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# Chaos: Butterflies also Generate Phase Transitions

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**Abstract** We exhibit examples of mixing subshifts of finite type and of continuous potentials such that there are phase transitions but the pressure is always strictly convex. More surprisingly, we show that the pressure can be analytic on some interval although there exist several equilibrium states.

**Keywords** Thermodynamic formalism · Equilibrium state · Phase transitions

**Mathematics Subject Classification** 37D35 · 37A60 · 82C26

## 1 Introduction

### 1.1 General Background

In this paper, we deal with the notion of phase transitions. More precisely, adopting the Dynamical System viewpoint, we study the shape of the graph for the pressure function and (non-)equivalence between analyticity and presence of several equilibrium states.

It is noteworthy that depending on the viewpoint (Statistical Mechanics, Probability Theory or Dynamical Systems), the settings, the questions and the interests concerning phase transitions are different. Furthermore, and this may be source of confusion, the several viewpoints share the same terminology and vocabulary for sensibly different objects or notions.

In Statistical Mechanics and in Probability Theory, one usually considers lattices with interaction energy between the sites. Often, the geometry of the lattice and the decay of correlation of interactions are the main issues. In Dynamical Systems, one usually consider one-dimensional lattice $\Gamma^{\mathbb{Z}}$ (with natural $\mathbb{Z}$-actions) and the main issue is the regularity of

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the potential considered as a function on $\Gamma^{\mathbb{Z}}$ (see e.g. Sect. 2.4.4 in [25] for a discussion on that point).

The definition of phase transition also naturally depends on the viewpoint. The Ising model (see e.g. [10,13]) and the Potts model are studied in Probability Theory, either in mean fields case (see [1,4,11]), or via percolation theory (see [3,15]). In both cases, a phase transition means the co-existence of several probability measures resolving or resulting from some optimization.

In Physics (see e.g. [25] Sect. 2.6.5), a phase transition can either mean a singularity of some thermodynamic quantity (Ehrenfest classification) or a change in the number of macro states (Gibbs classification). A more recent definition involves the Gâteaux differentiability of the pressure function (see e.g. [8,9,14,20,22]).

The topic is actually relatively new in Dynamical Systems (compared to Statistical mechanics and Probability Theory), and one usually considers that a phase transition occurs when the pressure function stops to be analytic (see [5,7,16,19]).

Our main result (Theorem B) is that both definitions (regularity of the pressure vs co-existence of several equilibria) are not equivalent. Of course, a first-order phase transition, that is when the pressure function is not $\mathcal{C}^1$, yields co-existence of several equilibria. But we show here that the converse is not true: there are mixing systems such that the pressure function is analytic on some interval despite the existence of several equilibria simultaneously.

Theorem A deals with the possible shape of the graph of the pressure function when a phase transition occurs. It turns out that most of the known examples of phase transitions in Dynamical Systems are "freezing" phase transitions, that is that the pressure function is eventually affine. Such transitions are known in Statistical Mechanics as the Fisher–Felderhof models (see e.g. [12] for a one-dimensional lattice case). Physically, this means that for some positive temperature, the system reaches its/one ground state and then stops to change.

It was thus natural to inquire about the possibility to get phase transitions in Dynamical Systems which are not freezing, that is, that after the transition the pressure is non-flat. This is the purpose of our Theorem A.

It is not clear whether there exist dictionaries between the different viewpoints to study phase transition. We hope that beyond the mathematical problems we solve here, the present paper could help clarify similarities and differences between the viewpoints.

On the other hand, we believe that several viewpoints is also source of fruitful transfer of knowledge. The thermodynamic formalism was introduced in Dynamical Systems during the 1970s, mostly by Sinai, Ruelle and Bowen (see [2,22–24]), initially for hyperbolic systems. Over the years, mathematicians became more interested in developing this formalism for less regular systems, also motivated by its applications to dimension theory and multifractal analysis. As a result, the natural questions in mathematics became further remote from physicists' interests. Due to the development of the ergodic optimization around 2000, mathematicians gradually rediscovered notions already known to physicists, such as e.g. the *ground states*.

We believe that the mathematical tools already developed in the thermodynamic formalism by mathematicians can be useful to physicists. Our way to prove existence of phase transition here is based on the study of some inducing scheme and on operator theory. This theory, which includes conformal measures, eigen-functions and subactions (for the zero temperature case) could be used in physics.

## 1.2 General Settings

We consider a subshift of finite type $\Sigma$ on a finite alphabet $\mathcal{A}$. An element of $\mathcal{A}$ can either be called a letter, or a digit or a symbols. Different alphabets will be considered depending on the theorem.

A point $x$ in $\Sigma$ is a sequence $x_0, x_1, \dots$ (also called an infinite word) where $x_i$ are letters of $\mathcal{A}$. Admissible transitions are given by an oriented graph. Two cases will be considered (see Figs. 1, 7). Most of the times we shall use the notation $x = x_0x_1x_2\dots$.

The distance between two points $x = x_0x_1\dots$ and $y = y_0y_1\dots$ is given by

$$d(x, y) = \frac{1}{2^{\min\{n,\ x_n \neq y_n\}}}.$$

A finite string of symbols $x_0\dots x_{n-1}$ is also called a *word*, of length $n$. For a word $w$, its length is $|w|$. A *cylinder* (of length $n$) is denoted by $[x_0\dots x_{n-1}]$. It is the set of points $y$ such that $y_i = x_i$ for $i = 0, \dots n-1$.

If $i$ is a digit in $\mathcal{A}$, $x = i^n*$ means that $x = \underbrace{i\dots i}_{n \text{ digits}}\, j$ where $j$ is any digit $\neq i$ such that $ij$ is admissible in $\Sigma$. In other words, this means that $x \in [\underbrace{i\dots i}_{n \text{ digits}}]\backslash[\underbrace{i\dots i}_{n+1 \text{ digits}}]$.

The alphabet will depend on some integer parameter $L$. It will be either

$$\{1_0, 1_1, 1_2, \dots, 1_L, 2, 3, 4\} \text{ or } \{1_0, 1_1, 1_2, \dots, 1_L, 2, 3, 4, 3', 4'\},$$

and $L$ may be equal to 0. For simplicity we set $1_0 =: 1$.

Let us consider positive real numbers $\alpha, \gamma, \delta$ and $\varepsilon$ *considered as fixed parameters*. The potential $\phi$ is defined by

$$\phi(x) = \begin{cases} -\alpha < 0 \text{ if } x_0 \in \{1, 1_1, 1_2, \dots, 1_L\} \\ -\log\left(\frac{n+1}{n}\right) \text{ if } x = 2^n*,\ 1 \leqslant n \leqslant +\infty \\ \gamma - \varepsilon\log\left(\frac{n+1}{n}\right) \text{ if } x_0 = 3 \text{ or } 3' \text{ and } n = \min\{j \geqslant 1,\ x_j = 2\}, \\ \gamma + \delta - \varepsilon\log\left(\frac{n+1}{n}\right) \text{ if } x_0 = 4 \text{ or } 4' \text{ and } n = \min\{j \geqslant 1,\ x_j = 2\}. \end{cases}$$

Hence, if $x = 2*$, $\phi(x) = -\log 2$. If $x = 3\underbrace{x_1\dots x_{n-2}}_{n-2 \text{ digits}=3,4}32\dots$, then $\phi(x) = \gamma - \varepsilon\log\left(\dfrac{n+1}{n}\right)$. If $x = 4\underbrace{x_1\dots x_{n-2}}_{n-2 \text{ digits}=3,4}32\dots$, then $\phi(x) = \gamma + \delta - \varepsilon\log\left(\dfrac{n+1}{n}\right)$. By convention $\dfrac{+\infty+1}{+\infty} = 1$, which defines $\phi(x)$ for $x$ in e.g. [3] with no digit 2. Note that $\phi$ is continuous.

We recall that for $\beta \geqslant 0$, the pressure function $\mathcal{P}(\beta)$ is defined by

$$\mathcal{P}(\beta) = \max_{\mu\ T-inv}\left\{h_\mu + \beta\int\phi\, d\mu\right\},$$

where $h_\mu$ is the Kolmogorov entropy of the measure $\mu$. We refer the reader to [2] for classical results on thermodynamic formalism of the shift. A measure for which the maximum is attained in the above equality is called an *equilibrium state* for $\beta.\phi$. Existence of an equilibrium state simply follows from the continuity of $\phi$ and the upper semi-continuity of $\mu \mapsto h_\mu$.

**Definition 1** We say that $\mathcal{P}(\beta)$ (or equivalently that the potential $\phi$) has a phase transition (at $\beta_c$) if $\beta \mapsto \mathcal{P}(\beta)$ is not analytic at $\beta = \beta_c$.

### 1.3 Results

We first consider the case $\mathcal{A} = \{1, 1_1, 1_2, \dots, 1_L, 2, 3, 4\}$. The transitions are given by Fig. 1. This gives a "butterfly" with two wings, each one tending to be autonomous. Both wings $\{1, 1_1, \dots, 1_L\}^{\mathbb{N}}$ and $\{3, 4\}^{\mathbb{N}}$ are full shifts of finite type. The unique exit letters from the wings are respectively 1 and 3, and any transition from one wing to the other must pass through 2. Digits 1 and 3 are also the unique entrance digits into the wings.

We emphasize that the system is irreducible but has several subsystems. In particular we shall consider $\Sigma_{34} := \{3, 4\}^{\mathbb{N}} \subset \Sigma$ and $\Sigma_{234} := \{2, 3, 4\}^{\mathbb{N}} \cap \Sigma$, the restriction of $\Sigma$ to the invariant set of infinite words containing only the letters 2, 3 or 4. For the same potential $\phi$, we shall consider the associated pressure functions, $\mathcal{P}_{34}(\beta)$ and $\mathcal{P}_{234}(\beta)$. We leave it to the reader to check that $\mathcal{P}_{34}(\beta) = \gamma.\beta + \log(1 + e^{\delta.\beta})$.

**Theorem A** *There exist two positive real numbers $\beta_1 < \beta_c$ such that $\mathcal{P}(\beta)$ and $\mathcal{P}_{234}(\beta)$ have a phase transition at $\beta_c$ and $\beta_1$ respectively. More precisely (see Fig. 2),*

(1) *the pressure function $\mathcal{P}_{34}(\beta)$ is analytic and strictly convex,*

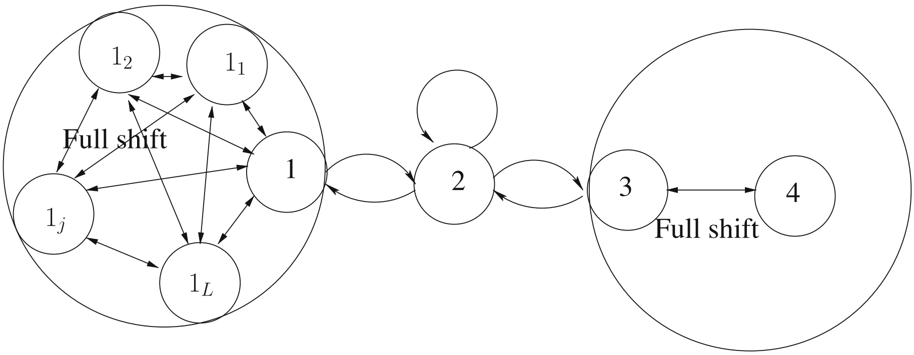
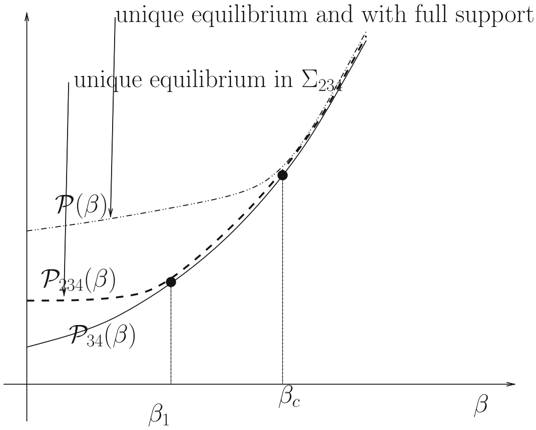



**Fig. 1** Dynamics for Theorem A

**Fig. 2** Different graphes



(2) *for $\beta < \beta_c$, the pressure function $\mathcal{P}(\beta)$ is analytic and satisfies $\mathcal{P}(\beta) > \mathcal{P}_{34}(\beta)$,*
(3) *for $\beta \geqslant \beta_c$, the pressure function $\mathcal{P}(\beta)$ satisfies $\mathcal{P}(\beta) = \mathcal{P}_{34}(\beta)$,*
(4) *for $\beta < \beta_1$, the pressure function $\mathcal{P}_{234}(\beta)$ is analytic and satisfies $\mathcal{P}_{234}(\beta) > \mathcal{P}_{34}(\beta)$,*
(5) *for $\beta \geqslant \beta_1$, the pressure function $\mathcal{P}_{234}(\beta)$ satisfies $\mathcal{P}_{234}(\beta) = \mathcal{P}_{34}(\beta)$.*

*For $\beta \leqslant \beta_1$, there is a unique equilibrium state for $\Sigma_{234}$ and it has full support in $\Sigma_{234}$.*

*For $\beta < \beta_c$, there is a unique equilibrium state for $\Sigma$ and it has full support. For $\beta > \beta_c$ there is a unique equilibrium state and it has support in $\Sigma_{34}$.*

*For $\beta = \beta_c$, there are two ergodic equilibrium states for $\Sigma$ if and only if $\varepsilon\beta_c > 2$. This inequality can be realized (depending of the parameters $\alpha$, $\varepsilon$, $\gamma$ and $\delta$) or non-realized.*

*At each phase transition ($\beta = \beta_1$ or $\beta = \beta_c$) the entropy is positive.*

*The pressure is differentiable at the critical value if and only if $\varepsilon.\beta_c \leqslant 2$.*

As it was said above, the main motivation for Theorem A was to build phase transitions with a non-flat pressure function after the transition. Actually such an example was already known in [6]. However, in that case, the map is a skew product over a Horseshoe and the potential, the logarithm of the derivative in the central direction (equivalent to the direction of fibers). Then, when it is projected onto the Horseshoe, the potential is not a function anymore. Hence, that phase transition cannot be realized as a continuous potential defined on a subshift of finite type.

To be complete concerning the shape of the pressure, we also have to mention [16]. There, they also show that the pressure may be non-flat after the transition. However, and this is the main difference, in their construction they always need some interval where the pressure is flat. In their words, for this interval the system is transient, which means that it has no conformal measure. In our example, at any $\beta$ there exists a conformal measure and the pressure is always strictly convex.

Let us now present the main result of the paper. As we said above, the regularity of the pressure function is an indicator of the uniqueness of the equilibrium state. This regularity can be understood in two different ways.

First, regularity of the map $\beta \mapsto \mathcal{P}(\beta)$ is studied. Conversely, non-regularity yields different order of phase transitions as was seen above.

Another point is the Gâteaux-differentiability of the functional $\mathcal{P}(\phi + .)$ on $\mathcal{C}(\Sigma)$. In [26, Cor.2], it is shown that Gâteaux-differentiability at $\phi$ is equivalent to uniqueness of the equilibrium state for $\phi$. Of course the latter point is more subtle than the regularity of $\mathcal{P}(\beta)$. It was usually expected that the analyticity for $\mathcal{P}(\beta)$ would insure the uniqueness of the equilibrium state. We prove here that this is not the case.

**Theorem B** *There exist irreducible subshifts of finite type and continuous potentials such that their pressure function is analytic on some interval $]\beta'_c, +\infty[$ but there coexist several equilibrium states.*

*There exist an infinite-dimensional space of functions $\varphi$ such that for every $\beta > \beta'_c$, $\varphi \mapsto \mathcal{P}(\phi + \varphi)$ is Gâteaux-differentiable in the $\varphi$-direction.*

We recall that a function $f$ is Gâteaux-differentiable at $x$ in the direction $y$ if

$$\lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}$$

exists.

## 1.4 Overview of the Paper. Main Tool. Heuristic Explanation of the Phase Transitions

### *1.4.1 Overview of the Paper*

The main tool is the notion of local equilibrium state as it was introduced in [17] and developed in further works by the author. We briefly recall below the principal points of that theory.

In Sect. 2 we prove most of the results concerning Theorem A and the phase transition for $\mathcal{P}(\beta)$. In Sect. 3 we prove the "small" phase transition for $\Sigma_{234}$ and finish the proof of Theorem A. In Sect. 4, we prove Theorem B. To do so, we add a new wing $\{3', 4'\}^{\mathbb{N}}$ copying the wing $\{3, 4\}^{\mathbb{N}}$. We show that the effect of this addition is just to move $\beta_c$ but does not change that fact that $\mathcal{P}(\beta)$ eventually equals $\mathcal{P}_{34}(\beta)$. But $\mathcal{P}_{34}(\beta) = \mathcal{P}_{3'4'}(\beta)$ and then two equilibria coexist, one in $\Sigma_{34}$, the other in $\Sigma_{3'4'}$.

### *1.4.2 Local Equilibrium*

We refer to [18] for a synthesis concerning the notion of local equilibrium. The principal points are the following.

Consider a cylinder $J$ and denote by $\tau_J(x)$ the first return time into the cylinder $J$. Let $g$ denote the first return map into $J$, *i.e.,* $g(x) := \sigma^{\tau_J(x)}(x)$. Note that it is not defined *everywhere* but the inverse branches are well-defined. We recall $\phi + \ldots + \phi \circ \sigma^{n-1}$ is denoted by $S_n(\phi)$. Given $Z \in \mathbb{R}$, the *Induced Transfer Operator on the cylinder J* for the first return time map and for the potential

$$\beta\phi(x) + \cdots + \beta\phi \circ \sigma^{\tau_J(x)-1}(x) - \tau_J(x)Z,$$

is defined by

$$\mathcal{L}_{Z,\beta,J}(\psi)(x) := \sum_{y:\, g(y)=x} e^{S_{\tau_J(y)}(\beta.\phi)(y) - Z\tau_J(y)}\psi(y),$$

where $\psi : J \to \mathbb{R}$ is continuous.

The main point is that for every $\beta$, there exists $Z_c(\beta)$ such that $\mathcal{L}_{Z,\beta,J}$ is well-defined for $Z > Z_c(\beta)$ (and sometimes also for $Z = Z_c(\beta)$) and $\mathcal{L}_{Z,\beta,J}(\mathbb{1}_J)(x)$ diverges for every $x$ and $Z < Z_c(\beta)$. For these $Z$'s, if $\lambda_{Z,\beta,J}$ denotes the spectral radius of $\mathcal{L}_{Z,\beta,J}$, then, the existence of a global equilibrium state is related to the value $\lim\limits_{Z \downarrow Z_c(\beta)} \lambda_{Z,\beta,J} =: \lambda_c(\beta)$:

(1) If $\lambda_c(\beta) > 1$, there exists a unique equilibrium state for $\beta.\phi$ and it has full support.
(2) If $\lambda_c(\beta) < 1$, no equilibrium state for $\beta.\phi$ gives positive weight to $J$.
(3) The case $\lambda_c = 1$ is the critical one. There exists an equilibrium state with full support if and only if $\mathcal{L}_{Z_c(\beta),\beta,J}(\tau_J) < +\infty$.

This machinery works if $\phi$ is such that the induced potential $\beta\phi(x) + \cdots + \beta\phi \circ \sigma^{\tau_J(x)-1}(x) - \tau_J(x)Z$ has bounded variations with respect to the induced map $g$. We shall check this point. More precisely, we shall also choose $J$ properly such that the computation of $\lambda_{Z,\beta,J}$ is easy. In particular we will consider $J$ satisfying $\mathcal{L}_{Z,\beta,J}(\mathbb{1}_J)(x) = \lambda_{Z,\beta,J}$ for every $x$ in $J$.

### *1.4.3 Why Do Phase Transitions Arise*

Continuing with the same notations, we can show that $Z_c(\beta) \leqslant \mathcal{P}(\beta)$ (see [18, Prop.2.2]). Then, roughly speaking, there exists an equilibrium state $\mu_\beta$ for $\beta.\phi$ satisfying $\mu_\beta(J) > 0$

if and only if $\lambda_{\mathcal{P}(\beta),\beta,J} = 1$. If this later condition does not hold, then an equilibrium state has support in the dotted system, that is, the set of trajectories which avoid the cylinder $J$.

In our case, we emphasize that $\phi$ is very negative on the left wing, lightly negative on [2] and extremely positive on the right wing. For $\beta = 0$, the entropy makes the difference, but as $\beta$ increases, entropy is not sufficient to balance the sign of $\beta.\phi$. At some point, the potential is too positive in the dotted system and too negative on the left wing to compensate the difference of entropy. This gives the phase transition at $\beta_c$ where the pressure of subsystem $\Sigma_{234}$ equals the global pressure.

Choosing $J$ properly, we show that this also happens for $\Sigma_{34}$ with respect to $\Sigma_{234}$ at $\beta_1$.

*Remark 1* This is e.g. exactly what happens for the Manneville–Pomeau map and the Hofbauer potential. □

## 2 The Phase Transition for $\mathcal{P}(\beta)$

### 2.1 The Induced Operator on [1]

We study the induced transfer operator on the cylinder [1]. Denote by $\tau_{[1]}(x)$ the first return time into the cylinder [1] and by $g$ the first return map. We set

$$\mathcal{L}_{Z,\beta,[1]}(\psi)(x) := \sum_{y:\, g(y)=x} e^{S_{\tau_{[1]}(y)}(\beta.\phi)(y) - Z\tau_{[1]}(y)}\psi(y).$$

Trajectories returning to [1] after having left it are of the following form:

(1) They leave [1], visit $\{1_1, \ldots, 1_L\}$ for a while and then come back into [1].
(2) They leave [1], visit [2] for a while and then come back into [1].
(3) They leave [1], visit [2] for some time, visit {3, 4}, and more generally, alternate visit to [2] and to the right hand side wing, and eventually come back to [1] after a last visit to [2].

A counting argument yields

$$\begin{aligned}\mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]})(x) &= \sum_{n=1}^{+\infty} e^{-n\beta\alpha - nZ + (n-1)\log L} + \sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-nZ}\cdot e^{-\alpha\beta - Z}\\ &\quad\times\left[\sum_{k=0}^{+\infty}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-nZ}\cdot\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\varepsilon\beta} L_n(\beta,[3])e^{-nZ}\right)^{k}\right],\end{aligned} \tag{1}$$

where

$$L_n(\beta,[3]) = \sum_{w\in\{3,4\}^n,\, w_0=w_{n-1}=3} e^{\beta S_n(\phi)(w2)}. \tag{2}$$

The first summand corresponds to the first kind of trajectories. They are of the form $x = 1x_2x_3\ldots x_{n-1}1$ with $x_i \in \{1_1, \ldots, 1_L\}$. For those points we have that $\beta\phi(x) + \cdots + \beta\phi\circ\sigma^{\tau_{[1]}(x)-1}(x) - \tau_{[1]}(x)Z = -n\beta\alpha - nZ$. Moreover, since the system restricted to $\{1, 1_1, \ldots, 1_L\}$ is a full-shift on $L$ symbols, for each $n$ there are exactly $L^{n-1}$ different words of that form. Hence the factor $(n-1)\log L$.

The second term concerns points of the form $x = 12^n 1\dots$. In this case $e^{-\beta\alpha - Z}$ takes into account the potential at $x$ and the contribution of the string of 2's yields the term $\left(\frac{1}{n+1}\right)^{\beta}$.

Finally, trajectories of the third kind correspond to words of the form

1(string of 2's) (intermittence of eligible strings of 3's or 4's and strings of 2's) 1 ...

Note that an eligible string of 3's and 4's starts and finishes by 3, and such a string is necessarily followed by a string of 2's before returning to 1. Moreover, we emphasize that $\phi$ is constant on cylinders of the form $[\omega 2]$ where $\omega$ is an eligible word of finite length of 3's and 4's. The sum over $k$ in (1) is for $k$ visits (and possibly $k = 0$ for trajectories of the second form) to the right hand side wing.

Now, if $\omega = \omega_0 \dots \omega_{n-1}$ is a word of length $n$ with $k$ letters equal to 4 such that $\omega 2$ is eligible, then $S_n(\phi)(\omega 2)$ is equal to $n\beta + k\delta$ and $k \leqslant n-2$ because $\omega_0 = \omega_{n-1} = 3$. Therefore, the computation for trajectories in the right hand side wing gives

$$\begin{aligned} L_n(\beta, [3]) &= e^{n\beta\gamma} \sum_{k=0}^{n-2} e^{k\beta\delta} \binom{n-2}{k} \\ &= e^{n\beta\gamma}(1+e^{\beta\delta})^{n-2} = \frac{\left(e^{\gamma\beta} + e^{(\gamma+\delta)\beta}\right)^n}{(1+e^{\delta\beta})^2} = \frac{1}{(1+e^{\delta\beta})^2} e^{n\mathcal{P}_{34}(\beta)}. \end{aligned} \tag{3}$$

Let us set for simplicity

$$\Sigma_1 = \Sigma_1(Z,\beta) := \sum_{n=1}^{+\infty} e^{-n\beta\alpha - nZ + (n-1)\log L}$$

$$\Sigma_2 = \Sigma_2(Z,\beta) := \sum_{n=1}^{+\infty} \left(\frac{1}{n+1}\right)^{\beta} e^{-nZ}$$

$$\Sigma_3 = \Sigma_3(Z,\beta) := \frac{1}{(1+e^{\delta\beta})^2} \sum_{n=1}^{+\infty} \left(\frac{1}{n+1}\right)^{\varepsilon\beta} e^{n(\mathcal{P}_{34}(\beta) - Z)}.$$

With these notations we get that for every $x$ in [1]

$$\begin{aligned} \mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]})(x) &= \Sigma_1 + \Sigma_2 e^{-\alpha\beta - Z} \sum_{k=0}^{+\infty} \left(\Sigma_2\Sigma_3\right)^k \\ &= \Sigma_1 + \frac{\Sigma_2 e^{-\alpha\beta - Z}}{1 - \Sigma_2\Sigma_3}. \end{aligned}$$

### 2.2 Critical Value $Z_c(\beta)$ for $\mathcal{L}_{Z,\beta,[1]}$

The quantity $\mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]})$ is well-defined if and only if

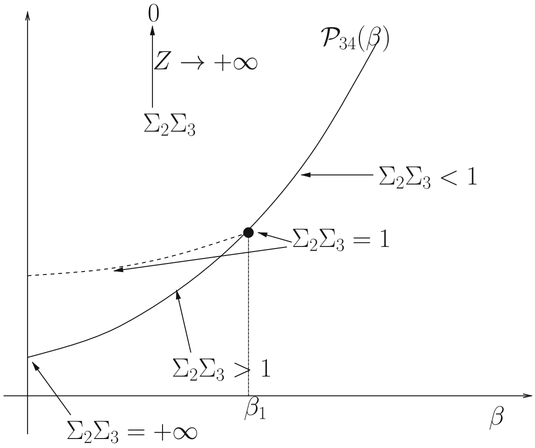
$$\Sigma_1 < +\infty, \tag{4a}$$

$$\Sigma_2\Sigma_3 < 1. \tag{4b}$$

Condition (4a) is satisfied if only if $Z > \log L - \alpha\beta$.

Let us now study Condition (4b). First, we emphasize that a necessary condition is $Z \geqslant \mathcal{P}_{34}(\beta)$. Now we have:

**Fig. 3** Existence of critical $\beta_1$




**Lemma 2** *There exists a maximal* $\beta_1 > 0$ *such that the implicit equation*

$$\Sigma_2\Sigma_3 = 1 \textit{ with constraint } Z \geqslant \mathcal{P}_{34}(\beta)$$

*admits a (unique) solution* $\widetilde{Z}_c(\beta)$ *for every* $0 \leqslant \beta \leqslant \beta_1$ *(See Fig. 3).*

*Proof* Both $\Sigma_2$ and $\Sigma_3$ are decreasing in $Z$ for fixed $\beta$, which yields uniqueness (if it exists) of $\widetilde{Z}_c(\beta)$ satisfying

$$\Sigma_2(Z, \beta)\Sigma_3(Z, \beta) = 1.$$

Note that both $\Sigma_2$ and $\Sigma_3$ go to 0 if $Z$ goes to $+\infty$. Then, the existence of $\widetilde{Z}_c(\beta)$ follows from the value of $\Sigma_2(\mathcal{P}_{34}(\beta), \beta)\Sigma_3(\mathcal{P}_{34}(\beta), \beta)$. If it is larger than 1 then $\widetilde{Z}_c(\beta)$ exits, if it is smaller than 1, then $\widetilde{Z}_c(\beta)$ does not exist. Now,

$$\Sigma_2(\mathcal{P}_{34}(\beta), \beta)\Sigma_3(\mathcal{P}_{34}(\beta), \beta) = \sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-n\mathcal{P}_{34}(\beta)}\frac{\zeta(\varepsilon\beta) - 1}{(1 + e^{\delta\beta})^2},$$

$\beta \mapsto \mathcal{P}_{34}(\beta)$ increases in $\beta$, thus $\beta \mapsto \Sigma_2(\mathcal{P}_{34}(\beta), \beta)\Sigma_3(\mathcal{P}_{34}(\beta), \beta)$ decreases in $\beta$. It goes to $+\infty$ if $\beta$ goes to 0 and to 0 if $\beta$ goes to $+\infty$.

Therefore, there exists a unique $\beta_1$ such that

$$\Sigma_2(\mathcal{P}_{34}(\beta_1), \beta_1)\Sigma_3(\mathcal{P}_{34}(\beta_1), \beta_1) = \sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta_1} e^{-n\mathcal{P}_{34}(\beta_1)}\frac{\zeta(\varepsilon\beta_1) - 1}{(1 + e^{\delta\beta_1})^2} = 1 \quad (5)$$

*Remark 2* We point out that $\varepsilon\beta_1 > 1$ because the Zeta function in (5) converges. □

Note that for every $\beta < \beta_1$, $\widetilde{Z}_c(\beta) > \mathcal{P}_{34}(\beta)$ and it is given by the following implicit formula

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-n\widetilde{Z}_c(\beta)}\frac{1}{(1 + e^{\delta\beta})^2}\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\varepsilon\beta} e^{-n(\widetilde{Z}_c(\beta) - \mathcal{P}_{34}(\beta))} = 1.$$

It shows that $\widetilde{Z}_c(\beta)$ is analytic in $\beta$ for $\beta < \beta_1$.

As a consequence of the proof of Lemma 2, it follows that:

(1) for every $0 \leqslant \beta < \beta_1$

   (a) $\widetilde{Z}_c(\beta) > \mathcal{P}_{34}(\beta)$,
   (b) $\Sigma_2\Sigma_3 < 1$ for $Z > \widetilde{Z}_c(\beta)$,
   (c) $\Sigma_2\Sigma_3 > 1$ for $\mathcal{P}_{34}(\beta) \leqslant Z < \widetilde{Z}_c(\beta)$,
   (d) $\Sigma_2\Sigma_3 = 1$ for $Z = \widetilde{Z}_c(\beta)$.

(2) For every $\beta > \beta_1$, Condition (4b) holds for every $Z \geqslant \mathcal{P}_{34}(\beta)$.
(3) For $\beta = \beta_1$ (4b) holds for every $Z > \mathcal{P}_{34}(\beta_1)$.

*Remark 3* We will see below that $\widetilde{Z}_c(\beta) = \mathcal{P}_{234}(\beta)$. See also [18, Th 1] □

The function $\beta \mapsto \log L - \alpha\beta$ is decreasing and goes to $-\infty$ if $\beta \to +\infty$, whereas $\mathcal{P}_{34}(\beta)$ increases to $+\infty$. Therefore, there eventually exists $\beta_2 \geqslant 0$ such that $\mathcal{P}_{34}(\beta) \geqslant \log L - \alpha\beta$ if and only if $\beta \geqslant \beta_2$. Consequently (see Fig. 6, p. 16),

(1) if $\beta_2 \geqslant \beta_1$, then $Z_c(\beta) = \log L - \alpha.\beta$ for $\beta \leqslant \beta_2$ and $Z_c(\beta) = \mathcal{P}_{34}(\beta)$ for $\beta > \beta_2$.
(2) if $\beta_2 < \beta_1$ then $Z_c(\beta) = \log L - \alpha\beta$ for some (possibly empty) interval $[0, \widetilde{\beta}]$, then $Z_c(\beta) = \widetilde{Z}_c(\beta)$ for $\widetilde{\beta} \leqslant \beta \leqslant \beta_1$ and $Z_c(\beta) = \mathcal{P}_{34}(\beta)$ for $\beta \geqslant \beta_1$.

### 2.3 Spectral Radius of $\mathcal{L}_{Z,\beta,[1]}$

We emphasize that the induced potential is constant on cylinders for the induced map on [1]. Consequently it satisfies the Bowen condition (H2) of [18]. Moreover, for every $x$ in [1], $\mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]})(x)$ is equal to the spectral radius $\lambda_{Z,\beta,[1]}$:

$$\lambda_{Z,\beta,[1]} = \mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]})(x) = \Sigma_1 + \frac{\Sigma_2 e^{-\alpha\beta - Z}}{1 - \Sigma_2\Sigma_3}. \tag{6}$$

We are interested by the level curve $\lambda_{Z,\beta,[1]} = 1$ because it partially determines the implicit function $Z = \mathcal{P}(\beta)$.

**Lemma 3** *There exists $\beta_c > \max(\beta_1, \beta_2)$ such that for every $\beta < \beta_c$, there exists a unique $Z_c(\beta) < Z = Z(\beta)$ such that $\lambda_{Z(\beta),\beta,[1]} = 1$ (see Fig. 4).*

*Moreover, for every $\beta > \beta_c$ and for every $Z \geqslant Z_c(\beta) = \mathcal{P}_{34}(\beta)$, $\lambda_{Z,\beta,[1]} < 1$.*

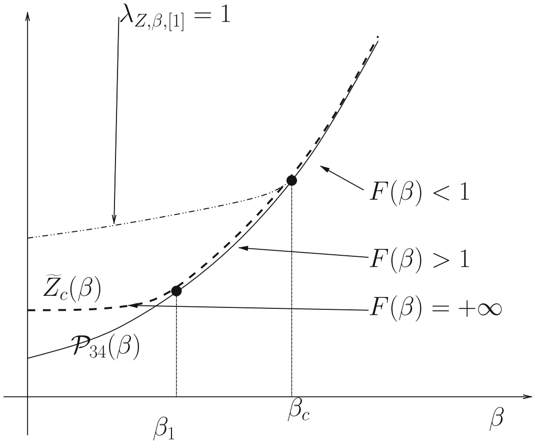



**Fig. 4** Implicit curve for spectral radius equal to 1, with $L = 0$

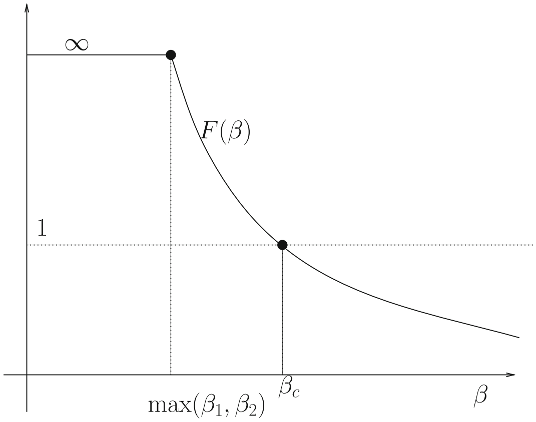



**Fig. 5** The graph of $F(\beta)$

*Proof* We emphasize that $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ are decreasing in $Z$ for fixed $\beta$. They go to 0 as $Z$ goes to $+\infty$.

Therefore, the existence of a $Z$ such that $\lambda_{Z,\beta,[1]} = 1$ follows from the value of

$$F(\beta) := \lim_{Z\downarrow Z_c(\beta)} \lambda_{Z,\beta,[1]}.$$

• Let us first consider the case $L = 0$. Then, $Z_c(\beta) = \widetilde{Z}_c(\beta)$ for $\beta \leqslant \beta_1$ and $Z_c(\beta) = \mathcal{P}_{34}(\beta)$ for $\beta \geqslant \beta_1$. Let us first consider the case $\beta \leqslant \beta_1$. For a fixed $\beta$, if $Z \downarrow \widetilde{Z}_c(\beta)$, $\lambda_{Z,\beta,[1]}$ increases to $+\infty$ because $\Sigma_2\Sigma_3$ goes to 1. On the other hand, if $Z$ goes to $+\infty$, $\lambda_{Z,\beta,[1]}$ goes to 0.

Therefore, there exists a unique $Z$ such that $\lambda_{Z,\beta,[1]} = 1$.

Let us now assume $\beta > \beta_1$. Then,

$$F(\beta) := e^{-\alpha\beta-\mathcal{P}_{34}(\beta)} + \frac{\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-n\mathcal{P}_{34}(\beta)} e^{-\alpha\beta-\mathcal{P}_{34}(\beta)}}{1-\frac{1}{(1+e^{\delta\beta})^2}\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-n\mathcal{P}_{34}(\beta)}\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\varepsilon\beta}} \tag{7}$$

and the existence of a solution for $\lambda_{Z,\beta,[1]} = 1$ is thus a consequence of the inequality $F(\beta) \geqslant 1$. As $\beta \mapsto \mathcal{P}_{34}(\beta)$ increases, we claim that $F(\beta)$ decreases in $\beta$. If $\beta \downarrow \beta_1$, $\Sigma_2\Sigma_3$ goes to 1 and then $F(\beta)$ goes to $+\infty$. If $\beta \to +\infty$, $F(\beta) \to 0$.

Consequently, there exists a unique $\beta_c$, such that $F(\beta_c) = 1$ (see Fig. 5). We have $\beta_c > \beta_1$ because $\lim_{\beta\downarrow\beta_1} F(\beta) = +\infty$.

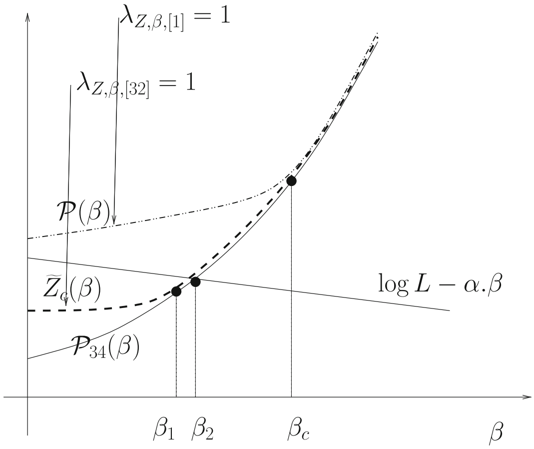
For $\beta < \beta_c$, the implicit equation $\lambda_{Z,\beta,[1]} = 1$ has a unique solution, $Z = \mathcal{P}(\beta)$. For $\beta > \beta_c$ there are no solutions.

• The case $L \geqslant 1$. In that case $\Sigma_1$ is equal to

$$\Sigma_1 := \sum_{n=1}^{+\infty} e^{-n\beta.\alpha-nZ+(n-1)\log L} = \frac{e^{-\beta.\alpha-Z}}{1-e^{-\beta.\alpha-Z+\log L}}, \tag{8}$$

and it decreases in $\beta$ (for fixed $Z$), decreases in $Z$ (for fixed $\beta$) and diverges if $Z = \log L - \beta.\alpha$. The shape of $F(\beta)$ is the same as in the previous case, whatever the relative positions of $\beta_1$ and $\beta_2$ are : $F(\beta) = +\infty$ for every $\beta \leqslant \max(\beta_1, \beta_2)$ and then $F(\beta)$ decreases for $\beta \geqslant \max(\beta_1, \beta_2)$ because for these $\beta$'s $Z_c(\beta) = \mathcal{P}_{34}(\beta)$. Therefore, there exists a unique $\beta_c$ such that $F(\beta_c) = 1$. □

**Fig. 6** Principal curves




## 2.4 Thermodynamic Formalism for $\beta < \beta_c$

If $\beta < \beta_c$ we use Th. 4, Th. 2 and Remark 3 in [18]. The pressure function $\mathcal{P}(\beta)$ is given by an implicit function inside the interior of the domain of analyticity in both variables $Z$ and $\beta$ for $\lambda_{Z,\beta,[1]}$. Analyticity for $\beta \mapsto \mathcal{P}(\beta)$ for $\beta < \beta_c$ thus follows from analyticity of implicit function (see e.g. [21]).

By construction, $\mathcal{P}(\beta) > Z_c(\beta) \geqslant \mathcal{P}_{34}(\beta)$ (see Fig. 6).

We point out that increasing $L$ does not affect the existence of $\beta_c$ but may only shift the level curve $\lambda_{Z,\beta,[1]} = 1$ up, and therefore increase $\beta_c$ (see Fig. 6, p. 16). As a by-product, this shows that $\varepsilon\beta_c$ can be made as big as wanted if $L$ is increased.

**Lemma 4** $\lim_{\beta\to\beta_c^-} \mathcal{P}(\beta) = \mathcal{P}_{34}(\beta_c)$.

*Proof* Remember that $\mathcal{P}(\beta)$ is given by the implicit formula $\lambda_{Z,\beta,[1]} = 1$. On the other hand, for every $\beta_1 < \beta < \beta_c$ and $Z = \mathcal{P}_{34}(\beta)$ we set

$$\lambda_{Z,\beta,[1]} = F(\beta).$$

Furthermore, $F(\beta) > 1$ for $\beta < \beta_c$ and it goes to 1 if $\beta \to \beta_c$ (by definition of $\beta_c$).

As for any fixed $\beta$, $Z \mapsto \lambda_{Z,\beta,[1]}$ decreases, for $\beta = \beta_c$, $Z = \mathcal{P}_{34}(\beta)$ is the unique solution for

$$\lambda_{Z,\beta,[1]} = 1 \quad \text{(see Fig. 5)},$$

thus $\mathcal{P}(\beta_c) = \mathcal{P}_{34}(\beta_c)$. □

## 2.5 Thermodynamic Formalism for $\beta > \beta_c$ and Number of Equilibrium States at $\beta_c$

Due to [18, Th. 4] no equilibrium state can give a positive weight to [1] for $\beta > \beta_c$.

The next lemma shows that an equilibrium cannot give a positive weight to the left hand side wing without giving weight to [1].

**Lemma 5** *If* $\mu$ *is an equilibrium state for* $\beta.\phi$ *and* $\mu([1]) = 0$*, then* $\mu([1_1]) = \ldots = \mu([1_L]) = 0$.

*Proof* Assume $\mu$ is an equilibrium state for $\beta.\phi$ and $\mu([1]) = 0$. By considering an ergodic component of $\mu$, we can assume that $\mu$ is ergodic.

Then, if $\mu([1_1]\cup\ldots\cup[1_L]) > 0$, by ergodicity, $\mu([1_1]\cup\ldots\cup[1_L]) = 1$. As the potential $\phi$ is constant on $[1_1]\cup\ldots\cup[1_L]$, $\mu$ is the measure with maximal entropy supported in $\{1_1,\ldots,1_L\}^{\mathbb{N}}$ and $\mathcal{P}(\beta) = \log L - \beta\alpha$.

In that case, the measure of maximal entropy supported in $\{1, 1_1,\ldots,1_L\}^{\mathbb{N}}$ has a pressure $\log(L+1) - \beta\alpha > \mathcal{P}(\beta)$ which is in contradiction with the definition of the pressure. □

Consequently, for $\beta > \beta_c$, any equilibrium state has its support in $\{2, 3, 4\}^{\mathbb{N}} \cap \Sigma$. The thermodynamic formalism for $\beta > \beta_c$ is thus a consequence of the results in Sect. 3.

At the transition, $\beta = \beta_c$, $\mathcal{P}(\beta_c) = \mathcal{P}_{34}(\beta_c)$ follows from Lemma 4, which yields that the unique equilibrium state in $\Sigma_{34}$ for $\beta.\phi$ is an equilibrium state for the global system. Then, existence of an equilibrium state giving positive weight to [1] is related to the condition $\mathcal{L}_{\mathcal{P}_{34}(\beta_c),\beta_c,[1]}(\tau_{[1]}) < +\infty$ (see again [18, Th. 4]). A simple computation shows

$$\mathcal{L}_{\mathcal{P}_{34}(\beta_c),\beta_c,[1]}(\tau_{[1]}) = \left|\frac{\partial \mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]})(x)}{\partial Z}_{|Z=\mathcal{P}_{34}(\beta_c)}\right|.$$

Hence we have

**Proposition 6** *If $\varepsilon\beta_c > 2$, then there are at least two ergodic equilibrium states for $\beta = \beta_c$ and only one of them has full support. If $\varepsilon\beta_c \leqslant 2$, then no equilibrium state for $\beta = \beta_c$ gives positive weight to* [1].

*Proof* We recall that for every $x$ in [1]

$$\lambda_{Z,\beta,[1]} = \mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]})(x) = \Sigma_1(Z,\beta) + \frac{\Sigma_2(Z,\beta)e^{-\alpha\beta-Z}}{1-\Sigma_2(Z,\beta)\Sigma_3(Z,\beta)}.$$

Therefore we have to compute $\frac{\partial\lambda_{Z,\beta,[1]}}{\partial Z}_{|Z=\mathcal{P}_{34}(\beta)}$. Using the chain rule for computation of the derivative of product function, $\frac{\partial\lambda_{Z,\beta,[1]}}{\partial Z}_{|Z=\mathcal{P}_{34}(\beta)}$ involves $\Sigma_1$ $\Sigma_2$, $\Sigma_3$, $\frac{\partial\Sigma_1}{\partial Z}$, $\frac{\partial\Sigma_2}{\partial Z}$ and $\frac{\partial\Sigma_3}{\partial Z}$.

All these terms are series, and more precisely power series in $e^{-Z}$. Therefore, $\frac{\partial\lambda_{Z,\beta,[1]}}{\partial Z}_{|Z=\mathcal{P}_{34}(\beta)}$ converges if and only if all these series converge.

We have already seen that at the transition, $Z_c(\beta) = \mathcal{P}_{34}(\beta_c) > \log L - \beta_c.\alpha$ which yields the convergence of $\Sigma_1$ and $\frac{\partial\Sigma_1}{\partial Z}_{|Z=\mathcal{P}_{34}(\beta)}$. We have also seen that for $\beta > \beta_c > \beta_1$, $\Sigma_2\Sigma_3 < 1$ which yields convergences for both $\Sigma_2$ and $\Sigma_3$. We also claim that $\frac{\partial\Sigma_2}{\partial Z}_{|Z=\mathcal{P}_{34}(\beta_c)}$ converges because $\mathcal{P}_{34}(\beta_c) > 0$.

Therefore, the global convergence is equivalent to the convergence of $\frac{\partial\Sigma_3}{\partial Z}_{|Z=\mathcal{P}_{34}(\beta_c)}$, that is

$$\sum_n \left(\frac{1}{n+1}\right)^{\varepsilon\beta} n < +\infty. \tag{9}$$

This holds if and only if $\varepsilon\beta_c > 2$. □

*Remark 4* As was said above, if $\varepsilon\beta_c > 2$, by [18, Th.4] there exists a unique equilibrium state which gives a positive weight to [1]. It is also the unique equilibrium with full support. □

*Remark 5* Existence of at most 2 equilibrium states will follow from the uniqueness of the equilibrium state in $\{2,3,4\}^{\mathbb{N}} \cap \Sigma$ for $\beta > \beta_1$. □

# 3 Phase Transition for $\mathcal{P}_{234}(\beta)$. End of the Proof of Theorem A

## 3.1 First Inequalities and Preliminary Results

One of the main difficulties is that we do not know, at that stage, whether the pressure for $\beta > \beta_c$ is strictly bigger than $\mathcal{P}_{34}(\beta)$ or not.

Let $\mathcal{P}_{234}(\beta)$ be the pressure for the sub-system $\{2,3,4\}^{\mathbb{N}} \cap \Sigma$ of points in $\Sigma$ with no symbols in $\{1, 1_1, \ldots, 1_L\}$ and for the potential $\beta.\phi$. As it is a subsystem of the global one, $\mathcal{P}_{234}(\beta) \leqslant \mathcal{P}(\beta)$. Conversely, $\{3,4\}^{\mathbb{N}}$ is a subsystem of $\{2,3,4\}^{\mathbb{N}} \cap \Sigma$ and then $\mathcal{P}_{234}(\beta) \geqslant \mathcal{P}_{34}(\beta)$. Therefore Lemma 4 yields that

$$\mathcal{P}_{234}(\beta_c) = \mathcal{P}(\beta_c) = \mathcal{P}_{34}(\beta_c)$$

at the transition. The main question is to know whether for $\beta > \beta_c$ an equilibrium state (for $\Sigma$) gives a positive weight to the cylinder [2] or not.

Since we only have to show that for $\beta > \beta_c$, $\mathcal{P}_{234}(\beta) = \mathcal{P}_{34}(\beta)$ holds, from now on untill the end of this section, equilibrium states are with respect to the system $\Sigma_{234}$. We recall that $\phi$ is continuous and the entropy is upper semi-continuous. Thus, there exist at least one equilibrium state, say $\widehat{\mu}_\beta$, in $\Sigma_{234}$.

**Lemma 7** *For every $\beta$, for every equilibrium state $\widehat{\mu}_\beta$, $\widehat{\mu}_\beta([3]) > 0$.*

*Proof* If $\widehat{\mu}_\beta([3]) = 0$, then any ergodic component of $\widehat{\mu}_\beta$ is either $\delta_{2^\infty}$ or $\delta_{4^\infty}$. In the first case the pressure is 0, in the second case it is $\beta.(\delta+\gamma)$. In both cases, the value is strictly lower than $\mathcal{P}_{34}(\beta) \leqslant \mathcal{P}_{234}(\beta)$. □

**Lemma 8** *There exists an equilibrium state, $\widehat{\mu}_\beta$, for $\beta.\phi$ satisfying $\widehat{\mu}_\beta([32]) = 0$ if and only if $\mathcal{P}_{234}(\beta) = \mathcal{P}_{34}(\beta)$.*

*Proof* Let $\widehat{\mu}_\beta$ be an equilibrium state such that $\widehat{\mu}_\beta([32]) = 0$. Then $\sigma$-invariance immediately yields that every cylinder of the form $[i_0 i_1 \ldots i_{n-1} 32]$ has null $\widehat{\mu}_\beta$-measure. As $\widehat{\mu}_\beta([3]) > 0$, this shows that $\widehat{\mu}_\beta(\Sigma_{34}) = 1$, thus $\mathcal{P}_{234}(\beta) \leqslant \mathcal{P}_{34}(\beta)$. The converse inequality holds as was seen above: the unique equilibrium state in $\Sigma_{34}$ (for $\beta.\phi$) is an equilibrium state for $\Sigma_{234}$ and it gives no weight to [32]. □

For our purpose we will thus induce on the cylinder [32]. To avoid heavy notations, the first return time will simply be denoted by $\tau$ and the first return map by $T$.

## 3.2 Induced Operator in [32]

Consider a point in [32] say $x := 32x_2x_3\ldots$. Any $y$ satisfying $T(y) = x$ is of the form

$$y = 3\underbrace{2\ldots2}_{\text{at least one 2}}3\omega 32x_2x_3\ldots,$$

where $\omega$ is a word in 3 and 4.

If $x' := 32x'_2x'_3\ldots$ $y := 32^n3\omega32x_2x_3\ldots$ and $y' := 32^n3\omega32x'_2x'_3\ldots$, we emphasize that

$$S_{n+1+|\omega|+1}(\phi)(y') = S_{n+1+|\omega|+1}(\phi)(y) \tag{10}$$

holds. In other words, if we induce on [32], the induced potential satisfies the Bowen condition (H2) of [18] and we can apply the same machinery that was used for the cylinder [1].

In that case, the family of transfer operators for $T$ and $\beta.\phi$ is defined by

$$\mathcal{L}_{Z,\beta,[32]}(\psi)(x) := \frac{1}{2^{\varepsilon\beta}}e^{\gamma\beta-Z}\sum_{n=1}^{+\infty}\sum_{\omega}\left(\frac{1}{n+1}\right)^{\beta} \times e^{-nZ}\left(\frac{2}{|\omega|+2}\right)^{\varepsilon\beta} e^{|\omega|\gamma\beta+(\#4\in\omega).\delta\beta-|\omega|Z}\psi(32^n\omega x), \qquad (11)$$

where $\psi$ belongs to $\mathcal{C}^0([32])$, $x$ starts with $32\ldots$, $\omega$ is a (possibly empty) word with digits 3 and 4 and starting with 3 and $\#4\in\omega$ is the number of 4's in $\omega$. The computation gives[1]

$$\mathcal{L}_{Z,\beta,[32]}(\mathbb{1}_{[32]}) = \frac{1}{(1+e^{\delta\beta})^2}\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-nZ}\sum_{m=1}^{+\infty}\left(\frac{1}{m+1}\right)^{\varepsilon\beta} e^{m(\mathcal{P}_{34}(\beta)-Z)}.$$

Note that the induced potential is constant on cylinders associated to $T$ (in [32]), hence $\lambda_{Z,\beta,[32]} = \mathcal{L}_{Z,\beta,[32]}(\mathbb{1}_{[32]})(x)$ for every $x$ in [32]. We emphasize that $\mathcal{L}_{Z,\beta,[32]}(\mathbb{1}_{[32]})(x)$ is actually equal to $\Sigma_2\Sigma_3$. Therefore, the implicit equation

$$\lambda_{Z,\beta,[32]} = 1,$$

is exactly verified for $Z = \widetilde{Z}_c(\beta)$ (see Lemma 2) and holds if and only if $\beta \leqslant \beta_1$. By definition of $\beta_1$, for every $\beta > \beta_1$, $\Sigma_2\Sigma_3 < 1$. Therefore, from [18] we get:

- $\mathcal{P}_{234}(\beta) = \widetilde{Z}_c(\beta)$ for $\beta < \beta_1$,
- there is a unique equilibrium state for $\beta < \beta_1$, and it is fully supported in $\Sigma_{234}$,
- there is a unique equilibrium state for $\beta > \beta_1$ and it is the one in $\Sigma_{34}$,
- there are two ergodic equilibrium states for $\beta = \beta_1$ if and only if $\varepsilon\beta_1 > 2$ (to get convergence for $\frac{\partial\Sigma_3}{\partial Z}$).

At that point, all the results stated in Theorem A are proved except $\varepsilon.\beta_1 < 2$, the fact that conditions $\varepsilon.\beta_c \lesseqgtr 2$ can be realized and (non-)differentiability for $\mathcal{P}(\beta)$ at the transition $\beta = \beta_c$

## 3.3 End of the Proof of Theorem A

### *3.3.1 Proof that $\varepsilon\beta_1 < 2$*

We recall that $\beta_1$ is defined by the implicit formula (5):

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta_1} e^{-n\mathcal{P}_{34}(\beta_1)}\frac{\zeta(\varepsilon\beta_1)-1}{(1+e^{\delta\beta_1})^2} = 1,$$

with $\mathcal{P}_{34}(\beta) = \gamma\beta + \log(1+e^{\delta\beta})$. Note that $\mathcal{P}_{34}(\beta)$ is always bigger than $\log 2$, thus for every choices of the parameters

$$\zeta(\varepsilon\beta_1) \geqslant (1+e^{\delta\beta_1})^2 + 1 > 5.$$

Now, $\zeta(2) = \dfrac{\pi^2}{6}$, which shows $\varepsilon\beta_1 < 2$.

[1] see the computation of $L_n(\beta, [3])$ page 8 for how to deal with $\#4\in\omega$.

### 3.3.2 Values for $\varepsilon\beta_c$

We remind that $\beta_c$ is given by the implicit formula derived from Equality (7):

$$\sum_{n=1}^{+\infty} e^{-n\alpha\beta_c - n\mathcal{P}_{34}(\beta_c)+(n-1)\log L} + \frac{\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta_c} e^{-n\mathcal{P}_{34}(\beta_c)} e^{-\alpha\beta_c-\mathcal{P}_{34}(\beta_c)}}{1-\frac{1}{(1+e^{\delta\beta_c})^2}\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta_c} e^{-n\mathcal{P}_{34}(\beta_c)}\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\varepsilon\beta_c}} = 1. \tag{12}$$

We have already seen that increasing $L$ is a simple way to force $\varepsilon\beta_c > 2$ to hold.

Remind that $\beta_c > \beta_1$ and $\varepsilon.\beta_1 > 1$ and $\mathcal{P}_{34}(\beta) = \beta.\gamma + \log(1+e^{\beta.\delta})$. Then, assume that $\delta \to +\infty$, $\varepsilon$ being fixed, this yields $\delta\beta_c \to +\infty$. Substituting this in (12), the first summand and the numerator of the fraction go to 0 if $\delta \to +\infty$. Consequently, the denominator must also tend to 0, and as $\delta\beta_c$ tends to $+\infty$, we must have

$$\varepsilon\beta_c \to 1.$$

It is thus lower than 2 if $\delta$ is sufficiently large.

### 3.3.3 Non-differentibality at $\beta_c$ for $\varepsilon.\beta_c > 2$

The right-derivative for $\mathcal{P}(\beta)$ at the transition $\beta_c$ is equal to $\mathcal{P}_{34}'(\beta) = \gamma + \delta.\frac{e^{\delta.\beta}}{1+e^{\delta.\beta}}$. This is a positive real number.

On the other hand, for $\beta \leqslant \beta_c$, $\mathcal{P}(\beta)$ is given by the implicit formula

$$\mathcal{P}(\beta) = Z \text{ and } \lambda_{Z,\beta,[1]} = \Sigma_1(Z,\beta) + \frac{\Sigma_2(Z,\beta)e^{-\alpha\beta - Z}}{1-\Sigma_2(Z,\beta)\Sigma_3(Z,\beta)} = 1.$$

Therefore, the left-derivative for $\mathcal{P}(\beta)$ is given by

$$\mathcal{P}_l'(\beta) = -\frac{\partial\lambda_{Z,\beta,[1]}}{\partial\beta}.\frac{1}{\frac{\partial\lambda_{Z,\beta,[1]}}{\partial Z}}. \tag{13}$$

We emphasize that the partial derivatives on the right-hand side are well-defined for $\beta < \beta_c$ (see Sect. 3.3.4) but also for $\beta = \beta_c$ because $\varepsilon.\beta_c > 2$. We remind equalities

$$\Sigma_1 = \Sigma_1(Z,\beta) := \sum_{n=1}^{+\infty} e^{-n\beta\alpha - nZ + (n-1)\log L}$$

$$\Sigma_2 = \Sigma_2(Z,\beta) := \sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\beta} e^{-nZ}$$

$$\Sigma_3 = \Sigma_3(Z,\beta) := \frac{1}{(1+e^{\delta\beta})^2}\sum_{n=1}^{+\infty}\left(\frac{1}{n+1}\right)^{\varepsilon\beta} e^{n(\mathcal{P}_{34}(\beta)-Z)},$$

which easily show that $Z \mapsto \lambda_{Z,\beta,[1]}$, $\beta \mapsto \Sigma_1(Z,\beta)$ and $\beta \mapsto \Sigma_2(Z,\beta)$ decrease (thus have negative derivative).

Computing terms in the right-hand side of (13), we claim (and let the reader check) that $\frac{\partial\lambda_{Z,\beta,[1]}}{\partial\beta}$ is of the form $A + \mathcal{P}_{34}'(\beta)S$ and $\frac{\partial\lambda_{Z,\beta,[1]}}{\partial Z}$ is of the form $B - S$, where $A$ and $B$ are negative and $S$ is positive. This yields

$$\mathcal{P}_l'(\beta) = -\frac{A + \mathcal{P}_{34}'(\beta)S}{B - S} = \mathcal{P}_{34}'(\beta) + \frac{\mathcal{P}_{34}'(\beta)B + A}{S - B} < \mathcal{P}_{34}'(\beta). \tag{14}$$

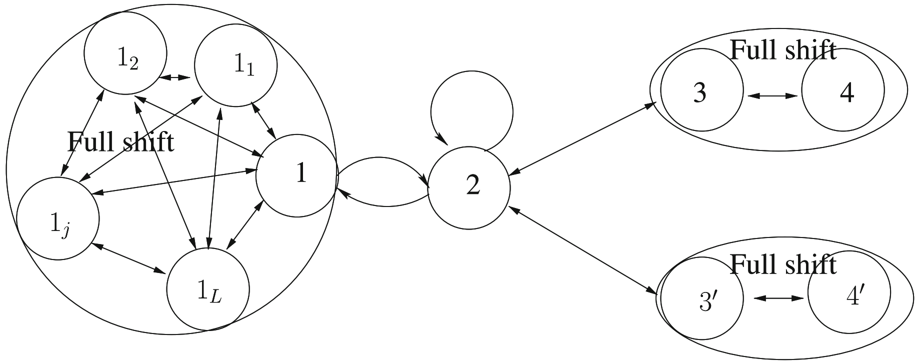



**Fig. 7** Dynamics for Theorem B

### *3.3.4 Differentiability at $\beta_c$ if $\varepsilon.\beta_c \leqslant 2$*

We claim that equality (13) still holds for $\beta < \beta_c$ because the diverging term for $\beta = \beta_c$ is

$$S = \frac{1}{(1+e^{\delta.\beta})^2}\sum_{n=1}^{+\infty} n\left(\frac{1}{n+1}\right)^{\varepsilon.\beta} e^{n(\mathcal{P}_{34}(\beta)-\mathcal{P}(\beta))}.$$

This term diverges for $\beta = \beta_c$ but converges for $\beta < \beta_c$ because $\mathcal{P}(\beta) > \mathcal{P}_{34}(\beta)$. Then,

$$\mathcal{P}_l'(\beta) = -\frac{A + \mathcal{P}_{34}'(\beta)S}{B - S} = \mathcal{P}_{34}'(\beta) + \frac{\mathcal{P}_{34}'(\beta)B + A}{S - B}$$

still holds[2] for $\beta < \beta_c$. Therefore, if $\beta \uparrow \beta_c$, $S$ goes to $+\infty$. We let the reader check that $A$ and $B$ are bounded thus, $\mathcal{P}_l'(\beta)$ goes to $\mathcal{P}_{34}'(\beta_c)$ as $\beta \uparrow \beta_c$.

## 4 Proof of Theorem B

For proving Theorem B we consider the next subshift of finite type (Fig. 7):

The right hand side wing has been replaced by two copies of itself. The left hand side wing is a full shift with $L+1$ symbols, each wing at the right hand side is also a full shift with two symbols. The unique exit digits of the wings are also the unique entrance digits and are the symbols 1, 3 and $3'$. To go from one of these symbols to another one, one must pass through 2.

### 4.1 Phase Transition for $\mathcal{P}(\beta)$

We now explain how to adapt the results from the proof of Theorem A to this new case. Inducing in [1], orbits of the third form (see the enumeration before Equality (1)) leave [1], visit [2] for a while and then can either visit $\Sigma_{34}$ or visit $\Sigma_{3'4'}$. More precisely, after a string of 2's we can either get a word of the form $3\omega3$ where $\omega$ is a word with digits 3 or 4, or a word of the form $3'\omega'3'$ with $\omega'$ a word with digits $3'$ and $4'$.

By symmetry of the potential, any sum $\Sigma_3$ has thus to be replaced by $2\Sigma_3 = \Sigma_3 + \Sigma_3$, one for a string in 3 and 4 and one for a string in $3'$ and $4'$.

[2] Actually this is the exact derivative because the pressure is analytic.

Consequently

$$\mathcal{L}_{Z,\beta,[1]}(\mathbb{1}_{[1]}) = \Sigma_1 + \Sigma_2 \sum_{k=0}^{+\infty} (2\Sigma_3\Sigma_2)^k,$$

and Condition (4b) has to be replaced by

$$\Sigma_2\Sigma_3 < \frac{1}{2}. \tag{15}$$

The implicit formula (similar to (5)) we have to consider is

$$\Sigma_2(\mathcal{P}_{34}(\beta_3), \beta_3)\Sigma_3(\mathcal{P}_{34}(\beta_3), \beta_3) = \frac{1}{2},$$

where $\beta_3$ replaces $\beta_1$: for $\beta < \beta_3$, $\widetilde{Z}_c(\beta)$ is strictly larger than $\mathcal{P}_{34}(\beta)$. For $\beta \geqslant \beta_3$, $\widetilde{Z}_c(\beta) = \mathcal{P}_{34}(\beta)$.

Similarly, the new value for the spectral radius $\lambda_{Z,\beta,[1]}$ is

$$\Sigma_1 + \frac{\Sigma_2 e^{-\alpha\beta - Z}}{1 - 2\Sigma_2\Sigma_3},$$

and there exists $\beta_c' > \max(\beta_2, \beta_3)$ such that for every $\beta > \beta_c'$, $\lambda_{Z,\beta,[1]} < 1$. Then, for $\beta > \beta_c'$, $\mathcal{P}(\beta) = \mathcal{P}_{23434'}(\beta)$, and any equilibrium state has support in $\Sigma_{2343'4'}$ which is $\Sigma$ restricted to words without digits $1, 1_1, \ldots 1_L$.

### 4.2 Phase Transition for $\mathcal{P}_{2343'4'}(\beta)$

Again, we shall induce on the cylinder [32]. In that new case, the orbits leaving [32] and then returning back to [32] have the form: 32(string of 2's) (intermittence of strings[3] of 3′'s or 4′'s and strings of 2's) (3 and one string of 3's or 4's)32. This yields that the new equation to consider for $\lambda_{Z,\beta,[32]}$ is

$$\begin{aligned}
\lambda_{Z,\beta,[32]} &= \Sigma_2(Z,\beta)\left(\sum_{n=0}^{+\infty}(\Sigma_2(Z,\beta)\Sigma_3(Z,\beta))^n\right)\Sigma_3(Z,\beta) \\
&= \frac{\Sigma_2(Z,\beta)\Sigma_3(Z,\beta)}{1 - \Sigma_2(Z,\beta)\Sigma_3(Z,\beta)},
\end{aligned}$$

where we took into account the symmetry of $\Sigma_{34}$ and $\Sigma_{3'4'}$.

Note that $z \mapsto \dfrac{z}{1-z}$ increases for $z < 1$, and $\dfrac{z}{1-z} < 1 \iff z < \dfrac{1}{2}$. Then,

$$\lambda_{Z,\beta,[32]} < 1 \iff \Sigma_2(Z,\beta)\Sigma_3(Z,\beta) < \frac{1}{2}.$$

This shows that $\beta_3$ is a transition parameter for $\Sigma_{2343'4'}$: for $\beta < \beta_3$ there exists a unique equilibrium state in $\Sigma_{2343'4'}$ and it is fully supported. For $\beta > \beta_3$, no equilibrium state gives weight to [32]. For symmetry reasons, no equilibrium state gives a positive weight to [3′2], and thus, there are two equilibrium states which are the ones in $\Sigma_{34}$ and in $\Sigma_{3'4'}$.

For $\beta > \beta_c'$, there is no more a single global equilibrium state (for $\Sigma$ and $\mathcal{P}(\beta)$) but there are two "smaller" equilibria in $\Sigma_{34}$ and in $\Sigma_{3'4'}$. Nevertheless, $\mathcal{P}(\beta) = \mathcal{P}_{34}(\beta)$ is analytic for $\beta > \beta_c'$.

[3] We only consider eligible strings, that is of the form $3'\omega 3'$.

### 4.3 Gâteaux-Differentiability in Other Directions

To use the vocabulary from [16], the potential $\phi$ is a kind of grid function: it is constant on cylinders of the form [1], [$1_i$], [$2^n*$], [$3\omega 32$] with $\omega \in \{3, 4\}^n$ for some $n$ and [$3'\omega 3'2$] with $\omega \in \{3', 4'\}^n$ for some $n$.

Let $\mathcal{V}$ be the set of such functions, which are in addition Hölder continuous and totally symmetric in $3 \leftrightarrow 3'$ and $4 \leftrightarrow 4'$. $\mathcal{V}$ is infinite-dimension in $\mathcal{C}(\Sigma)$.

It we pick some $\varphi$ in $\mathcal{V}$, for $\beta > \beta_c'$, and for $t \in (-\eta, \eta)$ with $\eta \approx 0^+$, the induced transfer operators for $\beta.\phi + t.\varphi$ and $\beta.\phi$ have close spectrum.

As things are totally symmetric in $\Sigma_{34}$ or $\Sigma_{3'4'}$, there will still be two equilibrium states and the pressure is differentiable in direction $\varphi$ because we are into the domains with spectral radiuses $<1$.

*Remark 6* It is actually highly likely that $P(\beta.\phi + .)$ is Gâteaux differentiable in the direction of any $\varphi$ which is Hölder and totally symmetric in $3 \leftrightarrow 3'$ and $4 \leftrightarrow 4'$ (and not necessarily a grid function). □

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