t) \mathcal{E}_{N_0,l_0}^1(t) + (\mathcal{E}_N(t) + \varepsilon) \mathcal{D}_N(t). \tag{3.23}
 \end{aligned}$$

which combining with (3.2) and (4.35) give the proof of (3.20) under the assumptions (1.10). □

Lemma 3.4 *Let N_0, N and ϑ satisfy (1.21) and (1.10) respectively,, we can deduce that*

$$\frac{d}{dt} \mathcal{E}_{N,l}(t) + \mathcal{D}_{N,l}(t) \lesssim (1+t)^{\frac{1-3\vartheta}{2}} \{\mathcal{E}_{N_0,l-N_0+2}^1(t)\}^{1/2} \mathcal{D}_{N,l}(t) + \sum_{|\alpha|=N} \|\partial^\alpha E\| \|\mu^\delta \partial^\alpha f\| \tag{3.24}$$

holds for all $0 \leq t \leq T$.

Proof The proof of this lemma is divided into two steps. The first step is to deduce the desired energy type estimates on the derivatives of $f(t, x, v)$ with respect to the x -variable only. For this purpose, the standard energy estimate on $\partial^\alpha f$ with $1 \leq |\alpha| \leq N$ weighted by the time-velocity dependent function $w_l(\alpha, 0) = w_l(\alpha, 0)(t, v)$ gives

$$\begin{aligned} & \frac{d}{dt} \sum_{1 \leq |\alpha| \leq N} \|w_l(\alpha, 0)\partial^\alpha f\|^2 + \sum_{1 \leq |\alpha| \leq N} \|w_l(\alpha, 0)\partial^\alpha f\|_D^2 + \frac{\vartheta q}{(1+t)^{1+\vartheta}} \|\langle v \rangle w_l(\alpha, 0)\partial^\alpha f\|^2 \\ & \lesssim \sum_{1 \leq |\alpha| \leq N} \|\partial^\alpha f\|_D^2 + \sum_{1 \leq |\alpha| \leq N} \|\partial^\alpha E\| \|\mu^\delta \partial^\alpha f\| + \underbrace{\sum_{1 \leq |\alpha| \leq N} \left| \left(\partial^\alpha (E \cdot v f), w_l^2(\alpha, 0)\partial^\alpha f \right) \right|}_{J_4} \\ & + \underbrace{\sum_{1 \leq |\alpha| \leq N} \left| \left(\partial^\alpha [(E + v \times B) \cdot \nabla_v f], w_l^2(\alpha, 0)\partial^\alpha f \right) \right|}_{J_5} + \underbrace{\sum_{1 \leq |\alpha| \leq N} \left| \left(\partial^\alpha \Gamma(f, f), w_l^2(\alpha, 0)\partial^\alpha f \right) \right|}_{J_6}. \end{aligned} \tag{3.25}$$

As to the estimates on $J_4, J_5,$ and $J_6,$ we can deduce by following exactly the argument used above to control $J_1, J_2,$ and J_3 that

$$J_4 + J_5 \lesssim (1+t)^{1+\vartheta} (\|E\|_{L^\infty} + \|\nabla_x^2(E, B)\|_{H^{N_0-2}}) \mathcal{D}_{N,l}(t) + \mathcal{E}_N(t)\mathcal{E}_{N_0,l_0}^1(t) + \mathcal{E}\mathcal{D}_{N,l}(t), \tag{3.26}$$

and using Lemma 2.2 gives

$$J_6 \lesssim (1+t)^{\frac{1-3\vartheta}{2}} \mathcal{E}_{N_0,l-N_0+2}^1(t)^{1/2} \mathcal{D}_{N,l}(t) + \mathcal{E}_{N_0,N_0}^{1/2}(t) \mathcal{D}_{N,l}(t), \tag{3.27}$$

Here we choose that $l_0 \geq l + \frac{1}{2} - \frac{1-s}{\gamma} - N_0$ and $N \leq 2N_0$ in the estimates on the term J_4 and J_5 . Collecting the above estimates gives the desired weighted energy type estimates on the derivatives of $f(t, x, v)$ with respect to the x -variables only as follows

$$\begin{aligned} & \frac{d}{dt} \sum_{1 \leq |\alpha| \leq N} \|w_l(\alpha, 0)\partial^\alpha f\|^2 + \sum_{1 \leq |\alpha| \leq N} \|w_l(\alpha, 0)\partial^\alpha f\|_D^2 + \frac{\vartheta q}{(1+t)^{1+\vartheta}} \|\langle v \rangle^{1/2} w_l(\alpha, 0)\partial^\alpha f\|^2 \\ & \lesssim \sum_{1 \leq |\alpha| \leq N} \|\partial^\alpha f\|_D^2 + \sum_{1 \leq |\alpha| \leq N} \|\partial^\alpha E\| \|\mu^\delta \partial^\alpha f\| + \mathcal{E}_{N_0,N_0}^{1/2}(t) \mathcal{D}_{N,l}(t) \\ & + (1+t)^{1+\vartheta} (\|(E, B)\|_{L^\infty} + \|\nabla_x^2(E, B)\|_{H^{N_0-2}}) \mathcal{D}_{N,l}(t) + (1+t)^{\frac{1-3\vartheta}{2}} \mathcal{E}_{N_0,l-N_0+2}^{1/2}(t) \mathcal{D}_{N,l}(t). \end{aligned} \tag{3.28}$$

Next, by applying the microscopic projection $\{\mathbf{I} - \mathbf{P}\}$ to the first equation of (1.4), we can get that

$$\partial_t \{\mathbf{I} - \mathbf{P}\}f + v \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\}f - E \cdot v \mu^{1/2} q_1 + Lf = \{\mathbf{I} - \mathbf{P}\}g + \mathbf{P}(v \cdot \nabla_x f) - v \cdot \nabla_x \mathbf{P}f. \tag{3.29}$$

From (3.29), one has the weighted energy estimate on $\{\mathbf{I} - \mathbf{P}\}f$

$$\begin{aligned} & \frac{d}{dt} \|w_l(0, 0)\{\mathbf{I} - \mathbf{P}\}f\|^2 + \|w_l(0, 0)\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \frac{\vartheta q}{(1+t)^{1+\vartheta}} \|\langle v \rangle^{1/2} w_l(0, 0)\{\mathbf{I} - \mathbf{P}\}f\|^2 \\ & \lesssim \|\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|E\|^2 + \|\nabla f\|_D^2 + \mathcal{E}_N^{1/2}(t) \mathcal{D}_{N,l}(t) + (1+t)^{1+\vartheta} \|(E, B)\|_{L^\infty} \mathcal{D}_{N,l}(t). \end{aligned} \tag{3.30}$$

By the similar way, for the weighted energy estimate on $\{\mathbf{I} - \mathbf{P}\} \partial_\beta^\alpha f$ with $|\alpha| + |\beta| \leq N$ and $|\beta| \geq 1$, we have

$$\begin{aligned} & \frac{d}{dt} \sum_{m=1}^N C_m \sum_{\substack{|\beta|=m, \\ |\alpha|+|\beta|\leq N}} \|w_l(\alpha, \beta) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\} f\|^2 + \kappa \sum_{\substack{|\alpha|+|\beta|\leq N, \\ |\beta|\geq 1}} [\|w_l(\alpha, \beta) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_D^2 \\ & + \frac{q}{(1+t)^{1+\vartheta}} \| \langle v \rangle^{1/2} w_l(\alpha, \beta) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\} f \|^2] \\ & \lesssim \sum_{|\alpha|\leq N} \|w_l(\alpha, 0) \partial^\alpha \{\mathbf{I} - \mathbf{P}\} f\|_D^2 + \sum_{|\alpha|\leq N-1} (\|\nabla_x \mathbf{P} f\|^2 + \|\partial^\alpha E\|^2) \\ & + (1+t)^{1+\vartheta} (\|E, B\|_{L^\infty} + \|\nabla_x^2(E, B)\|_{H^{N_0-2}}) \mathcal{D}_{N,l}(t) \\ & + \mathcal{E}_{N_0, N_0}^{1/2}(t) \mathcal{D}_{N,l}(t) + (1+t)^{\frac{1-3\vartheta}{2}} \mathcal{E}_{N_0, l-N_0+2}^{1/2}(t) \mathcal{D}_{N,l}(t). \end{aligned} \tag{3.31}$$

Here we used the fact that $(v \times B) \cdot \partial_\beta^\alpha \nabla_v \{\mathbf{I} - \mathbf{P}\} f, w_{l-|\beta|}^2 \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\} f) = 0$. In addition, the estimate on the term $(\partial_\beta^\alpha (v \cdot \{\mathbf{I} - \mathbf{P}\} f), \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\} f)$ can be seen [5], that is why we restrict $\frac{1}{2} \leq s \leq 1$. The above other estimates are similar as (3.28), we omit it. Therefore, a proper linear combination of (4.35), (3.28), (3.30) and (3.31) implies (3.24). \square

Similar with Lemma 3.4, we also have

Lemma 3.5 *It holds that*

$$\frac{d}{dt} \mathcal{E}_{N-1,l}(t) + \mathcal{D}_{N-1,l}(t) \lesssim \mathcal{E}_{N_0, l_0}^1(t) \mathcal{E}_{N-1,l}(t) + \sum_{|\alpha|=N-1} \|\partial^\alpha E\| \|\mu^\delta \partial^\alpha f\|. \tag{3.32}$$

for all $0 \leq t \leq T$.

The following lemma is concerned with the weighted energy estimates on $\bar{\mathcal{E}}_{N_0, \ell}(t)$.

Lemma 3.6 *For any ℓ with $\ell = l_0 + l^*$ with $l^* = l' + \frac{3(\gamma+2s)}{2\gamma} + \chi_{N_0 \geq 4} \left(\frac{1}{2} - \frac{1-s}{\gamma}\right) (N_0 - 4)$ with l' being given in Lemma 4.1, if the assumptions of Lemma 3.1 hold, we have*

$$\frac{d}{dt} \bar{\mathcal{E}}_{N_0, \ell}(t) + \bar{\mathcal{D}}_{N_0, \ell}(t) \lesssim \sum_{|\alpha|=N_0} \varepsilon \|\partial^\alpha E\|^2. \tag{3.33}$$

Furthermore,

$$\sup_{0 \leq \tau \leq t} \bar{\mathcal{E}}_{N_0, l_0+l^*}(\tau) \lesssim Y_0^2 + \varepsilon \int_0^t \|\nabla^{N_0} E\|^2 ds. \tag{3.34}$$

Proof From the completely same procedure to obtain the energy inequality (3.24) for $\mathcal{E}_{N,l}(t)$, we notice the derivatives of the electromagnetic field of order up to N_0 decays in time, and use the Cauchy inequality, the weighted estimate on the highest order N_0 for the term $E \cdot v \mu^{1/2}$ can be dominated by

$$\sum_{|\alpha|=N_0} (\partial^\alpha E \cdot v \mu^{1/2}, w_\ell^2(\alpha, 0) \partial^\alpha f) \lesssim \sum_{|\alpha|=N_0} (\varepsilon \|\partial^\alpha E\|^2 + C_\varepsilon \|\mu^\delta \partial^\alpha f\|),$$

it follows that

$$\frac{d}{dt} \mathcal{E}_{N_0, \ell}(t) + \mathcal{D}_{K, \ell}(t) \lesssim \varepsilon \sum_{|\alpha|=N_0} \|\partial^\alpha E\|^2. \tag{3.35}$$

The proper linear combination of (5.1), (5.6), (5.8) and (3.35) gives (3.33), therefore (3.34) follows by the time integration of (3.33). This completes the proof of Lemma 3.6. \square

Now we are ready to obtain the closed estimates on $\mathcal{E}_N(t)$ and $\mathcal{E}_{N,l}(t)$ of the time-weighted energy norm $X(t)$ in the following:

Lemma 3.7 *It holds that*

$$\sup_{0 \leq s \leq t} \left\{ \mathcal{E}_N(s) + (1+s)^{-\frac{1+\epsilon_0}{2}} \mathcal{E}_{N,l}(t) \right\} + \int_0^t \mathcal{D}_N(s) ds \lesssim Y_0^2. \tag{3.36}$$

Proof Multiplying (3.20) by $(1+t)^{-\epsilon_0}$ gives

$$\begin{aligned} & \frac{d}{dt} \left\{ (1+t)^{-\epsilon_0} \mathcal{E}_N(t) \right\} + \epsilon_0 (1+t)^{-1-\epsilon_0} \mathcal{E}_N(t) + (1+t)^{-\epsilon_0} \mathcal{D}_N(t) \\ & \lesssim \frac{\delta}{(1+t)^{1+\epsilon_0+\vartheta}} \mathcal{D}_{N,l}(t) + (1+t)^{-\epsilon_0} \mathcal{E}_N(t) \mathcal{E}_{N_0,l_0}^1(t). \end{aligned} \tag{3.37}$$

Multiplying (3.24) by $(1+t)^{-(1+\epsilon_0)/2}$ yields that

$$\begin{aligned} & \frac{d}{dt} \left\{ (1+t)^{-\frac{\epsilon_0+1}{2}} \mathcal{E}_{N,l}(t) \right\} + \frac{\epsilon_0+1}{2} (1+t)^{-\frac{\epsilon_0+3}{2}} \mathcal{E}_{N,l}(t) + (1+t)^{-\frac{\epsilon_0+1}{2}} \mathcal{D}_{N,l}(t) \\ & \lesssim (1+t)^{-\frac{\epsilon_0+1}{2}} \mathcal{E}_N(t) \mathcal{E}_{N_0,l_0}^1(t) + (1+t)^{-\frac{\epsilon_0+1}{2}} \|\nabla^N E\| \|\mu^\delta \nabla^N f\| \\ & \lesssim (1+t)^{-\frac{\epsilon_0+1}{2}} \mathcal{E}_N(t) \mathcal{E}_{N_0,l_0}^1(t) + (1+t)^{-1-\epsilon_0} \|\nabla^N E\|^2 + \varepsilon \|\mu^\delta \nabla^N f\|^2. \end{aligned} \tag{3.38}$$

The proper linear combination of (3.20), (3.37) and (3.38) and taking the time integration yield

$$\begin{aligned} & \mathcal{E}_N(t) + (1+t)^{-\frac{\epsilon_0+1}{2}} \mathcal{E}_{N,l}(t) + \int_0^t \left\{ (1+s)^{-1-\epsilon_0} \mathcal{E}_N(s) + \mathcal{D}_N(s) \right\} ds \\ & + \int_0^t \left\{ (1+s)^{-\frac{\epsilon_0+3}{2}} \mathcal{E}_{N,l}(s) + (1+s)^{-\frac{\epsilon_0+1}{2}} \mathcal{D}_{N,l}(s) \right\} ds \lesssim Y_0^2. \end{aligned}$$

Here we used the fact that if $\varrho \in (\frac{1}{2}, 1]$, we have taken $l_0 = l_{0,1} = l_{0,0}$ such that

$$\mathcal{E}_{N_0,l_0}^1(t) = \mathcal{E}_{N_0,l_{0,1}}^1(t) \lesssim (1+t)^{-1-\varrho} X(t),$$

while for the case of $\varrho \in (1, \frac{3}{2})$, we have taken $l_0 = l_{0,0}$ such that

$$\mathcal{E}_{N_0,l_0}^1(t) \lesssim \mathcal{E}_{N_0,l_{0,0}}^0(t) \lesssim (1+t)^{-\varrho} X(t).$$

This completes the proof of Lemma 3.7. \square

3.1 The Proof of Theorem 1.1

Recall $X(t)$ -norm, combining Lemma 3.6 with Lemma 3.7 yields that

$$X(t) \lesssim Y_0^2.$$

The global existence follows further from the local existence and the continuity argument in the usual way. This completes the proof of Theorem 1.1.

3.2 The Proof of Theorem 1.3

Based on Theorem 1.1, it follows from Lemma 3.1 that

$$\mathcal{E}_{N_0}^k(t) \lesssim Y_0^2(1+t)^{-(k+\varrho)}, \quad k = 0, 1, 2, \dots, N_0 - 2$$

which gives (1.23). From Lemma 3.2, let $0 \leq i \leq k \leq N_0 - 3$ be an integer, it holds that

$$\mathcal{E}_{N_0, l_0, k + \frac{\gamma+2s}{2\gamma}i}^k(t) \lesssim Y_0^2(1+t)^{-k-\varrho+i}, \quad i = 0, 1, \dots, k.$$

Furthermore, when $\varrho \in [1, 3/2)$,

$$\mathcal{E}_{N_0, l_0, k + \frac{(\gamma+2s)(k+1)}{2\gamma}}^k(t) \lesssim Y_0^2(1+t)^{1-\varrho}.$$

(1.24) and (1.25) follow from the above two inequalities. To prove (1.26), by using the interpolation method, when $N_0 + 1 \leq |\alpha| \leq N - 1$, combining the time decay of $\|\nabla^{N_0} f\|$ and the bound of $\|\nabla^N f\|$ gives

$$\|\partial^\alpha f\|^2 \lesssim \|\nabla^N f\|^{\frac{|\alpha|-N_0}{N-N_0}} \|\nabla^{N_0} f\|^{\frac{N-|\alpha|}{N-N_0}} \lesssim Y_0^2(1+t)^{-\frac{(N-|\alpha|)(N_0-2+\varrho)}{N-N_0}}.$$

Thus we have completed the proof of Theorem 1.3.

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Compliance with ethical standards

Conflict of interest No conflict of interest.

Appendix 1: The Proof of (3.2)

To obtain the temporal decay estimates on the energy functional $\mathcal{E}_{N_0}^k(t)$, we need the following two lemmas:

Lemma 4.1 *Let $N_0 \geq 3$ and $\frac{1}{2} < \varrho < \frac{3}{2}$, then there exist positive constants l' and \bar{m} which depend only on N_0, s , and γ such that the following estimates hold:*

(1) *For $k = 0, 1, \dots, N_0 - 2$, it holds that*

$$\begin{aligned} & \frac{d}{dt} (\|\nabla^k f\|^2 + \|\nabla^k(E, B)\|^2) + \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \lesssim \max \{ \mathcal{E}_{\bar{m}, \bar{m}}(t), \bar{\mathcal{E}}_{N_0-1, N_0-1+l'}(t) \} \\ & \quad \times \left(\|\nabla^{k+1}(E, B)\|^2 + \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\nabla^{k+1}f\|_D^2 \right) \\ & \quad + \varepsilon \left(\|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\nabla^{k+1}f\|_D^2 \right), \end{aligned} \tag{4.1}$$

where $1 \leq j \leq k$.

(2) *If $k = N_0 - 1$, it holds that*

$$\begin{aligned} & \frac{d}{dt} (\|\nabla^{N_0-1} f\|^2 + \|\nabla^{N_0-1}(E, B)\|^2) + \|\nabla^{N_0-1}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \lesssim \max \left\{ \mathcal{E}_{3, \frac{7}{2} + \frac{s-1}{\gamma}}(t), \mathcal{E}_{\bar{m}, \bar{m}}(t), \bar{\mathcal{E}}_{N_0-1, N_0-1+l'}(t) \right\} \\ & \quad \times \left(\|\nabla^{N_0-1}(E, B)\|^2 + \|\nabla^{N_0-2}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\nabla^{N_0-1}f\|_D^2 \right) + \varepsilon \|\nabla^{N_0-1}f\|_D^2. \end{aligned} \tag{4.2}$$

(3) For $k = N_0 \geq 3$, if we take $n > \frac{2}{3}N_0 - \frac{5}{3}$, one has

$$\begin{aligned} & \frac{d}{dt} (\|\nabla^{N_0} f\|^2 + \|\nabla^{N_0}(E, B)\|^2) + \|\nabla^{N_0}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \lesssim \max \{ \mathcal{E}_{N_0+n}(t), \mathcal{E}_{\bar{m}, \bar{m}}(t), \bar{\mathcal{E}}_{N_0, N_0+l'}(t) \} \\ & \quad \times \left(\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-2}\{I - P\}f\|_D^2 + \|\nabla^{N_0-1} f\|_D^2 + \|\nabla^{N_0} f\|_D^2 \right) \\ & \quad + \varepsilon \left\{ \|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0-2}\{I - P\}f\|_{H_x^1 L_v^2}^2 + \|\nabla^{N_0} f\|_D^2 \right\}. \end{aligned} \tag{4.3}$$

Here $\bar{m} = \max \left\{ \frac{2\varrho+5}{2\varrho+3}N_0 - \frac{6}{2\varrho+3}, 3 + \frac{N_0-2+\varrho}{(\varrho+1)n+3-N_0} \right\}$ and $l' \geq \max \left\{ \frac{1}{2} + \frac{s-1}{\gamma}, \frac{\hat{l}}{-\gamma} \right\}$ where $\hat{l} = \max \{ \hat{l}_1, \hat{l}_2, \dots, \hat{l}_{19} \}$ where \hat{l}_i will be specified in the proofs of Lemma 4.1.

Proof For the case $k = 0$, multiplying (1.4) by f and integrating the resulting identity with respect to x and v over $\mathbb{R}_x^3 \times \mathbb{R}_v^3$, we have that

$$\frac{d}{dt} (\|f\|^2 + \|(E, B)\|^2) + \|\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \lesssim |(v \cdot Ef, f)| + |(\Gamma(f, f), \{\mathbf{I} - \mathbf{P}\}f)|, \tag{4.4}$$

For the first term on the right hand of (4.4), we have

$$\begin{aligned} & |(v \cdot Ef, f)| \\ & \lesssim |(v \cdot Ef, \mathbf{P}f)| + |(v \cdot Ef, \{\mathbf{I} - \mathbf{P}\}f)| \\ & \lesssim \|E\|_{L_x^2} \|\mu^\delta f\|_{L_x^3 L_v^2} \|\mu^\delta f\|_{L_x^\varrho L_v^2} + \|E\|_{L_x^\varrho} \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^3 L_v^2} \\ & \quad \times \|\langle v \rangle^{\frac{\gamma}{2}+s} \{\mathbf{I} - \mathbf{P}\}f\|_{L_x^2 L_v^2} \\ & \lesssim \|E\| \|\mu^\delta f\|_{L_v^3 L_x^3} \|\mu^\delta f\|_{L_v^2 L_x^\varrho} + \|E\|_{L_v^2 L_x^\varrho} \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} \{\mathbf{I} - \mathbf{P}\}f\|_{L_v^3 L_x^3} \|\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \|\Lambda^{-\frac{1}{2}} E\|^{\frac{2}{3}} \|\nabla_x E\|^{\frac{1}{3}} \|\Lambda^{-\frac{1}{2}} f\|^{\frac{2}{3}} \|\nabla_x f\|_D^{\frac{2}{3}} \|\nabla_x f\|_D + \|\nabla_x E\| \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} \{\mathbf{I} - \mathbf{P}\}f\|_{H_x^1} \\ & \quad \times \|\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \bar{\mathcal{E}}_{1, \frac{3}{2} + \frac{s-1}{\gamma}}(t) (\|\nabla_x E\|^2 + \|\nabla_x f\|_D^2) + \varepsilon (\|\nabla_x f\|_D^2 + \|\{\mathbf{I} - \mathbf{P}\}f\|_D^2). \end{aligned}$$

The other term can be estimated as follows

$$\begin{aligned} & |(\Gamma(f, f), \{\mathbf{I} - \mathbf{P}\}f)| \\ & \lesssim \varepsilon \|\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|f\|_{L_{\gamma/2+s}^2} \|f\|_{L_D^2} + \|f\|_{L_v^2} \|f\|_{L_{\gamma/2+s}^2} \\ & \lesssim \varepsilon \|\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|f\|_{L_x^\varrho}^2 \|f\|_{L_x^\varrho L_v^2}^2 \lesssim \varepsilon \|\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|f\|_{H_x^1}^2 \|\nabla_x f\|_D^2. \end{aligned}$$

Collecting the above estimates gives

$$\frac{d}{dt} (\|f\|^2 + \|(E, B)\|^2) + \|\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \lesssim \bar{\mathcal{E}}_{1, \frac{3}{2} + \frac{s-1}{\gamma}}(t) (\|\nabla_x E\|^2 + \|\nabla_x f\|_D^2) + \varepsilon \|\nabla_x f\|_D^2 \tag{4.5}$$

which gives (4.1) with $k = 0$.

For $k = 1, 2, \dots, N_0 - 2$, applying ∇^k to (1.4), multiplying the resulting identity by $\nabla^k f$, and then integrating the final result with respect to x and v over $\mathbb{R}_x^3 \times \mathbb{R}_v^3$, we have that

$$\begin{aligned} & \frac{d}{dt} \left(\|\nabla^k f\|^2 + \|\nabla^k(E, B)\|^2 \right) + \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\|_\sigma^2 \\ & \lesssim \underbrace{\left| \left(\nabla^k(v \cdot Ef), \nabla^k f \right) \right|}_{I_1} + \underbrace{\left| \left(\nabla^k(E \cdot \nabla_v f), \nabla^k f \right) \right|}_{I_2} \\ & + \underbrace{\left| \left(\nabla^k(v \times B \cdot \nabla_v f), \nabla^k f \right) \right|}_{I_3} + \underbrace{\left| \left(\nabla^k \mathbf{\Gamma}(f, f), \nabla^k f \right) \right|}_{I_4}. \end{aligned} \tag{4.6}$$

By applying the macro-micro decomposition, one can deduce that

$$\begin{aligned} I_1 & \lesssim \underbrace{\left| \left(\nabla^k(v \cdot Ef), \nabla^k \mathbf{P}f \right) \right|}_{I_{1,1}} + \underbrace{\left| \left(\nabla^k(v \cdot E\mathbf{P}f), \nabla^k\{\mathbf{I} - \mathbf{P}\}f \right) \right|}_{I_{1,2}} \\ & + \underbrace{\left| \left(\nabla^k(v \cdot E\{\mathbf{I} - \mathbf{P}\}f), \nabla^k\{\mathbf{I} - \mathbf{P}\}f \right) \right|}_{I_{1,3}}. \end{aligned} \tag{4.7}$$

Applying Lemma 2.5, $I_{1,1}$ and $I_{1,2}$ can be dominated by

$$\begin{aligned} I_{1,1} + I_{1,2} & \lesssim \sum_{j \leq k-1} \|\nabla^j E\|_{L_x^6} \|\nabla^{k-1-j}(\mu^\delta f)\|_{L_x^3 L_v^2} \|\nabla^{k+1}(\mu^\delta f)\| \\ & \lesssim \sum_{j \leq k-1} \|\Lambda^{-\frac{1}{2}} E\|_{L_x^6}^{\frac{2k-2j}{2k+3}} \|\nabla^{k+1} E\|_{L_x^3 L_v^2}^{\frac{2j+3}{2k+3}} \|\Lambda^{-\frac{1}{2}}(\mu^\delta f)\|_{L_x^3 L_v^2}^{\frac{2j+3}{2k+3}} \|\nabla^{k+1}(\mu^\delta f)\|_{L_x^3 L_v^2}^{\frac{2k-2j}{2k+3}} \\ & \quad \times \|\nabla^{k+1}(\mu^\delta f)\| \\ & \lesssim \bar{\mathcal{E}}_{0,0}(t) \left(\|\nabla^{k+1} E\|_D^2 + \|\nabla^{k+1} f\|_D^2 \right) + \varepsilon \|\nabla^{k+1} f\|_D^2. \end{aligned}$$

For $I_{1,3}$, it holds from the Cauchy inequality and Holder inequalities that

$$\begin{aligned} I_{1,3} & \lesssim \sum_{j \leq k} \|\nabla^j E \nabla^{k-j}\{\mathbf{I} - \mathbf{P}\}f\langle v \rangle^{1-\frac{\gamma+2s}{2}}\| \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\langle v \rangle^{\frac{\gamma+2s}{2}}\| \\ & \lesssim \underbrace{\sum_{j \leq k} \|\nabla^j E \nabla^{k-j}\{\mathbf{I} - \mathbf{P}\}f\langle v \rangle^{1-\frac{\gamma+2s}{2}}\|}_{I_{1,3,1}}^2 + \varepsilon \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\|_D^2. \end{aligned}$$

To deal with $I_{1,3,1}$, when $j = 0$, we have from Lemma 2.5 that

$$\begin{aligned} I_{1,3,1} & \lesssim \left(\|E\|_{L_x^\infty} \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\langle v \rangle^{1-\frac{\gamma+2s}{2}}\| \right)^2 \\ & \lesssim \left(\|\Lambda^{-\varrho} E\|^{1-\theta} \|\nabla^{k+1} E\|^\theta \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\langle v \rangle^{\frac{\gamma+2s}{2}}\|^{1-\theta} \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\langle v \rangle^{\hat{l}_1}\|^\theta \right)^2 \\ & \lesssim \bar{\mathcal{E}}_{k,k-\frac{\hat{l}_1}{\gamma}}(t) \left(\|\nabla^{k+1} E\|^2 + \|\nabla^k\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \right). \end{aligned}$$

Here $\theta = \frac{3+2\varrho}{2(k+1+\varrho)}$ and $\hat{l}_1 = \frac{1-\gamma-2s}{\theta} + \frac{\gamma+2s}{2} = \frac{2(1-\gamma-2s)(k+1+\varrho)}{3+2\varrho} + \frac{\gamma+2s}{2}$.

While for the case $j \neq 0$, we can deduce from Lemma 2.5 that

$$\begin{aligned}
 I_{1,3,1} &\lesssim \sum_{1 \leq j \leq k} \left(\left\| \nabla^j E \right\|_{L_x^6} \left\| \nabla^{k-j} \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{1 - \frac{\gamma+2s}{2}} \right\|_{L_v^2 L_x^3} \right)^2 \\
 &\lesssim \sum_{1 \leq j \leq k} \left(\left\| \Lambda^{-\varrho} E \right\|^{1-\theta_j} \left\| \nabla^{k+1} E \right\|^{\theta_j} \left\| \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{1 - \frac{\gamma+2s}{2}} \right\|^{1-\alpha_j} \right. \\
 &\quad \left. \times \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{1 - \frac{\gamma+2s}{2}} \right\|^{\alpha_j} \right)^2 \\
 &\lesssim \sum_{1 \leq j \leq k} \left(\left\| \Lambda^{-\varrho} E \right\|^{1-\theta_j} \left\| \nabla^{k+1} E \right\|^{\theta_j} \left\| \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{1 - \frac{\gamma+2s}{2}} \right\|^{1-\alpha_j} \right. \\
 &\quad \left. \times \left(\left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{\hat{l}_{2j}} \right\|^{1-\beta_j} \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{\frac{\gamma+2s}{2}} \right\|^{\beta_j} \right)^{\alpha_j} \right)^2 \\
 &\lesssim \sum_{1 \leq j \leq k} \left(\left\| \Lambda^{-\varrho} E \right\|^{1-\theta_j} \left\| \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{1 - \frac{\gamma+2s}{2}} \right\|^{1-\alpha_j} \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{\hat{l}_{2j}} \right\|^{\alpha_j(1-\beta_j)} \right. \\
 &\quad \left. \times \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \langle v \rangle^{\frac{\gamma+2s}{2}} \right\|^{\alpha_j \beta_j} \left\| \nabla^{k+1} E \right\|^{\theta_j} \right)^2 \\
 &\lesssim \bar{\mathcal{E}}_{k, k - \frac{\hat{l}_2}{\gamma}}(t) \left(\left\| \nabla^{k+1} E \right\|^2 + \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right\|_D^2 \right).
 \end{aligned}$$

Here we can deduce from Lemma 2.5 that $\theta_j = \frac{j+\varrho+1}{k+1+\varrho}$, $\alpha_j = \frac{2k-2j+1}{2k}$, $\frac{\gamma+2s}{2} \cdot \beta_j + \hat{l}_{2j}(1 - \beta_j) = 1 - \frac{\gamma+2s}{2}$. If we set $\alpha_j \beta_j = 1 - \theta_j$, then we can get that $\hat{l}_{2j} = \frac{\gamma+2s}{2} + \frac{1-\gamma-2s}{1-\beta_j} = \frac{\gamma+2s}{2} + \frac{(1-\gamma-2s)(2k-2j+1)(k+1+\varrho)}{(2k-2j+1)(k+1+\varrho)-2k(k-j)}$ and we take $\hat{l}_2 = \max_{1 \leq j \leq k} \{\hat{l}_{2j}\}$.

Consequently

$$\begin{aligned}
 I_1 &\lesssim \bar{\mathcal{E}}_{k, k - \frac{\max\{\hat{l}_j, \hat{l}_2\}}{\gamma}}(t) \left(\left\| \nabla^{k+1} E \right\|^2 + \left\| \nabla^{k+1} f \right\|_D^2 + \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right\|_D^2 \right) + \varepsilon \left(\left\| \nabla^{k+1} f \right\|_D^2 \right. \\
 &\quad \left. + \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right\|_D^2 \right).
 \end{aligned}$$

For I_3 on the right-hand side of (4.6), we can get

$$\begin{aligned}
 I_3 &\lesssim \sum_{1 \leq j \leq k} \left| \left(v \times \nabla^j B \cdot \nabla_v \nabla^{k-j} f, \nabla^k f \right) \right| \\
 &= \underbrace{\sum_{1 \leq j \leq k} \left| \left(v \times \nabla^j B \cdot \nabla_v \nabla^{k-j} \mathbf{P} f, \nabla^k f \right) \right|}_{I_{3,1}} \\
 &\quad + \underbrace{\sum_{1 \leq j \leq k} \left| \left(v \times \nabla^j B \cdot \nabla_v \nabla^{k-j} \{\mathbf{I} - \mathbf{P}\} f, \nabla^k \mathbf{P} f \right) \right|}_{I_{3,2}} \\
 &\quad + \underbrace{\sum_{1 \leq j \leq k} \left| \left(v \times \nabla^j B \cdot \nabla_v \nabla^{k-j} \{\mathbf{I} - \mathbf{P}\} f, \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right) \right|}_{I_{3,3}}.
 \end{aligned} \tag{4.8}$$

Applying the same trick as the estimate on $I_{1,1}$, $I_{3,1}$ and $I_{3,2}$ can be bounded by

$$\bar{\mathcal{E}}_{0,0}(t) \left(\|\nabla^{k+1} B\|^2 + \|\nabla^{k+1} f\|_D^2 \right) + \varepsilon \|\nabla^{k+1} f\|_D^2.$$

For $I_{3,3}$, when $j = k$,

$$\begin{aligned} I_{3,3} &\lesssim \left\| \nabla^k B \right\|_{L_x^6} \left\| \langle v \rangle^{1-\frac{\gamma}{2}-s} \nabla_v \{ \mathbf{I} - \mathbf{P} \} f \right\|_{L_v^2 L_x^3} \left\| \langle v \rangle^{\frac{\gamma}{2}+s} \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|_{L_v^2 L_x^2} \\ &\lesssim \left\| \langle v \rangle^{1-\frac{\gamma}{2}-s} \nabla_v \{ \mathbf{I} - \mathbf{P} \} f \right\|_{L_v^2 L_x^3} \left\| \nabla^{k+1} B \right\| \left\| \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|_D \\ &\lesssim \bar{\mathcal{E}}_{2, \frac{5}{2} + \frac{s-1}{\gamma}}(t) \left\| \nabla^{k+1} B \right\|^2 + \varepsilon \left\| \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|_D^2. \end{aligned} \tag{4.9}$$

While for the case $1 \leq j \leq k - 1$,

$$\begin{aligned} I_{3,3} &\lesssim \sum_{1 \leq j \leq k-1} \left\| \nabla^j B \right\|_{L_x^\infty} \left\| \nabla_v \nabla^{k-j} \{ \mathbf{I} - \mathbf{P} \} f \right\| \left\| \langle v \rangle \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\| \\ &\lesssim \sum_{1 \leq j \leq k-1} \left\| \Lambda^{-\varrho} B \right\|^{\frac{2k-2j-1}{2(k+1+\varrho)}} \left\| \nabla^{k+1} B \right\|^{\frac{2j+2\varrho+3}{2(k+1+\varrho)}} \left\| \nabla_v^{m_{1j}+1} \nabla^{k-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{1}{m_{1j}+1}} \\ &\quad \times \left\| \nabla^{k-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_{1j}}{m_{1j}+1}} \left\| \langle v \rangle \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\| \\ &\lesssim \sum_{1 \leq j \leq k-1} \left\| \Lambda^{-\varrho} B \right\|^{\frac{2k-2j-1}{2(k+1+\varrho)}} \left\| \nabla^{k+1} B \right\|^{\frac{2j+2\varrho+3}{2(k+1+\varrho)}} \left\| \nabla_v^{m_{1j}+1} \nabla^{k-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{1}{m_{1j}+1}} \\ &\quad \times \left\| \Lambda^{-\varrho} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_{1j}j}{(m_{1j}+1)(k+\varrho)}} \left\| \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_{1j}(k-j+\varrho)}{(m_{1j}+1)(k+\varrho)}} \left\| \langle v \rangle \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\| \\ &\lesssim \sum_{1 \leq j \leq k-1} \left\| \Lambda^{-\varrho} B \right\|^{\frac{2k-2j-1}{2(k+1+\varrho)}} \left\| \nabla^{k+1} B \right\|^{\frac{2j+2\varrho+3}{2(k+1+\varrho)}} \left\| \nabla_v^{m_{1j}+1} \nabla^{k-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{1}{m_{1j}+1}} \\ &\quad \times \left\| \Lambda^{-\varrho} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_{1j}j}{(m_{1j}+1)(k+\varrho)}} \left\| \langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_{1j}(k-j+\varrho)\beta_j}{(m_{1j}+1)(k+\varrho)}} \\ &\quad \times \left\| \langle v \rangle^{\hat{l}_{3j}} \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_{1j}(k-j+\varrho)(1-\beta_j)}{(m_{1j}+1)(k+\varrho)}} \left\| \langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\beta_j} \\ &\quad \times \left\| \langle v \rangle^{\hat{l}_{4j}} \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|^{1-\beta_j} \\ &\lesssim \max \left\{ \bar{\mathcal{E}}_{k+m_1, 1+m_1}(t), \bar{\mathcal{E}}_{k, -\frac{\hat{l}_4}{\gamma}}(t) \right\} \left(\left\| \nabla^{k+1} B \right\|^2 + \left\| \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|_D^2 \right) \\ &\quad + \varepsilon \left\| \nabla^k \{ \mathbf{I} - \mathbf{P} \} f \right\|_D^2. \end{aligned} \tag{4.10}$$

Here we have used the fact that there exists a positive constant $\beta_j \in (0, 1)$ such that

$$\frac{2j + 2\varrho + 3}{2(k + 1 + \varrho)} + \frac{m_{1j}(k - j + \varrho)\beta_j}{(m_{1j} + 1)(k + \varrho)} + \beta_j = 2$$

holds for $1 \leq j \leq k - 1$. A necessary and sufficient condition to guarantee the existence of such a β is

$$\frac{2j + 2\varrho + 3}{2(k + 1 + \varrho)} + \frac{m_{1j}(k - j + \varrho)}{(m_{1j} + 1)(k + \varrho)} + 1 > 2, \quad 1 \leq j \leq k - 1,$$

from which one can deduce that $m_{1j} > \frac{2k^2 + 2\varrho k - 2jk - k - 2j\varrho - \varrho}{2k\varrho + 3k + 2\varrho^2 + 3\varrho - 2j}$ holds for $1 \leq j \leq k - 1$. Noticing that $\frac{1}{2} < \varrho < \frac{3}{2}$, it is easy to see that we can take

$$m_1 = \max_{1 \leq j \leq k-1} m_{1j} = \frac{2k}{2\varrho + 3} + \frac{2\varrho - 3}{2\varrho + 3} > \frac{2k^2 + 2\varrho k - 3k - 3\varrho}{2k\varrho + 3k + 2\varrho^2 + 3\varrho - 2}.$$

Consequently, $k - j + 1 + m_{1j} \leq k + m_1 = \frac{2\varrho + 5}{2\varrho + 3}k + \frac{2\varrho - 3}{2\varrho + 3} \leq \frac{2\varrho + 5}{2\varrho + 3}N_0 - \frac{2\varrho + 13}{2\varrho + 3}$ with $N_0 \geq 4$.

Moreover, since \hat{l}_{3j} and \hat{l}_{4j} satisfy $\frac{\gamma + 2s}{2}\beta_j + \hat{l}_{3j}(1 - \beta_j) = 0$ and $\frac{\gamma + 2s}{2}\beta_j + \hat{l}_{4j}(1 - \beta_j) = 1$ with $0 < \beta_j < 1$ respectively, one can deduce that $\hat{l}_{3j} = \frac{\gamma + 2s}{2} - \frac{\gamma + 2s}{2(1 - \beta_j)}$ and $\hat{l}_{4j} = \frac{\gamma + 2s}{2} - \frac{\gamma + 2s - 2}{2(1 - \beta_j)}$, from which we can see that $\hat{l}_{4j} > \hat{l}_{3j}$ where

$$\beta_j = \frac{(4k + 1 + 2\varrho - 2j)(m_{1j} + 1)(k + \varrho)}{(k + 2 + 2\varrho)(k + \varrho + 2km_{1j} + 2\varrho m_{1j} - jm_{1j})}.$$

Here we take $\hat{l}_4 = \max_{1 \leq j \leq k-1} \{\hat{l}_{4j}\}$.

Consequently,

$$\begin{aligned} I_3 \lesssim & \max \left\{ \bar{\mathcal{E}}_{k+m_1, 1+m_1}(t), \bar{\mathcal{E}}_{k, -\frac{\hat{l}_4}{\gamma}}(t), \bar{\mathcal{E}}_{2, \frac{5}{2} + \frac{s-1}{\gamma}}(t) \right\} \\ & \times \left(\left\| \nabla^{k+1} B \right\|^2 + \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right\|_D^2 + \left\| \nabla^{k+1} f \right\|_D^2 \right) \\ & + \varepsilon \left(\left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right\|_D^2 + \left\| \nabla^{k+1} f \right\|_D^2 \right). \end{aligned}$$

By the same way as the estimate on I_3 , the term I_2 containing E on the right-hand side of (4.6) can be bounded by

$$\begin{aligned} I_2 \lesssim & \max \left\{ \bar{\mathcal{E}}_{k+m_1, 1+m_1}(t), \bar{\mathcal{E}}_{k, -\frac{\hat{l}_4}{\gamma}}(t), \bar{\mathcal{E}}_{2, \frac{5}{2} + \frac{s-1}{\gamma}}(t) \right\} \\ & \times \left(\left\| \nabla^{k+1} E \right\|^2 + \left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right\|_D^2 + \left\| \nabla^{k+1} f \right\|_D^2 \right) \\ & + \varepsilon \left(\left\| \nabla^k \{\mathbf{I} - \mathbf{P}\} f \right\|_D^2 + \left\| \nabla^{k+1} f \right\|_D^2 \right). \end{aligned}$$

For the last term I_4 on the right-hand side of (4.6), it follows that

$$\begin{aligned}
 I_4 &\lesssim \sum_{j \leq k} |(\Gamma(\nabla^j f, \nabla^{k-j} f), \nabla^k \{\mathbf{I} - \mathbf{P}\} f)| \\
 &\lesssim \sum_{j \leq k} \int_{\mathbb{R}_x^3} \left\{ |\nabla^j f|_{L^2_{\gamma+s/2}} |\nabla^{k-j} f|_D + |\nabla^{k-j} f|_{L^2_{\gamma+s/2}} |\nabla^j f|_D \right. \\
 &\quad \left. + \min \left\{ |\nabla^j f|_{L^2} |\nabla^{k-j} f|_{L^2_{\gamma+s/2}}, |\nabla^{k-j} f|_{L^2} |\nabla^j f|_{L^2_{\gamma+s/2}} \right\} \right\} |\mathbf{I} - \mathbf{P}\} f|_D dx \\
 &\lesssim \underbrace{\sum_{j \leq k} \int_{\mathbb{R}_x^3} |\nabla^j \mathbf{P} f|_{L^2} |\nabla^{k-j} f|_D |\mathbf{I} - \mathbf{P}\} f|_D dx}_{I_{4,1}} \\
 &\quad + \underbrace{\sum_{j \leq k} \int_{\mathbb{R}_x^3} |\nabla^j \{\mathbf{I} - \mathbf{P}\} f|_{L^2} |\nabla^{k-j} f|_D |\mathbf{I} - \mathbf{P}\} f|_D dx}_{I_{4,2}}
 \end{aligned} \tag{4.11}$$

The first term $I_{4,1}$ on the right-hand side of (4.11) can be estimated as follows:

$$\begin{aligned}
 I_{4,1} &\lesssim \sum_{j \leq k} \|\nabla^j \mathbf{P} f\|_{L^2_{\nu} L^6_x} \|\nabla^{k-j} f\|_{L^2_D L^3_x} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim \sum_{j \leq k} \|\nabla^{m'} \mathbf{P} f\|^{1-\theta} \|\nabla^{k+1} \mathbf{P} f\|^\theta \|f\|_D^\theta \|\nabla^{k+1} f\|_D^{1-\theta} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim \max \{ \mathcal{E}_{N_0-1}(t), \mathcal{E}_{1,1}(t) \} \|\nabla^{k+1} f\|_D^2 + \varepsilon \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^2.
 \end{aligned} \tag{4.12}$$

From Lemma 2.5, we can obtain

$$\theta = \frac{2j + 1}{2(k + 1)}, \quad m' = \frac{k + 1}{2j + 1} \leq k + 1 \leq N_0 - 1.$$

When $j \leq k - 1$, the second term $I_{4,2}$ on the right-hand side of (4.11) can be bounded by

$$\begin{aligned}
 I_{4,2} &\lesssim \sum_{j \leq k-1} \|\nabla^j \{\mathbf{I} - \mathbf{P}\} f\|_{L^2_{\nu} L^6_x} \|\nabla^{k-j} f\|_{L^2_D L^3_x} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim \sum_{j \leq k-1} \|\{\mathbf{I} - \mathbf{P}\} f\|_D^{\frac{k-j-1}{k}} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^{\frac{j+1}{k}} \|f\|_D^{\frac{2j+1}{2(k+1)}} \|\nabla^{k+1} f\|_D^{\frac{2k-2j}{2(k+1)}} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim \sum_{j \leq k-1} \|\{\mathbf{I} - \mathbf{P}\} f\|_D^{\frac{k-j-1}{k}} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^{\frac{j}{2} + s} \|f\|_D^{\frac{j+1}{k} \beta} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^{\hat{l}_{5j}} \|\{\mathbf{I} - \mathbf{P}\} f\|_D^{\frac{j+1}{k} (1-\beta)} \\
 &\quad \times \|f\|_D^{\frac{2j+1}{2(k+1)}} \|\nabla^{k+1} f\|_D^{\frac{2k-2j}{2(k+1)}} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim \max \left\{ \mathcal{E}_{N_0-1, N_0-1-\frac{\hat{l}_{5j}}{\gamma}}(t), \mathcal{E}_{1,1}(t) \right\} \|\nabla^{k+1} f\|_D^2 + \varepsilon \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^2.
 \end{aligned} \tag{4.13}$$

Here $\frac{\gamma+2s}{2} \cdot \beta_j + \hat{l}_{5j}(1 - \beta_j) = 0$. If we set $\frac{j+1}{k} \beta_j = \frac{2j+1}{2(k+1)}$, then we can get that $\hat{l}_{5j} = \frac{\gamma+2s}{2} + \frac{-\gamma-2s}{2(1-\beta_j)} = \frac{\gamma+2s}{2} + \frac{(-\gamma-2s)(k+1)(j+1)}{2(k+1)(j+1)-k(2j+1)}$ and we take $\hat{l}_5 = \max_{j \leq k-1} \hat{l}_{5j}$.

When $j = k$,

$$\begin{aligned}
 I_{4,2} &\lesssim \sum_{j=k} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 L_x^2} \|f\|_{L_D^2 L_x^\infty} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim \sum_{j=k} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\langle v \rangle^{\frac{\gamma}{2} + s}\|^\theta \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\langle v \rangle^{\hat{l}_6}\|^{1-\theta} \|f\|_D^\theta \|\nabla^{k+1} f\|_D^{1-\theta} \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D \\
 &\lesssim \max \left\{ \mathcal{E}_{N_0-1, N_0-1-\frac{\hat{l}_6}{\gamma}}(t), \mathcal{E}_{1,1}(t) \right\} \left(\|\nabla^{k+1} f\|_D^2 + \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^2 \right) + \varepsilon \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^2.
 \end{aligned}$$

Here $\theta = \frac{2k-1}{2(k+1)}, \frac{\gamma+2s}{2} \cdot \theta + \hat{l}_6(1-\theta) = 0$ from Lemma 2.5 which deduce that $\hat{l}_6 = -\frac{(2k-1)(\gamma+2s)}{6}$.

Consequently,

$$\begin{aligned}
 I_4 &\lesssim \max \left\{ \mathcal{E}_{N_0-1, N_0-1-\frac{\max(\hat{l}_5, \hat{l}_6)}{\gamma}}(t), \mathcal{E}_{1,1}(t) \right\} \left(\|\nabla^{k+1} f\|_D^2 + \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^2 \right) \\
 &\quad + \varepsilon \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^2.
 \end{aligned} \tag{4.14}$$

Collecting the above estimate gives (4.1). When $k = N_0 - 1 \geq 2$, (4.6) tells us that

$$\begin{aligned}
 &\frac{d}{dt} \left(\|\nabla^{N_0-1} f\|^2 + \|\nabla^{N_0-1}(E, B)\|^2 \right) + \|\nabla^{N_0-1} \{\mathbf{I} - \mathbf{P}\} f\|_\sigma^2 \\
 &\lesssim \underbrace{\left| \left(\nabla^{N_0-1}(E \cdot v f), \nabla^{N_0-1} f \right) \right|}_{I_5} + \underbrace{\left| \left(\nabla^{N_0-1}(E \cdot \nabla_v f), \nabla^{N_0-1} f \right) \right|}_{I_6} \\
 &\quad + \underbrace{\left| \left(\nabla^{N_0-1}(v \times B \cdot \nabla_v f), \nabla^{N_0-1} f \right) \right|}_{I_7} + \underbrace{\left| \left(\nabla^{N_0-1} \Gamma(f, f), \nabla^{N_0-1} f \right) \right|}_{I_8}.
 \end{aligned} \tag{4.15}$$

To estimate I_5 , due to

$$\begin{aligned}
 I_5 &\lesssim \underbrace{\left| \left(E \cdot v \nabla^{N_0-1} f, \nabla^{N_0-1} f \right) \right|}_{I_{5,1}} + \underbrace{\left| \left(\nabla^{N_0-1} E \cdot v f, \nabla^{N_0-1} f \right) \right|}_{I_{5,2}} \\
 &\quad + \underbrace{\sum_{1 \leq j \leq N_0-2} \left| \left(\nabla^j E \cdot v \nabla^{N_0-1-j} f, \nabla^{N_0-1} f \right) \right|}_{I_{5,3}},
 \end{aligned} \tag{4.16}$$

From Lemma 2.5, using $L^\infty - L^2 - L^2$ Soblev inequality implies that

$$\begin{aligned}
 I_{5,1} &\lesssim \|E\|_{L_x^\infty} \|\nabla^{N_0-1} f\langle v \rangle^{1-\frac{\gamma+2s}{2}}\| \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \|\Lambda^{-\varrho} E\|^{1-\theta} \|\nabla^{N_0-1} E\|^\theta \|\nabla^{N_0-1} f\langle v \rangle^{\hat{l}_7}\|^\theta \|\nabla^{N_0-1} f\langle v \rangle^{\frac{\gamma+2s}{2}}\|^{1-\theta} \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \tilde{\mathcal{E}}_{N_0-1, N_0-1-\frac{\hat{l}_7}{\gamma}}(t) \left(\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \varepsilon \|\nabla^{N_0-1} f\|_D^2,
 \end{aligned}$$

where $\theta = \frac{3+2\varrho}{2(N_0-1+\varrho)}$ and $\hat{l}_7 = \frac{\gamma+2s}{2} + \frac{1-\gamma-2s}{\theta} = \frac{\gamma+2s}{2} + \frac{2(N_0-1+\varrho)(1-\gamma-2s)}{3+2\varrho}$,

$$\begin{aligned}
 I_{5,2} &\lesssim \|\nabla^{N_0-1} E\| \|f\langle v \rangle^{-\gamma/2-s+1}\|_{L_x^\infty} \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \mathcal{E}_{2, \frac{5}{2} + \frac{s-1}{\gamma}}(t) \|\nabla^{N_0-1} E\|^2 + \varepsilon \|\nabla^{N_0-1} f\|_D^2.
 \end{aligned}$$

Applying $L^6 - L^3 - L^2$ Soblev inequality yields that

$$\begin{aligned}
 I_{5,3} &\lesssim \sum_{1 \leq j \leq N_0-2} \|\nabla^j E\|_{L_x^6} \|\nabla^{N_0-1-j} f\langle v \rangle^{-\gamma/2-s+1}\|_{L_x^3} \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \sum_{1 \leq j \leq N_0-2} \|\Lambda^{-\varrho} E\|^{1-\theta_j} \|\nabla^{N_0-1} E\|^{\theta_j} \|\nabla^{N_0-1} f\langle v \rangle^{-\gamma/2-s+1}\|^{\alpha_j} \|f\langle v \rangle^{-\gamma/2-s+1}\|^{1-\alpha_j} \\
 &\quad \times \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \sum_{1 \leq j \leq N_0-2} \|\Lambda^{-\varrho} E\|^{1-\theta_j} \|\nabla^{N_0-1} E\|^{\theta_j} [\|\nabla^{N_0-1} f\langle v \rangle^{\hat{l}_{8j}}\|^{1-\beta_j} \|\nabla^{N_0-1} f\langle v \rangle^{\frac{\gamma}{2}+s}\|^{\beta_j}]^{\alpha_j} \\
 &\quad \times \|f\langle v \rangle^{-\gamma/2-s+1}\|^{1-\alpha_j} \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \sum_{1 \leq j \leq N_0-2} \|\Lambda^{-\varrho} E\|^{1-\theta_j} \|f\langle v \rangle^{-\gamma/2}\|^{1-\alpha_j} \|\nabla^{N_0-1} f\langle v \rangle^{\hat{l}_{8j}}\|^{\alpha_j-\alpha_j\beta_j} \|\nabla^{N_0-1} E\|^{\theta_j} \\
 &\quad \times \|\nabla^{N_0-1} f\langle v \rangle^{\frac{\gamma}{2}+s}\|^{\alpha_j\beta_j} \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \bar{\mathcal{E}}_{N_0-1, N_0-1-\frac{\hat{l}_8}{\gamma}}(t) \left(\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \varepsilon \|\nabla^{N_0-1} f\|_D^2.
 \end{aligned} \tag{4.17}$$

Here $\theta_j = \frac{j+\varrho+1}{N_0-1+\varrho}$ and $\alpha_j = \frac{2N_0-1-j}{2(N_0-1)}$ by using Lemma 2.5. Meanwhile, we set $\alpha_j\beta_j = 1 - \theta_j$ and \hat{l}_{8j} satisfies

$$\frac{\gamma + 2s}{2} \cdot \beta_j + \hat{l}_{8j}(1 - \beta_j) = -\frac{\gamma}{2} - s + 1,$$

it is easy to obtain that $\hat{l}_{8j} = \frac{1-\gamma-2s}{1-\beta_j} + \frac{\gamma+2s}{2}$ where $\beta_j = \frac{2(N_0-j-2)(N_0-1)}{(N_0-1+\varrho)(2N_0-1-j)}$. We take $\hat{l}_8 = \max_{1 \leq j \leq N_0-2} \{\hat{l}_{8j}\}$.

Consequently,

$$\begin{aligned}
 I_5 &\lesssim \max \left\{ \mathcal{E}_{2, \frac{5}{2} + \frac{s-1}{\gamma}}(t), \bar{\mathcal{E}}_{N_0-1, N_0-1-\frac{\max\{\hat{l}_7, \hat{l}_8\}}{\gamma}}(t) \right\} \left(\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) \\
 &\quad + \varepsilon \|\nabla^{N_0-1} f\|_D^2
 \end{aligned} \tag{4.18}$$

The third term I_7 on the right-hand side of (4.15) can be estimated as follows:

$$\begin{aligned}
 I_7 &\lesssim \underbrace{|(v \times \nabla^{N_0-1} B \cdot \nabla_v f, \nabla^{N_0-1} f)|}_{I_{7,1}} + \underbrace{|(v \times \nabla^{N_0-2} B \cdot \nabla_x \nabla_v f, \nabla^{N_0-1} f)|}_{I_{7,2}} \\
 &\quad + \underbrace{\sum_{1 \leq j \leq N_0-3} |(\nabla^j E \cdot \nabla^{N_0-1-j} \nabla_v f, \nabla^{N_0-1} f)|}_{I_{7,3}}.
 \end{aligned} \tag{4.19}$$

$I_{7,1}$ and $I_{7,2}$ on the right-hand side of (4.19) can be bounded by

$$\begin{aligned}
 I_{7,1} + I_{7,2} &\lesssim \|\nabla_v \nabla f\langle v \rangle^{1-\gamma/2-s}\|_{H^1} \|\nabla^{N_0-1} B\| \|\nabla^{N_0-1} f\|_D \\
 &\lesssim \mathcal{E}_{3, \frac{7}{2} + \frac{s-1}{\gamma}}(t) \|\nabla^{N_0-1} B\|^2 + \varepsilon \|\nabla^{N_0-1} f\|_D^2.
 \end{aligned} \tag{4.20}$$

$I_{7,3}$ on the right-hand side of (4.19) can be bounded by

$$\begin{aligned}
 I_{7,3} &\lesssim \sum_{1 \leq j \leq N_0-3} \left| \left(v \times \nabla^j B \cdot \nabla^{N_0-1-j} \nabla_v \mathbf{P} f, \nabla^{N_0-1} f \right) \right| \\
 &\quad + \sum_{1 \leq j \leq N_0-3} \left| \left(v \times \nabla^j B \cdot \nabla^{N_0-1-j} \nabla_v \{ \mathbf{I} - \mathbf{P} \} f, \nabla^{N_0-1} f \right) \right| \\
 &\lesssim \bar{\mathcal{E}}_{0,0}(t) \left(\left\| \nabla^{N_0-1} B \right\|^2 + \left\| \nabla^{N_0-1} f \right\|_D^2 \right) + \varepsilon \left\| \nabla^{N_0-1} f \right\|_D^2 \\
 &\quad + \sum_{1 \leq j \leq N_0-3} \left\| \Lambda^{-\varrho} B \right\|^{\frac{2N_0-2j-5}{2(N_0-1+\varrho)}} \left\| \nabla^{N_0-1} B \right\|^{\frac{2j+2\varrho+3}{2(N_0-1+\varrho)}} \left\| \nabla_v^{m_2j+1} \nabla^{N_0-1-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{1}{m_2j+1}} \\
 &\quad \times \left\| \nabla^{N_0-1-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_2j}{m_2j+1}} \left\| \langle v \rangle \nabla^{N_0-1} f \right\| \\
 &\lesssim \bar{\mathcal{E}}_{0,0}(t) \left(\left\| \nabla^{N_0-1} B \right\|^2 + \left\| \nabla^{N_0-1} f \right\|_D^2 \right) + \varepsilon \left\| \nabla^{N_0-1} f \right\|_D^2 \\
 &\quad + \sum_{1 \leq j \leq N_0-3} \left\| \Lambda^{-\varrho} B \right\|^{\frac{2N_0-2j-5}{2(N_0-1+\varrho)}} \left\| \nabla^{N_0-1} B \right\|^{\frac{2j+2\varrho+3}{2(N_0-1+\varrho)}} \left\| \nabla_v^{m_2j+1} \nabla^{N_0-1-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{1}{m_2j+1}} \\
 &\quad \times \left\| \Lambda^{-\varrho} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_2j}{(m_2j+1)(N_0-1+\varrho)}} \left\| \nabla^{N_0-1} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_2j(N_0-1+\varrho-j)}{(m_2j+1)(N_0-1+\varrho)}} \left\| \langle v \rangle \nabla^{N_0-1} f \right\| \\
 &\lesssim \bar{\mathcal{E}}_{0,0}(t) \left(\left\| \nabla^{N_0-1} B \right\|^2 + \left\| \nabla^{N_0-1} f \right\|_D^2 \right) + \varepsilon \left\| \nabla^{N_0-1} f \right\|_D^2 \\
 &\quad + \sum_{1 \leq j \leq N_0-3} \left\| \Lambda^{-s} B \right\|^{\frac{2N_0-2j-5}{2(N_0-1+\varrho)}} \left\| \nabla^{N_0-1} B \right\|^{\frac{2j+2\varrho+3}{2(N_0-1+\varrho)}} \left\| \nabla_v^{m_2j+1} \nabla^{N_0-1-j} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{1}{m_2j+1}} \\
 &\quad \times \left\| \Lambda^{-\varrho} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_2j}{(m_2j+1)(N_0-1+\varrho)}} \left\| \langle v \rangle \hat{I}_{\varrho_j} \nabla^{N_0-1} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_2j(N_0-1+\varrho-j)(1-\beta_j)}{(m_2j+1)(N_0-1+\varrho)}} \\
 &\quad \times \left\| \langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^{N_0-1} \{ \mathbf{I} - \mathbf{P} \} f \right\|^{\frac{m_2j(N_0-1+\varrho-j)\beta_j}{(m_2j+1)(N_0-1+\varrho)}} \left\| \langle v \rangle \hat{I}_{10_j} \nabla^{N_0-1} f \right\|^{1-\beta_j} \left\| \langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^{N_0-1} f \right\|^{\beta_j} \\
 &\lesssim \max \left\{ \mathcal{E}_{3, \frac{7}{2} + \frac{s-1}{\gamma}}(t), \mathcal{E}_{N_0-1+m_2, m_2+1}(t), \bar{\mathcal{E}}_{N_0-1, N_0-1-\frac{\hat{I}_{10}}{\gamma}}(t) \right\} \\
 &\quad \times \left(\left\| \nabla^{N_0-1} B \right\|^2 + \left\| \nabla^{N_0-1} f \right\|_D^2 \right) + \varepsilon \left\| \nabla^{N_0-1} f \right\|_D^2.
 \end{aligned} \tag{4.21}$$

Here we have used the fact that there exists a positive constant $\beta_j \in (0, 1)$ such that

$$\frac{2j + 2\varrho + 3}{2(N_0 - 1 + \varrho)} + \frac{m_2j(N_0 - 1 + \varrho - j)\beta_j}{(m_2j + 1)(N_0 - 1 + \varrho)} + \beta_j = 2$$

holds for $1 \leq j \leq N_0 - 3$. A necessary and sufficient condition to guarantee the existence of such a β is

$$\frac{2j + 2\varrho + 3}{2(N_0 - 1 + \varrho)} + \frac{m_2j(N_0 - 1 + \varrho - j)}{(m_2j + 1)(N_0 - 1 + \varrho)} + 1 > 2, \quad 1 \leq j \leq N_0 - 3,$$

from which one can deduce that $m_{2j} > \frac{2N_0-5-2j}{2\varrho+3}$ holds for $1 \leq j \leq N_0 - 3$. Noticing that $\frac{1}{2} < \varrho < \frac{3}{2}$, it is easy to see that we can take

$$m_2 = \max_{1 \leq j \leq k-1} m_{2j} = \frac{2N_0}{2\varrho+3} - \frac{6}{2\varrho+3} > \max_{1 \leq j \leq k-1} \frac{2N_0 - 5 - 2j}{2\varrho+3}.$$

Consequently, $N_0 - j + m_{2j} \leq N_0 + m_2 = \frac{2\varrho+5}{2\varrho+3}N_0 - \frac{6}{2\varrho+3}$ with $N_0 \geq 4$.

Moreover, since \hat{l}_{9j} and \hat{l}_{10j} satisfy $\frac{\gamma+2s}{2}\beta_j + \hat{l}_{9j}(1 - \beta_j) = 0$ and $\frac{\gamma+2s}{2}\beta_j + \hat{l}_{10j}(1 - \beta_j) = 1$ with $0 < \beta_j < 1$ respectively, one can deduce that $\hat{l}_{9j} = \frac{\gamma+2s}{2} - \frac{\gamma+2s}{2(1-\beta_j)}$ and $\hat{l}_{10j} = \frac{\gamma+2s}{2} - \frac{\gamma+2s-2}{2(1-\beta_j)}$, from which we can see that $\hat{l}_{10j} > \hat{l}_{9j}$ where

$$\beta_j = \frac{(m_{2j} + 1)(4N_0 + 2\varrho - 2j - 7)}{2(N_0 - 1 + \varrho + m_{2j}(2N_0 - 2 + 2\varrho - j))}.$$

Here we take $\hat{l}_{10} = \max_{1 \leq j \leq N_0-3} \{\hat{l}_{10j}\}$. Consequently,

$$I_7 \lesssim \max \left\{ \mathcal{E}_{N_0-1+m_2, m_2+1}(t), \bar{\mathcal{E}}_{N_0, N_0-\frac{\hat{l}_{10}}{\gamma}}(t) \right\} \left(\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \varepsilon \|\nabla^{N_0-1} f\|_D^2.$$

Applying the same trick as I_7 by replacing $v \times B$ with E , one can obtain

$$I_6 \lesssim \max \left\{ \mathcal{E}_{N_0-1+m_2, m_2+1}(t), \bar{\mathcal{E}}_{N_0, N_0-\frac{\hat{l}_{10}}{\gamma}}(t) \right\} \left(\|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \varepsilon \|\nabla^{N_0-1} f\|_D^2.$$

For the last term I_8 on the right hand side of (4.15), as (4.11), one has

$$\begin{aligned} I_8 &\lesssim |(\Gamma(f, \nabla^{N_0-1} f), \nabla^{N_0-1} f)| + |(\Gamma(\nabla^{N_0-1} f, f), \nabla^{N_0-1} f)| \\ &\quad + |(\Gamma(\nabla f, \nabla^{N_0-2} f), \nabla^{N_0-1} f)| + |(\Gamma(\nabla^{N_0-2} f, \nabla f), \nabla^{N_0-1} f)| \\ &\quad + \sum_{2 \leq j \leq N_0-3} |(\Gamma(\nabla^j f, \nabla^{N_0-1-j} f), \nabla^{N_0-1} f)| \\ &\lesssim \mathcal{E}_{N_0-1, N_0-1+\nu}(t) \left(\|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_D^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \varepsilon \|\nabla^{N_0-1} f\|_D^2. \end{aligned} \tag{4.22}$$

Collecting the above estimates gives (4.2). Now we turn to the last case $k = N_0 \geq 3$, we have

$$\begin{aligned} &\frac{d}{dt} (\|\nabla^{N_0} f\|^2 + \|\nabla^{N_0}(E, B)\|^2) + \|\nabla^{N_0} \{\mathbf{I} - \mathbf{P}\} f\|_D^2 \\ &\lesssim \underbrace{\left| (\nabla^{N_0}(E \cdot v f), \nabla^{N_0} f) \right|}_{I_9} + \underbrace{\left| (\nabla^{N_0}(E \cdot \nabla_v f), \nabla^{N_0} f) \right|}_{I_{10}} \\ &\quad + \underbrace{\left| (\nabla^{N_0}(v \times B \cdot \nabla_v f), \nabla^{N_0} f) \right|}_{I_{11}} + \underbrace{\left| (\nabla^{N_0} \Gamma(f, f), \nabla^{N_0} f) \right|}_{I_{12}}. \end{aligned} \tag{4.23}$$

For the first term I_9 on the right hand of the above inequality,

$$\begin{aligned}
 I_9 \lesssim & \underbrace{|(E \cdot v \nabla^{N_0} f, \nabla^{N_0} f)|}_{I_{9,1}} + \underbrace{|(\nabla E \cdot v \nabla^{N_0-1} f, \nabla^{N_0} f)|}_{I_{9,2}} + \underbrace{|(\nabla^{N_0-1} E \cdot v \nabla f, \nabla^{N_0} f)|}_{I_{9,3}} \\
 & + \underbrace{|(\nabla^{N_0} E \cdot v f, \nabla^{N_0} f)|}_{I_{9,4}} + \underbrace{\sum_{2 \leq j \leq N_0-2} |(\nabla^j E \cdot v \nabla^{N_0-j} f, \nabla^{N_0} f)|}_{I_{9,5}}.
 \end{aligned} \tag{4.24}$$

$I_{9,1}$ and $I_{9,2}$ can be bounded by

$$\begin{aligned}
 I_{9,1} + I_{9,2} & \lesssim \|E\|_{L^\infty} \|\nabla^{N_0} f\langle v \rangle^{-\frac{\gamma}{2}-s+1}\| \|\nabla^{N_0} f\|_D + \|\nabla E\|_{L^3} \|\nabla^{N_0-1} f\langle v \rangle^{-\frac{\gamma}{2}}\|_{L^6} \|\nabla^{N_0} f\|_D \\
 & \lesssim \|\Lambda^{-\varrho} E\|^{1-\theta} \|\nabla^{N_0-1} E\|^\theta \|\nabla^{N_0} f\langle v \rangle^{\hat{l}_{11}}\|^\theta \|\nabla^{N_0} f\langle v \rangle^{\frac{\gamma}{2}+s}\|^{1-\theta} \|\nabla^{N_0} f\|_D \\
 & \lesssim \bar{\mathcal{E}}_{N_0, N_0 - \frac{\hat{l}_{11}}{\gamma}}(t) (\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0} f\|_D^2) + \varepsilon \|\nabla^{N_0} f\|_D^2.
 \end{aligned}$$

Here we can deduce that $\theta = \frac{3+2\varrho}{2(N_0-1+\varrho)}$ from Lemma 2.5, and \hat{l}_{11} satisfies the following equation $\hat{l}_{11}\theta + \frac{\gamma+2s}{2}(1-\theta) = -\frac{\gamma}{2} - s + 1$ which deduces that $\hat{l}_{11} = \frac{\gamma+2s}{2} - \frac{\gamma+2s-1}{\theta} = \frac{\gamma+2s}{2} - \frac{2(N_0-1+\varrho)(\gamma+2s-1)}{3+2\varrho}$.

Using $L^2 - L^\infty - L^2$ gives

$$\begin{aligned}
 I_{9,3} & \lesssim \|\nabla^{N_0-1} E\| \|\nabla f\langle v \rangle^{1-\frac{\gamma}{2}-s}\|_{L^\infty} \|\nabla^{N_0} f\|_D \\
 & \lesssim \mathcal{E}_{3, \frac{7}{2} - \frac{1-s}{\gamma}}(t) \|\nabla^{N_0-1} E\|^2 + \varepsilon \|\nabla^{N_0} f\|_D^2.
 \end{aligned}$$

Applying the similar method as (4.17), the last term $I_{9,5}$ on the right-hand side of (4.24) can be dominated by

$$\begin{aligned}
 I_{9,5} & \lesssim \sum_{2 \leq j \leq N_0-2} \|\nabla^j E\|_{L^6} \|\nabla^{N_0-j} f\langle v \rangle^{-\frac{\gamma}{2}-s+1}\|_{L^3} \|\nabla^{N_0} f\|_D \\
 & \lesssim \sum_{2 \leq j \leq N_0-2} \|\Lambda^{-\varrho} E\|^{1-\theta_j} \|\nabla^{N_0-1} E\|^{\theta_j} \|\nabla^{N_0-1} f\langle v \rangle^{-\frac{\gamma}{2}-s+1}\|^{\alpha_j} \|f\langle v \rangle^{-\frac{\gamma}{2}-s+1}\|^{1-\alpha_j} \\
 & \quad \times \|\nabla^{N_0} f\|_D \\
 & \lesssim \sum_{2 \leq j \leq N_0-2} \|\Lambda^{-\varrho} E\|^{1-\theta_j} \|\nabla^{N_0-1} E\|^{\theta_j} [\|\nabla^{N_0-1} f\langle v \rangle^{\hat{l}_{12j}}\|^{1-\beta_j} \|\nabla^{N_0-1} f\langle v \rangle^{\frac{\gamma}{2}+s}\|^{\beta_j}]^{\alpha_j} \\
 & \quad \times \|f\langle v \rangle^{-\frac{\gamma}{2}-s+1}\|^{1-\alpha_j} \|\nabla^{N_0} f\|_D \\
 & \lesssim \sum_{2 \leq j \leq N_0-2} \|\Lambda^{-\varrho} E\|^{1-\theta_j} \|f\langle v \rangle^{-\frac{\gamma}{2}-s+1}\|^{1-\alpha_j} \|\nabla^{N_0-1} f\langle v \rangle^{\hat{l}_{12j}}\|^{\alpha_j - \alpha_j \beta_j} \|\nabla^{N_0-1} E\|^{\theta_j} \\
 & \quad \times \|\nabla^{N_0-1} f\langle v \rangle^{\frac{\gamma}{2}+s}\|^{\alpha_j \beta_j} \|\nabla^{N_0} f\|_D \\
 & \lesssim \bar{\mathcal{E}}_{N_0-1, N_0-1 - \frac{\hat{l}_{12}}{\gamma}}(t) (\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-1} f\|_D^2) + \varepsilon \|\nabla^{N_0} f\|_D^2.
 \end{aligned} \tag{4.25}$$

Here $\theta_j = \frac{j+1+\varrho}{N_0-1+\varrho}$ and $\alpha_j = \frac{2N_0-2j+1}{2(N_0-1)}$ by using Lemma 2.5. Meanwhile, we set $\alpha_j \beta_j = 1 - \theta_j$ and \hat{l}_{12j} satisfies

$$\frac{\gamma + 2s}{2} \cdot \beta_j + \hat{l}_{12j}(1 - \beta_j) = -\frac{\gamma}{2} - s + 1,$$

it is easy to obtain that $\hat{l}_{12j} = -\frac{\gamma+2s-1}{1-\beta_j} + \frac{\gamma+2s}{2}$ where $\beta_j = \frac{2(N_0-2-j)(N_0-1)}{(2N_0-2j+1)(N_0-1+\varrho)}$. Here we take $\hat{l}_{12} = \max_{2 \leq j \leq N_0-2} \hat{l}_{12j}$.

For the most important term $I_{9,4}$, if $n > \frac{2}{3}N_0 - \frac{5}{3}$, we have from Lemma 2.5 that

$$\begin{aligned}
 I_{9,4} &\lesssim \|\nabla^{N_0-1} E\|_{\frac{n}{n+1}} \|\nabla^{N_0+n}\|_{\frac{1}{n+1}} \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} f\|_{\frac{2N_0-5}{2N_0-2}} \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} \nabla^{N_0-1} f\|_{\frac{3}{2N_0-2}} \\
 &\quad \times \|\nabla^{N_0} f\|_D \\
 &\lesssim \|\nabla^{N_0-1} E\|_{\frac{n}{n+1}} \|\nabla^{N_0+n}\|_{\frac{1}{n+1}} \left[\|\langle v \rangle^{\hat{l}_{13}} \nabla^{N_0-1} f\|^{1-\alpha} \|\langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^{N_0-1} f\|^\alpha \right]_{\frac{3}{2N_0-2}} \\
 &\quad \times \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} f\|_{\frac{2N_0-5}{2N_0-2}} \|\nabla^{N_0} f\|_D \\
 &\lesssim \|\nabla^{N_0-1} E\|_{\frac{n}{n+1}} \|\nabla^{N_0+n}\|_{\frac{1}{n+1}} \|\langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^{N_0-1} f\|_{\frac{3\alpha}{2N_0-2}} \|\langle v \rangle^{\hat{l}_{13}} \nabla^{N_0-1} f\|_{\frac{3(1-\alpha)}{2N_0-2}} \\
 &\quad \times \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} f\|_{\frac{2N_0-5}{2N_0-2}} \|\nabla^{N_0} f\|_D \\
 &\lesssim \|\nabla^{N_0-1} E\|_{\frac{n}{n+1}} \|\nabla^{N_0+n}\|_{\frac{1}{n+1}} \|\langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^{N_0-1} f\|_{\frac{3\alpha}{2N_0-2}} \|\langle v \rangle^{\hat{l}_{13}} \nabla^{N_0-1} f\|_{\frac{3(1-\alpha)}{2N_0-2}} \\
 &\quad \times \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} f\|_{\frac{2N_0-5}{2N_0-2}} \|\nabla^{N_0} f\|_D \\
 &\lesssim \|\nabla^{N_0-1} E\|_{\frac{n}{n+1}} \|\nabla^{N_0+n}\|_{\frac{1}{n+1}} \|\langle v \rangle^{\frac{\gamma+2s}{2}} \nabla^{N_0-1} f\|_{\frac{1}{n+1}} \|\langle v \rangle^{\hat{l}_{13}} \nabla^{N_0-1} f\|_{\frac{3(1-\alpha)}{2N_0-2}} \\
 &\quad \times \|\langle v \rangle^{-\frac{\gamma}{2}-s+1} f\|_{\frac{2N_0-5}{2N_0-2}} \|\nabla^{N_0} f\|_D \\
 &\lesssim \max\{\mathcal{E}_{N_0+n}(t), \bar{\mathcal{E}}_{N_0-1, N_0-1-\frac{\hat{l}_{13}}{\gamma}}(t)\} \left(\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \varepsilon \|\nabla^{N_0} f\|_D^2.
 \end{aligned} \tag{4.26}$$

Here we require that $\frac{3\alpha}{2N_0-2} = \frac{1}{n+1}$ which deduces that $\alpha = \frac{2N_0-2}{3(n+1)}$. We set $\frac{\gamma+2s}{2} \cdot \alpha + \hat{l}_{13}(1-\alpha) = 1 - \frac{\gamma}{2} - s$ which yields that $\hat{l}_{13} = \frac{1-\frac{\gamma}{2}-s-\frac{\gamma+2s}{2}\alpha}{1-\alpha} = \frac{\gamma+2s}{2} + \frac{1-\gamma-2s}{1-\alpha} = \frac{\gamma+2s}{2} + \frac{3(1-\gamma-2s)(n+1)}{3n-2N_0+1}$.

Consequently, one has

$$\begin{aligned}
 I_9 &\lesssim \max \left\{ \mathcal{E}_{3, \frac{7}{2}-\frac{1-s}{\gamma}}(t), \mathcal{E}_{N_0+n}(t), \bar{\mathcal{E}}_{N_0-1, N_0-\frac{\max\{\hat{l}_{11}, \hat{l}_{12}, \hat{l}_{13}\}}{\gamma}}(t) \right\} \\
 &\quad \times \left(\|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \varepsilon \|\nabla^{N_0} f\|_D^2.
 \end{aligned}$$

For the second term I_{11} on the right-hand side of (4.23),

$$\begin{aligned}
 I_{11} &\lesssim \underbrace{\sum_{1 \leq j \leq N_0} \left| \left(v \times \nabla^j B \cdot \nabla_v \nabla^{N_0-j} \mathbf{P} f, \nabla^{N_0} f \right) \right|}_{I_{11,1}} + \underbrace{\left(v \times \nabla^{N_0-2} B \cdot \nabla_v \nabla_x^2 \{ \mathbf{I} - \mathbf{P} \} f, \nabla^{N_0} f \right)}_{I_{11,2}} \\
 &\quad + \underbrace{\left(v \times \nabla^{N_0-1} B \cdot \nabla_v \nabla_x \{ \mathbf{I} - \mathbf{P} \} f, \nabla^{N_0} f \right)}_{I_{11,3}} + \underbrace{\left(v \times \nabla^{N_0} B \cdot \nabla_v \{ \mathbf{I} - \mathbf{P} \} f, \nabla^{N_0} f \right)}_{I_{11,4}} \\
 &\quad + \underbrace{\sum_{1 \leq j \leq N_0-3} \left(v \times \nabla^j B \cdot \nabla_v \nabla^{N_0-j} \{ \mathbf{I} - \mathbf{P} \} f, \nabla^{N_0} f \right)}_{I_{11,5}}
 \end{aligned} \tag{4.27}$$

By the similar way as (4.24), $I_{11,1}$ can be bounded by

$$C \bar{\mathcal{E}}_{N_0+n}(t) \{ \|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0-1} f\|_D^2 + \|\nabla^{N_0} f\|_D^2 \} + \varepsilon \|\nabla^{N_0} f\|_D^2.$$

For the second term $I_{11,2}$ and $I_{11,3}$ on the right-hand side of (4.23),

$$\begin{aligned}
 I_{11,2} + I_{11,3} &\lesssim \|\nabla^{N_0-1} B\| \|\nabla_v \nabla_x \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 L_x^\infty} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\quad + \|\nabla^{N_0-2} B\|_{L_x^6} \|\nabla_v \nabla_x^2 \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 L_x^3} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0-1} B\| \|\nabla_v \nabla_x^2 \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0-1} B\| \|\nabla_v^2 \nabla_x^2 \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{1/2} \|\nabla_x^2 \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{1/2} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0-1} B\| \|\nabla_v^2 \nabla_x^2 \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{1/2} \|\Lambda^{-\varrho} \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{\frac{N_0-2+s}{2(N_0-1+s)}} \\
 &\quad \times \|\nabla_x^{N_0-1} \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{\frac{1}{2(N_0-1+s)}} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0-1} B\| \|\nabla_v^2 \nabla_x^2 \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{1/2} \|\Lambda^{-\varrho} \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{\frac{N_0-2+s}{2(N_0-1+s)}} \\
 &\quad \times \|\langle v \rangle \hat{I}_{14} \nabla_x^{N_0-1} \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{\frac{1-\beta}{2(N_0-1+s)}} \|\langle v \rangle^{\frac{\gamma}{2}+s} \nabla_x^{N_0-1} \{\mathbf{I} - \mathbf{P}\} f\|_{L_v^2 H_x^1}^{\frac{\beta}{2(N_0-1+s)}} \\
 &\quad \times \|\langle v \rangle \hat{I}_{15} \nabla^{N_0} f\|^{1-\beta} \|\langle v \rangle \hat{I}_{15} \nabla^{N_0} f\|^\beta \\
 &\lesssim \bar{\mathcal{E}}_{N_0, N_0 - \frac{\hat{I}_{15}}{\gamma}}(t) \{ \|\nabla^{N_0-1} \{\mathbf{I} - \mathbf{P}\} f\|_D^2 + \|\nabla^{N_0} f\|_D^2 \} + \varepsilon \|\nabla^{N_0-1} B\|^2
 \end{aligned}
 \tag{4.28}$$

Here we can get $\hat{I}_{14} = \frac{\gamma+2s}{2} - \frac{\gamma+2s}{2(1-\beta)}$ and $\hat{I}_{15} = \frac{\gamma+2s}{2} - \frac{\gamma+2s-2}{2(1-\beta_j)}$ from $\frac{\gamma+2s}{2} \cdot \beta + \hat{I}_{14}(1-\beta) = 0$ and $\frac{\gamma+2s}{2} \cdot \beta + \hat{I}_{15}(1-\beta) = 1$. Moreover we require that $\frac{\beta}{2(N_0-1+s)} + \beta = 1$ which yields that $\beta = \frac{2(N_0-1+s)}{2N_0-1+2s}$.

As the estimate on $I_{9,4}$, we have

$$\begin{aligned}
 I_{11,4} &\lesssim \|\nabla^{N_0+n} B\|^{\frac{1}{1+n}} \|\nabla^{N_0-1} B\|^{\frac{n}{1+n}} \|\nabla_v \{\mathbf{I} - \mathbf{P}\} f\|_{L_x^\infty L_v^2} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0+n} B\|^{\frac{1}{1+n}} \|\nabla^{N_0-1} B\|^{\frac{n}{1+n}} \|\nabla_x \nabla_v \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0+n} B\|^{\frac{1}{1+n}} \|\nabla^{N_0-1} B\|^{\frac{n}{1+n}} \|\nabla_x \nabla_v^{m_3+1} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{1}{1+m_3}} \\
 &\quad \times \|\nabla_x \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{m_3}{1+m_3}} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0+n} B\|^{\frac{1}{1+n}} \|\nabla^{N_0-1} B\|^{\frac{n}{1+n}} \|\nabla_x \nabla_v^{m_3+1} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{1}{1+m_3}} \\
 &\quad \times \|\Lambda^{-\varrho} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{m_3(N_0-3)}{(1+m_3)(N_0-2+\varrho)}} \|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{m_3(1+\varrho)}{(1+m_3)(N_0-2+\varrho)}} \|\langle v \rangle \nabla^{N_0} f\| \\
 &\lesssim \|\nabla^{N_0+n} B\|^{\frac{1}{1+n}} \|\nabla^{N_0-1} B\|^{\frac{n}{1+n}} \|\nabla_x \nabla_v^{m_3+1} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{1}{1+m_3}} \\
 &\quad \times \|\Lambda^{-\varrho} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{m_3(N_0-3)}{(1+m_3)(N_0-2+\varrho)}} \|\langle v \rangle \hat{I}_{16} \nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{m_3\beta(1+\varrho)}{(1+m_3)(N_0-2+\varrho)}} \\
 &\quad \times \|\langle v \rangle^{\frac{\gamma}{2}+s} \nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_v^2}^{\frac{m(1-\beta)(1+\varrho)}{(1+m_\gamma)(N_0-2+\varrho)}} \|\langle v \rangle \hat{I}_{17} \nabla^{N_0} f\|^{1-\beta} \|\langle v \rangle^{\frac{\gamma}{2}+s} \nabla^{N_0} f\|^\beta \\
 &\lesssim \max \left\{ \mathcal{E}_{N_0+n}(t), \mathcal{E}_{3+m_3, 3+m_3}(t), \bar{\mathcal{E}}_{N_0, N_0 - \frac{\hat{I}_{17}}{\gamma}}(t) \right\}
 \end{aligned}$$

$$\begin{aligned} & \times \left\{ \|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0-2}\{\mathbf{I} - \mathbf{P}\}f\|_{H_x^1 L_D^2}^2 \right\} \\ & + \varepsilon \left\{ \|\nabla^{N_0-2}\{\mathbf{I} - \mathbf{P}\}f\|_{H_x^1 L_D^2}^2 + \|\nabla^{N_0} f\|_D^2 \right\}. \end{aligned} \tag{4.29}$$

Here we need to ask $\frac{n}{1+n} + \frac{m_3(1+\varrho)\beta}{(1+m_3)(N_0-2+\varrho)} + \beta = 2$ which deduces that $m_3 > \frac{N_0-2+\varrho}{\varrho n+n+3-N_0}$, we can get $\hat{l}_{19j} = \frac{\gamma+2s}{2} - \frac{\gamma+2s}{2(1-\beta)}$ and $\hat{l}_{20} = \frac{\gamma+2s}{2} - \frac{\gamma+2s-2}{2(1-\beta)}$ from $\frac{\gamma+2s}{2} \cdot \beta + \hat{l}_{19j}(1-\beta) = 0$ and $\frac{\gamma+2s}{2} \cdot \beta + \hat{l}_{20}(1-\beta) = 1$. We can choose m_3 suitably such that $3+m_3 \leq 3 + \frac{N_0-2+\varrho}{(\varrho+1)n+3-N_0}$.

While for the case $1 \leq j \leq N_0 - 3$, one has

$$\begin{aligned} I_{11,5} & \lesssim \sum_{1 \leq j \leq N_0-3} \|\nabla^j B\|_{L_x^\infty} \|\nabla^{N_0-j} \nabla_v \{\mathbf{I} - \mathbf{P}\}f\| \|\langle v \rangle \nabla^{N_0} f\| \\ & \lesssim \sum_{1 \leq j \leq N_0-3} \|\Lambda^{-\varrho} B\|_{\frac{2(N_0-2j-5)}{2(N_0-1+\varrho)}} \|\nabla^{N_0-1} B\|_{\frac{2j+2\varrho+3}{2(N_0-1+\varrho)}} \|\nabla_v^{m_{4j}+1} \nabla^{N_0-j} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{1}{m_{4j}+1}} \\ & \quad \times \|\nabla^{N_0-j} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{m_{4j}}{m_{4j}+1}} \|\langle v \rangle \nabla^{N_0} f\| \\ & \lesssim \sum_{1 \leq j \leq N_0-3} \|\Lambda^{-\varrho} B\|_{\frac{2(N_0-2j-5)}{2(N_0-1+\varrho)}} \|\nabla^{N_0-1} B\|_{\frac{2j+2\varrho+3}{2(N_0-1+\varrho)}} \|\nabla_v^{m_{4j}+1} \nabla^{N_0-j} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{1}{m_{4j}+1}} \\ & \quad \times \|\Lambda^{-\varrho} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{m_{4j}(j-1)}{(m_{4j}+1)(N_0-1+\varrho)}} \|\nabla^{N_0-1} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{m_{4j}(N_0+\varrho-j)}{(m_{4j}+1)(N_0-1+\varrho)}} \|\langle v \rangle \nabla^{N_0} f\| \\ & \lesssim \sum_{1 \leq j \leq N_0-3} \|\Lambda^{-\varrho} B\|_{\frac{2(N_0-2j-5)}{2(N_0-1+\varrho)}} \|\nabla^{N_0-1} B\|_{\frac{2j+2\varrho+3}{2(N_0-1+\varrho)}} \|\nabla_v^{m_{4j}+1} \nabla^{N_0-j} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{1}{m_{4j}+1}} \\ & \quad \times \|\Lambda^{-\varrho} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{m_{4j}(j-1)}{(m_{4j}+1)(N_0-1+\varrho)}} \|\langle v \rangle \hat{l}_{18j} \nabla^{N_0-1} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{m_{4j}(N_0+\varrho-j)(1-\beta_j)}{(m_{4j}+1)(N_0-1+\varrho)}} \\ & \quad \times \|\langle v \rangle^{\frac{\gamma}{2}+s} \nabla^{N_0-1} \{\mathbf{I} - \mathbf{P}\}f\|_{\frac{m_{4j}(N_0+\varrho-j)\beta_j}{(m_{4j}+1)(N_0-1+\varrho)}} \|\langle v \rangle \hat{l}_{19j} \nabla^{N_0} f\|^{1-\beta_j} \|\langle v \rangle^{\frac{\gamma}{2}+s} \nabla^{N_0} f\|^{\beta_j} \\ & \lesssim \max \left\{ \mathcal{E}_{N_0+m_4, N_0+m_4}(t), \bar{\mathcal{E}}_{N_0, N_0-\frac{\hat{l}_{19}}{\gamma}}(t) \right\} \left\{ \|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0} f\|_D^2 \right\} \\ & \quad + \varepsilon \left\{ \|\nabla^{N_0-1} \{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\nabla^{N_0} f\|_D^2 \right\}. \end{aligned} \tag{4.30}$$

Here we have used the fact that there exist positive constants m_{8j} and $\beta_j \in (0, 1)$ such that $\frac{2j+2\varrho+3}{2(N_0-1+\varrho)} + \frac{m_{4j}(N_0+\varrho-j)\beta_j}{(m_{4j}+1)(N_0-1+\varrho)} + \beta_j = 2$ holds for $1 \leq j \leq N_0 - 3$. Similar to the way to determine of m_{1j} , it is sufficient to take

$$m_4 = \max_{1 \leq j \leq N_0-3} m_{4j} = \frac{2}{2\varrho+5} N_0 - \frac{6}{2\varrho+5} > \max_{1 \leq j \leq N_0-3} \frac{2N_0-5-2j}{2\varrho+5}.$$

Consequently, $N_0 - j + 1 + m_{4j} \leq N_0 + m_4 = \frac{2\varrho+7}{2\varrho+5} N_0 - \frac{6}{2\varrho+5}$ with $N_0 \geq 4$. Moreover, since \hat{l}_{18j} and \hat{l}_{19j} satisfy $\frac{\gamma+2s}{2} \beta_j + \hat{l}_{18j}(1-\beta_j) = 0$ and $\frac{\gamma+2s}{2} \beta_j + \hat{l}_{19j}(1-\beta_j) = 1$ with $0 < \beta_j < 1$ respectively, one can deduce that, $\hat{l}_{18j} = \frac{\gamma+2s}{2} - \frac{\gamma+2s}{2(1-\beta_j)}$ and $\hat{l}_{19j} = \frac{\gamma+2s}{2} - \frac{\gamma+2s-2}{2(1-\beta_j)}$ from which we can see that $\hat{l}_{19j} > \hat{l}_{18j}$ where

$$\beta_j = \frac{(m_{4j} + 1)(4N_0 + 2\varrho - 2j - 7)}{2(N_0 - 1 + \varrho + m_{4j}(2N_0 + 2\varrho - j - 1))}.$$

Here we take $\hat{l}_{19} = \max_{1 \leq j \leq N_0-3} \{\hat{l}_{19j}\}$.

Consequently, one has

$$\begin{aligned} I_{11} &\lesssim \max \left\{ \mathcal{E}_{N_0+n}(t), \mathcal{E}_{N_0+\max\{m_3, m_4\}, N_0+\max\{m_3, m_4\}}(t), \bar{\mathcal{E}}_{N_0, N_0 - \frac{\max\{\hat{l}_{15}, \hat{l}_{17}, \hat{l}_{19}\}}{\gamma}}(t) \right\} \\ &\quad \times \left\{ \|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_D^2 + \|\nabla^{N_0-1} f\|_D^2 + \|\nabla^{N_0} f\|_D^2 \right\} \quad (4.31) \\ &\quad + \varepsilon \left\{ \|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_D^2}^2 + \|\nabla^{N_0} f\|_D^2 \right\}. \end{aligned}$$

Applying the same trick as I_{11} , we have

$$\begin{aligned} I_{10} &\lesssim \max \left\{ \mathcal{E}_{N_0+n}(t), \mathcal{E}_{N_0+\max\{m_3, m_4\}, N_0+\max\{m_3, m_4\}}(t), \bar{\mathcal{E}}_{N_0, N_0 - \frac{\max\{\hat{l}_{15}, \hat{l}_{17}, \hat{l}_{19}\}}{\gamma}}(t) \right\} \\ &\quad \times \left\{ \|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_D^2 + \|\nabla^{N_0-1} f\|_D^2 + \|\nabla^{N_0} f\|_D^2 \right\} \quad (4.32) \\ &\quad + \varepsilon \left\{ \|\nabla^{N_0-1} E\|^2 + \|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_{H_x^1 L_D^2}^2 + \|\nabla^{N_0} f\|_D^2 \right\}. \end{aligned}$$

For the last term on the right-hand side of (4.23), by the similar way as (4.22), one has

$$I_{12} \lesssim \bar{\mathcal{E}}_{N_0, N_0+l'}(t) \left\{ \|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\} f\|_D^2 + \|\nabla^{N_0-1} f\|_D^2 + \|\nabla^{N_0} f\|_D^2 \right\} + \varepsilon \|\nabla^{N_0} f\|_D^2.$$

Thus collecting the above estimates gives (4.3). This completes the proof of Lemma 4.1. \square

The next lemma is concerned with the macro dissipation $\mathcal{D}_{N,mac}(t)$ defined by

$$\mathcal{D}_{N,mac}(t) \sim \|\nabla_x(a_{\pm}, b, c)\|_{H^{N-1}}^2 + \|a_+ - a_-\|^2 + \|E\|_{H^{N-1}}^2 + \|\nabla_x B\|_{H^{N-2}}^2.$$

Applying the argument of Lemma 3.2 in [15, p 3727] and Lemma 3.3 in [15, p 3731], we easily have the following lemma:

Lemma 4.2 *For the macro dissipation estimates on $f(t, x, v)$, we have the following results:*

(i) *For $k = 0, 1, 2, \dots, N_0 - 2$, there exists interactive energy functionals $G_f^k(t)$ satisfying*

$$G_f^k(t) \lesssim \|\nabla^k(f, E, B)\|^2 + \|\nabla^{k+1}(f, E, B)\|^2 + \|\nabla^{k+2} E\|^2$$

such that

$$\begin{aligned} &\frac{d}{dt} G_f^k(t) + \|\nabla^k(E, a_+ - a_-)\|_{H^1}^2 + \|\nabla^{k+1}(\mathbf{P}f, B)\|^2 \\ &\quad \lesssim \bar{\mathcal{E}}_{N_0-1,0}(t) \left(\|\nabla^{k+1}(E, B)\|^2 + \|\nabla^{k+1} f\|_D^2 \right) + \|\nabla^k \{\mathbf{I} - \mathbf{P}\} f\|_D^2 \quad (4.33) \\ &\quad + \|\nabla^{k+1} \{\mathbf{I} - \mathbf{P}\} f\|_D^2 + \|\nabla^{k+2} \{\mathbf{I} - \mathbf{P}\} f\|_D^2; \end{aligned}$$

(ii) *For $k = N_0 - 1$, there exists an interactive energy functional $G_f^{N_0-1}(t)$ satisfying*

$$G_f^{N_0-1}(t) \lesssim \|\nabla^{N_0-2}(f, E, B)\|^2 + \|\nabla^{N_0-1}(f, E, B)\|^2 + \|\nabla^{N_0}(f, E)\|^2$$

such that

$$\begin{aligned} & \frac{d}{dt} G_f^{N_0-1}(t) + \|\nabla^{N_0-2}(E, a_+ - a_-)\|_{H^1}^2 + \|\nabla^{N_0-1} B\|^2 + \|\nabla^{N_0} \mathbf{P}f\|^2 \\ & \lesssim \bar{\mathcal{E}}_{N_0,0}(t) \left(\|\nabla^{N_0-1}(E, B)\|^2 + \|\nabla^{N_0-1} f\|_D^2 \right) + \|\nabla^{N_0-2} \{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \quad + \|\nabla^{N_0-1} \{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\nabla^{N_0} \{\mathbf{I} - \mathbf{P}\}f\|_D^2; \end{aligned} \tag{4.34}$$

(iii) As [4], there exists an interactive energy functional $\mathcal{E}_N^{int}(t)$ satisfying

$$\mathcal{E}_N^{int}(t) \lesssim \sum_{|\alpha| \leq N} \|\partial^\alpha (f, E, B)\|^2$$

such that

$$\frac{d}{dt} \mathcal{E}_N^{int}(t) + \mathcal{D}_{N,mac}(t) \lesssim \sum_{|\alpha| \leq N} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \mathcal{E}_N(t) \mathcal{D}_N(t) \tag{4.35}$$

holds for any $t \in [0, T]$.

Appendix 2: The Estimates in the Negative Space

To close the first part of the a priori assumption (3.1), we have the following lemma: the first one is about the estimate of $\|(f, E, B)(t)\|_{\dot{H}^{-e}}$.

Lemma 5.1 *Under the assumptions stated above, we have for $\varrho \in (\frac{1}{2}, \frac{3}{2})$ that*

$$\begin{aligned} & \frac{d}{dt} (\|\Lambda^{-\varrho} f\|^2 + \|\Lambda^{-\varrho}(E, B)\|^2) + \|\Lambda^{-\varrho} \{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \lesssim (\bar{\mathcal{E}}_{0,0}(t))^{1/2} \left(\|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}}(E, B)\|^2 + \|\Lambda^{\frac{3}{4}-\frac{\varrho}{2}} f\|_D^2 \right) \\ & \quad + \bar{\mathcal{E}}_{2, \max\{\frac{3}{2} + \frac{\varrho-1}{\gamma}, 2\}}(t) \|\Lambda^{\frac{3}{2}-\varrho}(f, E, B)\|^2. \end{aligned} \tag{5.1}$$

Proof We have by taking Fourier transform of (1.4) with respect to x , multiplying the resulting identity by $|\xi|^{-2s} \hat{f}_\pm$ with \hat{f}_\pm which is the complex conjugate of \hat{f}_\pm , and integrating the final result with respect to ξ and v over $\mathbb{R}_\xi^3 \times \mathbb{R}_v^3$ that

$$\begin{aligned} & \left(\partial_t \hat{f}_\pm + v \cdot \mathcal{F}[\nabla_x f_\pm] \pm \mathcal{F}[(E + v \times B) \cdot \nabla_v f_\pm] \mp \frac{1}{2} v \cdot \mathcal{F}[E f_\pm] \mp \hat{E} \cdot v \mu^{\frac{1}{2}} \right. \\ & \quad \left. + \mathcal{F}[L_\pm f] - \mathcal{F}[\Gamma_\pm(f, f)] \right) |\xi|^{-2\varrho} \hat{f} = 0. \end{aligned} \tag{5.2}$$

Recalling that throughout this manuscript, $\mathcal{F}[g](t, \xi, v) = \hat{g}(t, \xi, v)$ denotes the Fourier transform of $g(t, x, v)$ with respect to x . (5.2) together with (2.1) yields

$$\begin{aligned} & \frac{d}{dt} \left(\|\Lambda^{-\ell} f\|^2 + \|\Lambda^{-\ell}(E, B)\|^2 \right) + \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_{\sigma}^2 \\ & \lesssim \underbrace{\sum_{\pm} \left| \left(\mathcal{F}[E \cdot \nabla_v f_{\pm}] \mid |\xi|^{-2\ell} \hat{f}_{\pm} \right) \right|}_{J_7} + \underbrace{\sum_{\pm} \left| \left(\mathcal{F}[v \times B \cdot \nabla_v f_{\pm}] \mid |\xi|^{-2\ell} \hat{f}_{\pm} \right) \right|}_{J_8} \\ & + \underbrace{\sum_{\pm} \left| \left(v \cdot \mathcal{F}[E f_{\pm}] \mid |\xi|^{-2\ell} \hat{f}_{\pm} \right) \right|}_{J_9} + \underbrace{\sum_{\pm} \left| \left(\mathcal{F}[\Gamma_{\pm}(f, f)] \mid |\xi|^{-2\ell} \hat{f}_{\pm} \right) \right|}_{J_{10}}. \end{aligned} \tag{5.3}$$

To estimate J_j ($j = 7, 8, 9$), we have from Lemmas 2.1, 2.5, and 2.6 that

$$\begin{aligned} J_7 & \lesssim |(\mathcal{F}[E \cdot \nabla_v f], |\xi|^{-2\ell} \mathcal{F}[\mathbf{P}f])| + |(\mathcal{F}[E \cdot \nabla_v f], |\xi|^{-2\ell} \mathcal{F}[\{\mathbf{I} - \mathbf{P}\}f])| \\ & \lesssim \|\Lambda^{-\ell}(E \cdot \mu^{\delta} f)\| \|\Lambda^{-\ell}(\mu^{\delta} f)\| + \|\Lambda^{-\ell}(E \nabla_v f(v)^{-\frac{\gamma}{2}-s})\| \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \|E \cdot \mu^{\delta} f\|_{L_x^{\frac{6}{3+2\ell}}} \|\Lambda^{-\ell}(\mu^{\delta} f)\| + \|E \nabla_v f(v)^{-\frac{\gamma}{2}-s}\|_{L_x^{\frac{6}{3+2\ell}}} \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \|E\|_{L_x^{\frac{12}{3+2\ell}}} \|\mu^{\delta} f\|_{L_x^{\frac{12}{3+2\ell}}} \|\Lambda^{-\ell}(\mu^{\delta} f)\| + (\|E\|_{L_x^{\frac{3}{\ell}}} \|\nabla_v f(v)^{-\frac{\gamma}{2}-s}\|)^2 \\ & \quad + \varepsilon \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \lesssim \|\Lambda^{\frac{3}{4}-\frac{\ell}{2}} E\| \|\Lambda^{\frac{3}{4}-\frac{\ell}{2}}(\mu^{\delta} f)\| \|\Lambda^{-\ell}(\mu^{\delta} f)\| + \|\Lambda^{\frac{3}{2}-\ell} E\|^2 \|\nabla_v f(v)^{-\frac{\gamma}{2}-s}\|^2 \\ & \quad + \varepsilon \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \lesssim (\bar{\mathcal{E}}_{0,0}(t))^{1/2} (\|\Lambda^{\frac{3}{4}-\frac{\ell}{2}} E\|^2 + \|\Lambda^{\frac{3}{4}-\frac{\ell}{2}} f\|_D^2) + \bar{\mathcal{E}}_{1, \frac{3}{2}+\frac{s}{\gamma}}(t) \|\Lambda^{\frac{3}{2}-\ell} E\|^2 \\ & \quad + \varepsilon \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D^2. \end{aligned}$$

For J_8 and J_9 , we have by repeating the argument used in deducing the estimate on J_7 that

$$\begin{aligned} J_8 + J_9 & \lesssim (\bar{\mathcal{E}}_{0,0}(t))^{1/2} \left(\|\Lambda^{\frac{3}{4}-\frac{\ell}{2}}(E, B)\|^2 + \|\Lambda^{\frac{3}{4}-\frac{\ell}{2}} f\|_D^2 \right) \\ & \quad + \bar{\mathcal{E}}_{1, \frac{3}{2}+\frac{s}{\gamma}}(t) \|\Lambda^{\frac{3}{2}-\ell}(E, B)\|^2 + \varepsilon \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D^2. \end{aligned} \tag{5.4}$$

J_{10} can be bounded from Lemma 2.3 by

$$\begin{aligned} J_{10} & = (\mathcal{F}[\Gamma(f, f)], |\xi|^{-2\ell} \mathcal{F}\{\mathbf{I} - \mathbf{P}\}f) \\ & \lesssim \|\Lambda^{-\ell}(\langle v \rangle^{-\gamma/2-s} \Gamma(f, f))\| \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \|\langle v \rangle^{-\gamma/2-s} \Gamma(f, f)\|_{L_x^p L_v^2} \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \left\| |\langle v \rangle^{-\gamma/2-s} f|_{L_{\gamma/2+s}^2} \mid \langle v \rangle^{-\gamma/2-s} f \mid_{H_{\gamma/2+s}^2} \right\|_{L_x^p} \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \|\langle v \rangle^{-\gamma/2-s} f\|_{L_x^{\frac{3}{\ell}} L_{\gamma/2+s}^2} \|\langle v \rangle^{-\gamma/2-s} f\|_{L_x^2 H_{\gamma/2+s}^2} \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \|f\|_{L_x^2 L_x^{\frac{3}{\ell}}} \|f\|_{L_x^2 H_v^2} \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D \\ & \lesssim \bar{\mathcal{E}}_{2,2}(t) \|\Lambda^{\frac{3}{2}-\ell} f\|^2 + \varepsilon \|\Lambda^{-\ell}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \end{aligned} \tag{5.5}$$

Substituting the estimates on $J_j (j = 7, 8, 9, 10)$ into (5.3) yields

$$\begin{aligned} & \frac{d}{dt} (\|\Lambda^{-e} f\|^2 + \|\Lambda^{-e}(E, B)\|^2) + \|\Lambda^{-e}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 \\ & \lesssim (\bar{\mathcal{E}}_{0,0}(t))^{1/2} \left(\|\Lambda^{\frac{3}{4}-\frac{e}{2}}(E, B)\|^2 + \|\Lambda^{\frac{3}{4}-\frac{e}{2}}f\|_D^2 \right) \\ & \quad + \bar{\mathcal{E}}_{2,\max\{\frac{3}{2}+\frac{e-1}{\nu}, 2\}}(t) \|\Lambda^{\frac{3}{2}-e}(f, E, B)\|^2. \end{aligned}$$

Thus we complete this proof of lemma 5.1. □

For the macro dissipation estimate on $\|\Lambda^{1-e}\mathbf{P}f\|^2$, applying the argument of Lemma 3.2 in [15, page3727] and Lemma 3.3 in [15, page3731], we easily have the following lemma:

Lemma 5.2 *Let $\varrho \in (\frac{1}{2}, \frac{3}{2})$, there exists an interactive functional $G_f^{-\varrho}(t)$ satisfying*

$$G_f^{-\varrho}(t) \lesssim \|\Lambda^{-\varrho} f\|^2 + \|\Lambda^{1-\varrho} f\|^2$$

such that

$$\begin{aligned} & \frac{d}{dt} G_f^{-\varrho}(t) + \|\Lambda^{1-\varrho}\mathbf{P}f\|^2 + \|\Lambda^{-\varrho}(a_+ - a_-)\|^2 \\ & \lesssim \|\Lambda^{-\varrho}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\Lambda^{1-\varrho}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t) \end{aligned} \tag{5.6}$$

holds for any $0 \leq t \leq T$. Moreover, there exists an interactive functional $G_{E,B}(t)$ satisfying

$$G_{E,B}(t) \lesssim \|\Lambda^{1-\varrho}(f, E, B)\|^2 + \|\Lambda^{-\varrho}(f, E, B)\|^2 + \|\Lambda^{2-\varrho}E\|^2 \tag{5.7}$$

such that

$$\begin{aligned} & \frac{d}{dt} G_{E,B}(t) + \|\Lambda^{1-\varrho}(E, B)\|^2 + \|\Lambda^{-\varrho}E\|^2 + \|\Lambda^{-\varrho}(a_+ - a_-)\|_{H^1}^2 \\ & \lesssim \|\Lambda^{-\varrho}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\Lambda^{1-\varrho}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \|\Lambda^{2-\varrho}\{\mathbf{I} - \mathbf{P}\}f\|_D^2 + \bar{\mathcal{E}}_{1,0}(t)\bar{\mathcal{D}}_{2,0}(t) \end{aligned} \tag{5.8}$$

holds for any $0 \leq t \leq T$.

References

1. Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., Yang, T.: Regularizing effect and local existence for non-cutoff Boltzmann equation. Arch. Ration. Mech. Anal. **198**(1), 39–123 (2010)
2. Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., Yang, T.: Global existence and full regularity of the Boltzmann equation without angular cutoff. Commun. Math. Phys. **304**(2), 513–581 (2011)
3. Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., Yang, T.: The Boltzmann equation without angular cutoff in the whole space: I, global existence for soft potential. J. Funct. Anal. **263**(3), 915–1010 (2012)
4. Duan, R.-J.: Global smooth dynamics of a fully ionized plasma with long-range collisions. Ann. Inst. H. Poincaré Anal. Non Linéaire **31**(4), 751–778 (2014)
5. Duan, R.-J., Liu, S.-Q., Yang, T., Zhao, H.-J.: Stability of the nonrelativistic Vlasov–Maxwell–Boltzmann system for angular non-cutoff potentials. Kinet. Relat. Models **6**(1), 159–204 (2013)
6. Duan, R.-J., Strain, R.M.: Optimal large-time behavior of the Vlasov–Maxwell–Boltzmann system in the whole space. Commun. Pure. Appl. Math. **24**(11), 1497–1546 (2011)
7. Gressman, P.T., Strain, R.M.: Global classical solutions of the Boltzmann equation without angular cut-off. J. Am. Math. Soc. **24**(3), 771–847 (2011)
8. Guo, Y.: The Vlasov–Poisson–Boltzmann system near Maxwellians. Commun. Pure. Appl. Math. **55**(9), 1104–1135 (2002)

9. Guo, Y.: The Vlasov–Maxwell–Boltzmann system near Maxwellians. *Invent. Math.* **153**(3), 593–630 (2003)
10. Guo, Y.: The Boltzmann equation in the whole space. *Indiana Univ. Math. J.* **53**, 1081–1094 (2004)
11. Guo, Y., Wang, Y.J.: Decay of dissipative equation and negative sobolev spaces. *Commun. Partial Differ. Equ.* **37**, 2165–2208 (2012)
12. Helander, P., Sigmar, D.J.: *Collisional Transport in Magnetized Plasmas*. Cambridge University Press, Cambridge, MA (2002)
13. Hosono, T., Kawashima, S.: Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system. *Math. Models Methods Appl. Sci.* **16**(11), 1839–1859 (2006)
14. Krall, N.A., Trivelpiece, A.W.: *Principles of Plasma Physics*. McGraw-Hill, New York (1973)
15. Lei, Y.-J., Zhao, H.-J.: Negative Sobolev spaces and the two-species Vlasov–Maxwell–Landau system in the whole space. *J. Funct. Anal.* **267**(10), 3710–3757 (2014)
16. Liu, T.-P., Yang, T., Yu, S.-H.: Energy method for the Boltzmann equation. *Physica D* **188**, 178–192 (2004)
17. Stein, E.M.: *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, NJ (1970)
18. Strain, R.M.: The Vlasov–Maxwell–Boltzmann system in the whole space. *Commun. Math. Phys.* **268**(2), 543–567 (2006)
19. Strain, R.M.: Optimal time decay of the non cut-off Boltzmann equation in the whole space. *Kinet. Relat. Models* **5**(3), 583–613 (2012)
20. Villani, C.: A review of mathematical topics in collisional kinetic theory. In: North-Holland, Amsterdam, *Handbook of Mathematical Fluid Dynamics*, vol. I, pp. 71–305 (2002)
21. Wang, Y.-J.: Global solution and time decay of the Vlasov–Poisson–Landau system in \mathbb{R}_x^3 . *SIAM J. Math. Anal.* **44**(5), 3281–3323 (2012)