

Examples of DLR States Which are Not Weak Limits of Finite Volume Gibbs Measures with Deterministic Boundary Conditions

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Abstract We review what is known about the structure of the set of weak limiting states of the Ising and Potts models at low enough temperature, and in particular we prove that the mixture $\frac{1}{2}(\mu^\pm + \mu^\mp)$ of two reflection-symmetric Dobrushin states of the 3-dimensional Ising model at low enough temperature is a Gibbs state which is not a limit of finite-volume measures with deterministic boundary conditions. Finally we point out what the issues are in order to extend the analysis to the Potts model, and give a few conjectures.

Keywords Gibbs states · Ising model · Potts model · Boundary conditions

Mathematics Subject Classification 82B05 · 82B20 · 82B26

1 Introduction

In the end of the 60s, the seminal works of Dobrushin and Lanford-Ruelle [13, 29] describe the equilibrium states of a lattice model of statistical mechanics in the thermodynamic limit as probability measures μ that are solutions of the DLR equation:

$$\mu(\cdot) = \int d\mu(\omega) \gamma_\Lambda(\cdot | \omega), \quad \text{for all finite subsets } \Lambda \text{ of the lattice,}$$

where the probability kernel γ_Λ is the Gibbsian specification associated to the system; see [17]. Under very weak assumptions (at least for bounded spins), it can be shown that the set \mathcal{G} of all DLR states is a non-empty simplex, which contains the (a priori non-convex) set of weak limits of finite-volume Gibbs measures, denoted \mathcal{W} . Moreover, extremal measures of \mathcal{G} , the set of which is denoted $\text{ex}\mathcal{G}$, have the extra property to be weak limits of finite volume measures with boundary conditions that are typical for it, which implies that $\text{ex}\mathcal{G} \subset \mathcal{W}$.

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The analysis of $\text{ex}\mathcal{G}$ is in general a very hard problem which remains essentially open in dimensions 3 and higher, for any nontrivial model, even in perturbative regimes.

This article focuses on the relationship between \mathcal{G} and \mathcal{W} . Although it is clear that $\mathcal{W} \subseteq \mathcal{G}$, it is harder to determine whether $\mathcal{W} = \mathcal{G}$. For example, in [3] the question is mentioned as an open problem. Here we will settle the question by showing that it is not the case. Indeed we will exhibit a (non-extremal) infinite-volume measure of the 3-dimensional Ising model which belongs to $\mathcal{G} \setminus \mathcal{W}$.

Note added. After submitting this paper, I was informed by Y. Higuchi that the result of Theorem 1 was independently found before, and privately communicated to him, by M. Miyamoto, who afterwards also mentioned it in his textbook (only in Japanese) [33].

We now introduce some further notation and define the sets \mathcal{G} and \mathcal{W} in detail for the Ising and Potts models. Let $q, d \in \mathbb{N} \setminus \{0, 1\}$ and $\Omega = \{1, \dots, q\}^{\mathbb{Z}^d}$ be the space of configurations. Let Λ be a finite subset of \mathbb{Z}^d , and $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$ be its complement. The finite-volume Gibbs measure in Λ for the q -state Potts model with boundary conditions $\omega \in \{0, 1, \dots, q\}^{\mathbb{Z}^d}$ and at inverse-temperature $\beta > 0$ is the probability measure on Ω (with the associated product σ -algebra) defined by

$$\mathbb{P}_{q,\beta,\Lambda}^\omega(\sigma) = \begin{cases} \frac{1}{Z_{q,\beta,\Lambda}^\omega} e^{-\beta H_\Lambda^\omega(\sigma)} & \text{if } \sigma_i = \omega_i \text{ for all } i \in \Lambda^c \\ 0 & \text{otherwise,} \end{cases}$$

where the normalization constant $Z_{q,\beta,\Lambda}^\omega$ is the partition function. The Hamiltonian in Λ is given by

$$H_\Lambda^\omega(\sigma) = - \sum_{\substack{i \sim j \\ \{i,j\} \cap \Lambda \neq \emptyset}} \delta_{\sigma_i, \sigma_j}$$

where $i \sim j$ if i and j are nearest neighbors in \mathbb{Z}^d . In the case of pure boundary condition $i \in \{1, \dots, q\}$, meaning that $\omega_x = i$ for every $x \in \Lambda^c$, we denote the measure by $\mathbb{P}_{q,\beta,\Lambda}^i$. In the case of free boundary condition, $\omega_x = 0$ for every $x \in \Lambda^c$, we denote the measure by $\mathbb{P}_{q,\beta,\Lambda}^\emptyset$.

Below we write $\mu_{\beta,\Lambda}^\omega$ for the Ising measure on $\{-1, +1\}^{\mathbb{Z}^d \cap \Lambda}$ with boundary condition ω , that is for $\mathbb{P}_{2,\beta/2,\Lambda}^\omega$ with states 1, 2 identified with $-1, +1$ (the constant $1/2$ in front of β comes from the identity $\delta_{\sigma_i, \sigma_j} = (1 + \sigma_i \sigma_j)/2$ when $\sigma_i, \sigma_j \in \{-1, +1\}$).

For an arbitrary subset A of \mathbb{Z}^d , let \mathcal{F}_A be the σ -algebra generated by spins in A .

Definition 1 A probability measure \mathbb{P} on Ω is an infinite-volume DLR state for the q -state Potts model at inverse temperature β if and only if it satisfies the following DLR condition:

$$\mathbb{P}(\cdot | \mathcal{F}_{\Lambda^c})(\omega) = \mathbb{P}_{q,\beta,\Lambda}^\omega \quad \text{for } \mathbb{P} - \text{ a.e. } \omega, \text{ and all finite subsets } \Lambda \text{ of } \mathbb{Z}^d. \tag{1}$$

Let $\mathcal{G}_{q,\beta}$ be the space of infinite-volume DLR states for the q -state Potts model. This set being a simplex [17], let $\text{ex}\mathcal{G}_{q,\beta}$ denote the set of its extremal points. Let $\text{tr}\mathcal{G}_{q,\beta}$ denote the set of translation invariant DLR states, namely measures $\mathbb{P} \in \mathcal{G}_{q,\beta}$ such that $\mathbb{P}(f \circ \tau) = \mathbb{P}(f)$ for all local functions f and all translations τ of \mathbb{Z}^d .

We also formally define the (in principle smaller) set of Gibbs states which can be obtained via boundary conditions as follows:

Definition 2 A probability measure \mathbb{P} on Ω is a weak-limiting Gibbs state for the q -state Potts model at inverse temperature β if:

$$\text{for all local functions } f, \quad \mathbb{P}(f) = \lim_{\Lambda_n \uparrow \mathbb{Z}^d} \mathbb{P}_{q,\beta,\Lambda_n}^{\omega_n}(f)$$

for some sequence of finite volumes $(\Lambda_n)_n \uparrow \mathbb{Z}^d$ and of deterministic boundary conditions $(\omega_n)_n \in \Omega$. We write $\mathbb{P} = \lim_{n \rightarrow \infty} \mathbb{P}_{q,\beta,\Lambda_n}^{\omega_n}$. Let $\mathcal{W}_{q,\beta}$ be the space of weak-limiting Gibbs states for the q -state Potts model.

The non-emptiness of the set of DLR states follows from a compactness argument in general [17], but for the Potts model this can be proved constructively. For $i \in \{1, \dots, q\}$, the weak limits $\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}_{q,\beta,\Lambda}^i$ exist and belong to $\mathcal{G}_{q,\beta}$ (in particular, the limit does not depend on the sequence of boxes chosen); this follows easily, e.g., from the random cluster representation [22]. We denote by $\mathbb{P}_{q,\beta}^i$ the corresponding limit. It can be checked [18, Prop. 6.9] that the phases $\mathbb{P}_{q,\beta}^i$ are translation invariant.

When β is less than the critical inverse temperature $\beta_c = \beta_c(q, d)$ (which is non-trivial for $d \geq 2$), it is known that there exists a unique infinite-volume Gibbs measure. The relevant values of β for a study of $\mathcal{G}_{q,\beta}$ are thus $\beta \geq \beta_c(q, d)$.

1.1 The Case of the Ising Model

Let μ^+ and μ^- be the two pure phases (that is, translation-invariant extremal Gibbs measures) of the Ising model. Let μ^\pm denote the limiting infinite-volume Gibbs state for the Dobrushin boundary condition ω^\pm such that $\omega_{(x,y,z)}^\pm = +1$ for $z \geq 0$ and -1 for $z < 0$. We write μ^\mp for the ‘‘spin flip’’ of μ^\pm , namely the measure symmetric with respect to the plane $z = -1/2$.

1.1.1 Dimension 2

In the beginning of the 80s, Aizenman [2] and Higuchi [25] proved independently that the DLR states of the 2d Ising model are all convex combination of the pure phases, namely, for any $\beta \geq 0$,

$$\mathcal{G}_{2,\beta} = \{\alpha\mu^+ + (1 - \alpha)\mu^- : \alpha \in [0, 1]\}. \tag{2}$$

In particular, all the DLR states are translation invariant: $\text{tr}\mathcal{G}_{2,\beta} = \mathcal{G}_{2,\beta}$.

Gallavotti [16] proved, by studying the fluctuations of the Dobrushin interface, that the corresponding weak limiting state μ^\pm is the mixture $\frac{1}{2}(\mu^+ + \mu^-)$. This was refined by Higuchi [24], who proved that the interface, after diffusive scaling, weakly converges to a Brownian bridge at sufficiently low temperatures. These two results were then pushed to all subcritical temperatures by, respectively, Messager and Miracle-Sole [31] and Greenberg and Ioffe [21].

By exploiting the Gaussian scaling of the Dobrushin interface, Abraham and Reed [1] produced a set of deterministic boundary conditions $(\omega_\alpha)_{\alpha \in (0,1)}$ such that $\lim_{n \rightarrow \infty} \mu_{\Lambda_n}^{\omega_\alpha} = \alpha\mu^+ + (1 - \alpha)\mu^-$. Basically, they shift up the Dobrushin boundary condition by an amount $C_\alpha\sqrt{n}$ around the cubic box of size n , and choose the right constant C_α to get the mixture with proportion α of μ^+ . These results imply that the weak limiting states and the DLR states of the Ising model coincide in 2 dimensions: for any $\beta \geq 0$,

$$\mathcal{W}_{2,\beta} = \mathcal{G}_{2,\beta}.$$

Note that the behavior of the macroscopic interfaces induced by an arbitrary boundary condition was studied in [12]. We refer to [7] for a review on the microscopic theory of equilibrium crystal shapes.

1.1.2 Dimension 3 (and More)

The existence of non-translation invariant states in dimension 3 and more was discovered by Dobrushin [14]. He proved that, at low enough temperatures, the interface created under $\mu_{\Lambda_n}^\pm$ is rigid, namely given by a plane with local defects, and the corresponding weak limiting Gibbs state is extremal. This implies in particular the existence of a countable number of extremal DLR states in dimension $d \geq 3$ at low enough temperature, which are in bijection with all the hyperplanes of \mathbb{Z}^d orthogonal to any coordinate axis. It is however widely believed that the 3-dimensional system has a “roughening- temperature” $1/\beta_R$ above which the horizontal interface is no longer sharp, and the corresponding Gibbs state is translation invariant.

The horizontal Dobrushin states are conjectured to be the only extremal non-translation invariant states in 3 dimensions. We quote [31]: “there can only be planes parallel to the faces of the lattice cubes at finite distance, and no angles, corners, or diagonal planes as rigid interfaces.” For example, the 3d Dobrushin interface orthogonal to the vector $(1, 1, 1)$ is believed to be delocalized, and to have $O(\sqrt{\log n})$ fluctuations in finite volume at low temperature, where n is the side length of the box. The result is currently known only¹ at zero temperature [28, Theorem 15]. Note that the similar diagonal Dobrushin interface in 4 and more dimensions (orthogonal to the vector $(1, 1, 1, \dots, 1)$) is rigid at low enough temperature [31], which enriches the set $\text{ex}\mathcal{G}$.

It is an interesting question to determine what the typical fluctuations of the interfaces enforced by general boundary conditions are, in particular those giving rise to non-planar limiting shapes. This is in general already an open problem at zero temperature, and for an isotropic surface tension. The best known results in this direction are large deviation principles. Cerf and Pisztora [9] proved that, in dimensions $d \geq 3$, for a given “macroscopic” boundary condition,² asymptotically as the mesh size of the box tends to zero, the law of the so-called phase partition (i.e. the partition of the space according to the value of the locally³ dominant spin) is determined by a variational problem. More precisely, the empirical phase partition is εn -close to some partition which is compatible with the boundary condition and minimizes the surface tension. It is conjectured that, as $\beta \downarrow \beta_c$, the (rescaled) surface tension becomes more and more isotropic and so the solution of the variational problem should approach the solution of the classical (isotropic) Plateau problem.

Concerning $\text{tr}\mathcal{G}$, Bodineau [5] proved that for any $d \geq 3$ all the translation invariant Gibbs states of the Ising model are convex combinations of the pure phases μ^+ and μ^- .

Let us now summarize which consequences these known results have on the sets \mathcal{W} and \mathcal{G} in dimension 3; see Fig. 1.

If the conjecture about the fluctuations of the low-temperature tilted Dobrushin interface is true, then the corresponding Gibbs state in the thermodynamic limit is translation invariant, and an argument à la Abraham and Reed [1] allows to construct a sequence of boundary conditions which have $\alpha\mu^+ + (1 - \alpha)\mu^-$ as weak limit, for any $\alpha \in (0, 1)$. One has to shift

¹ We emphasize that the natural monotonicity of the fluctuations that we could expect with respect to the temperature is not true in general. Indeed, positive temperature may result in reduction of the fluctuations [6].

² The boundary of a fixed region $\Omega \subset \mathbb{R}^d$ must be partitioned in such a way that for each n , on the boundary of $\Omega_n = \Omega \cap \frac{1}{n}\mathbb{Z}^d$, the number of nearest-neighbor pairs of vertices having different spins is $o(n^{d-1})$.

³ More precisely in a region of size $f(n)$ such that $\log n \ll f(n) \ll n^{1/(d-1)}$.

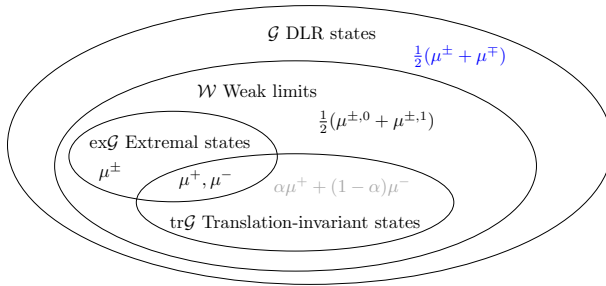
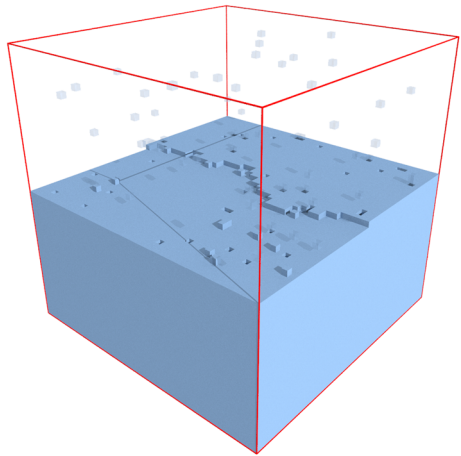


Fig. 1 Inclusion of properties, and examples for the Ising model on \mathbb{Z}^3 at low temperature. It is still a conjecture that all mixtures of μ^+ and μ^- are weak limits. We prove the outmost result

Fig. 2 A realization of the “one-step boundary condition” at low temperature. Minus spins are *full* (and translucent) cubes, and plus spins are *transparent*. Simulation due to V. Beffara



up the plane by an amount $C_\alpha \sqrt{\log n}$. Together with Bodineau’s characterization [5] of the translation invariant states, this would imply that $\text{tr}\mathcal{G} \subset \mathcal{W}$.

Note that there exist mixtures of non-translation invariant states which are reachable with boundary conditions. Let us denote by $\mu^{\pm,z}$ the Ising measure with horizontal Dobrushin boundary condition, parallel to the plane xy and at height z , then $\mu = \frac{1}{2}(\mu^{\pm,0} + \mu^{\pm,1})$ is the weak limit of the “one-step boundary condition”:

$$\omega_{(x,y,z)} = \begin{cases} +1 & \text{if } z \geq 0 \text{ and if } z \geq -1 \text{ and } x \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

Indeed, at low enough temperature, the horizontal Dobrushin interfaces are localized, and so the typical interface induced by the “one-step boundary condition” consist of a plane at height -1 inside the half-space $x \geq 0$, a plane at height 0 inside the half-space $x < 0$, both with local defects, and a one-dimensional step between the two which undergoes Brownian fluctuations; see Fig. 2. The associated Gibbs state is invariant under the translations parallel to the xy plane. General “step boundary conditions” and their link with facets of the equilibrium crystal are studied in [32], see in particular Remark 7.

In this paper, we prove that there also exist mixtures of non-translation invariant states which are not reachable with boundary conditions. The proof is presented in Sect. 2.

Theorem 1 *In dimension $d \geq 3$, for β large enough (depending on d),*

$$\mu = \frac{1}{2}(\mu^\pm + \mu^\mp) \in \mathcal{G}_{2,\beta} \setminus \mathcal{W}_{2,\beta}.$$

Namely μ cannot be reached by a sequence of finite volume measures with boundary conditions.

1.1.3 Random Boundary Conditions

The Ising model with boundary conditions sampled from the symmetric i.i.d. field $\{-1, 1\}^{\mathbb{Z}^2}$ has been studied by van Enter et al. [37,38]. A corollary of their results is that for a typical boundary condition, the probability of the set of configurations containing an interface tends to zero in the infinite-volume limit, which is a stronger result than the absence of translationally non-invariant states (2). When \mathbb{Z}^2 is replaced by \mathbb{Z}^d , for $d \geq 4$, it is expected that $\{\mu^-, \mu^+\}$ is the almost sure set of limit measures (along the regular sequence of cubes). For $d = 2, 3$, they conjecture that this set is $\text{tr}\mathcal{G} = \{\alpha\mu^+ + (1 - \alpha)\mu^- : \alpha \in [0, 1]\}$.

An interesting result concerning the biased setting can be found in [23]. Higuchi proved that μ^- is the only limiting Gibbs state corresponding to a sequence of boundary conditions ω_n such that the density n_+ of + spins is smaller than $3/8$ on $\partial\Lambda_n$ for every ω_n . The fraction $3/8$ is optimal in the sense that for any $\theta > 3/8$, there exists a sequence of boundary conditions such that $3/8 < n_+ \leq \theta$ and for which the limiting Gibbs state is μ^+ .

1.1.4 Global Markov Property

It is worth noting that a mixture of Dobrushin measures similar to $(\mu^\pm + \mu^\mp)/2$ provides an example of a DLR state failing to satisfy the global Markov property. We refer to [3] for a review of the role of this property in statistical mechanics. However, there are extremal Gibbs measures constructed by Israel [26] which also lack the global Markov property, and thus the two properties (lacking the global Markov property and not being a weak limit state) are not the same. A state \mathbb{P} is said to satisfy the global Markov property if

$$\mathbb{P}(\cdot | \mathcal{F}_{\Lambda^c})(\omega) = \mathbb{P}(\cdot | \mathcal{F}_{\partial\Lambda})(\omega) \text{ for any (not necessarily finite) set } \Lambda \tag{3}$$

For spin systems with nearest-neighbor interaction it is a generalized version of (1). Let $\tilde{\mu}^\mp$ be the Gibbs states obtained from μ^\pm by the reflection $(x, y, z) \rightarrow (x, y, -z)$. Note that $\tilde{\mu}^\mp \neq \mu^\mp$, since this reflection is the identity on the plane $z = 0$, so that $\tilde{\mu}^\mp$ agrees with μ^\pm on $\mathcal{F}_{\{z=0\}}$. Then the article [20] explains that $\tilde{\mu} := \frac{1}{2}(\mu^\pm + \tilde{\mu}^\mp)$ does not satisfy (3) for $\Lambda = \{z > 0\}$, since $\tilde{\mu}^\mp$ and μ^\pm agree on $\partial\Lambda$ but are mutually singular on Λ^c . For the proof of Theorem 1, we also use the idea that specifying σ on one side of the box determines whether σ is a configuration of the first or the second phase of the mixture, but we need more input.

1.2 The Case of the Potts Model

1.2.1 Dimension 2

The set $\mathcal{G}_{q,\beta}$ for $\beta > \beta_c$ has been recently proved to be the simplex with the q pure phases as extremal measures [11]. In particular all the Gibbs states are translation invariant.

$$\mathcal{G}_{q,\beta} = \left\{ \sum_{i=1}^q \alpha_i \mathbb{P}_{q,\beta}^i : \alpha_1, \dots, \alpha_q \geq 0, \sum_{i=1}^q \alpha_i = 1 \right\} \tag{4}$$

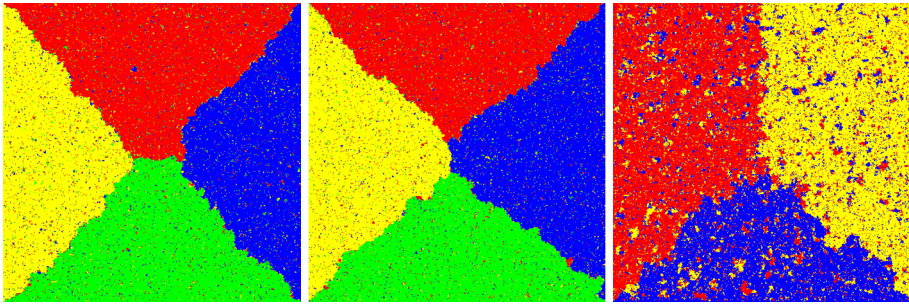


Fig. 3 (Left, middle) Two realizations of the 1-2-3-4 boundary condition for the $q = 4$ Potts model. (Right) A realization of the 1-2-3 boundary condition for $q = 3$ below the critical temperature. Simulation due to V. Beffara

We summarize here the main results of the above work.

Although an arbitrary boundary condition ω_n can a priori enforce the presence of $O(n)$ interfaces, we proved that, uniformly in ω_n , only a finite number of them penetrate up to the half box with high probability. Moreover, these macroscopic interfaces are in a δn neighborhood of the graphs which are solutions of the so-called Steiner problem: link the endpoints in a way which is compatible with the boundary condition and which minimizes surface tension.

These minimal graphs are called Steiner forests (they are collections of disjoint trees). Due to the uniform convexity of the surface tension, proved in [8] for all $q \geq 2$, and a general geometric argument exposed in [4], each inner node of the trees has degree 3, and there exists an $\eta > 0$ such that the angle between two edges incident to an inner node is always larger than $\pi/2 + \eta$.

As a consequence, the possible local configurations of the system in the εn neighborhood of the origin are either a pure phase, or two phases separated by a straight interface (which undergoes Brownian fluctuations [8]), or three phases separated by a “tripod-like” interface (whose triple point and legs undergo Brownian fluctuations). The archetypical illustration is the 1-2-3-4 boundary condition which gives rise to two possible Steiner trees; see Fig. 3.

Given these results, an argument à la Abraham and Reed should achieve to reach mixtures of *three* pure phases with boundary conditions : take the 1-2-3 boundary condition, and shift it with respect to the origin by a vector $\mathbf{C}_\alpha \sqrt{n}$, with a well-chosen $\mathbf{C} = \mathbf{C}(\alpha_1, \alpha_2, \alpha_3)$ in order to bias the limiting measure towards $\alpha_1 \mathbb{P}^1 + \alpha_2 \mathbb{P}^2 + \alpha_3 \mathbb{P}^3$; see Fig. 3 on the right.

This would imply that for any $\beta \geq 0$,

$$\mathcal{W}_{q,\beta} = \mathcal{G}_{q,\beta}, \text{ for } q = 2, 3.$$

Starting at $q = 4$, asking what the structure of \mathcal{W} is (and if $\mathcal{W} = \mathcal{G}$) becomes a difficult question. The study of Steiner forests gives one way to construct non-trivial convex combinations of pure phases: we can look for symmetric domains and boundary conditions which give rise to several possible Steiner trees intersecting at some location. However, on the one hand quite little is known about the structure of Steiner forests for a general norm on the plane, and on the other hand, it is not clear if we can get all the weak limiting states with this method.

However, by adding a slowly growing number of boundary spins to the free boundary condition, it might be possible to obtain continuous changes of weights (thus biasing the mixture $\frac{1}{q} \sum_{i=1}^q \mathbb{P}^i$). Therefore, it seems to be reasonable to conjecture that $\mathcal{W}_{q,\beta} = \mathcal{G}_{q,\beta}$ for

the 2-dimensional Potts model for any q . This question will be addressed in a forthcoming paper.

1.2.2 Dimension 3 (and More)

The large deviations results of Cerf and Pisztora [9] for the empirical phase partition, which we already mentioned in the previous subsection, are valid for the Potts models for all $q \geq 2$ in dimension 3 below the critical temperature. For $q \leq 4$, it is conjectured that the (rescaled) surface tension converges to the Euclidean ball as $\beta \downarrow \beta_c(q)$, whereas for q large (conjecturally up to $q = 4$), this should not be the case as the phase transition is of first order.

The macroscopic phase separation surfaces are minimizing the surface tension τ . Note that the geometry of interfaces is much more complicated in systems with more than two phases. Moreover, very little is known about the surface tension in dimension 3, although the following properties are widely believed to be true: τ satisfies the sharp simplex inequality (that is τ is uniformly convex), the value of τ is minimal in axis directions, and τ increases as the normal vector moves from $(0, 0, 1)$ to $(1, 1, 1)$. See the introduction of [9].

The localization of the horizontal Dobrushin interface for the Potts model ($\omega_{(x,y,z)} = i$ if $z \geq 0$ and j if $z < 0$) at low enough temperature for $d \geq 3$ has been proved by Gielis and Grimmett [19]. A non translation invariant measure, corresponding to the coexistence of the ordered and disordered phases, is obtained by Černý and Kotecký [10] for the random cluster model at the criticality $p_c(q)$ with sufficiently large q (limit of finite volume measures with free-wired Dobrushin boundary conditions).

Concerning $\text{tr}\mathcal{G}$, Martirosian [30] proved that for any $d \geq 2$ and q large enough (depending on d) all the translation invariant Gibbs states of the Potts model at $\beta > \beta_c(q, d)$ are convex combinations of the pure phases $\mathbb{P}_{q,\beta}^i, i = 1, \dots, q$.

The state of the art concerning \mathcal{W} is thus even more restricted than in the case of the Ising model. A natural guess is that a similar result as Theorem 1, namely $\frac{1}{2}(\mathbb{P}_{q,\beta}^{12} + \mathbb{P}_{q,\beta}^{21}) \in \mathcal{G}_{q,\beta} \setminus \mathcal{W}_{q,\beta}$, is true for the Potts model. Nevertheless, we argue in Section 3 why it is much more difficult to prove (or disprove). Both outcomes would of course be interesting.

2 Proof of Theorem 1

Write as above $\mu = \frac{1}{2}(\mu^\pm + \mu^\mp)$. We first use the localization of the Dobrushin interface [14] in dimension $d \geq 3$ to deduce positive association of flipped spins across the symmetry plane $z = 0$. Using the bound of Van Beijeren [35] on the magnetization of a spin at height 0, and the FKG inequality, we have

$$m_2^\star \leq \mu^\pm(\sigma_{(x,y,z)}) \leq m_3^\star \quad \text{for } z \in \mathbb{N}^+, \tag{5}$$

where $m_d^\star = \mu^+(\sigma_0)$ in dimension d . Note that $m_d^\star \rightarrow 1$ as $\beta \rightarrow \infty$ for all d . Moreover, by symmetry,

$$\mu^\pm(\sigma_{(x,y,z)} = -1) = \mu^\mp(\sigma_{(x,y,z)} = +1),$$

and hence

$$\mu(\sigma_{(x,y,z)} = +1) = \mu(\sigma_{(x,y,z)} = -1) = 1/2. \tag{6}$$

Let $\mathbf{z} = (0, 0, z)$ for some $z \in \mathbb{N}^+$ and $\hat{\mathbf{z}} = (0, 0, -z - 1)$, two points which are symmetric with respect to the plane $z = -1/2$. Note that by symmetry $\mu^\pm(\sigma_{\mathbf{z}}) = -\mu^\pm(\sigma_{\hat{\mathbf{z}}})$. By a union bound and (6), we have

$$\begin{aligned} \mu(\sigma_{\mathbf{z}} = +1 | \sigma_{\hat{\mathbf{z}}} = -1) &= \frac{\mu(\sigma_{\mathbf{z}} = +1, \sigma_{\hat{\mathbf{z}}} = -1)}{\mu(\sigma_{\hat{\mathbf{z}}} = -1)} \geq \frac{\frac{1}{2}\mu^\pm(\sigma_{\mathbf{z}} = +1, \sigma_{\hat{\mathbf{z}}} = -1)}{1/2} \\ &\geq 1 - \mu^\pm(\sigma_{\mathbf{z}} = -1) - \mu^\pm(\sigma_{\hat{\mathbf{z}}} = +1) \\ &= 1 - \frac{1 - \mu^\pm(\sigma_{\mathbf{z}})}{2} - \frac{1 + \mu^\pm(\sigma_{\hat{\mathbf{z}}})}{2} = \mu^\pm(\sigma_{\mathbf{z}}) \geq m_2^* \end{aligned} \tag{7}$$

Now suppose that μ is a weak limit of finite-volume measures, i.e. $\mu = \lim_{n \rightarrow \infty} \mu_{\Lambda_n}^{\omega_n}$ for some deterministic sequence of boundary conditions $(\omega_n)_n$ and $\Lambda_n \uparrow \mathbb{Z}^d$. As every $\mu_{\Lambda_n}^{\omega_n}$ satisfy the FKG inequality, so does μ . Which implies,

$$\mu(\sigma_{\mathbf{z}} = +1 | \sigma_{\hat{\mathbf{z}}} = -1) \leq \mu(\sigma_{\mathbf{z}} = +1) = 1/2. \tag{8}$$

This is a contradiction with (7) as soon as β is large enough that $m_2^*(\beta) > 1/2$. Note that if we take z large, we can actually replace the bound in (7) by $m_3^*(1 - \varepsilon)$, with some $\varepsilon = \varepsilon(\beta, z) \rightarrow 0$ as $z \rightarrow \infty$. The contradiction holds then as soon as β is large enough that $m_3^*(\beta) > 1/2$. \square

Remark 1 We could have used “negative association of the same value of spin” across the Dobrushin interface, namely the following inequality holds as well:

$$\mu(\sigma_{\mathbf{z}} = +1 | \sigma_{\hat{\mathbf{z}}} = +1) \leq 1 - m_2^* \tag{9}$$

and is in contradiction with the FKG inequality:

$$\mu(\sigma_{\mathbf{z}} = +1 | \sigma_{\hat{\mathbf{z}}} = +1) \geq \mu(\sigma_{\mathbf{z}} = +1) = 1/2.$$

As we will see in the next section, this “second proof” gives a priori two hopes of extending the result to the Potts model. However, none of them works.

Remark 2 The result holds for all vertical translates and axis-symmetry of the Dobrushin boundary condition, as well as for $\mu = \alpha\mu^\pm + (1 - \alpha)\mu^\mp$, with $\alpha \in (0, 1)$ to be chosen such that

$$\frac{m_2^*}{\frac{1+m_3^*}{2} + \frac{1-\alpha}{\alpha} \frac{1-m_2^*}{2}} > \frac{1}{2}.$$

Remark 3 It is possible to make the contradiction hold up to the roughening temperature β_R of the 3-dimensional Ising model, by looking at the proportion of + spins in large but finite boxes in the two half-spaces. Recall that by [35] we have $0 < \beta_c(3) \leq \beta_R < \beta_c(2)$.

Let $\Lambda_m(\mathbf{z})$ be the box of (odd) side-length $m < z$ centered at \mathbf{z} . Denote by $M_{\mathbf{z}}^m$ the majority of the spins inside $\Lambda_m(\mathbf{z})$, namely

$$M_{\mathbf{z}}^m = \begin{cases} +1 & \text{if } \#\{i \in \Lambda_m(\mathbf{z}) : \sigma_i = +1\} > \#\{i \in \Lambda_m(\mathbf{z}) : \sigma_i = -1\} \\ -1 & \text{else,} \end{cases}$$

where $\#X$ denotes the cardinality of the set X . Then, by the same computation as in (7), we have on the one hand

$$\mu(M_{\mathbf{z}}^m = +1, M_{\hat{\mathbf{z}}}^m = -1) \geq \frac{1}{2}(1 - 2\mu^\pm(M_{\mathbf{z}}^m = -1)) \geq \frac{1}{2}(1 - 2\varepsilon),$$

with $\varepsilon = \varepsilon(z, m)$ being small in z and m as soon as $\beta > \beta_R$. And on the other hand, always by symmetry, $\mu(M_{\hat{\mathbf{z}}}^m = -1) = \frac{1}{2}$. So that

$$\mu(M_{\mathbf{z}}^m = +1 | M_{\hat{\mathbf{z}}}^m = -1) \geq 1 - 2\varepsilon. \tag{10}$$

As the event $\{M_z^m = +1\}$ (resp. $\{M_z^m = -1\}$) is increasing (resp. decreasing), the FKG inequality implies

$$\mu(M_z^m = +1 \mid M_z^m = -1) \leq \mu(M_z^m = +1) \leq \frac{1}{2},$$

which is in contradiction with (10) as soon as $\beta > \beta_R$, if z and m are taken sufficiently large. \square

3 The State of Affairs for the Potts Model

3.1 Absence of Certain Spin Correlation Inequalities in Presence of Boundary Conditions

The direct generalization of the proof of Theorem 1 to the Potts model is not possible, because the needed correlation inequalities break down for non-free boundary conditions. Indeed, we provide counter-examples to the FKG inequality for the fuzzy Potts measure with non-free boundary conditions; see [27] for the case of free boundary conditions.

First, for the Potts model with free boundary conditions, on any finite graph Λ , Schonmann [34] proved that the following correlation inequality holds, which we could name “negative association of different kinds of spins”. For any $i \neq j \in \{1, \dots, q\}$, and $A, B \subset \Lambda$,

$$\mathbb{P}_{q,\beta,\Lambda}^\emptyset \left(\prod_{x \in A} \mathbb{1}_{[\sigma_x=i]} \mid \prod_{y \in B} \mathbb{1}_{[\sigma_y=j]} \right) \leq \mathbb{P}_{q,\beta,\Lambda}^\emptyset \left(\prod_{x \in A} \mathbb{1}_{[\sigma_x=i]} \right). \tag{11}$$

Proposition 1 *There exist boundary conditions ω such that (11) does not hold for $\mathbb{P}_{q,\beta,\Lambda}^\omega$.*

Proof Here is a counter-example, based on the analysis of subcritical Gibbs states of the Potts model on \mathbb{Z}^2 . For the boundary condition $\omega = 1\text{-}2\text{-}3\text{-}4$ depicted in Fig. 4, at fixed supercritical β , in a sufficiently large box, the typical interfaces are concentrated around two possible deterministic Steiner trees, and undergo Brownian fluctuations around these objects; see also Fig. 3.

Indeed, by uniform convexity of the Wulff shape [8], these two trees are shorter than the spanning minimal tree (consisting of three sides of the box), which would have 90 degrees between its branches; see [4]. Therefore, for some x in the region A and some y in the region B , at large enough n , we have :

$$\mathbb{P}_{\Lambda_n}^{1234}(\sigma_x = 1 \mid \sigma_y = 3) \geq 1 - \varepsilon \quad \text{but} \quad \mathbb{P}_{\Lambda_n}^{1234}(\sigma_x = 1) \leq \frac{1}{2} + \varepsilon < 1, \tag{12}$$

with $\varepsilon = \varepsilon(\beta) \rightarrow 0$ as $\beta \rightarrow \infty$, which contradicts (11), and also shows that the analogue of the “first” proof for the Ising model cannot be extended to the Potts model. Counter-examples of this kind exist for $q = 3$. \square

Secondly, Schonmann [34] also proves “positive association of the same kind of spins” for the Potts model with free boundary conditions on any finite graph. That is, for any $i \in \{1, \dots, q\}$,

$$\mathbb{P}_{q,\beta,\Lambda}^\emptyset \left(\prod_{x \in A} \mathbb{1}_{[\sigma_x=i]} \mid \prod_{y \in B} \mathbb{1}_{[\sigma_y=i]} \right) \geq \mathbb{P}_{q,\beta,\Lambda}^\emptyset \left(\prod_{x \in A} \mathbb{1}_{[\sigma_x=i]} \right). \tag{13}$$

Fig. 4 The two possible Steiner trees (solid and dashed lines) for the 1-2-3-4 boundary condition

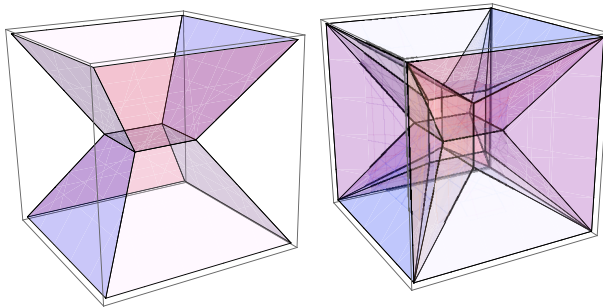
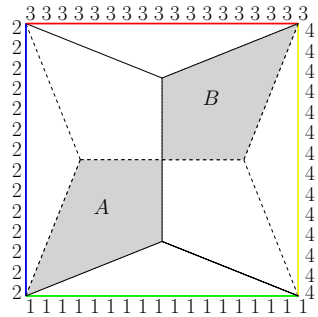


Fig. 5 (Left) the soap film in direction z ; (Right) the three possible soap films in directions x, y and z

However, if soap-film-like surfaces are minimal surfaces for the surface tension of the 3 dimensional Potts model (which is widely believed to be true), then (13) does not hold for the 1-2-3-4-5-6 as boundary condition, namely a different color on each face of a cube, see Fig. 5. The typical interfaces should be concentrated around one of three possible minimal surfaces, each one having a little square aligned with one coordinate axis. For some x and y in well-chosen regions (more precisely in the interior of two different parts of the symmetric difference between the locations of a phase in two Steiner surfaces), we can get $\mathbb{P}_{\Lambda_n}^{123456}(\sigma_x = 1 | \sigma_y = 1) \leq 1/2 + \varepsilon$ whereas $\mathbb{P}_{\Lambda_n}^{123456}(\sigma_x = 1) \geq 2/3 - \varepsilon > 1/2$ which contradicts (13), and shows that the analogue of the “second” proof for the Ising model, mentioned in Remark 1, cannot be extended to the Potts model.

Note that the existence of at least two Steiner trees for which the locations of a pure phase have a non-empty symmetric difference is enough to provide a 2d counter-example to (13).

3.2 Correlation Inequalities for Specific Boundary Conditions and Exclusion of Certain Weak Limits

In [36], van den Berg et al. proved some conditional correlation inequalities for the random cluster model on finite graphs $\Lambda = (V, E)$; see [22] for the definition and a review on the random cluster model. For $S \subset V$, let C_S denote the set of edges belonging to open paths starting at vertices of S . They show that for $q \geq 1$, if S and T are disjoint sets of vertices and f and g functions of the clusters of S and T , written (C_S, C_T) , each increasing in C_S and decreasing in C_T , then,

$$\phi_{q,p,\Lambda}(fg|S \leftrightarrow T) \geq \phi_{q,p,\Lambda}(f|S \leftrightarrow T) \cdot \phi_{q,p,\Lambda}(g|S \leftrightarrow T) \tag{14}$$

where $\phi_{q,p}$ denotes the random cluster model with parameters q and p on the graph Λ .

We prove here that this result implies the correlation inequality (11) in the Potts model with certain specific boundary conditions.

Proposition 2 For $q \geq 2$, subsets $A, B \subset \Lambda \in \mathbb{Z}^d$, for any $i \neq j \in \{1, \dots, q\}$ and any bicolor boundary condition $\omega \in \{i, j, \emptyset\}^{\partial\Lambda}$, we have

$$\mathbb{P}_{q,\beta,\Lambda}^\omega \left(\prod_{x \in A} \mathbb{1}_{[\sigma_x=i]} \mid \prod_{y \in B} \mathbb{1}_{[\sigma_y=j]} \right) \leq \mathbb{P}_{q,\beta,\Lambda}^\omega \left(\prod_{x \in A} \mathbb{1}_{[\sigma_x=i]} \right). \tag{15}$$

Proof The well-known Edwards-Sokal coupling [15] implies that the Potts measure $\mathbb{P}_{q,\beta,\Lambda}^\omega$ is coupled to the random cluster measure $\phi_{q,p,\Lambda}(\cdot \mid \cap_{i \neq j} \{E_i(\omega) \leftrightarrow E_j(\omega)\})$ with $p = 1 - e^{-\beta}$ and $E_i(\omega) = \{x \in \partial\Lambda : \omega_x = i\}$. Let us write $cond(\omega) = \cap_{i \neq j} \{E_i(\omega) \leftrightarrow E_j(\omega)\}$ and $\kappa(X)$ for the number of connected components of the set X . Then,

$$\begin{aligned} \mathbb{P}_{q,\beta,\Lambda}^\omega \left(\prod_{x \in A} \mathbb{1}_{[\sigma_x=i]} \prod_{y \in B} \mathbb{1}_{[\sigma_y=j]} \right) &= \phi_{q,p,\Lambda} (f_A \cdot f_B \mid cond(\omega)) \\ \text{with } f_A &= \sum_{X \subset A} \sum_{c=0}^{|A \setminus X|} \frac{1}{q^c} \mathbb{1}_{[X \leftrightarrow E_i]} \mathbb{1}_{[A \setminus X \leftrightarrow E_i]} \mathbb{1}_{[\kappa(A \setminus X)=c]} \\ f_B &= \sum_{Y \subset B} \sum_{c'=0}^{|B \setminus Y|} \frac{1}{q^{c'}} \mathbb{1}_{[Y \leftrightarrow E_j]} \mathbb{1}_{[B \setminus Y \leftrightarrow E_j]} \mathbb{1}_{[\kappa(B \setminus Y)=c']} \end{aligned}$$

The key remark is that, for any $q \geq 2$, for any bicolor boundary condition $\omega \in \{i, j, \emptyset\}^{\partial\Lambda}$, we have $cond(\omega) = \{E_i \leftrightarrow E_j\}$, and so (14) ensures that f_A and f_B are negatively correlated. Indeed, one can check that f_A is ‘‘increasing in the connectedness’’ of the graph $A \cup E_i$ (resp. f_B is ‘‘increasing in the connectedness’’ of the graph $B \cup E_j$), which implies that f_A is increasing in C_{E_i} (and decreasing in C_{E_j}), and that f_B is increasing in C_{E_j} (and decreasing in C_{E_i}). Therefore, using (14), we get (15). \square

Proposition 2 gives a partial result for the Potts model concerning the existence of non-weak limit states, counterpart of Theorem 1.

Proposition 3 The measure $\mathbb{P} = \frac{1}{2}(\mathbb{P}^{12} + \mathbb{P}^{21})$, mixture of Dobrushin states for the 3-dimensional Potts model, is not a weak limit of finite-volume measures with boundary conditions $\omega \in \{1, 2, \emptyset\}^{\partial\Lambda}$.

Proof We adapt the ‘‘first’’ proof for the Ising model, and keep the same notations. Localization of the Dobrushin interface at low enough temperature is also known for the Potts model [19]. Therefore,

$$\mathbb{P}(\sigma_{\mathbf{z}} = 1 \mid \sigma_{\hat{\mathbf{z}}} = 2) \geq 1 - \varepsilon \tag{16}$$

On the other hand, suppose that \mathbb{P} is a weak limit of finite-volume measures $\mathbb{P}_{q,\beta,\Lambda_n}^{\omega_n}$ for some deterministic sequence of boundary conditions $(\omega_n)_n \in \{1, 2, \emptyset\}^{\partial\Lambda}$ and some boxes $\Lambda_n \uparrow \mathbb{Z}^d$. By (15), every $\mathbb{P}_{q,\beta,\Lambda}^\omega$ with $\omega \in \{1, 2, \emptyset\}^{\partial\Lambda}$ satisfies

$$\mathbb{P}_{q,\beta,\Lambda}^\omega (\sigma_{\mathbf{z}} = 1 \mid \sigma_{\hat{\mathbf{z}}} = 2) \leq \mathbb{P}_{q,\beta,\Lambda}^\omega (\sigma_{\mathbf{z}} = 1). \tag{17}$$

This inequality being preserved by weak limits, the measure \mathbb{P} satisfies it as well, hence

$$\mathbb{P}(\sigma_{\mathbf{z}} = 1 | \sigma_{\hat{\mathbf{z}}} = 2) \leq \mathbb{P}(\sigma_{\mathbf{z}} = 1) \approx \frac{1}{2} \left(\mathbb{P}_{q,\beta}^1(\sigma_{\mathbf{z}} = 1) + \mathbb{P}_{q,\beta}^2(\sigma_{\mathbf{z}} = 1) \right) \quad (18)$$

which converges to $1/2$ as $\beta \rightarrow \infty$, providing a contradiction with (16). \square

Despite this result, it is important to mention that there is enough structure in the Potts model to possibly allow mixtures of localized states. The measure $\mathbb{P} = \frac{1}{2}(\mathbb{P}^{12} + \mathbb{P}^{21})$ might still be reachable by a sequence of finite-volume measures with well-chosen boundary conditions, for example having a non-trivial structure in 2 directions, allowing the intersection of different possible Steiner trees, and being translation invariant in the 3rd direction, in order to localize them. This is a work in progress.

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