J Stat Phys (2015) 160:29–59
DOI 10.1007/s10955-015-1240-y

Multiscale Conservation Laws Driven by Lévy Stable and Linnik Diffusions: Asymptotics, Shock Creation, Preservation and Dissolution

B. Gunaratnam[1] · **W. A. Woyczyński**[2]

Received: 3 November 2014 / Accepted: 12 March 2015 / Published online: 28 March 2015

Abstract Recent work has shown that the solutions of the fractal conservation laws driven by Lévy α-stable diffusions exhibit shocks for bounded, odd, and convex on the positive half-line, initial data when the parameter $\alpha < 1$. We study the analogous situation for the Lévy α-Linnik diffusions in which case the local behavior is strikingly different, although we are able to establish analytically that the large time behavior of the two types of conservation laws are similar. But the main new insights obtained via large-scale numerical experiments is that, for any $0 < \alpha \leq 2$, the conservation laws driven by α-Linnik diffusions display shocks that do not dissipate over time, while those for α-stable diffusion ($0 < \alpha \leq 1$) do. We formulate rigorous conjectures based on these numerical experiments.

Keywords Multiscale · Conservation laws · Lévy stable diffusions · Anomalous diffusion · Linnik diffusions · Shocks

1 Introduction

Over the past several decades, the subject of linear and nonlinear diffusion equations driven by non-local, pseudo-differential operators witnessed a considerable research activity well represented in both mathematical and physical literature, see, e.g., [4–8,18,20,30,33,39,40], with a wide range of applications to anomalous diffusion phenomena, mathematical finance

✉ W. A. Woyczyński
waw@case.edu
http://sites.google.com/a/case.edu/waw

B. Gunaratnam
b0guna01@louisville.edu

1 Department of Bioinformatics and Biostatistics, University of Louisville, Louisville, KY 40202, USA

2 Department of Mathematics, Applied Mathematics and Statistics, and Center for Stochastic and Chaotic Processes in Science and Technology, Case Western Reserve University, Cleveland, OH 44106, USA

and dislocation dynamics, see, .e.g., [10,29,31,36]. Moreover, it has evolved into a vital part of the theory of conservation laws, and more generally nonlocal nonlinear evolution equations [17,22,23].

It is known from recent work [1,12] that the solutions of fractal conservation laws driven by Lévy α-stable diffusions exhibit shocks for bounded, odd, and convex on $\mathbb{R}^+$ initial data when the parameter $\alpha < 1$. We study the analogous situation for the Lévy α-Linnik diffusions, in which case the shock behavior is strikingly different although, conceptually, they are not too far from α-stable diffusions. For nonlinear nonlocal evolution equations driven by α-Linnik diffusions no general rigorous mathematical results are available, but we are able to derive some asymptotic results based on perturbation of α-stable diffusion framework. But the main new insights came from our simulations and large-scale numerical schemes for α-Linnik conservation laws with respect to Riemann, piecewise linear, and smooth initial conditions. The main discovery here is that, despite their similar heavy tail behavior, the Linnik diffusions produce shocks that do not dissipate in time; such shock do dissipate in the case of α-stable diffusions. Thus we are able to formulate rigorous conjectures based on these numerical experiments.

The Lévy α-stable distributions and processes have been well studied in the mathematical and physical literature (under the name “anomalous diffusions” in the latter), see, e.g., [2,3,34,35,38]. The fundamental reason for the their importance is the general central limit theorem of probability theory: The rescaled sums $X_1 + \cdots + X_n$, of independent, identically distributed random variables have asymptotically, as $n \to \infty$, α-stable distributions, with the Gaussian case corresponding to the situation when the summands have the finite second moment (essentially). But, if the fixed range of summation n , is replaced by a randomly distributed integer N, the situation changes, and in the particular case of the geometric distribution of N, the normalized sum $X_1 + \cdots + X_N$ of the random number of independent random α-Linnik summands has the α-Linnik distribution. This random stability fact was the main motivation behind our taking up this work; the finite but random number of summands may provide, in certain situations, a better description of the real-world situations faced by scientists and economists. We note that Fokker–Planck-type equations driven by α-Linnik diffusion (called there the geometric-stable noise) were studied in [9].

The paper is organized as follows: In the preliminary Sect. 2, we give a brief overview and introduce the notation for the classical α-stable distribution and, less classical, α-Linnik distribution (both are special cases of the general Lévy infinitely divisible distributions), including properties of the density function and the tail probabilities behavior, and relationship between the two classes. We also show how the parameters of the α-Linnik distributions can be estimated. Additionally, we study the fractional lower order moments and the density of the Lévy measure for the α-Linnik distributions. Section 3 sets up formally the nonlinear nonlocal evolution equations driven by α-stable and α-Linnik diffusions and describes the basic asymptotics derived from the fact that, in some ways, α-Linnik diffusions can be viewed as perturbations of α-stable diffusions. Section 4 summarizes the known results about the shock creation for fractal Burgers equations, that is the conservation laws with a quadratic nonlinearity. Section 5 contains our numerical results on non-dissipative nature of shocks appearing in the Linnik context. Section 6 describes our conclusions, conjectures and proposed future work. A detailed description of our numerical scheme employed in Section 5 is provided in the Appendix.

2 Preliminaries: General Lévy, α-Stable, and α-Linnik Distributions and Processes

2.1 Lévy Processes and Infinitely Divisible Distributions

Let us begin by recalling that a stochastic process $X = \{X_t;\ t \geq 0\}$ is called a *Lévy process* (see [2,3,20,34,35,38] for the detailed expositions)[1] if :

(i) $X_0 = 0$, with probability 1,
(ii) It has independent increments, i.e., for any $n \in \mathbb{N}$, and any $0 < t_1 < t_2 < \cdots < t_n$, the random variables $X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent,
(iii) It has stationary increments, i.e., for any $s < t$, $X_t - X_s$ is equal in distribution to X_{t-s}
(iv) It is right continuous with left limits, with probability 1.

In this paper we will concentrate on the on-dimensional case, and it is easy to see that the characteristic functions (CF) $\phi(\xi, t) = \mathbb{E}\exp(i\xi X(t))$ of the 1-D distributions of the Lévy processes are *infinitely divisible* (ID), that is, for every $n \geq 1$, $t \geq 0$,

$$\phi(\xi, t) = [\phi(\xi, t/n)]^n, \quad \xi \in \mathbb{R}. \tag{2.1}$$

A more detailed description of the structure of characteristic functions of infinitely divisible distributions is given by the following *Lévy-Khinchine Representation Theorem*:

Theorem 2.1 *A random variable X has an infinitely divisible distribution if, and only if, there exist $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and a nonnegative measure Λ on $\mathbb{R}\backslash\{0\}$ satisfying the condition $\int_{\mathbb{R}}(1 \wedge |x|^2)\Lambda(dx) < \infty$, such that*

$$\mathbb{E}[e^{i\xi X}] = e^{-\psi(\xi)}, \tag{2.2}$$

where the characteristic exponent

$$\psi(\xi) = -i\mu\xi + \frac{(\sigma\xi)^2}{2} - \int_{\mathbb{R}\backslash\{0\}} \left(e^{i\xi x} - 1 - i\xi x\mathbf{I}_{|x|<1}\right)\Lambda(dx). \tag{2.3}$$

The triplet (μ, σ, Λ) is called the *characteristic triplet* of X, $\psi(u)$—the characteristic exponent of X, $\mu \in \mathbb{R}$— the drift coefficient, $\sigma > 0$—the Gaussian, or diffusion coefficient, and Λ— the Lévy measure of X. The Lévy measure describes the "intensity" of jumps of a certain height of a Lévy process in a time interval of length 1.

In view of the stationarity of increments, the characteristic function of the Lévy process $X(t)$ is of the form

$$\phi(\xi, t) = (\phi(\xi, 1))^t = e^{-t\psi(\xi)}, \tag{2.4}$$

where $\psi(u)$ is the characteristic exponent of X_1. For any infinitely divisible random variable X (that is, a random variable with an infinitely divisible distribution), we can construct a Lévy process $(X_t)_{t\geq 0}$ such that $X_1 \stackrel{d}{=} X$.

[1] In what follows, in view of the physical context of the discussions, we will also use the term Lévy diffusions, instead of Lévy processes.

2.2 α-Stable Distributions

The self-similar (symmetric) Lévy distributions form a class of the classical α-stable distribution (denoted $S(\alpha, c)$ in this paper) with the characteristic functions of the form

$$\phi_S(\xi; \alpha, c) = \exp(-|c\,\xi|^\alpha), \tag{2.5}$$

where the constant $0 < \alpha \leq 2$, is called the index of stability, and c is the scale parameter. In general, probability density functions (PDFs), $f_S(x; \alpha, c)$, of α-stable distributions do not have clean closed-form representations in terms of elementary functions although a lot of effort has been expanded on developing "explicit" expressions in terms of other known special functions. Apart from the obvious Gaussian case of $\alpha = 2$, we would like to mention the Cauchy case, $\alpha = 1$, when the PDF is of the form

$$f_S(x; 1, c) = \frac{c}{\pi\left(c^2 + x^2\right)}. \tag{2.6}$$

This particular case displays an interesting duality with the similarly parametrized 1-Linnik distribution discussed in Sect. 2.3.

The Lévy measure Λ_S of the α-stable distribution $S(\alpha, c)$ is absolutely continuous and its density $\lambda_S(x) = \Lambda_S(dx)/dx$ of the Lévy measure of the α-stable distribution

$$\lambda_S(x) = \frac{\Lambda_S(dx)}{dx} = \frac{C}{|x|^{\alpha+1}}, \tag{2.7}$$

with the easily determined constant $C = c^\alpha(2\int_0^\infty(\cos v - 1)v^{-(1+\alpha)}dv)^{-1}$.

2.3 α-Linnik Distributions and Their Basic Properties

In this subsection, we introduce the (symmetric) α-Linnik distributions and, since they are less well know in the mathematical physics community than the α-stable distributions (and the related anomalous diffusions), we also discuss their statistical, analytic and asymptotic properties. The name, although unorthodox, emphasizes analogies and contrasts between the properties of α-stable and α-Linnik distributions for the same parameter α; in most of the literature the latter are just called Linnik distributions.

The univariate Linnik distribution (denoted $L(\alpha, \gamma)$ in this paper) with index $\alpha \in (0, 2]$, and the scale parameter $\gamma > 0$, is defined by its characteristic function [28],

$$\phi_L(\xi; \alpha, \gamma) = \frac{1}{1 + |\gamma\xi|^\alpha}, \quad t \in \mathbb{R}, \quad \gamma > 0. \tag{2.8}$$

Let us denote its density function by $f_L(x; \alpha, \gamma)$. In the extreme case $\alpha = 2$ and $\gamma = 1$, the Linnik distribution corresponds to the standard Laplace (two-sided exponential) distribution with the density function,

$$f(x; 2, 1) = \frac{1}{2}e^{-|x|/2}; \quad x \in \mathbb{R}. \tag{2.9}$$

By inverting the characteristic function $\phi_L(x; \alpha, 1)$ one can immediately see [24,25] that $f_L(x; \alpha, 1) = f_L(-x; \alpha, 1)$, and

$$\begin{aligned} f_L(x; \alpha, 1) &= \frac{1}{\pi}\int_0^\infty \frac{1}{1 + t^\alpha}\cos(xu)du \\ &= \frac{\sin(\pi\alpha/2)}{\pi}\int_0^\infty \frac{t^\alpha e^{-u|x|}du}{1 + t^{2\alpha} + 2t^\alpha\cos(\pi\alpha/2)}. \end{aligned}$$

Note that, for $1 < \alpha \leq 2$, $L(0; \alpha, 1) < \infty$, but, for $0 < \alpha \leq 1$, the density has a singularity at the origin, $L(0; \alpha, 1) = \infty$. Also, the absolute first moment is finite only for $1 < \alpha \leq 2$, and the variance is well defined only in the case $\alpha = 2$. The distribution is unimodal, and [11] provides an elegant method of generating Linnik random variables, and contains a constructive proof that $(1 + |u|^{\alpha})^{-1}$ is a characteristic function (see Remark 2.2). A few sample PDFs are shown in Fig. 1.

For $\alpha \in (0, 2)$, the α-Linnik distributions, like the corresponding α-stable distributions, have 'fat tails' (no longer exponential). More precisely, they have the following asymptotic behavior at infinity:

Theorem 2.2 ([24]) *For any* $\alpha \in (0, 2)$,

$$f_L(x; \alpha, 1) \sim \frac{1}{\pi} \sum_{k=1}^{\infty} \left\{ (-1)^{k-1} \Gamma(1 + \alpha k) \sin\left(\frac{\pi \alpha k}{2}\right) \right\} |x|^{-1-\alpha k}, \quad as \quad x \to \infty, \tag{2.10}$$

or, equivalently,

$$f_L(x; \alpha, 1) = \frac{1}{\pi} \sum_{k=1}^{n} \left\{ (-1)^{k-1} \Gamma(1 + \alpha k) \sin\left(\frac{\pi \alpha k}{2}\right) \right\} |x|^{-1-\alpha k} + R_{n,\alpha}(x), \tag{2.11}$$

where

$$|R_{n,\alpha}(x)| \leq \frac{\Gamma\left(1 + \alpha(n + 1)\right)}{\pi \sin(\pi \alpha/2)} |x|^{-1-\alpha(n+1)}. \tag{2.12}$$

Consequently, for any $\alpha \in (0, 2)$, *the density* $f_L(x; \alpha, 1)$ *decreases at* ∞ *at the rate of the power function* $x^{-(1+\alpha)}$. *More precisely,*

$$f_L(x; \alpha, 1) \sim \frac{1}{\pi} \left\{ \Gamma(1 + \alpha) \sin\left(\frac{\pi \alpha}{2}\right) \right\} |x|^{-(1+\alpha)}, \quad as \quad x \to \infty. \tag{2.13}$$

The importance of α-stable distributions comes from their universality as approximants of the distributions of sums of a fixed but large numbers of independent, identically distributed random variables. Indeed, the fundamental central limit theorem states that if $X_1, X_2, \ldots, X_n$ are independent, identically distributed (i.i.d.) random variables then their sums $S_n = X_1 + X_2 + \cdots + X_n$ converge in distribution (after some rescaling) to an α-stable distribution for some $\alpha \in (0, 2]$. This fact is directly related to the "stability" property: if $X_1, X_2, \ldots, X_n$ are symmetric α-stable themselves then $Y = n^{-1/\alpha}(X_1 + X_2 + \cdots + X_n)$ has the same distribution as each of the $X_i' s$.

The α-Linnik distributions have a similar stability property, but under random geometric summation. More precisely [26], we have the following simple, but revealing, result:

Proposition 2.1 *If* $X, X_1, X_2, \ldots$ *are i.i.d. random variables and* N *is an independent of* $X_1, X_2, \ldots$ *random variable with the geometric distribution with mean* $1/p, 0 < p < 1$, *that is,* $\mathbb{P}(N = n) = p(1 - p)^{n-1}, n = 1, 2, \ldots,$ *then the following two statements are equivalent:*

(i) *X is α-stable with respect to geometric summation, i.e.,*

$$p^{1/\alpha} \sum_{i=1}^{N} X_i \stackrel{d}{=} X,$$

(ii) *X has the α-Linnik distribution.*

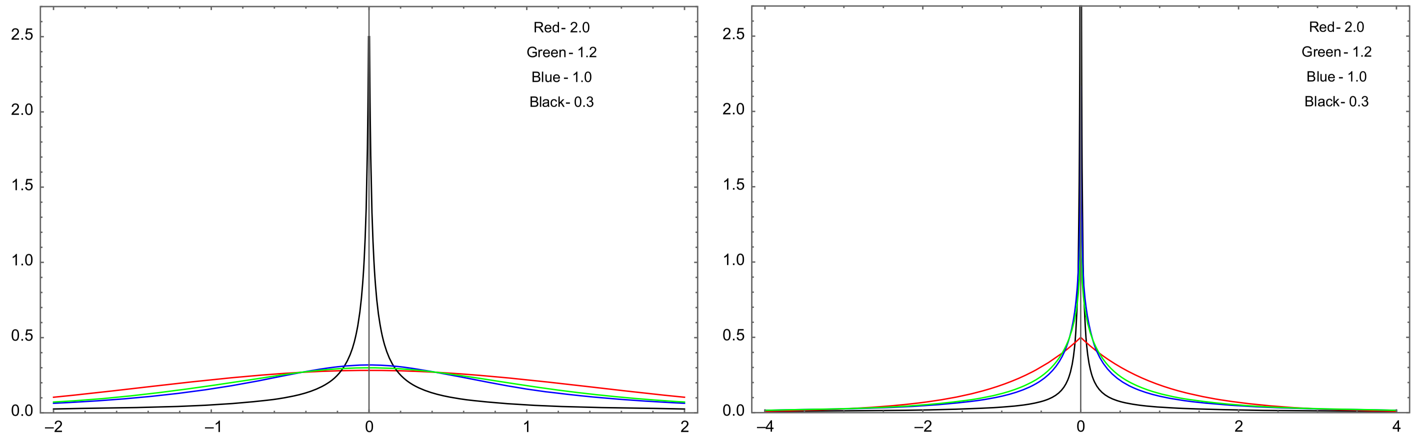


Fig. 1 PDFs of α-stable (*left*), and α-Linnik distributions (*right*) for $\alpha = 2$ (*red*), 1.25 (*green*), 1.0 (*blue*), and 0.3 (*grey*). Note the singularity at zero for $\alpha = 1$, and 0.3, for the α-Linnik distribution (Color figure online)

Proof The verification is straightforward via the characteristic function:

$$\begin{aligned}
&\mathbb{E}\exp[i\xi p^{1/\alpha}(X_1+X_2+\cdots+X_N)]\\
&=\sum_{n=1}^{\infty}\mathbb{E}\big(\exp[i\xi p^{1/\alpha}(X_1+X_2+\cdots+X_N)]|N=n\big)p(1-p)^{n-1}\\
&=\sum_{n=1}^{\infty}\left(\frac{1}{1+|\gamma\xi p^{1/\alpha}|^{\alpha}}\right)^{n}p(1-p)^{n-1}=\frac{1}{1+|\gamma\xi|^{\alpha}}.
\end{aligned}$$

□

Two elementary, but crucial observations are in order here:

Remark 2.1 The α-Linnik distributions are in the domain of attraction of the α-stable distribution, that is, if $Z_n = n^{-1/\alpha}(X_1 + X_2 + \cdots + X_n)$ are i.i.d. and $X_i \sim L(\alpha, \gamma)$ for all i, then the distribution of Z_n converges to a, α-stable distribution as n tends to infinity. Indeed, working again with characteristic functions, we get

$$\lim_{n\to\infty}\mathbb{E}(\exp(i\xi Z_n))=\lim_{n\to\infty}(1+|\xi\gamma|^{\alpha}/n)^{-n}=\exp(-|\gamma\xi|^{\alpha}).$$

Remark 2.2 If Z is the standard exponential random variable, and $Y \sim S(\alpha, c)$ is an independent α-stable random variable then

$$X = Z^{1/\alpha}Y \tag{2.14}$$

is a Linnik random variable with the distribution $L(\alpha, \gamma)$, and $\gamma = c$, see [11]. Indeed,

$$\mathbb{E}\,e^{i\xi Z^{1/\alpha}Y}=\iint e^{i\xi z^{1/\alpha}y}f_Y(y)e^{-z}\,dy\,dz=\int_0^{\infty}e^{-z(1+|c\xi|^{\alpha})}dz=\frac{1}{1+|c\xi|^{\alpha}}.$$

2.4 Comparison of Tails and Lower-Order Fractional Moments of α-Stable and α-Linnik Distributions

For $0 < \alpha < 2$, the tail behavior of the distribution of $Y \sim S(\alpha, c)$ is as follows [34]:

$$\lim_{y\to\infty} y^{\alpha}\mathbb{P}(|Y|>y)=C_{\alpha}c^{\alpha}, \tag{2.15}$$

where

$$C_{\alpha}=\frac{\sin(\pi\alpha/2)\Gamma(\alpha)}{\pi}, \tag{2.16}$$

In the α-Linnik case we have a similar result.

Proposition 2.2 *$X \sim L(\alpha, \gamma)$, $0 < \alpha < 2$, then*

$$\lim_{y\to\infty} y^{\alpha}\mathbb{P}(|X|>y)=C_{\alpha}\gamma^{\alpha}. \tag{2.17}$$

Proof Indeed, in view of Remark 2.2,

$$\begin{aligned}
\lim_{x\to\infty}x^{\alpha}\mathbb{P}(|X|>x)&=\lim_{x\to\infty}x^{\alpha}\mathbb{P}\big(|Y|>xZ^{-1/\alpha}\big)\\
&=\lim_{x\to\infty}x^{\alpha}\iint_{\{|y|>xz^{-1/\alpha}\}}f_Y(y)e^{-z}\,dy\,dz
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \left(\lim_{x\to\infty} (xz^{-1/\alpha})^\alpha \int_{\{|y|>xz^{-1/\alpha}\}} f_Y(y)\,dy \right) z e^{-z}\,dz \\
&= C_\alpha \gamma^\alpha \int_0^\infty z e^{-z}\,dz = C_\alpha \gamma^\alpha .
\end{aligned}
$$

□

Given the above asymptotic results (see also Theorem 2.2), both, the α-Linnik amd α-stable random variables with $0 < \alpha < 2$ have finite moments of fractional order less than α; the higher-order moments are infinite. More precisely, if $Y \sim S(\alpha, c)$, and $0 < p < \alpha$, then [34]

$$
\mathbb{E}|Y|^p = \frac{2^{p+1} c^p \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(-\frac{p}{\alpha}\right)}{\alpha\sqrt{\pi}\,\Gamma\left(-\frac{p}{2}\right)}. \tag{2.18}
$$

On the other hand, for the α-Linnik distributions we have the following result:

Proposition 2.3 *Let $X \sim L(\alpha, \gamma)$, with $0 < \alpha < 2$. Then, for every p, $0 < p < \alpha$,*

$$
\mathbb{E}|X|^p = \frac{2^{p+1} p\gamma^p \Gamma\big(\frac{p+1}{2}\big)\Gamma\big(-\frac{p}{\alpha}\big)\Gamma\big(\frac{p}{\alpha}\big)}{\alpha^2\sqrt{\pi}\,\Gamma\big(-\frac{p}{2}\big)}. \tag{2.19}
$$

Proof In view of Remark 2.2,

$$
\mathbb{E}|X|^p = \mathbb{E}Z^{p/\alpha} \cdot \mathbb{E}|Y|^p = \int_0^\infty z^{p/\alpha} e^{-z}\,dz \cdot \mathbb{E}|Y|^p = \Gamma\left(\frac{p}{\alpha}+1\right)\mathbb{E}|Y|^p,
$$

which gives (2.19) as a result of (2.18), and the identity $\Gamma(x+1) = x\Gamma(x)$. □

Remark 2.3 Thus the α-stable and α-Linnik ditributions have exactly matching probability distribution tails if $c = \gamma$. We will exploit this fact in the numerical work presented in Sect. 5. However, the exactly matching tails of $S(\alpha, c)$ and $L(\alpha, \gamma)$ in the case $c = \gamma$ do not imply that their fractional moments match. To assure the equality of the p-th order fractional lower order moment of the α-Linnik distribution $L(\alpha, \gamma)$, and the α-stable distribution $S(\alpha, c, \beta, \mu)$, which may be also useful in simulations, their parameters have to be related via the following equality

$$
\frac{p\gamma^p \Gamma\,(p/\alpha)}{\alpha} = c^p, \qquad 0 < p < \alpha. \tag{2.20}
$$

For example, if $\alpha = 3/2$, then to assure the equality of the first-order moments we must have $c = 0.902745\gamma$.

2.5 Lévy Measure of the Linnik Distribution

There is an obvious relationship between the characteristic function $\phi_S(\xi; \alpha, 1)$ of the symmetric stable distribution, and the characteristic function $\phi_L(\xi; \alpha, 1)$ of the Linnik distribution:

$$
\phi_L(\xi) = \frac{1}{1 - \log \phi_S(\xi)}. \tag{2.21}
$$

This fact permits expressing the density of the Lévy measure of Λ_L of the α-Linnik distribution in terms of the α-stable distribution [27]:

$$\frac{\Lambda_L(dx)}{dx} = \frac{\alpha}{2|x|}\mathbb{E}\exp\left(-\left|\frac{x}{X}\right|^{\alpha}\right), \tag{2.22}$$

where $X \sim S(\alpha, 1)$.

Finally, using the tail asymptotics (2.16) of the $S(\alpha, 1c)$, one obtains the following asymptotics for the Lévy measure of the α-Linnik distribution:

$$\lim_{x\to\infty} x^{\alpha}\Lambda_L([x,\infty)) = \frac{1}{2\Gamma(1-\alpha)\cos(\pi\alpha/2)}; \qquad 0 < \alpha < 2. \tag{2.23}$$

2.6 Generalized Linnik Distributions

Several authors, see, e.g., [15,32], defined the generalized Linnik distributions as distributions with the characteristic functions

$$\phi_{GL}(\xi, \alpha, \gamma, t) = \frac{1}{(1+|\gamma\xi|^{\alpha})^t}, \tag{2.24}$$

where $\alpha \in (0, 2]$, and $\gamma, t > 0$. For integer values of t, the generalized Linnik distribution is simply the convolution of regular Linnik distributions. For arbitrary $t > 0$, they are the distributions of the corresponding Lévy process, see (2.4), or, equivalently, the distributions of the semigroup generated by the basic Linnik distribution, $L(\alpha, \gamma)$. They will play a natural role in the subsequent discussion of conservation laws driven by Linnik diffusions.

3 Conservation Laws Driven by α-Stable and α-Linnik Diffusions

Mathematical conservation laws are integro-differential evolution equations, such as Navier–Stokes and Burgers equations, expressing the physical principles of conservation of mass, energy, momentum, enstrophy, etc., in different dynamical situations. For general theory of conservation laws we refer to the monograph [37].

3.1 Infinitesimal Generators of α-Stable and α-Linnik Diffusions

We are now turning to a study of 1-D evolution equations, for a function $u = u(t, x)$, of the form

$$u_t + \mathcal{L}u + (f(u))_x = 0, \tag{3.1}$$

where $\mathcal{L}$ is an infinitesimal generator of the semigroup associated with a Lévy process and $f : \mathbb{R} \mapsto \mathbb{R}$ is a (nonlinear) function. Such equations are often called fractal, or anomalous conservation laws (see, e.g., [5,6]). The operators $\mathcal{L}$ are easiest to describe in terms of their actions in the Fourier domain; they are both so-called Fourier multiplier operators. Let's begin by recalling the basic terminology and establishing the notation.

Like any Markov processes,[2] the Lévy process, $X_t, t > 0$, has associated with it a semigroup, $P_t, P_{t+s} = P_t P_s, t, s, > 0$, of convolution operators acting on a bounded function $f(x)$ via the formula

$$P_t f(x) = \mathbb{E}^x(f(X(t)) = \int_{\mathbb{R}} f(x+y)\, P(X(t) \in dy). \tag{3.2}$$

[2] See, e.g., [8,18], for basic information in this area.

The *infinitesimal generator* $\mathcal{L}$ of such a semigroup is defined by the formula

$$\mathcal{L} = \lim_{h\to 0} \frac{P_h - P_0}{h} \tag{3.3}$$

and the family of functions (densities) $v(t,x) = P_t f(x)$ satisfies clearly the (generalized) Fokker–Planck evolution equation

$$v_t = \mathcal{L}v, \tag{3.4}$$

because

$$\lim_{h\to 0} \frac{P_{t+h} - P_t}{h} = \lim_{h\to 0} \frac{P_h - P_0}{h} P_t = \mathcal{L}P_t. \tag{3.5}$$

In the case of a general Lévy processes X_t, we have the identity

$$\mathcal{F}(\mathcal{L}f)(\xi) = -\psi(\xi)\mathcal{F}f(\xi) \tag{3.6}$$

where $\mathcal{F}$ stands for the Fourier transform [3], and $\psi(u)$ is the characteristic exponent of $X(1)$ defined in (2.3). So, $\mathcal{L}$ is the pseudo-differential operator with the Fourier multiplier $\psi(\xi)$, which is also called the *symbol* of the semigroup (P_t). Indeed,

$$\begin{aligned}\mathcal{F}(P_t f)(\xi) &= \left(\int_{\mathbb{R}} e^{-i\xi x}\mathbb{E}f(X(t)+x)\,dx\right) = \mathbb{E}\left(\int_{\mathbb{R}} e^{-i\xi(y-X(t))} f(y)\,dy\right)\\ &= \mathbb{E}e^{i\xi X(t)}\int_{\mathbb{R}} e^{-i\xi y} f(y)\,dy = \exp(-t\psi(\xi)\mathcal{F}f(\xi),\end{aligned}$$

which implies (3.6), in view of of (3.3).

In the case of the usual Brownian motion the infinitesimal operator $\mathcal{L}$ is just the classical Laplacian Δ. For the α-stable process with $X(1) \sim S(\alpha, c)$,

$$\mathcal{F}(\mathcal{L}f)(\xi) = -|c\xi|^\alpha \mathcal{F}f(\xi), \tag{3.7}$$

And, for the α-Linnik process, $X(1) \sim L(\alpha,\gamma)$, in view of (2.8),

$$\mathcal{F}(\mathcal{L}f)(\xi) = -\log(1 + |\gamma\xi|^\alpha)(\mathcal{F}f(\xi), \tag{3.8}$$

For general theory, including Feller processes, see, e.g., [8,18,19].

Now we are ready to state the results about the existence, uniqueness and the asymptotic behavior of the Lévy and Linnik conservation laws driven by stable and Linnik diffusions.

3.2 Asymptotics of Solutions of α-Stable and α-Linnik Conservation Laws with Supercritical Nonlinearity ($\alpha > 1$)

In this subsection we describe several asymptotic results for the 1-D Cauchy problem for nonlinear pseudodifferential equations of the form

$$u_t + \mathcal{L}u + \nabla\mathcal{N}u = 0, \quad u(x,0) = u_0(x), \tag{3.9}$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, $u : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$, $-\mathcal{L}$ is a generator of a Lévy semigroup with symbol $\psi(\xi)$, and $\mathcal{N}$ is a nonlinear operator to be specified later. All these equations are generalizations of the classical Burgers equation

$$u_t - u_{xx} + (u^2)_x = 0, \tag{3.10}$$

[3] In this paper we use the convention, $(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx$, and $(\mathcal{F}^{-1}g)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} g(\xi)\,d\xi$.

The main point here is the observation that for the α-stable and α-Linnik diffusions the large time behavior of the solutions is similar. This is in contrast to the phenomena observed in the next section where, for these two types of diffusions and the Riemann- type initial data, shocks behave dramatically differently.

The solutions to the Cauchy problem (3.9) have to be understood in some weak sense and several options are here available, and have been studied in detail in [4–7,23]. For the sake of this presentation let us just say that as the solution to (3.9) we mean a *mild* solution of the integral equation,

$$u(t) = e^{-t\mathcal{L}}u_0 - \int_0^t \nabla \cdot e^{-(t-\tau)\mathcal{L}}(\mathcal{N}u)(\tau)\,d\tau, \tag{3.11}$$

motivated by the classical Duhamel formula. The regularity of the solutions is expressed in terms of the Sobolev space $W^{2,2}$.

Theorem 3.1 (see, [6])

(i) *Assume that $f \in C^1(\mathbb{R}, \mathbb{R}^d)$ and $\mathcal{L}$ is the infinitesimal generator of a Lévy process and its symbol (characteristic exponent), $\psi(\xi)$, satisfies the condtion*

$$\limsup_{|\xi|\to\infty} \frac{\psi(\xi) - a_0|\xi|^2}{|\xi|^{\widetilde{\alpha}}} < \infty \ \textit{for some}\ 0 < \widetilde{\alpha} < 2,\ \textit{and}\ a_0 > 0. \tag{3.12}$$

Given $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique solution $u \in \mathcal{C}([0,\infty);\ L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ of the problem

$$u_t + \mathcal{L}u + \nabla \cdot f(u) = 0, \qquad u(x,0) = u_0(x). \tag{3.13}$$

This solution is regular, $u \in C((0,\infty); W^{2,2}(\mathbb{R})) \cap C^1((0,\infty); L^2(\mathbb{R}))$, satisfies the conservation of integral property, $\int u(x,t)\,dx = \int u_0(x)\,dx$, and the contraction property,

$$\|u(t)\|_p \le \|u_0\|_p, \tag{3.14}$$

for each $p \in [1,\infty]$ and all $t > 0$. Moreover, the maximum and minimum principles hold: $\operatorname{ess\,inf} u_0 \le u(x,t) \le \operatorname{ess\,sup} u_0$, *a.e. x, t, as well as the comparison principle for $u_0 \le v_0 \in L^1(\mathbb{R})$:*

$$u(x,t) \le v(x,t)\ \textit{a.e.}\ x,\ t,\ \textit{and}\ \|u(t) - v(t)\|_1 \le \|u_0 - v_0\|_1. \tag{3.15}$$

(ii) *If*

$$0 < \liminf_{\xi\to 0} \frac{\psi(\xi)}{|\xi|^\alpha} \le \limsup_{\xi\to 0} \frac{|\psi(\xi)|}{|\xi|^\alpha} < \infty, \quad 0 < \inf_\xi \frac{|\psi(\xi)|}{|\xi|^2}, \tag{3.16}$$

for some $0 < \alpha < 2$, then the bound

$$\|u(t)\|_p \le C_p \min\left(t^{-n(1-1/p)/2}, t^{-n(1-1/p)/\alpha}\right)\|u_0\|_1 \tag{3.17}$$

holds for all $1 \le p \le \infty$. Moreover, if $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then

$$\|u(t)\|_p \le C(1+t)^{-n(1-1/p)/\alpha} \tag{3.18}$$

with a constant C which depends on $\|u_0\|_1$ and $\|u_0\|_p$.

(iii) *Assume that u is a solution of the Cauchy problem* (3.13) *with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $e^{-t\mathcal{L}}$ satisfies* (3.12) *and* (3.16) *with some* $0 < \alpha < 2$. *Furthermore, suppose that* $f \in C^1$, $\limsup_{s \to 0} |f(s)|/|s|^r < \infty$ *for some*

$$r > \max((\alpha, 1)). \tag{3.19}$$

Then, for every $1 \le p \le \infty$, *the relation*

$$\lim_{t \to \infty} t^{(1-1/p)/\alpha} \|u(t) - e^{-t\mathcal{L}} u_0\|_p = 0 \tag{3.20}$$

holds.

Remark 3.1 It may be worthwhile to observe that the third condition in (3.16) means that the Gaussian part is nontrivial, that is, $\sigma^2 > 0$. Indeed , it is well known that $\sigma^2 = \lim_{|\xi| \to \infty} |\psi(\xi)|/|\xi|^2 \ge \inf_\xi |\psi(\xi)|/|\xi|^2$. Under the condition given in the first part of (3.16) the asymptotics of $\psi(\xi)$ at $\xi = 0$ is like $|\xi|^\alpha$, so, effectively, (3.16) is equivalent to saying that

$$0 < \liminf_{\xi \to 0} \frac{\psi(\xi)}{|\xi|^\alpha} \le \limsup_{\xi \to 0} \frac{|\psi(\xi)|}{|\xi|^\alpha} < \infty, \quad \text{and} \quad \sigma^2 > 0.$$

Recall that $e^{-t\mathcal{L}} u_0$ denotes the action of the Lévy semigroup on the function u_0, that is a solution of the linear equation $u_t + \mathcal{L}u = 0$, with the initial data u_0 and that for the linear equation the asymptotics is clear: there exists a nonnegative function $\eta \in L^\infty(0, \infty)$ satisfying $\lim_{t \to \infty} \eta(t) = 0$, and such that

$$\left\| e^{t\mathcal{L}} * u_0 - \int_{\mathbb{R}} u_0(x)\, dx \cdot p_{\mathcal{L}}(t) \right\|_p \le t^{-(1-1/p)/\alpha} \eta(t),$$

where $p_{\mathcal{L}}(t)$ is the kernel of the operator $\mathcal{L}$ in (6). Higher order asymptotics is also available, (see, [5]).

The above general result has direct consequences for *multifractal conservation laws*

$$u_t + \mathcal{L}u + f(u)_x = 0, \tag{3.21}$$

driven by stable and Linnik diffusions.[4] Recall that the *multifractal stable operator* is defined as follows:

$$\mathcal{L} = -a_0 \Delta + \sum_{j=1}^{k} a_j (-\Delta)^{\alpha_j/2}, \tag{3.22}$$

$0 < \alpha_j < 2$, $a_j > 0, j = 0, 1, \ldots, k$, where $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian defined as the Fourier multiplier operator

$$((-\Delta)^{\alpha/2} v) = \mathcal{F}^{-1}(|\xi|^\alpha (\mathcal{F}v)(\xi)). \tag{3.23}$$

Similarly, the *multifractal Linnik operator* will be understood here as the operator of the form

$$\mathcal{L} = -a_0 \Delta + \sum_{j=1}^{k} a_j L_{\alpha_j}, \tag{3.24}$$

[4] The particle approximations and the propagation of chaos results for such systems have been studied in [22].

where

$$L_\alpha v = \mathcal{F}^{-1}(\log(1 + |\xi|^\alpha)(\mathcal{F}v)(\xi)). \tag{3.25}$$

Note the parabolic regularization included in the operator. To avoid confusion we will assume that all the α_j's are different.

Corollary 3.1 *All the statements of Theorem 3.1 are valid for the conservation laws (3.21) driven by multifractal stable and Linnik diffusions with*

$$\alpha = \min(\alpha_1, \ldots, \alpha_k).$$

In particular, if u is a solution of the Cauchy problem (3.21) with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $e^{-t\mathcal{L}}$ satisfies (3.12) and (3.16) with some $0 < \alpha < 2$, and $f \in C^1$, $\limsup_{s\to 0} |f(s)|/|s|^r < \infty$, for some

$$r > \max((\alpha, 1)), \tag{3.26}$$

then, for every $1 \le p \le \infty$, the relation

$$\lim_{t\to\infty} t^{(1-1/p)/\alpha} \|u(t) - e^{-t\mathcal{L}} u_0\|_p = 0 \tag{3.27}$$

holds. Moreover,

$$\left\| e^{t\mathcal{L}} * u_0 - \int_{\mathbb{R}} u_0(x)\,dx \cdot p_{\mathcal{L}}(t) \right\|_p \le t^{-(1-1/p)/\alpha} \eta(t),$$

where $p_{\mathcal{L}}(t)$ is the kernel of the operator $\mathcal{L}$ in (3.21).

Proof All that is required is verification the conditions (3.12) and (3.16) are satisfied. Indeed, for the mutifractal stable case (3.22) with the symbol,

$$\psi(\xi) = a_0|\xi|^2 + \sum_{j=1}^k a_j |\xi|^{\alpha_j}, \tag{3.28}$$

we have, for $\alpha^* = \max(\alpha_1, \ldots, \alpha_k)$, and $\alpha_{j^*} = \alpha^*$

$$\limsup_{|\xi|\to\infty} \frac{\psi(\xi) - a_0|\xi|^2}{|\xi|^{\alpha^*}} = a_{j^*} < \infty$$

and with $\alpha_* = \min(\alpha_1, \ldots, \alpha_k)$

$$0 < \lim_{\xi\to 0} \frac{\psi(\xi)}{|\xi|^{\alpha_*}} = a_{j_*} < \infty,$$

where $\alpha_{j_*} = \alpha_*$, and

$$\inf_{\xi} \frac{\psi(\xi)}{|\xi|^2} = a_0 > 0,$$

For the mutifractal Linnik case (3.24) with the symbol,

$$\psi(\xi) = a_0|\xi|^2 + \sum_{j=1}^k a_j \log(1 + |\xi|^{\alpha_j}),$$

verification of condition (3.12) also gives

$$\limsup_{|\xi|\to\infty} \frac{\psi(\xi) - a_0|\xi|^2}{|\xi|^{\alpha^*}} = \limsup_{|\xi|\to\infty} \frac{\sum_{j=1}^k a_j \log(1+|\xi|^{\alpha_j})}{|\xi|^{\alpha^*}} = a_{j^*} < \infty,$$

while

$$0 < \lim_{\xi\to 0} \frac{a_0|\xi|^2 + \sum_{j=1}^k a_j \log(1+|\xi|^{\alpha_j})}{|\xi|^{\alpha}} = a_{j_*} < \infty,$$

and, similarly, for condition (3.16) we have

$$\inf_{\xi} \frac{a_0|\xi|^2 + \sum_{j=1}^k a_j \log(1+|\xi|^{\alpha_j})}{|\xi|^2} \geq a_0 > 0.$$

□

Remark 3.2 Explicit asymptotic expressions. The explicit representationss of the kernels of some of those multifractal operators in terms of special functions are being studied in [16].

3.3 Asymptotics of Solutions of α-Stable and α-Linnik Conservation Laws with Critical Nonlinearity ($\alpha > 1$)

By contrast with the results of the previous section, let us note, (see, [41]), that the first order asymptotics of solutions to the Cauchy problem for the Burgers equation (3.10) is described by the relation

$$t^{(1-1/p)/2}\|u(t) - U_M(t)\|_p \to 0, \qquad \text{as} \qquad t\to\infty,$$

where

$$U_M(x,t) = t^{-1/2}\exp(-x^2/4t)\left(K(M) + \frac{1}{2}\int_0^{x/2\sqrt{t}} \exp(-\xi^2/4)\, d\xi\right)^{-1}$$

is the, so-called, source solution such that $u(x,0) = M\delta_0$. It is easy to verify that this solution is self-similar, i.e., $U_M(x,t) = t^{-1/2}U(xt^{-1/2},1)$. Thus, the long time behavior of solutions to the classical Burgers equation is genuinely nonlinear, i.e., it is not determined by the asymptotics of the linear heat equation.

As it turns out that genuinely nonlinear behavior of the Burgers equation is due to the precisely matched balancing influence of the regularizing Laplacian diffusion operator and the gradient-steepening quadratic inertial nonlinearity.

The next result finds such a matching critical nonlinearity exponent for the nonlocal multifractal conservation law so that the solutions of (3.13) behave asymptotically like self-similar source solutions U of (3.13) with singular initial data $M\delta_0$.

Theorem 3.2 *(see, [6]) Let u be a solution of the Cauchy problem (3.13) with the operator $\mathcal{L} = (-\Delta)^{\alpha/2} + \mathcal{K}$, where $1 < \alpha < 2$, and $\mathcal{K}$ is an infinitesimal generator of a Lévy process whose symbol k satisfies the condition*

$$\lim_{\xi\to 0} \frac{k(\xi)}{|\xi|^{\alpha}} = 0, \tag{3.29}$$

and $u_0 \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} u_0(x)\,dx = M > 0$. Assume that the nonlinearity f is such that

$$\lim_{s\to 0} \frac{f(s)}{s|s|^{(\alpha-1)/n}} \in \mathbb{R}. \tag{3.30}$$

Then, for each $1 \le p \le \infty$,

$$\lim_{t\to\infty} t^{n(1-1/p)/\alpha}\|u(t)-U(t)\|_p = 0, \tag{3.31}$$

where $U = U_M$ *is the unique solution of the problem* (3.13) *with* $r = \max((\alpha-1)/n+1, 1))$ *and the initial data* $M\delta_0$. *Moreover,* U *is of self-similar form* $U(x,t) = t^{-n/\alpha}U(xt^{-1/\alpha}, 1)$, $\int_{\mathbb{R}^d} U(x,1)\,dx = M$, *and* $U \ge 0$.

Analogous to Corollary 3.1., we also have the following result in the case of multifractal conservation laws with critical nonlinearities.

Corollary 3.2 *All the statements of Theorem* 3.2 *are valid for conservation laws* (3.13) *driven by multifractal stable, and multifractal Linnik diffusions, with the infinitesimal generator with the symbol*

$$\psi(\xi) = \sum_{j=1}^{n} a_j|\xi|^{\alpha_j}, \qquad a_j > 0, \quad j = 1, 2, \ldots, n,$$

in the stable case,

$$\psi(\xi) = \sum_{j=1}^{n} a_j \log(1+|\xi|^{\alpha_j}), \qquad a_j > 0, \quad j = 1, 2, \ldots, n,$$

in the Linnik case, and

$$\alpha = \alpha_* \equiv \min(\alpha_1, \ldots, \alpha_k).$$

Proof In this context, denoting $a_{j_*} = \alpha_* = \alpha$, the perturbation $\mathcal{K}$ of the operator $\mathcal{L}$ has the symbol

$$k(\xi) = \sum_{j=1}^{n} a_j|\xi|^{\alpha_j} - a_{j_*}|\xi|^{\alpha_{j*}},$$

in the stable case, and

$$k(\xi) = \sum_{j=1}^{n} a_j \log(1+|\xi|^{\alpha_j}) - a_{j_*}\log(1+|\xi|^{\alpha_{j*}}),$$

in the Linnik case. Now, remembering that the a_j's were assumed earlier to be all different, the verification of the condition (3.29) is immediate. □

Remark 3.3 Note that, in contrast to Corollary 3.1, the parabolic regularization (inclusion of the Gaussian term) is not necessary in the above critical case.

4 Shock Creation for Fractal Burgers Equations: Analytical Results

In the remainder of this paper we consider conservation laws with quadratic nonlinearity and α-stable and α-Linnik driving diffusions, $\alpha \in (0, 2)$. The first type was introduced and studied as fractal Burgers equation in [4] and, in the α-stable case, shock appearance for their solutions was studied in [1,12,13]. The main purpose of this section is to set up the stage for a numerical study (see, Sect. 6) expanding on the work in [12], which shows that the behavior of shocks is significantly different in the α-Linnik case than in the α-stable case.

Let us begin with the standard definition of an entropy solution for a general 1-D fractal conservation law (see, e.g., [1]),

$$u_t + \mathcal{L}u + (f(u))_x = 0, \qquad u(0,x) = u_0(x), \tag{4.1}$$

where $u = u(t,x), t > 0, \quad x \in \mathbb{R}, \quad u_0 \in L^\infty(\mathbb{R}), \quad f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, and $\mathcal{L}$ is the α-stable infinitesimal generator, $\alpha \in (0,2)$, defined as the Fourier multiplier, with the integral representation

$$[\mathcal{L}u](x) = -c_\alpha \left(\int_{|z|\le r} \frac{u(x+z) - u(x) - u^{'}(x)z}{|z|^{1+\alpha}} dz + \int_{|z|>r} \frac{u(x+z) - u(x)}{|z|^{1+\alpha}} dz \right) \tag{4.2}$$

where $c_\alpha = \alpha\Gamma(\frac{1+\alpha}{2})/(2\sqrt{\pi}\pi^\alpha\Gamma(1-\frac{\alpha}{2}))$.

An entropy solution to (4.1) is a function $u \in L^\infty\left((0,\infty)\times\mathbb{R}\right)$, such that, for all non-negative $\varphi \in C_c^\infty\left((0,\infty)\times\mathbb{R}\right)$, for all smooth convex function $\eta : \mathbb{R} \to \mathbb{R}$, and all $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi^{'} = \eta^{'} f^{'}$, and all $r > 0$, we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (\eta(u)\partial_t\varphi + \phi(u)\partial_x\varphi) \\
&\quad + c_\alpha \int_0^\infty \int_{\mathbb{R}} \int_{|z|>r} \eta^{'}(u(t,x)) \frac{u(t,x+z) - u(t,x)}{|z|^{1+\alpha}} \varphi(t,x) dt dx dz \\
&\quad + c_\alpha \int_0^\infty \int_{\mathbb{R}} \int_{|z|\le r} \eta(u(t,x)) \frac{\varphi(t,x+z) - \varphi(t,x) - \partial_x\varphi(t,x).z}{|z|^{1+\alpha}} dt dx dz \\
&\quad + \int_{\mathbb{R}} \eta(u_0)\varphi(0,\cdot) \quad \ge 0.
\end{aligned} \tag{4.3}$$

In the special case of the quadratic nonlinearity, that is, of the 1-D fractal Burgers equation of the form

$$u_t + (u^2/2)_x + \mathcal{L}(u) = 0, \quad u(0,x) = u_o(x), \tag{4.4}$$

it was demonstrated in [1] that the solution of the equation (4.4) can exhibit shocks (i.e., jump discontinuities) for bounded, odd on $\mathbb{R}$, and convex on $\mathbb{R}^+$ initial data when $\alpha < 1$. No such effect is present in the case $\alpha > 1$, as in that case the fractional Laplacian has a regularizing effect, see, e.g., [4,6,7,14]. The basic analytical result on shock creation is as follows:

Theorem 4.1 (*see,* [1]) *Consider the fractal Burgers equation* (4.4) *driven by an α-stable generator of the form* (4.2) *with $\alpha \in (0,1)$. Then , locally in time, shocks in initial data are preserved, and with continuous initial data, shocks do appear, also locally in time, if the initial data and its derivative are simultaneously large; other wise no shocks are created. More precisely*

(a) (*Short term preservation of initial shocks*) *Let u_0 be discontinuous at* 0*, bounded, odd on* $\mathbb{R}$*, and convex on the positive half-line. If u is the unique entropy solution to* (4.4) *then $u \in C_b\left([0,\infty)\times R\setminus\{0\}\right)$ is odd and non-increasing with respect to space variable, and there exists $\epsilon > 0$ such that*

$$\inf_{t\in[0,0+\epsilon)} \{u(t,0^-) - u(t,0^+)\} > 0,$$

where $u(t,0^\pm)$ denote the limits $\lim_{x\to 0^\pm} u(t,x)$.

(b) (*Short term shock creation for initial continuous data*) *There exists* $S(\alpha) > 0$ *such that if* u_0 *is bounded, odd on* $\mathbb{R}$*, and convex on the positive half-line, and for some* $x_* > 0$ *we have* $u_0(x_*) < -S(\alpha)x_*^{1-\alpha}$*, then the unique entropy solution to* (4.4) $u \in C_b([0,\infty) \times R \setminus \{0\})$ *is odd and non-increasing with respect to space variable and there exist* $0 \le t_* < \infty$*, and* $\epsilon > 0$*, such that*

$$\inf_{t\in[t_*,t_*+\epsilon)} \{u(t,0^-) - u(t,0^+)\} > 0,$$

where $u(t,0^{\pm})$ *denote the limits* $\lim_{x\to 0^{\pm}} u(t,x)$.

(c) *(No creation of shocks) If* $u_0 \in W^{1,\infty}(\mathbb{R})$*, and*

$$\|u_0'\|_{L^\infty(\mathbb{R})}^{1-\alpha} \cdot \|u_0\|_{L^\infty(\mathbb{R})}^{\alpha} < \frac{\Gamma(\frac{1+\alpha}{2})}{2^{\alpha+1}\pi^{\frac{1}{2}+\alpha}\Gamma(1-\frac{\alpha}{2})},$$

then the entropy solution u *of* (4.4) *belongs to the Sobolev space* $W^{1,\infty}((0,\infty),\mathbb{R})$*, and, for all* $t > 0$*, satisfies the inequalities*

$$\|u(t,.)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}, \qquad and \qquad \|\partial_x u(t,.)\|_{L^\infty(\mathbb{R})}\|u_0'\|_{L^\infty(\mathbb{R})}.$$

5 Shock Creation, Persistence and Dissolution for α-Stable and α-Linnik Burgers Equation: Numerical Results

In this section we present results of numerical studies of solutions of the α-stable and α-Linnik conservation laws with quadratic nonlinearity (fractal Burgers equations) contrasting dramatic difference in their time evolution despite the similar distributional tail behavior of the corresponding diffusions. The multiscale case is not considered, and the constants in the basic Lévy distributions underlying the two diffusions are taken to be 1. Computations have been carried out for the following three types of odd, decreasing, and convex on the positive half-line initial conditions:

(i) Riemann-type initial data:

$$u_0(x) = \begin{cases} 1, & \text{for} \quad x \le 0; \\ -1, & \text{for} \quad x > 0, \end{cases}$$

(ii) Piecewise linear (but continuous) data:

$$u_0(x) = \min(1, \max(-10x, -1)),$$

(iii) Smooth, infinitely differentiable data:

$$u_0(x) = (-2/\pi)\arctan(x).$$

The plots in Figs. 2, 3, 4, 5, 6, and 7 compare the evolution of the initial profile $u(0,x) = u_0(x)$ for the α-stable case (top pictures) and the α-Linnik case (bottom pictures) for the above three initial conditions. Two values of α are considered, $\alpha = 0.3 < 1$ and $\alpha = 1.25 > 1$[5]. The evolution is traced for the times $T = 1, 5, 10,$and 25. The red, dash-dot-dash lines indicate the initial data and continuous, colored curves track the solutions at different time instants. The complete numerical scheme is described in detail in Appendix.

[5] The Matlab code, as well as the results of numerical computations for other values of α can be found on the webpage of the second-named author: https://sites.google.com/a/case.edu/waw.

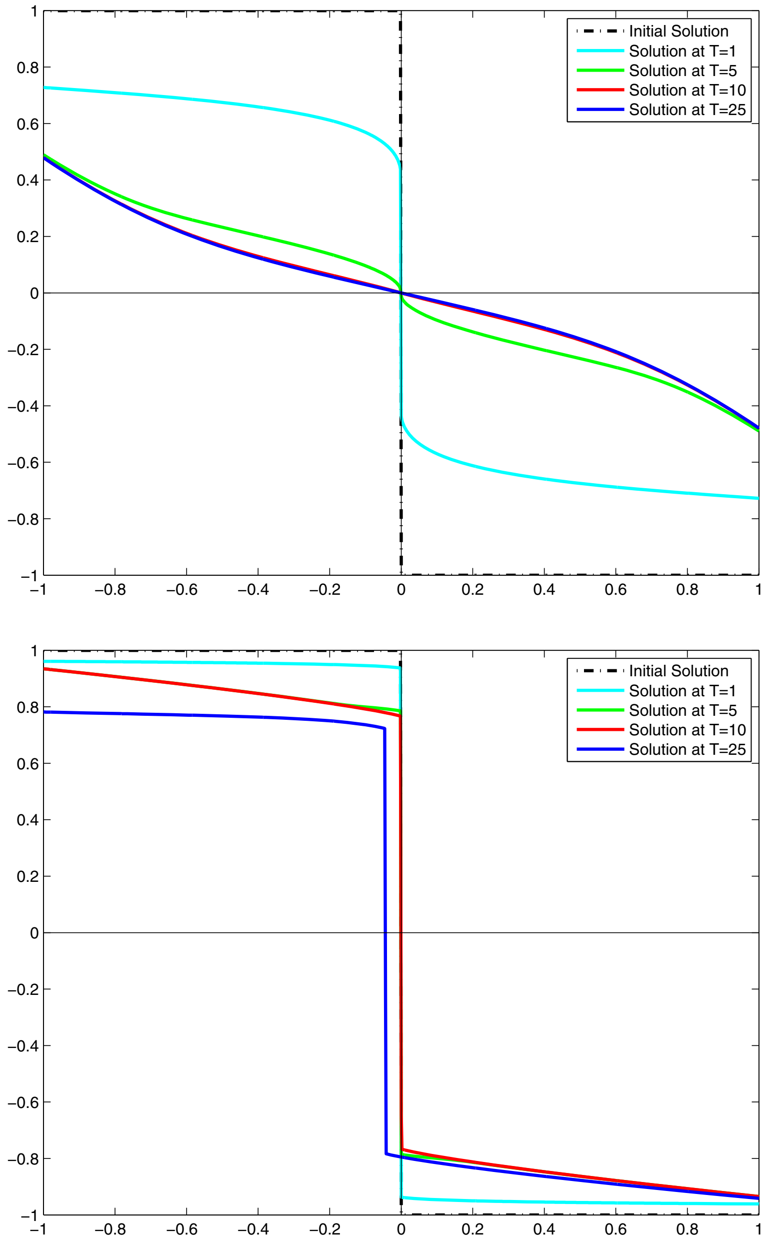


Fig. 2 (*Top*) The solution of α-stable fractional equations (3.21), and (3.22), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a Riemann initial condition and $\alpha = 0.3$. (*Bottom*) The solution of α-Linnik fractional equations (3.21), and (3.24), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a Riemann initial condition and $\alpha = 0.3$

For $\alpha = 0.3 < 1$, (Figs. 2, 3, 4) all three initial conditions ((i)–(iii), lead to shock creation at finite times both, in the α-stable and the α-Linnik case. In the α-stable case this phenomenon has been established rigorously in Theorem 4.1. In the α-Linnik case we have only a numerical results and the resulting plots.

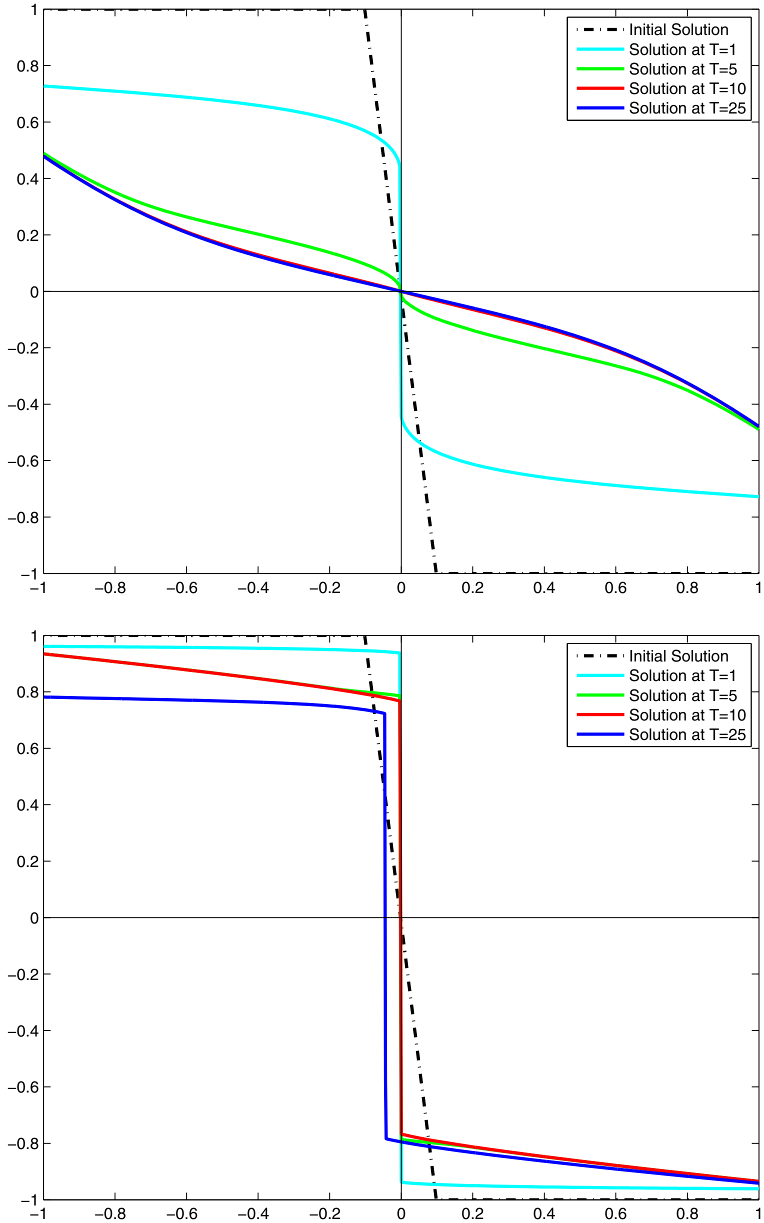


Fig. 3 (*Top*) The solution of α-stable fractional equations (3.21), and (3.22), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a continuous but nondifferentialble piecewise linear initial condition and $\alpha = 0.3$. (*Bottom*) The solution of α-Linnik fractional equations (3.21), and (3.24), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a continuous but nondifferentialble piecewise linear initial condition and $\alpha = 0.3$

However, for large times, our numerical calculation show that in the α-stable case the shocks dissolve by the time $T = 10$, and the solutions stabilize thereafter. In the α-Linnik case the shocks seem to persist indefinitely (at least in the time interval we investigated), a phenomenon, we believe has not been observed before. The situation is summarized in Fig. 8

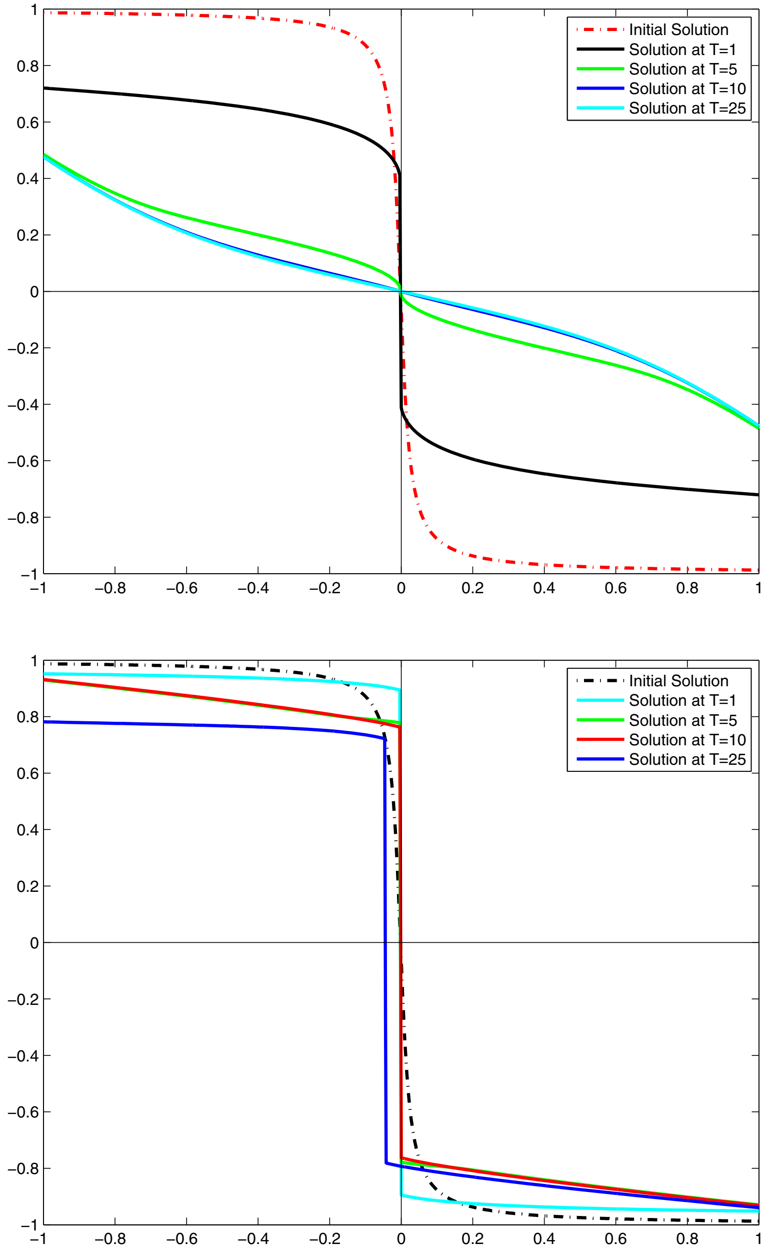


Fig. 4 (*Top*) The solution of α-stable fractional equations (3.21), and (3.22), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a smooth initial condition and $\alpha = 0.3$. (*Bottom*) The solution of α-Linnik fractional equations (3.21), and (3.24), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a smooth initial condition and $\alpha = 0.3$

which shows the eventual decay to zero of the shock size in the α-stable case, while the shock size in the α-Linnik case initially decreases but then, after $T \approx 6$, it stabilizes at the positive value of about 1.5.

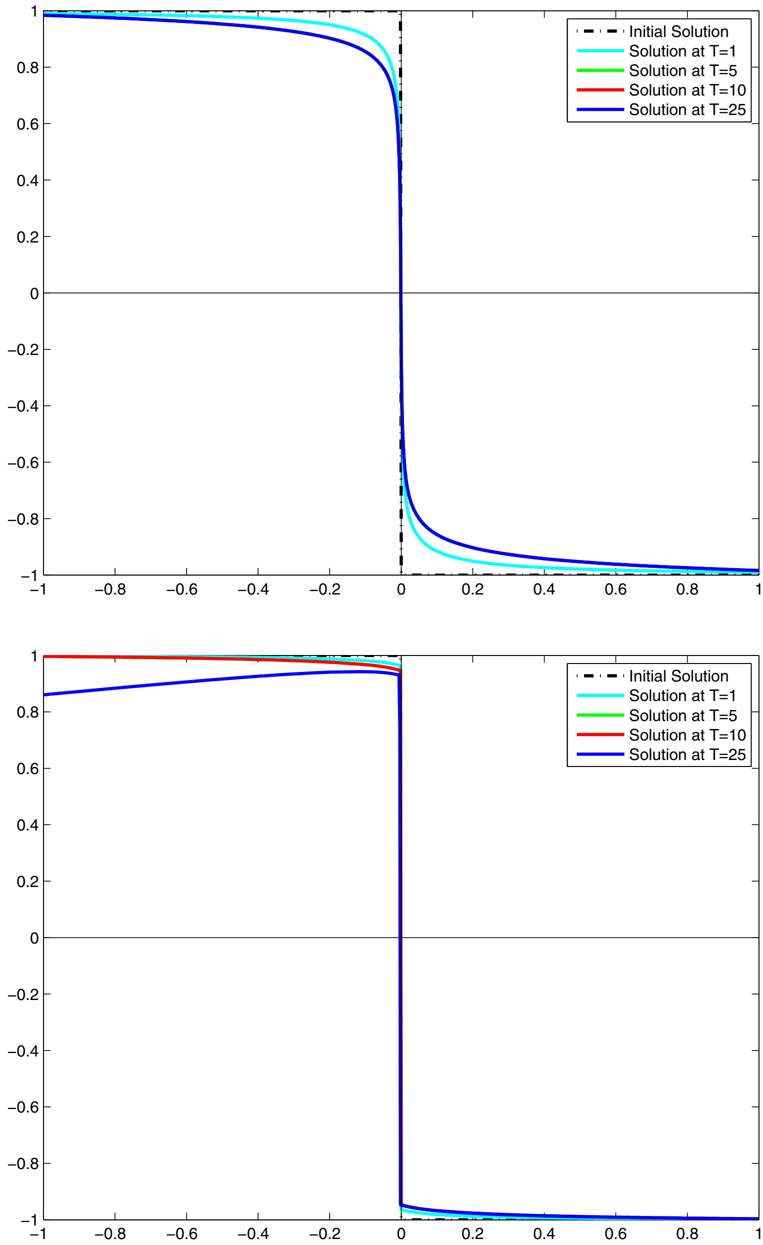
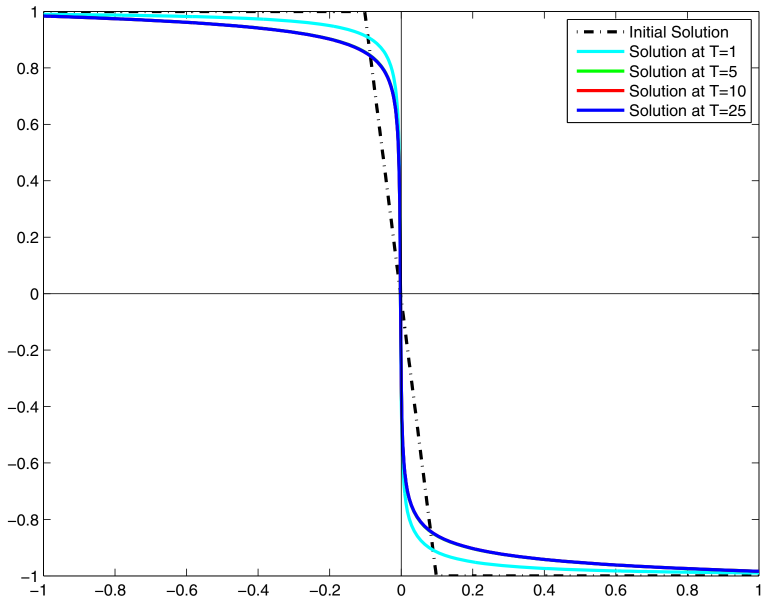


Fig. 5 (*Top*) The solution of α-stable fractional equations (3.21), and (3.22), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a Riemann initial condition and $\alpha = 1.25$. (*Bottom*) The solution of α-Linnik fractional equations (3.21), and (3.24), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a Riemann initial condition and $\alpha = 1.25$

Moreover, the shocks in the α-stable case stay put at $x = 0$ until they dissolve. But, surprisingly, in the α-Linnik case, the shocks, initially located at $x = 0$, begin to recede to the left (towards the negative x's).



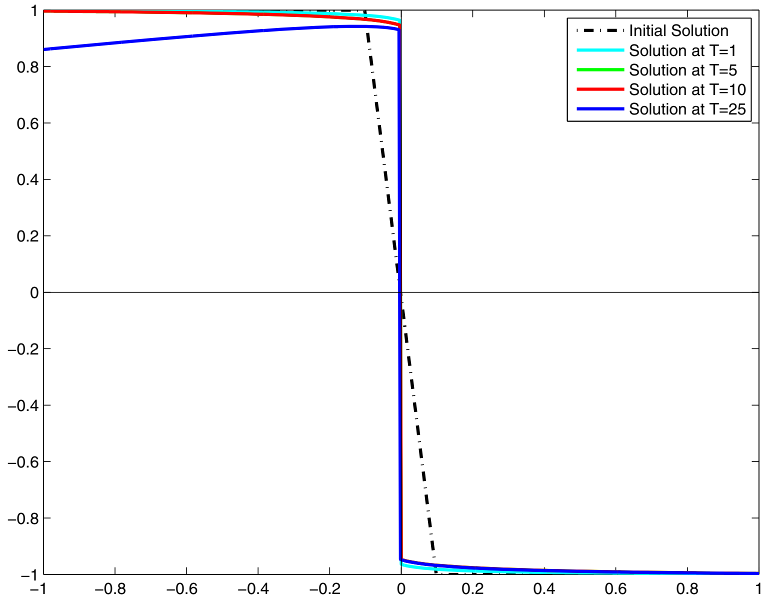


Fig. 6 (*Top*) The solution of α-stable fractional equations (3.21), and (3.22), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a continuous but nondifferentialble piecewise linear initial condition and $\alpha = 1.25$. (*Bottom*) The solution of α-Linnik fractional equations (3.21), and (3.24), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a continuous but nondifferentialble piecewise linear initial condition and $\alpha = 1.25$

For $\alpha = 1.25$, (Figs. 5, 6, 7, and 5.8) , the α-stable diffusion is strong enough to obliterate the initial shock instantaneously and the solutions (as established in Sect. 3) are smooth for all $t > 0$, regardless of the level of smoothness of the initial conditions. Not so, for the α-Linnik

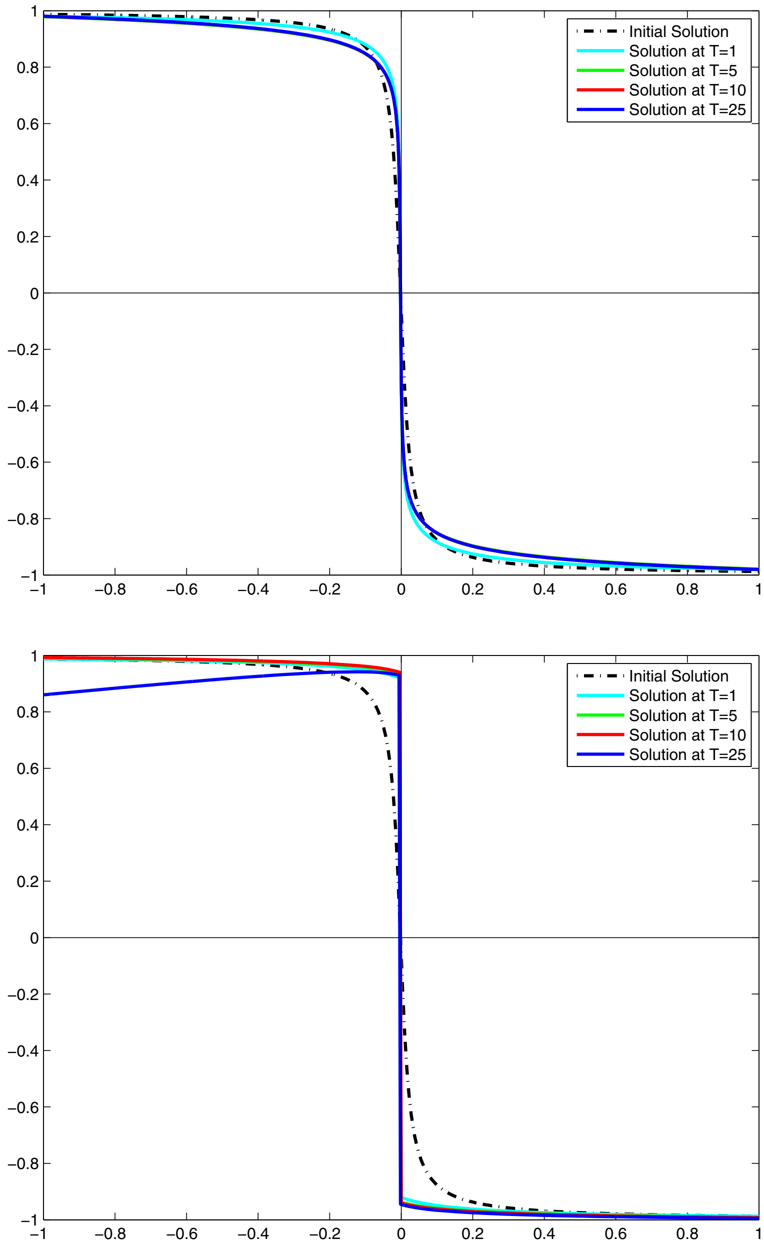
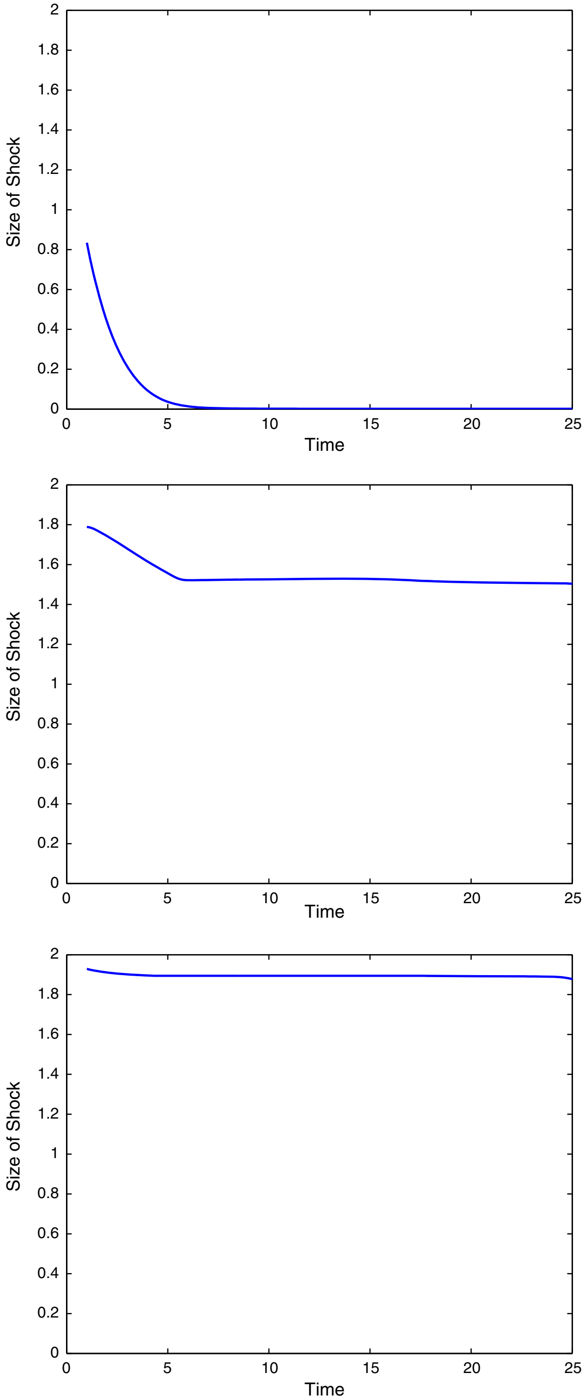


Fig. 7 (*Top*) The solution of α-stable fractional equations (3.21), and (3.22), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a smooth initial condition and $\alpha = 1.25$. (*Bottom*) The solution of α-Linnik fractional equations (3.21), and (3.24), with the quadratic linearity at times $t = 1, 5, 10, 25$ for a smooth initial condition and $\alpha = 1.25$

driving diffusion. Not only the shocks are created in finite time and they persist , with the shock size stabilizing at the positive value after $T \approx 6$ (like in the case $\alpha = 0.3 < 1$), but a new phonomenon appears, on the negative x-axis the solutions are no longer decraasing. However, the shock no longer recedes to the left, which was the case for $\alpha = 0.3 < 1$.

Fig. 8 Shock size dependence on time for α-stable (fractal) conservation law, and α-Linnik conservation laws with $\alpha = 0.3 < 1$ (*left*, and *center*) and for α-Linnik conservation laws with $\alpha = 1.25$ (*right*). In the 0.3-stable case (*top*) the shock appears at about $T \approx 1$, and dissolves completely by the time $T \approx 6$. For the 0.3-Linnik conservation law (*middle*), the shock is again created in finite time and its size initially decreases but it stabilizes at $T \approx 6$. For the 1.25-Linnik conservation law (*bottom*), the shock is again created in finite time and its size initially decreases, but it stabilizes at $T \approx 5$. The summary plot was created for smooth initial data (iii). In the 1.25-stable case there are no shocks even for the Riemann initial data



6 Conclusions, Conjectures and Future Work

The paper introduces a new type of the conservation laws driven by what we call α-Linnik diffusions and conducts a systematic comparison of their solutions with the solutions of better understood α-stable diffusions. Both are special cases of general Lévy diffusion, also known in the physical literature as anomalous diffusions.

After preliminary materials in Sects. 1 and 2, in Sect. 3 we described similar asymptotic large-time behavior for both α-stable (fractal) and α-Linnik conservation laws, and in Sect. 4 we summarized known facts about shock creation for α-stable conservation laws.

Surprisingly, the shock behavior for α-Linnik conservation laws is dramatically different than for α-stable laws, and the new phenomena appear. Our investigations here are numerical, and given that the tail behavior of the two types of distributions is similar, the temptation is to explain the differences by the strikingly different behavior of the α-stable and α-Linnik distributions at the origin. Whereas the first class is smooth at $x = 0$, the second has a singularity at the same point.

We do not have rigorous proofs of these facts at this point and resolving these issues is part of our future plans. In this context we would like to pose the following formal conjectures (for the type of initial data discussed in Sect. 5) :

Conjecture 1 For an α-stable (fractional) conservation laws with $\alpha < 1$, there exists $t_c > 0$ (obviously greater than the time t_0 of shock creation in Theorem 4.1) such that the solution becomes continuous (and smooth) for all $t > t_c$.

Conjecture 2 For a solution of the α-Linnik conservation law with $\alpha < 1$,there exists a time t_0 such that, at $t > t_0$, the shock is created and its size begins decreasing, but at another time $t_s > t_0$ the size of the shock stops decreasing and the shock begins to move to the left.

Estimating the critical times t_0, and t_s, as well as the speed of the shock movement to the left for $t > t_s$ would also constitute a worthwhile future project.

Conjecture 3 For a solution of the α-Linnik conservation law with $\alpha > 1$,there exists a time t_0 such that, at $t > t_0$, the shock is created and its size begins decreasing, but at another time $t_s > t_0$ the size of the shock stops decreasing.

Another important future project is to study the "anomalous turbulence" problem in the spirit of [39] (or, [21]), that is the behavior of solutions of stable and Linnik conservation laws when the initial data are random fields.

Acknowledgments The authors are grateful to the anonymous reviewers whose thorough reading and extensive comments helped improve the final version of this paper. One of the referees also suggested some very interesting possible avenues of further research in the area that are certainly worth pursuing. The authors are grateful to René Schilling of Technische Universitaet Dresden for careful reading of the manuscript and comments that led to significant improvements of the article.

Appendix: A Numerical Method for Stable and Linnik Conservation Laws

In this section, we present the details of the numerical method used to produce results of Sect. 6, adapting to the Linnik case the methodology developed in [12].

Let $\delta_t > 0$, and $\delta x > 0$ be the time and space steps. The scheme consists in computing approximate values u_i^n of the solution to (4.1) on the lattice $[n\delta t, (n+1)\delta t) \times [i\delta x, (i+1)\delta x)$,

$n \in \mathbb{N}$, and $i \in \mathbb{Z}$,

$$u_i^0 = \frac{1}{\delta x} \int_{i\delta x}^{(i+1)\delta x} u_o(x)dx, \tag{6.1}$$

$$\frac{\delta t}{\delta x}\left(u_i^{n+1} - u_i^n\right) + F\left(u_i^n, u_{i+1}^n\right) - F\left(u_{i-1}^n, u_i^n\right) + \delta x \mathcal{L}^{\delta x}[u^{n+1}]_i = 0, \tag{6.2}$$

where F is a numerical Burgers flux corresponding to the continuous flux f, and $\mathcal{L}^{\delta x}$ is a discretization of the non-local term $\mathcal{L}$, where $\mathcal{L}^{\delta x}$ is the discretization of $\mathcal{L}$, and the numerical flux is defined as follows:

$$F(a, b) = \begin{cases} \min\limits_{a \le u \le b} \dfrac{u^2}{2}, & \text{if } a \le b; \\ \max\limits_{b \le u \le a} \dfrac{u^2}{2}, & \text{if } a > b. \end{cases}$$

The assumptions on $\mathcal{L}^{\delta x}$ are as follows:

(i) $l^\infty(\mathbb{Z}) \ni \nu \mapsto \mathcal{L}^{\delta x}[\nu] \in l^\infty(\mathbb{Z})$ is linear;

(ii) $\forall \nu \in l^\infty(\mathbb{Z})$, if $(i_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{k\to\infty} \nu_{i_k} = \sup_{j\in\mathbb{Z}} \nu_j$, then $\liminf_{k\to\infty} \mathcal{L}^{\delta x}[\nu]_{i_k} \ge 0$;

(iii) If $\tau : l^\infty(\mathbb{Z}) \mapsto l^\infty(\mathbb{Z})$ is the left translation $\tau(\nu)_i = \nu_{i+1}$, then $\tau \mathcal{L}^{\delta x} = \mathcal{L}^{\delta x} \tau$;

(iv) $\exists A^{\delta x} > 0$ such that, for all $\nu \in l^\infty(\mathbb{Z})$, $\mathcal{L}^{\delta x}[\nu]_0$ only depends on $\left(\nu_j\right)_{|j| \le A^{\delta x}}$.

A detailed description of the implementation of the numerical algorithm is provided below.

For a chosen space step $\delta x > 0$, formula (4.2) makes it easy to write a discretization of $\mathcal{L}$: we approximate each integral using the basic quadratic rule on the mesh $([j\delta x, (j+1)\delta x))_{j\in\mathbb{Z}}$, and we use the finite difference approximation of the derivative. However, such an approximation would use all of $(\nu_j)_{j\in\mathbb{Z}}$ in order to compute $\mathcal{L}^{\delta x}[\nu]_i$; in practical applications, the considered functions are usually constant near $-\infty$ and $+\infty$. We take this into account when discretizating $\mathcal{L}$ and use the mesh $([j\delta x, (j+1)\delta x))_{j\in\mathbb{Z}}$ only up to $|z| = J_{\delta x}\delta x$ (for some integer $J_{\delta x}$ such that $J_{\delta x}\delta x \to \infty$ as $\delta x \to 0$), approximating the remaining parts with two unbounded space steps $(-\infty, -J_{\delta x}\delta x]$ and $[J_{\delta x}\delta x, +\infty)$. This leads to the scheme,

$$\begin{aligned} \mathcal{L}^{\delta x}[\nu]_i = &-c(\alpha) \sum_{0<|j|<r/\delta x} \delta x \left(\nu_{i+j} - \nu_i - \frac{\nu_{i+1} - \nu_{i-1}}{2\delta x}\right) \Lambda'(j\delta x) \\ &-c(\alpha) \sum_{r/\delta x<|j|\le J_{\delta x}} \delta x \left(\nu_{i+j} - \nu_i\right) \Lambda'(j\delta x) \\ &-c(\alpha) \sum_{j<-J_{\delta x}} \delta x \left(\nu_{i-J_{\delta x}-1} - \nu_i\right) \Lambda'(j\delta x) \\ &-c(\alpha) \sum_{j>J_{\delta x}} \delta x \left(\nu_{i+J_{\delta x}+1} - \nu_i\right) \Lambda'(j\delta x) \end{aligned} \tag{6.3}$$

where

$$\Lambda'(j\delta x) = \frac{\alpha}{2|j\delta x|} \mathbb{E} \exp\left(-\left|\frac{j\delta x}{X}\right|^\alpha\right),$$

and X has a α-stable distribution.

Additionally, we can estimate the approximate value of $\sum_{j>J_{\delta x}} \Lambda^{'}(j\delta x)$ using the formula (2.22) for the Lévy measure of α-Linnik distribution,

$$\begin{aligned}
&\int_{J_{\delta x}}^{\infty}\int_{0}^{\infty} f_\alpha\left(\frac{j\delta x}{w^{1/\alpha}}\right).\frac{e^{-w}}{w^{1+1/\alpha}}dwdj \\
&= \int_{0}^{\infty}\int_{J_{\delta x}}^{\infty} f_\alpha\left(\frac{j\delta x}{w^{1/\alpha}}\right)dj.\frac{e^{-w}}{w^{1+1/\lambda}}dw \\
&\sim \int_{0}^{\infty}\int_{J_{\delta x}}^{\infty} \left(\frac{w^{1/\alpha}}{\delta x}\right) f_\alpha\left(\frac{j\delta x}{w^{1/\alpha}}\right)d\left(\frac{j\delta x}{w^{1/\alpha}}\right)\frac{e^{-w}}{w^{1+1/\alpha}}dw \\
&= \int_{0}^{\infty}\left(\frac{w^{1/\lambda}}{\delta x}\right).\left(\frac{k_\alpha}{\left(\frac{j\delta x}{w^{1/\alpha}}\right)^{\alpha}}\right).\frac{e^{-w}}{w^{1+1/\alpha}}dw \\
&= \int_{0}^{\infty}\frac{k_\alpha}{\delta x^{\alpha+1}J_{\delta x}^{\alpha}}e^{-w}dx \\
&= \frac{k_\alpha}{\delta x^{\alpha+1}J_{\delta x}^{\alpha}},
\end{aligned}$$

where $k_\alpha = \sin(\frac{\pi\alpha}{2})\Gamma(\alpha)/\pi$. Therefore, the approximate value of $\mathcal{L}^{\delta x}[\nu]_i$ is given by the following formula,

$$\begin{aligned}
\mathcal{L}^{\delta x}[\nu]_i = &-c(\alpha)\sum_{0<|j|<r/\delta x}\delta x\left(\nu_{i+j}-\nu_i\right)\frac{\alpha}{2|j\delta x|}\mathbb{E}\exp\left(-\left|\frac{j\delta x}{X}\right|^{\alpha}\right) \\
&-c(\alpha)\left(\nu_{i-J_{\delta x}-1}-\nu_i\right)\frac{k_\alpha}{(\delta x J_{\delta x})^{\alpha}}-c(\alpha)\left(\nu_{i+J_{\delta x}+1}-\nu_i\right)\frac{k_\alpha}{(\delta x J_{\delta x})^{\alpha}}
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\mathcal{L}^{\delta x}[\nu]_i = &-c(\alpha)\sum_{0<|j|<r/\delta x}\delta x\left(\nu_{i+j}-\nu_i\right)\frac{\alpha}{2|j\delta x|}\mathbb{E}\exp\left(-\left|\frac{j\delta x}{X}\right|^{\alpha}\right). \\
&-c(\alpha)\left(\nu_{i-J_{\delta x}-1}-\nu_i\right)\frac{\sin(\frac{\pi\alpha}{2})\Gamma(\alpha)}{\pi(\delta x J_{\delta x})^{\alpha}}-c(\alpha)\left(\nu_{i+J_{\delta x}+1}-\nu_i\right)\frac{\sin(\frac{\pi\alpha}{2})\Gamma(\alpha)}{\pi(\delta x J_{\delta x})^{\alpha}}.
\end{aligned} \tag{6.4}$$

The step-by-step numerical algorithm for computation of the solution of the Linnik conservation law is as follows:

Step 1: Define $\nu,\ W,\ \mathcal{L}^{\delta x}$ and $G^{\delta x}$, by the following formulae:

(i)

$$\nu = \left(h_i^n 1_{\mathbb{Z}\setminus[-m,m]}(i)\right)_{|i|\in\mathbb{Z}}.$$

For example, for $m = 2$,

$$\nu^T = \left[\ldots h_{-4}^n, h_{-3}^n, 0, 0, 0, 0, 0, h_3^n, h_4^n, \ldots\right]$$

so that $\nu_2 = 0,\quad \nu_{-4} = h_{-4}^n$, etc.

(ii)

$$W = \left(U_i^{n+1}\right)_{|i|\le m} = \begin{pmatrix} U_{-m}^{n+1} \\ \vdots \\ U_0^{n+1} \\ \vdots \\ U_m^{n+1} \end{pmatrix}_{(2m+1)\times 1}$$

(iii)

$$\begin{aligned}
\mathcal{L}^{\delta x}[\nu]_i &= -c(\alpha) \sum_{0<|j|\le J_{\delta x}} \delta x(\nu_{i+j} - \nu_i)\Lambda^{'}(j\delta x) \\
&\quad -c(\alpha)(\nu_{i-J_{\delta x}-1} - \nu_i)\frac{\sin(\frac{\pi\alpha}{2})\Gamma(\alpha)}{\pi(\delta x J_{\delta x})^{\alpha}} \\
&\quad -c(\alpha)(\nu_{i+J_{\delta x}+1} - \nu_i)\frac{\sin(\frac{\pi\alpha}{2})\Gamma(\alpha)}{\pi(\delta x J_{\delta x})^{\alpha}} \\
&= -c(\alpha) \sum_{0<|j|\le J_{\delta x}} \delta x(h_{i+j}^n - h_i^n)\Lambda^{'}(j\delta x) - c(\alpha)(h_{i-J_{\delta x}-1}^n - h_i^n)\frac{\sin(\frac{\pi\alpha}{2})\Gamma(\alpha)}{\pi(\delta x J_{\delta x})^{\alpha}} \\
&\quad -c(\alpha)(h_{i+J_{\delta x}+1}^n - h_i^n)\frac{\sin(\frac{\pi\alpha}{2})\Gamma(\alpha)}{\pi(\delta x J_{\delta x})^{\alpha}} \\
&= c_1 \sum_{0<|j|\le J_{\delta x}} (h_{i+j}^n - h_i^n)\Lambda^{'}(j\delta x) + c_2\left(h_{i-J_{\delta x}-1}^n - h_i^n\right) \\
&\quad + c2\left(h_{i+J_{\delta x}+1}^n - h_i^n\right),
\end{aligned}$$

where $c_1 = -c(\alpha)$, and $c_2 = -c(\alpha)\sin(\frac{\pi\alpha}{2})\Gamma(\alpha)/(\pi(\delta x J_{\delta x})^{\alpha})$.

(iv) The formula for the 1st column of the symmetric Toeplitz matrix $G^{\delta x}$ is given below. Recall that an $(n \times n)$ matrix A is said to be *Toeplitz* if it has the form

$$A = [a_{j-k}]_{j,k=1}^n \tag{6.5}$$

The entries along each diagonal of a Toeplitz matrix are constant. The 1st column of a symmetric Toeplitz matrix $G^{\delta x}$ is as follows:

$$G(1) = \begin{pmatrix} -2c1\sum_{j=1}^{1500}\Lambda^{'}(j\delta x) - 2c2 \\ \Lambda^{'}(1.\delta x) \\ \Lambda^{'}(2.\delta x) \\ \Lambda^{'}(3.\delta x) \\ \vdots \\ \Lambda^{'}(1500.\delta x) \end{pmatrix}_{1501\times 1}$$

and the 1st column of the symmetric, positive definite Toeplitz matrix $\left(I + \delta t G^{\delta x}\right)$

$$\left(I + \delta t G^{\delta x}\right)(1) = \begin{pmatrix} 1 - 2c1.\delta t \sum_{j=1}^{1500} \Lambda^{'}(j\delta x) - 2c2.\delta t \\ \delta t.\Lambda^{'}(1.\delta x) \\ \delta t.\Lambda^{'}(2.\delta x) \\ \delta t.\Lambda^{'}(3.\delta x) \\ \vdots \\ \delta t.\Lambda^{'}(1500.\delta x) \end{pmatrix}_{1501\times 1}$$

Now, the complete symmetric Toeplitz matrix now can be found because it is determined by the first column.

(v) With the numerical Burgers flux defined at the beginning of this Appendix, we have:

$$F(a,b) = \begin{cases} a^2/2, & \text{if} a, b > 0; \\ b^2/2, & \text{if} a, b < 0; \\ 0, & \text{if} a < 0, b > 0; \\ b^2/2, & \text{if } a > 0, b < 0, |a| < |b|; \\ a^2/2, & \text{otherwise.} \end{cases}$$

Step 2: Assume

$$u_i^0 = \begin{cases} -1, & \text{if } i \geq 0; \\ 1, & \text{if } i < 0. \end{cases}$$

and $\delta t = 0.00167, \ m = 750.$

Step 3: For each i, find h_i^0 , and use the equation

$$h_i^n = \begin{cases} u_i^n + \frac{\delta t}{\delta x} F(u_{i-1}^n, u_i^n) - \frac{\delta t}{\delta x} F(u_i^n, u_{i+1}^n), & \text{if } 3001 \leq i \leq 4501; \\ u_{i+1}^n, & \text{otherwise.} \end{cases}$$

Since u_i^0 are known for all i, we can calculate h_i^0 for all i.

If ν and W are defined as in **Step 1**, the above equation reduces to a square system of size $2m + 1$ on W:

$$W + \delta t G^{\delta x} W = \left(h_i^0 - \delta t g^{\delta x}[\nu]_i\right)_{|i| \leq 750}$$

Step 4: Find u_i^1, for all i. A Toeplitz matrix A (see, Step 1) is said to be circulant if it has the form

$$a_{-k} = a_{n-k}, \quad 1 \leq k \leq n-1. \tag{6.6}$$

Obviously, any circulant matrix is uniquely determined by its first column,

$$a = [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}]^T.$$

Next, we employ the equation (6.5) to calculate u_i^1, for all i, using the standard preconditioned conjugate gradient method, where A is the symmetric positive definite matrix $I + \delta t G^{\delta x}$, of dimension (1501 × 1501) (defined in Step 1), and **x= W** , is as defined in the Step 1. Taking $n = 0$, we have

$$\mathbf{x} = \left(U_i^1\right)_{|i|\leq 750} = \begin{pmatrix} U_{-750}^1 \\ \vdots \\ U_0^1 \\ \vdots \\ U_{750}^1 \end{pmatrix}_{1501\times 1}$$

The vector

$$\mathbf{b} = \left(h_i^0 - \delta t \mathcal{L}^{\delta x}[\nu]_i\right)_{|i|\leq 750},$$

defined by equation (6.5) has the dimension 1501×1. Here, we know the values of h_i^0 for all i, and the formula for $\mathcal{L}^{\delta x}[\nu]_i$ is defined in Step 1. For example,

$$\begin{aligned}\mathcal{L}^{\delta x}[\nu]_{-750} &= c1 \sum_{0<|j|\leq 1500} \left(h_{-750+j}^0 - h_{-750}^0\right) \Lambda^{'}(j\delta x) + c2\left(h_{-750-1500-1}^0 - h_{-750}^0\right) \\ &\quad + c2\left(h_{-750+1500+1}^0 - h_{-750}^0\right)\end{aligned}$$

Step 5: Find h_i^1, for all i. Since we already know u_i^1, the values for h_i^1, for all i, can be calculated by going back to Step 3. Then we move forward to Step 4 to calculate u_i^2 for all i. Finally we need repeat the procedure to calculate the values of u_i, $i = 1, \ldots, n$, for $n = 300$.

References

1. Alibaud, N., Droniou, J., Vovelle, J.: Occurrence and non-appearance of shocks in fractal Buergers equations. J. Hyperbolic Diff. Equ. **4**(3), 479–499 (2007)
2. Applebaum, D.: Lévy Processes and Stochastic Calculus, 2nd edn. Cambridge University Press, Cambridge (2009)
3. Bertoin, J.: Lévy Processes: Theory and Applications. Cambridge University Press, Cambridge (1996)
4. Biler, P., Funaki, T., Woyczyński, W.A.: Fractal Burgers equations. J. Differ. Equ. **148**, 9–46 (1998)
5. Biler, P., Karch, G., Woyczyński, W.A.: Asymptotics of multifractal conservation laws. Studia Math. **135**, 231–252 (1999)
6. Biler, P., Karch, G., Woyczyński, W.A.: Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws. Ann. Inst. H. Poincaré Nonlineaire Anal. **18**, 613–637 (2001)
7. Biler, P., Karch, G., Woyczyński, W.A.: Asymptotics for conservation laws involving Lévy diffusion generators. Studia Math. **148**, 171–192 (2001)
8. Böttcher, B., Schilling, R.L., Wang, J.: Lévy Matters III. Lévy-Type processes: Construction, Approximation and Sample Path Properties. Springer-Verlag, Berlin, New York (2013)
9. Burnecki, K., Janczura, J., Magdziarz, M., Weron, A.: Can one see a competition between subdiffusion and Lévy flights? A case of geometric-stable noise. Acta Phys. Pol. B **39**, 1043–1053 (2008)
10. Cont, R., Tankov, P.: Financial Modelling with Jump Processes. Chapman & Hall, Boca Raton (2012)
11. Devroye, L.: A note on Linnik's distributions. Stat. Probab. Lett. **9**, 305–306 (1990)
12. Droniou, J.: A numerical method for fractal conservation laws. Math. Comput. **79**(269), 95–124 (2010)
13. Droniou, J., Imbert, C.: Fractal first order partial differential equations. Arch. Ration. Mech. Anal. **182**, 299–331 (2006)
14. Droniou, J., Gallouet, T., Vovelle, J.: Global solutions and smoothing effect for a non-local regularization of an hyperbolic equation. J. Evol. Equ. **3**, 499–521 (2003)
15. Erdogan, M.B., Ostrovskii, I.V.: Analytic and asymptotic properties of generalized Linnik probability densities. J. Math. Anal. Appl. **217**, 555–578 (1998)
16. Górska, K., Woyczyński, W.A.: Explicit representations for multiscale Lvy processes, and asymptotics of multifractal conservation laws. arXiv:1502.0027 (2015)
17. Imbert, C., Souganidis, P.E.: Phasefield theory for fractional diffusion-reaction equations and applications. arXiv:0907.5524 (2009)
18. Jacob, N.: Pseudo-Differential Operators and Markov Processes, vol. 1–3. Imperial College Press, London (2002–2005)

19. Jacob, N., Schilling, R.: Lévy-type processes and pseudo differential operators. In: Barndorff-Nielsen, O., Mikosch, T., Resnick, S. (eds.) Lévy Processes. Bikhäuser, Boston (2001)
20. Janicki, A., Weron, A.: Simulation and Chaotic Behavior of α-Stable Stochastic Processes. Marcel Dekker Inc, New York (1994)
21. Janicki, A., Woyczynski, W.A.: Hausdorff dimension of regular points in stochastic Burgers flows with Lévy alpha-stable initial data. J. Stat. Phys. **88**, 277–299 (1997)
22. Jourdain, B., Méléard, S., Woyczynski, W.A.: Probabilistic approximation and inviscid limits for one-dimensional fractional conservation laws. Bernoulli **11**, 689–714 (2005)
23. Karch, G., Woyczynski, W.A.: Fractal Hamilton-Jacobi-KPZ equations. Trans. Am. Math. Soc. **360**, 2423–2442 (2008)
24. Kotz, S., Kozubowski, T.J., Hayfavi, A.: Analytic and asymptotic properties of Linnik's probability densities. J. Math. Anal. Appl. **193**, 353–371, 497–521 (1995)
25. Kotz, S., Kozubowski, T.J., Podgórski, K.: The Laplace Distribution and Generalizations. Springer, New York (2001)
26. Kozubowski, T.J.: The inner characterization of geometric stable laws. Stat. Decis. **12**, 307–321 (1994)
27. Kozubowski, T.J., Rachev, S.T.: Univariate geometric stable laws. J. Comput. Anal. Appl. **1**(2), 177–217 (1999)
28. Linnik, J.V.: Linear forms and statistical criteria, I, II. Select. Transl. Math. Statist. Probab **3**, 1–90 (1963)
29. Mann, J.A., Woyczyński, W.A.: Growing fractal interfaces in the presence of self-similar hopping surface diffusion. Phys. A **291**, 159–183 (2001)
30. Meerschaert, M.M., Sikorskii, A.: Stochastic Models for Fractional Calculus. De Gruyter Studies in Mathematics, vol. 43. Walter de Gruyter, Berlin (2012)
31. Metzler, R., Klafter, J.: The random walk's guide to anomalous diffusion; a fractional dynamics approach. Phys. Rep. **339**, 1–77 (2000)
32. Pakes, A.G.: Mixture representations for symmetric generalized Linnik laws. Statist. Probab. Lett. **37**, 213–221 (1998)
33. Piryatinska, A., Saichev, A.I., Woyczynski, W.A.: Models of anomalous diffusion: the subdiffusive case. Phys. A **349**, 375–424 (2005)
34. Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance. Chapman & Hall, New York (1994)
35. Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999)
36. Schoutens, W.: Levy Processes in Finance. Wiley, Chichester (2003)
37. Smoller. J.: Shock Waves and Reaction—Diffusion Equations, 2nd ed. Springer-Verlag, New York (1994)
38. Uchaykin, V.V., Zolotarev, V.M.: Chance and Stability. Stable Distributions and Their Applications. VSP, Utrecht (1999)
39. Woyczyński, W.A.: Burgers-KPZ Turbulence: Göttingen Lectures. Springer, New York (1998)
40. Woyczyński, W.A.: Lévy Processes in the physical sciences. In: Barndorff-Nielsen, O., Mikosch, T., Resnick, S. (eds.) Lévy Processes. Bikhäuser, Boston (2001)
41. Zuazua, E.: Weakly nonlinear large time behavior in scalar convection-diffusion equations. Differ. Integr. Equ. **6**, 1481–1491 (1993)