

Diffraction Theory of Point Processes: Systems with Clumping and Repulsion

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Abstract We discuss several examples of point processes [all taken from Hough et al. (Zeros of Gaussian analytic functions and determinantal point processes, 2009)] for which the autocorrelation and diffraction measures can be calculated explicitly. These include certain classes of determinantal and permanental point processes, as well as an isometry-invariant point process that arises as the zero set of a Gaussian random analytic function.

Keywords Point processes · Diffraction spectrum · Gaussian analytic function

1 Introduction

Mathematical diffraction theory deals with the relationship between the structure of point configurations in space and the associated autocorrelation and diffraction measures, one of the main questions being how the order properties of the point configuration translate into properties of the diffraction measure. There exist many results about deterministic point configurations (periodic and aperiodic tilings, model sets, substitution systems, compare [4] and the references therein), but in recent years random point configurations have also been considered. In particular, reference [3] provides a general framework for the investigation of point configurations within the theory of point processes, along with a number of examples, most of them closely connected to renewal and Poisson processes. However, beyond the i.i.d. situation, the number of explicit examples is still rather small, and it seems that more examples are needed for a better understanding of the problem and a further development of

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the theory. In particular, it seems desirable to have examples with some effective interaction for which the autocorrelation and diffraction measures may be calculated explicitly. Some of the few exceptional cases which have been considered here are the Ising model on the square lattice and the dimer model on the triangular lattice; see [5].

The aim of this paper is to extend the list of these explicit examples by discussing various point processes (all taken from [13]) from the viewpoint of mathematical diffraction theory. All these examples are simple, stationary and ergodic (see Sect. 2 for definitions). Moreover, the numbers of points in neighbouring subsets of Euclidean space may be either positively correlated (“clumping”) or negatively correlated (“repulsion”). We discuss determinantal and permanent point processes as (classes of) examples for systems with repulsion and clumping, respectively. As a further example for a system with repulsion, we consider the zero set of a certain Gaussian random analytic function in the complex plane. More precisely, we take the unique Gaussian random analytic function, up to scaling, such that the zero set is translation-invariant (in fact, even isometry-invariant) in distribution; see Sect. 6 for details. Furthermore, we also briefly look at Cox processes.

Our main results support the widespread expectations about the diffraction measures of generic random point configurations (with good mixing properties, say). For instance, in most of our examples, the diffraction measure is absolutely continuous apart from the (trivial) Bragg peak at the origin. (Here, by a *Bragg peak*, one understands a point mass contained in the diffraction measure.) Moreover, for a certain class of determinantal and permanent point processes, it turns out that the “diffraction spectrum” is equivalent to the “dynamical spectrum”; see Remark 3.16 for details. For general point processes, the question under what conditions this is true seems to be open, but for the above examples, this is perhaps not too surprising in view of their excellent mixing properties.

2 Preliminaries

This section contains some background information on point processes, Fourier transforms, and mathematical diffraction theory. As they are sufficient for our discussion here, we restrict ourselves to positive measures, and refer to [3] and the references therein for the general case.

2.1 Point Processes

A measure on \mathbb{R}^d is a measure on the Borel σ -field (or σ -algebra) \mathbb{B}^d of \mathbb{R}^d . A measure ω on \mathbb{R}^d is *locally finite* if $\omega(A) < \infty$ for any bounded Borel set A . A *point measure* on \mathbb{R}^d is a measure on \mathbb{R}^d taking values in $\mathbb{N}_0 \cup \{\infty\}$. A point measure ω on \mathbb{R}^d is *simple* if $\omega(\{x\}) \leq 1$ for any $x \in \mathbb{R}^d$. We write $\mathcal{M}(\mathbb{R}^d)$ for the space of locally finite measures on \mathbb{R}^d and $\mathcal{N}(\mathbb{R}^d)$ for the subspace of locally finite point measures on \mathbb{R}^d . It is well known (compare [10, Appendix A2] and [11, Sect. 9]) that there exists a metric such that $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{N}(\mathbb{R}^d)$ are complete separable metric spaces, the induced topology is that of vague convergence, and the induced Borel σ -fields $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{N}(\mathbb{R}^d)$ are the smallest σ -fields such that the mappings $\omega \mapsto \omega(A)$, with $A \in \mathbb{B}^d$, are measurable. A *random measure* is a random variable taking values in $\mathcal{M}(\mathbb{R}^d)$, and a *point process* is a random variable taking values in $\mathcal{N}(\mathbb{R}^d)$. A point process is called *simple* if its realisation is simple with probability 1. If ω is a random measure or a point process such that $\mathbb{E}(\omega(A)) < \infty$ for any bounded Borel set A , we say that the expectation measure of ω exists, and call the measure $A \mapsto \mathbb{E}(\omega(A))$ the *expectation measure* of ω .

A locally finite point measure ω on \mathbb{R}^d may be written in the form $\omega = \sum_{i \in I} \delta_{x_i}$, where I is a countable index set and $(x_i)_{i \in I}$ is a family of points in \mathbb{R}^d with at most finitely points in any bounded Borel set. Then, for any $k \geq 1$, the locally finite point measures ω^k and $\omega^{\bullet k}$ on $(\mathbb{R}^d)^k$ are defined by

$$\omega^k := \sum_{i_1, \dots, i_k \in I} \delta_{(x_{i_1}, \dots, x_{i_k})} \quad \text{and} \quad \omega^{\bullet k} := \sum_{\substack{i_1, \dots, i_k \in I \\ \text{distinct}}} \delta_{(x_{i_1}, \dots, x_{i_k})}.$$

Note that ω^k is simply the k -fold product measure of ω . If ω is a point process such that $\mathbb{E}(\omega(A)^k) < \infty$ for any bounded Borel set A , the expectation measures $\mu^{(k)}$ and $\mu^{\bullet(k)}$ of ω^k and $\omega^{\bullet k}$ are called the k th moment measure of ω and the k th factorial moment measure of ω , respectively. If $\mu^{\bullet(k)}$ is absolutely continuous with respect to Lebesgue measure on $(\mathbb{R}^d)^k$, its density is denoted by ϱ_k and called the k -point correlation function of the point process ω .

Remark 2.1 It is easy to see that the absolute continuity of the second factorial moment measure implies that the underlying point process is simple; compare [10, Proposition 5.4.6] for details. \diamond

For any $x \in \mathbb{R}^d$, we write T_x for the translation by x on \mathbb{R}^d (with $T_x(u) := u + x$). The notation T_x is extended to the induced translation actions on sets, functions, and measures in the standard way: $T_x(A) := x + A := \{T_x(y) : y \in A\}$, $(T_x f)(y) := f(T_{-x}(y)) = f(-x + y)$, and $(T_x \omega)(A) := \omega(T_{-x}(A)) = \omega(-x + A)$. Note that the latter is consistent with translation of an indicator function $\mathbf{1}_A$ (also interpretable as a measure) so that $T_x(\mathbf{1}_A) = \mathbf{1}_{T_x(A)}$. Finally, we also write T_x for the induced translation on sets of measures: $T_x(B) := \{T_x(\omega) : \omega \in B\}$.

A set $B \in \mathcal{M}(\mathbb{R}^d)$ is called *invariant* if $T_x^{-1}(B) = B$ for any $x \in \mathbb{R}^d$. A random measure ω on \mathbb{R}^d is called *stationary* (or *translation-invariant*) if for any $x \in \mathbb{R}^d$, ω and $T_x(\omega)$ have the same distribution. A random measure ω on \mathbb{R}^d is called *ergodic* if it is stationary and if, for any invariant set $B \in \mathcal{M}(\mathbb{R}^d)$, one has $\mathbb{P}(\omega \in B) \in \{0, 1\}$. A random measure ω on \mathbb{R}^d is called *mixing* if it is stationary and if, for any sets $B_1, B_2 \in \mathcal{M}(\mathbb{R}^d)$, one has $\mathbb{P}(\omega \in T_x^{-1}(B_1) \cap B_2) \rightarrow \mathbb{P}(\omega \in B_1)\mathbb{P}(\omega \in B_2)$ as $|x| \rightarrow \infty$. Here, $|x|$ denotes the Euclidean norm of x . It is well known that mixing implies ergodicity; see e.g. [11, Sect. 12.3].

If ω is a stationary point process such that $\mathbb{E}(\omega(A)^k) < \infty$ for any bounded Borel set A , the *reduced k th moment measure* $\mu_{\text{red}}^{(k)}$ of ω and the *reduced k th factorial moment measure* $\mu_{\text{red}}^{\bullet(k)}$ of ω are the (unique) locally finite measures on $(\mathbb{R}^d)^{k-1}$ such that

$$\int_{(\mathbb{R}^d)^k} f(x_1, x_2, \dots, x_k) d\mu^{(k)}(x) = \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{k-1}} f(x, x + y_1, \dots, x + y_{k-1}) d\mu_{\text{red}}^{(k)}(y) dx$$

and

$$\int_{(\mathbb{R}^d)^k} f(x_1, x_2, \dots, x_k) d\mu^{\bullet(k)}(x) = \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{k-1}} f(x, x + y_1, \dots, x + y_{k-1}) d\mu_{\text{red}}^{\bullet(k)}(y) dx$$

for any bounded measurable function f on $(\mathbb{R}^d)^k$ with bounded support, compare [11, Proposition 12.6.3]. Here, the first reduced (factorial) moment measure is regarded as a constant ϱ , which is also called the *mean density* of the point process. Furthermore, if $\mu^{\bullet(k)}$ is absolutely continuous with respect to Lebesgue measure on $(\mathbb{R}^d)^k$, we may assume its density $\varrho_k(x_1, \dots, x_k)$ to be *translation-invariant*, i.e.

$$\varrho_k(x_1 + t, \dots, x_k + t) = \varrho_k(x_1, \dots, x_k) \tag{2.1}$$

for any $t \in \mathbb{R}^d$. Then, $\mu_{\text{red}}^{\bullet(k)}$ is absolutely continuous with respect to Lebesgue measure on $(\mathbb{R}^d)^{k-1}$, with density $\varrho_k(0, y_1, \dots, y_{k-1})$. In particular, $\varrho_1(0) = \varrho$.

2.2 Fourier Transforms

For the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$, we use the convention

$$\widehat{f}(y) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i xy} \, dx, \tag{2.2}$$

where xy is the standard inner product on \mathbb{R}^d . Let us note that, with this convention, Fourier inversion takes the form

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(y)e^{2\pi i xy} \, dy \tag{2.3}$$

for any continuous function $f \in L^1(\mathbb{R}^d)$ such that $\widehat{f} \in L^1(\mathbb{R}^d)$. The special case of radially symmetric functions will be discussed later when we need it.

Besides Fourier transforms of L^1 -functions, we will also use Fourier transforms of L^2 -functions. The Fourier transform on L^2 is defined as the (unique) continuous extension of the Fourier transform on L^1 restricted to $L^1 \cap L^2$, viewed as a mapping from $L^1 \cap L^2 \subset L^2$ to L^2 . It is well known that the Fourier transform on L^2 is an isometry.

Moreover, we will also use Fourier transforms of translation-bounded measures. A measure μ on \mathbb{R}^d is *translation-bounded* if, for any bounded Borel set $B \in \mathbb{B}^d$, one has $\sup_{x \in \mathbb{R}^d} \mu(x + B) < \infty$. A translation-bounded measure μ on \mathbb{R}^d is *transformable* if there exists a translation-bounded measure $\widehat{\mu}$ on \mathbb{R}^d such that

$$\int_{\mathbb{R}^d} f(x) \, d\widehat{\mu}(x) = \int_{\mathbb{R}^d} \widehat{f}(x) \, d\mu(x)$$

holds for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$. In this case, the measure $\widehat{\mu}$ is unique, and it is called the *Fourier transform* of the measure μ . Indeed, these definitions may even be extended to *signed* measures; see [3] for details.

Given a locally integrable function f on \mathbb{R}^d , we write $f \lambda^d$ or $f(x) \lambda^d$ for the signed measure on \mathbb{R}^d given by $(f \lambda^d)(B) := \int_B f(x) \, dx$, with $B \in \mathbb{B}^d$ bounded. Note that if f is integrable, $(f \lambda^d)^\wedge = \widehat{f} \lambda^d$. Note further that, if f is the Fourier transform of some integrable function φ , it follows by standard Fourier inversion that $(f \lambda^d)^\wedge = \varphi_- \lambda^d$ where $\varphi_-(x) := \varphi(-x)$ denotes the reflection of φ at the origin.

2.3 Mathematical Diffraction Theory

Let ω be a locally finite measure on \mathbb{R}^d , and let $\widetilde{\omega}(A) := \omega(-A)$ be its reflection at the origin. Write B_n for the open ball of radius n around the origin and $\lambda^d(B_n)$ for its d -dimensional volume. The *autocorrelation measure* of ω is defined by

$$\gamma := \lim_{n \rightarrow \infty} \frac{\omega|_{B_n} * \widetilde{\omega|_{B_n}}}{\lambda^d(B_n)},$$

provided that the limit exists in $\mathcal{M}(\mathbb{R}^d)$ with respect to the vague topology. In this case, the *diffraction measure* of ω is the Fourier transform of γ . Let us note that $\widehat{\gamma}$ exists due to the fact that γ is a positive and positive-definite measure, and that $\widehat{\gamma}$ is also a positive and positive-definite measure; see [3, 4, 8] or [10, Sect. 8.6] for details.

In each of the following examples, ω will be given by the realisation of a stationary and ergodic simple point process on \mathbb{R}^d . The following result from [3], see also [15], shows that

the autocorrelation and diffraction measures exist and, moreover, almost surely do not depend on the realisation.

Theorem 2.2 ([12, Theorem 1.1], [3, Theorem 3]) *Let ω be a stationary and ergodic point process such that the reduced first moment measure $\mu_{red}^{(1)} = \varrho$ and the reduced second moment measure $\mu_{red}^{(2)}$ exist. Then, almost surely, the autocorrelation measure γ of ω exists and satisfies*

$$\gamma = \mu_{red}^{(2)} = \varrho\delta_0 + \mu_{red}^{\bullet(2)}.$$

□

Remark 2.3 In [12, Theorem 1.1] and [3, Theorem 3], the preceding result is stated in a slightly different form, namely that the autocorrelation measure of ω is equal to the first moment measure of the so-called Palm measure of ω . However, the latter coincides with the reduced second moment measure of ω under the assumptions of the theorem; see e.g. [3, Equation (47)]. The second equality in Theorem 2.2 is a well-known relation between $\mu_{red}^{(2)}$ and $\mu_{red}^{\bullet(2)}$; see e.g. [10, Sect. 8.1]. ◇

We will often use Theorem 2.2 in the following form.

Corollary 2.4 *Suppose that, in addition to the assumptions of Theorem 2.2, the factorial moment measure $\mu^{\bullet(2)}$ is absolutely continuous with a translation-invariant density of the form*

$$\varrho_2(x_1, x_2) = \varrho^2 + g(x_2 - x_1) \quad \text{with } g \in L^1(\mathbb{R}^d). \tag{2.4}$$

Then, almost surely, the autocorrelation and diffraction measures of ω exist and are given by

$$\gamma = \varrho\delta_0 + (\varrho^2 + g)\lambda^d \tag{2.5}$$

and

$$\widehat{\gamma} = \varrho^2\delta_0 + (\varrho + \widehat{g})\lambda^d, \tag{2.6}$$

respectively.

If the argument of the function needs to be specified, we usually write $g(x)\lambda^d$ in (2.5) and $\widehat{g}(t)\lambda^d$ in (2.6). Note that, under the assumptions of Corollary 2.4, the diffraction measure is absolutely continuous apart from the Bragg peak at the origin.

Proof of Corollary 2.4 Equation (2.5) is immediate from Theorem 2.2 and our comments around Eq. (2.1). Equation (2.6) then follows by taking the Fourier transform and using the relations $\widehat{\delta}_0 = \lambda^d$, $\widehat{\lambda^d} = \delta_0$, and $\widehat{g\lambda^d} = \widehat{g}\lambda^d$. □

3 Determinantal Point Processes

Determinantal point processes are used to model particle configurations with repulsion; see [19], [13, Sect. 4.2] or [2, Sect. 4.2] for background information. See also [10, 16, Example 5.4(c)], where these point processes are called *fermion processes*.

In the sequel, we shall always assume the following:

$$\text{The kernel } K : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C} \text{ is continuous, Hermitian and positive-definite.} \tag{3.1}$$

Here, K is *Hermitian* if, for all $x, y \in \mathbb{R}^d$, $K(x, y) = \overline{K(y, x)}$, and *positive-definite* if, for all $k \in \mathbb{N}$ and all $x_1, \dots, x_k \in \mathbb{R}^d$, $\det(K(x_i, x_j)_{1 \leq i, j \leq k}) \geq 0$.

A point process ω on \mathbb{R}^d is called *determinantal* with kernel K if, for any $k \in \mathbb{N}$, the k -point correlation function exists and is given by

$$\varrho_k(x_1, \dots, x_k) = \det(K(x_i, x_j)_{1 \leq i, j \leq k}). \tag{3.2}$$

It is well known that, if there exists a determinantal point process with a given kernel K , its distribution is uniquely determined; compare [13, Lemma 4.2.6]. As regards existence, note first that, if K is a kernel as in Eq. (3.1), then, for any compact subset $B \subset \mathbb{R}^d$, we have an integral operator $\mathcal{K}_B : L^2(B) \rightarrow L^2(B)$ defined by

$$(\mathcal{K}_B f)(x) := \int_B K(x, y)f(y) \, dy \quad (x \in B). \tag{3.3}$$

It is well known that this operator is bounded, self-adjoint, positive-definite and of trace class; see [2, Lemma 4.2.13] and references given there. Furthermore, there is the following criterion for the existence of an associated determinantal point process.

Theorem 3.1 ([19, Theorem 3], [13, Theorem 4.5.5], [2, Corollary 4.2.22]) *Let K be a kernel as in Eq. (3.1). Then K defines a determinantal point process on \mathbb{R}^d if and only if, for any compact subset $B \subset \mathbb{R}^d$, the spectrum of the operator \mathcal{K}_B is contained in the interval $[0, 1]$.* □

Remark 3.2 Let us mention that in part of the above-mentioned literature it is assumed that the kernel K is measurable, locally square-integrable, Hermitian, positive-definite, and locally of trace class. Indeed, all the results stated above continue to hold under this weaker assumption. However, we will only be interested in stationary determinantal point processes, and the assumption of continuity is satisfied in all our examples. Furthermore, continuous kernels are convenient in that they give rise to (unique) continuous correlation functions. ◇

Henceforward, we shall always assume that the determinantal point process ω is *stationary* with mean density 1. Then, by Eq. (2.1), (the continuous versions of) the first and second correlation functions satisfy

$$\varrho_1(x) = K(x, x) = 1 \quad \text{and} \quad \varrho_2(x, y) = 1 - |K(x, y)|^2 = 1 - g(x - y) \tag{3.4}$$

for all $x, y \in \mathbb{R}^d$, where $g(x) := |K(0, x)|^2$. Note that g is positive and positive-definite; for the latter, use that $g(x - y) = |K(x, y)|^2 = K(x, y)\overline{K(x, y)}$ and that the pointwise (or Hadamard) product of positive-definite kernels is also positive-definite.

Remark 3.3 Let us emphasise that the stationarity of the determinantal point process entails the translation-invariance of the correlation functions and of the modulus of the kernel, but not necessarily that of the kernel itself, as for the Ginibre process (see Example 3.13). ◇

Lemma 3.4 *Let ω be a stationary determinantal point process with a kernel K as specified in Eq. (3.1) and with mean density 1, and let g be as in Eq. (3.4). Then, g is integrable with $\int_{\mathbb{R}^d} g(y) \, dy \leq 1$.*

Proof We use the same argument as in the proof of [2, Lemma 4.2.32]. It follows from the definitions of the ordinary and the factorial moment measures and Eq. (3.4) that, for any bounded Borel set A , one has

$$\begin{aligned}
 0 \leq \text{Var}(\omega(A)) &= \mu^{(2)}(A \times A) - (\mu^{(1)}(A))^2 = \mu^{(1)}(A) + \mu^{\bullet(2)}(A \times A) - (\mu^{(1)}(A))^2 \\
 &= \int_A 1 \, dx + \int_A \int_A (\varrho_2(x, y) - 1) \, dy \, dx = \lambda^d(A) - \int_A \int_{A+x} g(y) \, dy \, dx.
 \end{aligned}$$

Taking $A = B_n$, the ball of radius n around the origin, we get

$$\frac{1}{\lambda^d(B_n)} \int_{B_n} \int_{B_n+x} g(y) \, dy \, dx \leq 1$$

for any $n \in \mathbb{N}$, from which it follows that $\int_{\mathbb{R}^d} g(y) \, dy \leq 1$. □

Combining Lemma 3.4 with Corollary 2.4, we get the following result.

Proposition 3.5 *Let ω be a stationary and ergodic determinantal point process with a kernel K as in Eq. (3.1) and with mean density 1. Then, the autocorrelation and diffraction measures of ω are given by*

$$\gamma = \delta_0 + (1 - g)\lambda^d \quad \text{and} \quad \widehat{\gamma} = \delta_0 + (1 - \widehat{g})\lambda^d,$$

with g as in Eq. (3.4). Moreover, we have $0 \leq \widehat{g}(t) \leq 1$ for all $t \in \mathbb{R}^d$, with $\widehat{g}(t) = 1$ at most for $t = 0$. In particular, the absolutely continuous part of the diffraction measure is equivalent to Lebesgue measure.

Proof The statements about γ and $\widehat{\gamma}$ are immediate from Corollary 2.4, Eq. (3.4), and Lemma 3.4. The statements about \widehat{g} follow from the positivity and positive-definiteness of the function g and well-known properties of the Fourier transform. □

Let us now turn to the case that the kernel K itself is *translation-invariant*, which means that there exists a function $K : \mathbb{R}^d \rightarrow \mathbb{C}$ such that $K(x, y) = K(x - y)$ for all $x, y \in \mathbb{R}^d$. (By slight abuse of notation, we use the same symbol for the function and for the associated kernel.) More precisely, we will assume the following:

We have $K(x, y) = K(x - y)$ for all $x, y \in \mathbb{R}^d$, where the function $K : \mathbb{R}^d \rightarrow \mathbb{C}$ on the right-hand side is the Fourier transform of a probability density φ on \mathbb{R}^d with values in $[0, 1]$. (3.5)

Note that Condition (3.5) entails Condition (3.1).

Remark 3.6 It can be shown that, if a kernel K as in Eq. (3.1) is translation-invariant, it is the kernel of a (stationary) determinantal point process with mean density 1 if and only if it is of the form in Eq. (3.5); cf. [14] for a similar result. ◇

Let us sketch the argument why a kernel K as in Eq. (3.5) defines a stationary and ergodic determinantal point process with mean density 1. Suppose that Eq. (3.5) holds. Then, $\varphi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\widehat{\varphi} = K \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $\widehat{K} = \varphi_-$ by Fourier inversion (in the L^2 -sense), where $\varphi_-(x) = \varphi(-x)$ as before. Moreover, since $K \in L^2(\mathbb{R}^d)$, the convolution

$$(\mathcal{K}f)(x) := (K * f)(x) := \int_{\mathbb{R}^d} K(x - y)f(y) \, dy \quad (x \in \mathbb{R}^d)$$

is well-defined for any $f \in L^2(\mathbb{R}^d)$ by the Cauchy-Schwarz inequality. Furthermore, for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have

$$\widehat{\mathcal{K}f} = \widehat{K * f} = \widehat{K} \cdot \widehat{f} = \varphi_- \cdot \widehat{f}$$

and therefore

$$\mathcal{K}f = \mathcal{F}^{-1}M_{\varphi_-}\mathcal{F}f, \tag{3.6}$$

where \mathcal{F} denotes the Fourier transform on $L^2(\mathbb{R}^d)$, \mathcal{F}^{-1} its inverse, and M_{φ_-} the multiplication operator $g \mapsto g\varphi_-$ on $L^2(\mathbb{R}^d)$. By continuity, this extends to all of $L^2(\mathbb{R}^d)$, and \mathcal{K} is a bounded, self-adjoint, positive-definite convolution operator on $L^2(\mathbb{R}^d)$ with spectrum $S_\varphi \subset [0, 1]$, where

$$S_\varphi := \{y \in [0, 1] : \lambda^d(\{x \in \mathbb{R}^d : |\varphi(x) - y| < \varepsilon\}) > 0 \text{ for all } \varepsilon > 0\}$$

is the essential range of φ . Thus, the operators \mathcal{K}_B defined in Eq. (3.3) must also have spectra contained in $[0, 1]$, and it follows from Theorem 3.1 that K defines a determinantal point process ω .

Since the correlation functions determine the distribution of ω , ω is clearly stationary with mean density $K(0) = 1$. Moreover, since $K = \widehat{\varphi}$ with $\varphi \in L^1(\mathbb{R}^d)$, it follows from the Riemann–Lebesgue lemma that $K(x) \rightarrow 0$ as $|x| \rightarrow \infty$, and this implies that ω is mixing and ergodic; see [19, Theorem 7] or [2, Theorem 4.2.34].

We therefore obtain the following consequence of Proposition 3.5.

Corollary 3.7 *If K is a kernel as in Eq. (3.5), the autocorrelation and diffraction measures of the associated determinantal point process are given by*

$$\gamma = \delta_0 + (1 - |K|^2)\lambda^d \quad \text{and} \quad \widehat{\gamma} = \delta_0 + (1 - |\widehat{K}|^2)\lambda^d.$$

Equivalently,

$$\gamma = \delta_0 + (1 - |\widehat{\varphi}|^2)\lambda^d \quad \text{and} \quad \widehat{\gamma} = \delta_0 + (1 - (\varphi * \varphi_-))\lambda^d,$$

where $\varphi_-(x) := \varphi(-x)$ as above. □

Remark 3.8 (Self-reproducing kernels) Suppose that, in the situation of Proposition 3.5, the kernel K is *self-reproducing* in the sense that

$$\int_{\mathbb{R}^d} K(x, y)K(y, z) \, dy = K(x, z) \tag{3.7}$$

for all $x, z \in \mathbb{R}^d$ or, equivalently, the associated integral operator \mathcal{K} on $L^2(\mathbb{R}^d)$ (defined similarly as in (3.3), but with $B = \mathbb{R}^d$) is a projection, i.e. $\mathcal{K}^2 = \mathcal{K}$. (Let us mention without proof that \mathcal{K} is indeed a well-defined operator on $L^2(\mathbb{R}^d)$, as follows from Theorem 3.1 and Lemma 3.4.) Then, with g as in Eq. (3.4), we have

$$\widehat{g}(0) = \int |K(0, x)|^2 \, dx = \int K(0, x)K(x, 0) \, dx = K(0, 0) = 1,$$

so that the density of the absolutely continuous part of the diffraction measure equals zero at the origin.

Moreover, for a translation-invariant kernel as in Eq. (3.5), the converse is also true. Indeed, in this case, we have $|\widehat{K}|^2(0) = 1$ if and only if $\widehat{K}(t)$ is an indicator function, as already pointed out in [19]. For the convenience of the reader, let us reproduce the argument here: Using that $\widehat{f_1 f_2} = \widehat{f_1} * \widehat{f_2}$ for $f_1, f_2 \in L^2(\mathbb{R}^d)$ and that $\widehat{K} = \varphi_-$ is $[0, 1]$ -valued, we obtain

$$|\widehat{K}|^2(0) = (\widehat{K} * \widehat{K})(0) = \int \widehat{K}(t)\overline{\widehat{K}(t)} \, dt = \int (\widehat{K}(t))^2 \, dt \leq \int \widehat{K}(t) \, dt = K(0) = 1,$$

with equality if and only if $\widehat{K} = 1$ holds a.e. on the set $\{\widehat{K} \neq 0\}$. Since indicator functions correspond to projection operators by Eq. (3.6), this proves the claim. ◇

By [2, Corollary 4.2.23], ‘thinnings’ of determinantal point processes are again determinantal point processes. This leads to the following observation.

Remark 3.9 (Thinned determinantal point processes) Let ω be a determinantal point process on \mathbb{R}^d with a kernel K as in Eq. (3.1), let $0 < p \leq 1$, and let ω_p denote the point process obtained from ω by (i) deleting each point with probability $1 - p$, independently of one another, and (ii) rescaling the resulting point process so that the mean density becomes 1. Then, ω_p is the determinantal point process associated with the kernel $K_p(x, y) := K(x/p^{1/d}, y/p^{1/d})$, as follows from [2, Corollary 4.2.23]. Thus, each determinantal point process ω gives rise to an entire family $(\omega_p)_{0 < p \leq 1}$ of determinantal point processes.

Furthermore, if ω is stationary and ergodic, ω_p is also stationary and ergodic, and if g is defined as in Eq. (3.4), and g_p is the analogous function for ω_p , we have $g_p(x) = g(x/p^{1/d})$ and $\widehat{g}_p(t) = p\widehat{g}(tp^{1/d})$. Therefore, by Proposition 3.5, the autocorrelation and diffraction measures of ω_p are given by

$$\gamma_p = \delta_0 + (1 - g(x/p^{1/d}))\lambda^d \quad \text{and} \quad \widehat{\gamma}_p = \delta_0 + (1 - p\widehat{g}(tp^{1/d}))\lambda^d.$$

As $p \rightarrow 0$, the repulsion between the points decreases, and the point process converges in distribution to the homogeneous Poisson process with intensity 1.

Finally, note that if K is a translation-invariant kernel as in Eq. (3.5), the same holds for K_p . More precisely, if K is the Fourier transform of the probability density $\varphi(t)$ (with values in $[0, 1]$), then K_p is the Fourier transform of the probability density $\varphi_p(t) := p\varphi(tp^{1/d})$ (with values in $[0, p]$). In this case, the formulas for the autocorrelation and diffraction measures reduce to

$$\gamma = \delta_0 + (1 - |K|^2(x/p^{1/d}))\lambda^d \quad \text{and} \quad \widehat{\gamma} = \delta_0 + (1 - p|\widehat{K}|^2(tp^{1/d}))\lambda^d,$$

as can easily be checked. ◇

Evidently, the construction below Eq. (3.5) gives rise to a large number of examples. Let us mention some particularly interesting cases.

Example 3.10 (Sine process) An important example is given by the sine process, which corresponds to $d = 1$, $K(x) = \frac{\sin(\pi x)}{\pi x}$ and $\varphi(t) = \mathbf{1}_{[-1/2, +1/2]}(t)$. In this case, the autocorrelation and diffraction measures are given by

$$\gamma = \delta_0 + \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)\lambda \quad \text{and} \quad \widehat{\gamma} = \delta_0 + (1 - \max\{0, 1 - |t|\})\lambda.$$

This example arises in connection with the local eigenvalue statistics of the Gaussian Unitary Ensemble (GUE) in random matrix theory [2, 13, 19], and is discussed from the viewpoint of diffraction theory in [6].

By Remark 3.9, the sine process gives rise to a whole family of determinantal point processes, with $K_p(x) = \frac{\sin(\pi x/p)}{\pi x/p}$ and $\varphi_p(t) = p\mathbf{1}_{[-1/(2p), +1/(2p)]}(t)$, where $0 < p \leq 1$. The autocorrelation and diffraction measures are now given by

$$\gamma_p = \delta_0 + \left(1 - \left(\frac{\sin(\pi x/p)}{\pi x/p}\right)^2\right)\lambda \quad \text{and} \quad \widehat{\gamma}_p = \delta_0 + (1 - p\max\{0, 1 - p|t|\})\lambda.$$

See Fig. 1. Note that for $p > 1$, the function K_p does not give rise to a determinantal point process, as the condition $0 \leq \varphi_p \leq 1$ is violated. Thus, the sine process ($p = 1$) is the point process with the strongest repulsion in this determinantal family, and it seems to be the only member of this family arising in random matrix theory. ◇

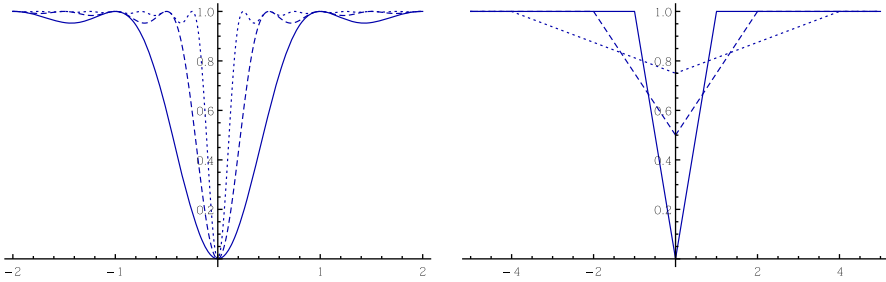


Fig. 1 The absolutely continuous parts of the autocorrelation (left) and diffraction (right) measures of the thinned sine process for $p = 1$ (normal), $p = 0.5$ (dashed) and $p = 0.25$ (dotted)

Example 3.11 Let $d \in \mathbb{N}$, let φ denote the density of the uniform distribution on the d -dimensional ball of volume 1 centered at the origin, and let $K := \widehat{\varphi}$. Then it is well known that $K(x) = \alpha^{-1/2}|x|^{-d/2}J_{d/2}(2\pi\alpha^{-1/d}|x|)$, where $\alpha := \lambda^d(B_1)$ denotes the volume of the d -dimensional unit ball B_1 . Thus, the autocorrelation and diffraction measures are given by

$$\gamma = \delta_0 + \left(1 - \alpha^{-1}|x|^{-d} (J_{d/2}(2\pi\alpha^{-1/d}|x|))^2\right)\lambda^d$$

and

$$\widehat{\gamma} = \delta_0 + \left(1 - (\varphi * \varphi)(t)\right)\lambda^d.$$

Here, $J_{d/2}$ is the Bessel function of the first kind of order $d/2$. Note that for $d = 1$, we recover the sine process. ◇

Here is another example of a rotation-invariant kernel.

Example 3.12 Take $d \in \mathbb{N}$, $K(x) = e^{-\pi|x|^2}$ and $\varphi(t) = e^{-\pi|t|^2}$. In this case, the autocorrelation and diffraction measures are given by

$$\gamma = \delta_0 + \left(1 - e^{-2\pi|x|^2}\right)\lambda^d \quad \text{and} \quad \widehat{\gamma} = \delta_0 + \left(1 - \left(\frac{1}{2}\right)^{d/2} e^{-\pi|t|^2/2}\right)\lambda^d,$$

by an application of Corollary 3.7. ◇

Note that the pair $(\gamma, \widehat{\gamma})$ comes close to being self-dual here. It seems natural to try to obtain a genuinely self-dual pair $(\gamma, \widehat{\gamma})$ by appropriate rescaling. However, this would require the transformations $e^{-\pi|x|^2} \rightarrow e^{-\pi|x|^2/2}$ for the function K and $e^{-\pi|t|^2} \rightarrow 2^{d/2}e^{-2\pi|t|^2}$ for its Fourier transform \widehat{K} , and this is not allowed as the spectrum of the corresponding convolution operator \mathcal{K} is no longer contained in the interval $[0, 1]$.

Nevertheless, at least for $d = 2$, there does exist a stationary determinantal point process with a self-dual pair $(\gamma, \widehat{\gamma})$ of the desired form, although one not coming from a translation-invariant kernel.

Example 3.13 (Ginibre process) On $\mathbb{R}^2 \simeq \mathbb{C}$, consider the kernel

$$K(z, w) = \exp\left(-\frac{1}{2}\pi|z|^2 - \frac{1}{2}\pi|w|^2 + \pi z\bar{w}\right).$$

This kernel is *not* translation-invariant, but one can show that it still defines a determinantal point process that is stationary and ergodic. By Proposition 3.5, the autocorrelation and diffraction measures are given by

$$\gamma = \delta_0 + \left(1 - e^{-\pi|x|^2}\right)\lambda^2 \quad \text{and} \quad \widehat{\gamma} = \delta_0 + \left(1 - e^{-\pi|t|^2}\right)\lambda^2.$$

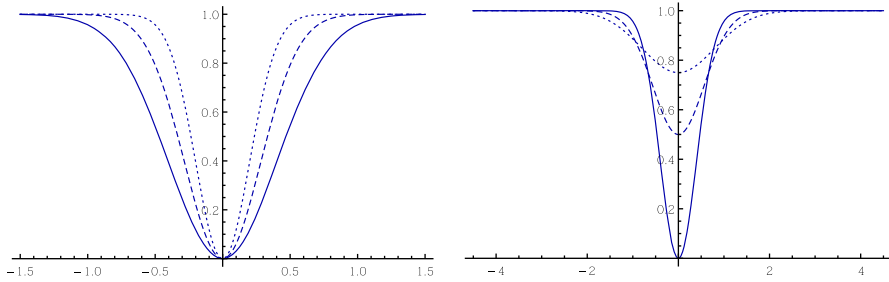


Fig. 2 The absolutely continuous parts of the autocorrelation (*left*) and diffraction (*right*) measures (viewed along a line through the origin) of the thinned Ginibre process for $p = 1$ (normal), $p = 0.5$ (*dashed*) and $p = 0.25$ (*dotted*)

Note that the pair $(\gamma, \widehat{\gamma})$ is self-dual here. Note also that the diffraction density vanishes at the origin. (Indeed, the integral operator \mathcal{K} determined by the kernel K is a projection operator here.) This example arises in connection with the local eigenvalue statistics of the Ginibre Ensemble in random matrix theory [2, 13, 19], and is discussed from the viewpoint of diffraction theory in [6].

Similarly as above, by Remark 3.9, we may also consider thinned versions of the Ginibre process. Here, the autocorrelation and diffraction measures are given by

$$\gamma_p = \delta_0 + (1 - e^{-\pi|x|^2/p})\lambda^2 \quad \text{and} \quad \widehat{\gamma}_p = \delta_0 + (1 - pe^{-\pi p|t|^2})\lambda^2,$$

via the usual reasoning (Fig. 2). ◇

Example 3.14 (Renewal process) Another interesting example is given by the class of those stationary determinantal point processes which are simultaneously renewal processes; see [19, Sect. 2.4] and references given there. Here, $d = 1$, $K(x) = \exp(-|x|/\alpha)$ and $\varphi(t) = \frac{2\alpha}{1+(2\pi\alpha t)^2}$, where $0 < \alpha \leq \frac{1}{2}$. The density of the increments of the associated renewal process is given by

$$f_\alpha(x) = \frac{2}{\sqrt{1-2\alpha}} e^{-x/\alpha} \sinh(\sqrt{1-2\alpha}(x/\alpha)) \mathbf{1}_{(0,\infty)}(x),$$

see [19, Eq. 2.42]. The autocorrelation and diffraction measures are given by

$$\gamma_\alpha = \delta_0 + (1 - \exp(-2|x|/\alpha))\lambda \quad \text{and} \quad \widehat{\gamma}_\alpha = \delta_0 + \left(1 - \frac{\alpha}{1+\pi^2\alpha^2 t^2}\right)\lambda.$$

See Fig. 3. Of course, this can also be obtained from the density f_α and [3, Theorem 1], which provides formulas for the autocorrelation and diffraction measures of general renewal processes.

Similarly to what we saw above, this family of point processes approaches the homogeneous Poisson process as $\alpha \rightarrow 0$, while the kernel does not define a determinantal point process for $\alpha > 1/2$. Note also that for $\alpha = 1/2$ the distribution of the increments is the gamma distribution with the density $4xe^{-2x}$ and that all other members of the family can be obtained from the associated determinantal point process by the thinning procedure described in Remark 3.9. ◇

Example 3.15 Let Q_1 be the Poisson distribution with parameter 1, let Q_2 be the compound Poisson distribution with parameter 1 and compounding distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$, and let

$$\varphi_1(x) := \int_{\mathbb{R}} \mathbf{1}_{[-1/2,+1,2]}(x-y) dQ_1(y) \quad \text{and} \quad \varphi_2(x) := \int_{\mathbb{R}} \mathbf{1}_{[-1/2,+1,2]}(x-y) dQ_2(y).$$

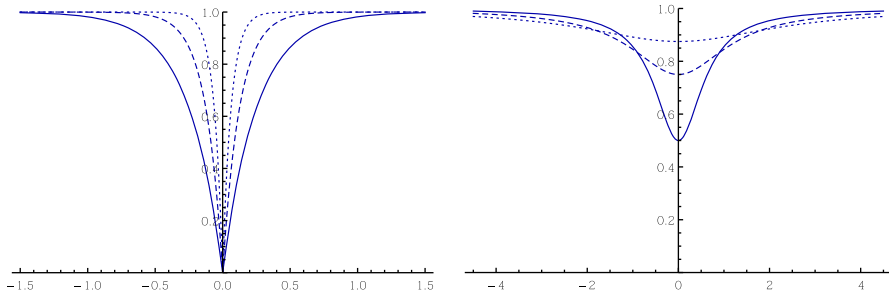


Fig. 3 The absolutely continuous parts of the autocorrelation (left) and diffraction (right) measures of the renewal process for $\alpha = 0.5$ (normal), $\alpha = 0.25$ (dashed) and $\alpha = 0.125$ (dotted)

Then, φ_1 and φ_2 are probability densities bounded by 1, and the associated functions K_1 and K_2 read

$$K_1(x) = \exp(e^{-2\pi i x} - 1) \frac{\sin(\pi x)}{\pi x} \quad \text{and} \quad K_2(x) = \exp(\cos(2\pi x) - 1) \frac{\sin(\pi x)}{\pi x}.$$

Since

$$|K_1(x)|^2 = \exp(2 \cos(2\pi x) - 2) \left(\frac{\sin \pi x}{\pi x}\right)^2 = |K_2(x)|^2,$$

it follows that the associated stationary determinantal point processes have the same autocorrelation and diffraction measures. However, the point processes themselves are *not* the same, as can be verified by comparing the 3-point correlation functions.

Thus, even within the restricted class of determinantal point processes with a translation-invariant kernel as in Eq. (3.5), the *inverse problem* to reconstruct the distribution of a point process from its diffraction measure does not have a unique solution; see [4] for background information and other examples. \diamond

Remark 3.16 (Diffraction spectrum versus dynamical spectrum) Let ω be a stationary and ergodic point process for which the first and second moment measures exist. Then, the diffraction measure (or rather its equivalence class) is also called the *diffraction spectrum* of ω , whereas the *dynamical spectrum* of ω is the spectrum of the dynamical system defined by the shift operators $T_x, x \in \mathbb{R}^d$, on $(\mathcal{N}(\mathbb{R}^d), \mathcal{N}(\mathbb{R}^d), \mathbb{P}_\omega)$. More precisely, the dynamical spectrum may be defined as the maximal spectral type of the group of unitary operators $f \mapsto f \circ T_x$ on $L^2(\mathcal{N}(\mathbb{R}^d), \mathcal{N}(\mathbb{R}^d), \mathbb{P}_\omega)$; compare [9] or [17] for details. It is of interest in diffraction theory to clarify the relationship between the diffraction spectrum and the dynamical spectrum; see [7] and references therein for background information.

Let us consider the diffraction spectrum and the dynamical spectrum for a determinantal point process with a translation-invariant kernel as in Eq. (3.5). On the one hand, as we have seen above, the diffraction spectrum is equivalent to $\delta_0 + \lambda^d$. On the other hand, the dynamical spectrum is also equivalent to $\delta_0 + \lambda^d$. Indeed, it was shown in [19] that the determinantal point process is absolutely continuous (when viewed as a dynamical system), so that the dynamical spectrum is dominated by $\delta_0 + \lambda^d$. Furthermore, as also shown in [19], the centred linear statistics $\omega \mapsto \int f(x) d\omega(x) - \mathbb{E}(\int f(x) d\omega(x))$, where $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, possess the spectral measure $(1 - |\widehat{K}|^2(t)) |\widehat{f}(t)|^2 \lambda^d$, which implies that the dynamical spectrum must be equivalent to $\delta_0 + \lambda^d$. Thus, the ‘diffraction spectrum’ and the ‘dynamical spectrum’ are equivalent here.

It seems interesting to ask whether this is a general property of translation-invariant point processes with “good” mixing properties. Let us mention here that the determinantal point process is not only mixing, but even mixing of all orders; see [19]. \diamond

4 Permanent Point Processes

We now turn to *permanent point processes* where the correlation functions are given by the permanent instead of the determinant of a certain kernel. For such processes, the particles tend to form clumps, whereas they repel one another for determinantal point processes. See [13, Sect. 4.9] for background information. See also [16] and [10, Example 6.2 (b)], where these point processes are called *boson processes*.

Let K be a kernel as in Eq. (3.1). A point process ω on \mathbb{R}^d is called *permanent* with kernel K if, for any $k \in \mathbb{N}$, the k -point correlation function exists and is given by

$$\varrho_k(x_1, \dots, x_k) = \text{per}(K(x_i, x_j)_{1 \leq i, j \leq k}), \tag{4.1}$$

where per denotes the permanent; compare [13, Definition 2.1.5].

We refer to [13, Sect. 4.9] for the proof of the following result.

Proposition 4.1 ([13, Corollary 4.9.3]) *For any kernel K as in Eq. (3.1), there exists a permanent point process with kernel K on \mathbb{R}^d .* \square

Furthermore, it is not hard to see that if there exists a permanent point process ω with a given kernel K as in Eq. (3.1), its distribution is uniquely determined. This follows from the observation that, for any bounded Borel set A , the probability generating function defined by $z \mapsto \mathbb{E}(z^{\omega(A)})$ exists in an open neighborhood of the unit ball; see [16, Theorem 6].

Remark 4.2 Let us note here that the probability generating function $z \mapsto \mathbb{E}(z^{\omega(A)})$ is an entire function for determinantal, but generally not for permanent point processes. The reason for this difference is that determinants satisfy Hadamard’s inequality, while there is no comparable estimate for permanents. \diamond

Of course, we will be interested in permanent point processes which are also stationary with mean density 1. For brevity, let us directly turn to permanent point processes with translation-invariant kernels. More precisely, we shall assume the following:

$$\begin{aligned} K(x, y) = K(x - y) \text{ holds for all } x, y \in \mathbb{R}^d, \text{ where the function } K : \mathbb{R}^d \rightarrow \mathbb{C} \\ \text{on the right-hand side is the Fourier transform of a probability density} \\ \varphi \text{ on } \mathbb{R}^d. \end{aligned} \tag{4.2}$$

Note that, in contrast to Condition (3.5), the probability density need not be bounded here. However, Condition (4.2) still implies Condition (3.1).

Now suppose that Condition (4.2) holds. Let ω denote the associated permanent point process, which exists by Proposition 4.1. Since the correlation functions determine the distribution of ω , ω is clearly stationary with mean density $K(0) = 1$. Moreover, by the Riemann–Lebesgue lemma, we have $K(x) \rightarrow 0$ as $|x| \rightarrow \infty$, and a variation of the proof of [19, Theorem 7] or [2, Theorem 4.2.34] shows that ω is also mixing and hence ergodic.

Furthermore, by Eq. (2.1), (the continuous versions of) the first and second correlation functions of ω satisfy

$$\varrho_1(x_1) = K(0) = 1 \text{ and } \varrho_2(x_1, x_2) = 1 + |K(x_1 - x_2)|^2. \tag{4.3}$$

Therefore, similarly as in Corollary 2.4, we have the following result:

Proposition 4.3 *Let K be a kernel as in Eq. (4.2). Then, the autocorrelation and diffraction measures of the associated permanental point process ω are given by*

$$\gamma = \delta_0 + (1 + |K|^2)\lambda^d \quad \text{and} \quad \widehat{\gamma} = \delta_0 + (1 + (\varphi * \varphi_-))\lambda^d,$$

where $\varphi_-(x) := \varphi(-x)$ as before.

Note that the autocorrelation and diffraction densities are larger than 1 here, in line with the clumping picture. Also, under the above-mentioned assumptions, the diffraction measure is absolutely continuous apart from the Bragg peak at the origin, and the absolutely continuous part of the diffraction measure is equivalent to Lebesgue measure.

Note that if K were square-integrable, we could also write $\widehat{|K|^2}$ instead of $(\varphi * \varphi_-)$ in the result for the diffraction measure, similarly as in Corollary 3.7. However, in general, K need not be square-integrable here.

Proof of Proposition 4.3 This follows from the proof of Corollary 2.4 and the observation that $(|K|^2 \lambda)^\widehat{=} = (\varphi * \varphi_-) \lambda$.

To check this observation, note that $K = \widehat{\varphi}$ implies $|K|^2 = (\varphi * \varphi_-)^\widehat{=}$. Thus, $((\varphi * \varphi_-) \lambda)^\widehat{=} = |K|^2 \lambda$, and the desired relation follows by Fourier inversion in the space of positive and positive-definite measures, and the fact that $(\varphi * \varphi_-) \lambda$ is a symmetric measure. \square

Clearly, all examples for determinantal point processes translate into examples for permanental point processes. In particular, the thinning procedure described in Remark 3.9 also extends to permanental point processes. However, the kernel K_p introduced there may now be considered also for $p > 1$, where it still defines a permanental point process ω_p . (Of course, the probabilistic description in terms of thinning breaks down in this region. However, at least for natural numbers p , ω_p may be viewed as the superposition of p independent copies of ω . In fact, this is not surprising in view of the representation as a Cox process to be mentioned below.) Thus, any permanental point process ω gives rise to a whole family of permanental point processes $(\omega_p)_{0 < p < \infty}$. These families interpolate between the homogeneous Poisson process (for $p \rightarrow 0$) and the *non-ergodic* mixed Poisson process with the exponential distribution with parameter 1 as mixing distribution (for $p \rightarrow \infty$).

Remark 4.4 A careful analysis of the arguments in [19] shows that, with the obvious modifications, Remark 3.16 continues to hold for permanental point processes as in Proposition 4.3. In particular, both the diffraction spectrum and the dynamical spectrum are equivalent to $\delta_0 + \lambda^d$ here. \diamond

Remark 4.5 As mentioned in the last section, our Condition (3.5) in the investigation of determinantal point processes is essential in the sense that it must be satisfied for any determinantal point process with a *translation-invariant* kernel satisfying Eq. (3.1). In contrast, our Condition (4.2) in the investigation of permanental point processes could be relaxed.

For instance, we could start from the assumption that the function $K : \mathbb{R}^d \rightarrow \mathbb{C}$ in Eq. (4.2) is the Fourier transform of a continuous (but not absolutely continuous) probability measure Q on \mathbb{R}^d . Let ω be the associated permanental point process, which exists by Prop. 4.1. Then, it is not necessarily true that $K(x) \rightarrow 0$ as $|x| \rightarrow \infty$, but one can convince oneself that ω is still stationary and ergodic; see Sect. 5 for details. Hence, an argument similar to the proof of Prop. 4.3 leads to the conclusion that the autocorrelation and diffraction measures of ω are given by

$$\gamma = \delta_0 + (1 + |K|^2)\lambda^d \quad \text{and} \quad \widehat{\gamma} = \delta_0 + \lambda^d + (Q * Q_-),$$

where $Q_-(A) := Q(-A)$ denotes the reflection of Q at the origin. Of course, the diffraction measure may now contain a singular continuous component.

Note that we required the probability measure Q to be continuous. If the probability measure Q is not continuous, the corresponding Fourier transform K still gives rise to a stationary permanental point process ω , but this point process need not be ergodic anymore. A simple (counter)example is given by the kernel $K \equiv 1$, for which the associated permanental point process is a mixed Poisson process with directing measure $Z \lambda$, where Z has an exponential distribution with parameter 1. This point process is stationary but not ergodic, and the autocorrelation measure is equal to $\gamma = Z \delta_0 + Z^2 \lambda$, thus depends on the realisation.

5 Cox Processes

Recall the definition of a Cox process from [10, Sect. 6.2]. Let ω_0 be a random measure on \mathbb{R}^d . A point process ω on \mathbb{R}^d is called *Cox process directed by ω_0* if, conditionally on ω_0 (i.e. when ω_0 is regarded as fixed), ω is a Poisson process with intensity measure ω_0 .

It is a standard result that a Cox process on \mathbb{R}^d is simple if and only if the directing measure is continuous. Furthermore, it is well known that a Cox process on \mathbb{R}^d is stationary [ergodic, mixing] if and only if the directing measure is stationary [ergodic, mixing]; compare [11, Proposition 12.3.7].

For the formulation of the next result, let us recall that, for a general random measure ω , the k th moment measure $\mu^{(k)}$ is defined as the expectation measure (if it exists) of the product measure ω^k , and for a stationary random measure ω , the k th reduced moment measure $\mu_{\text{red}}^{(k)}$ is then defined similarly as in Sect. 2. Furthermore, for a stationary random measure with mean density 1, we may define the *reduced covariance measure* by

$$\kappa_{\text{red}}^{(2)} = \mu_{\text{red}}^{(2)} - \lambda^d. \tag{5.1}$$

For a stationary point process with mean density 1, we may additionally define the *reduced factorial covariance measure* by

$$\kappa_{\text{red}}^{\bullet(2)} = \mu_{\text{red}}^{\bullet(2)} - \lambda^d. \tag{5.2}$$

From Theorem 2.2, we obtain the following result.

Proposition 5.1 *Let ω be a stationary and ergodic Cox process with a directing measure ω_0 for which the first and second moment measures exist, and suppose that ω_0 , and hence ω , have mean density 1. Then, almost surely, the autocorrelation and diffraction measures of ω are given by*

$$\gamma = \delta_0 + \lambda^d + \kappa_0 \quad \text{and} \quad \widehat{\gamma} = \delta_0 + \lambda^d + \widehat{\kappa}_0,$$

where κ_0 is the reduced covariance measure of ω_0 , and $\widehat{\kappa}_0$ its Fourier transform.

Let us note that the reduced covariance measure κ_0 is a positive-definite measure, so that the Fourier transform $\widehat{\kappa}_0$ exists as a positive measure. Also, let us note that $\widehat{\kappa}_0$ is also known as the *Bartlett spectrum* of ω_0 in the literature; see e.g. [10, Chapters 8.1 and 8.2] for more information. (More precisely, the Bartlett spectrum is defined as the inverse Fourier transform of the reduced covariance measure. However, as the reduced covariance measure is symmetric, the Fourier transform and the inverse Fourier transform coincide, at least for our definition of the Fourier transform.)

Proof of Proposition 5.1 By [10, Proposition 6.2.2], the reduced factorial covariance measure of the Cox process ω equals the reduced covariance measure of the directing measure ω_0 , i.e. $\kappa_{\text{red}}^{\bullet(2)}(\omega) = \kappa_{\text{red}}^{(2)}(\omega_0)$. Here, the measures in the brackets indicate which measures the moment measures belong to. It therefore follows from Theorem 2.2 and Eq. (5.2) that the autocorrelation of ω is given by

$$\gamma = \delta_0 + \mu_{\text{red}}^{\bullet(2)}(\omega) = \delta_0 + \lambda^d + \kappa_{\text{red}}^{\bullet(2)}(\omega) = \delta_0 + \lambda^d + \kappa_{\text{red}}^{(2)}(\omega_0).$$

Taking the Fourier transform completes the proof. □

Remark 5.2 It is well known that permanental point processes are special cases of Cox processes; see [16] or [13, Proposition 4.9.2]. For the convenience of the reader, and since we can use this connection to establish the ergodicity of permanental point processes, let us outline the argument in a simple situation.

Suppose that the kernel K is *translation-invariant* and satisfies Eq. (3.1) and $K(0, 0) = 1$. Then, the underlying function $K : \mathbb{R}^d \rightarrow \mathbb{C}$ is continuous, Hermitian, and positive-definite with $K(0) = 1$, and hence the covariance function of a stationary complex Gaussian process $(X_t)_{t \in \mathbb{R}^d}$. To avoid technical issues, let us assume that $(X_t)_{t \in \mathbb{R}^d}$ has continuous sample paths. Then, it is not difficult to check (using Wick’s formula for the moments of complex Gaussian random variables; see e.g. [13, Lemma 2.1.7]) that the Cox process ω directed by $\omega_0 := |X_t|^2 \lambda^d$ is a permanental point process with kernel K .

Furthermore, if $(X_t)_{t \in \mathbb{R}^d}$ is stationary [ergodic, mixing], then ω_0 is also stationary [ergodic, mixing], being a factor in the sense of ergodic theory, and this implies that ω is stationary [ergodic, mixing] by the above-mentioned results on Cox processes. Thus, we obtain useful sufficient conditions for ergodicity and mixing of Cox processes from the well-known theory of stationary Gaussian processes: ω is ergodic if the spectral measure of $(X_t)_{t \in \mathbb{R}^d}$ is continuous, and ω is mixing if $K(x) \rightarrow 0$ as $|x| \rightarrow \infty$. ◇

Let us end this section with an example demonstrating that stationary and ergodic Cox processes can have additional Bragg peaks apart from the origin.

Example 5.3 Consider the stochastic process $X = (X_t)_{t \in \mathbb{R}}$ with $X_t = 1 + \cos(2\pi(t + U))$, where U is uniformly distributed on $[0, 1]$. Let $\omega_0 := X_t \lambda$ be the random measure with density X_t , and let ω be the Cox process directed by ω_0 . Then, one can check that ω_0 , and hence ω , is stationary and ergodic. Furthermore, it is easy to check that the reduced covariance measure of ω_0 is given by $\kappa_0 = \frac{1}{2} \cos(2\pi x) \lambda$. It therefore follows from Proposition 5.1 that the autocorrelation and diffraction measures of ω are given by

$$\gamma = \delta_0 + \lambda + \frac{1}{2} \cos(2\pi x) \lambda \quad \text{and} \quad \widehat{\gamma} = \delta_0 + \lambda + \frac{1}{4}(\delta_{-1} + \delta_{+1}),$$

respectively. ◇

6 Zeros of Gaussian Random Analytic Functions

A *Gaussian random analytic function* is a random variable f whose values are analytic functions on \mathbb{C} with the property that, for all $n \in \mathbb{N}$ and for all choices of $z_1, \dots, z_n \in \mathbb{C}$, the n -tuple $(f(z_1), \dots, f(z_n))$ has a complex Gaussian distribution with mean 0. One such example is the Gaussian random analytic function f given by

$$f(z) := \sum_{n=0}^{\infty} a_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n, \tag{6.1}$$

where L is a positive constant and the a_n are i.i.d. standard complex Gaussian random variables. We are interested in the zero set of f viewed as a point process on $\mathbb{C} \simeq \mathbb{R}^2$. By [13, Proposition 2.3.4], the distribution of the zero set of f is invariant under translations (and also under rotations), and by [13, Corollary 2.5.4], f is essentially the only Gaussian random analytic function with this property. Furthermore, by the proof of [13, Proposition 2.3.7], the zero set of f defines an ergodic point process with respect to the group of translations. Indeed, the zero set of f is even mixing:

Proposition 6.1 *The point process given by the zero set of the Gaussian random analytic function f in Eq. (6.1) is mixing.*

Proof We use a similar argument as in the proof of [13, Proposition 2.3.7]. For convenience, let us suppose that $L = 1$. Then, using that the covariance kernel of the complex Gaussian process f is given by $K(z, w) = \exp(z\bar{w})$ (cf. Equation (6.4) below), it is straightforward to check that, for any $\zeta \in \mathbb{C}$, the complex Gaussian processes $(f(z + \zeta)e^{-|z + \zeta|^2/2}e^{-i \operatorname{Im}(z\bar{\zeta})})_{z \in \mathbb{C}}$ and $(f(z)e^{-|z|^2/2})_{z \in \mathbb{C}}$ have the same distribution. As a consequence, the stochastic process $(v(z))_{z \in \mathbb{C}}$ given by

$$v(z) := |f(z)|e^{-|z|^2/2}$$

is stationary, i.e. for any $\zeta \in \mathbb{C}$, $(v(z + \zeta))_{z \in \mathbb{C}}$ and $(v(z))_{z \in \mathbb{C}}$ have the same distribution. Furthermore, the stochastic process $(v(z))_{z \in \mathbb{C}}$ is mixing, i.e. for any events $A, B \in \mathcal{B}(\mathcal{C}(\mathbb{C}))$,

$$\mathbb{P}((v(z + \zeta))_{z \in \mathbb{C}} \in A \wedge (v(z))_{z \in \mathbb{C}} \in B) \xrightarrow{|\zeta| \rightarrow \infty} \mathbb{P}((v(z))_{z \in \mathbb{C}} \in A) \mathbb{P}((v(z))_{z \in \mathbb{C}} \in B). \tag{6.2}$$

Here, $\mathcal{C}(\mathbb{C})$ denotes the space of continuous functions $\varphi : \mathbb{C} \rightarrow \mathbb{R}$ (endowed with the topology of locally uniform convergence), and $\mathcal{B}(\mathcal{C}(\mathbb{C}))$ denotes its Borel σ -field, which coincides with the Borel σ -field generated by the projections π_z , with $z \in \mathbb{C}$. By standard arguments, it suffices to check (6.2) for events A and B of the form $A = \bigcap_{j=1}^m \pi_{z_j}^{-1}(A_j)$ and $B = \bigcap_{k=1}^n \pi_{w_k}^{-1}(B_k)$, where $m, n \in \mathbb{N}$, $z_j, w_k \in \mathbb{C}$, and $A_j, B_k \subset \mathbb{R}$ are Borel sets. But now, again using that the covariance kernel of f is given by $K(z, w) = \exp(z\bar{w})$, it is easy to see that the random vectors $(f(z_j + \zeta)e^{-|z_j + \zeta|^2/2}e^{-i \operatorname{Im}(z_j\bar{\zeta})})_{j=1, \dots, m}$ and $(f(w_k)e^{-|w_k|^2/2})_{k=1, \dots, n}$ are asymptotically independent as $|\zeta| \rightarrow \infty$. Therefore,

$$\begin{aligned} &\mathbb{P}(|f(z_j + \zeta)|e^{-|z_j + \zeta|^2/2} \in A_j \forall j \wedge |f(w_k)|e^{-|w_k|^2/2} \in B_k \forall k) \\ &\xrightarrow{|\zeta| \rightarrow \infty} \mathbb{P}(|f(z_j)|e^{-|z_j|^2/2} \in A_j \forall j) \mathbb{P}(|f(w_k)|e^{-|w_k|^2/2} \in B_k \forall k), \end{aligned}$$

and (6.2) is proved. Since the zero set of the Gaussian random analytic function f may be represented as a factor (in the sense of ergodic theory) of the stochastic process $(v(z))_{z \in \mathbb{C}}$, this establishes Proposition 6.1. \square

Since the point process of zeros is ergodic and the moment measures of any order exist (see below for details), the autocorrelation and diffraction measures exist by Theorem 2.2. By [13, Corollary 3.4.2], the k -point correlation functions of the zero set of f are given by

$$\varrho_k(z_1, \dots, z_k) = \operatorname{per}(C - BA^{-1}B^*) / \det(\pi A), \tag{6.3}$$

where the $k \times k$ matrices A, B, C have the entries

$$A(i, j) := \mathbb{E}(f(z_i)\overline{f(z_j)}), \quad B(i, j) := \mathbb{E}(f'(z_i)\overline{f'(z_j)}), \quad C(i, j) := \mathbb{E}(f'(z_i)\overline{f'(z_j)}),$$

and B^* denotes the conjugate transpose of B . Straightforward calculations yield

$$\mathbb{E}(f(z)\overline{f(w)}) = \sum_{n=0}^{\infty} \mathbb{E}(|a_n|^2) \frac{L^n}{n!} z^n \overline{w}^n = \exp(Lz\overline{w}), \tag{6.4}$$

$$\mathbb{E}(f'(z)\overline{f'(w)}) = \sum_{n=0}^{\infty} \mathbb{E}(|a_n|^2) \frac{L^n}{n!} nz^{n-1}\overline{w}^n = L\overline{w} \exp(Lz\overline{w}), \tag{6.5}$$

$$\mathbb{E}(f'(z)\overline{f'(w)}) = \sum_{n=0}^{\infty} \mathbb{E}(|a_n|^2) \frac{L^n}{n!} nz^{n-1}n\overline{w}^{n-1} = (L^2z\overline{w} + L) \exp(Lz\overline{w}). \tag{6.6}$$

If we insert this into Eq. (6.3) (for $k = 1$ and $k = 2$), we find after some calculation that

$$\varrho_1(z) = \frac{L}{\pi}$$

and

$$\begin{aligned} \varrho_2(z_1, z_2) &= \frac{L^2 \exp(L|z_1 - z_2|^2) \left(1 - \exp(L|z_1 - z_2|^2) + L|z_1 - z_2|^2\right)^2}{\pi^2 \left(\exp(L|z_1 - z_2|^2) - 1\right)^3} \\ &+ \frac{L^2 \left(1 - \exp(L|z_1 - z_2|^2) + L|z_1 - z_2|^2 \exp(L|z_1 - z_2|^2)\right)^2}{\pi^2 \left(\exp(L|z_1 - z_2|^2) - 1\right)^3}. \end{aligned}$$

In particular, the 2-point correlation function depends on z_1 and z_2 only via their distance $r := |z_1 - z_2|$, as it should. From now on, we will always set $L = \pi$, so that the mean density of the point process is equal to 1. Then, expressing the two-point correlation function in terms of r , we obtain

$$\varrho_2(0, r) = 1 - g(r),$$

where

$$g(r) := \frac{e^{-\pi r^2} (-2 + 4\pi r^2 - \pi^2 r^4) + e^{-2\pi r^2} (4 - 4\pi r^2 - \pi^2 r^4) - 2e^{-3\pi r^2}}{(1 - e^{-\pi r^2})^3}. \tag{6.7}$$

Moreover, an explicit calculation with the Fourier transform of a radially symmetric function shows that the Fourier transform of g is given by

$$h(s) := 1 + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{(k-2)!} \pi^k s^{2k} \zeta(k+1); \tag{6.8}$$

see below for details. Therefore, Corollary 2.4 implies the following result.

Theorem 6.2 *Let ω be the point process given by the zeros of the Gaussian random analytic function in Eq. (6.1), with $L = \pi$. Then, the autocorrelation and diffraction measures of ω are given by*

$$\gamma = \delta_0 + (1 - g(r))\lambda^2 \quad \text{and} \quad \widehat{\gamma} = \delta_0 + (1 - h(s))\lambda^2,$$

where $r \equiv r(x_1, x_2) := \sqrt{x_1^2 + x_2^2}$, $s \equiv s(t_1, t_2) := \sqrt{t_1^2 + t_2^2}$, and $g(r)$ and $h(s)$ are the functions defined in Eqs. (6.7) and (6.8).

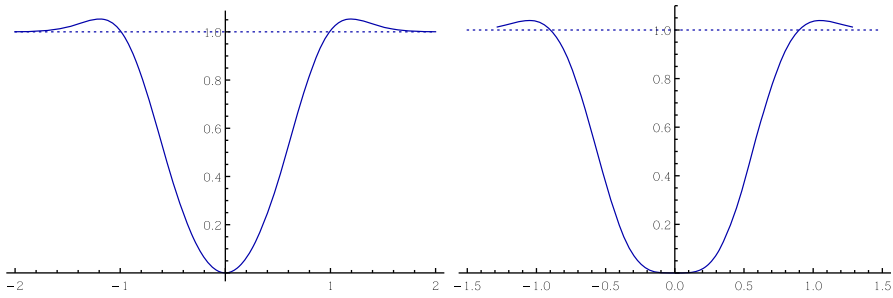


Fig. 4 The autocorrelation (left) and diffraction (right) density (viewed along a line through the origin) of the point process that derives from the zeros of the Gaussian random analytic function (6.1)

We can see from Fig. 4 that the diffraction density exceeds 1 for $s \approx 1$. Consequently, as already observed in [13], the zero set of the Gaussian random analytic function f is not a determinantal point process.

Proof of Theorem 6.2 Let

$$\varphi(u) := \frac{e^{-u}(-2 + 4u - u^2) + e^{-2u}(4 - 4u - u^2) - 2e^{-3u}}{(1 - e^{-u})^3}, \quad u > 0. \tag{6.9}$$

A straightforward Taylor expansion shows that $\varphi(u) = 1 + \mathcal{O}(u)$ as $u \rightarrow 0$. Thus, $\varphi(u)$ has a continuous extension at zero, with $\varphi(0) = 1$. Moreover, there exists a constant $C > 0$ such that $|\varphi(u)| \leq Ce^{-u/2}$ for all $u \geq 0$. It therefore follows that the function $g(r) = \varphi(\pi r^2)$ (regarded as a radially symmetric function on \mathbb{R}^2) is integrable on \mathbb{R}^2 .

We can now compute the Fourier transform of g . In general, when $g \in L^1(\mathbb{R}^2)$ is radially symmetric, which means that it depends only on the Euclidean norm $r = |x|$, the analogous property holds for \widehat{g} . Writing $g(r)$ and $\widehat{g}(s)$ instead of $g(x)$ and $\widehat{g}(t)$, respectively, and using polar coordinates, one obtains

$$\widehat{g}(s) = \int_0^\infty \int_0^{2\pi} e^{-2\pi i r s \cos(\vartheta)} d\vartheta g(r) r dr = 2\pi \int_0^\infty r g(r) J_0(2\pi r s) dr, \tag{6.10}$$

which is essentially the Hankel transform in one dimension. Here, we have employed the classic identity

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i z \cos(\vartheta)} d\vartheta = J_0(z),$$

where J_0 is the Bessel function of the first kind of order 0, with series expansion

$$J_0(z) = \sum_{m=0}^\infty (-1)^m \frac{z^{2m}}{4^m (m!)^2}.$$

Clearly, $J_0(0) = 1$, while $J_0(r) = \mathcal{O}(r^{-1/2})$ as $r \rightarrow \infty$; compare [1]. Applying the identity (6.10) to the function $g(r) = \varphi(\pi r^2)$ from (6.7), we obtain, after a change of variables,

$$\widehat{g}(s) = \int_0^\infty \varphi(u) J_0(\sqrt{4\pi u} s) du = \sum_{m=0}^\infty \frac{(-1)^m \pi^m s^{2m}}{(m!)^2} \int_0^\infty u^m \varphi(u) du. \tag{6.11}$$

The exchange of integration and summation is justified by dominated convergence, using the estimate $|\varphi(u)| \leq Ce^{-u/2}$.

Lemma 6.3 *The function $I : [0, \infty) \rightarrow \mathbb{R}$ defined by $I(\alpha) := \int_0^\infty u^\alpha \varphi(u) \, du$ is continuous on $[0, \infty)$, with*

$$I(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0, \\ \alpha(1 - \alpha)\Gamma(\alpha + 1)\zeta(\alpha + 1), & \text{if } \alpha > 0, \end{cases}$$

where Γ is the gamma function and ζ is Riemann’s zeta function.

We thus have $I(0) = 1$, $I(1) = 0$ and $I(m) = -m(m - 1)m!\zeta(m + 1)$ for $m \in \mathbb{N}$, $m \geq 2$. Inserting this into (6.11) gives

$$\widehat{g}(s) = 1 + \sum_{m=2}^\infty \frac{(-1)^{m+1}}{(m - 2)!} (\pi s^2)^m \zeta(m + 1),$$

as claimed.

Proof of Lemma 6.3 Using the estimate $|\varphi(u)| \leq Ce^{-u/2}$, it is straightforward to see that $u \mapsto u^\alpha \varphi(u)$ is integrable for any $\alpha \geq 0$ and, by dominated convergence, $\alpha \mapsto I(\alpha)$ is continuous at any $\alpha \geq 0$.

Observe that $(1 - e^{-u})^{-3} = \sum_{n=0}^\infty \frac{1}{2}(n + 1)(n + 2)e^{-nu}$ for any $u > 0$. Inserting this into (6.9) gives

$$\varphi(u) = - \sum_{n=1}^\infty (n^2u^2 - 4nu + 2)e^{-nu}$$

for any $u > 0$. Since $\int_0^\infty u^{x-1}e^{-u} \, du = \Gamma(x)$ and $\Gamma(x + 1) = x\Gamma(x)$ for any $x > 0$, we find for any fixed $\alpha > 0$ that

$$\begin{aligned} I(\alpha) &= - \sum_{n=1}^\infty \frac{\Gamma(\alpha + 3)}{n^{\alpha+1}} + 4 \sum_{n=1}^\infty \frac{\Gamma(\alpha + 2)}{n^{\alpha+1}} - 2 \sum_{n=1}^\infty \frac{\Gamma(\alpha + 1)}{n^{\alpha+1}} \\ &= \alpha(1 - \alpha)\Gamma(\alpha + 1)\zeta(\alpha + 1). \end{aligned}$$

Here, the termwise integration is justified by dominated convergence as long as $\alpha > 0$.

Recalling that $\alpha\zeta(\alpha + 1) = 1 + \mathcal{O}(1)$ as $\alpha \rightarrow 0$ and using the continuity of $\alpha \mapsto I(\alpha)$ at $\alpha = 0$, we finally obtain

$$I(0) = \lim_{\alpha \rightarrow 0} I(\alpha) = \Gamma(1) = 1,$$

which completes the argument. □

Although we have seen in Proposition 6.1 that the point process of zeros of the Gaussian random analytic function f is mixing, any realisation of it displays an amazing amount of structure. Indeed, the zero set can be given a remarkable visual interpretation as tilings. The details can be found in [13, Chapter 8], but briefly, the tilings arise as the basins of descent of the ‘potential function’ u on \mathbb{C} defined by

$$u(z) := \log |f(z)| - \frac{1}{2}|z|^2.$$

This function goes to $-\infty$ at the zeros of f , and if one follows the gradient curves defined by the equation

$$\frac{dZ(t)}{dt} = \nabla u(Z(t)),$$

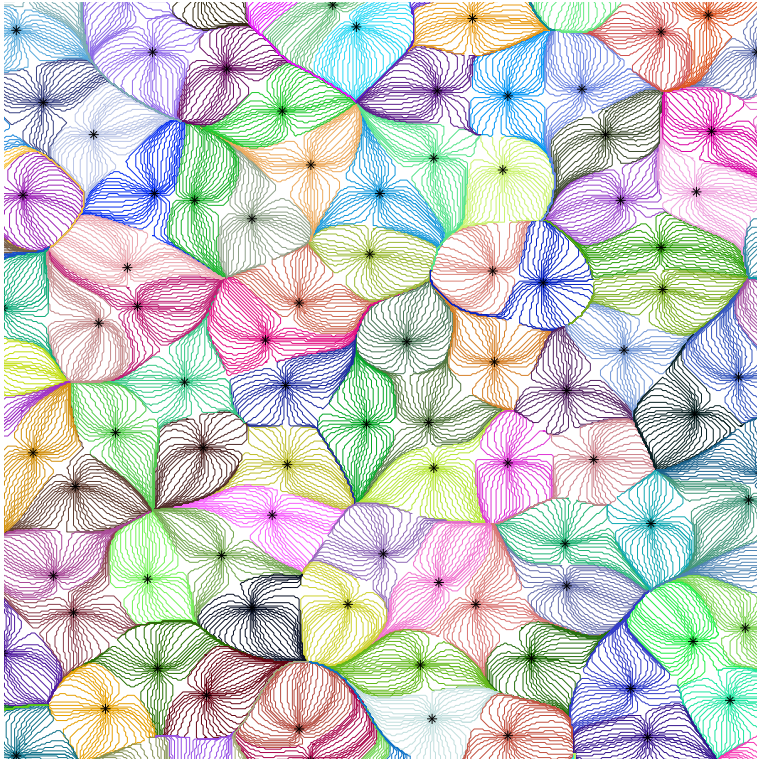


Fig. 5 The point process of zeros of the Gaussian random analytic function (6.1) and the associated basins of attraction. Thanks to Manjunath Krishnapur and Ron Peled for making the MatLab program available to us

they can be thought of as paths of descent under u as a ‘gravitational’ attraction. The basins of attraction of each zero of f then lead to a tiling (Fig. 5). Remarkably, these tiles almost surely have the same area π/L and are bounded by finitely many smooth curves; see [18] or [13, Theorem 8.2.7]. Technically, this is an example of an *allocation*, by which area is associated to each point of the point process.

Clearly, we have no long-range order that manifests itself as non-trivial Bragg peaks, and also none that would lead to singular continuous components. Yet, there are lots of remarkable patterns that almost repeat (at random locations, of course), and one might wonder to what extent spectral theory is able to capture such features. This might be an interesting point for future investigations.

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