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Golden–Thompson's Inequality for Deformed Exponentials

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Abstract Deformed logarithms and their inverse functions, the deformed exponentials, are important tools in the theory of non-additive entropies and non-extensive statistical mechanics. We formulate and prove counterparts of Golden–Thompson's trace inequality for q-exponentials with parameter q in the interval $[1, 3]$.

Keywords Deformed exponentials · Golden–Thompson's trace inequality · Concave trace function

Mathematics Subject Classification 47A63

1 Introduction and Main Result

Tsallis [7] generalised in 1988 the standard Bolzmann–Gibbs entropy to a non-extensive quantity S_q depending on a parameter q. In the quantum version it is given by

$$S_q(\rho) = \frac{1 - \operatorname{Tr} \rho^q}{q - 1} \qquad q \neq 1,$$

where ρ is a density matrix. It has the property that $S_q(\rho) \to S(\rho)$ for $q \to 1$, where $S(\rho) = -\operatorname{Tr} \rho \log \rho$ is the von Neumann entropy. The Tsallis entropy may be written on a similar form

$$S_q(\rho) = -\operatorname{Tr} \rho \log_q(\rho),$$

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where the deformed logarithm $\log_q$ is given by

$$\log_q x = \int_1^x t^{q-2}\,dt = \begin{cases} \dfrac{x^{q-1}-1}{q-1} & q > 1 \\ \log x & q = 1 \end{cases}$$

for $x > 0$. The deformed logarithm is also denoted the q-logarithm. The inverse function $\exp_q$ is called the q-exponential and is given by

$$\exp_q(x) = (x(q-1)+1)^{1/(q-1)} \qquad \text{for} \quad x > \frac{-1}{q-1}\,.$$

The q-logarithm and the q-exponential functions converge, respectively, to the logarithmic and the exponential functions for $q \to 1$.

The aim of this article is to generalise Golden–Thompson's trace inequality [2,6] to deformed exponentials. The main result is the following:

Theorem 1.1 *Let A and B be positive definite matrices.*

(i) *If $1 \le q < 2$ then*

$$\operatorname{Tr}\,\exp_q(A+B) \le \operatorname{Tr}\,\exp_q(A)^{2-q}\big(A(q-1)+\exp_q B\big).$$

(ii) *If $2 \le q \le 3$ then*

$$\operatorname{Tr}\,\exp_q(A+B) \ge \operatorname{Tr}\,\exp_q(A)^{2-q}\big(A(q-1)+\exp_q B\big).$$

Notice that for $q = 1$ we recover Golden–Thomson's trace inequality

$$\operatorname{Tr}\,\exp(A+B) \le \operatorname{Tr}\,\exp(A)\exp(B).$$

This inequality is valid for arbitrary self-adjoint matrices A and B. However, it is sufficient to know the inequality for positive definite matrices, since the general form follows by multiplication with positive numbers.

2 Preliminaries

We collect a few well-known results that we are going to use in the proof of the main theorem. The q-logarithm is a bijection of the positive half-line onto the open interval $(-(q-1)^{-1}, \infty)$, and the q-exponential is consequently a bijection of the interval $(-(q-1)^{-1}, \infty)$ onto the positive half-line. For $q > 1$ we may thus safely apply both the q-logarithm and the q-exponential to positive definite operators. We also notice that

$$\frac{d}{dx}\log_q(x) = x^{q-2} \qquad \text{and} \qquad \frac{d}{dx}\exp_q(x) = \exp_q(x)^{2-q}\,. \tag{1}$$

The proof of the following lemma is rather easy and may be found in [4, Lemma 5].

Lemma 2.1 *Let $\varphi\colon \mathcal{D} \to \mathcal{A}_{sa}$ be a map defined in a convex cone $\mathcal{D}$ in a Banach space X with values in the self-adjoint part of a C^*-algebra $\mathcal{A}$. If φ is Fréchet differentiable, convex and positively homogeneous then*

$$d\varphi(x)h \le \varphi(h).$$

for $x, h \in \mathcal{D}$.

Let H be any $n \times n$ matrix. The map

$$A \to \mathrm{Tr}\ \left(H^* A^p H\right)^{1/p},$$

defined in positive definite $n \times n$ matrices, is concave for $0 < p \le 1$ and convex for $1 \le p \le 2$, cf. [1, Theorem 1.1]. By a slight modification of the construction given in Remark 3.2 in the same reference, cf. also [3], we obtain that the mapping

$$(A_1, \ldots, A_k) \to \mathrm{Tr}\ \left(H_1^* A_1^p H_1 + \cdots + H_k^* A_k H_k\right)^{1/p}, \tag{2}$$

defined in k-tuples of positive definite $n \times n$ matrices, is concave for $0 < p \le 1$ and convex for $1 \le p \le 2$; for arbitrary $n \times n$ matrices $H_1, \ldots, H_k$.

3 Deformed Trace Functions

Theorem 3.1 *Let $H_1, \ldots, H_k$ be matrices with $H_1^* H_1 + \cdots + H_k^* H_k = 1$ and define the function*

$$\varphi(A_1, \ldots, A_k) = \mathrm{Tr}\ \exp_q\left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i\right) \tag{3}$$

in k-tuples of positive definite matrices. Then φ is positively homogeneous of degree one. It is concave for $1 \le q \le 2$ and convex for $2 \le q \le 3$.

Proof For $q > 1$ we obtain

$$\begin{aligned}
\varphi(A_1, \ldots, A_k) &= \mathrm{Tr}\ \exp_q\left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i\right) \\
&= \mathrm{Tr}\left(1 + (q-1)\sum_{i=1}^k H_i^* \log_q(A_i) H_i\right)^{1/(q-1)} \\
&= \mathrm{Tr}\left(1 + (q-1)\sum_{i=1}^k H_i^* \frac{A_i^{q-1} - 1}{q-1} H_i\right)^{1/(q-1)} \\
&= \mathrm{Tr}\left(1 + \sum_{i=1}^k H_i^* (A_i^{q-1} - 1) H_i\right)^{1/(q-1)} \\
&= \mathrm{Tr}\ \left(H_1^* A_1^{q-1} H_1 + \cdots + H_k^* A_k^{q-1} H_k\right)^{1/(q-1)}.
\end{aligned}$$

From this identity it follows that φ is positively homogeneous of degree one. The concavity for $1 < q \le 2$ and the convexity for $2 \le q \le 3$ now follows from (2). The statement for $q = 1$ follows by letting q tend to one. □

Corollary 3.2 *Let L be a positive definite matrix, and let $H_1, \ldots, H_k$ be matrices such that $H_1^* H_1 + \cdots + H_k^* H_k \le 1$. Then the function*

$$\varphi(A_1, \ldots, A_k) = \mathrm{Tr}\ \exp_q\left(L + H_1^* \log_q(A_1) H_1 + \cdots + H_k^* \log_q(A_k) H_k\right),$$

defined in k-tuples of positive definite matrices, is concave for $1 \le q \le 2$ and convex for $2 \le q \le 3$.

Proof We may without loss of generality assume $H_1^* H_1 + \cdots + H_k^* H_k < 1$ and put $H_{k+1} = \big(1 - (H_1^* H_1 + \cdots + H_k^* H_k)\big)^{1/2}$. We then have

$$H_1^* H_1 + \cdots + H_k^* H_k + H_{k+1}^* H_{k+1} = 1$$

and may use the preceding theorem to conclude that the function

$$(A_1, \ldots, A_{k+1}) \to \operatorname{Tr} \exp_q \left(H_1^* \log_q(A_1) H_1 + \cdots + H_{k+1}^* \log_q(A_{k+1}) H_{k+1}\right)$$

of $k+1$ variables is concave for $1 \le q \le 2$ and convex for $2 \le q \le 3$. Since H_{k+1} is invertible we may choose

$$A_{k+1} = \exp_q \left(H_{k+1}^{-1} L H_{k+1}^{-1}\right)$$

which makes sense since $H_{k+1}^{-1} L H_{k+1}^{-1}$ is positive definite. Concavity for $1 \le q \le 2$ and convexity for $2 \le q \le 3$ in the first k variables of the above function then yields the result. □

Setting $q = 1$ we recover in particular [5, Theorem 3].

Corollary 3.3 *Let $H_1, \ldots, H_k$ be matrices with $H_1^* H_1 + \cdots + H_k^* H_k \le 1$, and let L be self-adjoint. The trace function*

$$(A_1, \ldots, A_k) \to \operatorname{Tr} \exp\left(L + H_1^* \log(A_1) H_1 + \cdots + H_k^* \log(A_k) H_k\right)$$

is concave in positive definite matrices.

Corollary 3.4 *The trace function φ defined in* (3) *satisfies*

$$\varphi(B_1, \ldots, B_k) \le \operatorname{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i\right)^{2-q} \sum_{j=1}^k H_j^* (d\log_q(A_j) B_j) H_j$$

for $1 \le q \le 2$ and

$$\varphi(B_1, \ldots, B_k) \ge \operatorname{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i\right)^{2-q} \sum_{j=1}^k H_j^* (d\log_q(A_j) B_j) H_j$$

for $2 \le q \le 3$, where $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ are positive definite matrices.

Proof For $1 \le q \le 2$ we obtain

$$d\varphi(A_1, \ldots, A_k)(B_1, \ldots, B_k) \ge \varphi(B_1, \ldots, B_k)$$

by Lemma 2.1. By the chain rule for Fréchet differentiable mappings between Banach spaces we therefore obtain

$$\begin{aligned}
\varphi(B_1, \ldots, B_k) &\le \sum_{j=1}^k d_j \varphi(A_1, \ldots, A_k) B_j \\
&= \sum_{j=1}^k \operatorname{Tr} d\exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i\right) H_j^* (d\log_q(A_j) B_j) H_j \\
&= \sum_{j=1}^k \operatorname{Tr} \exp_q \left(\sum_{i=1}^k H_i^* \log_q(A_i) H_i\right)^{2-q} H_j^* (d\log_q(A_j) B_j) H_j
\end{aligned}$$

where we used the identity $\operatorname{Tr} df(A)B = \operatorname{Tr} f'(A)B$ valid for differentiable functions. This proves the first assertion. The result for $2 \le q \le 3$ follows similarly. □

4 Proof of the Main Theorem

In order to prove Theorem 1.1 i we set $k = 2$ in Corollary 3.4 and obtain

$$\varphi(B_1, B_2) \le \mathrm{Tr}\; \exp_q(X)^{2-q} \left(H_1^*(d\log_q(A_1)B_1)H_1 + H_2^*(d\log_q(A_2)B_2)H_2\right)$$

for $1 \le q \le 2$ and positive definite matrices A_1, A_2 and B_1, B_2 where

$$X = H_1^* \log_q(A_1)H_1 + H_2^* \log_q(A_2)H_2\,.$$

If we set $A_1 = B_1$ and $A_2 = 1$ the inequality reduces to

$$\varphi(B_1, B_2) \le \mathrm{Tr}\; \exp_q(H_1^* \log_q(B_1)H_1)^{2-q} \Big(H_1^* B_1^{q-1} H_1 + H_2^* B_2 H_2\Big)\,.$$

We now set $H_1 = \varepsilon^{1/2}$ for $0 < \varepsilon < 1$, and to fixed positive definite matrices L_1 and L_2 we choose B_1 and B_2 such that

$$\begin{aligned} L_1 &= H_1^* \log_q(B_1)H_1 = \varepsilon \log_q(B_1)\\ L_2 &= H_2^* \log_q(B_2)H_2 = (1-\varepsilon)\log_q(B_2). \end{aligned}$$

It follows that

$$B_1 = \exp_q\left(\varepsilon^{-1}L_1\right) \qquad \text{and} \qquad B_2 = \exp_q\left((1-\varepsilon)^{-1}L_2\right).$$

Inserting in the inequality we obtain

$$\begin{aligned} \mathrm{Tr}\; \exp_q(L_1 + L_2) &\le \mathrm{Tr}\; \exp_q(L_1)^{2-q} \left(\varepsilon \exp_q(\varepsilon^{-1}L_1)^{q-1} + (1-\varepsilon)\exp_q((1-\varepsilon)^{-1}L_2)\right)\\ &= \mathrm{Tr}\; \exp_q(L_1)^{2-q} \left(L_1(q-1) + \varepsilon + (1-\varepsilon)\exp_q((1-\varepsilon)^{-1}L_2)\right). \end{aligned}$$

This expression decouble L_1 and L_2 and reduces the minimisation problem over ε to the commutative case. We furthermore realise that minimum is obtained by letting ε tend to zero and that

$$\lim_{\varepsilon\to 0}(1-\varepsilon)\exp_q\left((1-\varepsilon)^{-1}L_2\right) = \exp_q(L_2).$$

We finally replace L_1 and L_2 with A and B. This proves the first statement in Theorem 1.1.

The proof of the second statement is virtually identical to the proof of the first. Since now $2 \le q \le 3$ the second inequality in Corollary 3.4 applies. Setting $k = 2$ and applying the same substitutions as in the proof of the first statement we arrive at the inequality

$$\mathrm{Tr}\; \exp_q(L_1 + L_2) \ge \mathrm{Tr}\; \exp_q(L_1)^{2-q} \left(L_1(q-1) + \varepsilon + (1-\varepsilon)\exp_q((1-\varepsilon)^{-1}L_2)\right).$$

Since $2 \le q \le 3$ the function

$$\varepsilon \to \varepsilon + (1-\varepsilon)\exp_q\left((1-\varepsilon)^{-1}L_2\right)$$

is now decreasing, and we thus maximise the right hand side in the above inequality by letting ε tend to zero. This proves the second statement in Theorem 1.1.

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