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Quantum Algebra Symmetry of the ASEP with Second-Class Particles

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Abstract We consider a two-component asymmetric simple exclusion process (ASEP) on a finite lattice with reflecting boundary conditions. For this process, which is equivalent to the ASEP with second-class particles, we construct the representation matrices of the quantum algebra $U_q[\mathfrak{gl}(3)]$ that commute with the generator. As a byproduct we prove reversibility and obtain in explicit form the reversible measure. A review of the algebraic techniques used in the proofs is given.

Keywords Asymmetric simple exclusion process · Second class particles · Quantum algebras

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1 Introduction

The standard asymmetric simple exclusion process (ASEP) [12,13,21] defined on a finite lattice with reflecting boundary conditions is a reversible process with explicitly known invariant measures. They are not translation invariant, in contrast to the non-reversible uniform measures for periodic boundary conditions, and form the finite-size analogs of reversible blocking measures [12]. It has been shown in [18] that these measures can be constructed by using a non-Abelian symmetry property of the generator, viz. its commutativity with the generators of the quantum algebra $U_q[\mathfrak{gl}(2)]$.

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A related process of great interest is the ASEP with second class particles [9]. For periodic boundary conditions the invariant measures, which are non-reversible, can be computed in principle using the matrix product ansatz [6,7,17], or alternatively using methods from queuing theory [8]. However, they have a complicated non-uniform structure and no closed-form expression for the stationary probabilities as a function of the particle configurations is known.

The invariant measures for the ASEP with second class particles with reflecting boundary have not been studied yet. However, it has been known for a long time that the generator of this process has a quantum algebra symmetry, viz. its generator commutes with the generators of the quantum algebra $U_q[\mathfrak{gl}(3)]$ [1]. However, the corresponding representation matrices were never computed and the invariant measures for reflecting boundary conditions, which are expected to be reversible measures, have remained unknown. In this work we construct the matrix representations of the generators of $U_q[\mathfrak{gl}(3)]$ which commute with the generator of the ASEP with second-class particles. This approach provides a constructive method to obtain in explicit form reversible measures. We also review the algebraic tools required in the proofs.

2 Definitions and Notation for the Two-Component ASEP

2.1 State Space and Configurations

We consider the finite integer lattice $\Lambda := \{1, 2, \dots, L\}$ of L sites and local occupation variables $\eta(k) \in \mathbb{S} = \{0, 1, 2\}$. We say that a site $k \in \Lambda$ is occupied by a particle of type A (B) if $\eta(k) = 0(2)$ or that it is empty (represented by the symbol 0) if $\eta(k) = 1$. These local occupation variables define the configuration $\eta = \{\eta(1), \dots, \eta(L)\} \in \mathbb{S}^L$ of the particle system. The fact that a site can be occupied by at most one particle of any type is the exclusion principle.

The following functions of configurations η will play a role. For $1 \leq k \leq L$ we define the cyclic flip operation

$$\gamma^k(\eta) = \{\eta(1), \dots \eta(k-1), \eta(k) + 1 (\mathrm{mod}\ 3), \eta(k+1), \dots, \eta(L)\}. \tag{1}$$

We define $\eta^{k\pm} := (\gamma^k)^{\pm 1}(\eta)$ and observe that $(\gamma^k)^{-1} = (\gamma^k)^2$. For $1 \leq k \leq L-1$ We also define the local permutation

$$\sigma^{kk+1}(\eta) = \{\eta(1), \dots \eta(k-1), \eta(k+1), \eta(k), \eta(k+2), \dots, \eta(L)\} =: \eta^{kk+1}. \tag{2}$$

We also define local occupation number variables

$$a_k := \delta_{\eta(k),0}, \quad \upsilon_k := \delta_{\eta(k),1}, \quad b_k := \delta_{\eta(k),2}. \tag{3}$$

where $\delta_{k,l}$ is the usual Kronecker symbol with k, l from any set. In particular, we define the particle numbers

$$N(\eta) = \sum_{k=1}^{L} a_k, \quad M(\eta) = \sum_{k=1}^{L} b_k, \quad V(\eta) = \sum_{k=1}^{L} \upsilon_k. \tag{4}$$

Notice that $N(\eta) + M(\eta) + V(\eta) = L$. Occasionally we denote configurations with a fixed number N particles of type A and M particles of type B by $\eta_{N,M}$.[1]

[1] The occupation numbers can be formally regarded as families of mappings $a_k : \mathbb{S}^L \mapsto \{0, 1\}$, $b_k : \mathbb{S}^L \mapsto \{0, 1\}$ and should thus be understood as functions $a_k(\eta)$, $b_k(\eta)$ of η. Since the functional argument η will

Another useful way to specify a configuration η uniquely is by indicating the particle positions on the lattice. We write $\mathbf{z}(\eta) = \{\mathbf{x}, \mathbf{y}\}$ with $\mathbf{x} := \{x : \eta(x) = 0\}$, $\mathbf{y} := \{y : \eta(y) = 2\}$. We call this notation the position representation. Since the order of A-particles is conserved we may label them consecutively from left to right by 1 to N, and similarly we may label the B-particles by 1 to M. By the exclusion principle one has $\mathbf{x} \cap \mathbf{y} = \emptyset$ and by conservation of ordering $x_1 < x_2 < \cdots < x_N$, $1 \le y_1 < y_2 < \cdots < y_M \le L$. In multiple sums over x_i and/or y_i such sums will be understood as respecting all these exclusion constraints.[2] We note the trivial, but frequently used identities

$$N(\eta) \equiv N(\mathbf{z}) = |\mathbf{x}|, \quad M(\eta) \equiv M(\mathbf{z}) = |\mathbf{y}| \tag{5}$$

$$a_k = \sum_{i=1}^{N(\mathbf{z})} \delta_{x_i,k}, \quad b_k = \sum_{i=1}^{M(\mathbf{z})} \delta_{y_i,k}. \tag{6}$$

For a configuration $\eta \equiv \mathbf{z}$ we also define the number $N_k(\eta)$ of A-particles to the left of a particle at site k and analogously the number $M_k(\eta)$ of B-particles and vacancies $V_k(\eta)$ to the left of site k

$$N_k(\eta) := \sum_{i=1}^{k-1} a_i = \sum_{i=1}^{N(\eta)} \sum_{l=1}^{k-1} \delta_{x_i,l}, \quad M_k(\eta) := \sum_{i=1}^{k-1} b_i = \sum_{i=1}^{M(\eta)} \sum_{l=1}^{k-1} \delta_{y_i,l}. \tag{7}$$

Similarly we define $V_k(\eta) := \sum_{i=1}^{k-1} \upsilon_i$.

2.2 The Two-Component ASEP

Following [3] the two-component ASEP that we are going to study can be informally described as follows. Each bond $(k, k+1)$, $1 \le k \le L-1$ carries a clock which rings independently of all other clocks after an exponentially distributed random time with parameter τ_k where $\tau_k = wq$ if $\eta(k+1) > \eta(k)$, $\tau_k = wq^{-1}$ if $\eta(k+1) < \eta(k)$ and $\tau_k = \infty$ if $\eta(k) = \eta(k+1)$. When the clock rings the particle occupation variables are interchanged and the clock acquires the parameter corresponding to interchanged variables. Symbolically this process can be presented by the table of nearest neighbour particle jumps

$$\left.\begin{array}{l} A0 \to 0A \\ 0B \to B0 \\ AB \to BA \end{array}\right\} \text{with rate } wq, \quad \left.\begin{array}{l} 0A \to A0 \\ B0 \to 0B \\ BA \to AB \end{array}\right\} \text{with rate } wq^{-1}. \tag{8}$$

We consider reflecting boundary conditions, which means that no jumps from site 1 to the left and no jumps from site L to the right are allowed. We shall assume partially asymmetric hopping, i.e., $0 < q < \infty$. By interchanging the role of B-particles and vacancies this process turns into the ASEP with second-class particles [9].

More precisely, for functions $f : \mathbb{S}^L \to \mathbb{C}$ we define this Markov process η_t by the generator

$$\mathcal{L} f(\eta) := {\sum}'_{\eta' \in \mathbb{S}^L} w(\eta \to \eta')[f(\eta') - f(\eta)] \tag{9}$$

Footnote 1 continued
always be clear from context [as is the case e.g. in (4)], we do not write it explicitly. However, we shall usually write explicitly the argument for the particle number functions $N(\eta)$, $M(\eta)$ to contrast them with their numerical values N, M.

[2] We shall use interchangeably the arguments η, $\mathbf{z}$, $\{\mathbf{x}, \mathbf{y}\}$ for functions of the configurations. When the argument is clear from context it may be omitted.

with the transition rates

$$w(\eta \to \eta') = \sum_{k=1}^{L-1} w^{kk+1}(\eta)\delta_{\eta',\eta^{kk+1}} \tag{10}$$

defined in terms of the local hopping rates

$$w^{kk+1}(\eta) = wq\,(a_k\upsilon_{k+1} + \upsilon_k b_{k+1} + a_k b_{k+1}) + wq^{-1}\,(\upsilon_k a_{k+1} + b_k\upsilon_{k+1} + b_k a_{k+1}) \tag{11}$$

for a transition from a configuration η to a configuration $\eta' = \eta^{kk+1}$ defined by (2). The prime at the summation symbol (9) indicates the absence of the term $\eta' = \eta$ which is omitted since $w(\eta \to \eta)$ is not defined.[3]

We fix more notation and summarize some well-known basic facts from the theory of Markov processes. For a probability distribution $P(\eta)$ the expectation of a continuous measurable function $f(\eta)$ is denoted by $\langle\, f\,\rangle_P := \sum_\eta f(\eta)P(\eta)$. The transposed generator is defined by $\mathcal{L}^T f(\eta) := \sum'_{\eta'\in\mathbb{S}^L} f(\eta')\mathcal{L}\,\mathbf{1}_{\eta'}(\eta)$ where $\mathbf{1}_{\eta'}(\eta) = \delta_{\eta,\eta'}$. With this definition (9) yields for a probability distribution $P(\eta)$ the *master equation*

$$\mathcal{L}^T P(\eta) = \sum\nolimits'_{\eta'\in\mathbb{S}^L}[w(\eta' \to \eta)P(\eta') - w(\eta \to \eta')P(\eta)]. \tag{12}$$

The time-dependent probability distribution $P(\eta,t) := \mathrm{Prob}\,[\,\eta_t = \eta\,]$ follows from the semi-group property $P(\eta,t) = \mathrm{e}^{\mathcal{L}^T t}P_0(\eta)$ with initial distribution $P_0(\eta) := P(\eta,0)$. An invariant measure is denoted $\pi^*(\eta)$ and defined by

$$\mathcal{L}^T\pi^*(\eta) = 0, \quad \sum_\eta \pi^*(\eta) = 1. \tag{13}$$

A general stationary measure is denoted by π. It satisfies $\mathcal{L}^T\pi(\eta) = 0$, but no assumption on the normalization $\sum_\eta \pi(\eta)$ is made.

The time-reversed process is defined by

$$\mathcal{L}^{rev} f(\eta) := \sum_{\eta'}{}' w^{rev}(\eta \to \eta')[f(\eta') - f(\eta)] \tag{14}$$

with $w^{rev}(\eta \to \eta') = w(\eta' \to \eta)\pi(\eta')/\pi(\eta)$. The process is reversible if the rates satisfy the detailed balance condition $w^{rev}(\eta \to \eta') = w(\eta' \to \eta)$. We remark that

$$\pi(\eta)\mathcal{L}^{rev} f(\eta) = \mathcal{L}^T(\pi(\eta)f(\eta)) \tag{15}$$

which is a consequence of (13).

We define the transition matrix H of the process by the matrix elements

$$H_{\eta'\eta} = \begin{cases} -w(\eta \to \eta') & \eta \neq \eta' \\ \sum'_{\eta'} w(\eta \to \eta') & \eta = \eta'. \end{cases} \tag{16}$$

with $w(\eta \to \eta')$ given in (10). In slight abuse of language we shall also call H the generator of the process.

[3] We shall usually omit the set $\mathbb{S}^L$ in the summation symbol and simply write $\sum_\eta$.

2.3 The Quantum Algebra $U_q[\mathfrak{gl}(n)]$

The quantum algebra $U_q[\mathfrak{gl}(n)]$ is the q-deformed universal enveloping algebra of the Lie algebra $\mathfrak{gl}(n)$. This associative algebra over $\mathbb{C}$ is generated by $\mathbf{L}_i^{\pm 1}$, $i = 1, \ldots, n$ and $\mathbf{X}_i^{\pm}$, $i = 1, \ldots, n-1$ with the relations [4,10]

$$[\,\mathbf{L}_i\,,\,\mathbf{L}_j\,] = 0 \tag{17}$$

$$\mathbf{L}_i \mathbf{X}_j^{\pm} = q^{\pm(\delta_{i,j+1}-\delta_{i,j})/2} \mathbf{X}_j^{\pm} \mathbf{L}_i \tag{18}$$

$$[\,\mathbf{X}_i^{+}\,,\,\mathbf{X}_j^{-}\,] = \delta_{ij} \frac{(\mathbf{L}_{i+1}\mathbf{L}_i^{-1})^2 - (\mathbf{L}_{i+1}\mathbf{L}_i^{-1})^{-2}}{q - q^{-1}} \tag{19}$$

and, for $1 \le i, j \le n-1$, the quadratic and cubic Serre relations

$$[\,\mathbf{X}_i^{\pm}\,,\,\mathbf{X}_j^{\pm}\,] = 0 \quad \text{if } |i-j| \neq 1, \tag{20}$$

$$(\mathbf{X}_i^{\pm})^2 \mathbf{X}_j^{\pm} - (q + q^{-1}) \mathbf{X}_i^{\pm} \mathbf{X}_j^{\pm} \mathbf{X}_i^{\pm} + \mathbf{X}_j^{\pm} (X_i^{\pm})^2 = 0 \quad \text{if } |i-j| = 1. \tag{21}$$

Notice the replacement $q^2 \to q$ that we made in the definitions of [4].

3 Results

Before stating the results we introduce for $q,\ q^{-1} \neq 0$ and $x \in \mathbb{C}$ the symmetric q-number

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{22}$$

This definition extends straightforwardly to finite-dimensional matrices through the Taylor expansion of the exponential. We point out that $[-x]_q = -[x]_q$, $[x]_{q^{-1}} = [x]_q$ and $[x]_1 = x$. For integers one has the representation

$$[n]_q = \sum_{k=0}^{n-1} q^{2k-n+1} \tag{23}$$

and the q-factorial

$$[n]_q! := \begin{cases} 1 & n = 0 \\ \prod_{k=1}^{n} [k]_q & n \ge 1 \end{cases} \tag{24}$$

and the q-multinomial coefficients

$$C_L(N) = \frac{[L]_q!}{[N]_q![L-N]_q!}, \quad C_L(N, M) = \frac{[L]_q!}{[N]_q![M]_q![L-N-M]_q!}. \tag{25}$$

The first main result is a symmetry property of the generator.

Theorem 1 *Let H be the transition matrix (16) of the two-component ASEP defined by (9) with asymmetry parameter q and let $Y_i^{\pm}$, L_j, $i = 1, 2,\ j = 1, 2, 3$ be matrices with matrix elements*

$$(L_1)_{\eta'\eta} = q^{-N(\eta)/2} \delta_{\eta',\eta},\ (L_2)_{\eta'\eta} = q^{-V(\eta)/2} \delta_{\eta',\eta},\ (L_3)_{\eta'\eta} = q^{-M(\eta)/2} \delta_{\eta',\eta} \tag{26}$$

$$(Y_i^{\pm})_{\eta'\eta} = \sum_{k=1}^{L} (Y_i^{\pm}(k))_{\eta'\eta} \tag{27}$$

with

$$(Y_1^+(k))_{\eta'\eta} = q^{2V_k(\eta)-V_\eta} \upsilon_k \, \delta_{\eta',\eta^{k-}} \quad (Y_1^-(k))_{\eta'\eta} = q^{-2N_k(\eta)+N(\eta)} a_k \, \delta_{\eta',\eta^{k+}} \tag{28}$$
$$(Y_2^+(k))_{\eta'\eta} = q^{2M_k(\eta)-M(\eta)} b_k \, \delta_{\eta',\eta^{k-}} \quad (Y_2^-(k))_{\eta'\eta} = q^{-2V_k(\eta)+V_\eta} \upsilon_k \, \delta_{\eta',\eta^{k+}}. \tag{29}$$

and $\eta^{k\pm} = (\gamma^k)^{\pm 1}(\eta)$ *defined by* (1). *Then:*

(a) *The matrices* $Y_i^\pm,\ L_j,\ i = 1, 2,\ j = 1, 2, 3$ *form a representation of the quantum algebra* $U_q[\mathfrak{gl}(3)]$ (17)–(21).
(b) *The transition matrix* H *satisfies* $[\, H\ ,\ Y_i^\pm \,] = [\, H\ ,\ L_j \,] = 0$ *for* $i = 1, 2,\ j = 1, 2, 3$.

The second main result concerns reversibility.

Theorem 2 *The two-component exclusion process* η_t *defined by* (9) *with asymmetry parameter* q *is reversible with the reversible measure*

$$\pi(\eta) = q^{\sum_{k=1}^{L}(2k-L-1)(a_k-b_k)+\sum_{k=1}^{L-1}\sum_{l=1}^{k}(a_l b_{k+1}-b_l a_{k+1})}. \tag{30}$$

Remark 1 (i) In terms of position variables (6) we can write

$$\pi(\eta) = q^{\sum_{i=1}^{N(\eta)}[2x_i-L-1-M_{x_i}(\eta)]-\sum_{i=1}^{M(\eta)}[2y_i-L-1-N_{y_i}(\eta)]}. \tag{31}$$

(ii) In terms of conjugate occupation numbers $\bar{a}_k = 1 - a_k,\ \bar{b}_k = 1 - b_k$ we can use the identity $\sum_{k=1}^{L-1}\sum_{l=1}^{k}(x_{k+1} - x_l) = \sum_{k=1}^{L}(2k - L - 1)\, x_k$ to write

$$\pi(\eta) = q^{\sum_{k=1}^{L-1}\sum_{l=1}^{k}(\bar{a}_l\bar{b}_{k+1}-\bar{b}_l\bar{a}_{k+1})}. \tag{32}$$

(iii) For finite q one has

$$\pi(\eta) > 0 \quad \forall \eta \tag{33}$$

so that $\pi^{-1}(\eta)$ is finite.

4 Tools

Here we present a review of the tools that are used to prove the theorems. Some of these tools are not standard in the context of probability theory. The first subsection begins with simple facts included for the benefit of readers not familiar with the matrix representation of properties of a Markov chain [14,21]. The second subsection summarizes more advanced algebraic material from the theory of complete integrability of one-dimensional quantum systems.

4.1 Generator in Matrix Form

The defining equation (9) is linear and can therefore be written in matrix form using the transition matrix (16)

$$\mathcal{L} f(\eta) = -\sum_{\eta' \in \mathbb{S}^L} f(\eta') H_{\eta'\eta}. \tag{34}$$

Notice that the sum includes the term $\eta' = \eta$. In order to write the matrix H explicitly one has to choose an concrete basis, i.e., to each configuration η one assigns a canonical basis vector and defines the ordering $\iota(\eta)$ of the basis. The set of all basis vectors, which are denoted by $|\,\eta\,\rangle$, spans the complex vector space $\mathbb{C}^{|\mathbb{S}^L|}$. We work with a vector space over $\mathbb{C}$ rather than over $\mathbb{R}$ since in computations one may encounter eigenvectors and eigenvalues of H which may be complex since H is in general not symmetric.

Before defining a convenient ordering of the basis we make explicit the relation between the matrix representation (16) of the generator and the definition (9) of the process and rewrite in matrix form some of the Markov properties stated above.

4.1.1 Matrix Representation of the Markov Chain

Biorthogonal Basis, Inner Product and Tensor Product In our convention the basis vectors $|\,\eta\,\rangle$ of dimension $d = 3^L$ are represented as column vectors. We define also the row vector $\langle\,\eta\,| = |\,\eta\,\rangle^T$ with the biorthogonality property

$$\langle\,\eta\,|\,\eta'\,\rangle = \delta_{\eta\eta'}. \tag{35}$$

The superscript T on vectors or matrices denotes transposition.

Consider for an arbitrary set of states Ω of cardinality d and associated vector space $\mathfrak{V}$ of dimension d two vectors $\langle\,w\,|$ with components $w_i \in \mathbb{C}$ and $|\,v\,\rangle$ with components $v_i \in \mathbb{C}$. We define the inner product by

$$\langle\,w\,|\,v\,\rangle = \sum_{i=1}^{d} w_i v_i \tag{36}$$

without complex conjugation.

The tensor product $|\,v\,\rangle\langle\,w\,| \equiv |\,v\,\rangle \otimes \langle\,w\,|$ is a $d \times d$-matrix with matrix elements $(|\,v\,\rangle\langle\,w\,|)_{i,j} = v_i w_j$. This notation follows a convenient convention borrowed from quantum mechanics. Specifically, we have the representation

$$\mathbf{1} = \sum_{\eta\in\Omega} |\,\eta\,\rangle\langle\,\eta\,| \tag{37}$$

of the d-dimensional unit matrix. A function $\phi : \Omega \mapsto \Omega$ is represented by a non-diagonal matrix

$$\hat{\phi} := \sum_{\eta} |\,\phi\,\rangle\langle\,\eta\,| \tag{38}$$

which is an endomorphism $\mathfrak{V} \mapsto \mathfrak{V}$ satisfying $\hat{\phi}|\,\eta\,\rangle = |\,\phi\,\rangle$. We recall that for two tensor vectors $\langle\,W\,| = \langle\,w_1\,|\otimes\cdots\otimes\langle\,w_L\,|, |\,V\,\rangle = |\,v_1\,\rangle\otimes\cdots\otimes|\,v_L\,\rangle$ the inner product of factorizes:

$$\langle\,W\,|\,V\,\rangle = \prod_{k=1}^{L} \langle\,w_k\,|\,v_k\,\rangle. \tag{39}$$

Generator As a consequence of biorthogonality of the basis one has $H_{\eta'\eta} = \langle\,\eta'\,|H|\,\eta\,\rangle$ and therefore (9) takes the form

$$\mathcal{L}f(\eta) = -\langle\,f\,|H|\,\eta\,\rangle \tag{40}$$

where the row vector $\langle\,f\,| = \sum_\eta f(\eta)\langle\,\eta\,|$ has components $f(\eta)$. A probability measure $\mathrm{Prob}\,[\,\eta_t = \eta\,] \equiv P(\eta,t)$ is represented by the column vector

$$| P(t) \rangle = \sum_{\eta} P(\eta, t) | \eta \rangle. \tag{41}$$

The semigroup property of the Markov chain is reflected in the time-evolution equation

$$| P(t) \rangle = \mathrm{e}^{-Ht} | P_0 \rangle \tag{42}$$

of a probability measure $P_0(\eta) \equiv P(\eta_0)$.

Lowest Eigenvalue and Eigenvector We define the *summation vector*

$$\langle s | := \sum_{\eta} \langle \eta | \tag{43}$$

which is the row vector where all components are equal to 1. Normalization implies

$$\langle s | P(t) \rangle = 1 \quad \forall t \geq 0. \tag{44}$$

By taking the time-derivative one has as a consequence

$$\langle s | H = 0 \tag{45}$$

which means that the summation vector is a left eigenvector of H with eigenvector 0. This property follows from the fact that a diagonal element of $H_{\eta\eta}$ is by construction the sum of all transition rates that appear with negative sign in the same column η of H.

A stationary measure, denoted by $| \pi \rangle$, is a right eigenvector of H with eigenvalue 0, i.e.,

$$H | \pi \rangle = 0. \tag{46}$$

By the Perron-Frobenius theorem 0 is the eigenvalue of H with the lowest real part. The probability vector corresponding to a normalized stationary measure (44) is denoted by $| \pi^* \rangle$.

For the stationary distribution we define the diagonal matrices

$$\hat{\pi} := \sum_{\eta} \pi(\eta) | \eta \rangle \langle \eta |, \quad \hat{\pi}^* := \sum_{\eta} \pi^*(\eta) | \eta \rangle \langle \eta |. \tag{47}$$

For ergodic processes with finite state space one has $0 < \pi^*(\eta) \leq 1$ for all η. Then all powers $(\hat{\pi}^*)^{\alpha}$ exist. In terms of these diagonal matrices we can write the generator of the reversed dynamics as

$$H^{rev} = \hat{\pi} H^T \hat{\pi}^{-1} = \hat{\pi}^* H^T (\hat{\pi}^*)^{-1}. \tag{48}$$

This is the matrix form of (15).

Expectation Values The expectation $\langle f \rangle_P$ of a function $f : \mathfrak{V} \mapsto \mathbb{C}$ with respect to a probability distribution $P(\eta)$ is the inner product

$$\langle f \rangle_P = \langle f | P \rangle = \langle s | \hat{f} | P \rangle \tag{49}$$

where

$$\hat{f} := \sum_{\eta} f(\eta) | \eta \rangle \langle \eta | \tag{50}$$

is a diagonal matrix with diagonal elements $f(\eta)$. Notice that

$$f(\eta) = \langle \eta \,|\hat{f}|\, \eta \rangle = \langle s \,|\hat{f}|\, \eta \rangle. \tag{51}$$

4.1.2 The Tensor Basis

In order to define a convenient ordering of the basis for H we choose the tensor approach advocated in [14,21] for interacting particle systems.

Only One Site We begin with the basis for a single site where $\eta \in \mathbb{S}$. For a row vector of dimension 3 with components w_i we write $(w| = (w_1, w_2, w_3)$ and column vectors of dimension 3 with components v_i we write $|v) = (v|^T$. We define the inner product $(w|v) := w_1 v_1 + w_2 v_2 + w_3 v_3$. We choose for a single site the order $\iota_1(\eta) = 1 + \eta$ and denote the corresponding canonical basis vectors of $\mathbb{C}^3$

$$|\mathbf{A}) \equiv |1) := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\mathbf{0}) \equiv |2) := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\mathbf{B}) \equiv |3) := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{52}$$

With the dual basis $(\eta| = |\eta)^T$ we have the biorthogonality relation $(\eta|\eta') = \delta_{\eta,\eta'}$ The local summation vector is given by $(s| := (\mathbf{A}| + (\mathbf{0}| + (\mathbf{B}| = (1, 1, 1)$.

It is useful to introduce the following matrices with the convention $|\eta)(\eta'| \equiv |\eta) \otimes (\eta'|$:

$$a^+ := |\mathbf{A})(\mathbf{0}|, \quad b^+ := |\mathbf{B})(\mathbf{0}|, \quad c^+ := |\mathbf{A})(\mathbf{B}|, \tag{53}$$

$$a^- := |\mathbf{0})(\mathbf{A}|, \quad b^- := |\mathbf{0})(\mathbf{B}|, \quad c^- := |\mathbf{B})(\mathbf{A}|. \tag{54}$$

Having in mind the action of these operators to the right on a column vector, we call a^+ and b^+ creation operators, a^- and b^- are called annihilation operators and $c^\pm$ are particle exchange operators.

We also define the projectors

$$\hat{a} := |\mathbf{A})(\mathbf{A}|, \quad \hat{v} := |\mathbf{0})(\mathbf{0}|, \quad \hat{b} := |\mathbf{B})(\mathbf{B}|. \tag{55}$$

and the three-dimensional unit matrix

$$\mathbb{1} = \sum_\eta |\eta)(\eta| = \hat{a} + \hat{v} + \hat{b}. \tag{56}$$

They satisfy the following relations:

$$a^+\hat{a} = b^+\hat{a} = b^-\hat{a} = c^+\hat{a} = 0, \quad a^-\hat{a} = a^-, \quad c^-\hat{a} = c^- \tag{57}$$

$$a^-\hat{v} = b^-\hat{v} = c^+\hat{v} = c^-\hat{v} = 0, \quad a^+\hat{v} = a^+, \quad b^+\hat{v} = b^+ \tag{58}$$

$$a^+\hat{b} = a^-\hat{b} = b^+\hat{b} = c^-\hat{b} = 0, \quad b^-\hat{b} = b^-, \quad c^+\hat{b} = c^+ \tag{59}$$

and

$$\hat{a}a^- = \hat{a}b^+ = \hat{a}b^- = \hat{a}c^- = 0, \quad \hat{a}a^+ = a^+, \quad \hat{a}c^+ = c^+ \tag{60}$$

$$\hat{v}a^+ = \hat{a}b^+ = \hat{v}c^+ = \hat{v}c^- = 0, \quad \hat{v}a^- = a^-, \quad \hat{v}b^- = b^- \tag{61}$$

$$\hat{b}a^+ = \hat{b}a^- = \hat{b}b^- = \hat{b}c^+ = 0, \quad \hat{b}b^+ = b^+, \quad \hat{b}c^- = c^-. \tag{62}$$

With the occupation variables (3) for a single site we have the projector properties

$$\hat{a}|\eta) = a|\eta), \quad \hat{b}|\eta) = b|\eta), \quad \hat{v}|\eta) = v|\eta) \tag{63}$$

for $\eta \in \{0, 1, 2\}$ and $a \equiv a(\eta) = \delta_{\eta,0}$ and so on. Moreover,

$$(s|a^+ = (s|\hat{v}, \quad (s|b^+ = (s|\hat{v}, \quad (s|c^+ = (s|\hat{b}, \tag{64}$$

$$(s|a^- = (s|\hat{a}, \quad (s|b^- = (s|\hat{b}, \quad (s|c^- = (s|\hat{a}. \tag{65}$$

L Sites For a configuration $\eta \in \mathbb{S}^L$ it is natural and indeed convenient to choose the ternary ordering $\iota_L(\eta) = 1 + \sum_{k=1}^{L} \eta(k)3^{k-1}$ of the basis vectors. This corresponds to the tensor basis defined by

$$|\,\eta\,\rangle := |\eta_1) \otimes |\eta_2) \otimes \cdots \otimes |\eta_L), \quad \langle\,\eta\,| := (\eta_1| \otimes (\eta_2| \otimes \cdots \otimes (\eta_L| \tag{66}$$

spanning the vector space $(\mathbb{C}^3)^{\otimes L}$ of dimension 3^L. We shall also use the notations $|\,\mathbf{z}\,\rangle$ and $|\,\mathbf{x}, \mathbf{y}\,\rangle$ instead of $|\,\eta\,\rangle$. Furthermore, if a configuration η has N particles of type A and M particles of type B we may denote this fact by writing $|\,\eta_{N,M}\,\rangle$ for the corresponding basis vector.

The summation vector $\langle\,s\,|$ is given by

$$\langle\,s\,| := (s|^{\otimes L}. \tag{67}$$

This is the row vector of dimension 3^L where all components are equal to 1. The summation vector restricted to configurations with a fixed number N of particles of type A and M particles of type B is denoted by

$$\langle\,s_{N,M}\,| = \sum_{\eta_{N,M}} \langle\,\eta_{N,M}\,|. \tag{68}$$

The basis vector corresponding to the empty lattice is denoted by $|\,\emptyset\,\rangle := |\,\eta_{0,0}\,\rangle = \langle\,s_{0,0}\,|^T$.

For matrices M the expression $M^{\otimes j}$ will denote the j-fold tensor product of M withself if $j > 1$. For $j = 1$ we define $M^{\otimes 1} = M$ and for $j = 0$ we define $M^{\otimes 0} = 1$ with the c-number 1. For arbitrary 3×3-matrices u we define tensor operators

$$u_k := \mathbb{1}^{\otimes(k-1)} \otimes u \otimes \mathbb{1}^{\otimes(L-k)}. \tag{69}$$

Multilinearity of the tensor product yields $u_k v_{k+1} = \mathbb{1}^{\otimes(k-1)} \otimes [(u \otimes \mathbb{1})(\mathbb{1} \otimes v)] \otimes \mathbb{1}^{\otimes(L-k-1)} = \mathbb{1}^{\otimes(k-1)} \otimes (u \otimes v) \otimes \mathbb{1}^{\otimes(L-k-1)}$ and the commutator property $u_k v_l = v_l u_k$ for $k \neq l$.

For $u = \hat{a}$ or $\hat{b}$ we note the projector lemma [3] which will be used repeatedly below.

Lemma 1 *The tensor occupation operators $\hat{a}_k$, $\hat{b}_k$ act as projectors*

$$\hat{a}_k|\,\eta\,\rangle = a_k|\,\eta\,\rangle = \sum_{i=1}^{N(\eta)} \delta_{x_i,k}|\,\eta\,\rangle, \quad \hat{b}_k|\,\eta\,\rangle = b_k|\,\eta\,\rangle = \sum_{i=1}^{M(\eta)} \delta_{y_i,k}|\,\eta\,\rangle \tag{70}$$

with the occupation variables a_k and b_k (3) (or particle coordinates x_i and y_i respectively) understood as functions of η or $\mathbf{z} = \eta$.

One obtains the diagonal matrix $\hat{f}$ (50) of a function $f(\eta)$ by the following simple recipe: In $f(\eta)$ one substitutes η_i by the diagonal matrix $\hat{\eta}_i$ where $\hat{\eta}_i = \hat{a}_i$ if $\eta_i = 0$, $\hat{\eta}_i = \hat{v}_i$ if $\eta_i = 1$, and $\hat{\eta}_i = \hat{b}_i$ if $\eta_i = 2$.

The function $\gamma^k(\eta)$ defined in (1) is represented by the matrix

$$\hat{\gamma}_k := a_k^- + b_k^+ + c_k^+. \tag{71}$$

Multilinearity of the tensor product ensures that $\hat{\gamma}_k|\,\eta\,\rangle = |\,\eta^{k+}\,\rangle$ as defined in (1).

For $L = 2$ we define the permutation operator

$$Q = \hat{a}_1\hat{a}_2 + \hat{b}_1\hat{b}_2 + \hat{\upsilon}_1\hat{\upsilon}_2 + a_1^-a_2^+ + a_1^+a_2^- + b_1^+b_2^- + b_1^-b_2^+ + c_1^-c_2^+ + c_1^+c_2^-. \quad (72)$$

Embedding two neighbouring sites into a lattice of L sites yields for $1 \leq k \leq L-1$ the nearest-neighbour permutation operator

$$\hat{\pi}_{k,k+1} := \mathbb{1}^{\otimes(k-1)} \otimes Q \otimes \mathbb{1}^{\otimes(L-k-1)}. \quad (73)$$

One readily verifies that $\hat{\pi}_{k,k+1}|\,\eta\,\rangle = |\,\eta^{k,k+1}\,\rangle$ as defined in (2).

4.1.3 Construction of the Generator in the Tensor Basis

Consider first $L = 2$ and denote the transition matrix (16) for two sites by h. From the definition of the process one computes the off-diagonal part by observing that $|\,0A\,\rangle = |\,\pi^{12}(\{A0\})\,\rangle = (a^-|\mathbf{A})) \otimes (a^+|\mathbf{0})) = (a^- \otimes a^+)(|\mathbf{A}) \otimes |\mathbf{0})) = a_1^-a_2^+|\,A0\,\rangle$ and therefore $h_{\{0A\}\{A0\}} = qw\langle\,0A\,|a_1^-a_2^+|\,A0\,\rangle$ for the transition $A0 \to 0A$, and so on. The corresponding diagonal elements $h_{\eta\eta}$ follow from (11) with the substitution of the occupation variables by the respective projectors according to Lemma (1). With the tensor basis (66) for two sites one thus obtains the 9×9-matrix

$$h = w\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & -q^{-1} & 0 & 0 \\ 0 & -q & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & -q^{-1} & 0 \\ 0 & 0 & -q & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (74)$$

Next we embed this process on two neighbouring sites $(k, k+1)$ in Λ. By the multilinearity of the tensor product the generator becomes $h_{k,k+1} = \mathbb{1}^{\otimes(k-1)} \otimes h \otimes \mathbb{1}^{\otimes(L-k-1)}$ where the hopping matrices

$$\begin{aligned} h_{k,k+1} := wq\left(\hat{a}_k\hat{\upsilon}_{k+1} - a_k^-a_{k+1}^+ + \hat{\upsilon}_k\hat{b}_{k+1} - b_k^+b_{k+1}^- + \hat{a}_k\hat{b}_{k+1} - c_k^-c_{k+1}^+\right) \\ +wq^{-1}\left(\hat{\upsilon}_k\hat{a}_{k+1} - a_k^+a_{k+1}^- + \hat{b}_k\hat{\upsilon}_{k+1} - b_k^-b_{k+1}^+ + \hat{b}_k\hat{a}_{k+1} - c_k^+c_{k+1}^-\right) \end{aligned} \quad (75)$$

act non-trivially only on the subspace corresponding to sites $k, k+1$ in the tensor space. The off-diagonal elements of $h_{k,k+1}$ are the transition rates $h_{\eta^{kk+1}\eta}$ in the tensor basis (66). The diagonal elements of $h_{k,k+1}$ defined in (16) follow from probability conservation. The generator for the two-component ASEP on the lattice $\{1, \ldots, L\}$ then follows as

$$H = \sum_{k=1}^{L-1} h_{k,k+1}. \quad (76)$$

As will be seen below, the generator H is closely related to the Hamiltonian operator of a quantum spin chain. This is true for other interacting particle systems and motivates the terms “Quantum spin techniques” [14] or “quantum Hamiltonian formalism” [21] for the representation of the generator of an interacting particle system in the tensor basis.

Remark 2 Because of multilinearity (64), (65) are lifted to $\langle\, s\,| a_k^+ = \langle\, s\,|\hat{\upsilon}_k$, $\langle\, s\,| b_k^+ = \langle\, s\,|\hat{\upsilon}_k$, $\langle\, s\,| c_k^+ = \langle\, s\,|\hat{b}_k$ and $\langle\, s\,| a_k^- = \langle\, s\,|\hat{a}_k$, $\langle\, s\,| b_k^- = \langle\, s\,|\hat{b}_k$, $\langle\, s\,| c_k^- = \langle\, s\,|\hat{a}_k$. This yields $\langle\, s\,| h_{k,k+1} = 0$ which implies probability conservation (45) .

4.2 Quantum Algebra $U_q[\mathfrak{gl}(3)]$ and the Perk–Schultz Quantum Chain

Above we have defined the quantum algebra $U_q[\mathfrak{gl}(3)]$ (17)–(21). It is convenient to work also with the subalgebra $U_q[\mathfrak{sl}(3)]$.

4.2.1 Relation Between $U_q[\mathfrak{gl}(n)]$ and $U_q[\mathfrak{sl}(n)]$

We introduce generators $\tilde{\mathbf{H}}_i$, $1 \le i \le n$ and $\mathbf{H}_i$ $1 \le i \le n-1$ through

$$q^{-\tilde{\mathbf{H}}_i/2} = \mathbf{L}_i, \quad \mathbf{H}_i = \tilde{\mathbf{H}}_i - \tilde{\mathbf{H}}_{i+1}. \tag{77}$$

Then the quantum algebra $U_q[\mathfrak{sl}(n)]$ is the subalgebra generated by $q^{\pm \mathbf{H}_i/2}$, and $\mathbf{X}_i^{\pm}$, $i = 1, \ldots, n-1$ with relations (20), (21) and

$$q^{\mathbf{H}_i/2} q^{-\mathbf{H}_i/2} = q^{-\mathbf{H}_i/2} q^{\mathbf{H}_i/2} = I \tag{78}$$
$$q^{\mathbf{H}_i/2} q^{\mathbf{H}_j/2} = q^{\mathbf{H}_j/2} q^{\mathbf{H}_i/2} \tag{79}$$
$$q^{\mathbf{H}_i} \mathbf{X}_j^{\pm} q^{-\mathbf{H}_i} = q^{\pm A_{ij}} \mathbf{X}_j^{\pm} \tag{80}$$
$$[\, \mathbf{X}_i^{+} \,,\, \mathbf{X}_j^{-} \,] = \delta_{ij} [\mathbf{H}_i]_q . \tag{81}$$

with the unit I and the Cartan matrix

$$A_{ij} := \begin{cases} \;\;2 & \text{if } i = j \\ -1 & \text{if } j = i \pm 1 \\ \;\;0 & \text{else.} \end{cases} \tag{82}$$

of simple Lie algebras of type A_n. That $U_q[\mathfrak{sl}(n)]$ is a subalgebra of $U_q[\mathfrak{gl}(n)]$ can be seen by noticing that $\sum_{i=1}^n \tilde{\mathbf{H}}_i$ belongs to the center of $U_q[\mathfrak{gl}(n)]$ [10].

We remark that the commutation relations (78), (79) can be substituted by

$$[\, \mathbf{H}_i \,,\, \mathbf{H}_j \,] = 0 \tag{83}$$
$$[\, \mathbf{H}_i \,,\, \mathbf{X}_j^{\pm} \,] = \pm A_{ij} \mathbf{X}_j^{\pm} \tag{84}$$

and defining $q^{\pm \mathbf{H}_i/2}$ as a formal series through the Taylor expansion of the exponential.

4.2.2 Finite-Dimensional Representations for $n = 3$

For $n = 3$ the quadratic Serre relations (20) are void. By introducing [5]

$$\mathbf{X}_3^{\pm} := q^{1/2} \mathbf{X}_1^{\pm} \mathbf{X}_2^{\pm} - q^{-1/2} \mathbf{X}_2^{\pm} \mathbf{X}_1^{\pm} \tag{85}$$

the cubic Serre relations reduce to quadratic relations. One has instead of (21)

$$q^{-1/2} \mathbf{X}_1^{\pm} \mathbf{X}_3^{\pm} - q^{1/2} \mathbf{X}_3^{\pm} \mathbf{X}_1^{\pm} = 0 \tag{86}$$
$$q^{1/2} \mathbf{X}_2^{\pm} \mathbf{X}_3^{\pm} - q^{-1/2} \mathbf{X}_3^{\pm} \mathbf{X}_2^{\pm} = 0. \tag{87}$$

and also

$$[\, \mathbf{H}_i \,,\, \mathbf{X}_3^{\pm} \,] = \pm \mathbf{X}_3^{\pm}. \tag{88}$$

In order to distinguish the matrices corresponding to the three-dimensional fundamental representation from the abstract generators we use lower case letters. In terms of (53)–(55) the three-dimensional fundamental representation of $U_q[\mathfrak{gl}(3)]$ is given by:

$$x_1^{\pm} = a^{\pm}, \quad x_2^{\pm} = b^{\mp} \tag{89}$$

$$\tilde{h}_1 = \hat{a}, \quad \tilde{h}_2 = \hat{\upsilon}, \quad \tilde{h}_3 = \hat{b}, \tag{90}$$

corresponding to

$$h_1 = \hat{a} - \hat{\upsilon}, \quad h_2 = \hat{\upsilon} - \hat{b}. \tag{91}$$

for the representation of the generators H_i of $U_q[\mathfrak{sl}(3)]$. We also mention the representation $x_3^{\pm} = \pm q^{\pm 1/2} c^{\pm}$.

Next we introduce the coproduct

$$\Delta(\mathbf{X}_i^{\pm}) = \mathbf{X}_i^{\pm} \otimes q^{\mathbf{H}_i/2} + q^{-\mathbf{H}_i/2} \otimes \mathbf{X}_i^{\pm} \tag{92}$$

$$\Delta(\mathbf{H}_i) = \mathbf{H}_i \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{H}_i. \tag{93}$$

By repeatedly applying the coproduct to the fundamental representation, we construct the tensor representations of $U_q[\mathfrak{sl}(3)]$, denoted by capital letters,

$$X_i^{\pm} = \sum_{k=1}^{L} X_i^{\pm}(k), \quad H_i = \sum_{k=1}^{L} H_i(k) \tag{94}$$

with

$$X_i^{\pm}(k) = q^{-H_i/2} \otimes \cdots \otimes q^{-H_i/2} \otimes X_i^{\pm} \otimes q^{H_i/2} \cdots \otimes q^{H_i/2}, \tag{95}$$

$$H_i(k) = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes H_i \otimes \mathbb{1} \cdots \otimes \mathbb{1}. \tag{96}$$

For the full quantum algebra $U_q[\mathfrak{gl}(3)]$ we have

$$\tilde{H}_1 = \sum_{k=1}^{L} \hat{a}_k =: \hat{N}, \quad \tilde{H}_2 = \sum_{k=1}^{L} \hat{\upsilon}_k =: \hat{V}, \quad \tilde{H}_3 = \sum_{k=1}^{L} \hat{b}_k =: \hat{M}. \tag{97}$$

Here $\hat{N}$ and $\hat{M}$ are the particle number operators satisfying

$$\hat{N} |\, \eta_{N,M} \,\rangle = N |\, \eta_{N,M} \,\rangle, \quad \hat{M} |\, \eta_{N,M} \,\rangle = M |\, \eta_{N,M} \,\rangle. \tag{98}$$

The unit I is represented by the 3^L-dimensional unit matrix $\mathbf{1} := \mathbb{1}^{\otimes L} = \hat{N} + \hat{V} + \hat{M}$.

Notice that $H_1(k) = \hat{a}_k - \hat{\upsilon}_k$ and $H_2(k) = \hat{\upsilon}_k - \hat{b}_k$. In the L-fold tensor product $X_i^{\pm}(k)$ $(H_i(k))$ the term $X_i^{\pm}$ (H_i) is the kth factor. Therefore (89) yields

$$X_1^{\pm}(k) = q^{-\frac{1}{2}\sum_{j=1}^{k-1}(\hat{a}_j - \hat{\upsilon}_j) + \frac{1}{2}\sum_{j=k+1}^{L}(\hat{a}_j - \hat{\upsilon}_j)} a_k^{\pm} \tag{99}$$

$$X_2^{\pm}(k) = q^{-\frac{1}{2}\sum_{j=1}^{k-1}(\hat{\upsilon}_j - \hat{b}_j) + \frac{1}{2}\sum_{j=k+1}^{L}(\hat{\upsilon}_j - \hat{b}_j)} b_k^{\mp}. \tag{100}$$

One has $(X_i^{\pm}(k))^2 = 0$, $X_i^{\pm}(k) X_j^{\mp}(k) = 0$ for $i \neq j$ and

$$X_i^{\pm}(k) X_i^{\pm}(l) = \begin{cases} q^{\pm 2} X_i^{\pm}(l) X_i^{\pm}(k) & k < l \\ 0 & k = l \\ q^{\mp 2} X_i^{\pm}(l) X_i^{\pm}(k) & k > l \end{cases} \tag{101}$$

$$X_i^{\pm}(k) X_j^{\mp}(l) = X_i^{\pm}(l) X_j^{\mp}(k) \text{ for } i \neq j. \tag{102}$$

Thus the spatial order in which particles are created (or annihilated) by applying the operators $X_i^{\pm}(k)$ gives rise to combinatorial issues when building many-particle configurations from the reference state corresponding to the empty lattice.

4.2.3 The Perk–Schultz Quantum Spin Chain

We introduce the integrable Perk–Schultz quantum spin chain [16]

$$G = \sum_{k=1}^{L-1} g_{k,k+1} \tag{103}$$

where $g_{k,k+1}$ is reminiscent of (75), but with all non-zero off-diagonal elements equal to -1, i.e.,

$$\begin{aligned}\frac{1}{w} g_{k,k+1} = q\left(\hat{a}_k \hat{v}_{k+1} + \hat{v}_k \hat{b}_{k+1} + \hat{a}_k \hat{b}_{k+1}\right) + q^{-1}\left(\hat{v}_k \hat{a}_{k+1} + \hat{b}_k \hat{v}_{k+1} + \hat{b}_k \hat{a}_{k+1}\right) \\ - a_k^- a_{k+1}^+ - b_k^+ b_{k+1}^- - c_k^- c_{k+1}^+ - a_k^+ a_{k+1}^- - b_k^- b_{k+1}^+ - c_k^+ c_{k+1}^-.\end{aligned} \tag{104}$$

Quantum algebras have an intimate connection with integrable quantum spin systems defined by a (parameter-dependent) R-matrix satisfying the Yang–Baxter equation [2]. This comes from the fact that the coproduct (92), (93) of a quantum algebra commutes with the associated $\check{R}$-matrix [10] that is defined in terms of the R-matrix for the local subspaces corresponding to sites $k, k+1$ by $\check{R}_{k,k+1} = R_{k,k+1}\hat{\pi}_{k,k+1}$ with the permutation operator (73).[4] The $\check{R}_{k,k+1}$-matrices corresponding to $U_q[\mathfrak{sl}(n)]$ can be expressed in terms of the generators T_k of a Hecke algebra defined by

$$T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \quad (T_k - q^{-1})(T_k + q) = 0, \tag{105}$$

$$[\, T_k\, ,\; T_l\,] = 0 \quad \text{for } |k-l| \geq 2. \tag{106}$$

One has $\check{R}_{k,k+1} = xT_k^{-1} - x^{-1}T_k$ [10] with a parameter x that is immaterial in the present context. For the present case of the Perk–Schultz chain with $n = 3$ (103) one has $\check{R}_{k,k+1} = (x - x^{-1})g_{k,k+1} + x^{-1}q - xq^{-1}$, corresponding to the representation $T_k = g_{k,k+1} - q$, $T_k^{-1} = g_{k,k+1} - q^{-1}$ of the Hecke algebra. The commutativity of $\check{R}_{k,k+1}$ with the tensor representations (94) of $U_q[\mathfrak{sl}(3)]$ therefore implies

$$\left[\, g_{k,k+1}\, ,\; X_i^{\pm}\, \right] = [\, g_{k,k+1}\, ,\; H_i\,] = 0. \tag{107}$$

Thus the Perk–Schultz quantum Hamiltonian is symmetric under the action of the quantum algebra $U_q[\mathfrak{sl}(3)]$ and then trivially also under $U_q[\mathfrak{gl}(3)]$.

We remark that from the defining relations (105) one finds the matrix relations $g_{k,k+1}g_{k+1,k+2}g_{k,k+1} - g_{k,k+1} = g_{k+1,k+2}g_{k,k+1}g_{k+1,k+2} - g_{k+1,k+2}$ and $(g_{k,k+1})^2 = (q + q^{-1})g_{k,k+1}$ for the Perk–Schultz chain (103). In addition the matrices $g_{k,k+1}$ satisfy further relations, the specific form of which are not relevant here, which define the $(3, 0)$-quotient of the Hecke algebra [15].

[4] The connection to integrable models, in particular the parameter dependence of R, the construction of the associated statistical mechanics transfer matrix, and its quantum Hamiltonian limit, is not important for the purposes of this work. We refer the interested reader to [10,11,19] for more details and to [2] for an introduction to the field.

5 Proofs

It was pointed out in [1] that the hopping matrices $h_{k,k+1}$ (75) for the ASEP with second-class particles satisfy the defining relations of the same $(3,0)$-quotient of the Hecke algebra as the $g_{k,k+1}$ of the Perk–Schultz quantum chain. This fact implies the existence of representation matrices of the generators of $U_q[\mathfrak{gl}(3)]$ with which the hopping matrices $h_{k,k+1}$ and hence the generator (76) commutes. However, in order to make this symmetry property useful for probabilistic and physical applications one must solve the main problem that was left open in [1], which is to actually construct this representation. This is the content of Theorem (1), proved below. It turns out that both Theorems (1) and (2) are consequences of a proposition that we first motivate and then prove.

5.1 Perk–Schultz Chain and ASEP with Second-Class Particles

The point in case is that G and H differ only by multiplicative factors q and q^{-1} in their off-diagonal elements. Therefore the following proposition is a natural working hypothesis.

Proposition 1 *Let H and G be the matrices defined in (76) and (103). There exists a similarity transformation R such that*

$$G = R^{-1} H R \tag{108}$$

with an invertible diagonal matrix R of dimension 3^L.

Proof In order to prove this proposition we use the quantum algebra symmetry of the Perk–Schultz chain to first construct a good candidate for such a transformation and then prove that it satisfies (108).

(1) Construction of a Candidate R: Fix the numbers N of particles of type A and M of particles of type B. For $N = M = 0$ one readily verifies that $\langle s_{0,0}|H = \langle s_{0,0}|G = 0$. Moreover, for configurations with N of particles of type A and M of particles of type B the symmetry (107) yields

$$\langle s_{0,0}|(X_1^-)^N (X_2^+)^M G = 0. \tag{109}$$

Let us now suppose that (108) is true for some matrix R. Then from (109) one obtains $\langle s_{0,0}|(X_1^-)^N (X_2^+)^M R^{-1} H = 0$. On the other hand, by ergodicity the summation vector $\langle s_{N,M}|$ (68) is the unique left eigenvector with eigenvalue 0 of H restricted to configurations with N particles of type A and M particles of type B, i.e., $\langle s_{0,0}|H = 0$. Thus $\langle s_{N,M}|$ must be proportional to $\langle s_{0,0}|(X_1^-)^N (X_2^+)^M R^{-1}$. More precisely, we can conclude that if R exists it has the property

$$\langle s_{N,M}| = Y_L(N,M)\langle 0,0|(X_1^-)^N (X_2^+)^M R^{-1} \tag{110}$$

with some normalization factor $Y_L(N,M)$. Notice that $\langle s| = \sum_{N=0}^{L}\sum_{M=0}^{L-N}\langle s_{N,M}|$. Therefore

$$\langle s|R = \sum_{N=0}^{L}\sum_{M=0}^{L-N}\langle s_{N,M}|R \tag{111}$$

$$= \sum_{N=0}^{L}\sum_{M=0}^{L-N} Y_L(N,M)\langle s_{0,0}|(X_1^-)^N (X_2^+)^M. \tag{112}$$

Now we make the diagonal ansatz $R = \sum_\eta R(\eta) |\, \eta \,\rangle\langle\, \eta \,|$. Thus we obtain from (112) that

$$R(\eta_{N,M}) = Y_L(N, M)\langle\, s_{0,0} \,|(X_1^-)^N (X_2^+)^M |\, \eta_{N,M} \,\rangle. \tag{113}$$

The normalization factors $Y_L(N, M)$ are arbitrary, since they can be absorbed by redefining $R = \tilde{R}E$ where E is a diagonal matrix with matrix elements $Y_L(N, M)$ in the block N, M. Since both G and H conserve particle number for both species one has $EHE^{-1} = H$ and $EGE^{-1} = G$. Therefore $G = \tilde{R}^{-1} H \tilde{R}$ which implies that $G = \tilde{R}^{-1} H \tilde{R}$. Hence the $Y_L(N, M)$ can indeed be chosen arbitrarily. It turns out to be convenient to choose

$$Y_L(N, M) = \left([N]_q![M]_q!\right)^{-1}. \tag{114}$$

This reduces the computation of the diagonal elements of R to the computation of the matrix elements $\langle\, 0, 0 \,|(X_1^-)^N (X_2^+)^M |\, \eta_{N,M} \,\rangle$ from the explicit representation (94).

In order to compute R we first set $M = 0$ and use

$$\langle\, s_{0,0} \,| \frac{(X_1^-)^N}{[N]_q!} = \sum_{\mathbf{x}} q^{\sum_{k=1}^N x_k - N \frac{L+1}{2}} \langle\, \mathbf{x}, \emptyset \,| \tag{115}$$

which is a simple adaptation of an analogous result for the standard single-species ASEP taken from [20]. The sum over $\mathbf{x}$ stands for the sum over all particle positions x_i ordered such that $x_i < x_j$ for $i < j$, which is the sum over all distinct sets of particle positions. Therefore $\sum_{\mathbf{x},\emptyset} \langle\, \mathbf{x}, \emptyset \,| = \langle\, s_{N,0} \,|$ which allows us to write

$$R(\mathbf{x}, \emptyset) = q^{\sum_{k=1}^{|\mathbf{x}|} x_k - |\mathbf{x}| \frac{L+1}{2}} \tag{116}$$

and, using (70),

$$\langle\, s_{0,0} \,| \frac{(X_1^-)^N}{[N]_q!} = \langle\, s_{N,0} \,| q^{\sum_{k=1}^L \left(k - \frac{L+1}{2}\right)\hat{a}_k}. \tag{117}$$

Next we apply $(X_2^+)^M$ (100) to this vector and observe that for the factors $q^{\pm H_2(k)/2}$ that appear in X_2^+ (instead of the $q^{\pm H_1(k)/2}$ that appear in X_1^-) any A-particle is like a non-existent site, since H_2 is build from projectors on B-particles and vacancies. Hence, with regard to the action of $(X_2^+)^M$, the existence of A-particles in a configuration $\eta_{N,M} \equiv z_{N,M}$ with N particles of type A behaves like a reduction of system size $L \to \tilde{L} = L - N$ and a coordinate shift

$$y_i \to \tilde{y}_i = y_i - N_{y_i}(\mathbf{z}) \tag{118}$$

for B-particles with $N_k(\mathbf{z})$ defined in (7). Therefore the action of $(X_2^+)^M/[M]_q!$ on $\langle\, s_{0,0} \,| \frac{(X_1^-)^N}{[N]_q!}$ yields a q-factor similar to the one in (116), but with N replaced by M, L replaced by $\tilde{L}$, x_k replaced by $\tilde{y}_k$ and q replaced by q^{-1}. We conclude

$$\langle\, s_{0,0} \,| \frac{(X_1^-)^N}{[N]_q!} \frac{(X_2^+)^M}{[M]_q!} = \sum_{\mathbf{z}_{N,M}} R(\mathbf{z}_{N,M}) \langle\, \mathbf{z}_{N,M} \,| \tag{119}$$

where the sum is over all distinct coordinate sets and

$$R(\mathbf{z}_{N,M}) = q^{\frac{1}{2}[(M-N)(L+1)-MN] + \sum_{i=1}^N x_i - \sum_{i=1}^M \tilde{y}_i}. \tag{120}$$

Next observe that for any configuration η

$$\sum_{i=1}^{M(\eta)} N_{y_i}(\eta) = \sum_{k=1}^{L}\sum_{l=1}^{k-1} a_l b_k = N(\eta)M(\eta) - \sum_{k=1}^{L}\sum_{l=1}^{k-1} b_l a_k$$
$$= N(\eta)M(\eta) - \sum_{i=1}^{N(\eta)} M_{x_i}(\eta). \tag{121}$$

For $\eta = \mathbf{z}_{N,M}$ this yields

$$R(\mathbf{z}_{N,M}) = q^{\frac{1}{2}\left[\sum_{i=1}^{N}(2x_i - L - 1 - M_{x_i}(\eta)) - \sum_{i=1}^{M}(2y_i - L - 1 - N_{y_i}(\eta))\right]}. \tag{122}$$

and with Lemma (1)

$$\langle\, s_{0,0}\,| \frac{(X_1^-)^N}{[N]_q!}\frac{(X_2^+)^M}{[M]_q!} = \langle\, s_{N,M}\,| q^{\frac{1}{2}\hat{U}} \tag{123}$$

with

$$\hat{U} = \sum_{k=1}^{L}(2k - L - 1)\left(\hat{a}_k - \hat{b}_k\right) + \sum_{k=1}^{L-1}\sum_{l=1}^{k}\left(\hat{a}_l\hat{b}_{k+1} - \hat{b}_l\hat{a}_{k+1}\right). \tag{124}$$

Notice that the matrix $\hat{U}$ does not depend on N and M. Taking the sum over N and M then yields from (112) the diagonal candidate matrix

$$R = q^{\frac{1}{2}\hat{U}}. \tag{125}$$

(2) Proof of the Transformation Property (108): We stress that the properties of R that we have used in its construction are only necessary conditions for the transformation property (108) to be valid. In order to prove this property we need two more technical ingredients. The first is a transformation lemma, proved in [3] (Lemma 5.1),

Lemma 2 *For any finite $p \neq 0$ we have*

$$p^{\hat{a}_l} a_x^{\pm} p^{-\hat{a}_l} = p^{\pm\delta_{l,x}} a_x^{\pm}, \quad p^{\hat{b}_l} a_x^{\pm} p^{-\hat{b}_l} = a_x^{\pm}, \tag{126}$$
$$p^{\hat{b}_l} b_x^{\pm} p^{-\hat{b}_l} = p^{\pm\delta_{l,x}} b_x^{\pm}, \quad p^{\hat{a}_l} b_x^{\pm} p^{-\hat{a}_l} = b_x^{\pm} \tag{127}$$
$$p^{\hat{a}_l\hat{b}_m} a_x^{\pm} p^{-\hat{a}_l\hat{b}_m} = p^{\pm\delta_{l,x}\hat{b}_m} a_x^{\pm}, \tag{128}$$
$$p^{\hat{a}_l\hat{b}_m} b_x^{\pm} p^{-\hat{a}_l\hat{b}_m} = p^{\pm\delta_{m,x}\hat{a}_l} b_x^{\pm}. \tag{129}$$

Applying these transformations yields the following auxiliary result.

Lemma 3 *The local creation and annihilation operators transform as follows:*

$$R a_x^{\pm} R^{-1} = q^{\mp\frac{1}{2}\sum_{k=1}^{x-1}(\hat{b}_k - \mathbb{1}) \pm \frac{1}{2}\sum_{k=x+1}^{L}(\hat{b}_k - \mathbb{1})} a_x^{\pm}, \tag{130}$$
$$R b_x^{\pm} R^{-1} = q^{\pm\frac{1}{2}\sum_{k=1}^{x-1}(\hat{a}_k - \mathbb{1}) \mp \frac{1}{2}\sum_{k=x+1}^{L}(\hat{a}_k - \mathbb{1})} b_x^{\pm} \tag{131}$$
$$R c_x^{\pm} R^{-1} = q^{\pm\frac{1}{2}\sum_{k=1}^{x-1}(\hat{v}_k + \mathbb{1}) \mp \frac{1}{2}\sum_{k=x+1}^{L}(\hat{v}_k + \mathbb{1})} c_x^{\pm} \tag{132}$$

Proof To prove these identities we use Lemma (2) and commutativity of the projection operators. This yields

$$\begin{aligned} &p^{\sum_{k=1}^{L-1}\sum_{l=1}^{k}(\hat{a}_l\hat{b}_{k+1}-\hat{b}_l\hat{a}_{k+1})}a_x^{\pm}p^{-\sum_{k=1}^{L-1}\sum_{l=1}^{k}(\hat{a}_l\hat{b}_{k+1}-\hat{b}_l\hat{a}_{k+1})}\\ &\quad = p^{\sum_{k=1}^{L-1}\sum_{l=1}^{k}(\pm\delta_{l,x}\hat{b}_{k+1}\mp\delta_{k+1,x}\hat{b}_l)}a_x^{\pm} = p^{\mp\sum_{k=1}^{x-1}\hat{b}_k\pm\sum_{k=x+1}^{L}\hat{b}_k)}a_x^{\pm} \end{aligned} \tag{133}$$

and

$$p^{\sum_{k=1}^{L}(2k-L-1)(\hat{a}_k-\hat{b}_k)}a_x^{\pm}p^{-\sum_{k=1}^{L}(2k-L-1)(\hat{a}_k-\hat{b}_k)} = p^{\pm(2x-L-1)}a_x^{\pm} \tag{134}$$

Joining both yields (130) and a similar computation yields (131). Finally, (132) follows from $c_k^+ = a_k^+ b_k^-$, $c_k^- = b_k^+ a_k^-$. □

Now we can prove (108). We split $g_{k,k+1} = g_{k,k+1}^{\mathrm{d}} - g_{k,k+1}^{\mathrm{o}}$ into its diagonal part $g_{k,k+1}^{\mathrm{d}}$ and its off-diagonal part $g_{k,k+1}^{\mathrm{o}}$ and similarly for $h_{k,k+1}$. Trivially one has $Rg_{k,k+1}^{\mathrm{d}}R^{-1} = g_{k,k+1}^{\mathrm{d}} = h_{k,k+1}^{\mathrm{d}}$.

For the offdiagonal parts, we consider first $a_k^{\pm}a_{k+1}^{\mp}$. Equation (130) in Lemma (3) yields

$$Ra_k^{\pm}a_{k+1}^{\mp}R^{-1} = a_k^{\pm}q^{\pm\frac{1}{2}(\hat{b}_{k+1}-\mathbb{1})}q^{\pm\frac{1}{2}(\hat{b}_k-\mathbb{1})}a_{k+1}^{\mp}. \tag{135}$$

The general projector property $p^{\hat{b}_m} = 1 + (p-1)\hat{b}_m$ together with (58) and (61) applied to the subspaces k and $k+1$ lead to

$$Ra_k^{\pm}a_{k+1}^{\mp}R^{-1} = q^{\mp 1}a_k^{\pm}a_{k+1}^{\mp}. \tag{136}$$

In the same fashion one proves

$$Rb_k^{\pm}b_{k+1}^{\mp}R^{-1} = q^{\pm 1}b_k^{\pm}b_{k+1}^{\mp}. \tag{137}$$

Finally, by similar arguments

$$\begin{aligned} Rc_k^{\pm}c_{k+1}^{\mp}R^{-1} &= c_k^{\pm}q^{\mp\frac{1}{2}(\hat{\upsilon}_{k+1}+\mathbb{1})}q^{\mp\frac{1}{2}(\hat{\upsilon}_k+\mathbb{1})}c_{k+1}^{\mp}\\ &= q^{\mp 1}c_k^{\pm}c_{k+1}^{\mp}. \end{aligned} \tag{138}$$

Comparing with (75) shows that $Rg_{k,k+1}^{\mathrm{o}}R^{-1} = h_{k,k+1}^{\mathrm{o}}$ and thus $RGR^{-1} = H$. □

5.2 Proof of Theorem (1)

Proof From (130), the commutativity of the projectors at different sites, and (91) lifted to the tensor space, one finds that the local generators $X_i^{\pm}(r)$ transform as follows:

$$Y_1^+(r) := RX_1^+(r)R^{-1} = q^{\sum_{k=1}^{r-1}\hat{\upsilon}_k-\sum_{k=r+1}^{L}\hat{\upsilon}_k}a_r^+, \tag{139}$$

$$Y_1^-(r) := RX_1^-(r)R^{-1} = q^{-\sum_{k=1}^{r-1}\hat{a}_k+\sum_{k=r+1}^{L}\hat{a}_k}a_r^-, \tag{140}$$

$$Y_2^+(r) := RX_2^+(r)R^{-1} = q^{\sum_{k=1}^{r-1}\hat{b}_k-\sum_{k=r+1}^{L}\hat{b}_k}b_r^- \tag{141}$$

$$Y_2^-(r) := RX^-(r)R^{-1} = q^{-\sum_{k=1}^{r-1}\hat{\upsilon}_k+\sum_{k=r+1}^{L}\hat{\upsilon}_k}b_r^+ \tag{142}$$

Moreover, since R and $\tilde{H}_i$ are all diagonal one has $R\tilde{H}_iR^{-1} = \tilde{H}_i$. Commutativity of the hopping matrices $h_{k,k+1}$ in H with $Y_i^{\pm}$ and L_j follows from Proposition (1) and the symmetry (107) of the Perk–Schultz quantum chain.

To prove the explicit expressions (26), (27) for the representations we focus on $Y_1^+(r)$. Using the fundamental representation and the factorization property (39) of the inner product of tensor vectors one finds $\langle \eta' | a_r^+ | \eta \rangle = \delta_{\eta', \eta^{r-}}$. The terms in the exponential follow trivially from Lemma (1) and the definitions (5), (7). Following similar arguments for the other generators yields the matrix elements of $Y_i^{\pm}$ and L_j (26), (27) as stated in (1). □

5.3 Proof of Theorem (2)

Proof Since G is symmetric and R is diagonal, Proposition (1) implies

$$H^T = (RR^T)^{-1} HRR^T = R^{-2} HR^2. \tag{143}$$

With (48) we thus have reversibility with a reversible measure $\hat{\pi} = R^2$ in the matrix form (47). By the projector Lemma (1) $\hat{\pi}$ yields the reversible measure (30) of Theorem (2). □

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Appendix

We display some explicit results for unnormalized stationary distributions for small lattices $L = 2, 3, 4$ and also L arbitrary with small particle numbers $N + M = 1, 2, 3, 4$.

$N = 0\ M = 0$	$\vert 00 \rangle$ $L = 2$
$N = 1\ M = 0$	$q^{-1} \vert A0 \rangle + q \vert 0A \rangle$
$N = 0\ M = 1$	$q \vert B0 \rangle + q^{-1} \vert 0B \rangle$
$N = 2\ M = 0$	$\vert AA \rangle$
$N = 1\ M = 1$	$q^{-1} \vert AB \rangle + q \vert BA \rangle$
$N = 0\ M = 2$	$\vert BB \rangle$

$N = 0\ M = 0$	$\vert 000 \rangle$ $L = 3$
$N = 1\ M = 0$	$q^{-2} \vert A00 \rangle + \vert 0A0 \rangle + q^2 \vert 00A \rangle$
$N = 0\ M = 1$	$q^{-2} \vert 00B \rangle + \vert 0B0 \rangle + q^2 \vert B00 \rangle$
$N = 2\ M = 0$	$q^{-2} \vert AA0 \rangle + \vert A0A \rangle + q^2 \vert 0AA \rangle$
$N = 1\ M = 1$	$q^{-3} \vert A0B \rangle + q^{-1}(\vert AB0 \rangle + \vert 0AB \rangle) + q(\vert BA0 \rangle + \vert 0BA \rangle) + q^3 \vert B0A \rangle$
$N = 0\ M = 2$	$q^{-2} \vert 0BB \rangle + \vert B0B \rangle + q^2 \vert BB0 \rangle$
$N = 3\ M = 0$	$\vert AAA \rangle$
$N = 2\ M = 1$	$q^{-2} \vert AAB \rangle + \vert ABA \rangle + q^2 \vert BAA \rangle$
$N = 1\ M = 2$	$q^{-2} \vert ABB \rangle + \vert BAB \rangle + q^2 \vert BBA \rangle$
$N = 0\ M = 3$	$\vert BBB \rangle$

$N=0\ M=0$	$\vert 0000\rangle\ L=4$
$N=1\ M=0$	$q^{-3}\vert A000\rangle+q^{-1}\vert 0A00\rangle+q\vert 00A0\rangle+q^{3}\vert 000A\rangle$
$N=0\ M=1$	$q^{-3}\vert 000B\rangle+q^{-1}\vert 00B0\rangle+q\vert 0B00\rangle+q^{3}\vert B000\rangle$
$N=2\ M=0$	$q^{-4}\vert AA00\rangle+q^{-2}\vert A0A0\rangle+\vert A00A\rangle+\vert 0AA0\rangle+q^{2}\vert 0A0A\rangle+q^{4}\vert 00AA\rangle$
$N=1\ M=1$	$q^{-5}\vert A00B\rangle+q^{-3}(\vert A0B0\rangle+\vert 0A0B\rangle)+q^{-1}(\vert AB00\rangle+\vert 0AB0\rangle+\vert 00AB\rangle)$ $+q(\vert BA00\rangle+\vert 0BA0\rangle+\vert 00BA\rangle)+q^{3}(\vert B0A0\rangle+\vert 0B0A\rangle)+q^{5}\vert B00A\rangle$
$N=0\ M=2$	$q^{-4}\vert 00BB\rangle+q^{-2}\vert 0B0B\rangle+\vert B00B\rangle+\vert 0BB0\rangle+q^{2}\vert B0B0\rangle+q^{4}\vert BB00\rangle$
$N=3\ M=0$	$q^{-3}\vert AAA0\rangle+q^{-1}\vert AA0A\rangle+q\vert A0AA\rangle+q^{3}\vert 0AAA\rangle$
$N=2\ M=1$	$q^{-5}\vert AA0B\rangle+q^{-3}(\vert AAB0\rangle+\vert A0AB\rangle)+q^{-1}(\vert 0AAB\rangle+\vert ABA0\rangle+\vert A0BA\rangle)$ $+q(\vert AB0A\rangle+\vert 0ABA\rangle+\vert BAA0\rangle)+q^{3}(\vert BA0A\rangle+\vert 0BAA\rangle)+q^{5}\vert B0AA\rangle$
$N=1\ M=2$	$q^{-5}\vert A0BB\rangle+q^{-3}(\vert AB0B\rangle+\vert 0ABB\rangle)+q^{-1}(\vert BA0B\rangle+\vert 0BAB\rangle+\vert ABB0\rangle)$ $+q(\vert 0BBA\rangle+\vert BAB0\rangle+\vert B0AB\rangle)+q^{3}(\vert BBA0\rangle+\vert B0BA\rangle)+q^{5}\vert BB0A\rangle$
$N=0\ M=3$	$q^{-3}\vert 0BBB\rangle+q^{-1}\vert B0BB\rangle+q\vert BB0B\rangle+q^{3}\vert BBB0\rangle$
$N=4\ M=0$	$\vert AAAA\rangle$
$N=3\ M=1$	$q^{-3}\vert AAAB\rangle+q^{-1}\vert AABA\rangle+q\vert ABAA\rangle+q^{3}\vert BAAA\rangle$
$N=2\ M=2$	$q^{-4}\vert AABB\rangle+q^{-2}\vert ABAB\rangle+\vert ABBA\rangle+\vert BAAB\rangle+q^{2}\vert BABA\rangle+q^{4}\vert BBAA\rangle$
$N=1\ M=3$	$q^{-3}\vert ABBB\rangle+q^{-1}\vert BABB\rangle+q\vert BBAB\rangle+q^{3}\vert BBBA\rangle$
$N=0\ M=4$	$\vert BBBB\rangle$

$\underline{N+M=1:}$

$$\pi^*(\{x\},\emptyset)\propto q^{2x-1}$$
$$\pi^*(\emptyset,\{y\})\propto q^{-2y+1}$$

$\underline{N+M=2:}$

$$\pi^*(\{x_1,x_2\},\emptyset)\propto q^{2x_1+2x_2-2}$$
$$\pi^*(\{x_1\},\{y_1\})\propto\begin{cases}q^{2x_1-2y_1-1} & y_1<x_1\\ q^{2x_1-2y_1+1} & y_1>x_1\end{cases}$$
$$\pi^*(\emptyset,\{y_1,y_2\})\propto q^{-2y_1-2y_2+2}$$

$\underline{N+M=3:}$

$$\pi^*(\{x_1,x_2,x_3\},\emptyset)\propto q^{2x_1+2x_2+2x_3-3}$$
$$\pi^*(\{x_1,x_2\},\{y_1\})\propto\begin{cases}q^{2x_1+2x_2-2y_1-3} & y_1<x_1,x_2\\ q^{2x_1+2x_2-2y_1-1} & x_1<y_1<x_2\\ q^{2x_1+2x_2-2y_1+1} & x_1,x_2<y_1\end{cases}$$
$$\pi^*(\{x_1\},\{y_1,y_2\})\propto\begin{cases}q^{2x_1-2y_1-2y_2-1} & y_1,y_2<x_1\\ q^{2x_1-2y_1-2y_2+1} & y_1<x_1<y_2\\ q^{2x_1-2y_1-2y_2+3} & x_1<y_1,y_2\end{cases}$$
$$\pi^*(\emptyset,\{y_1,y_2,y_3\})\propto q^{-2y_1-2y_2-2y_3+3}$$

$\underline{N+M=4:}$

$$\pi^*(\{x_1,x_2,x_3,x_4\},\emptyset) \propto q^{2x_1+2x_2+2x_3+2x_4-4}$$

$$\pi^*(\{x_1,x_2,x_3\},\{y_1\}) \propto \begin{cases} q^{2x_1+2x_2+2x_3-2y_1-5} & y_1 < x_1,x_2,x_3 \\ q^{2x_1+2x_2+2x_3-2y_1-3} & x_1 < y_1 < x_2,x_3 \\ q^{2x_1+2x_2+2x_3-2y_1-1} & x_1,x_2 < y_1 < x_3 \\ q^{2x_1+2x_2+2x_3-2y_1+1} & x_1,x_2,x_3 < y_1 \end{cases}$$

$$\pi^*(\{x_1,x_2\},\{y_1,y_2\}) \propto \begin{cases} q^{2x_1+2x_2-2y_1-2y_2-4} & y_1,y_2 < x_1,x_2 \\ q^{2x_1+2x_2-2y_1-2y_2-2} & y_1 < x_1 < y_2 < x_2 \\ q^{2x_1+2x_2-2y_1-2y_2} & y_1 < x_1,x_2 < y_2 \\ q^{2x_1+2x_2-2y_1-2y_2} & x_1 < y_1,y_2 < x_2 \\ q^{2x_1+2x_2-2y_1-2y_2+2} & x_1 < y_1 < x_2 < y_2 \\ q^{2x_1+2x_2-2y_1-2y_2+4} & x_1,x_2 < y_1,y_2 \end{cases}$$

$$\pi^*(\{x_1\},\{y_1,y_2,y_3\}) \propto \begin{cases} q^{2x_1-2y_1-2y_2-2y_3-1} & y_1,y_2,y_3 < x_1 \\ q^{2x_1-2y_1-2y_2-2y_3+1} & y_1,y_2 < x_1 < x_3 \\ q^{2x_1-2y_1-2y_2-2y_3+3} & y_1 < x_1 < y_2,y_3 \\ q^{2x_1-2y_1-2y_2-2y_3+5} & x_1 < y_1,y_2,y_3 \end{cases}$$

$$\pi^*(\emptyset,\{y_1,y_2,y_3,y_4\}) \propto q^{-2y_1-2y_2-2y_3-2y_4+4}$$

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