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Homogeneous Open Quantum Random Walks on a Lattice

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Abstract We study open quantum random walks (OQRWs) for which the underlying graph is a lattice, and the generators of the walk are homogeneous in space. Using the results recently obtained in Carbone and Pautrat (Ann Henri Poincaré, 2015), we study the quantum trajectory associated with the OQRW, which is described by a position process and a state process. We obtain a central limit theorem and a large deviation principle for the position process. We study in detail the case of homogeneous OQRWs on the lattice $\mathbb{Z}^d$, with internal space $\mathfrak{h} = \mathbb{C}^2$.

Keywords Quantum random walks · Open quantum random walks · Central limit theorem · Large deviations · Markov chains · Quantum dynamical semigroups

1 Introduction

Open quantum random walks (OQRWs) were defined by Attal et al. in [2]. They seem to be a good quantum analog of Markov chains, and, as such, are a very promising tool to model many physical problems (see [2,5] for an in-depth description of OQRWs, and [22,24,27] for applications and extensions).

Let us briefly describe open quantum random walks in a simple (if not the most general) situation. Our main object of interest is a random process $(X_p, \rho_p)_{p\in\mathbb{N}}$ where X_p belongs to a countable set V, and ρ_p is a positive, trace-class operator on a separable Hilbert space $\mathfrak{h}$. We view X_p as describing the position of a particle, and ρ_p as describing its (internal) quantum state. The evolution of $(X_p, \rho_p)_{p\in\mathbb{N}}$ is determined by a family $(L_{i,j})_{i,j\in V}$ of operators on $\mathfrak{h}$, satisfying the condition

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$$\sum_{i\in V} L_{i,j}^* L_{i,j} = \mathrm{Id} \quad \text{for all } j \in V$$

in the sense that, conditionally on $(X_p, \rho_p) = (j, \eta)$, the law of (X_{p+1}, ρ_{p+1}) is given by

$$(X_{p+1}, \rho_{p+1}) = \left(i, \frac{L_{i,j}\, \eta\, L_{i,j}^*}{\mathrm{Tr}(L_{i,j}\, \eta\, L_{i,j}^*)} \right) \quad \text{with probabiliy } \mathrm{Tr}(L_{i,j}\, \eta\, L_{i,j}^*).$$

The operators $(L_{i,j})_{i,j\in V}$ therefore entirely encode the transitions of (X_p, ρ_p). Note that what we call open quantum random walk itself is an operator determined by the $(L_{i,j})_{i,j\in V}$ and acting on a space of trace class operators, but we don't need to mention it for the moment.

In the paper [5], we described the notions of irreducibility and aperiodicity for OQRWs, and derived, in particular, convergence properties of the process $(X_p, \rho_p)_{p\in\mathbb{N}}$ for irreducible, or irreducible and aperiodic, OQRWs. In the same way as for classical Markov chains, those convergence results assumed the existence of an invariant state.

In the present paper we focus on space-homogeneous OQRWs on a lattice, i.e. on the case where V is an additive group and $L_{i,j}$ depends only on $j - i$. These OQRWs attracted special attention in many recent papers (see, for instance [1,3,16,21,28,30]). As we have shown in [5], they do not have an invariant state so that most of the convergence results from [5] are useless. We will show, however, that there exists an auxiliary map which allows to characterize many properties of the homogeneous open quantum random walk. With the help of the Perron-Frobenius theorem described in [5], we can obtain a central limit theorem and a large deviation principle for the position process $(X_p)_{p\in\mathbb{N}}$.

The immediate physical application of our results is quantum measurements, or more precisely, quantum repeated indirect measurements. In that framework, a given system $\mathcal{S}$ interacts sequentially with external systems $\mathcal{R}_p$, $p = 1, 2, \ldots$ representing measuring devices, and after each interaction, a measurement is done on $\mathcal{R}_p$. The connection with the present framework is that the sequence of measurement outcomes is given by the sequence $(M_p)_{p\in\mathbb{N}}$ where $M_p = X_p - X_{p-1}$, and the sequence $(\rho_p)_{p\in\mathbb{N}}$ represents the state of the physical system $\mathcal{S}$ after the first p measurements (for more details on repeated indirect measurement or "Kraus measurements", we refer the reader to [18,19]; for the connection with OQRWs, see [1]). Our results immediately give a law of large numbers, a central limit theorem and a large deviation principle for the statistics of the measurements $(M_p)_{p\in\mathbb{N}}$.

We will pay specific attention to the application of our results to the case where the internal state space of the particle (what is sometimes called the *coin space*) is two-dimensional. This will allow us to illustrate the full structure of space-homogeneous OQRWs, and in particular the notions of irreducibility, period, as well as the Baumgartner–Narnhofer decompositions (see [4]) discussed previously in [5].

Of the above cited articles, some give a central limit theorem for the position $(X_p)_{p\in\mathbb{N}}$ associated with an OQRW on $\mathbb{Z}^d$. The most general result so far is given in [1], and its proof is based on a central limit theorem for martingales and the Kümmerer-Maassen ergodic theorem (see [20]). Our proof is based on a completely different strategy, using a computation of the Laplace transform, and uses an irreducibility assumption which does not appear in existing central limit results. We will show, however, that the irreducibility assumption can be dropped in some situations, and that our central limit theorem contains the result of [1], but yields more general formulas.

In addition, we can prove a large deviation principle for the position process $(X_p)_p$ associated with an homogeneous OQRW on a lattice. The technique we used, based on the application of the Perron-Frobenius theorem to a suitable deformed positive map, goes back (to the best of our knowledge) to [13]. None of the articles cited above proves a large deviation principle. As we were completing this paper, however, we learnt of the recent article [29], which proves a similar result. We comment on this in Sect. 5.

The structure of the present paper is the following: in Sect. 2 we recall the main definitions of open quantum random walks specialized to the case where the underlying graph is a lattice in $\mathbb{R}^d$, and define the auxiliary map of an open quantum random walk. In Sect. 3 we recall standard results about irreducibility and period of completely positive maps. In Sect. 4 we characterize irreducibility and period of the open quantum random walk and its auxiliary map. In Sect. 5 we state our main results: the central limit theorem and the large deviation principle. In Sect. 6 we specialize to the situation where the underlying graph is $\mathbb{Z}^d$ and the internal state space is $\mathbb{C}^2$, and characterize each situation in terms of the transition operators. In Sect. 7 we study explicit examples.

2 Homogeneous Open Quantum Random Walks

In this section we recall basic results and notations about open quantum random walks. We essentially follow the notations of [5], but specialize to the space-homogeneous case. For a more detailed exposition we refer the reader to [2].

We consider a separable Hilbert space $\mathfrak{h}$ and a locally finite lattice $V \subset \mathbb{R}^d$, which we assume contains 0, and is positively generated by a set $S \neq \{0\}$, in the sense that any v in V can be written as $s_1 + \ldots + s_n$ with $s_1, \ldots, s_n \in S$. In particular, V is an infinite subgroup of $\mathbb{R}^d$. The canonical example is $V = \mathbb{Z}^d$, with $S = \{\pm v_1, \ldots, \pm v_d\}$ where $(v_1, \ldots, v_d)$ is the canonical basis of $\mathbb{R}^d$.

We denote by $\mathcal{H}$ the Hilbert space $\mathcal{H} = \mathfrak{h} \otimes \mathbb{C}^V$. We view $\mathcal{H}$ as describing the degrees of freedom of a particle constrained to move on V: the "V-component" describes the spatial degrees of freedom (the position of the particle) while $\mathfrak{h}$ describes the internal degrees of freedom of the particle. We describe the physical state of the system by a positive, trace-class operator ρ on $\mathcal{H}$ with trace one. Such an operator we will call a state.

We consider a map on the space $\mathcal{I}_1(\mathcal{H})$ of trace-class operators, given by

$$\mathfrak{M} \,:\, \rho \mapsto \sum_{j\in V} \sum_{s\in S} (L_s \otimes |j+s\rangle\langle j|)\ \rho\ \big(L_s^* \otimes |j\rangle\langle j+s|\big) \tag{2.1}$$

where the L_s, $s \in S$, are operators acting on $\mathfrak{h}$ satisfying

$$\sum_{s\in S} L_s^* L_s = \mathrm{Id}. \tag{2.2}$$

The L_s are thought of as encoding both the probability of a transition by the vector s, and the effect of that transition on the internal degrees of freedom. Equation (2.2) therefore encodes the "stochasticity" of the transitions.

Remark 2.1 It is worth underlining that the transition operators L_{ij}, mentioned in the introduction, depend here only on $i - j$ and are therefore replaced by the operators L_s, $s = i - j$. This is why we call the present open quantum random walks *homogeneous*.

Remark 2.2 The map $\mathfrak{M}$ defined above is a special case of a quantum Markov chain, as introduced by Gudder in [12]. See [5, Sect. 8] for more comments.

We associate with the OQRW $\mathfrak{M}$ the auxiliary map $\mathfrak{L}$ on the space $\mathcal{I}_1(\mathfrak{h})$ of trace-class operators on $\mathfrak{h}$ defined by

$$\mathfrak{L} \,:\, \rho \mapsto \sum_{s\in S} L_s\, \rho\, L_s^*. \tag{2.3}$$

Both (2.1) and (2.3) define trace-preserving (TP) maps, which are completely positive (CP), *i.e.* for any n in $\mathbb{N}^*$, the extensions $\mathfrak{M} \otimes \mathrm{Id}$ and $\mathfrak{L} \otimes \mathrm{Id}$ to $\mathcal{I}_1(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^n)$ and $\mathcal{I}_1(\mathfrak{h}) \otimes \mathcal{B}(\mathbb{C}^n)$, respectively, are positive. In particular, such maps transforms states into states. A completely-positive, trace-preserving map will be called a CP-TP map. We will call a map $\mathfrak{M}$, as defined by (2.1), an open quantum random walk, or OQRW; and we will call $\mathfrak{L}$ the auxiliary map of $\mathfrak{M}$. To be consistent with Remark 2.1, we should call $\mathfrak{M}$ an *homogeneous* OQRW, but since only such OQRWs will be considered, we drop the adjective "homogeneous" in the rest of this paper.

Let us recall that the topological dual $\mathcal{I}_1(\mathcal{H})^*$ can be identified with $\mathcal{B}(\mathcal{H})$ through the duality

$$(\rho, X) \mapsto \mathrm{Tr}(\rho X).$$

For this reason we will make no distinction between $\rho \in \mathcal{I}_1(\mathcal{H})$ and the map $\mathcal{B}(\mathcal{H}) \ni X \mapsto \mathrm{Tr}(\rho X)$. Note however that, strictly speaking, we should call such a map a *normal* state.

Remark 2.3 When $\Phi = \mathfrak{M}$ (respectively $\Phi = \mathfrak{L}$), the adjoint map Φ^* is a positive, unital (*i.e.* $\Phi^*(\mathrm{Id}) = \mathrm{Id}$) map on $\mathcal{B}(\mathcal{H})$ (respectively $\mathcal{B}(\mathfrak{h})$), and by the Russo-Dye theorem ([25]) one has $\|\Phi^*\| = \|\Phi^*(\mathrm{Id})\|$ where the latter is the operator norm on $\mathcal{B}(\mathcal{H})$ (respectively $\mathcal{B}(\mathfrak{h})$). This implies that trace-preserving positive maps have norm one, and in particular $\|\mathfrak{M}\| = 1$ and $\|\mathfrak{L}\| = 1$.

Remark 2.4 As noted in [2], classical Markov chains can be written as open quantum random walks. In the present case, if we have a subgroup V of $\mathbb{R}^d$ generated by a set S, and a Markov chain on V with translation-invariant transition matrix $P = (t_{i,j})_{i,j \in V}$ induced by the law $(t_s)_{s \in S}$ on S, then taking $\mathfrak{h} = \mathbb{C}$ and $L_s = \sqrt{t_s}$ induces the Markov chain with transition matrix P. This OQRW is called the minimal OQRW realization of the Markov chain (see [5] for a discussion of minimal and non-minimal OQRW realizations). Note that in this case the reduced map $\mathfrak{L}$ is trivial: $\mathfrak{L} = 1$.

A crucial remark is that, for any initial state ρ on $\mathcal{H}$, which is therefore of the form

$$\rho = \sum_{i,j \in V} \rho(i, j) \otimes |i\rangle\langle j|,$$

the evolved state $\mathfrak{M}(\rho)$ is of the form

$$\mathfrak{M}(\rho) = \sum_{i \in V} \mathfrak{M}(\rho, i) \otimes |i\rangle\langle i|, \quad \text{where } \mathfrak{M}(\rho, i) = \sum_{s \in S} L_s\, \rho(i - s, i - s)\, L_s^*. \tag{2.4}$$

Each $\mathfrak{M}(\rho, i)$ is a positive, trace-class operator on $\mathfrak{h}$ and $\sum_{i \in V} \mathrm{Tr}\, \mathfrak{M}(\rho, i) = 1$. We notice that off-diagonal terms $\rho(i, j)$, for $i \neq j$, do not appear in $\mathfrak{M}(\rho)$, and $\mathfrak{M}(\rho)$ itself is diagonal. For this reason, from now on, we will only consider states of the form $\rho = \sum_{i \in V} \rho(i) \otimes |i\rangle\langle i|$. Equation (2.4) remains valid, replacing $\rho(i, i)$ by $\rho(i)$.

We now describe the (classical) processes of interest associated with $\mathfrak{M}$. We begin with an informal discussion of these processes and their laws, and will only define the underlying probability space at the end of this section. We start from a state of the form $\rho = \sum_{i \in V} \rho(i) \otimes |i\rangle\langle i|$. We evolve ρ for a time p, obtaining the state $\mathfrak{M}^p(\rho)$ which, according to the previous discussion, is of the form

$$\mathfrak{M}^p(\rho) = \sum_{i \in V} \mathfrak{M}^p(\rho, i) \otimes |i\rangle\langle i|.$$

We then make a measurement of the position observable. According to standard rules of quantum measurement, we obtain the result $i \in V$ with probability $\mathrm{Tr}\,\mathfrak{M}^p(\rho, i)$. Therefore, the result of this measurement is a random variable Q_p, with law $\mathbb{P}(Q_p = i) = \mathrm{Tr}\,\mathfrak{M}^p(\rho, i)$ for $i \in V$. In addition, if the position $Q_p = i \in V$ is observed, then the state is transformed to $\frac{\mathfrak{M}^p(\rho,i)}{\mathrm{Tr}\,\mathfrak{M}^p(\rho,i)}$. This process $(Q_p, \frac{\mathfrak{M}^p(\rho,Q_p)}{\mathrm{Tr}\,\mathfrak{M}^p(\rho,Q_p)})$ we call the process "without measurement", to emphasize the fact that virtually only one measurement is done, at time p. Notice that, in practice, two values of this process at times $p < p'$ cannot be considered simultaneously as the measure at time p perturbs the system, and therefore subsequent measurements.

Now assume that we make a measurement at every time $p \in \mathbb{N}$, applying the evolution by $\mathfrak{M}$ between two measurements. Again assume that we start from a state ρ of the form $\sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$. Suppose that at time p, the position was measured at $X_p = j$ and the state (after the measurement) is $\rho_p \otimes |j\rangle\langle j|$. Then, after the evolution, the state becomes

$$\mathfrak{M}(\rho_p \otimes |j\rangle\langle j|) = \sum_{s\in S} L_s\, \rho_p\, L_s^* \otimes |j+s\rangle\langle j+s|,$$

so that a measurement at time $p+1$ gives a position $X_{p+1} = j+s$ with probability $\mathrm{Tr}\, L_s\, \rho_p\, L_s^*$, and then the state becomes

$$\rho_{p+1} \otimes |j+s\rangle\langle j+s| \quad \text{with} \quad \rho_{p+1} = \frac{L_s\, \rho_p\, L_s^*}{\mathrm{Tr}\, L_s\, \rho_p\, L_s^*}.$$

The sequence of random variables (X_p, ρ_p) is therefore a Markov process with transitions defined by

$$\mathbb{P}\Big((X_{p+1}, \rho_{p+1}) = \Big(j+s, \frac{L_s\, \eta\, L_s^*}{\mathrm{Tr}(L_s \eta L_s^*)}\Big) \Big| (X_p,\ \rho_p) = (j, \eta)\Big) = \mathrm{Tr}(L_s\, \eta\, L_s^*), \tag{2.5}$$

for any $j \in V$, $s \in S$ and $\eta \in \mathcal{I}_1(\mathfrak{h})$ and initial law

$$\mathbb{P}\left((X_0, \rho_0) = \left(i, \frac{\rho(i)}{\mathrm{Tr}\rho(i)}\right)\right) = \mathrm{Tr}\rho(i).$$

Note that the sequence $X_0 = i_0, \ldots, X_p = i_p$ is observed with probability

$$\mathbb{P}(X_0 = i_0, \ldots, X_p = i_p) = \mathrm{Tr}\left(L_{s_p} \ldots L_{s_1}\, \rho(i_0)\, L_{s_1}^* \ldots L_{s_p}^*\right) \tag{2.6}$$

if $i_1 - i_0 = s_1, \ldots i_p - i_{p-1} = s_p$ belong to S, and zero otherwise. In addition, this sequence completely determines the state ρ_p:

$$\rho_p = \frac{L_{s_p} \ldots L_{s_1}\, \rho(i_1)\, L_{s_1}^* \ldots L_{s_p}^*}{\mathrm{Tr}\, L_{s_p} \ldots L_{s_0}\, \rho(i_1)\, L_{s_0}^* \ldots L_{s_p}^*}. \tag{2.7}$$

As emphasized in [2], this implies that, for every p, the laws of X_p and Q_p are the same, *i.e.*

$$\mathbb{P}(X_p = i) = \mathbb{P}(Q_p = i) \quad \forall i \in V.$$

We now construct a probability space to carry the processes just described. Fixing an open quantum random walk $\mathfrak{M}$ on V defined by operators $(L_s)_{s\in S}$ we define the set $\Omega = V^{\mathbb{N}}$, equipped with the σ-field generated by cylinder sets. An element of Ω is denoted by $\omega = (\omega_p)_{p\in\mathbb{N}}$ and we denote by $(X_p)_{p\in\mathbb{N}}$ the coordinate maps. For any state ρ on $\mathcal{H}$ of the form $\rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$, we define a probability $\mathbb{P}_\rho^{(p)}$ on V^{p+1} by formula (2.6). One easily shows, using the stochasticity property (2.2), that the family $(\mathbb{P}_\rho^{(p)})_p$ is consistent, and can therefore be extended uniquely to a probability $\mathbb{P}_\rho$ on Ω. We denote by ρ_p the random variable

$$\rho_p = \frac{L_{X_p - X_{p-1}} \dots L_{X_1 - X_0}\, \rho(X_0)\, L^*_{X_1 - X_0} \dots L^*_{X_p - X_{p-1}}}{\mathrm{Tr}(L_{X_p - X_{p-1}} \dots L_{X_1 - X_0}\, \rho(X_0)\, L^*_{X_1 - X_0} \dots L^*_{X_p - X_{p-1}})}.$$

We will also denote $Q_p = X_p$, but will only use the notation Q_p when we consider "non-measurement" experiments, and in particular will never consider an event implying simultaneously outcomes Q_p and $Q_{p'}$ for $p \neq p'$. These processes reproduce the behaviour of the measurement outcomes and of the associated resulting states. In particular, equation (2.5) above holds in a mathematical sense with $\mathbb{P}_\rho$ replacing $\mathbb{P}$. From now on, we will usually drop the ρ in $\mathbb{P}_\rho$.

3 Irreducibility and Period: General Results

In this section we focus on the general notions of irreducibility and period for a completely positive (CP) map Φ on $\mathcal{I}_1(\mathcal{K})$, where $\mathcal{K}$ is a separable Hilbert space which, in practice, will be either $\mathfrak{h}$ or $\mathcal{H}$. We assume Φ is given in the form

$$\Phi(\rho) = \sum_{\kappa \in K} A_\kappa \rho A_\kappa^* \tag{3.1}$$

where K is a countable set, and the series $\sum_{\kappa \in K} A_\kappa^* A_\kappa$ is strongly convergent. This is the case for operators such as $\mathfrak{M}$ or $\mathfrak{L}$ and we actually know from the Kraus theorem that this is the case for any completely positive Φ (see [17] or [23], where this is called the operator-sum representation). We recall that such a map is automatically bounded as a linear map on $\mathcal{I}_1(\mathcal{K})$ (see e.g. [26, Lemma 2.2]), so that it is also weak-continuous. In most practical cases, we will additionally assume that $\|\Phi\| = 1$; this will be verified, in particular, if Φ is trace-preserving.

We give various equivalent definitions of the notion of irreducibility for Φ, which was originally defined by Davies in [6]. Note that this original definition holds for Φ positive, but for simplicity, we discuss it only for maps Φ which are completely positive (CP), and therefore have a Kraus decomposition (3.1). The equivalence between the different definitions, as well as the relevant references, are discussed in [5]. We recall some standard notations: an operator X on $\mathcal{K}$ is called positive, denoted $X \geq 0$, if, for $\phi \in \mathcal{K}$, one has $\langle \phi, X\phi \rangle \geq 0$. It is called strictly positive, denoted $X > 0$, if, for $\phi \in \mathcal{K} \setminus \{0\}$, one has $\langle \phi, X\phi \rangle > 0$.

Definition 3.1 The CP map Φ is called irreducible if one of the following equivalent conditions hold:

- for any $\rho \geq 0$, $\rho \neq 0$ in $\mathcal{I}_1(\mathcal{K})$, there exists t such that $\mathrm{e}^{t\Phi}(\rho) > 0$,
- for any non-zero $\phi \in \mathcal{K}$, the set $\mathbb{C}[A]\,\phi$ is dense in $\mathcal{K}$, where $\mathbb{C}[A]$ is the set of polynomials in A_κ, $\kappa \in K$,
- the only subspaces of $\mathcal{K}$ that are invariant by all operators A_κ are $\{0\}$ and $\mathcal{K}$.

We will also use the notion of regularity, which is evidently stronger than irreducibility:

Definition 3.2 The CP map Φ is called N-regular, for $N \in \mathbb{N}^*$, if one of the following equivalent conditions holds:

- for any $\rho \geq 0$, $\rho \neq 0$ in $\mathcal{I}_1(\mathcal{K})$, one has $\Phi^N(\rho) > 0$,
- for any non-zero $\phi \in \mathcal{K}$, the set $\{A_{\kappa_1} \dots A_{\kappa_N}\, \phi \mid \kappa_1, \dots, \kappa_N \in K\}$ is total in $\mathcal{K}$.

The map Φ is called regular if it is N-regular for some N in $\mathbb{N}^*$.

Remark 3.3 The following properties are immediate:

- If Φ is regular, then it is irreducible.
- If Φ is irreducible, then $\vee_{\kappa\in K}\operatorname{Ran}A_\kappa$, i.e. the closed vector space spanned by the ranges of the operators A_κ, coincides with $\mathcal{K}$ (while the converse is not true).
- If $\vee_{\kappa\in K}\operatorname{Ran}A_\kappa = \mathcal{K}$ and ρ is a faithful state, then $\Phi(\rho)$ is faithful. Indeed, we can write $\rho = \sum_j \rho_j |u_j\rangle\langle u_j|$, with $\rho_j > 0$ and $(u_j)_j$ an orthonormal basis for $\mathcal{K}$. Then $\Phi(\rho) = \sum_{j,\kappa} \rho_j |A_\kappa u_j\rangle\langle A_\kappa u_j|$, and the conclusion easily follows.
- If $\vee_{\kappa\in K}\operatorname{Ran}A_\kappa = \mathcal{K}$ and Φ in N-regular, $N \geq 1$, then Φ is $(N+n)-$regular for any $n \geq 0$. This is an immediate consequence of the previous point.

The following proposition, which is a Perron-Frobenius theorem for positive maps on $\mathcal{I}_1(\mathcal{K})$, essentially comes from [9] (for the finite dimensional case) and [26] (for the infinite dimensional case). To state it in sufficient generality, we need to recall the definition of the spectral radius of a map Φ:

$$r(\Phi) = \sup\{|\lambda|,\ \lambda \in \operatorname{Sp}\Phi\}$$

where $\operatorname{Sp}\Phi$ is the spectrum of Φ.

Proposition 3.4 *Assume a CP map Φ on $\mathcal{I}_1(\mathcal{K})$ has an eigenvalue λ of modulus $r(\Phi)$, with eigenvector ρ, and either* $\dim\mathcal{K} < \infty$ *or* $r(\Phi) = \|\Phi\|$. *Then:*

- *$|\lambda|$ is also an eigenvalue, with eigenvector $|\rho|$,*
- *if Φ is irreducible, then* $\dim\operatorname{Ker}(\Phi - \lambda\operatorname{Id}) = 1$.

In particular, if Φ is irreducible and has an eigenvalue of modulus $r(\Phi)$, then $r(\Phi)$ is an eigenvalue with geometric multiplicity one, with an eigenvector that is a strictly positive operator.

Remark 3.5 When Φ is a completely positive, trace-preserving map, one has $\|\Phi\| = 1$, so that the conclusion applies if λ is of modulus 1. In [5], this was enough, since we applied this result to the operator $\mathfrak{M}$. In Sect. 5, however, we will also need to apply it to a deformation of the operator $\mathfrak{L}$, which will no longer be trace-preserving.

Remark 3.6 The previous proposition gives in particular uniqueness and faithfulness of the invariant state, when it exists, for an irreducible map Φ. As one can expect, the converse result holds: if Φ admits a unique invariant state and that state is faithful, then Φ is irreducible (see [5, Sect. 7]).

We now turn to the notion of period for positive maps. From now on, quantities like $j+1$ or $j-1$ for $j = 0,\dots,d-1$ will always refer to addition or subtraction *modulo* d.

Definition 3.7 Let Φ be a completely positive, trace-preserving, irreducible map and consider a resolution of the identity $(P_0,\dots,P_{d-1})$, *i.e.* a family of orthogonal projections such that $\sum_{j=0}^{d-1} P_j = \operatorname{Id}$. One says that $(P_0,\dots,P_{d-1})$ is Φ-cyclic if $P_j A_\kappa = A_\kappa P_{j-1}$ for $j = 0,\dots,d-1$ and any κ. The supremum of all d for which there exists a Φ-cyclic resolution of the identity $(P_0,\dots,P_{d-1})$ is called the period of Φ. If Φ has period 1 then we call it aperiodic.

Remark 3.8 When Φ is the minimal OQRW realization of a classical Markov chain, Definition 3.7 coincides with the standard definition of the period, and the projectors P_i are simply the indicator functions of subsets of the state space. See Remark 4.7 in [5].

Remark 3.9 If $\dim \mathcal{K}$ is finite then the period is always finite.

The following proposition is the analog of a standard result for classical Markov chains:

Proposition 3.10 *Assume Φ is completely positive, irreducible, with finite period d, and denote by $P_0, \dots, P_{d-1}$ a Φ-cyclic resolution of the identity. Then:*

1. *we have the relation $\Phi(P_i\, \rho\, P_j) = P_{i+1}\, \Phi(\rho)\, P_{j+1}$,*
2. *for any $j = 0, \dots, d-1$, the restriction Φ_j^d of Φ^d to $P_j \mathcal{I}_1(\mathcal{K}) P_j$ is irreducible aperiodic,*
3. *if Φ has an invariant state ρ^{inv}, then Φ_j^d has a unique invariant state $\rho_j^{\text{inv}} \overset{\text{def}}{=} d \times P_j \rho^{\text{inv}} P_j$.*

Proof 1. The first relation is obvious, and shows that $P_j \mathcal{I}_1(\mathcal{K}) P_j$ is stable by Φ^d.

2. Consider a state $P_j \rho P_j$ in $P_j \mathcal{I}_1(\mathcal{K}) P_j$. By irreducibility of Φ, $\mathrm{e}^{t\Phi}(P_j \rho P_j)$ is faithful, so $P_j \mathrm{e}^{t\Phi}(P_j \rho P_j) P_j$ is faithful in $\operatorname{Ran} P_j$. But by the relation in point 1,

$$P_j \mathrm{e}^{t\Phi}(P_j \rho P_j) P_j = \sum_{n=0}^{\infty} \frac{t^{dn}}{(dn)!}\, \Phi^{dn}(P_j \rho P_j) = \sum_{n=0}^{\infty} \frac{t^{dn}}{(dn)!}\, (\Phi_j^d)^n (P_j \rho P_j).$$

   This shows that Φ_j^d is irreducible. Now, if Φ_j^d has a cyclic resolution of identity $(P_{j,0}, \dots, P_{j,\delta-1})$ then by the commutation relations this induces a Φ-cyclic resolution of the identity with $d \times \delta$ elements. Therefore, $\delta = 1$.
3. The invariance of ρ_j^{inv} is trivial by point 1, and the irreducibility of Φ_j^d implies the uniqueness of the invariant state (recall Remark 3.6). By Remark 4.8 in [5], $\operatorname{Tr}(P_j \rho^{\text{inv}} P_j)$ does not depend on j, so it is $1/d$.

□

The following results were originally proved by Fagnola and Pellicer in [10] (with partial results going back to [9] and [11]). We recall that the point spectrum of an operator is its set of eigenvalues, and that we denote by $\operatorname{Sp}_{pp} \Phi^*$ the point spectrum of Φ^*.

Proposition 3.11 *If Φ is an irreducible, completely positive, trace-preserving map on $\mathcal{I}_1(\mathcal{K})$ and has finite period d then:*

- *the set $\operatorname{Sp}_{pp} \Phi^* \cap \mathbb{T}$, is a subgroup of the circle group $\mathbb{T}$,*
- *the primitive root of unity $\mathrm{e}^{\mathrm{i}2\pi/d}$ belongs to $\operatorname{Sp}_{pp} \Phi^*$ if and only if Φ is d-periodic.*

An immediate consequence is the following:

Proposition 3.12 *If a completely positive, trace-preserving map Φ on $\mathcal{I}_1(\mathcal{K})$ is irreducible and aperiodic with invariant state ρ^{inv}, and $\mathcal{K}$ is finite-dimensional then*

- $\operatorname{Sp}_{\text{pp}} \Phi \cap \mathbb{T} = \{1\}$,
- *for any $\rho \in \mathcal{I}_1(\mathcal{K})$ one has $\Phi^p(\rho) \to \rho^{\text{inv}}$ as $p \to \infty$.*

4 Irreducibility and Period of $\mathfrak{M}$ and $\mathfrak{L}$

Now we turn to the case where the operator Φ is an open quantum random walk $\mathfrak{M}$ generated by L_s, $s \in S$, or the auxiliary map $\mathfrak{L}$ as defined by (2.3). We will study irreducibility and periodicity properties of both operators $\mathfrak{M}$ and $\mathfrak{L}$, and connections between them. This will explain why we focus on a study of $\mathfrak{L}$, when $\mathfrak{M}$ should intuitively be the object of interest.

For any v in V we denote

$$\mathcal{P}_\ell(v) = \left\{ \pi = (s_1, \dots, s_\ell) \in S^\ell \mid \sum_{p=1}^{\ell} s_p = v \right\}$$

and, in addition, we consider

$$\mathcal{P}(v) = \cup_{\ell \geq 1} \mathcal{P}_\ell(v), \qquad \mathcal{P}_\ell = \cup_{v \in V} \mathcal{P}_\ell(v) \qquad \mathcal{P} = \cup_{\ell \in \mathbb{N}} \mathcal{P}_\ell = \cup_{v \in V} \mathcal{P}(v).$$

In analogy with [5], we use the notation

$$L_\pi = L_{s_\ell} \cdots L_{s_1}, \qquad \text{for } \pi = (s_1, \dots, s_\ell) \in \mathcal{P}_\ell.$$

We remark that the notations for the paths and the set of paths are slightly different from our previous paper [5] since we can use homogeneity, which allows us to drop the dependence on the particular starting point.

The irreducibility of $\mathfrak{L}$ and $\mathfrak{M}$ are easily characterized in terms of paths. This is true in general for OQRWs (see [5], Proposition 3.9 in particular), but the following characterizations are specific to homogeneous OQRWs.

Proposition 4.1 *Let $\mathfrak{M}$ be an open quantum random walk defined by transition operators L_s, $s \in S$, and $\mathfrak{L}$ its auxiliary map.*

1. *The operator $\mathfrak{L}$ is irreducible if and only if the operators $\{L_s,\ s \in S\}$ have no invariant closed subspace in common, apart from $\{0\}$ and $\mathfrak{h}$.*
2. *The operator $\mathfrak{M}$ is irreducible if and only if the operators $\{L_{\pi_0},\ \pi_0 \in \mathcal{P}(0)\}$ have no invariant closed subspace in common, apart from $\{0\}$ and $\mathfrak{h}$.*

Proof Point 1 is proven by a direct application of Definition 3.1 (third condition).

To prove point 2, remark that irreducibility of $\mathfrak{M}$ amounts to the fact that, for any $x \otimes |w\rangle$, the set $\{L_\pi x \otimes |v + w\rangle, \pi \in \mathcal{P}(v)\}$ is dense in $\mathfrak{h}$ for any $v \in V$ (see the details in Proposition 3.9 of [5]). Now, if $\mathfrak{M}$ is not irreducible, then, for some $v \in V$, the closed space

$$\mathfrak{h}_v = \overline{\mathrm{Vect}\{L_\pi x,\ \pi \in \mathcal{P}(v)\}}$$

is nontrivial. Since the concatenation of $\pi_0 \in \mathcal{P}(0)$ with $\pi \in \mathcal{P}(v)$ gives an element of $\mathcal{P}(v)$, the space $\mathfrak{h}_v$ must be L_{π_0}-invariant, and this holds for any $\pi_0 \in \mathcal{P}(0)$. Conversely, if all operators $L_{\pi_0}, \pi_0 \in \mathcal{P}(0)$ have an invariant subspace $\mathfrak{h}'$ in common, then for any $x \in \mathfrak{h}'$ and $w = v = 0$, we have a contradiction to the above criterion for irreducibility of $\mathfrak{M}$. □

This proposition obviously implies the following result:

Corollary 4.2 *If $\mathfrak{M}$ is irreducible, then $\mathfrak{L}$ is irreducible.*

Proposition 4.1 also allows us to construct examples of OQRWs such that $\mathfrak{L}$ is irreducible, but not $\mathfrak{M}$.

Example 4.3 Let $V = \mathbb{Z}$, $\mathfrak{h} = \mathbb{C}^2$, $S = \{-1, +1\}$, denote $L_- = L_{-1}$, $L_+ = L_{+1}$ (this will be a particular case of the OQRWs treated in Example 6.11) and choose

$$L_+ = \begin{pmatrix} 0 & a_+ \\ b_+ & 0 \end{pmatrix} \qquad L_- = \begin{pmatrix} 0 & a_- \\ b_- & 0 \end{pmatrix}$$

with a_+, a_-, b_+, b_- positive, with $a_+^2 + b_+^2 = a_-^2 + b_-^2 = 1$ and $a_+ b_- \neq a_- b_+$. Then, by Proposition 4.1, $\mathfrak{L}$ is irreducible, but $\mathfrak{M}$ is not, since the vectors of the canonical basis are eigenvectors for any $L_\pi, \pi \in \mathcal{P}(0)$ (see also Proposition 6.12).

The following proposition is proved in [5]. We reprove it here.

Proposition 4.4 *Assume $\mathfrak{M}$ is irreducible. Then it has no invariant state.*

Proof By Corollary 4.2, $\mathfrak{L}$ is irreducible, so it has a unique invariant state ρ^{inv} on $\mathfrak{h}$, which is faithful. Assume $\mathfrak{M}$ has an invariant state; by irreducibility it is unique. Since $\mathfrak{M}$ is space-homogeneous, any translation of that state would be also invariant, so by unicity the invariant state must be of the form $\sum_{i\in V}\rho^{\mathrm{inv}}\otimes|i\rangle\langle i|$, but this has infinite trace, a contradiction. □

The only reason that prevents the proposed operator from being an invariant state for $\mathfrak{M}$ is the fact that its trace is infinite. This is similar to the situation for space-homogeneous classical Markov chains. In the same way that, for classical Markov chains, one can consider invariant, non necessarily finite, measures, we could extend the map $\mathfrak{M}$, using expression (2.4), to a wider domain and study invariant positive operators which are not trace class. If $\mathfrak{L}$ has an invariant state ρ^{inv} (which is the case if e.g. $\mathfrak{h}$ is finite), then $\sum_{i\in V}\rho^{\mathrm{inv}}\otimes|i\rangle\langle i|$ will be an invariant operator for the extension of $\mathfrak{M}$.

All ergodic convergence results for $\mathfrak{M}$ given in [5] assume the existence of an invariant state. However, some interesting asymptotic properties of $\mathfrak{M}$ can be studied in the absence of an invariant state, and this includes large deviations or central limit theorems. As we will see, such properties can be derived from the study of $\mathfrak{L}$. This is why, in the study of homogeneous OQRWs, the focus shifts from $\mathfrak{M}$ to $\mathfrak{L}$.

To avoid discussing trivial cases, in the rest of this paper we will usually make the following assumption, which by Remark 3.3 automatically holds as soon as $\mathfrak{L}$ (or $\mathfrak{M}$) is irreducible:

Assumption H1 one has the equality $\overline{\bigvee_{s\in S}\operatorname{Ran}L_s}=\mathfrak{h}$.

This assumption is a natural one, since after just one step, even in the reducible case, the system is effectively restricted to the space $\bigvee_{s\in S}\operatorname{Ran}L_s$. More precisely, for any positive operator ρ on $\mathfrak{h}$, one has, for any s,

$$\operatorname{supp}L_s\,\rho\,L_s^*\subset\operatorname{supp}\mathfrak{L}(\rho)\subset\overline{\bigvee_{s\in S}\operatorname{Ran}L_s}.$$

Note that we have not given results equivalent to Proposition 4.1 for the notion of regularity. We do this here:

Lemma 4.5 *The operator $\mathfrak{L}$ is N-regular if and only if for any $x\neq 0$ in $\mathfrak{h}$, the set $\{L_\pi x,\ |\,\pi\in\mathcal{P}_N\}$ is total in $\mathfrak{h}$. The operator $\mathfrak{M}$ can never be regular.*

Proof This is obtained by direct application of Definition 3.2, that shows the criterion for $\mathfrak{L}$. It also shows that $\mathfrak{M}$ is N-regular if and only if for any $x\neq 0$ in $\mathfrak{h}$, any v in V, the set $\{L_\pi x,\ |\,\pi\in\mathcal{P}_N(v)\}$ is total in $\mathfrak{h}$. However, if the distance from the origin to v is larger than N, then $\mathcal{P}_N(v)$ is empty. □

One could be tempted to consider a weaker version of regularity for $\mathfrak{L}$ where the index N can depend on ρ. The following result shows that, if $\mathfrak{h}$ is finite-dimensional, this is not weaker than regularity:

Lemma 4.6 *Assume $\mathfrak{h}$ is finite-dimensional. If for every $\rho\geq 0$ in $\mathcal{I}_1(\mathfrak{h})\setminus\{0\}$, there exists $N>0$ such that $\mathfrak{L}^N(\rho)$ is faithful, then there exists N_0 such that $\mathfrak{L}$ is N_0-regular.*

Proof First observe that $\mathfrak{L}$ is necessarily irreducible and so assumption H1 must hold. Besides, the current assumption implies that, for any x in $\mathfrak{h}$, there exists $N_x>0$ such that $\mathfrak{L}^{N_x}(|x\rangle\langle x|)$ is faithful. Since faithfulness of $\mathfrak{L}^{N_x}(|x\rangle\langle x|)$ is equivalent to the existence of a family $\pi_1, \ldots, \pi_{\dim \mathfrak{h}}$ of paths of length N_x, such that the determinant of $(L_{\pi_1} x, \ldots, L_{\pi_{\dim \mathfrak{h}}} x)$ is nonzero, there exist open subsets B_x of the unit ball, such that $x \in B_{x_0}$ implies that $\mathfrak{L}^{N_{x_0}}(|x\rangle\langle x|)$ is faithful. By compactness of the unit ball, there exists a finite covering by $B_{x_1} \cup \ldots \cup B_{x_p}$. Remark 3.3 then implies that if we let $N_0 = \sup_{i=1,\ldots,p} N_{x_i}$ one has $\mathfrak{L}^{N_0}(|x\rangle\langle x|)$ faithful for any nonzero x. This implies that $\mathfrak{L}$ is N_0-regular. □

Remark 4.7 Notice that the same result cannot be true when the dimension of $\mathfrak{h}$ is infinite: assume for example that $\mathfrak{h} = \mathbb{C}^{\mathbb{N}}$, and $\mathfrak{L}$ is itself a minimal OQRW realization of a classical Markov chain on $\mathbb{N}$ with stochastic matrix

$$t_{0j} > 0 \quad \forall j \in \mathbb{N}, \qquad t_{j,j-1} = t_{j,j+1} = \frac{1}{2} \quad \forall j \in \mathbb{N} \setminus \{0\}.$$

We now turn to the notion of period for $\mathfrak{L}$ and $\mathfrak{M}$. By Definition 3.7, a resolution of identity $(p_0, \ldots, p_{d-1})$ of $\mathfrak{h}$ will be $\mathfrak{L}$-cyclic if and only if

$$p_j L_s = L_s p_{j-1} \quad \text{for } j = 0, \ldots, d-1 \text{ and any } s \in S.$$

Consequently, by Proposition 3.10, we have

$$\mathfrak{L}(p_j \, \rho \, p_j) = p_{j+1} \, \mathfrak{L}(\rho) \, p_{j+1}. \tag{4.1}$$

Remark 4.8 Since the p_j's sum up to $\mathrm{Id}_{\mathfrak{h}}$, the period of $\mathfrak{L}$ cannot be greater than $\dim \mathfrak{h}$, a feature which will be extremely useful when $\dim \mathfrak{h}$ is small.

On the other hand, as we observed in [5], a resolution of identity $(P_0, \ldots, P_{d-1})$ of $\mathcal{H}$ will be $\mathfrak{M}$-cyclic if and only if it is of the form

$$P_k = \sum_{i \in V} P_{k,i} \otimes |i\rangle\langle i| \quad \text{with} \quad P_{k,i} L_s = L_s P_{k-1,i+s}. \tag{4.2}$$

Remark 4.9 The $\mathfrak{M}$-cyclic resolutions of the identity are translation invariant, in the sense that, if $P_k = \sum_{i \in V} P_{k,i} \otimes |i\rangle\langle i|$, $k = 0, ...d-1$, is a $\mathfrak{M}$-cyclic resolution of the identity, then also $P_k' = \sum_{i \in V} P_{k,i+v} \otimes |i\rangle\langle i|$, $k = 0, ...d-1$, is a cyclic resolution for any v.

We will, however, make little use of $\mathfrak{M}$-cyclic resolutions of the identity in this paper. On the other hand, the periodicity of $\mathfrak{L}$ can be an easy source of information on $\mathfrak{M}$:

Proposition 4.10 *We have the following properties:*

1. *The period of $\mathfrak{M}$, when finite, is even.*
2. *If $\mathfrak{L}$ is irreducible and has even period d, then $\mathfrak{M}$ is reducible.*

Proof 1. Assume that $(P_0, \ldots, P_{d-1})$ is a $\mathfrak{M}$-cyclic resolution of identity associated with $\mathfrak{M}$. As we observed above, the P_k are of the form

$$P_k = \sum_{i \in V} P_{k,i} \otimes |i\rangle\langle i| \quad \text{with} \quad P_{k,i} L_s = L_s P_{k-1,i+s}.$$

Then if we call i in V odd or even depending on the parity of its distance to the origin, define

$$P_{k,\mathrm{odd}} = \sum_{i \text{ odd}} P_{k,i} \otimes |i\rangle\langle i| \quad \text{and} \quad P_{k,\mathrm{even}} = \sum_{i \text{ even}} P_{k,i} \otimes |i\rangle\langle i|.$$

Then $(P_{0,\mathrm{odd}}, P_{1,\mathrm{even}}, P_{2,\mathrm{odd}}, \ldots)$ is a $\mathfrak{M}$-cyclic resolution of identity.

2. Denote by $(p_0, \dots, p_{d-1})$ a $\mathfrak{L}$-cyclic resolution of identity. Define

$$p_{\text{odd}} = \sum_{k \text{ odd}} p_k \qquad p_{\text{even}} = \sum_{k \text{ even}} p_k.$$

It is obvious from relations (4.2) that Ran p_{odd} and Ran p_{even} are nontrivial invariant spaces for any L_{π_0}, $\pi_0 \in \mathcal{P}(0)$. We conclude by Proposition 4.1.

□

Last, we give an analog of a classical property of Markov chains with finite state space:

Lemma 4.11 *If $\mathfrak{h}$ is finite-dimensional, then the map $\mathfrak{L}$ is irreducible and aperiodic if and only if it is regular.*

Proof If $\mathfrak{L}$ is irreducible and aperiodic, then by Proposition 3.12 for any state ρ on $\mathfrak{h}$, one has $\mathfrak{L}^n(\rho) \underset{n\to\infty}{\longrightarrow} \rho^{\text{inv}}$ so that $\mathfrak{L}^n(\rho)$ is faithful for large enough n. By Lemma 4.6, this implies the regularity of $\mathfrak{L}$. Conversely, if $\mathfrak{L}$ is N-regular, then it is irreducible, and for any projection p, the operator $\mathfrak{L}^N(p)$ is faithful, so that p cannot be a member of a cyclic resolution of identity unless $p = \text{Id}$. □

5 Central Limit Theorem and Large Deviations

The Perron-Frobenius theorem for CP maps allows us to obtain a large deviations principle and a central limit theorem for the position process $(X_p)_{p\in\mathbb{N}}$ (or, equivalently, for the process $(Q_p)_{p\in\mathbb{N}}$) associated with an open quantum random walk $\mathfrak{M}$ and an initial state ρ (see Sect. 2). In most of our statements, we assume for simplicity that $\mathfrak{L}$ is irreducible. We discuss extensions of our results at the end of this section.

Before going into the details of the proof, we should mention that, as we were completing the present article, we learnt about the recent paper [29], which proves a large deviation result for empirical measures of outputs of quantum Markov chains, which can be viewed as the "steps" $M_p = X_p - X_{p-1}$ taken by an open quantum random walk. This result is similar to the statement in our Remark 5.6, and implies a level-1 large deviation result for the position $(X_p)_{p\in\mathbb{N}}$ when the OQRW is irreducible and aperiodic. In addition, the statement in [29] extends to a large deviations principle for empirical measures of m-tuples $(M_p, \dots, M_{p+m-1})_p$. Our (independent) result, however, treats the case where the OQRW is irreducible but not aperiodic, and can be extended beyond the irreducible case.

For the proofs of this section, it will be convenient to introduce some new notations. For u in $\mathbb{R}^d$ we define $L_s^{(u)} = \mathrm{e}^{\langle u,s\rangle/2} L_s$, and denote $\mathfrak{L}_u$ the map induced by the $L_s^{(u)}$, $s \in S$: for ρ in $\mathcal{I}_1(\mathfrak{h})$,

$$\mathfrak{L}_u(\rho) = \sum_s L_s^{(u)} \rho L_s^{(u)*}.$$

This operator is a deformation of $\mathfrak{L}$. It is still a completely positive map on $\mathcal{I}_1(\mathfrak{h})$ but, in general, it is not trace-preserving. The operators $\mathfrak{L}_u$ will be useful in order to treat the moment generating functions of the random variables $(X_p)_{p\in\mathbb{N}}$:

Lemma 5.1 *For any u in $\mathbb{R}^d$ one has*

$$\mathbb{E}(\exp\langle u, X_p - X_0\rangle) = \sum_{i_0\in V} \operatorname{Tr}\big(\mathfrak{L}_u^p(\rho(i_0))\big). \tag{5.1}$$

Proof For any k in $\mathbb{N}^*$ let $S_k = X_{k+1} - X_k$ and consider $u \in \mathbb{R}^d$. Then we have

$$\begin{aligned}
&\mathbb{E}(\exp\langle u, X_p - X_0\rangle) \\
&\quad = \sum_{i_0 \in V} \sum_{s_1,\ldots,s_p \in S^p} \mathbb{P}(X_0 = i_0, S_1 = s_1, \ldots, S_p = s_p) \exp\langle u, s_1 + \ldots + s_p\rangle \\
&\quad = \sum_{i_0 \in V} \sum_{s_1,\ldots,s_p \in S^p} \mathrm{Tr}(L_{s_p} \ldots L_{s_1}\, \rho(i_0)\, L_{s_1}^* \ldots L_{s_p}^*) \exp\langle u, s_1 + \ldots + s_p\rangle
\end{aligned}$$

and this gives formula (5.1). □

Remark 5.2 One also has

$$\mathbb{E}(\exp\langle u, X_p\rangle) = \mathbb{E}(\exp\langle u, Q_p\rangle) = \sum_{i_0 \in V} \exp\langle u, i_0\rangle \,\mathrm{Tr}\big(\mathfrak{L}_u^p(\rho(i_0))\big).$$

This will allow us to give results analogous to Theorems 5.4 and 5.12 for the process $(Q_p)_p$. Note that considering X_p or $X_p - X_0$ is essentially equivalent, but as we remarked in Sect. 2, Q_p and Q_0 should not be considered simultaneously.

The following lemma describes the properties of the largest eigenvalue of $\mathfrak{L}_u$:

Lemma 5.3 *Assume that $\mathfrak{h}$ is finite-dimensional and $\mathfrak{L}$ is irreducible. For any u in $\mathbb{R}$, the spectral radius $\lambda_u \stackrel{\text{def}}{=} r(\mathfrak{L}_u)$ of $\mathfrak{L}_u$ is an algebraically simple eigenvalue of $\mathfrak{L}_u$, and has an eigenvector ρ_u which is a faithful state. In addition, the map $u \mapsto \lambda_u$ can be extended to be analytic in a neighbourhood of $\mathbb{R}^d$.*

Proof By Proposition 4.1, if $\mathfrak{L}$ is irreducible, then so is any $\mathfrak{L}_u$ for $u \in \mathbb{R}^d$. Proposition 3.4, applied here specifically to an Hilbert space of finite dimension, gives the first sentence except for the algebraic simplicity of the eigenvector λ_u, as it implies only the geometric simplicity. If we can prove that, for all u in $\mathbb{R}^d$, the eigenvalue λ_u is actually algebraically simple then the theory of perturbation of matrix eigenvalues (see [15, Chapt. II]) will give us the second sentence. Now, in order to prove the missing point, consider the adjoint $\mathfrak{L}_u^*$ of $\mathfrak{L}_u$ on $\mathcal{B}(\mathfrak{h})$, which in this finite-dimensional setting, can be identified, with $\mathcal{I}_1(\mathfrak{h})$. It is easy to see from Definition 3.1 that $\mathfrak{L}_u^*$ is irreducible. Its largest eigenvalue is λ_u, with eigenvector M_u, which, by Proposition 3.4, is invertible. We can consider the map

$$\widetilde{\mathfrak{L}}_u : \rho \mapsto \frac{1}{\lambda_u} M_u^{1/2} \mathfrak{L}_u(M_u^{-1/2}\, \rho\, M_u^{-1/2}) M_u^{1/2}.$$

This $\widetilde{\mathfrak{L}}_u$ is clearly completely positive, and is trace-preserving since $\widetilde{\mathfrak{L}}_u^*(\mathrm{Id}) = \mathrm{Id}$. Proposition 3.4 shows that $\widetilde{\mathfrak{L}}_u$ has 1 as a geometrically simple eigenvalue, with a strictly positive eigenvector $\widetilde{\rho}_u$. Then 1 must also be algebraically simple, otherwise there exists η_u such that $\widetilde{\mathfrak{L}}_u(\eta_u) = \eta_u + \widetilde{\rho}_u$, but taking the trace of this equality yields $\mathrm{Tr}(\widetilde{\rho}_u) = 0$, a contradiction. This implies that $\mathfrak{L}_u$ has λ_u as a algebraically simple eigenvalue. □

We can now state our large deviation result:

Theorem 5.4 *Assume that $\mathfrak{h}$ is finite-dimensional and that $\mathfrak{L}$ is irreducible. Then the process $(\frac{1}{p}(X_p - X_0))_{p\in\mathbb{N}^*}$ associated with $\mathfrak{M}$ satisfies a large deviation principle with a good rate function I. Explicitly, the function $I : \mathbb{R}^d \to [0, +\infty]$ defined by*

$$I(x) = \sup_{u \in \mathbb{R}^d} \big(\langle u, x\rangle - \log \lambda_u\big),$$

(*where* λ_u *is defined in Lemma* 5.3) *is lower semicontinuous, has compact level sets* $\{x \mid I(x) \leq \alpha\}$, *and for any open* G *and closed* F *with* $G \subset F \subset \mathbb{R}^d$, *one has*

$$-\inf_{x\in G} I(x) \leq \liminf_{p\to\infty} \frac{1}{p} \log \mathbb{P}\left(\frac{X_p - X_0}{p} \in G\right) \leq \limsup_{p\to\infty} \frac{1}{p} \log \mathbb{P}\left(\frac{X_p - X_0}{p} \in F\right) \leq -\inf_{x\in F} I(x).$$

Remark 5.5 If we add the assumption that X_0 has an everywhere defined moment generating function, i.e. that the initial state ρ satisfies $\mathbb{E}(\exp\langle u, X_0\rangle) = \sum_{i_0\in V} \mathrm{e}^{\langle u,i_0\rangle} \mathrm{Tr}\rho(i_0) < \infty$ for all u in $\mathbb{R}^d$, then this theorem also holds for $(X_p)_{p\in\mathbb{N}}$, or equivalently $(Q_p)_{p\in\mathbb{N}}$, in place of $(X_p - X_0)_{p\in\mathbb{N}}$.

Remark 5.6 If φ is any function $S \to \mathbb{R}$ and $S_p = \sum_{k=1}^{p} \varphi(X_k - X_{k-1})$ then the process $(\frac{S_p}{p})_{p\in\mathbb{N}}$ also satisfies a large deviation principle, with rate function

$$I_\varphi(x) = \sup_{t\in\mathbb{R}} \big(t\,x - \log \lambda_{t\varphi}\big)$$

where $\lambda_{t\varphi}$ is the largest eigenvalue of

$$\mathfrak{L}_{t\varphi} : \rho \mapsto \sum_{s\in S} \mathrm{e}^{t\varphi(s)} L_s\, \rho L_s^*.$$

This is shown by an immediate extension of the proofs of Lemma 5.3 and Theorem 5.4, and immediately yields a level-2 large deviation result for the process $(M_p)_{p\in\mathbb{N}}$, where $M_p = X_p - X_{p-1}$. Using the techniques of [29], one can prove a large deviation property for the empirical law of m-tuples $(M_p, \ldots, M_{p+m-1})_{p\in\mathbb{N}}$. Under the same condition as in Remark 5.5, this implies a large deviation property for m-tuples $(X_p, \ldots, X_{p+m-1})_{p\in\mathbb{N}}$. All of these results can be derived under the assumption that $\mathfrak{L}$ is irreducible, not necessarily aperiodic. The irreducibility assumption can be relaxed, as we discuss at the end of this section.

Proof We start with Eq. (5.1). Since $\mathfrak{h}$ is finite-dimensional, if $\rho(i_0)$ is faithful, then, with $r_{u,i_0} = \inf \mathrm{Sp}(\rho(i_0)) > 0$ and $s_{u,i_0} = \frac{\mathrm{Tr}\rho(i_0)}{\inf \mathrm{Sp}(\rho_u)} > 0$,

$$r_{u,i_0}\, \rho_u \leq \rho(i_0) \leq s_{u,i_0}\, \rho_u. \tag{5.2}$$

Note that $r_{u,i_0} \leq \mathrm{Tr}\rho(i_0)$ so that both r_{u,i_0} and s_{u,i_0} are summable along i_0. Consequently, we shall have

$$r_{u,i_0}\, \lambda_u^p\, \rho_u \leq \mathfrak{L}_u^p\big(\rho(i_0)\big) \leq s_{u,i_0}\, \lambda_u^p\, \rho_u. \tag{5.3}$$

Using these bounds in relation (5.1), we immediately obtain, for all $u \in \mathbb{R}^d$,

$$\lambda_u^p \sum_{i_0\in V} r_{u,i_0}\, \rho_u \leq \mathbb{E}(\exp\langle u, X_p - X_0\rangle) \leq \lambda_u^p \sum_{i_0\in V} s_{u,i_0}\, \rho_u$$

where the sums are finite and strictly positive; so that

$$\lim_{p\to\infty} \frac{1}{p} \log \mathbb{E}(\exp\langle u, X_p\rangle) = \log \lambda_u. \tag{5.4}$$

Now, if $\rho(i_0)$ is not faithful, but $\mathfrak{L}$ is aperiodic, due to Proposition 3.12, then $\mathfrak{L}^N(\rho(i_0))$ is faithful for large enough N, and (5.2) holds with $\mathfrak{L}_u^N(\rho(i_0))$ in place of $\rho(i_0)$ and (5.3) holds with $(p - N)$ instead of p in the exponents of λ_u. We still recover (5.4).

Finally, if $\rho(i_0)$ is not faithful and $\mathfrak{L}$ has period $d > 1$, then, considering a cyclic decomposition of identity $(p_0, \ldots, p_{d-1})$, we can consider the single blocks of the form $p_j \rho(i_0) p_j$. By Proposition 3.10, $\mathfrak{L}^d$ is irreducible aperiodic when restricted to each $p_j \mathcal{I}_1(\mathfrak{h}) p_j$ and $\mathfrak{L}_u^d(p_j \rho_u p_j) = \lambda_u^d p_j \rho_u p_j$. Then, by the regularity of the restrictions of $\mathfrak{L}^d$, using Remark 3.3 and the obvious extension of (4.1) to $\mathfrak{L}_u$, there exist $N \in \mathbb{N}$ and $r_{u,i_0}, s_{u,i_0} > 0$ such that, for any block $p_j \rho(i_0) p_j \neq 0$,

$$r_{u,i_0}\, p_j\, \rho_u\, p_j \leq p_j\, \mathfrak{L}_u^{dN}\big(\rho(i_0)\big)\, p_j \leq s_{u,i_0}\, p_j\, \rho_u\, p_j$$

and if $p = dN + r$, $r \in \{0, \ldots, d-1\}$,

$$r_{u,i_0}\, \lambda_u^{p-dN}\quad p_{j+r}\, \rho_u\, p_{j+r} \leq \mathfrak{L}_u^p\big(p_j\, \rho(i_0)\, p_j\big) \leq s_{u,i_0}\, \lambda_u^{p-dN}\quad p_{j+r}\, \rho_u\, p_{j+r}.$$

Summing over j, we recover Eq. (5.4) again.

In any case, we obtain (5.4) for all $u \in \mathbb{R}^d$. Lemma 5.3 shows that $u \mapsto \log \lambda_u$ is analytic on $\mathbb{R}$. We can now apply the Gärtner-Ellis theorem (see [7]) to conclude. □

Remark 5.7 As noted in Remark 2.4, when $\mathfrak{M}$ is the minimal OQRW realization of a classical Markov chain with transition probabilities $(t_s)_{s\in S}$, the map $\mathfrak{L}$ is trivial: it is just multiplication by 1 on $\mathbb{R}$. The maps $\mathfrak{L}_u$, however, are not trivial: they are multiplication by

$$\lambda_u = \sum_{s\in S} \exp\langle u, s\rangle\, t_s.$$

We therefore recover the same rate function as in the classical case, see e.g. [7, Sect. 3.1.1].

Remark 5.8 The technique of applying the Perron-Frobenius theorem to a u-dependent deformation of the completely positive map defining the dynamics, goes back (to the best of our knowledge) to [13], and is a non-commutative adaptation of a standard proof for Markov chains.

We denote by c the map $c : \mathbb{R}^d \ni u \mapsto \log \lambda_u$. As is well-known (see e.g. [8, Sect. II.6]), the differentiability of c at zero is related to a law of large numbers for the process $(X_p)_{p\in\mathbb{N}}$. Similarly, the second order differential will be relevant for the central limit theorem.

Corollary 5.9 *Assume that $\mathfrak{h}$ has finite dimension and that $\mathfrak{L}$ is irreducible. The function c on $\mathbb{R}^d$ is infinitely differentiable at zero. Denote by*

$$\mathfrak{L}_u' : \rho \mapsto \sum_{s\in S} \langle u, s\rangle\, L_s \rho L_s^* \quad \textit{and} \quad \mathfrak{L}_u'' : \rho \mapsto \sum_{s\in S} \langle u, s\rangle^2 L_s \rho L_s^*.$$

Then, denoting $\lambda_u' \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \lambda_{tu}$ and $\lambda_u'' \stackrel{\text{def}}{=} \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} \lambda_{tu}$, we have

$$\lambda_u' = \mathrm{Tr}\big(\mathfrak{L}_u'(\rho^{\mathrm{inv}})\big) \tag{5.5}$$

$$\lambda_u'' = \mathrm{Tr}\big(\mathfrak{L}_u''(\rho^{\mathrm{inv}})\big) + 2\mathrm{Tr}\big(\mathfrak{L}_u'(\eta_u)\big) \tag{5.6}$$

where η_u is the unique solution with trace zero of the equation

$$\big(\mathrm{Id} - \mathfrak{L}\big)(\eta_u) = \mathfrak{L}_u'(\rho^{\mathrm{inv}}) - \mathrm{Tr}\big(\mathfrak{L}_u'(\rho^{\mathrm{inv}})\big)\, \rho^{\mathrm{inv}}. \tag{5.7}$$

This implies immediately that

$$\mathrm{d}c(0)\,(u) = \lambda_u' \qquad \mathrm{d}^2 c(0)\,(u, u) = \lambda_u'' - {\lambda_u'}^2. \tag{5.8}$$

Proof Lemma 5.3 shows that c_u is infinitely differentiable at any $u \in \mathbb{R}^d$. In addition (again see [15, Chapt. II]), the largest eigenvalue λ_u of $\mathfrak{L}_u$ is an analytic perturbation of $\lambda_0 = 1$, and has an eigenvector ρ_u which we can choose to be a state, and this ρ_u is an analytic perturbation of ρ_0. Then one has

$$\lambda_{tu} = 1 + t\lambda'_u + \frac{t^2}{2}\lambda''_u + o(t^2)$$

$$\rho_{tu} = \rho^{\text{inv}} + t\,\eta_u + \frac{t^2}{2}\,\sigma_u + o(t^2)$$

$$\mathfrak{L}_{tu} = \mathfrak{L} + t\,\mathfrak{L}'_u + \frac{t^2}{2}\,\mathfrak{L}''_u + o(t^2)$$

and since every ρ_{tu} is a state then $\operatorname{Tr}\eta_u = \operatorname{Tr}\sigma_u = 0$. The relation $\mathfrak{L}_{tu}(\rho_{tu}) = \lambda_{tu}\,\rho_{tu}$ yields

$$\mathfrak{L}'_u(\rho^{\text{inv}}) + \mathfrak{L}(\eta_u) = \eta_u + \lambda'_u\,\rho^{\text{inv}}$$

$$\frac{1}{2}\mathfrak{L}(\sigma_u) + \mathfrak{L}'_u(\eta_u) + \frac{1}{2}\mathfrak{L}''_u(\rho^{\text{inv}}) = \frac{1}{2}\sigma_u + \lambda'_u\,\eta_u + \frac{1}{2}\lambda''_u\,\rho^{\text{inv}}.$$

Taking the trace of the first relation immediately yields relation (5.5). In addition, it yields relation (5.7). Since $\mathrm{Id} - \mathfrak{L}$ has kernel of dimension one, and range in the set of operators with zero trace, it induces a bijection on that state, so that (5.7) has a unique solution with trace zero. Then taking the trace of the second relation above, and using the fact that $\mathfrak{L}$ is trace-preserving gives relation (5.6). □

Corollary 5.10 *Assume that* $\mathfrak{h}$ *has finite dimension and* $\mathfrak{L}$ *is irreducible, and let* $m = \sum_s \operatorname{Tr}(L_s\rho^{\text{inv}}L_s^*)\,s$. *Then the process* $(\frac{1}{p}(X_p - X_0))_{p\in\mathbb{N}}$ *associated with* $\mathfrak{M}$ *converges exponentially to* m, i.e. *for any* $\varepsilon > 0$ *there exists* $N > 0$ *such that, for large enough* p,

$$\mathbb{P}\left(\left\|\frac{X_p - X_0}{p} - m\right\| > \varepsilon\right) \leq \exp(-pN).$$

This implies the almost-sure convergence of $(\frac{X_p}{p})_{p\in\mathbb{N}}$ *to* m.

Remark 5.11 The almost-sure convergence holds replacing X_p by Q_p.

Proof This is a standard result, see e.g. [8, Theorems II.6.3 and II.6.4]. □

Theorem 5.12 *Assume that* $\mathfrak{h}$ *is finite-dimensional and* $\mathfrak{L}$ *is irreducible. Denote by* m *the quantity defined in Corollary 5.10, and by* C *the covariance matrix associated with the quadratic form* $u \mapsto \lambda''_u - {\lambda'_u}^2$. *Then the position process* $(X_p)_{p\in\mathbb{N}}$ *associated with* $\mathfrak{M}$ *satisfies*

$$\frac{X_p - p\,m}{\sqrt{p}} \underset{p\to\infty}{\longrightarrow} \mathcal{N}(0, C)$$

where convergence is in law.

Proof Let us first consider the case where $\mathfrak{L}$ is irreducible and aperiodic. Equation (5.1) implies

$$\mathbb{E}(\exp\langle u, X_p - X_0\rangle) = \sum_{i_0\in V}\operatorname{Tr}\big(\mathfrak{L}_u^p(\rho(i_0))\big).$$

Now, considering the Jordan form of $\mathfrak{L}$ shows that, if

$$\delta \stackrel{\text{def}}{=} \sup\{|\lambda|, \lambda \in \operatorname{Sp}\mathfrak{L} \setminus \{1\}\},$$

then $\delta < 1$ and for u in a real neighbourhood of 0 and p in $\mathbb{N}$,

$$\mathfrak{L}_u^p = \lambda_u^p \big(\varphi_u(\cdot)\, \rho_u + O((\delta+\varepsilon)^p)\big) \tag{5.9}$$

for some ε such that $\delta + \varepsilon < 1$, where φ_u is a linear form on $\mathcal{I}_1(\mathfrak{h})$, analytic in u and such that $\varphi_0 = \operatorname{Tr}$ and the $O((\delta+\varepsilon)^p)$ is in terms of the operator norm on $\mathcal{I}_1(\mathfrak{h})$. This implies

$$\frac{1}{p} \log \sum_{i_0\in V} \operatorname{Tr}(\mathfrak{L}_u^p(\rho(i_0))) = \log\lambda_u + \frac{1}{p}\log \sum_{i_0\in V} \varphi_u(\rho(i_0)) + O((\delta+\varepsilon)^p) \tag{5.10}$$

for u in the above real neighbourhood of the origin. This and Lemma 5.3 implies that the identity

$$\lim_{p\to\infty} \frac{1}{p} \log \mathbb{E}(\exp\langle u, X_p - X_0\rangle) = \log\lambda_u \tag{5.11}$$

holds for u in a neighbourhood of the origin. In addition, by equation (5.10) and Corollary 5.9,

$$\lim_{p\to\infty} \frac{1}{p}\big(\nabla \log \mathbb{E}(\exp\langle u, X_p - X_0\rangle) - pm\big) = 0 \quad \lim_{p\to\infty} \frac{1}{p} \nabla^2 \log \mathbb{E}(\exp\langle u, X_p - X_0\rangle) = C.$$

By an application of the multivariate version of Bryc's theorem (see Appendix A.4 in [14]), we deduce that

$$\frac{X_p - X_0 - p\,m}{\sqrt{p}} \underset{p\to\infty}{\longrightarrow} \mathcal{N}(0, C)$$

and this proves our statement in the case where $\mathfrak{L}$ is irreducible aperiodic.

We now consider the case where $\mathfrak{L}$ is irreducible with period d. Let $p_0, \dots, p_{d-1}$ be a cyclic resolution of the identity; then, writing $p = qd + r$ we have for any $i_0 \in V$

$$\begin{aligned}\operatorname{Tr}\big(\mathfrak{L}_u^p(\rho(i_0))\big) &= \sum_{j=0}^{d-1} \operatorname{Tr}\Big(p_j\, \mathfrak{L}_u^{qd+r}(\rho(i_0))\, p_j\Big) \\ &= \sum_{j=0}^{d-1} \operatorname{Tr}\Big(\mathfrak{L}_u^{qd}(p_j\, \mathfrak{L}_u^r(\rho(i_0))\, p_j)\Big)\end{aligned}$$

by a straightforward extension of (4.1) to $\mathfrak{L}_u$. By Proposition 3.10, for any j, r and the previous discussion, one has

$$\lim_{q\to\infty} \frac{1}{qd} \log \operatorname{Tr}\big(\mathfrak{L}_u^{qd}(p_j\, \mathfrak{L}_u^r(\rho(i_0))\, p_j)\big) = \log\lambda_u$$

and one can extend all terms in this identity so that it holds in a complex neighbourhood of the origin. This finishes the proof of our statement. □

Remark 5.13 Again this result holds replacing X_p by Q_p.

Remark 5.14 The reader might wonder why we need to go through the trouble of considering relations (5.9) and (5.10) to derive the extension of (5.11) to complex u. This is because there is no determination of the complex logarithm that allows to consider $\log \mathbb{E}(\exp\langle u, X_p - X_0\rangle)$ for complex u and arbitrarily large p. This forces us to start by transforming $\frac{1}{p} \log \mathbb{E}(\exp\langle u, X_p - X_0\rangle)$.

Remark 5.15 The formulas for the mean and variance are the same as in [1] when $V = \mathbb{Z}^d$ and $S = \{\pm v_i,\ i = 1, \dots, d\}$ ($v_1, \dots, v_d$ is the canonical basis of $\mathbb{R}^d$). This can be observed from the fact that, if Y_u is the unique (up to a constant multiple of Id) solution of equation

$$(\mathrm{Id} - \mathfrak{L}^*)(Y_u) = \sum_{s\in S} \langle u, s\rangle L_s^* L_s - \langle u, m\rangle \,\mathrm{Id},$$

(note that this Y_u is the L_l of [1]) then

$$\mathrm{Tr}\big(\mathfrak{L}'_u(\eta_u)\big) = \mathrm{Tr}\big(\mathfrak{L}'_u(\rho^{\mathrm{inv}})Y_u\big) - \mathrm{Tr}\big(\mathfrak{L}'_u(\rho^{\mathrm{inv}})\big)\,\mathrm{Tr}\big(\rho^{\mathrm{inv}} Y_u\big)$$

and denoting $Y_i = Y_{v_i}$ we have

$$\begin{aligned}\langle u, Cu\rangle = \sum_{i,j=1}^{d} u_i u_j \Big(&\mathbb{1}_{i=j}\big(\mathrm{Tr}(L_{+i}\rho^{\mathrm{inv}}L_{+i}^*) + \mathrm{Tr}(L_{-i}\rho^{\mathrm{inv}}L_{-i}^*)\big)\\ &+ 2\mathrm{Tr}(L_{+i}\rho^{\mathrm{inv}}L_{+i}^*\,Y_j) - 2\mathrm{Tr}(L_{-i}\rho^{\mathrm{inv}}L_{-i}^*\,Y_j)\\ &- 2m_i\mathrm{Tr}(\rho^{\mathrm{inv}}\,Y_j) - m_i m_j\Big)\end{aligned}$$

which leads to the formula for C given in [1]:

$$\begin{aligned}C_{i,j} = \ &\mathbb{1}_{i=j}\big(\mathrm{Tr}(L_{+i}\rho^{\mathrm{inv}}L_{+i}^*) + \mathrm{Tr}(L_{-i}\rho^{\mathrm{inv}}L_{-i}^*)\big)\\ &+ \big(\mathrm{Tr}(L_{+i}\rho^{\mathrm{inv}}L_{+i}^*\,Y_j) + \mathrm{Tr}(L_{+j}\rho^{\mathrm{inv}}L_{+j}^*\,Y_i)\big)\\ &\ \ \big(\mathrm{Tr}(L_{-i}\rho^{\mathrm{inv}}L_{-i}^*\,Y_j) + \mathrm{Tr}(L_{-j}\rho^{\mathrm{inv}}L_{-j}^*\,Y_i)\big)\\ &- \big(m_i\mathrm{Tr}(\rho^{\mathrm{inv}}\,Y_j) + m_j\mathrm{Tr}(\rho^{\mathrm{inv}}\,Y_i)\big) - m_i m_j.\end{aligned}$$

Generalizations of Theorems 5.4 and 5.12 We finish with a discussion of possible generalizations of Theorems 5.4 and 5.12 beyond the case of irreducible $\mathfrak{L}$. To this aim, we introduce the following subspaces of $\mathfrak{h}$:

$$\mathcal{D} = \Big\{\phi \in \mathfrak{h} \mid \langle \phi, \mathfrak{L}^p(\rho)\,\phi\rangle \underset{p\to\infty}{\longrightarrow} 0 \text{ for any state } \rho\Big\} \quad \text{and} \quad \mathcal{R} = \mathcal{D}^{\perp}. \tag{5.12}$$

Alternatively, $\mathcal{R}$ can be defined as the supremum of the supports of $\mathfrak{L}$-invariant states, and $\mathcal{D}$ as $\mathcal{R}^{\perp}$. Note in particular that $\dim \mathcal{R} \geq 1$ and $\mathcal{R}$ is invariant by all operators L_s, $s \in S$. These subspaces are the Baumgartner–Narnhofer decomposition of $\mathfrak{h}$ associated with $\mathfrak{L}$ (see [4] or [5]). Note that, in [5], we only considered the subspaces $\mathcal{D}_{\mathfrak{M}}$ and $\mathcal{R}_{\mathfrak{M}}$ of $\mathcal{H}$ associated with $\mathfrak{M}$. In the present situation, the decomposition for $\mathfrak{M}$ plays no role and $\mathcal{R}_{\mathfrak{M}}$ is equal to $\{0\}$.

The following result will replace the Perron-Frobenius theorem when $\mathfrak{L}$ is not irreducible. The proof can be easily adapted from Proposition 3.4, Remark 3.6 and Lemma 5.3.

Proposition 5.16 *The following properties are equivalent:*

1. *the auxiliary map $\mathfrak{L}$ has a unique invariant state ρ^{inv},*
2. *the restriction $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R})}$ of $\mathfrak{L}$ to $\mathcal{I}_1(\mathcal{R})$ is irreducible,*
3. *the value 1 is an eigenvalue of $\mathfrak{L}$ with algebraic multiplicity one.*

If, in addition, $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R})}$ is aperiodic, then 1 is the only eigenvalue of modulus one, and for any state ρ, one has $\mathfrak{L}^p(\rho) \underset{p\to\infty}{\longrightarrow} \rho^{\mathrm{inv}}$.

This leads to an extension of Theorem 5.12 to the cases

- where $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R})}$ is irreducible (even if $\mathcal{R} \neq \mathfrak{h}$); by Proposition 5.16, this is equivalent to $\mathfrak{L}$ having a unique invariant state;
- when $\mathcal{R} = \mathfrak{h}$.

With these two extensions, our central limit theorem has the same generality as the one given in [1]: the first case is Theorem 5.2 of that reference, the second case is treated in [1, Sect. 7]. These extensions are proven observing that:

- by Proposition 5.16, the proof of Theorem 5.12 can immediately be extended to the situation where $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R})}$ is irreducible aperiodic, and from there to the situation where $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R})}$ is irreducible periodic;
- when $\mathcal{R} = \mathfrak{h}$, it admits a decomposition $\mathcal{R} = \oplus_k \mathcal{R}_k$ (see [5]), each $\mathcal{I}_1(\mathcal{R}_k)$ is stable by $\mathfrak{L}$, the restrictions $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R}_k)}$ are irreducible, and the non-diagonal blocks do not appear in a probability like (2.6).

We have seen in [5] that one can always decompose $\mathfrak{h}$ into $\mathfrak{h} = \mathcal{D} \oplus \bigoplus_{k\in K} \mathcal{R}_k$ with each $\mathcal{R}_k$ as discussed above. However, in the general case, we do not have a clear statement of Theorems 5.4 and 5.12 because if $\mathcal{D}$ is non-trivial and card $K \geq 2$, it is difficult to control how the mass of ρ_0 will flow from $\mathcal{D}$ into the different components $\mathcal{R}_k$.

Last, remark that the proof of Theorem 5.4 relies on the fact that $\mathfrak{L}_u$ is irreducible. This holds if $\mathfrak{L}$ is irreducible; the converse, however, is not true, and the two spaces $\mathcal{R}$ associated with $\mathfrak{L}$ and $\mathfrak{L}_u$ may be different. The proof of Theorem 5.4 can be extended to derive a lower large deviation bound in the case when $\mathcal{R} = \mathfrak{h}$ using the idea described above, but when $\mathfrak{L}$ is not irreducible, the quantity λ_u may not be analytic, in which case we *a priori* obtain only the upper large deviation bound, see Example 7.3.

6 Open Quantum Random Walks with Lattice $\mathbb{Z}^d$ and Internal Space $\mathbb{C}^2$

The goal of this section is to illustrate our various concepts, and give explicit formulas in the case where $V = \mathbb{Z}^d$ and $\mathfrak{h} = \mathbb{C}^2$. We start with a study of the operator $\mathfrak{L}$, a characterization of its (ir)reducibility, and of the associated decompositions of the state space in this specific situation.

We begin in Proposition 6.1 with a classification of the possible situations depending on the dimension of $\mathcal{R}$ (as defined in 5.12) and its possible decompositions. Then, in Lemma 6.3 we characterize those situations in terms of the form of the operators L_s. Later on, we also consider the period. To avoid discussing trivial cases, we will make a second assumption:

Assumption H2 the operators L_s are not all proportional to the identity.

This is equivalent to saying that we assume $\mathfrak{L} \neq \mathrm{Id}$.

Proposition 6.1 *Consider the operators L_s, $s \in S$, defining the open quantum random walk $\mathfrak{M}$, and suppose that assumptions* H1 *and* H2 *hold. Then we are in one of the following three situations.*

1. *If the L_s have no eigenvector in common, then $\mathfrak{L}$ is irreducible, there exists a unique $\mathfrak{L}$-invariant state which is faithful, and one has*
$$\mathcal{R} = \mathfrak{h} \qquad \mathcal{D} = \{0\}.$$
2. *If the L_s have only one (up to multiplication) eigenvector e_1 in common, then $\mathfrak{L}$ is not irreducible, the state $|e_1\rangle\langle e_1|$ (if $\|e_1\| = 1$) is the unique $\mathfrak{L}$-invariant state, and for any nonzero vector $e_2 \perp e_1$, one has*
$$\mathcal{R} = \mathbb{C}\, e_1 \qquad \mathcal{D} = \mathbb{C}\, e_2.$$

3. *If the L_s have two linearly independent eigenvectors e_1 and e_2 in common, any invariant state is of the form $\rho^{\text{inv}} = t\,|e_1\rangle\langle e_1| + (1-t)|e_2\rangle\langle e_2|$ for $t \in [0,1]$, and one has*

$$\mathcal{R} = \mathfrak{h} = \mathbb{C}\, e_1 \oplus \mathbb{C}\, e_2 \qquad \mathcal{D} = \{0\}.$$

Proof We recall that, by Definition 3.1, the map $\mathfrak{L}$ is irreducible if and only if the L_s do not have a common, nontrivial, invariant subspace. If $\mathfrak{h} = \mathbb{C}^2$ then this is equivalent to saying that the L_s do not have a common eigenvector.

Now assume that $\mathfrak{L}$ is not irreducible, so that the L_s have a common norm one eigenvector e_1, with $L_s\, e_1 = \alpha_s\, e_1$ for all s. Then $|e_1\rangle\langle e_1|$ is an invariant state. Complete (e_1) to an orthonormal basis (e_1, e_2). Then, if ρ is an invariant state, $\rho = \sum_{i,j=1,2} \rho_{i,j}\, |e_i\rangle\langle e_j|$, and

$$\mathfrak{L}(\rho) = \sum_{i,j=1,2} \sum_{s\in S} \rho_{i,j}\, |L_s e_i\rangle\langle L_s e_j|.$$

Then

$$\rho_{2,2} = \langle e_2, \rho\, e_2\rangle = \sum_{s\in S} \rho_{2,2}\, |\langle e_2, L_s e_2\rangle|^2$$

so that either $\rho_{2,2} = 0$ or $\sum_{s\in S} |\langle e_2, L_s e_2\rangle|^2 = 1$; but, since $\sum_{s\in S} \|L_s e_2\|^2 = 1$, this is possible only if e_2 is an eigenvector of all L_s, $s \in S$.

Now, $\rho \geq 0$ and $\rho_{2,2} = 0$ impose $\rho_{1,2} = \rho_{2,1} = 0$. Therefore, in situation 2, $|e_1\rangle\langle e_1|$ is the only invariant state. In situation 3, observe that if there existed an invariant state with $\rho_{1,2} = \overline{\rho_{2,1}} \neq 0$, then any state would be invariant and $\mathfrak{L}$ would be the identity operator, a case we excluded. □

Remark 6.2 In situations 2 and 3 we recover the fact, proven in [5] (and originally in [4]) that, if $|e_1\rangle\langle e_1|$ is an invariant state and $e_2 \neq 0$ is in $e_1^\perp \cap \mathcal{R}$ then $|e_2\rangle\langle e_2|$ is an invariant state. The above proposition gives an explicit Baumgartner–Narnhofer decomposition of $\mathfrak{h}$ (see [4] or [5, Sects. 6 and 7]). In the case where $\mathfrak{h} = \mathbb{C}^2$, it turns out that $\mathcal{R}$ can always be written in a unique way as $\mathcal{R} = \bigoplus \mathcal{R}_k$ with $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R}_k)}$ irreducible (except for the trivial case when $\mathfrak{L}$ is the identity map). This is not true in general and is a peculiarity related to the low dimension of $\mathfrak{h}$.

Next we study the explicit form of the operators L_s in each of the situations described by Proposition 6.1. We will use the standard notation that, for two families of scalars $(\alpha_s)_{s\in S}$ and $(\beta_s)_{s\in S}$, $\|\alpha\|^2$ is $\sum_{s\in S} |\alpha_s|^2$ and $\langle \alpha, \beta\rangle$ is $\sum_{s\in S} \overline{\alpha_s}\beta_s$.

Lemma 6.3 *With the assumptions and notations of Proposition 6.1:*

- *We are in situation 2 if and only if there exists an orthonormal basis of $\mathfrak{h} = \mathbb{C}^2$ in which*

$$L_s = \begin{pmatrix} \alpha_s & \gamma_s \\ 0 & \beta_s \end{pmatrix}$$

for every s with

$$\|\alpha\|^2 = \|\beta\|^2 + \|\gamma\|^2 = 1, \qquad \langle \alpha, \gamma\rangle = 0,$$
$$\sup_{s\in S} |\beta_s| > 0, \qquad \sup_{s\in S} |\gamma_s| > 0,$$

there exist $s \neq s'$ in S such that $(\alpha_s - \beta_s)\,\gamma_{s'} \neq (\alpha_{s'} - \beta_{s'})\,\gamma_s$.

- *We are in situation* 3 *if and only if there exists an orthonormal basis of* $\mathfrak{h} = \mathbb{C}^2$ *in which*

$$L_s = \begin{pmatrix} \alpha_s & 0 \\ 0 & \beta_s \end{pmatrix}$$

for every s, with

$$\|\alpha\|^2 = \|\beta\|^2 = 1,$$
$$\text{there exists } s \text{ in } S \text{ such that } \alpha_s \neq \beta_s.$$

Proof This is immediate by examination. □

Remark 6.4 In situation 2, let ρ be any state. One has

$$\langle e_2, \mathfrak{L}^p(\rho)\, e_2\rangle = \operatorname{Tr}\big(\rho\, \mathfrak{L}^{*\,p}(|e_2\rangle\langle e_2|)\big) = \|\beta\|^{2p}\, \langle e_2, \rho\, e_2\rangle \underset{p\to\infty}{\longrightarrow} 0$$

by the observation that $\|\beta\|^2 < 1$. We recover the fact that $\mathcal{D} = \mathbb{C}\, e_2$.

We now turn to the study of periodicity for the operator $\mathfrak{L}$. We start with a simple remark:

Remark 6.5 Whenever the operators L_s have a common eigenvector e, then the restriction of $\mathfrak{L}$ to $\mathcal{I}_1(\mathbb{C}e)$ is aperiodic. In particular, if $\mathfrak{L}$ is not irreducible but has a unique invariant state, then by Proposition 6.1, $\mathcal{R}$ is one-dimensional and $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R})}$ must be aperiodic.

In greater generality, because $\dim \mathfrak{h} = 2$, by Remark 4.8, any irreducible $\mathfrak{L}$ has period either one or two. The following lemma characterizes those L_s defining an operator $\mathfrak{L}$ with period 2:

Lemma 6.6 *The map* $\mathfrak{L}$ *is irreducible periodic if and only if there exists a basis of* $\mathfrak{h}$ *for which every operator* L_s *is of the form* $\begin{pmatrix} 0 & \gamma_s \\ \nu_s & 0 \end{pmatrix}$. *In that case, for any* $s \neq s'$, *one has* $\gamma_s\, \nu_{s'} \neq \gamma_{s'}\, \nu_s$ *and* $\|\gamma\|^2 = \|\nu\|^2 = 1$, *and the unique invariant state of* $\mathfrak{L}$ *is* $\frac{1}{2}\,\mathrm{Id}$.

Proof If the period of $\mathfrak{L}$ is two, then the cyclic resolution of identity must be of the form $|e_1\rangle\langle e_1|$, $|e_2\rangle\langle e_2|$ and the cyclicity imposes the relations

$$L_s e_1 \in \mathbb{C}\, e_2, \quad L_s e_2 \in \mathbb{C}\, e_1 \quad \text{for any } s \in S.$$

This gives the form of the L_s. The condition $\sum_s |\gamma_s|^2 = \sum_s |\nu_s|^2 = 1$ simply follows by the trace preservation property. Now observe that the eigenvalues of L_s are solutions of $\lambda_s^2 = \gamma_s \nu_s$. Fix one solution λ_s, the other being $-\lambda_s$. Then a vector ${}^t(x, y)$ is an eigenvector if and only if $\gamma_s y = \pm\lambda_s x$. Therefore, two operators L_s and $L_{s'}$ have an eigenvector in common if and only if $\nu_s\, \lambda_{s'} = \pm\nu_{s'}\, \lambda_s$. This is easily seen to be equivalent to $\gamma_s\, \nu_{s'} = \gamma_{s'}\, \nu_s$. Last, one easily sees that the equation

$$\sum_{s\in S} L_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} L_s^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is equivalent to $a = d$, $b = \langle \nu, \gamma\rangle\, c$ and $c = \langle \gamma, \nu\rangle\, b$. Moreover, $|\langle \gamma, \nu\rangle| = 1$ would imply that the vectors $(\gamma_s)_{s\in S}$ and $(\nu_s)_{s\in S}$ are proportional, which is forbidden by irreducibility. Therefore $a = d$ and $b = c = 0$. □

The following theorem is a central limit theorem for all open quantum random walks satisfying H1 and H2. It gives more explicit expressions for the parameters of the limiting Gaussian, except when $\mathfrak{L}$ is irreducible aperiodic, in which case the parameters of the Gaussian are given in Theorem 5.12.

Theorem 6.7 *Assume an open quantum random walk with* $V = \mathbb{Z}^d$ *and* $\mathfrak{h} = \mathbb{C}^2$ *satisfies assumptions* H1, H2*. Then there exist* $m \in \mathbb{C}^d$ *and* C *a* $d \times d$ *positive semi-definite matrix such that we have the convergence in law*

$$\frac{X_p - p\,m}{\sqrt{p}} \underset{p\to\infty}{\longrightarrow} \mathcal{N}(0, C).$$

Following the notation of Lemmas 6.3 and 6.6 we have:

- *In situation 1, if* $\mathfrak{L}$ *is periodic, consider two random variables* A *and* B *with* $\mathbb{P}(A = s) = |\nu_s|^2$ *and* $\mathbb{P}(B = s) = |\gamma_s|^2$*. Then*

$$m = \frac{1}{2}(\mathbb{E}(A) + \mathbb{E}(B)) \qquad C = \frac{1}{2}(\mathrm{var}(A) + \mathrm{var}(B)).$$

- *In situation 2, consider a classical random variable* A *with* $\mathbb{P}(A = s) = |\alpha_s|^2$*. Then*

$$m = \mathbb{E}(A) \qquad C = \mathrm{var}(A).$$

- *In situation 3, consider two classical random variables* A *and* B *with* $\mathbb{P}(A = s) = |\alpha_s|^2$ *and* $\mathbb{P}(B = s) = |\beta_s|^2$*, and denote* $p = \sum_{i\in V} \langle e_1, \rho(i)\, e_1\rangle$*, where* ρ *is the initial state. Then*

$$m = p\,\mathbb{E}(A) + (1 - p)\,\mathbb{E}(B) \qquad C = p\,\mathrm{var}(A) + (1 - p)\,\mathrm{var}(B).$$

Proof If $\mathfrak{L}$ is irreducible periodic, for any $\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}$, we have

$$L_s \sigma L_s^* = \begin{pmatrix}\sigma_{22}|\gamma_s|^2 & \sigma_{21}\gamma_s\bar{\nu}_s\\ \sigma_{12}\bar{\gamma}_s\nu_s & \sigma_{11}|\nu_s|^2\end{pmatrix}.$$

By direct examination of the equation $\mathfrak{L}_u(\sigma) = \lambda_u\,\sigma$ we obtain

$$\lambda_u = \sqrt{\mathbb{E}(\exp\langle u, A\rangle)}\,\sqrt{\mathbb{E}(\exp\langle u, B\rangle)}\,. \tag{6.1}$$

We immediately deduce

$$\lambda'_u = \left\langle u\,\frac{1}{2}(\mathbb{E}(A) + \mathbb{E}(B))\right\rangle \qquad \lambda''_u - {\lambda'_u}^2 = \left\langle u\,\frac{1}{2}(\mathrm{var}\,A + \mathrm{var}\,B)\,u\right\rangle.$$

In situation 2, we can use the extension discussed at the end of Sect. 5 with $P_{\mathcal{R}} = |e_1\rangle\langle e_1|$, and apply the formulas of Theorem 5.12 with $\mathfrak{L}$ replaced by $\mathfrak{L}_{\mathcal{I}_1(\mathbb{C}e_1)}$. We see easily that the largest eigenvalue of $\mathfrak{L}_u$ is

$$\lambda_u = \max\left(\sum_{s\in S} \mathrm{e}^{\langle u,s\rangle}|\alpha_s|^2, \sum_{s\in S} \mathrm{e}^{\langle u,s\rangle}|\beta_s|^2\right) \tag{6.2}$$

and in a neighbourhood of zero, the first term is the largest, so that

$$\lambda'_u = \sum_{s\in S}\langle u, s\rangle\,|\alpha_s|^2 \quad \text{and} \quad \lambda''_u = \sum_{s\in S}\langle u, s\rangle^2\,|\alpha_s|^2.$$

In situation 3, we again use the extension discussed at the end of Sect. 5 with $\mathcal{R}_1 = \mathbb{C}e_1$ and $\mathcal{R}_2 = \mathbb{C}e_2$. The limit parameters for each corresponding restriction are computed in the previous point and correspond to those for the random variables A and B. Since for any initial state ρ, a probability $\mathbb{P}(X_0 = i_0, \ldots, X_n = i_n)$ equals

$$\langle e_1, \rho(i_0)\, e_1\rangle \prod_{k=1}^{n} |\alpha_{i_k - i_{k-1}}|^2 + \langle e_2, \rho(i_0)\, e_2\rangle \prod_{k=1}^{n} |\beta_{i_k - i_{k-1}}|^2$$

and we recover the parameters given in the statement above. □

Remark 6.8 The irreducible periodic case described above can be understood in terms of a classical random walk, in a similar way to situation 3. Indeed, call a site i in V odd or even depending on the parity of its distance to the origin. Then exchanging the order of the basis vectors e_1 and e_2 at odd sites only is equivalent to considering a non-homogeneous OQRW with

$$L_{i,i+s} = \begin{pmatrix} \nu_s & 0 \\ 0 & \gamma_s \end{pmatrix} \quad \text{if } i \text{ is even,} \qquad L_{i,i+s} = \begin{pmatrix} \gamma_s & 0 \\ 0 & \nu_s \end{pmatrix} \quad \text{if } i \text{ is odd}$$

(strictly speaking, such OQRWs do not enter into the framework of this article, but in the general case studied in [5]). Then, we define $(A_p)_{p\in\mathbb{N}}$ and $(B_p)_{p\in\mathbb{N}}$ to be two i.i.d. sequences with same law as A, B respectively, and, if for example $X_0 = 0$ is even, we define a random variable π to take the values 1 and 2 with probabilities $p = \langle e_1, \rho(i_0)\, e_1\rangle$, $1 - p$ respectively. Then, conditioned on $\pi = 1$, the variable $X_p - X_0$ has the same law as $A_1 + B_2 + A_3 + \ldots$ (where the sum stops at step p). This explains the formulas given in Theorem 6.7 for situation 1, with $\mathfrak{L}$ periodic, as well as the next proposition.

For the case of irreducible, periodic $\mathfrak{L}$ we also have a simpler explicit formula for the rate function of large deviations:

Lemma 6.9 *Assume an open quantum random walk with $V = \mathbb{Z}^d$ and $\mathfrak{h} = \mathbb{C}^2$ satisfies assumptions* H1, H2 *and is irreducible periodic. Then, with the same notation as in Theorem 6.7, the position process $(X_p - X_0)_p$ satisfies a full large deviation principle, with rate function*

$$c(u) = \frac{1}{2}(\log \mathbb{E}(\exp\langle u, A\rangle)) + \log \mathbb{E}(\exp\langle u, B\rangle))).$$

Proof This follows immediately from Theorem 5.4 and Eq. (6.1) giving λ_u. □

Remark 6.10 In situation 2 of Lemma 6.3, one sees that the largest eigenvalue λ_u is given by (6.2). For u in a neighbourhood of zero, one has $\|\alpha_u\| > \|\beta_u\|$, but, if there exists u such that $\|\alpha_u\| = \|\beta_u\|$, then λ_u may not be differentiable and the large deviations principle may break down: see Example 7.3. A similar phenomenon can also appear in situation 3.

In this section we have characterized the properties of $\mathfrak{L}$ in terms of the operators $(L_s)_{s\in S}$. The connection between $(L_s)_{s\in S}$ and $\mathfrak{M}$ is more complex; to illustrate this, and to give a positive result in this direction we finish this section with the following example:

Example 6.11 We consider the case $d = 1$, $\mathfrak{h} = \mathbb{C}^2$, $S = \{-1, +1\}$ and denote $L_- = L_{-1}$, $L_+ = L_{+1}$. We state the next two propositions without proofs, as these are lengthy. The extension of these statements to finite homogeneous open quantum random walks, as well as the proofs, will be given in a future note.

Proposition 6.12 Irreducibility. *Define*

$$W \stackrel{\text{def}}{=} \{\textit{common eigenvectors of } L_+L_- \text{ and } L_-L_+\}.$$

The homogeneous OQRW on $\mathbb{Z}$ is reducible if and only if one of the following facts holds

- *W contains an eigenvector of L_- or L_+*
- *$W = \mathbb{C}e_0 \cup \mathbb{C}e_1 \setminus \{0\}$, for some linearly independent vectors e_0 and e_1 satisfying $L_-e_0, L_+e_0 \in \mathbb{C}e_1$ and $L_-e_1, L_+e_1 \in \mathbb{C}e_0$.*

Proposition 6.13 Period. *Suppose that the open quantum random walk $\mathfrak{M}$ is irreducible. Its period can only be* 2 *or* 4. *It is* 4 *if and only if there exists an orthonormal basis of* $\mathbb{C}^2$ *such that the representation of the transition matrices in that basis is*

$$L_\varepsilon = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad L_{-\varepsilon} = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}$$

for some $\varepsilon \in \{+, -\}$, *where* $a, b, c, d \in \mathbb{C} \setminus \{0\}$ *are such that* $|a|^2 + |d|^2 = |b|^2 + |c|^2 = 1$.

7 Examples

All the examples of this section will live in the context of Example 6.11, that is, we consider OQRWs with $V = \mathbb{Z}$, $S = \{-1, +1\}$, $\mathfrak{h} = \mathbb{C}^2$ with canonical basis e_1, e_2. Every operator will be written in matrix form with respect to this basis, if not specified otherwise. The different models will be completely determined once the transition operators $L_- = L_{-1}$, $L_+ = L_{+1}$ are defined.

Example 7.1 We consider the standard example from [2], which is treated in [1, Sect. 5.3]. This OQRW is defined by the transition operators

$$L_+ = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad L_- = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The only eigenvector of L_+ is e_1, the only eigenvector of L_- is e_2, so that we are in situation 1 of Proposition 6.1 and $\mathfrak{L}$ is irreducible. Again L_+^2 and L_-^2 have no eigenvector in common, so by Lemma 6.6, we conclude that $\mathfrak{L}$ is aperiodic (and therefore regular, by Lemma 4.11). We observe that $\rho^{\text{inv}} = \frac{1}{2}\text{Id}$ is the invariant state of $\mathfrak{L}$. We compute the quantities m and $C \in \mathbb{R}_+$ from Theorem 5.12:

$$m = \operatorname{Tr}(L_+ L_+^*) - \operatorname{Tr}(L_- L_-^*) = 0.$$

To compute C we need to find the solution η of

$$(\text{Id} - \mathfrak{L})(\eta) = \frac{1}{6} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

satisfying $\operatorname{Tr} \eta = 0$. We find $\eta = \frac{1}{12} \begin{pmatrix} 5 & 2 \\ 2 & -5 \end{pmatrix}$, and we have

$$C = \operatorname{Tr}\big(L_+ \rho^{\text{inv}} L_+^* + L_- \rho^{\text{inv}} L_-^*\big) + 2 \operatorname{Tr}\big(L_+ \eta L_+^* - L_- \eta L_-^*\big) = \frac{8}{9}.$$

By Theorem 5.12, we have the convergence in law

$$\frac{X_p - X_0}{\sqrt{p}} \underset{p \to \infty}{\longrightarrow} \mathcal{N}\left(0, \frac{8}{9}\right).$$

By Theorem 5.4, the process $\left(\frac{X_p - X_0}{p}\right)_p$ satisfies a large deviation property with good rate function equal to the Legendre transform I of $u \mapsto \log \lambda_u$, where λ_u is the largest eigenvalue of $\mathfrak{L}_u$. This map $\mathfrak{L}_u$, written in the canonical basis of the set of two by two matrices, has basis

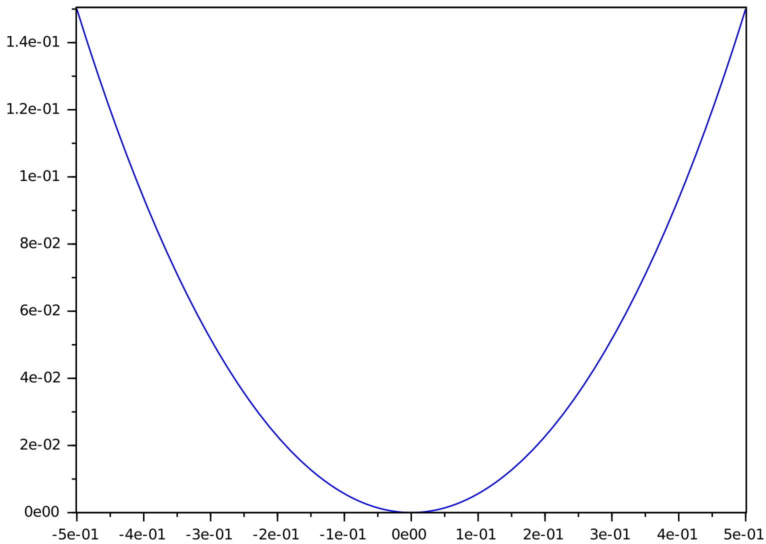

Fig. 1 Rate function for $\left(\frac{X_p - X_0}{p}\right)_p$ in Example 7.1

$$\frac{1}{3}\begin{pmatrix} e^u + e^{-u} & e^u & e^u & e^u \\ -e^{-u} & e^u + e^{-u} & 0 & e^u \\ -e^{-u} & 0 & e^u + e^{-u} & e^u \\ e^{-u} & -e^{-u} & -e^{-u} & e^u + e^{-u} \end{pmatrix}$$

and by a tedious computation, one shows that λ_u equals

$$\frac{1}{3}\Big(e^u + e^{-u} + (e^u + e^{-u} + \sqrt{e^{2u} + e^{-2u} + 3})^{1/3} - (e^u + e^{-u} + \sqrt{e^{2u} + e^{-2u} + 3})^{-1/3}\Big).$$

As expected from Lemma 5.3, this is a smooth and strictly convex function. Numerical computations prove that the rate function I has the form displayed in Fig. 1.

Example 7.2 We consider the OQRW with transition operators

$$L_+ = \begin{pmatrix} 0 & \sqrt{3}/2 \\ 1/\sqrt{2} & 0 \end{pmatrix} \qquad L_- = \begin{pmatrix} 0 & 1/2 \\ 1/\sqrt{2} & 0 \end{pmatrix}.$$

From Lemma 6.6, the map $\mathfrak{L}$ is irreducible and 2-periodic. Then, according to Theorem 6.7, defining A and B to be random variables with values in S satisfying

$$\mathbb{P}(A = +1) = \mathbb{P}(A = -1) = 1/2, \qquad \mathbb{P}(B = +1) = 1 - \mathbb{P}(B = -1) = 3/4,$$

with mean, variance, and cumulant generating function

$$m_A = 0, \quad C_A = 1, \quad c_A(u) = \log(e^u + e^{-u}) - \log 2,$$

$$m_B = 1/2, \quad C_B = 3/4, \quad c_B(u) = \log(3e^u + e^{-u}) - 2\log 2;$$

then, with the notations of Theorem 6.7,

$$m = (m_A + m_B)/2 = 1/4, \qquad C = (C_A + C_B)/2 = 7/8,$$

and one has the convergence in law

$$\frac{X_p - p/4}{\sqrt{p}} \underset{p\to\infty}{\longrightarrow} \mathcal{N}\left(0, \frac{7}{8}\right).$$

In addition, the process $\left(\frac{X_p - X_0}{p}\right)_{p\in\mathbb{N}}$ satisfies a large deviation property with a good rate function I obtained as the Legendre transform of

$$c(u) = \frac{1}{2}(c_A(u) + c_B(u)) = \frac{1}{2}\big(\log(\mathrm{e}^u + \mathrm{e}^{-u}) + \log(3\mathrm{e}^u + \mathrm{e}^{-u})\big) - \frac{3}{2}\log 2.$$

Explicitly, one finds that $I(t) = +\infty$ for $t \notin]-1, 1[$ and, for $t \in]-1, +1[$,

$$I(t) = t\,u_t + \frac{3}{2}\log 2 - \frac{1}{2}\big(\log(\mathrm{e}^{u_t} + \mathrm{e}^{-u_t}) + \log(3\mathrm{e}^{u_t} + \mathrm{e}^{-u_t})\big),$$

where $u_t = \frac{1}{2}\log\frac{2t+\sqrt{t^2+3}}{3(1-t)}$. This rate function has the profile displayed in Fig. 2.

Example 7.3 Consider the OQRW defined by the transition operators

$$L_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \qquad L_- = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 \end{pmatrix}.$$

First observe that the map $\mathfrak{L}$ is not irreducible in this case, as we are in situation 2 of Proposition 6.1. A straightforward computation shows that the largest eigenvalue of $\mathfrak{L}_u$ is

$$\lambda_u = \sup\left(\frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}, \frac{3\,\mathrm{e}^u}{4}\right).$$

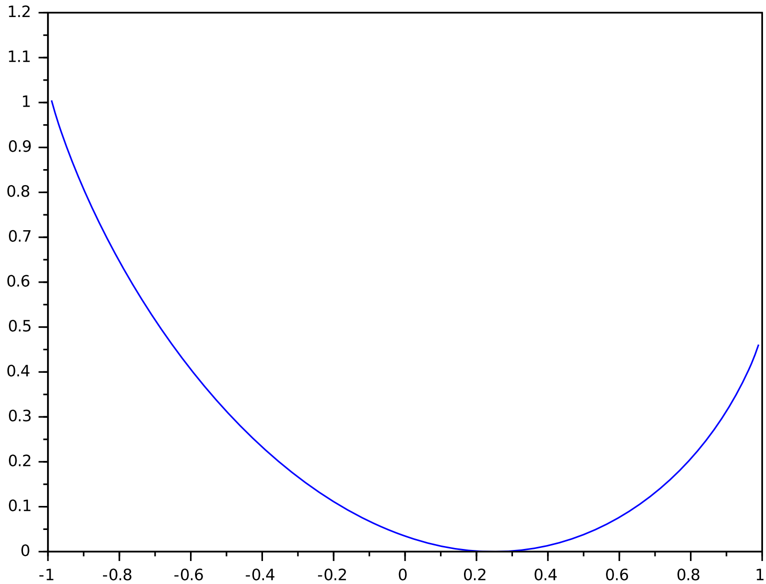

Fig. 2 Rate function for $\left(\frac{X_p - X_0}{p}\right)_p$ in Example 7.2

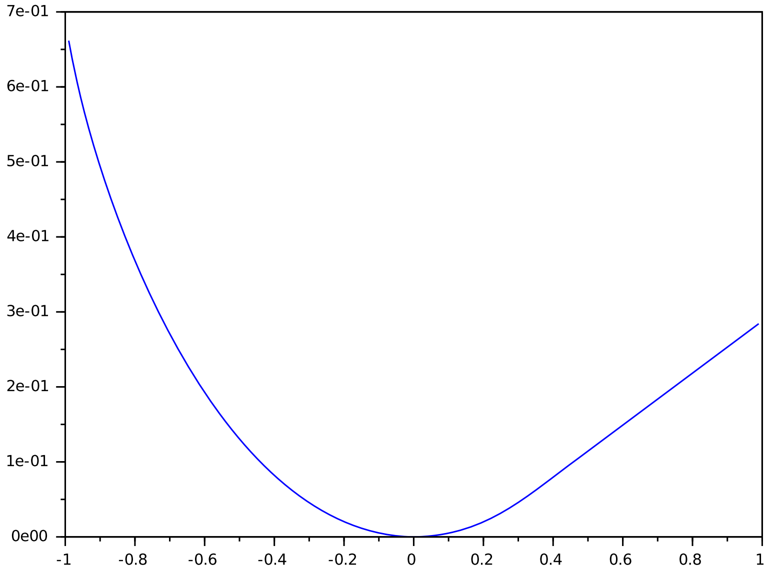

Fig. 3 Rate function for $\left(\frac{X_p - X_0}{p}\right)_p$ in Example 7.3

For u close to zero λ_u is $\frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}$ so that $\lambda'_u = 0$ and $\lambda''_u = 1$ for $u = 0$. We must therefore have

$$\frac{X_p - X_0}{\sqrt{p}} \underset{p \to \infty}{\longrightarrow} \mathcal{N}(0, 1).$$

Due to the generalizations discussed at the end of Sect. 5, we have $\mathcal{R} = \mathbb{C}e_1$, $\mathcal{D} = \mathbb{C}e_2$ and the central limit theorem holds: the behavior of the process $(X_p)_{p \in \mathbb{N}}$, associated with $\mathfrak{L}$, is the same as the one of the process $(\widetilde{X}_p)_{p \in \mathbb{N}}$ associated with the restriction $\mathfrak{L}_{|\mathcal{I}_1(\mathcal{R})}$.

As we commented previously, giving a large deviations result in this case is harder and we cannot use the general results we proved. The Gärtner-Ellis theorem could be applied by direct computation of the moment generating functions. In general, however, the rate function for the process $(X_p)_{p \in \mathbb{N}}$ will not coincide with the one for $(\widetilde{X}_p)_{p \in \mathbb{N}}$, since it will essentially depend on how much time the evolution spends in $\mathcal{D}$.

More precisely, for the transition matrices introduced above and taking the initial state $\rho = |e_2\rangle\langle e_2| \otimes |0\rangle\langle 0|$, we have, by relation (2.6),

$$P(X_p = p) = \mathrm{Tr}(|L_+^p e_2\rangle\langle L_+^p e_2|) = \left(\frac{3}{4}\right)^p + \left(\frac{1}{8}\right) 2^{1-p} \frac{\sqrt{3^p} - \sqrt{2^p}}{\sqrt{3} - \sqrt{2}}$$

and consequently

$$\lim_p \frac{1}{p} \log \mathbb{E}[e^{uX_p}] \geq \log\left(\frac{3}{4} e^u\right) \qquad \text{for all } u,$$

while $\lim_p \frac{1}{p} \log \mathbb{E}[e^{u\widetilde{X}_p}] = \log\left(\frac{e^u + e^{-u}}{2}\right)$, which for $u > \log 2$ is smaller than the bound $\log\left(\frac{3}{4} e^u\right)$.

This clarifies the fact that the large deviations will not depend only on $\mathfrak{L}_{|\mathcal{R}}$. Moreover, a second problem arises in this example, which is the lack of regularity of λ_u. Indeed, λ_u is the supremum of two quantities which coincide for $u_0 = \frac{1}{2}\log 2$, and $\log \lambda_u$ is not differentiable at u_0: the left derivative is equal to $\frac{1}{3}$ and the right derivative to 1. The restriction to $[-1, +1]$ of the Legendre transform of $\log \lambda_u$ is displayed in Fig. 3 (it is $+\infty$ outside of this interval) and we observe that it is not strictly convex.

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