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A Renormalisation Group Method. III. Perturbative Analysis

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Abstract This paper is the third in a series devoted to the development of a rigorous renormalisation group method for lattice field theories involving boson fields, fermion fields, or both. In this paper, we motivate and present a general approach towards second-order perturbative renormalisation, and apply it to a specific supersymmetric field theory which represents the continuous-time weakly self-avoiding walk on $\mathbb{Z}^d$. Our focus is on the critical dimension $d = 4$. The results include the derivation of the perturbative flow of the coupling constants, with accompanying estimates on the coefficients in the flow. These are essential results for subsequent application to the 4-dimensional weakly self-avoiding walk, including a proof of existence of logarithmic corrections to their critical scaling. With minor modifications, our results also apply to the 4-dimensional n-component $|\varphi|^4$ spin model.

Keywords Renormalisation group · Perturbation theory · Self-avoiding walk

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1 Introduction

Within theoretical physics, in the study of critical phenomena, or quantum field theory, or many-body theory, the calculation of physically relevant quantities such as critical exponents

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or particle mass is routinely carried out in a perturbative fashion. The perturbative calculations involve tracking the flow of coupling constants which parametrise a dynamical system evolving under renormalisation group transformations. In this paper, we present a general formalism for second-order perturbative renormalisation, and apply it to the continuous-time weakly self-avoiding walk.

This paper is the third in a series devoted to the development of a rigorous renormalisation group method. In part I of the series, we presented elements of the theory of Gaussian integration and defined norms and developed an analysis for performing analysis with Gaussian integrals involving both boson and fermion fields [11]. In part II, we defined and analysed a localisation operator whose purpose is to extract relevant and marginal directions in the dynamical system defined by the renormalisation group [12]. We now apply the formalism of parts I and II to the perturbative analysis of a specific supersymmetric field theory that arises as a representation of the continuous-time weakly self-avoiding walk [10]. Our development of perturbation theory makes contact with the standard technology of Feynman diagrams as it is developed in textbooks on quantum field theory, but our differences in emphasis prepare the ground for the control of non-perturbative aspects in parts IV and V [13,14].

The results of this paper are applied in [2,3], in conjunction with [5,13,14], to the analysis of the critical two-point function and susceptibility of the continuous-time weakly self-avoiding walk. They are also applied in [6] to the analysis of the critical behaviour of the 4-dimensional n-component $|\varphi|^4$ spin model. Our emphasis here is on the *critical dimension* $d = 4$, which is more difficult than dimensions $d > 4$. In this paper, we derive the second-order perturbative flow of the coupling constants, and prove accompanying estimates on the coefficients of the flow. The flow equations themselves are analysed in [2,5]. While the results of this paper are for the specific supersymmetric field theory representing the continuous-time self-avoiding walk, the principles are of wider validity, and apply in particular to the n-component $|\varphi|^4$ model.

The paper is organised as follows. We begin in Sect. 2 by motivating and developing a general approach to perturbation theory. Precise definitions are made in Sect. 3, and the main results are stated in Sect. 4. Proofs are deferred to Sects. 5 and 6. In addition, Sect. 6 contains a definition and analysis of the specific finite-range covariance decomposition that is important in and used throughout [2,3,13,14].

2 Perturbative Renormalisation

In this section, we present an approach to perturbative renormalisation that motivates the definitions of Sect. 3. The analysis is *perturbative*, meaning that it is valid as a formal power series but in this form cannot be controlled uniformly in the volume. We do not directly apply the contents of this section elsewhere, but they help explain why the definitions and results that follow in Sects. 3 and 4 are appropriate and useful. Also, the approach discussed here provides a perspective which guides related developments in part IV [13], and which together with part V [14] lead to remainder estimates that do apply uniformly in the volume. In particular, the proof of [13, Proposition 2.6], which goes beyond formal power series, relies on the principles presented here.

Given integers $L, N > 1$, let $\Lambda = \mathbb{Z}^d/L^N\mathbb{Z}$ denote the discrete torus of period L^N. Recall the definitions of the boson and fermion fields on Λ, and of the combined bosonic-fermionic Gaussian integration $\mathbb{E}_w$ with covariance w, from [11, Sect. 2] (for notational simplicity we write the bold-face covariances of [11] without bold face here). Recall also the definition of local monomial in [12, (1.7)]. Suppose we have a vector space $\mathcal{V}$ of local polynomials in the boson and/or fermion fields, whose elements are given by linear combinations of local monomials. The evaluation of the fields in an element V of $\mathcal{V}$ at a point $x \in \Lambda$ is denoted by V_x, and $V(X)$ denotes the sum

$$V(X) = \sum_{x \in X} V_x. \tag{2.1}$$

Supersymmetry plays no role in these considerations, so we do not assume in this section that the field theory is supersymmetric. For simplicity, we assume here that the elements of $\mathcal{V}$ are translation invariant on Λ. Observable terms, which break translation invariance, are handled by adapting what we do here to include the projections $\pi_\varnothing$ and π_* as in (3.20) below.

The main problem we wish to address is the computation of a Gaussian integral $\mathbb{E}_w e^{-V_0(\Lambda)}$, where $V_0 \in \mathcal{V}$, and where $w = w_N$ is a positive-definite covariance matrix indexed by Λ which approximates the inverse lattice Laplacian $[-\Delta_{\mathbb{Z}^d}]^{-1}$ in the infinite volume limit $N \to \infty$. We will see that divergences arise due to the slow decay of the covariance, but that perturbative renormalisation leads to expressions without divergences, provided the coupling constants are allowed to depend on scale. We consider the problem now at the level of formal power series in the coupling constants, working accurately to second order and with errors of order $O(V_0^3)$. The notation $O(V_0^n)$ signifies a series in the coupling constants whose lowest order terms have degree at least n, and we write $\approx$ to denote equality as formal power series up to an error $O(V_0^3)$. By expanding the exponentials, it is easy to verify that

$$\mathbb{E}_w e^{-V_0} \approx e^{-\mathbb{E}_w V_0 + \frac{1}{2}\mathbb{E}_w(V_0; V_0)}, \tag{2.2}$$

where the second term in the exponent on the right-hand side is the *truncated expectation* (or *variance*)

$$\mathbb{E}_w(V_0; V_0) = \mathbb{E}_w V_0^2 - (\mathbb{E}_w V_0)^2. \tag{2.3}$$

In (2.2)–(2.3), the abbreviation $V_0 = V_0(\Lambda) = \sum_{x \in \Lambda} V_x$ has left the Λ dependence implicit. Equation (2.2) gives the first two terms of the *cumulant expansion* and (2.3) is also referred to as an *Ursell function*.

The formula (2.2) provides a way to perform the integral, but it is not useful because in the infinite volume limit the covariance we are interested in decays in dimension $d = 4$ as $|x - y|^{-2}$, which is not summable in y, and this leads to divergent coefficients in (2.3). For example, suppose that there just one field, a real boson field ϕ, and that $V_0 = V_0(\Lambda) = \sum_{x \in \Lambda} \phi_x^2$. Evaluation of (2.3) in this case gives $\mathbb{E}_w(V_0; V_0) = |\Lambda| \sum_{x \in \Lambda} w(0, x)^2$. The volume factor $|\Lambda|$ is to be expected, but the sum over x is the *bubble diagram* and diverges in the infinite volume limit when $d = 4$. This is a symptom of worse divergences that occur at higher order.

A solution to this famous difficulty of infinities plaguing the functional integrals of physics is provided by the renormalisation group method. For the formulation we are using, we decompose the covariance as a sum $w = w_N = \sum_{j=1}^{N} C_j$. Then, as proved in [11, Proposition 2.6], the expectation can be performed progressively via iterated convolution:

$$\mathbb{E}_w e^{-V_0(\Lambda)} = \mathbb{E}_{C_N} \circ \mathbb{E}_{C_{N-1}} \theta \circ \cdots \circ \mathbb{E}_{C_1} \theta e^{-V_0(\Lambda)}, \tag{2.4}$$

with the operator θ as defined in [11, Definition 2.5] and discussed around (3.17) below. This is an extension of the elementary fact that if $X \sim N(0, \sigma_1^2 + \sigma_2^2)$ then we can evaluate $\mathbb{E}(f(X))$ progressively as

$$\mathbb{E}(f(X)) = \mathbb{E}(\mathbb{E}(f(X_1 + X_2) \mid X_2)), \tag{2.5}$$

with independent normal random variables $X_1 \sim N(0, \sigma_1^2)$ and $X_2 \sim N(0, \sigma_2^2)$.

An essential step is to understand the effect of a single expectation in the iterated expectation (2.4). For this, we seek a replacement

$$I_j(V, \Lambda) = e^{-V(\Lambda)}(1 + W_j(V, \Lambda)) \tag{2.6}$$

for $e^{-V(\Lambda)}$, with $W_j(V, \Lambda) = \sum_{x\in\Lambda} W_j(V, x)$ chosen to ensure that the form of $I_j(V, \Lambda)$ remains stable under expectation. By stability, we mean that given V_j, there exists V_{j+1} such that

$$\mathbb{E}_{C_{j+1}}\theta I_j(V_j, \Lambda) \approx I_{j+1}(V_{j+1}, \Lambda) \tag{2.7}$$

is correct to second order when both sides are expressed as power series in the coupling constants of V_j. In particular the coupling constants of V_{j+1} are power series in the coupling constants of V_j. The recursive composition of these power series expresses V_j as a series in the coupling constants of V_0 but, as explained above, this series has bad properties as $j \to \infty$. However, if V_{j+1} is instead expressed as a function of V_j as in (2.7), then this opens the door to the possibility to arrange that (2.7) holds uniformly in j and N. This is one of the great discoveries of theoretical physics—not in the sense of mathematical proof, but as a highly effective calculational methodology. Its first clear exposition in terms of progressive integration is due to Wilson [19], following earlier origins in quantum field theory [16].

We make several definitions, whose utility will become apparent below. According to [11, Proposition 2.6], for a polynomials A in the fields, the Gaussian expectation with covariance C can be evaluated using the Laplacian operator $\frac{1}{2}\Delta_C$, via $\mathbb{E}\theta A = e^{\frac{1}{2}\Delta_C} A$. For polynomials A, B in the fields, the truncated expectation is then given by

$$\mathbb{E}_C(\theta A; \theta B) = e^{\frac{1}{2}\Delta_C}(AB) - \big(e^{\frac{1}{2}\Delta_C} A\big)\big(e^{\frac{1}{2}\Delta_C} B\big). \tag{2.8}$$

Given A, B, we define

$$F_C(A, B) = e^{\frac{1}{2}\Delta_C}\big(e^{-\frac{1}{2}\Delta_C} A\big)\big(e^{-\frac{1}{2}\Delta_C} B\big) - AB, \tag{2.9}$$

and conclude that

$$\mathbb{E}_C(\theta A; \theta B) = F_C(\mathbb{E}_C\theta A, \mathbb{E}_C\theta B). \tag{2.10}$$

Also, for $X \subset \Lambda$, we define $W_j(V, X) = \sum_{x\in X} W_j(V, x)$ with

$$W_j(V, x) = \frac{1}{2}(1 - \mathrm{Loc}_x) F_{w_j}(V_x, V(\Lambda)), \tag{2.11}$$

with Loc the operator studied in [12], and we use this to define $W_j(V, \Lambda)$ in (2.6). Then we define $P(X) = P(V, X) = \sum_{x\in X} P_x$ by

$$P_x = \mathrm{Loc}_x \mathbb{E}_{C_{j+1}}\theta W_j(V, x) + \frac{1}{2}\mathrm{Loc}_x F_{C_{j+1}}(\mathbb{E}_{C_{j+1}}\theta V_x, \mathbb{E}_{C_{j+1}}\theta V(\Lambda)). \tag{2.12}$$

Finally, the local polynomial V_{pt} is defined in terms of V by

$$V_{\mathrm{pt}} = \mathbb{E}_{C_{j+1}}\theta V - P. \tag{2.13}$$

In Proposition 4.1 below, we present V_{pt} in full detail for the weakly self-avoiding walk.

The following is a version of [7, Proposition 7.1], with the observables omitted. Proposition 2.1 shows that the definitions above lead to a form of the interaction which is stable in the sense of (2.7). Its proof provides motivation for the definitions of W and V_{pt} made above.

Proposition 2.1 *As formal power series in V,*

$$\mathbb{E}_{C_{j+1}}\theta I_j(V,\Lambda) \approx I_{j+1}(V_{\mathrm{pt}},\Lambda), \tag{2.14}$$

with an error which is $O(V^3)$.

Proof The proof includes some motivational remarks that are not strictly necessary for the proof. Suppose that W_j is given; the initial condition is $W_0 = 0$. We initially treat W_j as an unknown sequence of *quadratic* functionals of V, of order $O(V^2)$, and we will discover that (2.11) is a good choice to achieve (2.14). We write $V = V(\Lambda)$ to simplify the notation. We use

$$e^{-V}(1+W) \approx e^{-V+W}, \tag{2.15}$$

together with (2.6) and (2.2), to obtain

$$\mathbb{E}_{C_{j+1}}\theta I_j(V,\Lambda) \approx e^{-\mathbb{E}_{C_{j+1}}\theta V}\left[1 + \mathbb{E}_{C_{j+1}}\theta W_j(V,\Lambda) + \frac{1}{2}\mathbb{E}_{C_{j+1}}(\theta V;\theta V)\right]. \tag{2.16}$$

The second-order term $\mathbb{E}_{C_{j+1}}\theta W_j + \frac{1}{2}\mathbb{E}_{C_{j+1}}(\theta V;\theta V)$ contains contributions which are marginal and relevant for the dynamical system on the space of functionals of the fields, generated by the maps $\mathbb{E}_{C_{j+1}}\theta$.

The idea of the renormalisation group is to track the flow explicitly on a finite-dimensional subspace of the full space of functionals of the fields. In our case, this subspace is the space $\mathcal{V}(\Lambda)$ of local polynomials, and we need to project onto this subspace of marginal and relevant directions. Call this projection Proj. Below, we will relate Proj to the operator Loc of [12]. For now, the one assumption about Proj we need is that

$$(1-\mathrm{Proj}) \circ \mathbb{E}\theta \circ \mathrm{Proj} = 0. \tag{2.17}$$

In other words, integration of relevant or marginal terms does not produce irrelevant terms, or, to put it differently, the space onto which Proj projects is $\mathbb{E}$-invariant. Then we define

$$P(\Lambda) = \mathrm{Proj}\left(\mathbb{E}_{C_{j+1}}\theta W_j(V,\Lambda) + \frac{1}{2}\mathbb{E}_{C_{j+1}}\big(\theta V;\theta V\big)\right). \tag{2.18}$$

It follows from (2.10) that

$$\mathbb{E}_C(\theta V(\Lambda);\theta V(\Lambda)) = F_C(\mathbb{E}_C\theta V(\Lambda), \mathbb{E}_C\theta V(\Lambda)), \tag{2.19}$$

and hence (2.18) is consistent with (2.12) when Proj is taken to be Loc. We then define $V_{\mathrm{pt}} = \mathbb{E}_{C_{j+1}}\theta V - P$ as in (2.13). From (2.16), dropping Λ from the notation, we now obtain

$$\mathbb{E}_{C_{j+1}}\theta I_j(V) \approx e^{-V_{\mathrm{pt}}}\left(1 + (1-\mathrm{Proj})\left(\mathbb{E}_{C_{j+1}}\theta W_j(V) + \frac{1}{2}\mathbb{E}_{C_{j+1}}(\theta V;\theta V)\right)\right). \tag{2.20}$$

In this way, the effect of the marginal and relevant terms in (2.16) has been incorporated into V_{pt}.

The demand that the form of the interaction remain stable under expectation now becomes

$$\mathbb{E}_{C_{j+1}}\theta I_j(V) \approx e^{-V_{\mathrm{pt}}}(1 + W_{j+1}(V_{\mathrm{pt}})), \tag{2.21}$$

with

$$W_{j+1}(V_{\mathrm{pt}}) \approx (1-\mathrm{Proj})\left(\mathbb{E}_{C_{j+1}}\theta W_j(V) + \frac{1}{2}\mathbb{E}_{C_{j+1}}(\theta V;\theta V)\right). \tag{2.22}$$

Let $V'_{j+1} = \mathbb{E}_{C_{j+1}}\theta V'_j$ with initial condition $V'_0 = V_0$. Since P and W are quadratic in V, it would be sufficient to solve

$$W_{j+1}(V'_{j+1}) \approx (1 - \text{Proj}) \left(\mathbb{E}_{C_{j+1}}\theta W_j(V'_j) + \frac{1}{2}\mathbb{E}_{C_{j+1}}(\theta V'_j; \theta V'_j) \right), \tag{2.23}$$

instead of (2.22). Thus we are led to the problem of showing that W as defined in (2.11) satisfies (2.23).

Starting with $j = 0$, for which $W_0 = 0$, we set

$$W_1(V'_1) = \frac{1}{2}(1 - \text{Proj})\, \mathbb{E}_{C_1}(\theta V'_0; \theta V'_0). \tag{2.24}$$

For $j = 1$, this leads to

$$\begin{aligned} W_2(V'_2) &\approx \frac{1}{2}(1 - \text{Proj}) \left(\mathbb{E}_{C_2}\theta(1 - \text{Proj})\, \mathbb{E}_{C_1}(\theta V'_0; \theta V'_0) + \mathbb{E}_{C_2}(\theta V'_1; \theta V'_1) \right) \\ &\approx \frac{1}{2}(1 - \text{Proj}) \left(\mathbb{E}_{C_2}\theta\mathbb{E}_{C_1}(\theta V'_0; \theta V'_0) + \mathbb{E}_{C_2}(\theta V'_1; \theta V'_1) \right), \end{aligned} \tag{2.25}$$

where in the second line we used (2.17). But by definition,

$$\mathbb{E}_{C_2}\theta\mathbb{E}_{C_1}(\theta V'_0; \theta V'_0(\Lambda)) + \mathbb{E}_{C_2}(\theta V'_1; \theta V'_1) = \mathbb{E}_{C_1+C_2}(\theta V'_0; \theta V'_0), \tag{2.26}$$

and hence

$$W_2(V'_2) \approx \frac{1}{2}(1 - \text{Proj})\, \mathbb{E}_{C_1+C_2}(\theta V_0, \theta V_0). \tag{2.27}$$

Iteration then leads to the stable form

$$W_j(V'_j) = \frac{1}{2}(1 - \text{Proj})\, \mathbb{E}_{w_j}(\theta V_0, \theta V_0) \qquad \text{with} \qquad w_j = \sum_{i=1}^{j} C_i. \tag{2.28}$$

By (2.28) and (2.10),

$$W_j(V'_j) = \frac{1}{2}(1 - \text{Proj})\, F_{w_j}(V'_j, V'_j). \tag{2.29}$$

In the above, Proj is applied to $F_{w_j}(V(\Lambda), V(\Lambda)) = \sum_{x\in\Lambda} F_{w_j}(V_x, V(\Lambda))$. Naively, we wish to define $\text{Proj} = \text{Loc}_\Lambda$, where Loc is the localisation operator of [12, Definition 1.17]. A difficulty with this is that Λ is not a coordinate patch in the sense used in [12], so Loc_Λ is not defined. This difficulty is easily overcome as, inspired by [12, Proposition 1.8], we can use the well-defined quantity $\sum_{x\in\Lambda} \text{Loc}_x F_{w_j}(V_x, V(\Lambda))$ instead of the ill-defined $\text{Loc}_\Lambda F_{w_j}(V(\Lambda), V(\Lambda))$. Thus we are led to define

$$\text{Proj } F_{w_j}(V(\Lambda), V(\Lambda)) = \sum_{x\in\Lambda} \text{Loc}_x F_{w_j}(V_x, V(\Lambda)). \tag{2.30}$$

In our application, it is shown in Lemma 5.2 below that $\mathbb{E}_C\theta$ maps the range of Loc into itself, and our assumption (2.17) is then a consequence of [12, (1.68)]. Finally, we observe that (2.29) is consistent with (2.11), and this completes the proof. □

We close this discussion with two further comments concerning W_j. First, although $e^{-V}(1 + W)$ and e^{-V+W} are equivalent as formal power series up to a third order error, they are by no means equivalent for the expectation. To illustrate this point with a single-variable example, if $V = \phi^4$ and $W = \phi^6$, then $e^{-V}(1 + W)$ is an integrable function of ϕ, but e^{-V+W} is certainly not. We keep W out of the exponent in I for reasons related to this phenomenon.

Second, in our applications we use a covariance decomposition with the finite-range property that $w_{j;x,y} = 0$ if $|x - y| > \frac{1}{2}L^j$, for some $L > 1$. This is discussed in detail in Sect. 6.1 below. With such a decomposition, although by definition it appears that $W_j(V, x)$ depends on $V(\Lambda)$ and hence on the fields at all points in space, it in fact depends only on V_y with $|x - y| \leq \frac{1}{2}L^j$.

3 Setup and Definitions

Now we adapt the discussion of Sect. 2 to the particular setting of the supersymmetric field theory representing the 4-dimensional weakly self-avoiding walk, and make precise definitions of the objects of study, including V_{pt}. The minor modifications required to study the n-component $|\varphi|^4$ spin model instead of the weakly self-avoiding walk are discussed in [6].

3.1 Fields and Observables

Let $d \geq 4$ and let $\Lambda = \mathbb{Z}^d/L^N\mathbb{Z}$ denote the discrete d-dimensional torus of side L^N, with $L > 1$ fixed. The field theory we consider consists of a complex boson field $\phi : \Lambda \to \mathbb{C}$ with its complex conjugate $\bar{\phi}$, and a pair of conjugate fermion fields $\psi, \bar{\psi}$. The fermion field is given in terms of the 1-forms $d\phi_x$ by $\psi_x = \frac{1}{\sqrt{2\pi i}} d\phi_x$ and $\bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\phi}_x$, where we fix some square root of $2\pi i$. This is the supersymmetric choice discussed in more detail in [11, Sects. 2.9–2.10].

In addition, we allow an optional *constant* complex observable boson field $\sigma \in \mathbb{C}$ with its complex conjugate $\bar{\sigma}$. The observable field is used in the analysis of the two-point function of the weakly self-avoiding walk in [2], and in the more extensive analysis of correlation functions presented in [18]. Readers only interested in *bulk* quantities, such as the susceptibility of the weakly self-avoiding walk, may skip any discussion of observables, or set $\sigma = 0$.

For the analysis of the two-point function, two particular points $a, b \in \Lambda$ are fixed. We then work with an algebra $\mathcal{N}$ which is defined in terms of a direct sum decomposition

$$\mathcal{N} = \mathcal{N}^{\varnothing} \oplus \mathcal{N}^{a} \oplus \mathcal{N}^{b} \oplus \mathcal{N}^{ab}. \tag{3.1}$$

The algebra $\mathcal{N}^{\varnothing}$ describes the bulk. Its elements are given by finite linear combinations of products of fermion fields with coefficients that are functions of the boson fields. The algebras $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$ account for contributions due to observables. Their elements are respectively given by elements of $\mathcal{N}^{\varnothing}$ multiplied by σ, by $\bar{\sigma}$, and by $\sigma\bar{\sigma}$. For example, $\phi_x\bar{\phi}_y\psi_x\bar{\psi}_x \in \mathcal{N}^{\varnothing}$, and $\sigma\bar{\phi}_x \in \mathcal{N}^a$. Thus $F \in \mathcal{N}$ has the expansion

$$F = F_{\varnothing} + F_a\sigma + F_b\bar{\sigma} + F_{ab}\sigma\bar{\sigma} \tag{3.2}$$

with components $F_{\varnothing}, F_a, F_b, F_{ab} \in \mathcal{N}_{\varnothing}$. There are canonical projections $\pi_\alpha : \mathcal{N} \to \mathcal{N}^\alpha$ for $\alpha \in \{\varnothing, a, b, ab\}$. We use the abbreviation $\pi_* = 1 - \pi_{\varnothing} = \pi_a + \pi_b + \pi_{ab}$. The algebra $\mathcal{N}$ is also discussed around [12, (1.60)] (there $\mathcal{N}$ is written $\mathcal{N}/\mathcal{I}$ but to simplify the notation we write $\mathcal{N}$ here instead).

3.2 Specification of Loc

As motivated in Sect. 2, to apply the renormalisation group method, we require an appropriate projection from $\mathcal{N}$ onto a finite-dimensional vector space $\mathcal{V}$ of local polynomials in the fields. This projection is the operator Loc_X defined and discussed in [12]. In the absence of observables, for any set $X \subset \Lambda$, the localisation operator $\mathrm{Loc}_X : \mathcal{N} \to \mathcal{V}$ of [12, Definition 1.17] is simply given by

$$\mathrm{Loc}_X F = \mathrm{loc}_X^{\varnothing} F_{\varnothing}, \tag{3.3}$$

with $\mathrm{loc}_X^{\varnothing}$ specified below. In the presence of observables, Loc_X is defined in a graded fashion by

$$\mathrm{Loc}_X F = \mathrm{loc}_X^{\varnothing} F_{\varnothing} + \sigma \mathrm{loc}^a_{X\cap\{a\}} F_a + \bar\sigma \mathrm{loc}^b_{X\cap\{b\}} F_b + \sigma\bar\sigma \mathrm{loc}^{ab}_{X\cap\{a,b\}} F_{ab}. \tag{3.4}$$

The definition of each loc^α requires: (i) specification of the scaling (or "engineering") *dimensions* of the fields, (ii) choice of a maximal monomial dimension $d_+ = d_+(\alpha)$ for each component of the range $\mathcal{V} = \mathcal{V}^{\varnothing} + \mathcal{V}^a + \mathcal{V}^b + \mathcal{V}^{ab}$ of Loc, and (iii) choice of covariant field polynomials $\hat P$ which form bases for the vector spaces $\mathcal{V}^\alpha$ (see [12, Definition 1.2]).

The dimensions of the boson and fermion fields are given by

$$[\phi] = [\bar\phi] = [\psi] = [\bar\psi] = \tfrac{d-2}{2} = 1. \tag{3.5}$$

By definition, the dimension of a monomial $\nabla^\alpha \varphi$ is equal to $|\alpha|_1 + [\phi]$, where α is a multi-index and φ may be any of $\phi, \bar\phi, \psi, \bar\psi$, and the dimension of a product of such monomials is the sum of the dimensions of the factors in the product.

For the restriction $\mathrm{loc}^{\varnothing}$ of Loc to $\mathcal{N}^{\varnothing}$, we take $d_+ = d = 4$, the spatial dimension. A natural way to choose the polynomials $\hat P$ and the space $\mathcal{V}$ they span is given in [12, (1.19)]. For $\mathrm{loc}^{\varnothing}$, we apply the choice given in [12, (1.19)] for all monomials in $\mathcal{M}_+$ with maximal dimension $d_+ = d = 4$, with one exception. The exception involves monomials containing a factor $\nabla^e\nabla^e\varphi$, where φ may be any of $\phi, \bar\phi, \psi, \bar\psi$. For these, we use the choice described in [12, Example 1.3], namely we define $\hat P$ by replacing $\nabla^e\nabla^e\varphi$ by $\nabla^{-e}\nabla^e\varphi$. The set $\mathcal{V}$ then has the Euclidean invariance property specified in [12, Proposition 1.4].

In the presence of observables, the specification of loc^a, loc^b and loc^{ab} depends on the scale j, and in particular depends on whether j is above or below the *coalescence scale* j_{ab} defined in terms of the two points $a, b \in \Lambda$ by

$$j_{ab} = \big\lfloor \log_L(2|a-b|) \big\rfloor. \tag{3.6}$$

We assume that $\pi_{ab} V_j = 0$ for $j < j_{ab}$, i.e., that V_j cannot have a $\sigma\bar\sigma$ term before the coalescence scale is reached. For loc^{ab} we take $d_+ = 0$. When Loc acts at scale k, for loc^a and loc^b we take $d_+ = [\phi] = \frac{d-2}{2} = 1$ if $k < j_{ab}$, and $d_+ = 0$ for $k \geq j_{ab}$. This choice keeps $\sigma\bar\phi$ in the range of Loc below coalescence, but not at or above coalescence. The above phrase "Loc acts at scale k" means that Loc produces a scale k object. For example, V_{pt} is a scale $j+1$ object, so the Loc occurring in P of (2.13) is considered to act on scale $j+1$. Thus the change in specification of Loc occurs for the first time in the formula for the scale j_{ab} version of V_{pt}.

Moreover, when restricted to $\pi_*\mathcal{N}$, according to (3.4), Loc_X is the zero operator when $X \cap \{a,b\} = \varnothing$. By definition, the map $\mathrm{loc}^a_{X\cap\{a\}}$ is zero if $a \notin X$, and if $a \in X$ it projects onto the vector space spanned by $\{1, \phi_a, \bar\phi_a, \psi_a, \bar\psi_a\}$ for $j < j_{ab}$, and by $\{1\}$ for $j \geq j_{ab}$. A similar statement holds for $\mathrm{loc}^b_{X\cap\{b\}}$, whereas the range of $\mathrm{loc}^{ab}_{X\cap\{a,b\}}$ is the union of the ranges of $\mathrm{loc}^a_{X\cap\{a\}}$ and $\mathrm{loc}^b_{X\cap\{b\}}$. As discussed in Sect. 5.2.2, in our application symmetry considerations reduce the range of Loc_X to the spans of $\mathbb{1}_a\sigma\bar{\phi}_a\}$, $\{\mathbb{1}_b\bar{\sigma}\phi_b\}$, and $\{\mathbb{1}_a\sigma\bar{\sigma}, \mathbb{1}_b\sigma\bar{\sigma}\}$, on $\mathcal{N}^a, \mathcal{N}^b$ and $\mathcal{N}^{ab}$, respectively; in fact Loc_X reduces to the zero operator on $\mathcal{N}^a, \mathcal{N}^b$, for $j \geq j_{ab}$.

This completes the specification of the operator Loc.

3.3 Local Polynomials

The range $\mathcal{V}$ of Loc consists of local polynomials in the fields. In this paper, we only encounter the subspace $\mathcal{Q} \subset \mathcal{V}$ of local polynomials defined as follows. To define this subspace, we first let $\mathcal{U}$ denote the set of $2d$ nearest neighbours of the origin in Λ, and, for $e \in \mathcal{U}$, define the finite difference operator $\nabla^e\phi_x = \phi_{x+e} - \phi_x$. We also set $\Delta = -\frac{1}{2}\sum_{e\in\mathcal{U}} \nabla^{-e}\nabla^e$. Then we define the 2-forms

$$\tau_x = \phi_x\bar{\phi}_x + \psi_x\bar{\psi}_x, \tag{3.7}$$

$$\tau_{\nabla\nabla,x} = \frac{1}{2}\sum_{e\in\mathcal{U}} \left((\nabla^e\phi)_x(\nabla^e\bar{\phi})_x + (\nabla^e\psi)_x(\nabla^e\bar{\psi})_x\right), \tag{3.8}$$

$$\tau_{\Delta,x} = \frac{1}{2}\left((-\Delta\phi)_x\bar{\phi}_y + \phi_x(-\Delta\bar{\phi})_y + (-\Delta\psi)_x\bar{\psi}_y + \psi_x(-\Delta\bar{\psi})_y\right). \tag{3.9}$$

The subspace $\mathcal{Q} \subset \mathcal{V}$ is then defined to consist of elements

$$V = g\tau^2 + \nu\tau + y\tau_{\nabla\nabla} + z\tau_\Delta + \lambda_a\,\sigma\bar{\phi} + \lambda_b\,\bar{\sigma}\phi + q_{ab}\sigma\bar{\sigma}, \tag{3.10}$$

where

$$\lambda_a = -\lambda^a\,\mathbb{1}_a, \qquad \lambda_b = -\lambda^b\,\mathbb{1}_b, \qquad q_{ab} = -\frac{1}{2}(q^a\mathbb{1}_a + q^b\mathbb{1}_b), \tag{3.11}$$

$g, \nu, y, z, \lambda^a, \lambda^b, q^a, q^b \in \mathbb{C}$, and the indicator functions are defined by the Kronecker delta $\mathbb{1}_{a,x} = \delta_{a,x}$.

The observable terms $\lambda_a\,\sigma\bar{\phi} + \lambda_b\,\bar{\sigma}\phi + q_{ab}\sigma\bar{\sigma}$ are discussed in further detail in Sect. 5.2.2 below. For the bulk, the following proposition shows that $\mathcal{Q}$ arises as a supersymmetric subspace of $\mathcal{V}$. To avoid a digression from the main line of discussion, the definition of supersymmetry is deferred to Sect. 5.2, where the proposition is also proved.

Proposition 3.1 *For the bulk, $\pi_\varnothing\mathcal{Q}$ is the subspace of $\pi_\varnothing\mathcal{V}$ consisting of supersymmetric local polynomials that are of even degree as forms and without constant term.*

The fact that constants are not needed in $\mathcal{Q}$ is actually a consequence of supersymmetry (despite the fact that constants *are* supersymmetric). This is discussed in Sect. 5.2.

3.4 Finite-Range Covariance Decomposition

Our analysis involves approximation of $\mathbb{Z}^d$ by a torus $\Lambda = \mathbb{Z}^d/L^N\mathbb{Z}^d$ of side length L^N, and for this reason we are interested in decompositions of the covariances $[\Delta_{\mathbb{Z}^d} + m^2]^{-1}$ and $[-\Delta_\Lambda + m^2]^{-1}$ as operators on $\mathbb{Z}^d$ and Λ, respectively. For $\mathbb{Z}^d$, this Green function exists for $d > 2$ for all $m^2 \geq 0$, but for Λ we must take $m^2 > 0$. For $\mathbb{Z}^d$, in Sect. 6.1 we follow [1] to define a sequence $(C_j)_{1\leq j<\infty}$ (depending on $m^2 \geq 0$) of positive definite covariances $C_j = (C_{j;x,y})_{x,y\in\mathbb{Z}^d}$ such that

$$[\Delta_{\mathbb{Z}^d} + m^2]^{-1} = \sum_{j=1}^{\infty} C_j \qquad (m^2 \geq 0). \tag{3.12}$$

The covariances C_j are Euclidean invariant, i.e., $C_{j;Ex,Ey} = C_{j;x,y}$ for any lattice automorphism $E : \mathbb{Z}^d \to \mathbb{Z}^d$ (see Sect. 5.2), and have the *finite-range* property

$$C_{j;x,y} = 0 \quad \text{if } |x-y| \geq \frac{1}{2}L^j. \tag{3.13}$$

For $j < N$, the covariances C_j can therefore be identified with covariances on Λ, and we use both interpretations. There is also a covariance $C_{N,N}$ on Λ such that

$$[-\Delta_\Lambda + m^2]^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N} \qquad (m^2 > 0). \tag{3.14}$$

Thus the finite volume decomposition agrees with the infinite volume decomposition except for the last term in the finite volume decomposition. The special covariance $C_{N,N}$ plays only a minor role in this paper. For $j \leq N$, let

$$w_j = \sum_{i=1}^{j} C_i. \tag{3.15}$$

3.5 Definition of V_{pt}

The finite range decomposition (3.14) is associated to a natural notion of *scale*, indexed by $j = 0, \ldots, N$. In our application in [2,3], we are led to consider a family of polynomials $V_j \in \mathcal{Q}$ indexed by the scale. Our goal here is to describe how, in the second order *perturbative* approximation, these polynomials evolve as a function of the scale, via the flow of their coefficients, or *coupling constants*.

Given a positive-definite matrix C whose rows and columns are indexed by Λ, we define the *Laplacian* (see [11, (2.40)])

$$\mathcal{L}_C = \frac{1}{2}\Delta_C = \sum_{u,v\in\Lambda} C_{u,v}\left(\frac{\partial}{\partial\phi_u}\frac{\partial}{\partial\bar{\phi}_v} + \frac{\partial}{\partial\psi_u}\frac{\partial}{\partial\bar{\psi}_v}\right). \tag{3.16}$$

The Laplacian is intimately related to Gaussian integration. To explain this, suppose we are given an additional boson field $\xi, \bar{\xi}$ and an additional fermion field $\eta, \bar{\eta}$, with $\eta = \frac{1}{\sqrt{2\pi i}}d\xi$, $\bar{\eta} = \frac{1}{\sqrt{2\pi i}}d\bar{\xi}$, and consider the "doubled" algebra $\mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}')$ containing the original fields and also these additional fields. We define a map $\theta : \mathcal{N}(\mathbf{\Lambda}) \to \mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}')$ by making the replacement in an element of $\mathcal{N}$ of ϕ by $\phi + \xi$, $\bar{\phi}$ by $\bar{\phi} + \bar{\xi}$, ψ by $\psi + \eta$, and $\bar{\psi}$ by $\bar{\psi} + \bar{\xi}$. According to [11, Proposition 2.6], for a polynomial A in the fields, the Gaussian super-expectation with covariance C can be evaluated using the Laplacian operator via

$$\mathbb{E}_C\theta A = e^{\mathcal{L}_C}A, \tag{3.17}$$

where the fields $\xi, \bar{\xi}, \eta, \bar{\eta}$ are integrated out by $\mathbb{E}_C$, with $\phi, \bar{\phi}, \psi, \bar{\psi}$ kept fixed, and where $e^{\mathcal{L}_C}$ is defined by its power series.

For polynomials V', V'' in the fields, we define

$$F_C(V', V'') = e^{\mathcal{L}_C}\left(e^{-\mathcal{L}_C}V'\right)\left(e^{-\mathcal{L}_C}V''\right) - V'V'', \tag{3.18}$$

By definition, $F_C(V', V'')$ is symmetric and bilinear in V' and V''. The map $e^{-\mathcal{L}_C}$ is equivalent to *Wick ordering* with covariance C [17], i.e., $e^{-\mathcal{L}_C}A = {:}A{:}_C$. In this notation, we could write F_C as a truncated expectation

$$F_C(V', V'') = \mathbb{E}_C\theta({:}V'{:}_C\,;\,{:}V''{:}_C), \tag{3.19}$$

but we will keep our expressions in terms of F_C.

To handle observables correctly, we also define

$$F_{\pi,C}(V', V'') = F_C(V', \pi_\varnothing V'') + F_C(\pi_* V', V''). \tag{3.20}$$

In particular $F_{\pi,C}$ is the same as F_C in the absence of observables, but not in their presence. When observables are present, if V' is expanded as $V' = \pi_\varnothing V' + \pi_* V'$, there are cross-terms $F_C(\pi_\varnothing V', \pi_* V'') + F(\pi_* V', \pi_\varnothing V'')$. The polynomial (3.20) is obtained from (3.18) by replacing these cross-terms by $2F_C(\pi_* V', \pi_\varnothing V'')$. This unusual bookkeeping accounts correctly for observables (it plays a role in the flow of the coupling constants λ, q and also in estimates in [13]).

For $X \subset \Lambda$ we define $W_j(V, X) = \sum_{x \in X} W_j(V, x)$ with

$$W_j(V, x) = \frac{1}{2}(1 - \mathrm{Loc}_x) F_{\pi, w_j}(V_x, V(\Lambda)), \tag{3.21}$$

where Loc_x ($= \mathrm{Loc}_{\{x\}}$) is the operator specified above, $V(\Lambda) = \sum_{x \in \Lambda} V_x$ as in (2.1), and w_j is given by (3.15). Let

$$P(X) = P_j(V, X) = \sum_{x \in X} \mathrm{Loc}_x \left(e^{\mathcal{L}_{j+1}} W_j(V, x) + \frac{1}{2} F_{\pi, C_{j+1}}(e^{\mathcal{L}_{j+1}} V_x, e^{\mathcal{L}_{j+1}} V(\Lambda)) \right), \tag{3.22}$$

where here and throughout the rest of the paper we write $\mathcal{L}_k = \mathcal{L}_{C_k}$. Finally, given V, we define

$$V_{\mathrm{pt}}(V, X) = e^{\mathcal{L}_{j+1}} V(X) - P_j(V, X), \tag{3.23}$$

where we suppress the dependence of V_{pt} on j in its notation. The subscript "pt" stands for "perturbation theory"—a reference to the formal power series calculations discussed in Sect. 2 that lead to its definition. Given $V \in \mathcal{V}$, the polynomial V_{pt} also lies in $\mathcal{V}$ by definition. The polynomial V_{pt} is the updated version of V as we move from scale j to scale $j+1$ via integration of the fluctuation fields with covariance C_{j+1}.

Remark 3.2 Recall from [11, (3.38)] that, for $X \subset \Lambda$, $\mathcal{N}(X)$ consists of those elements of $\mathcal{N}$ which depend on the fields only at points in X (for this purpose, we regard the external field σ as located at a and $\bar{\sigma}$ as located at b). A detail needed in the above concerns the $\mathcal{N}_X$ hypothesis in [12, Definition 1.6], which requires that we avoid applying Loc to elements of $\mathcal{N}(X)$ when X "wraps around" the torus. We are apparently applying Loc_x in (3.21)–(3.22) to field polynomials supported on the entire torus Λ. However, the finite-range property (3.13) ensures that the $\mathcal{N}_X$ hypothesis is satisfied for scales $j+1 < N$, so that Loc and V_{pt} are well-defined. For the moment, we do not define V_{pt} when $j+1 = N$, but we revisit this in Definition 4.2 below.

3.6 Further Definitions

To prepare for our statement of the explicit computation of V_{pt}, some definitions are needed. The following definitions are all in terms of the infinite volume decomposition (C_j) of (3.12). We write $C = C_{j+1}$ and $w = w_j = \sum_{i=1}^{j} C_i$. Given $g, \nu \in \mathbb{C}$, let

$$\eta' = 2C_{0,0}, \quad \nu_+ = \nu + \eta' g, \quad w_+ = w + C, \tag{3.24}$$

and, given a function $f = f(\nu, w)$, let

$$\delta[f(\nu, w)] = f(\nu_+, w_+) - f(\nu, w). \tag{3.25}$$

For a function $q_{0,x}$ of $x \in \mathbb{Z}^d$, we also define

$$(\nabla q)^2 = \frac{1}{2}\sum_{e\in\mathcal{U}}(\nabla^e q)^2, \quad q^{(n)} = \sum_{x\in\mathbb{Z}^d} q_{0,x}^n, \quad q^{(**)} = \sum_{x\in\mathbb{Z}^d} x_1^2 q_{0,x}. \tag{3.26}$$

All of the functions $q_{0,x}$ that we use are combinations of w that are invariant under lattice rotations, so that x_1^2 can be replaced by x_i^2 for any $i = 1, \ldots, d$ in (3.26). We then define

$$\beta = 8\delta[w^{(2)}], \qquad \theta = 2\delta\big[(w^3)^{(**)}\big], \tag{3.27}$$

$$\xi' = 4\big(\delta[w^{(3)}] - 3w^{(2)}C_{0,0}\big) + \tfrac{1}{4}\beta\eta', \qquad \pi' = 2\delta\big[(w\Delta w)^{(1)}\big], \tag{3.28}$$

$$\sigma = \delta\big[(w\Delta w)^{(**)}\big], \qquad \zeta = \delta\big[((\nabla w)^2)^{(**)}\big]. \tag{3.29}$$

The dependence on j in the above quantities has been left implicit.

We define a map $\varphi_{\rm pt} = \varphi_{{\rm pt},j} : \mathcal{Q} \to \mathcal{Q}$ as follows. Given V defined by coupling constants $(g, \nu, z, y, \lambda^a, \lambda^b, q^a, q^b)$, the polynomial $\varphi_{\rm pt}(V)$ has bulk coupling constants

$$g_{\rm pt} = g - \beta g^2 - 4g\delta\big[\nu w^{(1)}\big], \tag{3.30}$$

$$\nu_{\rm pt} = \nu + \eta'\big(g + 4g\nu w^{(1)}\big) - \xi' g^2 - \tfrac{1}{4}\beta g\nu - \pi' g(z+y) - \delta[\nu^2 w^{(1)}], \tag{3.31}$$

$$y_{\rm pt} = y + \sigma g z - \zeta g y - g\delta\big[\nu (w^2)^{(**)}\big], \tag{3.32}$$

$$z_{\rm pt} = z - \theta g^2 - \tfrac{1}{2}\delta\big[\nu^2 w^{(**)}\big] - 2z\delta[\nu w^{(1)}] - (y_{\rm pt} - y). \tag{3.33}$$

The observable coupling constants of $\varphi_{\rm pt}(V)$, with $(\lambda_{\rm pt}, \lambda)$ denoting either $(\lambda^a_{\rm pt}, \lambda^a)$ or $(\lambda^b_{\rm pt}, \lambda^b)$ and analogously for $(q_{\rm pt}, q)$, are given by

$$\lambda_{\rm pt} = \begin{cases} (1 - \delta[\nu w^{(1)}])\lambda & (j+1 < j_{ab}) \\ \lambda & (j+1 \ge j_{ab}), \end{cases} \tag{3.34}$$

$$q_{\rm pt} = q + \lambda^a \lambda^b C_{ab}. \tag{3.35}$$

4 Main Results

We now present our main results, valid for $d = 4$. In Sect. 4.1, we give the result of explicit computation of $V_{\rm pt}$ of (3.23). The form of $V_{\rm pt}$ can be simplified by a change of variables, and we discuss this transformation and its properties in Sect. 4.2. As explained in [3], the transformed flow equations for the coupling constants form part of an *infinite-dimensional* dynamical system which incorporates non-perturbative aspects in conjunction with the perturbative flow. This dynamical system can be analysed using the results of [5], which have been designed expressly for this purpose. To apply the results of [5], certain hypotheses must be verified, and the results of Sect. 4.2 also prepare for this verification.

4.1 Flow of Coupling Constants

The following proposition shows that for $j + 1 < N$, if $V \in \mathcal{Q}$ then $V_{\rm pt} \in \mathcal{Q}$, and gives the *renormalised coupling constants* $(g_{\rm pt}, \nu_{\rm pt}, y_{\rm pt}, z_{\rm pt}, \lambda^a_{\rm pt}, \lambda^b_{\rm pt}, q^a_{\rm pt}, q^b_{\rm pt})$ as functions of the coupling constants $(g, \nu, y, z, \lambda^a, \lambda^b, q^a, q^b)$ of V and of the covariances $C = C_{j+1}$ and $w = w_j = \sum_{i=1}^{j} C_i$.

Proposition 4.1 *Let $d = 4$ and $0 \le j < N-1$. If $V \in \mathcal{Q}$ then $V_{\rm pt} \in \mathcal{Q}$ and*

$$V_{\mathrm{pt},j+1} = \varphi_{\mathrm{pt},j} V. \tag{4.1}$$

In particular, $V_{\mathrm{pt},j+1}$ is independent of N for $j+1 < N$.

For the observable coupling constant q, in view of our assumption that $\pi_{ab} V_j = 0$ for $j < j_{ab}$, and since $C_{j+1;ab} = 0$ if $j+1 < j_{ab}$ by (3.6) and (3.13), when $V_{\mathrm{pt},j+1}$ is constructed from V_j we also have $q_{\rm pt} = 0$ for $j+1 < j_{ab}$, i.e., $\pi_{ab} V_{\mathrm{pt},j+1} = 0$. This lends consistency across scales to the assumption that $\pi_{ab} V_j = 0$ for $j < j_{ab}$. In fact $C_{j+1;ab} = 0$ if $j+1 = j_{ab}$, but we do not take advantage of this because it is sensitive to the choice of $\ge$ as opposed to $>$ in (3.13).

As mentioned in Remark 3.2, the definition of $V_{\rm pt}$ breaks down for $j+1 = N$ due to an inability to apply the operator Loc on the last scale, where the effect of the torus becomes essential. However, in view of Proposition 4.1, the following definition of $V_{\mathrm{pt},N}$ becomes natural. Moreover, when we prove nonperturbative estimates involving $V_{\rm pt}$ in [13, Proposition 2.6], we will see that this definition of $V_{\mathrm{pt},N}$ remains effective in implementing an analogue of Proposition 2.1.

Definition 4.2 We extend the definition of $V_{\mathrm{pt},j+1}$ to $j+1 = N$ by setting $V_{\mathrm{pt},N} = \varphi_{\mathrm{pt},N-1}$.

The equations (3.30)–(3.35) are called *flow equations* because they are applied recursively with $C = C_{j+1}$ and $w = w_j$ updated at each stage of the recursion. They define a j-dependent map $V \mapsto V_{\rm pt}$ for $j < N-1$. The proof of Proposition 4.1 is by explicit calculation of (3.23). The calculation is mechanical, so mechanical that it can be carried out on a computer. In fact, we have written a program [4] in the Python programming language to compute the polynomial P of (3.22), and this computer program leads to the explicit formulas given in Proposition 4.1. From that perspective, it is possible now to write "QED" for Proposition 4.1, but in Sect. 5 we nevertheless present a useful Feynman diagram formalism and use it to derive the coefficients (3.30) and (3.34)–(3.35) of $V_{\rm pt}$. The same formalism can be used for (3.31)–(3.33), but we do not present those details (several pages of mechanical computations). In Sect. 5, we also discuss consequences of supersymmetry for the flow equations.

4.2 Change of Variables and Dynamical System

The observable coupling constants do not appear in the flow of the bulk coupling constants. Thus the flow equations (3.30)–(3.35) have a *block triangular* structure; the flow of the bulk coupling constants is the same whether or not observables are present, whereas the observable flow does depend on the bulk flow. This structure is conceptually important and general; it persists *non-perturbatively* (see [2] and also [14]), and also holds for observables used in the analysis of correlation functions other than the two-point function [18].

We now discuss a change of variables that simplifies the bulk flow equations. In particular, the change of variables creates a system of equations that is itself triangular to second order. Unlike the block triangularity in bulk and observable variables, this triangularity *in the second-order approximation* of the bulk flow will be broken by higher-order corrections. Nonetheless, it provides an important structure in our analysis by enabling the application of [5].

In preparation of the definition of the change of variables, to counterbalance an exponential decay in ν_j, we define the rescaled coupling constant

$$\mu_j = L^{2j} \nu_j, \tag{4.2}$$

and also define normalised coefficients

$$\omega_j = L^2 \tfrac{1}{4} \beta_j, \quad \gamma_j = L^{2(j+1)} \gamma_j' \quad (\gamma = \eta, \xi, \pi), \tag{4.3}$$

$$\bar{w}_j^{(1)} = L^{-2j} w_j^{(1)}, \quad \bar{w}_j^{(**)} = L^{-4j} w_j^{(**)}. \tag{4.4}$$

The constants in (4.3)–(4.4) are all uniformly bounded, as we show in Lemma 6.2. Also, summation by parts on the torus gives $\sum_{x\in\Lambda} \tau_{\nabla\nabla,x} = \sum_{x\in\Lambda} \tau_{\Delta,x}$, and hence

$$z_{\mathrm{pt}} \sum_{x\in\Lambda} \tau_{\Delta,x} + y_{\mathrm{pt}} \sum_{x\in\Lambda} \tau_{\nabla\nabla,x} = (z_{\mathrm{pt}} + y_{\mathrm{pt}}) \sum_{x\in\Lambda} \tau_{\Delta,x}. \tag{4.5}$$

Boundary terms do arise if the sum over Λ is replaced by a sum over a proper subset of Λ, and in [13,14] we work with such smaller sums. Nevertheless, we are able to make use of (4.5) (our implementation occurs in [14, Sect. 6.2]). This suggests that $z_{\mathrm{pt}} + y_{\mathrm{pt}}$ should be a natural variable, so we define

$$z^{(0)} = y + z, \qquad z_{\mathrm{pt}}^{(0)} = y_{\mathrm{pt}} + z_{\mathrm{pt}}. \tag{4.6}$$

Taking the above into account, given V we define $V_{\mathrm{pt}}^{(0)}$ by

$$V_{\mathrm{pt}}^{(0)} = g_{\mathrm{pt}} \tau^2 + \mu_{\mathrm{pt}} L^{-2(j+1)} \tau + z_{\mathrm{pt}}^{(0)} \tau_\Delta. \tag{4.7}$$

The above definition is valid for *all* $0 \le j < \infty$, using the formulas (3.30)–(3.33) with coefficients computed from the decomposition $(C_j)_{1\le j<\infty}$ of $[-\Delta_{\mathbb{Z}^d} + m^2]^{-1}$. In view of (3.30)–(3.33), this leads us to consider the equations:

$$g_{\mathrm{pt}} = g - \beta_j g^2 - 4g\delta\big[\mu \bar{w}^{(1)}\big], \tag{4.8}$$

$$z_{\mathrm{pt}}^{(0)} = z^{(0)} - \theta_j g^2 - \tfrac{1}{2}\delta\big[\mu^2 \bar{w}^{(**)}\big] - 2z^{(0)}\delta\big[\mu \bar{w}^{(1)}\big], \tag{4.9}$$

$$\mu_{\mathrm{pt}} = L^2 \mu + \eta_j \big(g + 4g\mu \bar{w}^{(1)}\big) - \xi_j g^2 - \omega_j g\mu - \pi_j g z^{(0)} - \delta\big[\mu^2 \bar{w}^{(1)}\big], \tag{4.10}$$

and we define a map $\varphi_{\mathrm{pt}}^{(0)} = \varphi_{\mathrm{pt},j}^{(0)}$ on $\mathbb{R}^3$, for $1 \le j < \infty$, by

$$\varphi_{\mathrm{pt},j}^{(0)}\big(g, \mu, z^{(0)}\big) = \big(g_{\mathrm{pt}}, \mu_{\mathrm{pt}}, z_{\mathrm{pt}}^{(0)}\big). \tag{4.11}$$

The four terms involving δ on the right-hand sides of (4.8)–(4.10) can be eliminated by a change of variables. To describe the transformed system, we define a map $\bar{\varphi}_j$ on $\mathbb{R}^3$, for $1 \le j < \infty$, by

$$\bar{\varphi}_j(\bar{g}_j, \bar{z}_j, \bar{\mu}_j) = (\bar{g}_{j+1}, \bar{z}_{j+1}, \bar{\mu}_{j+1}), \tag{4.12}$$

where

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \tag{4.13}$$

$$\bar{z}_{j+1} = \bar{z}_j - \theta_j \bar{g}_j^2, \tag{4.14}$$

$$\bar{\mu}_{j+1} = L^2 \bar{\mu}_j + \eta_j \bar{g}_j - \xi_j \bar{g}_j^2 - \omega_j \bar{g}_j \bar{\mu}_j - \pi_j \bar{g}_j \bar{z}_j. \tag{4.15}$$

The change of variables is defined by a polynomial map $T_j : \mathbb{R}^3 \to \mathbb{R}^3$, which we write as $T_j(g, z, \mu) = (\check{g}, \check{z}, \check{\mu})$, where

$$\check{g} = g + 4g\mu\bar{w}_j^{(1)}, \tag{4.16}$$

$$\check{z} = z + 2z\mu\bar{w}_j^{(1)} + \tfrac{1}{2}\mu^2\bar{w}_j^{(**)}, \tag{4.17}$$

$$\check{\mu} = \mu + \mu^2\bar{w}_j^{(1)}. \tag{4.18}$$

The following proposition, which is proven in Sect. 6.2, gives the properties of the change of variables. We think that the existence of this change of variables may express an invariance property of the flow equations with respect to change of covariance decompositions, one that we do not fully understand.

Below (4.19) and in the remainder of the paper, $O(A^{-k})$ with k unspecified denotes a quantity that is bounded by k-dependent multiple of A^{-k} for arbitrary $k > 0$.

Proposition 4.3 *Let $d = 4$ and $\bar{m}^2 > 0$. There exist an open ball $B \subset \mathbb{R}^3$ centred at 0 (independent of $j \geq 1$ and $m^2 \in [0, \bar{m}^2]$), and analytic maps $\rho_{\mathrm{pt},j} : B \to \mathbb{R}^3$ such that, with the quadratic polynomials $T_j : \mathbb{R}^3 \to \mathbb{R}^3$ given by (4.16)–(4.18),*

$$T_{j+1} \circ \varphi_{\mathrm{pt},j}^{(0)} = \bar{\varphi}_j \circ T_j + \rho_{\mathrm{pt},j} \circ T_j, \tag{4.19}$$

$T_j(V) = V + O(|V|^2)$, the inverse T_j^{-1} to T_j exists on B and is analytic with $T_j^{-1}(V) = V + O(|V|^2)$, and $\rho_{\mathrm{pt},j}(V) = O((1 + m^2 L^{2j})^{-k}|V|^3)$. All constants are uniform in $j \geq 1$ and $m^2 \in [0, \bar{m}^2]$.

Our analysis of the dynamical system arising from the bulk flow equations (3.30)–(3.33) is based on an application of the main result of [5] to the transformed system $\bar{\varphi} + \rho_{\mathrm{pt}}$. The main result of [5] requires the verification of [5, Assumptions (A1–A3)]. We first recall the statements of [5, Assumptions (A1–A2)] in our present context, which are bounds on the coefficients in (4.13)–(4.15). These coefficients depend on the mass m^2 and decay as $j \to \infty$ if $m^2 > 0$.

This decay is naturally measured in terms of the *mass scale* j_m, defined by

$$j_m = \begin{cases} \lfloor \log_{L^2} m^{-2} \rfloor & (m^2 > 0) \\ \infty & (m^2 = 0). \end{cases} \tag{4.20}$$

However, [5, Assumptions (A1–A3)] are stated in a more general context, involving a quantity j_Ω which is closely related to j_m. To define j_Ω, we fix $\Omega > 1$, and set

$$j_\Omega = \inf\{k \geq 0 : |\beta_j| \leq \Omega^{-(j-k)}\|\beta\|_\infty \text{ for all } j\}, \tag{4.21}$$

with $j_\Omega = \infty$ if the infimum is over the empty set. In Proposition 4.4 below it is shown that $j_\Omega = j_m + O(1)$, uniformly in $m^2 \in (0, \delta]$, with $j_m = j_\Omega = \infty$ if $m^2 = 0$.

Assumption (A1) asserts that $\|\beta\|_\infty < \infty$ and that there exists $c > 0$ such that $\beta_j \geq c$ for all but c^{-1} values of $j \leq j_\Omega$, while Assumption (A2) asserts that each of $\theta_j, \eta_j, \xi_j, \omega_j, \pi_j$ is bounded above in absolute value by $O(\Omega^{-(j-j_\Omega)_+})$ (the coefficient ζ_j of Assumption (A2) is zero here). The result of [5] also takes into account non-perturbative aspects of the flow, which are discussed in [14]. The following proposition prepares the ground for the application of [5] by verifying that the transformed flow obeys [5, Assumptions (A1–A2)]. We write $\check{V} = T(V)$.

Proposition 4.4 *Let $d = 4$ and $\bar{m}^2 > 0$. Each coefficient in* (3.30)–(3.35) *and in* (4.13)–(4.15) *is a continuous function of $m^2 \in [0, \bar{m}^2]$. Fix any $\Omega > 1$. For $m^2 \in [0, \delta]$, with $\delta > 0$ sufficiently small, the map $\bar{\varphi}$ satisfies [5, Assumptions (A1–A2)]. In addition,*

$$j_\Omega = j_m + O(1) \tag{4.22}$$

uniformly in $m^2 \in (0, \delta]$, with $j_m = j_\Omega = \infty$ if $m^2 = 0$.

5 Flow Equations and Feynman Diagrams

As mentioned previously, we have written a computer program [4] in the Python programming language to compute $V_{\rm pt}$, and this program produces the equations of Proposition 4.1. In this section, we provide a Feynman diagram formalism, of independent interest, for an alternate computation of $V_{\rm pt}$. We use the formalism to derive the flow equations for $g_{\rm pt}, \lambda_{\rm pt}, q_{\rm pt}$ of (3.30) and (3.34)–(3.35). Using this formalism, it is possible also to derive (3.31)–(3.33), but we do not provide those details.

The polynomial $V_{\rm pt} = e^{\mathcal{L}} V - P$ is defined in (3.23). In Sect. 5.1, we develop the Feynman diagram approach that we use to calculate the τ^2 term of P, and compute the term $e^{\mathcal{L}} V$. In Sect. 5.2, we discuss the symmetries of the model and show that they ensure that $\pi_\varnothing V_{\rm pt}$ does not contain any terms that are not in $\mathcal{Q}$, and we prove Proposition 3.1. Then in Sect. 5.3 we complete the proof of (3.30) and (3.34)–(3.35). We assume throughout that $d = 4$.

5.1 Feynman Diagrams

A convenient way to carry out the computation of $V_{\rm pt}$ is via the Feynman diagram notation introduced in this section. Given $a, b \in \Lambda$, let

$$\tau_{ab} = \phi_a \bar{\phi}_b + \psi_a \bar{\psi}_b. \tag{5.1}$$

Lemma 5.1 *For $a, b \in \Lambda$,*

$$\mathcal{L}_C \tau_{ab} = 0, \tag{5.2}$$

and, for $a_i, b_i \in \Lambda$ and $n \ge 2$,

$$\mathcal{L}_C \prod_{i=1}^n \tau_{a_i b_i} = \sum_{i,j: i \ne j} C_{b_i, a_j} \tau_{a_i b_j} \prod_{k \ne i,j} \tau_{a_k b_k}. \tag{5.3}$$

Proof Equation (5.2) follows from (3.16) together with

$$\begin{aligned}\left(\frac{\partial}{\partial \phi_x} \frac{\partial}{\partial \bar{\phi}_y} + \frac{\partial}{\partial \psi_x} \frac{\partial}{\partial \bar{\psi}_y} \right) \tau_{ab} \\ = \frac{\partial}{\partial \phi_x} \frac{\partial}{\partial \bar{\phi}_y} \phi_a \bar{\phi}_b + \frac{\partial}{\partial \psi_x} \frac{\partial}{\partial \bar{\psi}_y} \psi_a \bar{\psi}_b = \delta_{xa} \delta_{yb} - \delta_{xa} \delta_{yb} = 0.\end{aligned}$$

Also, taking anti-commutativity into account, direct calculation gives

$$\mathcal{L}_C \, \tau_{a_1 b_1} \tau_{a_2 b_2} = C_{b_1, a_2} \tau_{a_1 b_2} + C_{b_2, a_1} \tau_{a_2 b_1}, \tag{5.4}$$

which is the $n = 2$ case of (5.3).

The general case of (5.3) can then be proved via induction on n, and we just sketch the idea. First, we write $\prod_{i=1}^n \tau_{a_i b_i} = \tau_{a_n b_n} \prod_{i=1}^{n-1} \tau_{a_i b_i}$. When $\mathcal{L}_C$ is applied to the product, there is a contribution of zero when it acts entirely on the factor $\tau_{a_n b_n}$ and the induction hypothesis can be applied to evaluate the contribution when $\mathcal{L}_C$ acts entirely on the factor $\prod_{i=1}^{n-1} \tau_{a_i b_i}$. What remains is the contribution where $\mathcal{L}_C$ acts jointly on both factors, and this can be seen to give rise to (5.3). □

This allows for a very simple calculation of the term $e^{\mathcal{L}_C} V$ in V_{pt}, as follows.

Lemma 5.2 *For $V \in \mathcal{Q}$,*

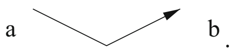
$$e^{\mathcal{L}_C} V_x = V_x + 2g C_{x,x} \tau_x. \tag{5.5}$$

Proof Since V is fourth order in the fields, we can expand $e^{\mathcal{L}_C}$ to second order in $\mathcal{L}_C$ to obtain

$$e^{\mathcal{L}_C} V = V + \mathcal{L}_C (g\tau_x^2 + \nu\tau_x + z\tau_{\Delta,x} + y\tau_{\nabla\nabla,x}) + \frac{1}{2!}\mathcal{L}_C^2 g\tau_x^2. \tag{5.6}$$

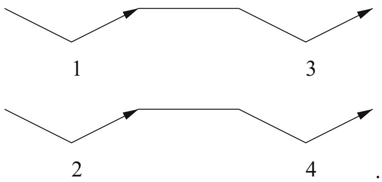
In the second term on the right-hand side, it follows from (5.2) that only $g\tau_x^2$ yields a nonzero contribution, and by the $n = 2$ case of (5.3) this contributes $2gC_{x,x}\tau_x$. A second application of (5.2) then shows that the final term on the right-hand side is zero. □

Lemma 5.1 shows, in particular, that products of τ_{ab} remain products of τ_{ab} under repeated application of the Laplacian $\mathcal{L}_C$. In (5.4), we say that (a_1, b_2) and (a_2, b_1) are *contractions* of (a_1, b_1) and (a_2, b_2). We visualise τ_{ab} as a vertex with an "in-leg" labelled a and an "out-leg" labelled b:



Contraction is then the operation of joining an out-leg of a vertex to an in-leg of a vertex, denoted:

Thus we regard (5.3) as the sum over all ways to contract two of the labelled pairs. For example, one term that arises in calculating $\mathcal{L}_C^2 \prod_{i=1}^4 \tau_{a_i b_i}$ is $C_{a_1,b_3}\tau_{a_1,b_3}C_{a_2,b_4}\tau_{a_2,b_4}$, which is denoted:



This Feynman diagram notation is useful in Sect. 5.3.

5.2 Symmetries

Next, we discuss the symmetries of the model, and prove Proposition 3.1. We first discuss the bulk, and prove in particular that if $V \in \mathcal{Q}$ then $\pi_\varnothing V_{\text{pt}} \in \mathcal{Q}$. Finally, we discuss the situation for observables.

5.2.1 Symmetry and the Bulk

Supersymmetry Supersymmetry is discussed in [10, Sect. 6], where the supersymmetry generator is defined in terms of the exterior derivative and interior product as $Q = d + \underline{i}$. It is convenient to define $\hat{Q} = (2\pi i)^{-1/2} Q$. In our present notation, $\hat{Q}$ can be written as the antiderivation on $\mathcal{N}$ defined by

$$\hat{Q} = \sum_{x\in\Lambda} \left(\psi_x \frac{\partial}{\partial \phi_x} + \bar{\psi}_x \frac{\partial}{\partial \bar{\phi}_x} - \phi_x \frac{\partial}{\partial \psi_x} + \bar{\phi}_x \frac{\partial}{\partial \bar{\psi}_x} \right). \tag{5.7}$$

In particular,

$$\hat{Q}\phi_x = \psi_x, \qquad \hat{Q}\bar{\phi}_x = \bar{\psi}_x, \qquad \hat{Q}\psi_x = -\phi_x, \qquad \hat{Q}\bar{\psi}_x = \bar{\phi}_x. \tag{5.8}$$

An element $F \in \mathcal{N}$ is said to be *supersymmetric* if $QF = 0$.
Gauge symmetry. The *gauge flow* on $\mathcal{N}$ is characterised by $q \mapsto e^{-2\pi i t} q$ for $q = \phi_x, \psi_x$ and $\bar{q} \mapsto e^{+2\pi i t} \bar{q}$ for $\bar{q} = \bar{\phi}_x, \bar{\psi}_x$, for all $x \in \Lambda$. An element $F \in \mathcal{N}$ is said to be *gauge invariant* if it is invariant under this flow.

As discussed in [10, Sect. 6], Q^2 is the generator of the gauge flow, so $F \in \mathcal{N}$ is gauge invariant if and only if $Q^2 F = 0$. In particular, supersymmetric elements are gauge invariant. Since the boson and fermion fields have the same dimension, Q maps $\mathcal{V}$ to itself. It is straightforward to verify that the monomials in $\pi_\varnothing \mathcal{Q}$ are all supersymmetric, hence gauge invariant.

We say that $V \in \mathcal{V}$ is an *even form* if it is a sum of monomials of even degree in $\psi, \bar{\psi}$, and we say that V is *homogeneous of degree n* if V lies in the span of monomials of degree n. For $d = 4$, with fields $\phi, \bar{\phi}, \psi, \bar{\psi}$ of dimension $[\phi] = 1$, and with $d_+ = d = 4$, the highest degree monomials have degree 4 and can have no spatial derivatives. Degree 2 monomials have at most two spatial derivatives. Gauge invariant monomials in $\phi, \bar{\phi}, \psi, \bar{\psi}$ must have even degree because for every field in the monomial, the conjugate of that field must also be in the monomial. The next lemma characterises the monomials in $\mathcal{V}$ that respect symmetries of the model, and shows that these are the ones that occur in $\pi_\varnothing \mathcal{Q}$.

Lemma 5.3 *If $V \in \mathcal{V}$ is even, supersymmetric, and degree 4, then $V = \alpha\tau^2$ for some $\alpha \in \mathbb{C}$. If $V \in \mathcal{V}$ is even, homogeneous of degree 2, and supersymmetric, then V is a linear combination of τ, $\tau_{\nabla\nabla}$ and τ_Δ.*

Proof The only gauge invariant, degree four monomials in $\mathcal{V}$ that are even forms are $(\phi\bar{\phi})^2$ and $\phi\bar{\phi}\psi\bar{\psi}$, because $\psi^2 = \bar{\psi}^2 = 0$. Therefore, for some $\alpha, \beta \in \mathbb{C}$,

$$V = \alpha(\phi\bar{\phi})^2 + \beta\phi\bar{\phi}\psi\bar{\psi}. \tag{5.9}$$

Recall that $\hat{Q}$ is an antiderivation. Since $\psi\bar{\psi}\hat{Q}(\phi\bar{\phi}) = 0$ and since V is supersymmetric,

$$0 = \hat{Q}V = 2\alpha\phi\bar{\phi}\hat{Q}(\phi\bar{\phi}) + \beta\phi\bar{\phi}\hat{Q}(\psi\bar{\psi}) = \phi\bar{\phi}\hat{Q}\big(2\alpha\phi\bar{\phi} + \beta\psi\bar{\psi}\big), \tag{5.10}$$

which by (5.8) implies that $\beta = 2\alpha$. Therefore, as required,

$$V = \alpha\big((\phi\bar{\phi})^2 + 2\phi\bar{\phi}\psi\bar{\psi}\big) = \alpha\big(\phi\bar{\phi} + \psi\bar{\psi}\big)^2 = \alpha\tau^2. \tag{5.11}$$

The monomials in $\mathcal{V}$ which are even, homogeneous of degree 2, and gauge invariant are given by

$$\phi\bar{\phi}, \quad \sum_{e\in\mathcal{U}} \nabla^e\phi\nabla^e\bar{\phi}, \quad \phi\Delta\bar{\phi} + (\Delta\phi)\bar{\phi}, \tag{5.12}$$

and the same with ϕ replaced by ψ (the fact that only Euclidean symmetric monomials occur in $\mathcal{V}$ is guaranteed by [12, Proposition 1.4].) If we now impose supersymmetry, by seeking linear combinations that are annihilated by $\hat{Q}$, the supersymmetric combination that contains $\phi\bar{\phi}$ is τ. Similarly $\tau_{\nabla\nabla}$ and τ_Δ are generated by the other two terms. □

Proof of Proposition 3.1 By Lemma 5.3, the monomials in $\pi_\varnothing \mathcal{V}$ that are supersymmetric, of even degree as forms, of even degree in the fields, and without constant term, are precisely those in $\pi_\varnothing \mathcal{Q}$. Since gauge symmetry eliminates monomials that are of odd degree in the fields, this completes the proof. □

Lemma 5.4 *If $V \in \mathcal{Q}$ then $\pi_\varnothing V_{\text{pt}} \in \mathcal{Q}$.*

Proof Since $\pi_\varnothing V_{\text{pt}}(V) = V_{\text{pt}}(\pi_\varnothing V)$ (because any σ or $\bar\sigma$ in V cannot disappear in creation of V_{pt}), we can and do assume that $V = \pi_\varnothing V$. In view of Lemmas 5.2–5.3, it suffices to show that P is supersymmetric but with no constant term (constants are certainly supersymmetric).

We begin by showing that P does not contain a nonzero constant term. Since we are assuming $\pi_* V = 0$, we may replace F_π by F in (3.22), and also in the definition of W in (3.21). To see that F contains no constant term, observe that in (3.18), A, B do not contain constant terms, and therefore, by Lemma 5.1, neither do $e^{-\mathcal{L}}A$ and $e^{-\mathcal{L}}B$. Hence, again by Lemma 5.1, F cannot contain any constant terms. Therefore, neither does $\text{Loc}_x F$. Similar reasoning shows that the W term in (3.22) cannot contain a nonzero constant term. Therefore P does not contain a nonzero constant term.

It remains to show that P is supersymmetric. Examination of (3.22) reveals that the supersymmetry of V will be inherited by P as long as the supersymmetry generator Q commutes with both $e^{\pm\mathcal{L}}$ and Loc_x. For the former, from (5.7), we obtain the commutator formulas

$$\left[\frac{\partial}{\partial\phi_u}\frac{\partial}{\partial\bar\phi_v}, \hat{Q}\right] = -\frac{\partial}{\partial\bar\phi_v}\frac{\partial}{\partial\psi_u} + \frac{\partial}{\partial\phi_u}\frac{\partial}{\partial\bar\psi_v}, \tag{5.13}$$

$$\left[\frac{\partial}{\partial\psi_u}\frac{\partial}{\partial\bar\psi_v}, \hat{Q}\right] = -\frac{\partial}{\partial\bar\psi_v}\frac{\partial}{\partial\phi_u} + \frac{\partial}{\partial\psi_u}\frac{\partial}{\partial\phi_v}, \tag{5.14}$$

and thereby conclude that $\hat{Q}$ commutes with $\mathcal{L}$ and hence also with $e^{\pm\mathcal{L}}$. Finally, the fact that Q commutes with Loc_x is a consequence of [12, Proposition 1.14]. This completes the proof. □

5.2.2 Symmetry and Observables

Next, we discuss symmetry of the observables, and the monomials in $\pi_*\mathcal{Q}$.

Recall from (3.2) that an element $F \in \mathcal{N}$ decomposes as $F = F_\varnothing + F_a\sigma + F_b\bar\sigma + F_{ab}\sigma\bar\sigma$, as a consequence of the direct sum decomposition $\mathcal{N} = \mathcal{N}^\varnothing \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab}$. The direct sum decomposition of $\mathcal{N}$ induces a decomposition $\mathcal{V} = \mathcal{V}^\varnothing \oplus \mathcal{V}^a \oplus \mathcal{V}^b \oplus \mathcal{V}^{ab}$. In particular, each $V \in \mathcal{Q} \subset \mathcal{V}$ is the sum of $\pi_\varnothing V = g\tau^2 + \nu\tau + z\tau_\Delta + y\tau_{\nabla\nabla}$, $\pi_a V = \lambda_a\,\sigma\bar\phi$, $\pi_b V = \lambda_b\,\bar\sigma\phi$, and $\pi_{ab} V = q_{ab}\sigma\bar\sigma$.

According to Sect. 3.2, the list of monomials in $\pi_*\mathcal{V}$, i.e., those that contain σ and/or $\bar\sigma$, is as follows. The monomials containing σ but not $\bar\sigma$ are given by σ multiplied by any element of $\{\mathbb{1}_a, \mathbb{1}_a\phi_a, \mathbb{1}_a\bar\phi_a, \mathbb{1}_a\psi_a, \mathbb{1}_a\bar\psi_a\}$ for $j < j_{ab}$, and σ multiplied by $\{\mathbb{1}_a\}$ for $j \geq j_{ab}$. The monomials containing $\bar\sigma$ but not σ consist of a similar list with σ replaced by $\bar\sigma$ and a replaced by b. The monomials containing $\sigma\bar\sigma$ are $\{\mathbb{1}_a\sigma\bar\sigma, \mathbb{1}_b\sigma\bar\sigma\}$.

We define the gauge group to act on σ and $\bar\sigma$ via $\sigma \mapsto e^{-2\pi it}\sigma$ and $\bar\sigma \mapsto e^{2\pi it}\bar\sigma$. If we now demand gauge invariance, and also rule out constants and forms of odd degree, the remaining monomials in $\pi_*\mathcal{V}$ are $\{\mathbb{1}_a\sigma\bar\phi_a, \mathbb{1}_b\bar\sigma\phi_b, \mathbb{1}_a\sigma\bar\sigma, \mathbb{1}_b\sigma\bar\sigma\}$ when $j < j_{ab}$, and $\{\mathbb{1}_a\sigma\bar\sigma, \mathbb{1}_b\sigma\bar\sigma\}$ when $j \geq j_{ab}$.

5.3 Calculation of P

It follows from Lemma 5.4 that $\pi_{\varnothing} V_{\mathrm{pt}} \in \mathcal{Q}$, and hence the bulk part of V_{pt} contains only the monomials listed in Lemma 5.3. Thus to compute the bulk part of V_{pt} it is only necessary to compute $g_{\mathrm{pt}}, \nu_{\mathrm{pt}}, y_{\mathrm{pt}}, z_{\mathrm{pt}}$. In this section, we complete the proof of (3.30) and (3.34)–(3.35). We prove (3.30) in Sect. 5.3.3, and then consider the observables in Sect. 5.3.4. The analysis is based on a formula for P obtained in Sect. 5.3.1.

5.3.1 Preliminary Identities

Since $e^{\mathcal{L}_C}$ reduces the dimension of a monomial in the fields, $e^{\mathcal{L}_C} : \mathcal{V} \to \mathcal{V}$, and since Loc_X acts as the identity on $\mathcal{V}$, it follows that

$$\mathrm{Loc}_X e^{\mathcal{L}_C} \mathrm{Loc}_X = e^{\mathcal{L}_C} \mathrm{Loc}_X . \tag{5.15}$$

The following lemma gives the formula we use to compute P.

Lemma 5.5 *For $x \in \Lambda$, for any local polynomial V, and for covariances C, w,*

$$P_x = \frac{1}{2} \sum_{y \in \Lambda} \big(\mathrm{Loc}_x F_{\pi, w+C}(e^{\mathcal{L}_C} V_x, e^{\mathcal{L}_C} V_y) - e^{\mathcal{L}_C} \mathrm{Loc}_x F_{\pi,w}(V_x, V_y) \big). \tag{5.16}$$

Proof The definition of P is given in (3.22), namely

$$P_x = \mathrm{Loc}_x e^{\mathcal{L}_C} W_j(V, x) + \frac{1}{2} \mathrm{Loc}_x F_{\pi,C}(e^{\mathcal{L}_C} V_x, e^{\mathcal{L}_C} V(\Lambda)). \tag{5.17}$$

By the definition of W_j in (3.21), this can be rewritten as

$$P_x = \frac{1}{2} \mathrm{Loc}_x \big(e^{\mathcal{L}_C} (1 - \mathrm{Loc}_x) F_{\pi,w}(V_x, V(\Lambda)) + F_{\pi,C}(e^{\mathcal{L}_C} V_x, e^{\mathcal{L}_C} V(\Lambda)) \big). \tag{5.18}$$

Application of (5.15) in (5.18) gives

$$\begin{aligned} P_x = &\frac{1}{2} \mathrm{Loc}_x \big(e^{\mathcal{L}_C} F_{\pi,w}(V_x, V(\Lambda)) + F_{\pi,C}(e^{\mathcal{L}_C} V_x, e^{\mathcal{L}_C} V(\Lambda)) \big) \\ &- \frac{1}{2} e^{\mathcal{L}_C} \mathrm{Loc}_x F_{\pi,w}(V_x, V(\Lambda)). \end{aligned} \tag{5.19}$$

By the definition of F in (3.18), for polynomials A, B in the fields,

$$\begin{aligned} F_{w+C}\big(e^{\mathcal{L}_C} A, e^{\mathcal{L}_C} B\big) &= e^{\mathcal{L}_C} e^{\mathcal{L}_w} \big(e^{-\mathcal{L}_w} A\big)\big(e^{-\mathcal{L}_w} B\big) - \big(e^{\mathcal{L}_C} A\big)\big(e^{\mathcal{L}_C} B\big) \\ &= e^{\mathcal{L}_C} F_w(A, B) + e^{\mathcal{L}_C} (AB) - \big(e^{\mathcal{L}_C} A\big)\big(e^{\mathcal{L}_C} B\big) \\ &= e^{\mathcal{L}_C} F_w(A, B) + F_C\big(e^{\mathcal{L}_C} A, e^{\mathcal{L}_C} B\big). \end{aligned} \tag{5.20}$$

By (3.20), (5.20) extends to

$$e^{\mathcal{L}_C} F_{\pi,w}(A, B) + F_{\pi,C}\big(e^{\mathcal{L}_C} A, e^{\mathcal{L}_C} B\big) = F_{\pi,w+C}\big(e^{\mathcal{L}_C} A, e^{\mathcal{L}_C} B\big). \tag{5.21}$$

With (5.21), (5.19) gives

$$P_x = \frac{1}{2} \mathrm{Loc}_x F_{\pi,w+C}\big(e^{\mathcal{L}_C} V_x, e^{\mathcal{L}_C} V(\Lambda)\big) - \frac{1}{2} e^{\mathcal{L}_C} \mathrm{Loc}_x F_{\pi,w}(V_x, V(\Lambda)), \tag{5.22}$$

and the right-hand side is equal to the right-hand side of (5.16). □

The first step in the evaluation of the right-hand side of (5.16) is to compute $F_w(V_x, V_y)$. We do this with the following lemma. Given a symmetric covariance w, and polynomials V', V'', we define

$$V' \overset{\leftrightarrow}{\mathcal{L}}_w V'' = \sum_{u,v\in\Lambda} w_{uv} \left(\frac{\partial V'}{\partial \phi_u} \frac{\partial V''}{\partial \phi_v} + \frac{\partial V'}{\partial \phi_v} \frac{\partial V''}{\partial \phi_u} + \frac{\partial V'}{\partial \psi_u} \frac{\partial V''}{\partial \bar{\psi}_v} + \frac{\partial V'}{\partial \psi_v} \frac{\partial V''}{\partial \bar{\psi}_u} \right). \quad (5.23)$$

For $n \geq 2$, we define $V'(\overset{\leftrightarrow}{\mathcal{L}}_w)^n V''$ analogously as a sum over $u_1, v_1, \ldots, u_n, v_n$, with n derivatives acting on each of V' and V'', with n factors w as in (5.23).

Lemma 5.6 *For $x, y \in \Lambda$, for a local polynomial V of degree A, and for a covariance w,*

$$F_w(V_x, V_y) = \sum_{n=1}^{A} \frac{1}{n!} V_x (\overset{\leftrightarrow}{\mathcal{L}}_w)^n V_y. \quad (5.24)$$

Proof By (3.18),

$$F_w(V_x, V_y) = e^{\mathcal{L}_w} \left(e^{-\mathcal{L}_w} V_x\right)\left(e^{-\mathcal{L}_w} V_y\right) - V_x V_y. \quad (5.25)$$

The Laplacian can be written as a sum of three contributions, one acting only on V_x, one only on V_y, and the cross term (5.23). The first two terms are cancelled by the operators $e^{-\mathcal{L}_w}$ appearing in (5.25), leading to

$$F_w(V_x, V_y) = V_x e^{\overset{\leftrightarrow}{\mathcal{L}}_w} V_y - V_x V_y. \quad (5.26)$$

Expansion of the exponential then gives (5.24). □

5.3.2 Localisation Operator

The computation of the flow equations requires the calculation of P, which involves the operator Loc as indicated in Lemma 5.5. An extensive discussion of the operator Loc is given in [12], and [12, Example 1.13] gives some sample calculations involving Loc. Given the specifications listed in Sect. 3.2, it follows from the definition of Loc that

$$\mathrm{Loc}_x \left[\tau_{a_1 b_1} \tau_{a_2 b_2}\right] = \tau_x^2, \quad (5.27)$$

and we use this repeatedly in our calculation of g_{pt} below. Also, for the calculation of λ_{pt} and q_{pt}, we use the fact that the monomials $\sigma \Delta \bar{\phi}$ and $\bar{\sigma} \Delta \phi$ are annihilated by Loc.

We do not provide the details of the calculation of ν_{pt}, y_{pt}, and z_{pt} here. As mentioned previously, their flow in (3.31)–(3.33) has been computed using a Python computer program. To help explain the nature of the terms that arise in these equations, we note the following facts about Loc, which extend [12, Example 1.13] and which are employed by the Python program. First, monomials of degree higher than 4 are annihilated by Loc. Less trivially, suppose that $q : \Lambda \to \mathbb{C}$ has range strictly less than the period of the torus and that it satisfies, for some $q^{(**)} \in \mathbb{C}$,

$$\sum_{x\in\Lambda} q(x) x_i = 0, \qquad \sum_{x\in\Lambda} q(x) x_i x_j = q^{(**)} \delta_{i,j}, \qquad i, j \in \{1, 2, \ldots, d\}. \quad (5.28)$$

Then

$$\mathrm{Loc}_x \left[\sum_{y \in \Lambda} q(x-y)\tau_y \right] = q^{(1)}\tau_x + q^{(**)}(\tau_{\nabla\nabla,x} - \tau_{\Delta,x}), \tag{5.29}$$

$$\mathrm{Loc}_x \left[\sum_{y \in \Lambda} q(x-y)(\tau_{xy} + \tau_{yx}) \right] = 2q^{(1)}\tau_x + q^{(**)}\tau_{\Delta,x}. \tag{5.30}$$

In particular, the coefficients θ, σ, ζ of (3.27) and (3.29) have their origin in (5.29)–(5.30). To simplify the result of the computation, we have also used the elementary properties that for any $q : \Lambda \to \mathbb{C}$,

$$\sum_{x \in \Lambda} (\nabla^e q)_x = 0, \quad \sum_{x \in \Lambda} (\Delta q)_x = 0, \tag{5.31}$$

as well as the fact that $\Delta x_1^2 = -2$, which, by summation by parts, implies that

$$\sum_{x \in \Lambda} (\Delta q)_x x_1^2 = -2 \sum_{x \in \Lambda} q_x = -2q^{(1)}. \tag{5.32}$$

5.3.3 Flow of g

We now prove the flow equation (3.30) for g_{pt}. As in Lemma 5.6, we write

$$F_w(V_x, V_y) = F_{xy} = \sum_{n=1}^{4} F_{n;xy} \quad \text{with} \quad F_{n;xy} = \frac{1}{n!} V_x (\overleftrightarrow{\mathcal{L}}_w)^n V_y. \tag{5.33}$$

The main work lies in proving the following lemma.

Lemma 5.7 *The τ_x^2 term in $\sum_{y \in \Lambda} \mathrm{Loc}_x F_{xy}$ is equal to*

$$\left(16g^2 w^{(2)} + 8g\nu w^{(1)}\right)\tau_x^2. \tag{5.34}$$

Before proving Lemma 5.7, we first note that it implies (3.30).

Proof of (3.30) By Lemmas 5.7 and 5.2, the τ_x^2 term in $e^{\mathcal{L}_C}\mathrm{Loc}_x F_w(V_x, V_y)$ is given by

$$\left(16g^2 w_{x,y}^2 + 8g\nu w_{x,y}\right)\tau_x^2. \tag{5.35}$$

Also, by (5.5), $e^{\mathcal{L}_C} V$ is equal to V with the coefficient ν replaced by

$$\nu_+ = \nu + 2gC_{0,0}, \tag{5.36}$$

so by Lemma 5.7 the τ^2 term in $\sum_{y \in \Lambda} \mathrm{Loc}_x F_{w+C}(e^{\mathcal{L}_C} V_x, e^{\mathcal{L}_C} V_y)$ is given by

$$\left(16g^2 w_+^{(2)} + 8g\nu_+ w_+^{(1)}\right)\tau_x^2, \tag{5.37}$$

where $w_+ = w + C$. By Lemma 5.5, the τ_x^2 term in P_x is therefore equal to

$$\left(8g^2\delta[w^{(2)}] + 4g\delta[\nu w^{(1)}]\right)\tau_x^2. \tag{5.38}$$

With the formula $V_{\mathrm{pt}} = e^{\mathcal{L}_C} V - P$ from (3.23), this implies that

$$g_{\mathrm{pt}} = g - 8g^2\delta[w^{(2)}] - 4g\delta[\nu w^{(1)}], \tag{5.39}$$

which is (3.30). □

Proof of Lemma 5.7 We compute the τ_x^2 term in $\mathrm{Loc}_x F_{n;x,y}$ for $n = 1, 2, 3, 4$. Since $F_{4;x,y}$ has degree zero it contains no τ_x^2 term, so it suffices to consider $n = 1, 2, 3$. The observables play no role in this discussion, and we can let

$$V = g\tau^2 + \nu\tau + z\tau_\Delta + y\tau_{\nabla\nabla}. \tag{5.40}$$

To compute $F_{n;x,y}$ for $n = 1, 2, 3$, we take the terms in (5.40) into account sequentially, starting with $g\tau^2$, then $\nu\tau$, then $z\tau_\Delta$, and finally $y\tau_{\nabla\nabla}$.
τ^2 *term.* We first study

$$F_{n;xy} = \frac{1}{n!} A_x (\overleftrightarrow{\mathcal{L}}_w)^n A_y \quad \text{with} \quad A = g\tau^2. \tag{5.41}$$

By (5.41), F_1 is a polynomial whose terms are degree 6 and therefore $\mathrm{Loc}_x F_{1;xy} = 0$. Also, $F_{3;xy}$ is a polynomial whose monomials have degree 2, and therefore we need not calculate them here. Thus we need only compute the τ^2 contribution to $F_{2;xy}$.

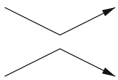
To make contact with Sect. 5.1, we replace A_x and A_y by

$$A_{12} = g\tau_{a_1 a_1}\tau_{a_2 a_2}, \quad A_{34} = g\tau_{a_3 a_3}\tau_{a_4 a_4}. \tag{5.42}$$

The labels 1, 2, 3, 4 help enumerate terms that result from carrying out the contractions in F, but after the enumeration many of these terms become the same when we return to the case at hand by setting

$$a_1 = a_2 = x, \quad a_3 = a_4 = y. \tag{5.43}$$

We represent A by



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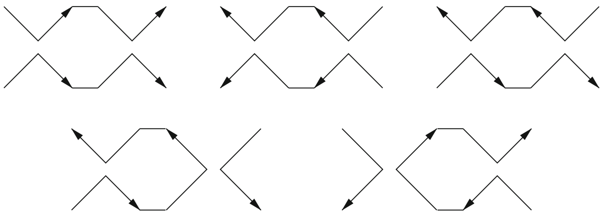
The diagrams for $F_{2;xy}$ are given by



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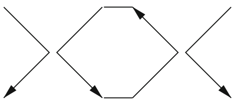
as well as the diagram



which contains a closed loop. The latter vanishes, because it arises, for example, from $\tau_1(\mathcal{L}^2\tau_2\tau_3)\tau_4$ which is 0 by Lemma 5.1. We claim that the five diagrams without closed loops amount to

$$F_{2;xy} = g^2 w^2(x, y)\big(2\tau_{xy}^2 + 2\tau_{yx}^2 + 4\tau_{xy}\tau_{yx} + 4\tau_x\tau_y + 4\tau_x\tau_y\big). \tag{5.44}$$

As a preliminary observation note that the prefactor of $\frac{1}{2!}$ in (5.41) cancels the 2! identical terms that arise from the order of the two contractions in applying (5.3) twice, so for each diagram we only count matchings: how many ways out-legs can be matched to in-legs. The five terms correspond to the five diagrams. These arise as follows:

First and second diagrams: each diagram has two matchings.
Third diagram: four matchings.
Fourth and fifth diagrams: each diagram has four matchings.

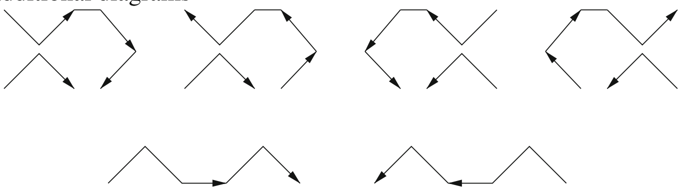
Since all terms on the right-hand side of (5.44) are fourth order in the fields, it is immediate from (5.27) that the τ^2 contribution to $\mathrm{Loc}_x F_{2;xy}$ is given by

$$\mathrm{Loc}_x F_{2;xy} = 16 g^2 w_{x,y}^2 \tau_x^2. \tag{5.45}$$

τ term. Now we consider the additional terms that arise when we add a $\nu\tau$ term so that

$$A = g\tau^2 + \nu\tau. \tag{5.46}$$

The additional terms in $F_{2,xy}$ are not needed since they are of degree 2, and there are no additional terms in $F_{3,xy}$. Repeating the calculations for $F_{1,xy}$ with the extra term in A we obtain the additional diagrams



Therefore $F_{1,xy}$ has the additional terms

$$2g\nu\tau_x w_{x,y}(\tau_{xy} + \tau_{yx}) + 2g\nu\tau_y w_{x,y}(\tau_{yx} + \tau_{xy}) + \nu^2 w_{x,y}(\tau_{xy} + \tau_{yx}). \tag{5.47}$$

Thus, by (5.27), the additional τ^2 contribution that arises here after localisation is:

$$8g\nu w_{x,y}\tau_x^2. \tag{5.48}$$

τ_Δ term. Now we consider the additional degree four terms that arise by adding $z\tau_\Delta$ to A, with τ_Δ defined in (3.9). These degree four terms arise from contractions between τ_x^2 and $\tau_{\Delta,y}$, and between $\tau_{\Delta,x}$ and τ_y^2. After localisation at x, these yield contributions involving $(\Delta_y w_{x,y})\tau_x^2$ and $(\Delta_x w_{x,y})\tau_x^2$. These both vanish after summation over $y \in \Lambda$. Thus there is no contribution to $\sum_y \mathrm{Loc}_x F_{1,xy}$ arising from the τ_Δ term.
$\tau_{\nabla\nabla}$ term. Now we consider the additional degree four terms that arise by adding $y\tau_{\nabla\nabla}$ to A, with $\tau_{\nabla\nabla}$ defined in (3.8). These contributions are similar to those for τ_Δ, and after localisation at x, produce contributions involving $\sum_{e\in\mathcal{U}}(\nabla_x^e \nabla_y^e w_{x,y})\tau_x^2$, which vanishes after summation over $y \in \Lambda$. Thus there is no contribution to $\sum_y \mathrm{Loc}_x F_{1,xy}$ arising from the $\tau_{\nabla\nabla}$ term.
The proof of Lemma 5.7 is now completed by combining (5.45) and (5.48). □

5.3.4 Flow of λ, q

We now prove the following lemma, which implies the flow equations (3.34)–(3.35).

Lemma 5.8 *For $j+1 < j_{ab}$, the observable part of $P = P_j$ as defined in (3.22) is given by*

$$\pi_* P_x = -\delta[\nu w^{(1)}](\lambda^a \sigma \bar{\phi}_a \mathbb{1}_{x=a} + \lambda^b \bar{\sigma} \phi_b \mathbb{1}_{x=b}) + \frac{1}{2} C_{ab} \lambda^a \lambda^b \sigma \bar{\sigma} (\mathbb{1}_{x=a} + \mathbb{1}_{x=b}), \tag{5.49}$$

while for $j+1 \geq j_{ab}$ (5.49) holds with the first term on the right-hand side replaced by zero.

Proof The distinction involving j_{ab} arises due to the change in d_+ discussed in Sect. 3.2, which stops λ^a, λ^b from evolving above the coalescence scale. Throughout the proof, we consider only the more difficult case of $j+1 < j_{ab}$.

We consider the effect on P of adding the observable terms $\pi_* V$ into V. The Laplacian annihilates the $\sigma\bar{\sigma}$ term and it cancels in the subtraction in (5.25), so can be dropped henceforth from V. Thus we wish to compute the new contributions that arise after adding the observable terms

$$A'_x = -\lambda^a \, \sigma \bar{\phi}_x \mathbb{1}_{x=a} - \lambda^b \, \bar{\sigma} \phi_x \mathbb{1}_{x=b} \tag{5.50}$$

to A_x and A_y. We will see, in particular, that there is no contribution from the observables to the flow of non-observable monomials.

Recall that $d_+ = [\phi] = 1$ in the definition of Loc restricted to $\pi_* \mathcal{N}$. We first consider the $\pi_a \mathcal{N}$ and $\pi_b \mathcal{N}$ terms. Writing $F = \sum_{n=1}^4 \frac{1}{n!} F_n$ as before, we need only consider the $n = 1$ term because the observables are degree one polynomials in $(\phi, \bar{\phi})$. Contractions with $g\tau^2$ give rise to monomials that are annihilated by Loc and therefore make no contribution. Contractions with $\nu\tau$ produce

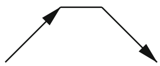

,

and, according to (3.20), the contribution of these diagrams to $F_{\pi,w}(A_x, A_y)$ is

$$A'_x \overset{\leftrightarrow}{\mathcal{L}}_w \, (\pi_\varnothing \nu\tau_y) + (\pi_* A'_x) \overset{\leftrightarrow}{\mathcal{L}}_w \, (\nu\tau_y) = -2\nu \, w_{x,y} \big(\lambda^a \sigma \bar{\phi}_y \mathbb{1}_{x=a} + \lambda^b \bar{\sigma} \phi_y \mathbb{1}_{x=b}\big). \tag{5.51}$$

These same diagrams also classify contractions between the observables and $z\tau_\Delta$ or $y\tau_{\nabla\nabla}$, but in this case make no contributions to $\mathrm{Loc}_x \sum_{y\in\Lambda} F_{xy}$ since, e.g., $\sigma \Delta \bar{\phi}$ is annihilated by Loc. Thus (5.51) constitutes the new terms arising from contractions between observable and non-observable terms in $F_{\pi,w}(A_x, A_y)$.

Next, we consider the $\pi_{ab}\mathcal{N}$ term. The contraction of the λ terms in A_x with those in A_y results in



which contributes

$$\lambda^a \lambda^b \, \sigma \bar{\sigma} \big(\mathbb{1}_{x=a} \mathbb{1}_{y=b} w_{x,y} + \mathbb{1}_{x=b} \mathbb{1}_{y=a} w_{y,x}\big). \tag{5.52}$$

Using $w_{a,b} = w_{b,a}$, and using (5.33), this makes a contribution

$$\lambda^a \lambda^b \, \sigma \bar{\sigma} \, w_{a,b} \, (\mathbb{1}_{x=a} + \mathbb{1}_{x=b}) \tag{5.53}$$

to $\mathrm{Loc}_x \sum_{y\in\Lambda} F_{1;x,y}(A_x, A_y)$. By Lemma 5.5, we find that the contribution to P_x is

$$- \delta[\nu w^{(1)}] \left(\lambda^a \sigma \bar{\phi}_x \mathbb{1}_{x=a} + \lambda^b \bar{\sigma} \phi_x \mathbb{1}_{x=b}\right) + \tfrac{1}{2} \lambda^a \lambda^b \, \sigma \bar{\sigma} \, \delta[w_{a,b}] \, (\mathbb{1}_{x=a} + \mathbb{1}_{x=b}). \tag{5.54}$$

Since $\delta[w_{a,b}] = C_{j+1;a,b}$, this completes the proof. □

6 Analysis of Flow Equations

In this section, we prove Propositions 4.3–4.4. This requires details of the specific covariance decompositions we use. In Sect. 6.1, we define the covariance decompositions, list their important properties, and use those properties to obtain estimates on the coefficients (3.27)–(3.29) of the flow equations. Then we prove Propositions 4.3–4.4 in Sects. 6.2–6.3, respectively.

6.1 Decomposition of Covariance

6.1.1 Definition of Decomposition

Let $d > 2$. We begin by describing the specific finite-range decomposition of the covariance $[-\Delta_{\mathbb{Z}^d} + m^2]^{-1}$ we use, from [1] (see also [8,9]). Recall from [1, Example 1.1] that for each $m^2 \geq 0$ there is a function $\phi_t^*(x, y; m^2)$ defined for $x, y \in \mathbb{Z}^d$ and $t > 0$ such that

$$[-\Delta_{\mathbb{Z}^d} + m^2]^{-1}_{x,y} = \int_0^\infty \phi_t^*(x, y; m^2)\, \frac{dt}{t}. \tag{6.1}$$

The function ϕ_t^* is positive definite as a function of x, y, has the finite-range property that $\phi_t^*(x, y; m^2) = 0$ if $|x - y| > t$ (this specific range can be achieved by rescaling in t), and is Euclidean invariant (this can be seen, e.g., from [1, (3.19)]). To obtain ϕ_t^* as a well-behaved function of m^2, it is necessary to restrict to a finite interval $m^2 \in [0, \bar{m}^2]$ and we make this restriction in the following. Further properties of ϕ_t^* are recalled in the proof of Proposition 6.1 below. Let

$$C_{j;x,y} = \begin{cases} \displaystyle\int_0^{\frac{1}{2}L} \phi_t^*(x, y; m^2)\, \frac{dt}{t} & (j = 1) \\ \displaystyle\int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t^*(x, y; m^2)\, \frac{dt}{t} & (j \geq 2). \end{cases} \tag{6.2}$$

Each C_j is a positive-definite $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix, is Euclidean invariant, has the finite-range property

$$C_{j;x,y} = 0 \quad \text{if} |x - y| \geq \frac{1}{2} L^j, \tag{6.3}$$

and, by construction,

$$C = [-\Delta_{\mathbb{Z}^d} + m^2]^{-1} = \sum_{j=1}^{\infty} C_j. \tag{6.4}$$

This is the covariance decomposition we employ in (3.12).

Next, we adapt (6.2) to obtain a decomposition for the torus $\Lambda = \mathbb{Z}^d / L^N \mathbb{Z}^d$. Let $L, N > 0$ be integers and $m^2 \in (0, \bar{m}^2)$. By (6.3), $C_{j;x,y+L^N z} = 0$ for $j < N$, $|x - y| < L^N$, and nonzero $z \in \mathbb{Z}^d$, and thus

$$C_{j;x,y} = \sum_{z \in \mathbb{Z}^d} C_{j;x,y+zL^N} \quad \text{for } j < N. \tag{6.5}$$

We therefore can and do regard C_j either as a $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix or as a $\Lambda \times \Lambda$ matrix if $j < N$. We also define

$$C_{N,N;x,y} = \sum_{z \in \mathbb{Z}^d} \sum_{j=N}^{\infty} C_{j;x,y+zL^N}. \tag{6.6}$$

Then C_j and $C_{N,N}$ are Euclidean invariant on Λ (i.e., invariant under automorphisms $E : \Lambda \to \Lambda$ as defined in Sect. 5.2). Since

$$[-\Delta_\Lambda + m^2]^{-1}_{x,y} = \sum_{z \in \mathbb{Z}^d} [-\Delta_{\mathbb{Z}^d} + m^2]^{-1}_{x,y+zL^N}, \tag{6.7}$$

it also follows that

$$[-\Delta_\Lambda + m^2]^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N}. \tag{6.8}$$

Therefore the effect of the torus is concentrated in the term $C_{N,N}$. This is the decomposition used in (3.14).

6.1.2 Properties of Decomposition

The following proposition provides estimates on the finite-range decomposition defined above. In its statement, given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write $\nabla_x^\alpha = \nabla_{x_1}^{\alpha_1} \cdots \nabla_{x_d}^{\alpha_d}$ where ∇_{x_k} denotes the finite-difference operator defined by $\nabla_{x_k} f(x, y) = f(x + e_k, y) - f(x, y)$. The number $[\phi]$ is equal to $\frac{1}{2}(d-2)$ as in (3.5).

Proposition 6.1 *Let $d > 2$, $L \geq 2$, $j \geq 1$, $\bar{m}^2 > 0$.*

(a) For multi-indices α, β with ℓ^1 norms $|\alpha|_1, |\beta|_1$ at most some fixed value p, and for any k, and for $m^2 \in [0, \bar{m}^2]$,

$$|\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}| \leq c(1 + m^2 L^{2(j-1)})^{-k} L^{-(j-1)(2[\phi]+(|\alpha|_1+|\beta|_1))}, \tag{6.9}$$

where $c = c(p, k, \bar{m}^2)$ is independent of m^2, j, L. The same bound holds for $C_{N,N}$ if $m^2 L^{2(N-1)} \geq \varepsilon$ for some $\varepsilon > 0$, with c depending on ε but independent of N.

(b) For $j > 1$, the covariance C_j is differentiable in $m^2 \in (0, \bar{m}^2)$, right-continuous at $m^2 = 0$, and there is a constant $c > 0$ independent of m^2, j, L such that

$$\left| \frac{\partial}{\partial m^2} C_{j;x,y} \right| \leq c\big(1 + m^2 L^{2(j-1)}\big)^{-k} \begin{cases} L^j & (d = 3) \\ \log L & (d = 4) \\ L^{-(d-4)(j-1)} & (d > 4). \end{cases} \tag{6.10}$$

Furthermore, C_1 is continuous in $m^2 \in (0, \bar{m}^2)$ and right-continuous at $m^2 = 0$, and $C_{N,N}$ is continuous in the open interval $m^2 \in (0, \bar{m}^2)$.

(c) Let $m^2 = 0$. There exists a smooth function $\rho : [0, \infty) \to [0, \infty)$ with $\int_0^\infty \rho(t)\, dt = 1$, such that the function $c_0 : \mathbb{R}^d \to \mathbb{R}$ defined by its Fourier transform $\hat{c}_0(\xi) = |\xi|^{-2} \int_{L^{-1}|\xi|}^{|\xi|} \rho(t)\, dt$ is smooth with compact support, and, as $j \to \infty$,

$$C_{j;x,y} = c_j(x - y) + O(L^{-(d-1)(j-1)}) \quad \text{for } m^2 = 0, \tag{6.11}$$

where $c_j(x) = L^{-(d-2)j} c_0(L^{-j}x)$.

Proof We use the results of [1, Example 1.1].
(a,b) Let $p, k \in \mathbb{N}$. For any α, β with $|\alpha|_1, |\beta|_1 \leq p$, [1, (1.35)] implies that there is $c = c(p, k)$ such that

$$|\nabla_x^\alpha \nabla_y^\beta \phi_t^*(x, y; m^2)| \leq c(1 + m^2 t^2)^{-k} t^{-(2[\phi]+|\alpha|_1+|\beta|_1)}, \tag{6.12}$$

$$\left| \frac{\partial}{\partial m^2} \phi_t^*(x, y; m^2) \right| \leq c(1 + m^2 t^2)^{-k} t^{-(2[\phi]-2)} \tag{6.13}$$

(using $\frac{1}{2}d > 1$ in [1, (1.36)] for the second bound). Consider first the case $j > 1$. In this case, we restrict to $t \in [\frac{1}{2}L^{j-1}, \frac{1}{2}L^j]$ and obtain upper bounds in (6.12)–(6.13) by replacing

t^2 by $L^{2(j-1)}$. Substitution of the resulting estimates into (6.2) imply (6.9)–(6.10) for $j > 1$, with constants independent of L. For example, $\log L$ in (6.10) arises as

$$\int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \frac{dt}{t} = \log L. \tag{6.14}$$

Moreover, since $\phi_t^*(x, y; m^2)$ is continuous in m^2 at $m^2 = 0^+$ and bounded for $t \in [\frac{1}{2}L^{j-1}, \frac{1}{2}L^j]$ by (6.12), for $j > 1$ the claimed right-continuity of $C_{j;x,y}$ at $m^2 = 0^+$ is a straightforward consequence of the dominated convergence theorem.

For $j = 1$, the bound (6.12) needs to be improved. To this end, we use the discrete heat kernel $p_t(x, y) = (\delta_y, e^{\Delta t}\delta_x)$. Since $e^{\Delta t}$ is a contraction on $L^2(\mathbb{Z}^d)$ and $\delta_x \in L^2(\mathbb{Z}^d)$, it follows that $p_t(x, y)$ is uniformly bounded, i.e., $p_t(x, x) \le ct^{-\alpha/2}$ with $\alpha = 0$. Thus [1, Theorem 1.1] and (6.12) imply that

$$|\phi_t^*(x, y; m^2)| \le c(t^{-2[\phi]} \wedge t^2). \tag{6.15}$$

It follows that

$$|C_{1;x,y}| = \left| \int_0^{\frac{1}{2}L} \phi_t^*(x, y; m^2) \frac{dt}{t} \right| \le c \int_0^1 t^2 \frac{dt}{t} + c \int_1^{\frac{1}{2}L} t^{-2[\phi]} \frac{dt}{t} \le \text{const.} \tag{6.16}$$

This proves (6.9) for $j = 1$ with $\alpha = \beta = 0$, and the estimates for $|\alpha|_1, |\beta|_1 \le p$ are an immediate consequence because the discrete difference operator is bounded on $L^\infty(\mathbb{Z}^d)$. For each $t > 0$, the integrand ϕ_t^* is continuous in m^2 and right-continuituous at $m^2 = 0$, and with the uniform bound (6.15), the claimed continuity of C_1 follows from the continuity of ϕ_t^* by the dominated convergence theorem as for $j > 1$.

Next we verify the claims for $C_{N,N}$. Let $\varepsilon > 0$ and $m^2 \ge \varepsilon L^{-2(N-1)}$. For $j \ge N$, we have $1 + m^2 L^{2(j-1)} \ge m^2 L^{2(j-1)} \ge \varepsilon L^{2(j-N)}$ and hence, with ε-dependent constant c,

$$(1 + m^2 L^{2(j-1)})^{-k-d} \le c(1 + m^2 L^{2(j-1)})^{-k} L^{-2d(j-N)}. \tag{6.17}$$

By (6.12) with k replaced by $k + d$, and by (6.2) and (6.6), it therefore follows that

$$\begin{aligned} |\nabla_x^\alpha \nabla_y^\beta C_{N,N;x,y}| &\le \sum_{j=N}^\infty \sum_{z\in\mathbb{Z}^d} |\nabla_x^\alpha \nabla_y^\beta C_{j;x,y+zL^N}| \\ &\le c\big(1 + m^2 L^{2(N-1)}\big)^{-k} \sum_{j=N}^\infty L^{d(j-N)} L^{-2d(j-N)} L^{-(j-1)(2[\phi]+|\alpha|_1+|\beta|_1)} \\ &\le c\big(1 + m^2 L^{2(N-1)}\big)^{-k} L^{-(N-1)(2[\phi]+(|\alpha|_1+|\beta|_1))}, \end{aligned} \tag{6.18}$$

where we have used the estimates

$$\sum_{z\in\mathbb{Z}^d} \mathbb{1}_{zL^N \le O(L^j)} = O\big(L^{d(j-N)}\big) \quad \text{and} \quad \sum_{j=N}^\infty L^{-(j-N)d} = \frac{1}{1 - L^{-d}} \le 2 \quad (\text{for } L \ge 2). \tag{6.19}$$

This shows that (6.9) holds also for $C_{N,N}$ if $m^2 L^{2(N-1)} \ge \varepsilon$ and thus completes the proof of (a).

To verify that $C_{N,N}$ is continuous in $m^2 \in (0, \bar{m}^2)$, let

$$C_{N,N;x,y}^M = \sum_{j=N}^M \sum_{z\in\mathbb{Z}^d} C_{j;x,y+zL^N}. \tag{6.20}$$

This is a finite sum (due to the finite range of C_j) of m^2-continuous functions, and thus is continuous in $m^2 \in (0, \bar{m}^2)$. Analogously to (6.18), it can be seen that, uniformly in $m^2 \in [\varepsilon L^{-2(N-1)}, \bar{m}^2)$,

$$|C_{N,N;x,y} - C^M_{N,N;x,y}| \to 0 \quad \text{as } M \to \infty. \tag{6.21}$$

As the uniform limit of a sequence of continuous functions, $C_{N,N;x,y}$ is thus continuous in $m^2 \in [\varepsilon L^{-2N}, \bar{m}^2)$. Since $\varepsilon > 0$ is arbitrary, $C_{N,N;x,y}$ is therefore continuous in $m^2 \in (0, \bar{m}^2)$. This completes the proof of (b).

(c) We make several references to [1]. By [1, (1.37)–(1.38)], there exist $c > 0$ and a function $\bar{\phi} \in C^\infty_c(\mathbb{R}^d)$ such that

$$\phi^*_t(x, y; 0) = (c/t)^{d-2}\bar{\phi}(c(x-y)/t) + O(t^{-(d-1)}) \tag{6.22}$$

(due to a typographical error, c^{d-2} is absent on the right-hand side of [1, (1.37)]). The function $\bar{\phi}$ is given in terms of another function W_1 in [1, (3.17)] as $\bar{\phi}(x) = \int_{\mathbb{R}^d} W_1(|\xi|^2)e^{ix\cdot\xi}d\xi$. By [1, Lemma 2.2, (2.22)], $W_1(\lambda) = \varphi(\lambda^{1/2})$ where $\varphi : [0,\infty) \to \mathbb{R}$ is a function such that $\int_0^\infty t\varphi(t)dt = 1$ and such that its Fourier transform $\hat{\varphi}(k) = (2\pi)^{-1}\int_{\mathbb{R}}\varphi(t)e^{ikt}dt$ has support in $[-1, 1]$ (we have chosen $C = 1$ as in [1, Remark 2.4]). Thus,

$$\begin{aligned}\phi^*_t(x, y; 0) &= (c/t)^{d-2}\int_{\mathbb{R}^d}\varphi(|\xi|)e^{ic\left(x-y\right)\cdot\xi/t}d\xi + O\left(t^{-(d-1)}\right)\\ &= (t/c)^2\int_{\mathbb{R}^d}\varphi(|\xi|t/c)e^{i(x-y)\cdot\xi}d\xi + O\left(t^{-(d-1)}\right).\end{aligned} \tag{6.23}$$

Set

$$\rho(s) = \left(\frac{s}{2c}\right)^2\varphi\left(\frac{s}{2c}\right)\frac{1}{s}. \tag{6.24}$$

By definition,

$$\hat{c}_j(\xi) = L^{2j}\hat{c}_0(L^j\xi) = \frac{1}{|\xi|^2}\int_{L^{j-1}|\xi|}^{L^j|\xi|}\rho(s)ds. \tag{6.25}$$

For $j \geq 2$, as in (6.2), interchange of integration (and the change of variables $s = 2t|\xi|$) gives

$$\int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j}\frac{dt}{t}(t/c)^2\int_{\mathbb{R}^d}d\xi\ \varphi(|\xi|t/c)e^{i(x-y)\cdot\xi} = \int_{\mathbb{R}^d}\hat{c}_j(\xi)e^{i(x-y)\cdot\xi}d\xi = c_j(x). \tag{6.26}$$

This completes the proof. □

6.1.3 Bounds on Coefficients

We now prove two lemmas which provide estimates for the coefficients of $\varphi_{\rm pt}$ (and hence $V_{\rm pt}$). The coefficients were defined in Sect. 3.4, in terms of the covariance decomposition (C_j) of $[-\Delta_{\mathbb{Z}^d} + m^2]^{-1}$ given by (6.2).

Lemma 6.2 *Let $d \geq 4$, $j \geq 0$, $\bar{m}^2 > 0$, $k \in \mathbb{R}$. The following bounds hold uniformly in $m^2 \in [0, \bar{m}^2]$ (with constants which may depend on $L, \bar{m}^2$ but not on j):*

$$\beta_j, \theta_j, \sigma_j, \zeta_j = O(L^{-(d-4)j}\left(1 + m^2L^{2j}\right)^{-k}), \tag{6.27}$$

$$\eta'_j, \pi'_j, \xi'_j = O\left(L^{-(d-2)j}(1 + m^2L^{2j})^{-k}\right), \tag{6.28}$$

$$\delta_j[(w^2)^{(**)}] = O\left(L^{-(d-6)j}(1 + m^2L^{2j}\right)^{-k}), \tag{6.29}$$

$$w_j^{(1)} = O(L^{2j}), \quad w_j^{(**)} = O(L^{4j}), \quad (w_j^2)^{(**)} = O(L^{2j}). \tag{6.30}$$

Moreover, the left-hand sides of (6.27)–(6.30) *are continuous in* $m^2 \in [0, \bar{m}^2]$.

Proof The continuity of the left-hand sides of (6.27)–(6.30) in m^2 is a consequence of their definitions together with the continuity of C_j given by Proposition 6.1(b). Thus it suffices to prove the estimates.

Fix $k \geq 0$. Within the proof, we set $M_j = (1 + m^2 L^{2j})^{-k}$, and all constants may depend on L but not on j. We use the uniform bounds (6.9) extensively without further comment. With the finite-range property, they imply

$$|\nabla^l C_{j,x}|, |\nabla^l C_{j+1,x}| \leq O(M_j L^{-(d-2)j} L^{-lj}) \mathbb{1}_{|x| \leq O(L^j)}, \quad l = 0, 1, 2. \tag{6.31}$$

The indicator functions in (6.31) give rise to volume factors in the estimates, i.e.,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{1}_{|x| \leq O(L^j)} \leq O(L^{dj}). \tag{6.32}$$

We also frequently bound a sum of exponentially growing terms by the largest term, i.e., for $s > 0$,

$$\sum_{l=1}^{j} L^{sl} \leq O(L^{sj}). \tag{6.33}$$

Finally, we recall the definitions (3.24)–(3.26) with $w = w_j = \sum_{l=1}^{j} C_l$ and $C = C_{j+1}$.
Bound on β_j. By definition, β_j is proportional to

$$\delta[w^{(2)}] = 2(wC)^{(1)} + C^{(2)}. \tag{6.34}$$

Using $dk - 2[\phi]k = 2k$ and $-2[\phi]j + 2j = -(d-4)j$,

$$(wC)^{(1)} = \sum_x C_{j+1,x} \sum_{k=1}^{j} C_{k,x} = O(M_j L^{-2[\phi]j}) \sum_{k=1}^{j} L^{dk} L^{-2[\phi]k} = O(L^{-(d-4)j} M_j), \tag{6.35}$$

and similarly,

$$C^{(2)} = \sum_x C_{j+1,x}^2 = O\big(M_j L^{dj} L^{-4j[\phi]}\big) = O\big(L^{-(d-4)j} M_j\big). \tag{6.36}$$

Bound on θ_j. By definition, θ_j is proportional to

$$\delta[(w^3)^{(**)}] = 3(w^2C)^{(**)} + 3(wC^2)^{(**)} + (C^3)^{(**)}. \tag{6.37}$$

With $dk + 2k - 2[\phi]k = 4k$ and $-4[\phi]j + 4j = -2(d-4)j \leq -(d-4)j$,

$$\begin{aligned}(wC^2)^{(**)} &= \sum_x \sum_{k=1}^{j} |x|^2 C_{k,x} C_{j+1,x}^2 \leq O(M_j L^{-4[\phi]j}) \sum_{k=1}^{j} L^{dk} L^{2k} L^{-2[\phi]k} \\ &= O(L^{-(d-4)j} M_j),\end{aligned} \tag{6.38}$$

and, with $-6[\phi]j + dj + 2j = -(2d-4)j \le -(d-4)j$,

$$(C^3)^{**} = \sum_x |x|^2 C^3_{j+1,x} \le O\big(M_j L^{-6[\phi]j} L^{dj} L^{2j}\big) \le O(L^{-(d-4)j} M_j). \tag{6.39}$$

Also,

$$(w^2 C)^{(**)} = 2 \sum_x \sum_{l=1}^{j} \sum_{k=1}^{l-1} |x|^2 C_{k,x} C_{l,x} C_{j+1,x} + \sum_x \sum_{k=1}^{j} |x|^2 C^2_{k,x} C_{j+1,x}. \tag{6.40}$$

The first sum in (6.40) is bounded, with $dk + 2k - 2[\phi]k = 4k$ and $4l - 2[\phi]l = (6-d)l$, by

$$\begin{aligned}\sum_x \sum_{l=1}^{j} \sum_{k=1}^{l-1} |x|^2 C_{k,x} C_{l,x} C_{j+1,x} &\le O(M_j L^{-2[\phi]j}) \sum_{l=1}^{j} \sum_{k=1}^{l-1} L^{dk} L^{2k} L^{-2[\phi]k} L^{-2[\phi]l} \\ &\le O(M_j L^{-2[\phi]j}) \sum_{l=1}^{j} L^{(6-d)l}.\end{aligned} \tag{6.41}$$

The sum in (6.41) is bounded by $O(L^{2j})$ if $d \ge 4$ so that, with $-2[\phi]j + 2j = -(d-4)j$,

$$\sum_x \sum_{l=1}^{j} \sum_{k=1}^{l-1} |x|^2 C_{k,x} C_{l,x} C_{j+1,x} \le O(L^{-(d-4)j} M_j) \tag{6.42}$$

as claimed. The second term in (6.40) is similarly bounded, with $dk+2k-4[\phi]k = (6-d)k$, as

$$\sum_x \sum_{k=1}^{j} |x|^2 C^2_{k,x} C_{j+1,x} \le O(M_j L^{-2[\phi]j}) \sum_{k=1}^{j} L^{dk} L^{2k} L^{-4[\phi]k} \le O(L^{-(d-4)j}) M_j). \tag{6.43}$$

This completes the proof of (6.27).
Bound on η'_j. It follows immediately from (6.9) that

$$\eta'_j = C_{j+1,0} \le O(M_j L^{-(d-2)j}). \tag{6.44}$$

Bound on ξ'_j. By definition, ξ'_j is the sum of three terms. The third term is trivially bounded by η'_j. The remaining two terms are proportional to

$$\begin{aligned}\delta[w^{(3)}] - 3w^{(2)}_j C_{j+1;0,0} &= \big(w^{(3)}_{j+1} - w^{(3)}_j\big) - 3w^{(2)}_j C_{j+1;0,0} \\ &= 3\Big((w^2_j C_{j+1})^{(1)} - w^{(2)}_j C_{j+1;0,0}\Big) + 3(w_j C^2_{j+1})^{(1)} + C^{(3)}_{j+1}.\end{aligned} \tag{6.45}$$

To bound the last two terms of (6.45), we use $-6[\phi]j + dj = -2dj + 6j \le -(d-2)j$ to obtain

$$C^{(3)}_{j+1} = \sum_y C^3_{j+1,x} \le O(M_j L^{dj} L^{-6[\phi]j}) \le O(L^{-(d-2)j} M_j). \tag{6.46}$$

Similarly, we use $dk - 2[\phi]k = 2k$ and $-4[\phi]j + 2j = -2dj + 6j \le -(d-2)j$ to obtain

$$(w_j C_{j+1}^2)^{(1)} \le O(M_j L^{-4[\phi]j}) \sum_{k=1}^{j} \sum_x C_{k,x} \le O(M_j L^{-4[\phi]j}) \sum_{k=1}^{j} L^{dk} L^{-2[\phi]k} \le O(L^{-(d-2)j} M_j). \tag{6.47}$$

The first term in (6.45) is proportional to

$$\left(w_j^2 (C_{j+1} - C_{j+1,0})\right)^{(1)} = \sum_{k=0}^{j-1} \sum_x \delta_k[w_x^2](C_{j+1,x} - C_{j+1,0}), \tag{6.48}$$

where we have used

$$w_{j,x}^2 = \sum_{k=0}^{j-1} \delta_k[w_x^2] \quad \text{with} \quad \delta_k[w_x^2] = w_{k+1,x}^2 - w_{k,x}^2. \tag{6.49}$$

The bounds

$$\left| C_{j+1,x} - C_{j+1,0} - \sum_{i=1}^{d} x_i (\nabla_i C)_0 \right| \le O(|x|^2 \|\nabla^2 C_{j+1}\|_\infty) \le O(M_j L^{-2[\phi]j} L^{-2j}) |x|^2, \tag{6.50}$$

$$\sum_x \delta_k[w_x^2] |x|^2 = O(L^{2k}) \sum_x \delta_k[w_x^2] = O(L^{2k} \beta_k) = O(L^{2k}), \tag{6.51}$$

and the identity (which follows from $w_{-x}^2 = w_x^2$)

$$\sum_x \sum_{i=1}^{d} \delta_k[w_x^2] x_i (\nabla_i C)_0 = -\sum_x \delta_k[w_x^2] x_i (\nabla_i C)_0 = 0 \tag{6.52}$$

then imply

$$\left(w_j^2 (C_{j+1} - C_{j+1,0})\right)^{(1)} \le O\left(M_j L^{-2j[\phi]} L^{-2j}\right) \sum_{k=0}^{j-1} L^{2k} = O(L^{-(d-2)j} M_j). \tag{6.53}$$

This gives the desired bound on ξ_j'.
Bound on σ_j. By definition,

$$\sigma = \delta[(w \Delta w)^{(**)}] = (C \Delta w)^{(**)} + (w \Delta C)^{(**)} + (C \Delta C)^{(**)}. \tag{6.54}$$

Since $dk - 2[\phi]k = 2k$ and $-2[\phi]j + 2j = -(d-4)j$,

$$(C \Delta w)^{(**)} = O(L^{-2[\phi]j} M_j) \sum_{k=1}^{j} L^{dk} L^{2k} L^{-2k} L^{-2[\phi]k} = O(L^{-(d-4)j} M_j). \tag{6.55}$$

Since $dk + 2k - 2[\phi]k = 4k$ and $-2[\phi]j - 2j + 4j = -(d-4)j$,

$$(w \Delta C)^{(**)} = O(L^{-2[\phi]j} L^{-2j} M_j) \sum_{k=1}^{j} L^{dk} L^{2k} L^{-2[\phi]k} = O(L^{-(d-4)j} M_j). \tag{6.56}$$

Since $-4[\phi]j - 2j + 6j = -2dj + 8j \le -(d-4)j$,

$$(C\Delta C)^{(**)} = O(L^{-4[\phi]j} L^{-2j} L^{dj} L^{2j} M_j) = O(L^{-(d-4)j} M_j). \tag{6.57}$$

Together the above three estimates give the required result.
Bound on ζ_j. The proof is analogous to the bound of σ_j and is omitted.
Bound on π'_j. By definition, π'_j is proportional to

$$\delta[(w\Delta w)^{(1)}] = \sum_{k=1}^{j+1} \sum_x \left(C_{k,x}\, \Delta C_{j+1,x} + \Delta C_{k,x}\, C_{j+1,x}\right) = 2 \sum_{k=1}^{j+1} \sum_x C_k(x)\, \Delta C_{j+1,x}. \tag{6.58}$$

With $dk - 2[\phi]k = 2k$,

$$\sum_{k=1}^{j+1} \sum_x C_k(x)\, \Delta C_{j+1,x} \le O(M_j L^{-2[\phi]j} L^{-2j}) \sum_{k=1}^{j+1} L^{dk} L^{-2[\phi]k} = O(L^{-(d-2)j} M_j), \tag{6.59}$$

as required.
Bound on $\delta[(w^2)^{**}]$ *and* $(w^2)^{**}$. By definition,

$$\delta[(w^2)^{**}] = 2(wC)^{(**)} + (C^2)^{(**)}. \tag{6.60}$$

With $dk + 2k - 2[\phi]k = 4k$ and $-2[\phi]j + 4j = (6-d)j$,

$$\begin{aligned} (wC)^{(**)} &= \sum_{k=1}^{j} \sum_x |x|^2 C_{k,x} C_{j+1,x} = O(M_j L^{-2[\phi]j}) \sum_{k=1}^{j} L^{dk} L^{2k} L^{-2[\phi]k} \\ &= O(M_j L^{(6-d)j}), \end{aligned} \tag{6.61}$$

and similarly,

$$(C^2)^{(**)} = \sum_x |x|^2 C_{j+1,x}^2 = O(M_j L^{-4[\phi]j} L^{dj} L^{2j}) = O(M_j L^{(6-d)j}). \tag{6.62}$$

Since $6 - d \le 2$ for $d \ge 4$, taking the sum over $\delta_k[(w^2)^{(**)}]$ also implies the bound $(w^2)^{(**)} = O(L^{2j})$.
Bound on $w_j^{(1)}$. By definition,

$$w^{(1)} = \sum_x \sum_{k=1}^{j} C_{k,x} = O(1) \sum_{k=1}^{j} L^{dk} L^{-2[\phi]k} = O(1) \sum_{k=1}^{j} L^{2k} = O(L^{2j}). \tag{6.63}$$

Bound on $w_j^{(**)}$. By definition,

$$w^{(**)} = \sum_x \sum_{k=1}^{j} |x|^2 C_{k,x} = O(1) \sum_{k=1}^{j} L^{dk} L^{2k} L^{-2[\phi]k} = O(1) \sum_{k=1}^{j} L^{4k} = O(L^{4j}). \tag{6.64}$$

This completes the proof. □

Lemma 6.3 *(a) For $m^2 = 0$, $\lim_{j\to\infty} \beta_j = 0$ for $d > 4$, whereas*

$$\lim_{j\to\infty} \beta_j = \frac{\log L}{\pi^2} \qquad \textit{for } d = 4. \tag{6.65}$$

(b) Let $d = 4$ and $\bar{m}^2 > 0$. There is a constant c' independent of j, L such that, for $m^2 \in (0, \bar{m}^2)$ and $j > 1$,

$$\left|\frac{\partial}{\partial m^2}\beta_j(m^2)\right| \leq c'(\log L)L^{d+2j}. \tag{6.66}$$

Proof (a) The conclusion for $d > 4$ follows immediately from (6.27), and we consider henceforth the case $d = 4$. In this proof, constants in error estimates may depend on L.

Let $c_0 \in C_c(\mathbb{R}^4)$ be the function defined by Proposition 6.1(c), and let $c_j(x) = L^{-2j}c_0(L^{-j}x)$, so that

$$C_{j,x} = c_j(x) + O(L^{-3j}). \tag{6.67}$$

We use the notation $(F, G) = \sum_{x\in\mathbb{Z}^4} F_x G_x$ for $F, G : \mathbb{Z}^4 \to \mathbb{R}$, and $\langle f, g\rangle = \int_{\mathbb{R}^4} fg\, dx$ for $f, g : \mathbb{R}^4 \to \mathbb{R}$. We first verify that

$$(C_j, C_{j+l}) - \langle c_0, c_l\rangle = O(L^{-j-2l}). \tag{6.68}$$

Let $R_{j,x} = C_{j,x} - c_j(x)$. Then

$$(C_j, C_{j+l}) = (c_j, c_{j+l}) + (c_j, R_{j+l}) + (c_{j+l}, R_j) + (R_j, R_{j+l}). \tag{6.69}$$

Riemann sum approximation gives

$$\begin{aligned}(c_j, c_{j+l}) - \langle c_0, c_l\rangle &= L^{-4j}\sum_{y\in L^{-j}\mathbb{Z}^d} c(y)c_l(y) - \int_{\mathbb{R}^d} c(y)c_l(y)\, dy\\ &= O(L^{-j})\|\nabla(cc_l)\|_{L^\infty} = O(L^{-j-2l}).\end{aligned} \tag{6.70}$$

The remaining terms are easily bounded using $|\mathrm{supp}(C_j)|, |\mathrm{supp}(R_j)| = O(L^{4j})$:

$$(c_j, R_{j+l}) \leq O(L^{4j})\|c_j\|_{L^\infty(\mathbb{Z}^4)}\|R_{j+l}\|_{L^\infty(\mathbb{Z}^4)} \leq O(L^{-j}L^{-3l}), \tag{6.71}$$

$$(c_{j+l}, R_j) \leq O(L^{4j})\|c_{j+l}\|_{L^\infty(\mathbb{Z}^4)}\|R_j\|_{L^\infty(\mathbb{Z}^4)} \leq O(L^{-j}L^{-2l}), \tag{6.72}$$

$$(R_j, R_{j+l}) \leq O(L^{4j})\|R_j\|_{L^\infty(\mathbb{Z}^4)}\|R_{j+l}\|_{L^\infty(\mathbb{Z}^4)} \leq O(L^{-2j}L^{-3l}), \tag{6.73}$$

and (6.68) follows.

From (6.68) we can now deduce that

$$\begin{aligned}\sum_{k=1}^{j}(C_k, C_{j+1}) &= \sum_{k=1}^{j}\langle c_0, c_{j+1-k}\rangle + \sum_{k=1}^{j} O(L^{-k-2(j-k)})\\ &= \sum_{k=1}^{j}\langle c_0, c_k\rangle + O(L^{-j}),\end{aligned} \tag{6.74}$$

$$(C_{j+1}, C_{j+1}) = \langle c_0, c_0\rangle + O(L^{-j}). \tag{6.75}$$

Thus, using $\langle c_0, c_k\rangle = \langle c_0, c_{-k}\rangle$, we obtain

$$w_{j+1}^{(2)} - w_j^{(2)} = 2(w_j, C_{j+1}) + (C_{j+1}, C_{j+1}) = \sum_{k=-j}^{j}\langle c_0, c_k\rangle + O(L^{-j}). \tag{6.76}$$

Application of $\|c_{-k}\|_{L^\infty} \le L^{2k}\|c_0\|_{L^\infty}$ and $\mathrm{supp}(c_{-k}) \subset B_{O(L^{-k})}$ gives

$$\sum_{k=j+1}^{\infty} |\langle c_0, c_k\rangle| = \sum_{k=j+1}^{\infty} |\langle c_0, c_{-k}\rangle| \le \|c_0\|_{L^\infty} \sum_{k=j+1}^{\infty} L^{2k} \int_{B_{O(L^{-k})}} |c_0(x)|\, dx$$
$$\le \|c_0\|_{L^\infty}^2 \sum_{k=j+1}^{\infty} O(L^{-2k}) \le O(L^{-2j}). \tag{6.77}$$

Thus we have obtained

$$\beta_j = 8(w_{j+1}^{(2)} - w_j^{(2)}) = \beta_\infty + O(L^{-j}) \quad \text{with} \quad \beta_\infty = 8 \sum_{k=-\infty}^{\infty} \langle c_0, c_k\rangle. \tag{6.78}$$

The constant β_∞ is determined as follows. By (6.78),

$$\beta_\infty = 8\langle c_0, v\rangle \quad \text{with} \quad v = \sum_{k\in\mathbb{Z}} c_k. \tag{6.79}$$

By Plancherel's theorem and (6.25),

$$\langle c_0, c_k\rangle = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} |\xi|^{-4} \left(\int_{L^{-1}|\xi|}^{|\xi|} \rho(t)\, dt\right)\left(\int_{L^{k-1}|\xi|}^{L^k|\xi|} \rho(t)\, dt\right) d\xi, \tag{6.80}$$

and hence, by Fubini's theorem, radial symmetry, and $\int_0^\infty \rho\, dt = 1$,

$$\langle c_0, v\rangle = \frac{\omega_3}{(2\pi)^4} \int_0^\infty \left(\int_{L^{-1}r}^{r} \rho(t)\, dt\right) \frac{dr}{r} = \frac{\omega_3}{(2\pi)^4} \int_0^\infty \left(\int_t^{Lt} \frac{dr}{r}\right) \rho(t)\, dt, \tag{6.81}$$

where $\omega_3 = 2\pi^2$ is the surface measure of the 3-sphere as a subset of $\mathbb{R}^4$. The inner integral in the last equation is equal to $\log L$. Thus, again using $\int_0^\infty \rho\, dt = 1$, we find that

$$\beta_\infty = \frac{8\omega_3}{(2\pi)^4} \log L = \frac{\log L}{\pi^2} \tag{6.82}$$

as claimed.
(b) In this proof, we set $d = 4$, and constants are independent of L. We write $f' = \frac{\partial}{\partial m^2} f$. Using the notation of (3.26), we have

$$\beta_j' = 16\Big((wC)^{(1)}\Big)' + 8\Big(C^{(2)}\Big)'. \tag{6.83}$$

By (6.9)–(6.10),

$$(C^{(2)})' = 2\sum_x C_{j+1,x}' C_{j+1,x} \le O(L^{-2j}\log L) O(L^{d(j+1)}) \le O(L^d (\log L) L^{2j}) \tag{6.84}$$

and, similarly,

$$\big((wC)^{(1)}\big)' = \sum_x \sum_{k=1}^{j} (C_{k,x} C_{j+1,x}' + C_{k,x}' C_{j+1,x}). \tag{6.85}$$

Again by (6.9)–(6.10),

$$\sum_x \sum_{k=1}^{j} C_{k,x} C'_{j+1,x} \le O(\log L) \sum_{k=1}^{j} L^{dk} L^{-2(k-1)}$$
$$\le O(L^d \log L) \sum_{k=1}^{j} L^{(d-2)(k-1)} \le O\big(L^d (\log L) L^{2j}\big), \tag{6.86}$$

$$\sum_x \sum_{k=1}^{j} C'_{k,x} C_{j+1,x} \le O(L^{-2j} \log L) \sum_{k=1}^{j} O(L^{dk}) \le O(L^d \big(\log L) L^{2j}\big). \tag{6.87}$$

This completes the proof. □

6.2 Proof of Proposition 4.3

Proof of Proposition 4.3 Let $\mu_+ = L^{2(j+1)} \nu_+$. By (3.25), (4.8)–(4.10) are equivalent to (we drop superscripts (0) on z and z_{pt})

$$\Big[g_{\mathrm{pt}} + 4g\mu_+ \bar{w}^{(1)}_{j+1}\Big] = \Big[g + 4g\mu \bar{w}^{(1)}_j\Big] - \beta_j g^2, \tag{6.88}$$

$$\big[z_{\mathrm{pt}} + 2z\mu \bar{w}^{(1)}_{j+1} + \tfrac{1}{2}\mu_+^2 \bar{w}^{(**)}_{j+1}\big] = \big[z + 2z\mu \bar{w}^{(1)}_j + \tfrac{1}{2}\mu^2 \bar{w}^{(**)}_j\big] - \theta_j g^2, \tag{6.89}$$

$$\big[\mu_{\mathrm{pt}} + \mu_+^2 \bar{w}^{(1)}_{j+1}\big] = L^2\big[\mu + \mu^2 \bar{w}^{(1)}_j\big] + \eta_j\big[g + 4g\mu \bar{w}^{(1)}_j\big]$$
$$- \xi_j g^2 - \omega_j g\mu - \pi g z. \tag{6.90}$$

The form of the rewritten equations (6.88)–(6.90) suggests that we define maps $T_j : \mathbb{R}^3 \to \mathbb{R}^3$ by $T_j(g, z, \mu) = (\check{g}, \check{z}, \check{\mu})$ where $(\check{g}, \check{z}, \check{\mu})$ are as in (4.16)–(4.18), i.e.,

$$\check{g} = g + 4g\mu \bar{w}^{(1)}_j, \tag{6.91}$$

$$\check{z} = z + 2z\mu \bar{w}^{(1)}_j + \tfrac{1}{2}\mu^2 \bar{w}^{(**)}_j, \tag{6.92}$$

$$\check{\mu} = \mu + \mu^2 \bar{w}^{(1)}_j. \tag{6.93}$$

By the inverse function theorem [15, (10.2.5)], there exists a ball $B_\epsilon(0) \subset \mathbb{R}^3$ such that T_j is an analytic diffeomorphism from $B_\epsilon(0)$ onto its image. Note that ϵ can be chosen uniformly in j and m^2 by the uniformity of the bounds on $\bar{w}^{(1)}_j$ and $\bar{w}^{(**)}_j$ in j and m^2 of Lemma 6.2. It also follows from the inverse function theorem that $T^{-1}(V) = V + O(|V|^2)$ with uniform constant.

The left-hand sides of (6.88)–(6.90) equal $T_{j+1}(\varphi^{(0)}_{\mathrm{pt},j}(V)) + O((1+m^2L^{2j})^{-k}|V|^3)$ and the right-hand sides are equal to $\bar{\varphi}_j(T_j(V)) + O((1+m^2L^{2j})^{-k}|V|^3)$. For example, with $T_{j+1}(V_{\mathrm{pt}}) = (\check{g}_{\mathrm{pt}}, \check{z}_{\mathrm{pt}}, \check{\mu}_{\mathrm{pt}})$, it follows from Lemma 6.2 that

$$g_{\mathrm{pt}} + 4g\mu_+ \bar{w}^{(1)}_{j+1} = \big[g_{\mathrm{pt}} + 4g_{\mathrm{pt}}\mu_{\mathrm{pt}} \bar{w}^{(1)}_{j+1}\big] + 4(g - g_{\mathrm{pt}})\mu_+ \bar{w}^{(1)}_{j+1} + 4g_{\mathrm{pt}}(\mu_+ - \mu_{\mathrm{pt}})\bar{w}^{(1)}_{j+1}$$
$$= \check{g}_{\mathrm{pt}} + 4\big(\beta_j g^2 + 4g\delta[\mu \bar{w}^{(1)}]\big)\mu_+ \bar{w}^{(1)}_{j+1}$$
$$+ 4g_{\mathrm{pt}}\Big(- 4\eta_j \bar{w}^{(1)} g\mu + \xi_j g^2 + \omega_j g\mu + \pi_j gz + \delta[\mu^2 \bar{w}^{(1)}]\Big)$$
$$= \check{g}_{\mathrm{pt}} + O\big((1+m^2L^{2j})^{-k}|V|^3\big). \tag{6.94}$$

Thus

$$T_{j+1}(\varphi^{(0)}_{\mathrm{pt},j}(V)) = \bar{\varphi}_j(T_j(V)) + O\big((1+m^2L^{-2j})^{-k}|V|^3\big) \tag{6.95}$$

as claimed. □

6.3 Proof of Proposition 4.4

Lemma 6.4 *Let $d = 4$, $\bar{m}^2 > 0$, and $m^2 \in [0, \bar{m}^2]$. For any $c < \pi^{-2}\log L$, there exists $n < \infty$ such that $\beta_j(m^2) \geq c$ for $n \leq j \leq j_m - n$, uniformly in $m^2 \in [0, \bar{m}^2]$.*

Proof Let $\varepsilon > 0$ satisfy $c+\varepsilon < \pi^{-2}\log L$. By (6.65), there exists n_0 such that $\beta_j(0) \geq c+\varepsilon$ if $j \geq n_0$. This is sufficient for the case $m^2 = 0$, where $j_m - n = \infty$.

Thus we consider $m^2 > 0$. With c' the constant in (6.66), choose n_1 such that $c'(\log L)L^{4-2n_1} \leq \varepsilon$. By definition in (4.20), $j_m = \lfloor \log_{L^2} m^{-2}\rfloor$, so $m^2L^{2j_m} \leq 1$. Thus, for $m^2 \in (0, \bar{m}^2]$, (6.66) implies that if $1 < j \leq j_m - n_1$ then

$$|\beta_j(0) - \beta_j(m^2)| \leq c'(\log L)L^{4+2j}m^2 \leq c'(\log L)L^{4-2n_1} \leq \varepsilon. \tag{6.96}$$

Therefore, $\beta_j(m^2) \geq c$ if $n_0 \leq j \leq j_m - n_1$, as claimed. □

Proof of Proposition 4.4 The continuity in $m^2 \in [0, \bar{m}^2]$ of the coefficients in (3.30)–(3.35) and in (4.13)–(4.15) is immediate from Proposition 6.1 and Lemma 6.2.

To verify that $\bar{\varphi}$ obeys [5, Assumptions (A1–A2)], we fix $\Omega > 1$, and recall from (4.21) the definition

$$j_\Omega = \inf\Big\{k \geq 0 : |\beta_j| \leq \Omega^{-(j-k)}\|\beta\|_\infty \text{ for all } j\Big\}. \tag{6.97}$$

Let k be such that $L^{2k} \geq \Omega$. Then, for all $j \geq 0$,

$$(1+m^2L^{2j})^{-k} \leq L^{-2k(j-j_m)_+} \leq \Omega^{-(j-j_m)_+}. \tag{6.98}$$

Fix c, n as in Lemma 6.4. Since $j_m \to \infty$ as $m \downarrow 0$, there is a δ such that $j_m > n$ when $m^2 \in [0, \delta]$. For such m, it follows from Lemma 6.4 that $\|\beta\|_\infty \geq c$. We apply Lemma 6.2 and (6.98) to see that there is a constant C such that

$$|\beta_j| \leq C\Omega^{-(j-j_m)_+} \leq \frac{C}{c}\Omega^{-(j-j_m)_+}\|\beta\|_\infty \leq \Omega^{-(j-k)_+}\|\beta\|_\infty \tag{6.99}$$

whenever $k \geq j_m + \log_\Omega(C/c)$. In particular, $j_\Omega \leq k$ and thus $j_\Omega \leq j_m + O(1)$. On the other hand, by Lemma 6.4, $\beta_{j_m-n} \geq c$, and the definition of j_Ω thus requires that $c \leq \beta_{j_m-n} \leq \Omega^{-(j_m-n-j_\Omega)_+}\|\beta\|_\infty$. Therefore, $j_m - n - j_\Omega \leq \log_\Omega(c^{-1}\|\beta\|_\infty)$, which implies that $j_\Omega \geq j_m - n - \log_\Omega(c^{-1}\|\beta\|_\infty)$. This completes the proof of (4.22). Also, the number of $j \leq j_\Omega$ with $\beta_j < c$ is bounded by $n + \log_\Omega(C/c)$. This proves [5, Assumption (A1)] and also shows

$$\Omega^{-(j-j_m)_+} = O(\Omega^{-(j-j_\Omega)_+}). \tag{6.100}$$

Then [5, Assumption (A2)] follows from Lemma 6.2, (6.98), and the previous sentence. This completes the proof. □

Finally, for use in [13], we note the following inequalities. First, it follows from Proposition 4.4 and [5, Lemma 2.1] that the sequence $(\bar{g}_j)$ solving (4.13) obeys (for sufficiently small $\bar{g}_0$)

$$\frac{1}{2}\bar{g}_{j+1} \leq \bar{g}_j \leq 2\bar{g}_{j+1}. \tag{6.101}$$

In addition, the combination of (6.9), (6.98), and (6.100) implies that there is an L-independent constant c such that for $m^2 \in [0, \delta]$ and $j = 1, \ldots, N-1$, and in the special case $C_j = C_{N,N}$ for $m^2 \in [\varepsilon L^{-2(N-1)}, \delta]$ with the constant c now depending on $\varepsilon > 0$,

$$|\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}| \le c\Omega^{-(j-j_\Omega)_+} L^{-(j-1)(2[\phi]+(|\alpha|_1+|\beta|_1))}. \tag{6.102}$$

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