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On Some Properties of the Landau Kinetic Equation

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Abstract We discuss some general properties of the Landau kinetic equation. In particular, the difference between the "true" Landau equation, which formally follows from classical mechanics, and the "generalized" Landau equation, which is just an interesting mathematical object, is stressed. We show how to approximate solutions to the Landau equation by the Wild sums. It is the so-called quasi-Maxwellian approximation related to Monte Carlo methods. This approximation can be also useful for mathematical problems. A model equation which can be reduced to a local nonlinear parabolic equation is also constructed in connection with existence of the strong solution to the initial value problem. A self-similar asymptotic solution to the Landau equation for large v and t is discussed in detail. The solution, earlier confirmed by numerical experiments, describes a formation of Maxwellian tails for a wide class of initial data concentrated in the thermal domain. It is shown that the corresponding rate of relaxation (fractional exponential function) is in exact agreement with recent mathematically rigorous estimates.

Keywords Boltzmann kinetic equation · Coulomb collisions ·
Nonlinear Landau–Fokker–Planck equation · Quasi-Maxwellian approximation

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1 Introduction

The Landau kinetic equation was first published in 1936 [1] as a version of the Boltzmann equation for Coulomb forces. It soon became (in combination with the Vlasov equation) the most important mathematical kinetic model in the theory of collisional plasma. The independent derivation of the Landau equation in the Fokker-Planck form in 1957 [2] should be also mentioned. Therefore this equation is often called the Landau–Fokker–Planck equation in plasma physics.

The first formal derivation of the Landau equation from the BBGKY hierarchy was done by Bogolyubov in 1946 in his book [3]. He introduced a more general view on the Landau equation as the kinetic equation for weakly interacting particles. The Landau collision terms for various "weak" potentials differ just by constant factor in front of the collision integral. Some recent related results and discussion on the validation problem can be found in [4]. It seems that the classical Landau equation can be in principle mathematically justified for smooth short-range potentials of the form $U(r) = \varepsilon^{-1/2}\varphi(r/\varepsilon)$ at the limit $\varepsilon = 0$.

The case of Coulomb forces is much more complicated. The Landau collision term appears in this case as a correction to the Vlasov equation. This term can be equally understood as an approximation of the Boltzmann collision integral or the Balescu–Lenard collision integral (see [5] for discussion of the "correct" collision integrals for Coulomb forces). Hence one can use the Vlasov–Landau equations as a reasonable kinetic model for some classes of physical problems, but even the mathematical formulation of the corresponding validation problem is not quite clear yet.

In this paper we consider only the spatially homogeneous Landau equation. It is important to distinguish the "true" Landau equation, discussed above, and, the "generalized" Landau equation. The latter was independently introduced by several authors (see e.g. [6–8]). It plays a role of a model of the Boltzmann equation for various potentials. This equation can be obtained as a limit of the Boltzmann equation with such model cross-section that corresponds to complete domination of the small-angle scattering (grazing collisions limit [8]).

On the other hand, this limit has no relation to physics in the non-Coulomb case. There is no intermolecular forces which lead to the Landau equation (here and below we usually omit the word "generalized") for e.g. power-like potentials $U(r) \sim r^{-n},\ n > 1$. These equations, however,are very interesting from mathematical point of view. They were intensively studied by mathematicians in last two decades. We mention a few important papers [9–15], where many other related references can be found. It is interesting that only weak existence theorems are proved for spatially homogeneous initial value problem (the existence of strong solution is proved only for indata close to Maxwellian [12]). A recent paper [13] by Desvillettes, partly based on earlier results of Villani [9], contains an important progress for true Landau equation, but the existence of strong solutions remains open.

Our goal in this paper is to clarify some properties of the Landau equation. In Sect. 2 we discuss the connection between Boltzmann and Landau equations. In particular, we introduce a quasi-Maxwellian approximation for the Landau equation with general cross-section and show that its solution can be (at least formally) approximated by the Wild sum. This property can be used in both theoretical study and the Monte Carlo methods for the Landau equation [16,17]. In Sect. 3 we consider the radial Landau equation and construct a model equation with similar properties. This equation, however, can be reduced to a local parabolic equation, which does not contain integrals. The strong existence theorem can be probably proved in this case by more or less standard means of the theory of nonlinear parabolic equations. In Sect. 4 we study a formation of Maxwellian tails for the case of so called very soft potentials and construct an asymptotic solution by using the approach of the paper [18]. Then we show that this solution converges to equilibrium with the rate $O[\exp(-ct^q)],\ 0 < q < 1$. The exponent q is the same, as in [19]. The approximate solution has the form of traveling wave (in the velocity space), it is confirmed by numerical results [18].

2 The Landau Equation and Its Connection with the Boltzmann Equation

We consider a spatially homogeneous gas of point particles with unit mass. Each particle is characterized at time $t \geq 0$ by its velocity $v \in \mathbb{R}^3$. Let $f(v,t)$ be a one-particle distribution function, i.e. a time-dependent probability density in the velocity space. Then the classical Landau equation (see [1,3]) for $f(v,t)$ reads

$$\frac{\partial f}{\partial t} = B \frac{\partial}{\partial v_i} \int_{\mathbb{R}^3} dw \; \frac{(u^2\delta_{ij} - u_i u_j)}{u^3} \left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j} \right) f(v) f(w), \tag{1}$$

$u = v - w$, with summation over repeating indices $i, j = 1, 2, 3$. The constant B depends on intermolecular potential $U(r)$, where $r > 0$ is the distance between two interacting particles. The formula for B reads

$$B = \frac{1}{8\pi} \int_0^\infty dr\, r^3\, \hat{U}(r)^2, \quad \hat{U}(|k|) = \int_{\mathbb{R}^3} dx U(|x|) e^{ik\cdot x}. \tag{2}$$

Some recent results on the derivation of this equation from BBGKY hierarchy in the weak coupling limit can be found in [4].

Note that $B = \infty$ for many typical molecular models, e.g. hard spheres or particles interacting via power-like potentials $U(r) \sim r^{-n},\ n \geq 1$. On the other hand, the first formal derivation of (1) was done by Landau [1] on the basis of the Boltzmann equation for the Coulomb potential $U(r) = e^2/r$ with angular cut-off at small and large angles. The key idea of Landau was to take into account only small-angle scattering in the Boltzmann collision integral. The connection between collision integrals of Boltzmann and Landau was later studied in more detail by several authors. In particular, it was shown in [6,7] that one can introduce a generalized Landau collision integral

$$Q_L(f,f) = \frac{1}{8} \frac{\partial}{\partial v_i} \int_{\mathbb{R}^3} dw \Phi(|u|) R_{ij}(u) \left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j} \right) f(v) f(w), \tag{3}$$

where

$$R_{ij}(u) = (|u|^2 \delta_{ij} - u_i u_j),\ \ \Phi(|u|) = 2\pi |u| \int_{-1}^{1} d\mu \sigma(|u|, \mu)(1-\mu),$$

$\sigma(|u|, \cos\theta)$ is a differential cross-section of scattering at the angle $0 \leq \theta \leq \pi$. The corresponding Boltzmann collision integral reads in similar notation

$$Q_B(f,f) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} dw d\omega g \left(|u|, \frac{u \cdot \omega}{|u|} \right) [f(v')f(w') - f(v)f(w)], \tag{4}$$

where $\omega \in \mathbb{S}^2$, $\quad g(|u|, \mu) = |u|\sigma(|u|, \mu)$,

$$v' = \frac{1}{2}(v + w + |u|\omega), \quad w' = \frac{1}{2}(v + w - |u|\omega).$$

The integral $Q_B(f, f)$ was presented in [6,7] in the form of infinite series whose first term coincides with $Q_L(f, f)$. Of course, the difference $\Delta Q = Q_B - Q_L$ is not small for arbitrary intermolecular potential $U(r)$. However, it can be made small if one considers a family of alternative kernels $g_\varepsilon(|u|, \mu)$ in (4), not necessarily with a physical meaning, satisfying

$$\lim_{\varepsilon\to 0} 2\pi \int_{-1}^{1} d\mu g_\varepsilon(|u|, \mu)(1-\mu)^k = \begin{cases} \Phi(|u|) & \text{if} \quad k = 1, \\ 0 \quad \text{if} \quad k > 1. \end{cases} \tag{5}$$

Roughly speaking, this is the so-called grazing collision limit of the Boltzmann equation. The mathematical study of related questions is done, in particular, in papers [8,20]. The most typical kernel in (5) as used in [8,9,20], and more recently in [21] is given by the product

$$g_\varepsilon(|u|, \mu) = |u|^\gamma g_\varepsilon(\mu), \tag{6}$$

where $\gamma = 1 - 4/n$, $n \geq 1$. It formally corresponds to the potential $U(r) \sim r^{-n}$, $n \geq 1$. Then we obtain $\Phi(|u|) = C_\gamma |u|^\gamma$, $-3 \leq \gamma < 1$. The traditional terminology relates the intervals (a) $0 \leq \gamma \leq 1$, (b) $-2 \leq \gamma < 0$, and (c) $\gamma < -2$ to respectively (a) hard, (b) moderately soft and (c) very soft potentials [9]. The differences are important from mathematical point of view: the case (c) is most difficult. On the other hand, only the Coulomb case $\gamma = -3$ is justified from physical (dynamical) point of view. Therefore we sometimes call Eq. (1) (with arbitrary constant B) the true Landau equation. All other cases of the Landau collision integral (3) with $\Phi(|u|) \neq |u|^{-3}$ are just interesting mathematical models.

To end this section we mention an interesting family of kernels, satisfying (5):

$$g_\varepsilon(|u|, \mu) = \frac{1}{2\pi\varepsilon}\delta[1 - \mu - \min(\varepsilon\Phi(|u|),\ 2)], \tag{7}$$

where $\Phi(|u|)$ is considered as a given monotone function (in particular, $\Phi = |u|^{-3}$ for the true Landau equation). Then we obtain a quasi-Maxwellian approximation of the Landau equation by Boltzmann equations, since the loss term in (4) is exactly the same as for pseudo-Maxwell molecules. This leads, in particular, to a very simple and efficient Monte Carlo method for the Landau equation (see [16] for details).

The general quasi-Maxwellian approximation of the Landau collision integral (3) by the Boltzmann collision integral (4) can be obtained by using any nonnegative kernel $g_\varepsilon(|u|, \mu)$, which satisfies the conditions (5) and, in addition, the condition

$$g_{tot,\varepsilon}(|u|) = 2\pi \int_{-1}^{1} d\mu g_\varepsilon(|u|, \mu) = \frac{1}{\varepsilon}. \tag{8}$$

An example of such kernel is given in (7). The condition (8) implies that

$$Q_B(f, f) = Q_\varepsilon{}^+(f, f) - \frac{\rho(f)}{\varepsilon} f, \quad \rho(f) = \int_{\mathbb{R}^3} dv f(v),$$

$$Q_\varepsilon{}^+(f_1, f_2) = \int dw d\omega\, g_\varepsilon\left(|u|, \frac{u\cdot\omega}{|u|}\right) f_1(v') f_2(w'),$$

in the notation of (4). Without loss of generality we assume that $\rho(f) = 1$. Then the solution $f^{(\varepsilon)}(v, t)$ of the initial value problem for the Boltzmann equation

$$f_t^{(\varepsilon)} + \frac{1}{\varepsilon} f^{(\varepsilon)} = Q_\varepsilon{}^+ \left(f^{(\varepsilon)}, f^{(\varepsilon)}\right), \quad f^{(\varepsilon)}|_{t=0} = f_0(v) \geq 0,$$

can be presented in the form of the Wild sum [22]

$$f^{(\varepsilon)}(v, t) = \sum_{n=0}^{\infty} e^{-\frac{t}{\varepsilon}} \left(1 - e^{-\frac{t}{\varepsilon}}\right)^n f_n^{(\varepsilon)}(v), \tag{9}$$

where

$$f_0^{(\varepsilon)} = f_0, \quad f_{n+1}^{(\varepsilon)} = \frac{\varepsilon}{n+1} \sum_{k=0}^{n} Q_\varepsilon{}^+ \left(f_k^{(\varepsilon)}, f_{n-k}^{(\varepsilon)}\right).$$

If $\rho(f_0) = 1$, then it is easy to verify that $\rho(f_n^{(\varepsilon)}) = 1$ for all $n = 0, 1, \ldots$ Hence, the series converges in $L_1(\mathbb{R}^3)$ and $\rho[f^{(\varepsilon)}(v, t)] = 1$, as follows from standard considerations [22]. We expect that $f^{(\varepsilon)}(v, t) \to f(v, t)$, as $\varepsilon \to 0$, where $f(v, t)$ solves the problem

$$f_t = Q_L(f, f), \quad f|_{t=0} = f_0(v), \tag{10}$$

in the notation of (3). To prove this is a difficult task, which is not considered in this paper. However, the general idea to use the Wild sum (9) as an approximate solution of the Landau equation can be of some interest from many viewpoints.

3 Radial Solutions and a Model Equation

We study below the Landau equation (10), (3) for radial solutions $f(v, t) = \tilde{f}(|v|, t)$. Then we obtain

$$Q_L(f, f) = div J(v), \quad J(v) = I(|v|)\, v,$$

$$I(|v|) = \frac{1}{8\,|v|^2} \int_{\mathbb{R}^3} dw \Phi(|u|) \left[|u|^2 v - (u \cdot v) u\right] \left[\frac{v}{|v|}\, \tilde{f}'(|v|)\, \tilde{f}(|w|) - \frac{w}{|w|}\, \tilde{f}(|v|)\, \tilde{f}'(|w|)\right],$$

where 'primes' denote differentiation. We omit tildes and simplify notation by denoting $|v| = v > 0$. Simple calculations yield

$$Q_L = \frac{1}{v^2} \frac{\partial}{\partial v} \int_0^\infty dw\; w K(v, w) \left(\frac{1}{v} \frac{\partial}{\partial v} - \frac{1}{w} \frac{\partial}{\partial w}\right) f(v) f(w), \tag{11}$$

where

$$K(v, w) = K(w, v) = \frac{\pi}{4} \int_{-1}^{1} d\mu (1 - \mu^2) \Phi\left(\sqrt{v^2 + w^2 - 2\mu v w}\right). \tag{12}$$

This form of Q_L in the radial case is very convenient for an obvious proof (by substitution $x = v^2$) of H-theorem. We can also transform Q_L to its more conventional form

$$Q_L(f, f) = \frac{1}{v^2}\frac{\partial}{\partial v}\left[A(v)\frac{\partial f}{\partial v} + B(v) f\right], \tag{13}$$

where

$$A(v) = \frac{1}{v}\int_0^\infty dw w K(v, w) f(w), \quad B(v) = \int_0^\infty dw f(w)\frac{\partial}{\partial w}K(v, w). \tag{14}$$

For brevity we do not indicate explicitly f-dependence of A and B. Note that

$$\frac{\partial}{\partial w}K(v, w) = \frac{\pi}{8}\frac{\partial}{\partial w}(vw)^2 \int_{(v-w)^2}^{(v+w)^2} dz\Phi(\sqrt{z})\left[1 - \left(\frac{v^2 + w^2 - z}{2vw}\right)^2\right]$$
$$= \frac{\pi}{8} w \int_{(v-w)^2}^{(v+w)^2} dz\Phi(\sqrt{z})(v^2 - w^2 + z).$$

In the case of power-like potentials we formally have $\Phi(|u|) = c_\gamma |u|^\gamma,\ 1 > \gamma \geq -3$, with some irrelevant constant $c_\gamma > 0$. Then

$$\begin{aligned} K(v, w) &= \frac{\pi c_\gamma}{32}\int_{(v-w)^2}^{(v+w)^2} dz\, z^{\gamma/2}\,[4v^2w^2 - (v^2 + w^2 - z)^2] \\ &= \frac{\pi c_\gamma}{16}\left\{\frac{2(v^2 + w^2)}{4 + \gamma}\left[(v + w)^{4+\gamma} - |v - w|^{4+\gamma}\right]\right. \\ &\quad - \frac{(v + w)^{6+\gamma} - |v - w|^{6+\gamma}}{6 + \gamma} - \\ &\quad \left.- \frac{(v^2 - w^2)^2}{2 + \gamma}\left[(v + w)^{2+\gamma} - |v - w|^{2+\gamma}\right]\right\}, \quad \gamma \neq -2. \end{aligned} \tag{15}$$

In the most important Coulomb case $\gamma = -3$ we obtain

$$K(v, w) = \frac{1}{3}\pi c_{-3} \min(v^3, w^3).$$

Then we can formally set $c_{-3} = \pi^{-1}$ and obtain

$$A(v) = \frac{1}{3v}\left[\int_0^v dw\, w^4 f(w) + v^3 \int_v^\infty dw\, w\, f(w)\right],$$
$$B(v) = \int_0^v dw\, w^2 f(w) \tag{16}$$

in the true Landau equation (13). In case $\gamma \neq -3$ we shall be interested below mainly in asymptotics of solutions for large values of v. Equation (15) yields for $\gamma \geq -3$

$$K_\gamma(v, w) \simeq \frac{\pi c_\gamma}{3} v^{3+\gamma} w^3, \quad \frac{\partial}{\partial w}K_\gamma(v, w) \simeq \pi c_\gamma v^{3+\gamma} w^2, \quad v \to \infty.$$

Hence, we obtain from (14) assuming that $\pi c_\gamma = 1$ for any $\gamma \geq -3$

$$A_\gamma(v) \approx \frac{1}{3} v^{2+\gamma} \int_0^\infty dw w^4 f(w),$$

$$B_\gamma(v) \approx v^{3+\gamma} \int_0^\infty dw w^2 f(w), \quad v \to \infty. \tag{17}$$

These asymptotic formulas will be used in Sect. 4.

Next we consider the true (radial) Landau equation

$$\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left[A(v) \frac{\partial f}{\partial v} + B(v) f \right], \quad f|_{t=0} = f_0(v), \tag{18}$$

where $A(v)$ and $B(v)$ are given in (16). It was already mentioned in Introduction that the problem of existence of classical solution $f(v,t)$ remains open. On the other hand, the existence of weak solution to the problem (18) (in its more general non radial version) is proved by Villani [9] and Desvillettes [13]. This fact together with many published numerical solutions of the problem (18) may be important arguments in favor of existence and uniqueness of the classical solution.

We derive below a related model equation, for which the existence of classical solution can be probably proved by standard means of the theory of nonlinear parabolic equations. Let us consider the right hand side of equation (18), (16) in the form (11) with $K(v,w) = \min(v^3, w^3)$.

Then we introduce a new variable $x = v^2$ and rewrite (18), (16) in the form

$$f_t = \frac{1}{x^\theta} \frac{\partial}{\partial x} \int_0^\infty dy \min(x^{1+\theta}, y^{1+\theta}) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) f(x) f(y),$$

$$f|_{t=0} = f_0(x), \tag{19}$$

where $\theta = 1/2$. This equation, however, can be also considered for any $\theta \in [0, 1/2]$. It has two conservation laws and H-theorem:

$$\frac{dm_0(t)}{dt} = \frac{dm_1(t)}{dt} = 0, \quad \frac{dH(t)}{dt} \leq 0,$$

where

$$m_2(t) = \int_0^\infty dx x^{\theta+n} f(x,t), \quad H(t) = \int_0^\infty dx x^\theta f(x,t) \ln f(x,t).$$

Thus the main properties of Eq. (19) do not depend on specific value of the parameter θ. Instead of the equation with "true" value $\theta = 1/2$ we can consider a model equation with $\theta = 0$.Then we obtain

$$f_t = \partial_x [D(x,t) f_x + F(x,t) f], \quad f|_{t=0} = f_0(x),$$

$$D(x,t) = \int_0^\infty dy f(y,t) \min(x,y), \quad F(x,t) = \int_0^x dy f(y,t),$$

where the time-dependence of D and F is indicated explicitly. Note that

$$\lim_{x\to\infty} D(x,t) = m_1(t) = const, \quad \lim_{x\to\infty} F(x,t) = m_0(t) = const.$$

Without loss of generality we can assume that $m_0 = m_1 = 1$. Then the corresponding Maxwellian is $M(x) = \exp(-x)$. We integrate the above equation on $[0, x]$ and obtain $F_t = D(x,t)F_{xx} + FF_x$. Note that $D_x = 1 - F$, therefore

$$-F_t = D_{xt} = DD_{xxx} + (1 - D_x)D_{xx} = \partial_x(DD_{xx} - D_x^2 + D_x).$$

Assuming that $D_x \to 0$, $D_{xx} \to 0$ as $x \to \infty$, we obtain a usual ("local") nonlinear parabolic equation for $D(x,t)$:

$$D_t = DD_{xx} - D_x^2 + D_x, \quad D|_{t=0} = \int_0^\infty dy f_0(y) \min(x, y).$$

A behavior of $D(x,t)$ is defined by conditions $D(0,t) = 0,\ D_x(0,t) = 1$. Moreover,

$$1 \geq D_x = \int_x^\infty dy f(y,t) \geq 0, \quad D_{xx} = -f \leq 0, \quad \lim_{x\to\infty} D(x,t) = 1.$$

The existence and uniqueness of a smooth solution $D(x,t)$ can be probably proved by more or less standard methods. We do not try to do it in this paper. Instead we just note that the above equation can be simplified by substitution $u = D^{-1}$. Then we obtain the equation

$$u_t = \partial_x\left(\frac{1}{u}u_x + u\right).$$

Its solution, however, must be singular at $x = 0$. In particular, we expect that $u(x,t) \to (1 - e^{-x})^{-1}$, as $t \to \infty$.

In the next section we study some asymptotic properties of solution to the Landau equation.

4 Formation of Maxwellian Tails and Large Time Asymptotics

The Landau equation (10), (3) with $\Phi(|u|) = c_\gamma |u|^\gamma$ can be written in the radial case as

$$f_t = \frac{1}{v^2}\frac{\partial}{\partial v}\Big[A_\gamma(v) f_v + B_\gamma(v) f\Big], \quad f|_{t=0} = f_0(v), \tag{20}$$

where $A_\gamma(v)$ and $B_\gamma(v)$ are given in (14), (15). We are mainly interested in the case $-3 \leq \gamma < 0$ of soft potentials in this section. Our goal is to construct (at the formal level) asymptotic solutions for large v, which describe a formation of Maxwellian tails for a class of initial data. The construction is based on ideas of the paper [18]. Then we will discuss an interesting connection of these asymptotic solutions with some rigorous mathematical results on rate of relaxation to equilibrium for the Landau equation [19].

To this end, we go back to (20) to note that there are two conservation laws

$$m_0 = \int_0^\infty dv f(v,t) v^2 = const, \quad m_1 = \int_0^\infty dv f(v,t) v^4 = const.$$

In addition, according to (17), the nonlocal coefficients take the form

$$A_\gamma(v) \simeq \frac{m_1}{3} v^{2+\gamma}, \quad B_\gamma(v) \simeq m_0 v^{3+\gamma}, \quad v \to \infty.$$

Without loss of generality we can assume that $m_0 = (4\pi)^{-1}, \; m_1 = 3m_0$. Then we scale the time in (20) by equality $\tilde{t} = |\gamma| t / 4\pi$ and obtain the following asymptotic equation:

$$|\gamma| f_t = \frac{1}{v^2} \frac{\partial}{\partial v} v^{3+\gamma} \left(\frac{1}{v} f_v + f \right), \quad v \gg 1, \tag{21}$$

where tildes are omitted. We usually assume the convergence to equilibrium, i.e. that

$$f \xrightarrow[t\to\infty]{} M(v) = (2\pi)^{-3/2} e^{-v^2/2}. \tag{22}$$

In the case $-3 \le \gamma < -2$ of very soft potentials (note that $\gamma = -3$ corresponds to the true Landau equation) all coefficients in the right hand side of (21) vanish for large v. Roughly speaking, this means that interaction is relatively weak for $v \gg 1$. Suppose that the initial function $f_0(v)$ is concentrated in the ball $|v| \le R, \; R = O(1)$. Then one can expect two different processes: (a) relatively fast relaxation to equilibrium for thermal velocities $v = O(1)$ and (b) relatively slow propagation of $f(v,t)$ to the domain of large v. This kind of behavior was clearly seen in early numerical experiments for the Landau equation in [23].

In order to construct a corresponding asymptotic solution we transform (21) by substitution $f(v,t) = M(v) u(x,t), \; x = v^\beta$, with some positive parameter β. Then we obtain

$$|\gamma| u_t = \beta^2 x^{2+\frac{\alpha-5}{\beta}} u_{xx} - \beta x^{1+\frac{\alpha-3}{\beta}} \left[1 - (\beta + \alpha - 2) x^{-2/\beta} \right] u_x, \quad \alpha = 3 + \gamma,$$

and choose the value $\beta = (5-\alpha)/2$. We can also neglect the small (for $x \to \infty$) term in brackets and obtain

$$|\gamma| u_t = \beta^2 u_{xx} - \beta x^{2/\beta - 1} u_x, \quad \beta = \frac{2-\gamma}{2}.$$

We consider below only negative values of γ (soft potentials) and rewrite this equation as

$$u_t + \frac{1}{p} x^{1-p} u_x = a^2 u_{xx}, \;\; p = -\frac{2\gamma}{2-\gamma}, \;\; a^2 = \frac{(2-\gamma)^2}{4|\gamma|}, \quad \gamma < 0.$$

If we neglect the second derivative, then the general solution reads: $u(x,t) = F(x^p - t)$, with arbitrary function F. This motivates the following transformation: $u(x,t) = \varphi(z,t), \; z = x - t^{1/p}$. Then we obtain

$$\varphi_t + \frac{t^{(1-p)/p}}{p} \left[\left(1 + \frac{z}{t^{1/p}} \right)^{1-p} - 1 \right] \varphi_z = a^2 \varphi_{zz}.$$

An asymptotic equation for large $t > 0$ and bounded $z \in [-R, R]$ reads

$$\varphi_t + \frac{(1-p)z}{pt} \varphi_z = a^2 \varphi_{zz}, \quad |z| \ll t^{1/p}. \tag{23}$$

Note that the function $u = \varphi = 1$ corresponds to the equilibrium state. We assume that the relaxation process is already finished for "thermal" values $x = O(1)$. On the other hand, we consider the initial data with compact support and expect relatively slow propagation to the domain $x \gg 1$. Hence, we can formally consider all real values of z and prescribe the following boundary conditions:

$$\lim_{z\to-\infty} \varphi(z,t) = 1, \quad \lim_{z\to\infty} \varphi(z,t) = 0.$$

It is natural to consider self-similar solutions of (23) such that $\varphi(z,t) = \psi(y), \quad y = z/\sqrt{t}$. The equation for $\psi(y)$ reads

$$a^2\psi'' + \left(1 + \frac{1}{\gamma}\right) y\psi' = 0, \quad a^2 = \frac{(2-\gamma)^2}{4|\gamma|}.$$

The boundary condition for large positive y can be fulfilled only for $\gamma < -1$. Then we obtain, by using both boundary conditions for $y \to \pm\infty$, a unique solution

$$\psi(y) = \frac{b}{\sqrt{\pi}} \int_y^\infty ds \exp(-b^2 s^2), \quad b^2 = \frac{2(|\gamma|-1)}{(2+|\gamma|)^2}, \quad \gamma < -1. \tag{24}$$

Thus, we obtain an approximate solution of Eq. (21) in the self-similar form

$$f(v,t) \simeq M(v)F(v,t), \quad F(v,t) = \psi\left[\frac{v^\beta - v_f^\beta(t)}{\sqrt{t}}\right],$$

$$v_f(t) = t^{1/\beta p} = t^{-1/\gamma}, \quad \beta = \frac{2-\gamma}{2}, \quad p = -\frac{\gamma}{\beta}, \quad \gamma < -1, \tag{25}$$

where $M(v)$ and $\psi(y)$ are given in (22) and (24). The function $F(v,t)$ looks like a traveling wave with asymptotic values $F_- = 1$ on the left and $F_+ = 1$ on the right. The front of the wave is defined by equality $F[v_f(t), t] = 1/2$, then we obtain $v_f(t) = t^{-1/\gamma}, \ \gamma < -1$.

A remarkable property of the wave is the stability of its structure. Indeed the width of the front can be characterized by a quantity

$$l_f(t) = F(v_f,t)/|F_v(v_f,t)| = \left(-2F_v(v_f,t)\right)^{-1}.$$

We obtain from (24), (25)

$$l_f^{-1}(t) = 2\psi'(0)\beta\, v_f^{\beta-1}(t)t^{-1/2} = \sqrt{\frac{2}{\pi}(|\gamma|-1)} = const, \quad \gamma < -1,$$

i.e. the width of does not depend on time. The above asymptotic solution of Eq. (21) is expected to be valid for large v and t provided $\gamma < -1$. A similar asymptotics for the Landau equation (20) is conjectured for a class of initial data concentrated in the thermal domain. This conjecture is in excellent agreement with numerical results (see [18] for details of comparison). It would be interesting to find a rigorous proof.

What can be said about the rate of convergence to equilibrium for these approximate solutions? Let us consider the difference

$$\Delta(v,t) = M(v) - f(v,t) = M(v)\psi_1[y(v,t)],$$

$$y(v,t) = \frac{v^\beta - v_f^\beta(t)}{\sqrt{t}}, \quad \psi_1(y) = \frac{b}{\sqrt{\pi}} \int_{-\infty}^{y} ds\, e^{-b^2 s^2}.$$

We are interested in the distance $\delta(t)$ between $f(v,t)$ and $M(v)$ given by

$$\delta(t) = \|\Delta(v,t)\| = \max_{v\geq 0} \Delta(v,t), \quad t \to \infty.$$

Note that

$$\Delta(v_f(t),t) = \frac{1}{2}M[v_f(t)] = \frac{1}{2(2\pi)^{3/2}} \exp(-t^{2/|\gamma|}).$$

Hence

$$\delta(t) \geq c_1 \exp\left(-\frac{1}{2}t^q\right), \quad q = \frac{2}{|\gamma|}, \quad c_1 = \frac{1}{2}(2\pi)^{3/2}, \quad \gamma < -1.$$

Moreover $\psi_1(y) < 1$, therefore

$$\delta_1(t) = \max_{v \geq v_f(t)} \Delta(v, t) \leq M[v_f(t)] = 2c_1 \exp\left(-\frac{t^q}{2}\right).$$

It remains to estimate $\delta(t)$ for $0 \leq v \leq v_f(t)$. We denote

$$x t^{1/|\gamma| + 1/2} = v_f^\beta - v^\beta, \; v_f = t^{1/|\gamma|}, \; \beta = \frac{2 + |\gamma|}{2}.$$

Then

$$v = \tau(1 - x)^{r/2}, \; 0 < x \leq 1, \; \tau = t^{1/|\gamma|}, \; r = 2/\beta,$$
$$\Delta(v, t) = g_r(x, \tau) = 2c_1 \psi(x\tau) \exp[-\frac{\tau^2}{2}(1 - x)^r]$$

in the notation of (24). Note that $r < 4/3$ if $\gamma < -1$, therefore $(1 - x)^r \geq (1 - x)^2$. Hence, $0 < g_r(x, \tau) \leq g_2(x, \tau)$. It is easy to show that $g_2(x, \tau)$ has a unique maximum at $x \simeq (1 + 2b^2)^{-1}$ for large τ. Finally we obtain the estimate

$$c_1 \exp\left(-\frac{t^q}{2}\right) \leq \delta(t) \leq c_2 \exp\left(-\theta \frac{t^q}{2}\right), \quad q = \frac{2}{|\gamma|},$$

with some constant $0 < \theta < 1$. Thus, for very soft potentials with $-3 \leq \gamma < -2$ we have an explicit example of approximate solution to the Landau equation, which rate of relaxation is defined by $\exp(-\lambda t^q)$, $0 < q < 1$. The exponent $q = 2/|\gamma|$ has precisely the value predicted earlier in Theorem 1 of the paper [19].

Below we briefly formulate the results of Sect. 4 for the sake of reader's convenience.

Proposition *The radial Landau equation* (20) *is formally reduced for large velocities* $v \gg 1$ *to linear parabolic equation* (21)*. This equation admits an approximate solution having the self-similar form* (25)*. In the case* $-3 \leq \gamma < -2$ *of very soft potentials the rate of decay to equilibrium for the approximate solution is exactly* $O(\exp(-ct^q))$ *with the value of* $0 < q < 1$ *predicted in* [19]*. It is expected (and confirmed numerically) that this self-similar solution describes an asymptotic behavior of a relatively large class of initial data for the Landau equation.*

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