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Ground States for Mean Field Models with a Transverse Component

Dmitry Ioffe · Anna Levit

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Abstract We investigate global logarithmic asymptotics of ground states for a family of quantum mean field models in transverse field. Our approach is based on a stochastic representation and a combination of large deviation and weak KAM techniques. We prove that asymptotic ground states are weak KAM (Fathi in C. R. Acad. Sci., Ser. I Math. 324(9):1043–1046, 1997; Fathi, The Weak KAM Theorem in Lagrangian Dynamics 10th Preliminary Version, 2008; Fathi in Nonlinear Differ. Equ. Appl. 14(1):1–27, 2007) and, in particular, viscosity (Capuzzo-Dolcetta and Lions in Trans. Am. Math. Soc. 318(2):643–683, 1990; Fathi, The Weak KAM Theorem in Lagrangian Dynamics 10th Preliminary Version, 2008, Chap. 7) solutions of certain stationary Hamilton-Jacobi equations. In general such solutions are not unique, and additional refined selection criteria are needed. The spin-$\frac{1}{2}$ model is worked out in more detail. We discuss phase transitions in the ground state as the strength of the transverse field varies. For a class of mean field interaction potentials this transition is of the first order. For all the models in question, asymptotic ground states with multiple wells necessarily develop shocks. A complete description of asymptotic ground states is derived for ferromagnetic p-body interactions.

Keywords Ground states · Quantum mean field model · Stochastic representation

1 Introduction

Stochastic representations/path integral approach frequently provides a useful intuition and insight into the structure of quantum spin states. Numerous examples include [1, 2, 8, 10, 12, 16, 21, 22, 26, 31]. In this work we rely on a path integral approach and related large

D. Ioffe
Faculty of Industrial Engineering, Technion, Haifa 3200, Israel
e-mail: ieioffe@ie.technion.ac.il

A. Levit (✉)
Department of Mathematics, UBC, Vancouver, BC V6T 1Z2, Canada
e-mail: anna.levit@math.ubc.ca

deviations techniques, and derive *global* logarithmic asymptotics of ground states for a class of quantum mean field models [17] in transverse field. These asymptotic limits are identified as weak KAM [18–20] type solutions of certain Hamilton-Jacobi equations. In principle, such solutions are not unique, and an additional refined analysis along the lines of [15, 23, 24] is needed for recovering the correct asymptotic ground state. This issue is addressed in more detail for the spin-$\frac{1}{2}$ case. In particular, our results imply logarithmic asymptotics of ground states for models with p-body interactions [5]. In the case of Laplacian with periodic potential a weak KAM approach to semi-classical asymptotics was already employed in [4].

Our stochastic representation gives rise to a family of continuous time Markov chains on a simplex Δ_d^N (defined below) of $\frac{1}{N}\mathbb{Z}^d$. The transition rates are enhanced by a factor of N, and the chain moves in a potential of the type NF. Ground states are Perron-Frobenius eigenfunctions of the corresponding generators. On the concluding stages of this work we have learned about the series of papers [27–29]. The models we consider here essentially fall into a much more general framework studied in these works. The authors of [27–29] extend an analysis of Schrödinger operators [15, 23, 24] on $\mathbb{R}^d$ to lattice operators on $\epsilon\mathbb{Z}^d$, and they develop powerful techniques, which go well beyond the scope of our work, and which enable a complete asymptotic expansion of low lying eigenvalues and eigenfunctions in *local* neighbourhoods of potential wells.

The paper is organized as follows: The class of models is described in Sect. 2.1, and the results are formulated in Sect. 2.3. Main steps of our approach are explained in Sect. 3, whereas some of the proofs are relegated to Sect. 4. The spin-$\frac{1}{2}$ case is studied in Sect. 5. Finally, in Appendix, we establish the required properties of the Lagrangian $\mathcal{L}_0$ in (2.12) and, accordingly, the required regularity properties of local minimizers.

Throughout the paper the symbol $\stackrel{\Delta}{=}$ means *by definition*.

2 The Model and the Result

2.1 Class of Models

Let $\mathbb{X}$ be a d-dimensional complex Hilbert space. For the rest of the paper we fix an orthonormal basis $\{|\alpha\rangle\}_{\alpha\in\mathcal{A}}$ of $\mathbb{X}$. We refer to the set $\mathcal{A}$ of cardinality d as the set of *classical labels*. The induced basis of $\mathbb{X}_N = \otimes_1^N \mathbb{X}$ is

$$|\underline{\alpha}\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle, \qquad \underline{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathcal{A}^N.$$

In the sequel we shall write $[\underline{\alpha}]_i$ for the i-th coordinate of $\underline{\alpha} \in \mathcal{A}^N$.

For $\alpha \in \mathcal{A}$ define projections $P_\alpha \stackrel{\Delta}{=} |\alpha\rangle\langle\alpha|$. The corresponding lifting to $\mathbb{X}_N$ of the projection operator acting on i-th component is $P_\alpha^i = I \otimes \cdots \otimes I \otimes P_\alpha \otimes I \otimes \cdots \otimes I$. For $\alpha \in \mathcal{A}$ set $M_\alpha^N = \frac{1}{N}\sum_i P_\alpha^i$. Let $\underline{M}^N$ be the d-dimensional vector with operator entries M_α^N.

We are ready to define the Hamiltonian H_N which acts on $\mathbb{X}_N$,

$$-\mathrm{H}_N = NF\big(\underline{M}^N\big) + \sum_i B_i. \tag{2.1}$$

Above, B_i-s are copies of a Hermitian matrix B on $\mathbb{X}$, B_i acts on the i-th component of $|\underline{\alpha}\rangle$.

We assume:

A1. F is a real polynomial of finite degree.

Let Δ_d be the simplex, $\Delta_d = \{\underline{m} \in \mathbb{R}_+^d : \sum m_i = 1\}$. In the sequel we shall write $\mathrm{int}(\Delta_d)$ for the relative interior of Δ_d. Accordingly, $\partial\Delta_d \stackrel{\Delta}{=} \Delta_d \setminus \mathrm{int}(\Delta_d)$.

Given $\underline{m} \in \Delta_d$ and a classical multi-label $\underline{\alpha} \in \mathcal{A}^N$ let us say that $\underline{\alpha} \sim \underline{m}$, or, equivalently, $\underline{m} = \underline{m}(\underline{\alpha})$, if

$$m_\alpha = \frac{\#\{i : \alpha_i = \alpha\}}{N} \quad \Leftrightarrow \quad M_\alpha^N |\underline{\alpha}\rangle = m_\alpha |\underline{\alpha}\rangle, \tag{2.2}$$

for all $\alpha \in \mathcal{A}$. Namely, $\underline{m}(\underline{\alpha})$ is the empirical distribution of the alphabet $\mathcal{A}$ in the sequence $\underline{\alpha}$. Define $\Delta_d^N = \Delta_d \cap \frac{1}{N}\mathbb{Z}^d$. In other words, $\underline{m} \in \Delta_d^N$ iff there exists a compatible $\underline{\alpha} \in \mathcal{A}^N$. In the above notation:

$$F\big(\underline{M}^N\big)|\underline{\alpha}\rangle = F\big(\underline{m}(\underline{\alpha})\big)|\underline{\alpha}\rangle. \tag{2.3}$$

A2. The transverse field B satisfies: For any $\alpha, \beta \in \mathcal{A}$,

$$\lambda_{\alpha\beta} = \lambda_{\beta\alpha} \stackrel{\Delta}{=} \langle \alpha | B | \beta \rangle \geq 0. \tag{2.4}$$

Without loss of generality we may assume that $\lambda \equiv 0$ on the diagonal. Indeed, the class of models we consider is stable under the transformation

$$F\big(\underline{M}^N\big) \mapsto F\big(\underline{M}^N\big) + \sum_\alpha \lambda_{\alpha\alpha} M_\alpha^N \quad \text{and} \quad B \mapsto B - \sum_\alpha \lambda_{\alpha\alpha} P_\alpha.$$

Furthermore, we shall assume that the (continuous time) Markov chain on $\mathcal{A}$ with jump rates $\{\lambda_{\alpha\beta}\}$ is irreducible.

Assumption **A2**, namely the fact that transverse fields have *real* representations in the basis of classical labels, and that we can readily interpret matrix entries $\{\lambda_{\alpha\beta}\}$ as jump rates of an irreducible continuous time Markov chain, is crucial for our approach.

2.2 An Example: Spin-s Models

The relation between the dimension d of $\mathbb{X}$ and the half-integer spin s is $d = 2\mathsf{s} + 1$. The set of classical labels is

$$\mathcal{A} = \{-\mathsf{s}, -\mathsf{s}+1, \ldots, \mathsf{s}\}.$$

The stochastic operators are $B_i = \lambda \mathsf{S}_i^x$. $\lambda \geq 0$ is the strength of the transverse field. Altogether, the Hamiltonian is

$$-\mathrm{H}_N = NF\big(M_{-\mathsf{s}}^N, M_{-\mathsf{s}+1}^N, \ldots, M_{\mathsf{s}}^N\big) + \lambda \sum_i \mathsf{S}_i^x. \tag{2.5}$$

For instance, the case of p-body ferromagnetic interaction corresponds to

$$F\big(M_{-\mathsf{s}}^N, M_{-\mathsf{s}-1}^N, \ldots, M_{\mathsf{s}}^N\big) = \Big(\sum_\alpha \alpha M_\alpha^N\Big)^p = \Big(\frac{1}{N}\sum_i \mathsf{S}_i^z\Big)^p. \tag{2.6}$$

The operators S^x act (under convention that $|\mathsf{s}+1\rangle = |-\mathsf{s}-1\rangle = 0$) on $\mathbb{X}$ as

$$\mathsf{S}^x|\alpha\rangle = \frac{1}{2}\sqrt{\mathsf{s}(\mathsf{s}+1) - \alpha(\alpha-1)}|\alpha-1\rangle + \frac{1}{2}\sqrt{\mathsf{s}(\mathsf{s}+1) - \alpha(\alpha+1)}|\alpha+1\rangle. \tag{2.7}$$

Consequently, the jump rates $\lambda_{\alpha\beta}$ are given by

$$\lambda_{\alpha\beta} = \begin{cases} \frac{\lambda}{2}\sqrt{\mathsf{s}(\mathsf{s}+1) - \alpha\beta}, & \text{if } |\alpha-\beta| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

2.3 The Result

In order to develop an asymptotic description of finite volume ground states we need to introduce some additional notation: For $\underline{m} \in \Delta_d^N$ set

$$c_N(\underline{m}) = \binom{N}{N\underline{m}} = \frac{N!}{\prod (N m_\alpha)!}.$$

The vectors $|\underline{m}\rangle \in \mathbb{X}_N$,

$$|\underline{m}\rangle \triangleq \frac{1}{\sqrt{c_N(\underline{m})}} \sum_{\underline{\alpha} \sim \underline{m}} |\underline{\alpha}\rangle \tag{2.9}$$

are normalized and orthogonal for different $\underline{m} \in \Delta_d^N$.

By the Perron-Frobenius theorem and Lemma 1 below the ground state of H_N is fully symmetrized, that is of the form

$$|h_N\rangle = \sum_{\underline{m} \in \Delta_d^N} h_N(\underline{m}) |\underline{m}\rangle, \tag{2.10}$$

and $h_N(\underline{m}) > 0$ for every $\underline{m} \in \Delta_d^N$. Let us represent

$$h_N(\underline{m}) = \mathrm{e}^{-N\psi_N(\underline{m})}. \tag{2.11}$$

It would be convenient to identify ψ_N with its linear interpolation (which is an element of the space of continuous functions $\mathsf{C}(\Delta_d)$).

Next introduce:

$$\begin{aligned} \mathcal{H}_0(\underline{m}, \underline{\theta}) &= \sum_{\alpha\beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} \big(\cosh(\theta_\beta - \theta_\alpha) - 1\big), \\ \mathcal{L}_0(\underline{m}, \underline{v}) &= \sup_{\underline{\theta}} \big\{ (\underline{v}, \underline{\theta}) - \mathcal{H}_0(\underline{m}, \underline{\theta}) \big\}. \end{aligned} \tag{2.12}$$

In (2.12) above $(\underline{v}, \underline{\theta})$ is the usual scalar product on $\mathbb{R}^d$. For $\underline{m} \in \Delta_d$ define

$$V(\underline{m}) = \frac{1}{2} \sum_{\alpha,\beta} \lambda_{\alpha\beta} (\sqrt{m_\beta} - \sqrt{m_\alpha})^2 - F(\underline{m}). \tag{2.13}$$

Finally set $\lambda_\alpha = \sum_\beta \lambda_{\alpha\beta}$,

$$\begin{aligned} \mathcal{H}(\underline{m}, \underline{\theta}) &= \mathcal{H}_0(\underline{m}, \underline{\theta}) - V(\underline{m}) \\ &= \sum_{\alpha\beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} \cosh(\theta_\beta - \theta_\alpha) - \sum_\alpha m_\alpha \lambda_\alpha + F(\underline{m}), \\ &\text{and} \\ \mathcal{L}(\underline{m}, \underline{v}) &= \mathcal{L}_0(\underline{m}, \underline{v}) + V(\underline{m}). \end{aligned} \tag{2.14}$$

Theorem 1 *Let E_N^1 be the lowest eigenvalue of $\mathcal{H}_N$. Set $\lambda = \sum_\alpha \lambda_\alpha$. Then the limit*

$$-\lambda + \mathsf{r}_1 \triangleq \lim_{N\to\infty} \frac{E_N^1}{N} \tag{2.15}$$

exists. Moreover,

$$\mathsf{r}_1 = \min_{\underline{m}} V(\underline{m}). \tag{2.16}$$

Furthermore, the sequence $\{\psi_N\}$, *defined in* (2.11), *is precompact in* $\mathsf{C}(\Delta_d)$. *Any subsequential limit* ψ *satisfies*: *For any* $T \geq 0$ *and any* $\underline{m} \in \Delta_d$,

$$\psi(\underline{m}) = \inf_{\gamma:\gamma(T)=\underline{m}} \left\{ \psi\big(\gamma(0)\big) + \int_0^T \mathcal{L}\big(\gamma(t), \gamma'(t)\big) \mathrm{d}t - T\mathsf{r}_1 \right\}, \tag{2.17}$$

where the infimum above is over all absolutely continuous curves $\gamma : [0, T] \to \Delta_d$. *Moreover, the set* $\mathcal{M}_\psi$ *of all local minima of* ψ *is a subset of* $\mathcal{M}_V \stackrel{\Delta}{=} \mathrm{argmin}(V) \stackrel{\Delta}{=} \{\underline{m} : V(\underline{m}) = \min V\} \subset \mathrm{int}(\Delta_d)$.

The proofs of (2.15) and (2.17) are given in Sect. 3.6 and the proof of (2.16) can be found in Sect. 4.3.

Remark 1 Hamiltonians $\mathcal{H}_0, \mathcal{H}$ are invariant under the shifts $\underline{\theta} \mapsto \underline{\theta} + c\mathbf{1}$, and, as a result, the Lagrangians $\mathcal{L}_0, \mathcal{L}$ are infinite whenever $(\underline{v}, \mathbf{1}) \neq 0$. Also, $\mathcal{L}_0(\underline{m}, 0) = 0 = \min_{\underline{v}} \mathcal{L}_0(\underline{m}, \underline{v})$. Consequently, $\mathcal{L}(\underline{m}, 0) = V(\underline{m}) = -\mathcal{H}(\underline{m}, 0)$, and (2.16) could be rewritten as

$$\mathsf{r}_1 = \min_{\underline{m},\underline{v}} \mathcal{L}(\underline{m}, \underline{v}) = -\max_{\underline{m}} \mathcal{H}(\underline{m}, 0). \tag{2.18}$$

Remark 2 Either of (2.16) and (2.17) unambiguously characterizes r_1, but not ψ. As we shall explain in the sequel, if ψ satisfies (2.17), then the weak KAM theory of Fathi [19] implies that ψ is a viscosity solution (see Sect. 3.7 for the precise statement) on $\mathrm{int}(\Delta_d)$ of the Hamilton-Jacobi equation

$$\mathcal{H}\big(\underline{m}, \nabla\psi(\underline{m})\big) = -\mathsf{r}_1. \tag{2.19}$$

Note that since ψ is a function on Δ_d, the gradients $\nabla\psi$ lie in the subspace

$$\mathbb{R}_0^d = \big\{\underline{v} : (\underline{v}, \mathbf{1}) = 0\big\}. \tag{2.20}$$

In general there might be many viscosity solutions of (2.19) which comply with the conclusions of Theorem 1. The solutions which are subsequential limits of $\{\psi_N\}$ are called *admissible*. Although we expect uniqueness of global admissible solutions for a large class of models, our approach does not offer a procedure for selecting the latter. The viscosity setup is important—at least for a large class of symmetric potentials the global admissible solutions are not smooth and develop shocks. A proper selection procedure should be related to a more refined analysis of the low lying spectra of H_N. As it was mentioned in the Introduction sharp asymptotics of eigenvalues and eigenfunctions in vicinity of potential wells were derived in a much more general context in [27–29]. In particular, it is explained therein how such asymptotics are related to (smooth) local solutions of (2.19). Implications of these results for a characterization of *global* admissible solutions is beyond the scope of this work and hopefully shall be addressed in full generality elsewhere. In the concluding Sect. 5 we work out a particular case of spin-$\frac{1}{2}$ models.

3 Structure of the Theory

3.1 Spectrum of H_N

For $\underline{\alpha} \in \mathcal{A}^N$ and a permutation π of $\{1, \ldots, N\}$ define $\pi\underline{\alpha}$ via $[\pi\underline{\alpha}]_i = \alpha_{\pi_i}$.

Lemma 1 *Let E_N be an eigenvalue of* H_N, *and let* $|b_N\rangle = \sum_{\underline{\alpha}\in\mathcal{A}^N} a_{\underline{\alpha}}|\underline{\alpha}\rangle$ *be a corresponding eigenfunction*; $\mathrm{H}_N|b_N\rangle = E_N|b_N\rangle$. *Then for every permutation π of* $\{1,\dots,N\}$ *the vector* $|\pi b_N\rangle \triangleq \sum_{\underline{\alpha}} a_{\pi\underline{\alpha}}|\underline{\alpha}\rangle$ *is also an eigenfunction of* H_N *corresponding to the same eigenvalue* E_N.

Proof Let $C(\underline{\alpha},\underline{\beta}) = \langle\underline{\beta}|\hat{B}|\underline{\alpha}\rangle$ be the matrix elements of $\hat{B} \triangleq \sum_i B_i$. Thus, $\hat{B}|\underline{\alpha}\rangle = \sum_{\underline{\beta}} C(\underline{\alpha},\underline{\beta})|\underline{\beta}\rangle$. The eigenfunction equation is recorded as: $\forall\,\underline{\beta}$

$$\sum_{\underline{\alpha}} a_{\underline{\alpha}} C(\underline{\alpha},\underline{\beta}) = \big(-E_N - NF\big(\underline{m}(\underline{\beta})\big)\big)a_{\underline{\beta}}.$$

Note that $C(\underline{\alpha},\underline{\beta}) = C(\pi\underline{\alpha},\pi\underline{\beta})$. Consequently, since in addition $\underline{m}(\underline{\beta}) = \underline{m}(\pi\underline{\beta})$,

$$\sum_{\underline{\alpha}} a_{\pi\underline{\alpha}} C(\underline{\alpha},\underline{\beta}) = \big(-E_N - NF\big(\underline{m}(\underline{\beta})\big)\big)a_{\pi\underline{\beta}}.$$

Therefore, $|\pi b_N\rangle \triangleq \sum_{\underline{\alpha}} a_{\pi\underline{\alpha}}|\underline{\alpha}\rangle$ is indeed an eigenfunction with the same eigenvalue E_N. □

By the Perron-Frobenius theorem it follows that if E_N^1 is the lowest eigenfunction of H_N, then the corresponding ground state $|h_N\rangle$ is necessarily fully symmetrized, namely there exists a strictly positive function h_N on Δ_d^N such that $|h_N\rangle \triangleq \sum_{\underline{m}\in\Delta_d^N} h_N(\underline{m})|\underline{m}\rangle$, and

$$\mathrm{H}_N|h_N\rangle = E_N^1|h_N\rangle. \tag{3.1}$$

In the sequel we shall restrict attention to those eigenvalues E_N of H_N which have non-trivial fully symmetrized eigenfunctions.

3.2 Stochastic Representation

Let $\alpha(t)$ be the continuous time Markov chain on $\mathcal{A}$ with jump rates $\lambda_{\alpha\beta}$. $\mathbb{P}_{\underline{\alpha}}^N$ is the path measure for N independent copies of such chain starting from $\underline{\alpha}$. Then the following representation of the entries of the density matrix holds [1, 26]: For any $T \geq 0$ and any $\underline{\alpha},\underline{\beta} \in \mathcal{A}^N$

$$\mathrm{e}^{-N\lambda T}\langle\underline{\beta}|\mathrm{e}^{-T\mathrm{H}_N}|\underline{\alpha}\rangle = \mathbb{E}_{\underline{\alpha}}^N\left(\exp\left\{N\int_0^T F\big(\underline{m}(t)\big)\mathrm{d}t\right\}\mathbb{1}_{\{\underline{\alpha}(T)=\underline{\beta}\}}\right). \tag{3.2}$$

Above $\underline{m}(t) = \underline{m}(\underline{\alpha}(t))$.

3.3 Mean Field Lumping

The process $\underline{m}_N(t) = \underline{m}(t) = \underline{m}(\underline{\alpha}(t))$ is a continuous time Markov chain on Δ_d^N with the generator

$$\mathcal{G}_N f(\underline{m}) = N\sum_{\alpha,\beta} m_\alpha\lambda_{\alpha\beta}\left(f\left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N}\right) - f(\underline{m})\right), \tag{3.3}$$

where δ_α and δ_β are Kronecker deltas. It is reversible with respect to the measure

$$\mu_N(\underline{m}) \triangleq \frac{c_N(\underline{m})}{d^N}. \tag{3.4}$$

Summing up in (3.2), first over all $\underline{\beta} \sim \underline{m}'$, and then over all $\underline{\alpha} \sim \underline{m}$, we, in view of the mean field nature of the interaction and by the definition (2.9) of vectors $|\underline{m}\rangle$, obtain:

$$\mathrm{e}^{-N\lambda T} \langle \underline{m}' | \mathrm{e}^{-T\mathrm{H}_N} | \underline{m} \rangle = \sqrt{\frac{\mu_N(\underline{m})}{\mu_N(\underline{m}')}} \mathbb{E}_{\underline{m}}^N \left(\exp\left\{ N \int_0^T F\big(\underline{m}(t)\big) \mathrm{d}t \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}'\}} \right), \tag{3.5}$$

for every $T \geq 0$ and every $\underline{m}, \underline{m}' \in \Delta_d^N$.

Using Girsanov's formula one can rewrite (3.5) in a variety of ways for different modifications of the jump rates in (3.3). Namely, let g be a positive function on Δ_d^N. Consider the modified rates

$$\lambda_{\alpha\beta}^{N,g} = \lambda_{\alpha\beta}^{N,g}(\underline{m}) = \frac{1}{g(\underline{m})} N m_\alpha \lambda_{\alpha\beta} g\big(\underline{m} + (\delta_\beta - \delta_\alpha)/N\big).$$

Let $\mathbb{P}_{\underline{m}}^{N,g}$ be the corresponding path measure. Then, the right hand side of (3.5) reads as

$$\sqrt{\frac{\mu_N(\underline{m}) g(\underline{m})^2}{\mu_N(\underline{m}') g(\underline{m}')^2}} \mathbb{E}_{\underline{m}}^{N,g} \left(\exp\left\{ N \int_0^T F_g\big(\underline{m}(t)\big) \mathrm{d}t \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}'\}} \right), \tag{3.6}$$

where

$$F_g(\underline{m}) = \sum_{\alpha\beta} \lambda_{\alpha\beta} m_\alpha \left(\frac{g(\underline{m} + (\delta_\beta - \delta_\alpha)/N)}{g(\underline{m})} - 1 \right) + F(\underline{m}). \tag{3.7}$$

Taking into consideration (3.5) and (3.6) a self-suggesting choice, which cancels the prefactor in the latter formula, is

$$g(\underline{m}) = \frac{1}{\sqrt{\mu_N(\underline{m})}} \quad \Rightarrow \quad \lambda_{\alpha\beta}^{N,g} = N\sqrt{m_\alpha(m_\beta + 1/N)}\, \lambda_{\alpha\beta}. \tag{3.8}$$

For the rest of the paper we fix g as in (3.8). The corresponding generator

$$\mathcal{G}_N^g f(\underline{m}) = \sum_{\alpha,\beta} \lambda_{\alpha\beta}^{N,g} \left(f\left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N} \right) - f(\underline{m}) \right) \tag{3.9}$$

is reversible with respect to the uniform measure on Δ_d^N. The function F_g in (3.7) equals

$$F_g(\underline{m}) = -V(\underline{m}) + \varXi_N(\underline{m}), \tag{3.10}$$

where V is precisely the function defined in (2.13), and the correction

$$\varXi_N(\underline{m}) = \sum_{\alpha\beta} \lambda_{\alpha\beta} \sqrt{m_\alpha} (\sqrt{m_\beta + 1/N} - \sqrt{m_\beta}). \tag{3.11}$$

All together, (3.5) reads as

$$\mathrm{e}^{-N\lambda T} \langle \underline{m}' \big| \mathrm{e}^{-T\mathrm{H}_N} | \underline{m} \rangle = \mathbb{E}_{\underline{m}}^{N,g} \left(\exp\left\{ -N \int_0^T (V - \varXi_N)\big(\underline{m}(t)\big) \mathrm{d}t \right\} \mathbb{1}_{\{\underline{m}(T) = \underline{m}'\}} \right). \tag{3.12}$$

As we shall see below it is convenient to work simultaneously with both representations (3.5) and (3.12).

Note that an immediate consequence of (3.12) is:

Lemma 2 *Let E_N be an eigenvalue of* H_N. *Then* $|u_N\rangle = \sum u_N(\underline{m})|\underline{m}\rangle$ *is a corresponding fully symmetrized normalized eigenfunction if and only if* u_N *is an eigenfunction of* $\mathcal{S}_N \stackrel{\Delta}{=} \mathcal{G}_N^g + NF_g = \mathcal{G}_N^g - N(V - \Xi_N)$ *with*

$$-R_N \stackrel{\Delta}{=} -(N\lambda + E_N) \tag{3.13}$$

being the corresponding eigenvalue.

Indeed, $\mathcal{S}_N$ is symmetric, whereas $|u_N\rangle$ is an eigenfunction of $\mathrm{e}^{-T\mathrm{H}_N}$ if and only if $\langle\underline{m}'|\mathrm{e}^{-T\mathrm{H}_N}|u_N\rangle \equiv \mathrm{e}^{-TE_N}u_N(\underline{m}')$.

3.4 The Eigenfunction Equation

Assumption **A.2** and Perron-Frobenius theorem imply that H_N has a non-degenerate ground state $|h_N\rangle = \sum_{\underline{m}} h_N(\underline{m})|\underline{m}\rangle$ with strictly positive entries $h_N(\underline{m}) > 0$. By Lemma 2, $h_N(\underline{m})$ is the principal eigenfunction of $\mathcal{G}_N^g + NF_g(\underline{m})$ with the corresponding top eigenvalue $-R_N^1 = -(N\lambda + E_N^1)$. The corresponding eigenfunction equation is: For any $T > 0$,

$$h_N(\underline{m}) = \mathbb{E}_{\underline{m}}^{N,g}\left(\exp\left\{-N\int_0^T (V - \Xi_N)\big(\underline{m}(t)\big)\mathrm{d}t + TR_N^1\right\}h_N\big(\underline{m}(T)\big)\right). \tag{3.14}$$

By reversibility (recall notation (3.10)),

$$\mathbb{E}_{\underline{m}}^{N,g}\left(\exp\left\{N\int_0^T F_g\big(\underline{m}(t)\big)\mathrm{d}t\right\}\mathbb{1}_{\{\underline{m}(T)=\underline{m}'\}}\right) = \mathbb{E}_{\underline{m}'}^{N,g}\left(\exp\left\{N\int_0^T F_g\big(\underline{m}(t)\big)\mathrm{d}t\right\}\mathbb{1}_{\{\underline{m}(T)=\underline{m}\}}\right).$$

Hence,

Lemma 3 *Functions* $\{h_N\}$ *satisfy*: *For every* $T \geq 0$, N *and* $\underline{m} \in \Delta_d^N$

$$h_N(\underline{m}) = \sum_{\underline{m}'} h_N\big(\underline{m}'\big)\mathbb{E}_{\underline{m}'}^{N,g}\left(\exp\left\{-N\int_0^T (V - \Xi_N)\big(\underline{m}(t)\big)\mathrm{d}t + TR_N^1\right\}\mathbb{1}_{\{\underline{m}(T)=\underline{m}\}}\right). \tag{3.15}$$

3.5 Compactness and Large Deviations

For $\underline{m}, \underline{m}' \in \Delta_d^N$ define

$$Z_T^{N,g}\big(\underline{m}', \underline{m}\big) \stackrel{\Delta}{=} \frac{1}{N}\log\mathbb{E}_{\underline{m}'}^{N,g}\left(\exp\left\{-N\int_0^T (V - \Xi_N)\big(\underline{m}(t)\big)\mathrm{d}t\right\}\mathbb{1}_{\{\underline{m}(T)=\underline{m}\}}\right). \tag{3.16}$$

In the sequel we shall identify $Z_T^{N,g}(\cdot,\cdot)$ with its continuous interpolation on $\Delta_d \times \Delta_d$.

Let $\mathcal{AC}_T$ be the family of all absolutely continuous trajectories $\gamma : [0, T] \mapsto \Delta_d$. For $\underline{m}, \underline{m}' \in \Delta_d$ define

$$Z_T^g\big(\underline{m}', \underline{m}\big) \stackrel{\Delta}{=} -\inf_{\substack{\gamma(0)=\underline{m}',\gamma(T)=\underline{m}\\ \gamma\in\mathcal{AC}_T}} \int_0^T \mathcal{L}\big(\gamma(t), \gamma'(t)\big)\mathrm{d}t, \tag{3.17}$$

where the Lagrangian $\mathcal{L}$ was defined in (2.14).

Theorem 2 *For all T sufficiently large the sequence of functions $\{Z_T^{N,g}\}$ is equi-continuous on $\Delta_d \times \Delta_d$. Moreover, the functions $\underline{m} \to Z_T^{N,g}(\underline{m}', \underline{m})$ are, uniformly in $\underline{m}'$, locally Lipschitz on* int(Δ_d). *Finally, for all T sufficiently large,*

$$\lim_{N\to\infty} \left| Z_T^{N,g}\big(\underline{m}', \underline{m}\big) - Z_T^{g}\big(\underline{m}', \underline{m}\big)\right| = 0, \tag{3.18}$$

for any $\underline{m}, \underline{m}' \in \Delta_d$.

Note that the equi-continuity of $\{Z_T^{N,g}\}$ in Theorem 2 implies that the convergence in (3.18) is actually uniform. Consequently, $Z_T^g(\cdot,\cdot)$ is continuous on $\Delta_d \times \Delta_d$ and locally Lipschitz on int$(\Delta_d \times \Delta_d)$.

Theorem 2 is a somewhat standard statement. Its proof will be sketched in Sect. 4.1.

3.6 Lax-Oleinik Semigroup

Recall the representation of the leading eigenfunction $h_N(\underline{m}) = \mathrm{e}^{-N\psi_N(\underline{m})}$. In the sequel we shall identify ψ_N with its (continuous) interpolation on Δ_d; $\psi_N \in \mathsf{C}(\Delta_d)$.

Theorem 3 *The sequence of numbers R_N^1/N is bounded in $\mathbb{R}$. The sequence of functions $\{\psi_N\}$ is equi-continuous on Δ_d and uniformly locally Lipschitz on* int(Δ_d).

Proof Since R_N^1 is the Perron-Frobenius eigenvalue,

$$\frac{R_N^1}{N} = -\frac{1}{N}\lim_{T\to\infty}\frac{1}{T}\log \mathbb{E}_{\underline{m}'}^{N,g}\left(\exp\left\{N\int_0^T F_g\big(\underline{m}(s)\big)\mathrm{d}s\right\}\right),$$

which is bounded since F_g is bounded on Δ_d. On the other hand, the equi-continuity and the uniform local Lipschitz property of $\{\psi_N\}$ is inherited from the corresponding properties of $\{Z_T^{N,g}\}$. Indeed, by (3.15), for any $\underline{m}, \underline{n} \in \Delta_d$,

$$\mathrm{e}^{-N\psi_N(\underline{m})} \le \mathrm{e}^{-N\psi_N(\underline{n}) + N\max_{\underline{m}'}(Z_T^{N,g}(\underline{m}',\underline{m}) - Z_T^{N,g}(\underline{m}',\underline{n}))}.$$ □

Proof of (2.15) and (2.17) of Theorem 1 Theorems 2, 3 and Lemma 3 imply that any accumulation point $(\mathsf{r}, \psi) \in \mathbb{R} \times \mathsf{C}(\Delta_d)$ of the sequence $\{\frac{1}{N}R_N^1, \psi_N\}$ satisfies: ψ is locally Lipschitz on int(Δ_d) and

$$\psi(\underline{m}) = \inf_{\gamma(T)=\underline{m}}\left\{\psi\big(\gamma(0)\big) + \int_0^T \mathcal{L}\big(\gamma(t), \gamma'(t)\big)\mathrm{d}t\right\} - T\mathsf{r} \stackrel{\Delta}{=} \mathcal{U}_T\psi(\underline{m}) - T\mathsf{r} \tag{3.19}$$

for all sufficiently large T (and hence, by the semi-group property of $\mathcal{U}_T$, for every $T \ge 0$) and each $\underline{m} \in \Delta_d$. In order to check (3.19) note that if ψ is a uniform subsequential limit of ψ_N, and r is a subsequential limit of $\frac{R_N}{N}$, then by (3.15) and (3.18),

$$\mathrm{e}^{-N\psi(\underline{m})(1+o(1))} = \sum_{\underline{m}'} \mathrm{e}^{-N(\psi(\underline{m}') + Z_T^g(\underline{m}',\underline{m}) - T\mathsf{r})},$$

and (3.19) follows by continuity of $\underline{m}' \to \psi(\underline{m}') + Z_T^g(\underline{m}', \underline{m})$.

Accumulation points of ψ_N are called *admissible solutions* of (3.19). Since $\mathcal{U}_T$ is non-expanding on $\mathsf{C}(\Delta_d)$, validity of Eq. (3.19) unambiguously determines r, which implies that the limit $\mathsf{r}_1 \stackrel{\Delta}{=} \lim_{N\to\infty}\frac{R_N^1}{N}$ indeed exists. In view of (3.13) we, therefore, have established (2.15) and (2.17) of Theorem 1. □

3.7 Viscosity and Weak KAM Solutions

Recall the definition (2.14) of $\mathcal{H}(\underline{m}, \underline{\theta})$. For r_1 defined as above consider the Hamilton-Jacobi equation

$$\mathcal{H}\big(\underline{m}, \nabla\psi(\underline{m})\big) = -\mathsf{r}_1. \tag{3.20}$$

Definition 1 For $\underline{m} \in \Delta_d$ and $\psi \in \mathsf{C}(\Delta_d)$ define lower and upper sub-differentials

$$D_-\psi(\underline{m}) = \left\{ \xi \in \mathbb{R}_0^n : \liminf_{\underline{m}' \to \underline{m}} \frac{\psi(\underline{m}') - \psi(\underline{m}) - (\xi, \underline{m}' - \underline{m})}{|\underline{m}' - \underline{m}|} \geq 0 \right\}, \tag{3.21}$$

and, similarly, $D_+\psi(\underline{m})$ with $\liminf$ changed to $\limsup$ and the sign of the inequality flipped.

A locally Lipschitz function ψ is said to be a viscosity supersolution of (3.20) at $\underline{m}$ if $\mathcal{H}(\underline{m}, \xi) \geq -\mathsf{r}_1$ for every $\xi \in D_-\psi(\underline{m})$. Similarly, it is said to be a viscosity subsolution of (3.20) at $\underline{m}$ if $\mathcal{H}(\underline{m}, \xi) \leq -\mathsf{r}_1$ for every $\xi \in D_+\psi(\underline{m})$. ψ is viscosity solution of (3.20) at $\underline{m}$ if it is both viscosity sub- and super-solution.

Definition 2 A locally Lipschitz function ψ is said to be a weak KAM solution of (3.20) if it satisfies (3.19) with $\mathsf{r} = \mathsf{r}_1$.

Theorem 4 *If ψ is a weak-KAM solution of* (3.20), *then it is a viscosity solution of* (3.20) *on* $\mathrm{int}(\Delta_d)$.

The proof of Theorem 4 is relegated to Sect. 4.2.

3.8 Minima of ψ

Theorem 5 *Let ψ be a weak KAM solution of* (3.20), *that is assume that ψ satisfies* (3.19). *Then all local minima of ψ lie in the interior* $\mathrm{int}(\Delta_d)$.

Theorem 5 will be proved in Sect. 4.3

3.9 Stochastic Representation of the Ground State

The eigenfunction equation (3.14) defines a Markovian semi-group

$$\widehat{\mathbb{E}}_T^N f(\underline{m}) = \frac{1}{h_N(\underline{m})} \mathbb{E}_{\underline{m}}^{N,g} \left(\exp\left\{ N \int_0^T F_g\big(\underline{m}(t)\big) \mathrm{d}t + T R_N^1 \right\} h_N\big(\underline{m}(T)\big) f\big(\underline{m}(T)\big) \right). \tag{3.22}$$

This corresponds to continuous time Markov chain with the generator

$$\widehat{\mathcal{G}}_N^g f(\underline{m}) = \frac{1}{h_N(\underline{m})} \sum_{\alpha,\beta} \lambda_{\alpha\beta}^{N,g} h_N\left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N}\right)\left(f\left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N}\right) - f(\underline{m})\right). \tag{3.23}$$

In the sequel we shall refer to $\widehat{\mathcal{G}}_N^g$ as to the generator of the ground state chain.

Lemma 4 *The generator $\widehat{\mathcal{G}}_N^g$ is reversible with respect to the probability measure $\nu_N(\underline{m}) \triangleq h_N^2(\underline{m}) = \mathrm{e}^{-2N\psi_N(\underline{m})}$. Furthermore, E_N is an eigenvalue of H_N if and only if $E_N^1 - E_N$ is an eigenvalue of $\widehat{\mathcal{G}}_N^g$.*

It is straightforward to check that $\widehat{\mathcal{G}}_N^g$ satisfies the detailed balance condition with respect to ν_N. It is equally straightforward to see from (3.22) that g_N is an eigenfunction of $\mathcal{G}_N^g + NF_g$, and hence by Lemma 2 of H_N, if and only if g_N/h_N is an eigenfunction of $\widehat{\mathcal{G}}_N^g$.

4 Proofs

4.1 Proof of Theorem 2

Consider the family of processes $\{\underline{m}(\cdot) = \underline{m}_N(\cdot)\}$ with generator $\mathcal{G}_N^g$ defined in (3.9). We shall identify $\underline{m}_N$ with its linear interpolation. For each $T > 0$, the family $\{\underline{m}_N(\cdot)\}$ is exponentially tight on $\mathsf{C}_{0,T}(\Delta_d)$.

Recall the definition of $\mathcal{H}_0$ and $\mathcal{L}_0$ in (2.12). They are related to limiting instantaneous currents for chains $\underline{m}_N(\cdot)$ with jump rates (3.8) in the following way: If $\{\mathcal{N}_{\alpha\beta}\}$ is a family of independent Poisson random variables with intensities $\sqrt{m_\alpha m_\beta}\lambda_{\alpha\beta}$, then $\mathcal{H}_0(\underline{m}, \underline{\theta})$ is the log-moment generating function of

$$\sum_\alpha \theta_\alpha \sum_\beta (\mathcal{N}_{\beta\alpha} - \mathcal{N}_{\alpha\beta}).$$

Large deviation results for Markov chains with generators of the type (3.9) were established in [30]. Alternatively, one can follow the approach of [14] and combine the Large Deviation Principle for projective limits [13] with the inverse contraction principle of [25] in order to conclude:

Lemma 5 *For each $T > 0$ and every initial condition $\underline{m} \in \Delta_d$ (where for each N we identify $\underline{m}$ with its discretization $\lfloor N\underline{m} \rfloor / N \in \Delta_d^N$) the family of processes $\{\underline{m}_N(t)\}$ satisfy a large deviations principle on $C_{0,T}(\Delta_d)$ with the following good rate function*

$$I_T(\gamma) = \begin{cases} \int_0^T \mathcal{L}_0(\gamma(s), \gamma'(s))\mathrm{d}s, & \textit{if } \gamma \textit{ is absolutely continuous,} \\ & \textit{and } \gamma(0) = \underline{m}, \\ \infty, & \textit{otherwise.} \end{cases} \tag{4.1}$$

By the upper bound in Varadhan's lemma,

$$\limsup_{N\to\infty} Z_T^{N,g}(\underline{m}', \underline{m}) \le Z_T^g(\underline{m}', \underline{m}).$$

On the other hand, by the lower bound in Varadhan's lemma, for each $\delta > 0$

$$Z_T^g(\underline{m}', \underline{m}) \le \liminf_{N\to\infty} \sup_{|\underline{m}_1 - \underline{m}| < \delta} Z_T^{N,g}(\underline{m}', \underline{m}_1).$$

Therefore, (3.18) is a consequence of the claimed continuity of $\underline{m} \to Z_T^{N,g}(\underline{m}', \underline{m})$. Let us proceed with establishing the asserted continuity properties of the family $Z_T^{N,g}(\cdot,\cdot)$. By reversibility,

$$Z_T^{N,g}(\underline{m}, \underline{m}') = Z_T^{N,g}(\underline{m}', \underline{m}), \tag{4.2}$$

so it would be enough to explore those in the second variable only.

An equivalent task is to check continuity properties of

$$Z_T^N\big(\underline{m}', \underline{m}\big) \triangleq \frac{1}{N} \log \mathbb{E}^N_{\underline{m}'}\left(\exp\left\{N \int_0^T F\big(\underline{m}(t)\big)\mathrm{d}t\right\}\mathbb{1}_{\{\underline{m}(T)=\underline{m}\}}\right). \tag{4.3}$$

Indeed, by (3.6)

$$Z_T^{N,g}\big(\underline{m}, \underline{m}'\big) = \sqrt{\frac{\mu_N(\underline{m})}{\mu_N(\underline{m}')}} Z_T^N\big(\underline{m}, \underline{m}'\big).$$

Now, under $\mathbb{P}_N$ the process $\underline{m}_N(t)$ is a superposition of N independent irreducible Markov chains on the finite set $\mathcal{A}$. Since F is bounded on Δ_d, the following claim is straightforward:

Lemma 6 *There exist $T_0 > 0$, $\epsilon_0 > 0$ and a constant $c_1 < \infty$ such that*

$$\mathrm{e}^{N Z^N_{T-\epsilon}(\underline{m}', \underline{m})} \geq \mathrm{e}^{-c_1 N\epsilon} \mathrm{e}^{N Z^N_T(\underline{m}', \underline{m})} \tag{4.4}$$

uniformly in N, $T \geq T_0$, $\epsilon \leq \epsilon_0$, $\underline{m}', \underline{m} \in \Delta_d$ and N.

Proof Indeed, trajectories $\underline{m}(\cdot)$ on $[0, T]$ and trajectories $\tilde{\underline{m}}(\cdot)$ on $[0, T - \varepsilon]$: are related by the following one to one map: $\tilde{\underline{m}}(t) = \underline{m}(t\frac{T}{T-\varepsilon})$. Since for some $c_2 = c_2(T_0) > 0$, up to exponentially small factors, the total number of jumps of all the particles is at most $c_2 N T$, the Radon-Nikodým derivative is under control and (4.4) follows. □

As a result, for any T, ϵ as above, and for any $\underline{m}', \underline{m}^1, \underline{m}^2 \in \Delta_d$,

$$Z_T^N\big(\underline{m}', \underline{m}^2\big) \geq Z_T^N\big(\underline{m}', \underline{m}^1\big) - c_1\epsilon + Z_\epsilon^N\big(\underline{m}^1, \underline{m}^2\big). \tag{4.5}$$

Fix $\underline{m}^1, \underline{m}^2$ and define $\mathcal{A}_+ = \mathcal{A}_+(\underline{m}^1, \underline{m}^2) = \{\alpha : m^1_\alpha > m^2_\alpha\}$. For $\alpha \in \mathcal{A}_+$ define $\eta_\alpha = m^1_\alpha - m^2_\alpha$. One way to drive $\underline{m}(\cdot)$ from $\underline{m}^1$ to $\underline{m}^2$ during ϵ units of time is to choose $N\eta_\alpha$ particles out of Nm^1_α for each $\alpha \in \mathcal{A}_+$, and to redistribute them during ϵ units of time into $\mathcal{A} \setminus \mathcal{A}_+$, without touching the rest of the particles. There is an obvious uniform lower bound $c_3\epsilon^n$ that a particle starting at the state α will be at state β at time ϵ. We infer:

$$\mathrm{e}^{N Z^N_\epsilon(\underline{m}^1, \underline{m}^2)} \geq \mathrm{e}^{-(\max_\alpha \sum_\beta \lambda_{\alpha,\beta} - \min F)\epsilon N} \prod_{\alpha\in\mathcal{A}_+} \binom{Nm^1_\alpha}{N\eta_\alpha} \big(c_3\epsilon^n\big)^{N\eta_\alpha}. \tag{4.6}$$

Hence,

$$Z_\epsilon^N\big(\underline{m}^1, \underline{m}^2\big) \geq -c_4\epsilon - c_5 \sum_{\alpha\in\mathcal{A}_+} \eta_\alpha \left(d \log\frac{1}{\epsilon} - \log\frac{m^1_\alpha}{\eta_\alpha}\right). \tag{4.7}$$

Both, the equi-continuity of $\underline{m} \to Z_T^N(\underline{m}', \underline{m})$ on Δ_d and its uniform local Lipschitz property on $\mathrm{int}(\Delta_d)$ readily follow from (4.5) and (4.7). □

4.2 Proof of Theorem 4

We follow the approach of [19]: Let $\underline{m} \in \mathrm{int}(\Delta)$ and assume that u is a smooth function such that $\{\underline{m}\} = \mathrm{argmin}\{u - \psi\}$ in a neighbourhood of $\underline{m}$. Then,

$$u(\underline{m}) \leq u\big(\gamma(-t)\big) + \int_{-t}^0 \mathcal{L}\big(\gamma, \gamma'\big)\mathrm{d}s - \mathsf{r}_1 t$$

for any $t \geq 0$ and for any smooth curve γ with $\gamma(0) = \underline{m}$. Let $\underline{v} = \gamma'(0)$. Then,

$$\big(\nabla u(\underline{m}), \underline{v}\big) - \mathcal{L}(\underline{m}, \underline{v}) \leq -\mathsf{r}_1.$$

Since the above holds for any $\underline{m} \in \mathbb{R}_0^n$ (see Remark 1, after Theorem 1), $\mathcal{H}(\underline{m}, \nabla u(\underline{m})) \leq -\mathsf{r}_1$ follows.

In order to check that ψ is a super-solution, note that by the upper and lower bounds on the Lagrangian $\mathcal{L}$ derived in Appendix and by the local Lipschitz property of (bounded and continuous) ψ the minimum

$$\min_{\gamma(t_0)=\underline{m}} \left\{ \psi\big(\gamma(0)\big) + \int_0^{t_0} \big(\mathcal{L}\big(\gamma, \gamma'\big) - \mathsf{r}_1\big) \mathrm{d}s \right\}$$

is attained at some γ_* with $\gamma_*(0) = \underline{m}'$ in a δ_0-neighbourhood of $\underline{m}$, for all t_0 and δ_0 appropriately small. As it is explained in Appendix, the minimizing curve γ_* is C^∞ and stays inside $\mathrm{int}(\Delta_d)$. Evidently,

$$\psi(\underline{m}) = \psi\big(\gamma_*(t)\big) + \int_t^{t_0} \big(\mathcal{L}\big(\gamma_*, \gamma_*'\big) - \mathsf{r}_1\big) \mathrm{d}s$$

for every $t \in [0, t_0]$. Assume that u is smooth and $\mathrm{argmax}\{u - \psi\} = \{\underline{m}\}$ in a δ_0 neighbourhood of $\underline{m}$. Then,

$$u(\underline{m}) = u\big(\gamma_*(t_0)\big) \geq u\big(\gamma_*(t)\big) + \int_t^{t_0} \big(\mathcal{L}\big(\gamma_*, \gamma_*'\big) - \mathsf{r}_1\big) \mathrm{d}s$$

for every $t \in [0, t_0]$. Set $\underline{v} = \gamma_*'(t_0)$. We infer:

$$\big(\nabla u(\underline{m}), \underline{v}\big) \geq \mathcal{L}(\underline{m}, \underline{v}) - \mathsf{r}_1.$$

Consequently, $\mathcal{H}(\underline{m}, \nabla u(\underline{m})) \geq -\mathsf{r}_1$. □

4.3 Proof of (2.16) of Theorem 1 and Theorem 5

By Theorem 3 and since $\nu_N(\underline{m}) = \mathrm{e}^{-2N\psi_N(\underline{m})}$ it follows that

$$\min_{\underline{m}} \psi(\underline{m}) = 0.$$

Let us rewrite (2.17) as

$$\psi(\underline{m}) = \inf_{\gamma(T)=\underline{m}} \left\{ \psi\big(\gamma(0)\big) + \int_0^T \big(\mathcal{L}\big(\gamma, \gamma'\big) - \mathsf{r}_1\big) \mathrm{d}t \right\} \tag{4.8}$$

Since $V(\underline{m}) = \mathcal{L}(\underline{m}, 0) \leq \mathcal{L}(\underline{m}, \underline{v})$, the above might be possible only if $\mathsf{r}_1 = \min_{\underline{m}} V(\underline{m})$.

Furthermore, the Lagrangian $\mathcal{L}$ is uniformly super-linear in the second variable: By (A.1) of Appendix for every $C > 0$ and $\delta > 0$ we can find $T > 0$ such that

$$\inf_{\mathrm{diam}(\gamma)>\delta} \int_0^T \big(\mathcal{L}\big(\gamma, \gamma'\big) - \mathsf{r}_1\big) \mathrm{d}t \geq C.$$

Which means that for $C > \max \psi$, the contribution to (4.8) for γ-s with the diameter larger than δ could be ignored.

Let, therefore, $\mathrm{diam}(\gamma) \leq \delta$. Recall that $\mathcal{L} = \mathcal{L}_0 + V$. Since $\mathcal{L}_0 \geq 0$,

$$\int_0^T \big(\mathcal{L}\big(\gamma, \gamma'\big) - \mathsf{r}_1\big) \mathrm{d}t \geq T \min_{\underline{m}' \in \gamma} \big(V(\underline{m}') - \mathsf{r}_1\big) \tag{4.9}$$

Using the assumption **A1**, we infer:

$$\psi(\underline{m}) \geq \min_{|\underline{m}'-\underline{m}|\leq\delta} \psi\big(\underline{m}'\big) + T \min_{|\underline{m}'-\underline{m}|\leq\delta} \big(V\big(\underline{m}'\big) - \mathsf{r}_1\big). \tag{4.10}$$

The claim of Theorem 5 follows as soon as we notice that all the minima of $\underline{m} \mapsto V(\underline{m})$ belong to $\mathrm{int}(\Delta_d)$. □

5 Results for Spin-$\frac{1}{2}$ Model

For general Spin-s models (2.5),

$$\mathsf{r}_1 = \min_{\underline{m}} V(\underline{m}) = \min_{\underline{m}} \left\{ \frac{\lambda}{4} \sum_{|\alpha-\beta|=1} \sqrt{\mathsf{s}(\mathsf{s}+1)-\alpha\beta}\,(\sqrt{m_\alpha}-\sqrt{m_\beta})^2 - F(\underline{m}) \right\}. \tag{5.1}$$

For Spin-$\frac{1}{2}$ Model it is convenient to take $\{-1, 1\}$ instead of $\{-\frac{1}{2}, \frac{1}{2}\}$ as a set of classical labels. The Hamiltonian is given by

$$-\mathrm{H}_N = NF\left(M_{-1}^N, M_1^N\right) + \lambda \sum_i \hat{\sigma}_i^x$$

where

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.2}$$

In this notation $\hat{\sigma}^z|\alpha\rangle = \alpha|\alpha\rangle$ and $\hat{\sigma}^x|\alpha\rangle = |-\alpha\rangle$ for $\alpha = \pm 1$.

The simplex Δ_d is just a segment $\{(m_{-1}, m_1) : m_{-1} + m_1 = 1\}$, parametrized by a single variable $m = m_1 - m_{-1} \in [-1, 1]$. Any vector $\underline{\theta} \in \mathbb{R}_0^2$ is of the form $\underline{\theta} = (\theta, -\theta)$. Define $F(m) = F(\frac{1-m}{2}, \frac{1+m}{2})$. Thus, in terms of m and θ, the Hamiltonian in (2.14) is

$$\mathcal{H}(m, \theta) = \lambda\sqrt{1-m^2}\cosh(2\theta) - \lambda + F(m). \tag{5.3}$$

Consequently, the effective potential

$$V(m) = -\mathcal{H}(m, 0) = \lambda - \left(\lambda\sqrt{1-m^2} + F(m)\right), \tag{5.4}$$

and the asymptotic leading eigenvalue r_1 is given by

$$\mathsf{r}_1 = \min_{m\in(-1,1)} V(m) = \lambda - \max_{m\in[-1,1]} \left\{\lambda\sqrt{1-m^2} + F(m)\right\}. \tag{5.5}$$

5.1 Minima of V

Let

$$\mathcal{M}_\lambda = \operatorname{argmim}(V) = \left\{m \in (-1, 1) : \mathcal{H}(m, 0) = -\mathsf{r}_1\right\}.$$

Since

$$V''(m) = \frac{\lambda}{(1-m^2)^{3/2}} - F''(m),$$

and since, by assumption **A1**, F is a polynomial of finite degree, there are at most $\deg(F)$ inflection points of V inside the interval $(-1, 1)$. Therefore, $\mathcal{M}_\lambda$ is finite and its cardinality is bounded above by $\frac{1}{2}\deg(F) + 1$.

In order to explore the minimization problem (5.5) it would be convenient to represent $F(m) = -G(\operatorname{sign}(m)\sqrt{1-m^2})$. Then,

$$\mathsf{r}_1 = \lambda - \max_{-1\le t\le 1} \left\{\lambda|t| - G(t)\right\}. \tag{5.6}$$

Indeed, as it clearly seen from (5.5) (and as it follows in general by Theorem 5), all the minima of V belong to $(-1, 1)$, a possible jump discontinuity of G at zero (if $F(-1) \neq F(1)$) plays no role for the computation of maxima. By convention, $G(-1) = G(1) = F(0)$.

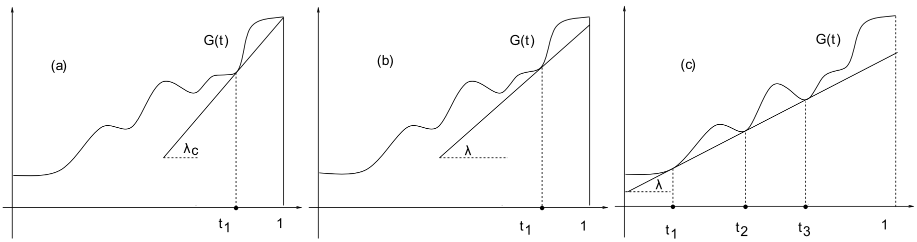


Fig. 1 (**a**) The critical λ_c and $\mathcal{T}^+_{\lambda_c} = \{t_1, 1\}$. (**b**) $\mathcal{T}^+_\lambda$ is a singleton. (**c**) $\mathcal{T}^+_\lambda = \{t_1, t_2, t_3\}$. There are at least two inflection points of G on (t_1, t_3)

In other words, let $\mathcal{T}_\lambda \subset [-1, 1]$ be the set of maximizers in (5.6). We set $\mathcal{T}_\lambda = \mathcal{T}^+_\lambda \cup \mathcal{T}^-_\lambda$, where $\mathcal{T}^+_\lambda = \mathcal{T}_\lambda \cap (0, 1)$. Then, for $m \neq 0$,

$$m \in \mathcal{M}_\lambda \quad \Leftrightarrow \quad t = \mathrm{sign}(m)\sqrt{1 - m^2} \in \mathcal{T}^{\mathrm{sign}(m)}_\lambda. \tag{5.7}$$

By convention, if $0 \in \mathcal{M}_\lambda$, then both $\pm 1 \in \mathcal{T}_\lambda$. Consequently, the set $\mathcal{T}_\lambda$ is finite, and its maximal cardinality is $\frac{1}{2}\deg(F) + 2$. Various options are depicted on Fig. 1 (for simplicity we depict only the $(0, 1]$ interval and, accordingly, the set $\mathcal{T}^+_\lambda$):

(a) First of all there exists $\lambda_c \in [0, \infty)$, such that $\mathcal{T}_\lambda = \{\pm 1\}$ on (λ_c, ∞).
(b) If $\lambda_c > 0$, then $\mathcal{T}_{\lambda_c}$ still contains ± 1. It could happen, however, that $\mathcal{T}_{\lambda_c} = \{-1, t_1, \ldots, t_k, 1\}$ contains at most $k \leq \deg(F)/2$ other points. In the latter case $(-1, 0) \cup (0, 1)$ necessarily contains at least $2k$ inflection points of G.
(c) There might be other exceptional values of $\lambda < \lambda_c$ for which either of $\mathcal{T}^\pm_\lambda$ is not a singleton. If, for instance $\mathcal{T}^+_\lambda = \{t^\lambda_1, \ldots, t^\lambda_k\}$ is not a singleton, then the interval $(t^\lambda_1, t^\lambda_k)$ contains at least $2(k - 1)$ inflection points of G.

Since $G(t) = -F(\mathrm{sign}(t)\sqrt{1 - t^2})$, there are at most $2\deg(F)$ inflection points of G all together in $(-1, 0) \cup (0, 1)$. Noting that intervals spanned by different $\mathcal{T}^\pm_\lambda$ are necessarily disjoint, we infer that $\mathcal{T}^\pm_\lambda$ is not a singleton for at most $\deg(F)$ values of λ.

Values of λ for which the cardinality of $\mathcal{T}_\lambda$ changes correspond to first order phase transitions in the ground state.

5.2 Ferromagnetic p-Body Interaction

In the usual Curie-Weiss case with pair interactions $G(t) = \frac{1}{2}(t^2 - 1)$, so that $\mathsf{r}_1 = -\frac{(\lambda-1)^2}{2}$ if $\lambda \leq 1$, and, accordingly, $\mathsf{r}_1 = 0$ if $\lambda \geq 1$. For $\lambda \geq 1$ the set $\mathcal{M}_\lambda = \{0\}$. For $\lambda \in (0, 1)$, $\mathcal{M}_\lambda = \{\pm\sqrt{1 - \lambda^2}\}$. No first order transition occurs.

In the $p > 2$-body ferromagnetic interaction case (2.6) the function

$$G(t) = -\,\mathrm{sign}(t)^p \left(1 - t^2\right)^{p/2}.$$

For odd p maximizers of $\lambda|t| - G(t)$ always lie in $(0, 1]$. For even p the set $\mathcal{M}_\lambda$ is symmetric. Thus in either case it is enough to consider

$$\mathsf{r}_1 = \lambda - \max_{t \in (0,1]} \left\{\lambda t + \left(1 - t^2\right)^{p/2}\right\}. \tag{5.8}$$

The crucial difference between the Curie-Weiss case $p = 2$ and $p > 2$ is that in the latter situation, $G'(1) = 0$, and G contains an inflection point $t_p = \sqrt{\frac{1}{p-1}}$ inside $(0, 1)$. An easy computation reveals that for $p > 2$,

$$\lambda_c = \frac{p}{p-1}\left(1 - \frac{1}{(p-1)^2}\right)^{\frac{p}{2}-1} \quad \text{and} \quad \mathcal{T}^+_{\lambda_c} = \left\{\frac{1}{p-1}, 1\right\}. \tag{5.9}$$

Accordingly, for even p,

$$\mathcal{M}_{\lambda_c} = \{0, \pm\hat{m}\} = \left\{0, \pm\sqrt{\frac{p(p-2)}{(p-1)^2}}\right\}, \tag{5.10}$$

whereas for odd p; $\mathcal{M}_{\lambda_c} = \{0, \hat{m}\} = \{0, \sqrt{\frac{p(p-2)}{(p-1)^2}}\}$. This is precisely formula (14) of [5]. For $\lambda > \lambda_c$ the set $\mathcal{M}_\lambda = \{0\}$. For $\lambda < \lambda_c$ there exists $m^* = m^*(\lambda, p) \in (\sqrt{\frac{p(p-2)}{(p-1)^2}}, 1)$ such that the set $\mathcal{M}_\lambda$ is a singleton $\{m^*\}$ in the odd case, whereas $\mathcal{M}_\lambda = \{\pm m^*\}$ in the even case. Thus, for mean-field models with p-body interaction, λ_c is the only value at which first order transition in the ground state occurs.

5.3 Asymptotic Ground States

Let us return to general polynomial interactions F. Fix $m \in (-1, 1)$ and consider the equation,

$$\mathcal{H}(m, \theta) = -\mathsf{r}_1. \tag{5.11}$$

The Hamiltonian $\mathcal{H}(m, \cdot)$ is strictly convex and symmetric. Hence, if $m \in \mathcal{M}_\lambda$, then $\theta = 0$ is the unique solution. If $m \notin \mathcal{M}_\lambda$, then necessarily $\mathcal{H}(m, 0) < -\mathsf{r}_1$. Hence, there exist $\theta(m) > 0$, such that $\pm\theta(m)$ are the unique solutions to (5.11). If F is symmetric, then $\theta(-m) = -\theta(m)$. In any case, however, the following holds:

Let ψ be an admissible solution of $\mathcal{H}(m, \psi') = -\mathsf{r}_1$. Since ψ is locally Lipschitz, it is a.e. differentiable. Consequently, $\psi'(m) = \pm\theta(m)$ a.e. on $(-1, 1)$. The proposition below relies only on the fact that ψ is a viscosity solution on $(-1, 1)$.

Proposition 1 *Record $\mathcal{M}_\lambda = \{-1 < m_1, \ldots, m_k < 1\}$ in the increasing order. Set $m_0 = -1$ and $m_{k+1} = 1$. Then on each of the intervals $[m_\ell, m_{\ell+1}]$ the gradient ψ' is of the following form: There exists $m^*_\ell \in [m_\ell, m_{\ell+1}]$, such that:*

$$\psi' = \theta \quad \text{on } \left[m_\ell, m^*_\ell\right) \quad \text{and} \quad \psi' = -\theta \quad \text{on } \left(m^*_\ell, m_{\ell+1}\right]. \tag{5.12}$$

Proof It would be enough to prove the following: If $m \in (m_\ell, m_{\ell+1})$ and $\psi'(m) = \theta(m)$, then for any $n \in (m_\ell, m)$,

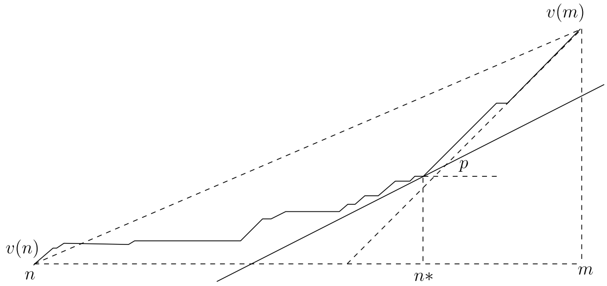
$$\psi(n) = \psi(m) - \int_n^m \theta(t)\mathrm{d}t. \tag{5.13}$$

Recall that since ψ is a viscosity solution on $(-1, 1)$, then

$$\liminf_{\varepsilon \to 0} \frac{\psi(n^* + \varepsilon) - \psi(n^*)}{|\varepsilon|} \geq \theta \quad \Rightarrow \quad \theta \notin \left(-\theta\left(n^*\right), \theta\left(n^*\right)\right) \tag{5.14}$$

for any $n^* \in (-1, 1)$. We shall show that if (5.13) is violated for some $n \in (m_\ell, m)$, then (5.14) is violated as well in the sense that there exists $n^* \in (n, m)$ and $\theta \in (-\theta(n^*), \theta(n^*))$ such that the right hand side of (5.14) holds.

Fig. 2 $p \in \partial v(n^*)$



Indeed, since $\psi'(t) = \pm\theta(t)$ a.e. on $(-1, 1)$ it always holds that

$$\psi(n) \geq \psi(m) - \int_n^m \theta(t)\mathrm{d}t, \quad n < m.$$

Let us assume strict inequality. For $k \in [n, m]$ define

$$v(k) = \int_n^k \frac{\psi'(t) + \theta(t)}{2\theta(t)}\mathrm{d}t.$$

By construction $v(n) = 0$, $v(m) \stackrel{\Delta}{=} p(m - n) < (m - n)$ and $v'(m) = 1$. There is no loss of generality to assume that $p > 0$. Hence there exists $n^* \in (n, m)$ such that $p \in \partial v(n^*)$ (see Fig. 2). By continuity of $\theta(m)$ this would mean that

$$\liminf_{\varepsilon \to 0} \frac{\psi(n^* + \varepsilon) - \psi(n^*)}{|\varepsilon|} \geq p\theta\big(n^*\big) - (1 - p)\theta\big(n^*\big) = (2p - 1)\theta\big(n^*\big).$$

Since for any inner point $n^* \in (m_\ell, m_{\ell+1})$; $\theta(n^*) > 0$, we arrived at a contradiction. □

Remark 3 Note that Proposition 1 implies that ground states ψ with more than one local minimum necessarily develop shocks. In particular, this is always the case whenever F is symmetric and $0 \notin \mathcal{M}_\lambda$.

If $m_\ell < m_\ell^* < m_{\ell+1}$ from Proposition 1, then m_ℓ^* is a local maximum of ψ, and m_ℓ^* is a shock location for the stationary Hamilton-Jacobi equation $\mathcal{H}(m, \psi) = -\mathsf{r}_1$.

If ψ is, in addition, a weak KAM solution (in particular, if ψ is admissible), then $\mathrm{argmin}\{\psi\} \subseteq \mathcal{M}_\lambda$. Consequently, by Theorem 5, in the latter case, $\psi' = -\theta$ on $(-1, m_1)$ and $\psi' = \theta$ on $(m_k, 1)$.

Admissible solutions are always normalized in the sense that $\min \psi = 0$ and the minimum is attained on $\mathcal{M}_\lambda \subset (-1, 1)$. It follows that admissible solutions are uniquely defined in the following two cases:

CASE. The set $\mathcal{M}_\lambda = \{m^*\}$ is a singleton. Then, $\psi' = -\theta$ on $(-1, m^*)$ and $\psi' = \theta$ on $(m^*, 1)$. Consequently,

$$\psi(m) = \left|\int_{m^*}^m \theta(t)\mathrm{d}t\right|. \tag{5.15}$$

CASE. The interaction F is symmetric and $\mathcal{M}_\lambda = \{\pm m^*\}$. Then, ψ is also symmetric; $\psi' = \theta$ on $(-m^*, 0) \cup (m^*, 1)$ and $\psi' = -\theta$ on $(-1, 0) \cup (0, m^*)$. That is $\psi(m) = \psi(-m)$ and ψ is still given by (5.15) for $m \geq 0$. Note that in this case ψ' has a jump at $m = 0$.

Let Λ^c be the set of λ which do not fall into one of the two cases above. As we have seen in Sect. 5.2, $\Lambda_c = \emptyset$ in the case of Curie-Weiss model, and $\Lambda_c = \{\lambda_c\}$ (see (5.9)) for general p-body interaction.

5.4 Multiple Wells

We shall refer to $\lambda \in \Lambda_c$ as to the case of multiple wells. Note first of all that there is a continuum of normalized solution of (2.17) as soon as the cardinality $|\mathcal{M}_\lambda| \geq 2$. Indeed, it is easy to see that any normalized ψ which complies with the conclusion of Proposition 1 will be a solution to (2.17).

One needs, therefore, an additional criterion to determine locations of shocks $\{m^*_\ell\}$ or, equivalently, to determine values $\{\psi(m_\ell)\}$ for admissible solutions. It would be tempting to derive location of shocks by some natural limiting procedure via stabilization of shock propagation along Rankine-Hugoniot curves. Since however, we arrived at (2.17) directly from the eigenvalue equation without recourse to a finite horizon problem, it was not clear to us which limit to consider.

Our selection of admissible solutions to (2.17) is based on a refined asymptotic analysis of Dirichlet eigenvalues in a vicinity of points belonging to the set $\mathcal{M}_\lambda$. Namely, the point $m_\ell \in \mathcal{M}_\lambda$ can be a local minimum of an admissible solution ψ only if there is an exponential splitting of the corresponding bottom eigenvalues. The precise result is formulated in Proposition 2 below.

The results of [27–29] enable one to explore asymptotic expansions of such eigenvalues with any degree of precision. In the simplest case we deduce the following corollary 1 which is explained in the concluding paragraph of this section.

Corollary 1 *Assume that*

$$\min_{m\in\mathcal{M}_\lambda} \chi_0(m) \stackrel{\Delta}{=} \min_{m\in\mathcal{M}_\lambda}\left\{\frac{\lambda}{(1-m^2)^2} - \frac{1}{\sqrt{1-m^2}}F''(m)\right\} \tag{5.16}$$

is attained at either a unique point m^* (non-symmetric potentials) or at a unique couple $\pm m^*$ (symmetric potentials). Then there is a unique admissible solution ψ, which is still given by (5.15).

For instance, in the critical ($\lambda = \lambda_c$) case of $p > 2$ body interaction,

$$\chi_0(0) = \lambda_c \quad \text{and} \quad \chi_0(\hat{m}) = (p-2)(p-1)^3\lambda_c. \tag{5.17}$$

The first of (5.17) is straightforward. In order to derive the expression for $\chi_0(\hat{m})$ we use (5.9) and (5.10) in the following way: Note that $\lambda_c = \frac{p}{p-1}\hat{m}^{p-2}$. Therefore, $F''(\hat{m}) = p(p-1)\hat{m}^{p-2} = (p-1)^2\lambda_c$. Since $1-\hat{m}^2 = \frac{1}{(p-1)^2}$, we infer:

$$\frac{\lambda_c}{(1-\hat{m}^2)^2} - \frac{F''(\hat{m})}{\sqrt{1-\hat{m}^2}} = \lambda_c\big((p-1)^4 - (p-1)^3\big) = (p-2)(p-1)^3\lambda_c.$$

Formula (5.17) implies $\chi_0(0) < \chi_0(\hat{m})$, for any integer $p > 2$ and $\lambda = \lambda_c(p)$. Consequently, even at $\lambda = \lambda_c$ there is still a unique admissible solution $\psi(m) = |\int_0^m \theta(t)\mathrm{d}t|$ with the unique minimum at $m^* = 0$.

Spectral Asymptotics and the Set $\mathcal{M}_\lambda$. Assume that $\lambda \in \Lambda_c$ and, as before, denote $\mathcal{M} = \{m_1, \ldots, m_k\}$.

Lemma 7 *For any $\delta > 0$ there exists $\epsilon > 0$ such that*

$$\min_{\mathrm{d}(m,\mathcal{M}_\lambda)\geq\delta} \psi(m) \geq \epsilon, \tag{5.18}$$

uniformly in normalized admissible solutions of (2.17).

Proof Let $m \in (m_l, m_{l+1})$. By Proposition 1

$$\psi(m) \geq \min\left\{\psi(m_l) + \int_{m_l}^{m} \theta(t)\mathrm{d}t, \psi(m_{l+1}) + \int_{m}^{m_{l+1}} \theta(t)\mathrm{d}t\right\},$$

and (5.18) follows. □

In the sequel $h_N = \mathrm{e}^{-N\psi_N}$ is the Perron-Frobenius eigenfunction of $\mathcal{G}_N^g + NF_g \triangleq \mathcal{S}_N$; $\mathcal{S}_N h_N = -R_N^1 h_N$. Recall:

$$\begin{aligned}\mathcal{S}_N f(m) &= N\big(F(m) - \lambda\big) f(m) \\ &\quad + \frac{N\lambda}{2}\sqrt{(1-m)\left(1+m+\frac{2}{N}\right)} f\left(m + \frac{2}{N}\right) \\ &\quad + \frac{N\lambda}{2}\sqrt{(1+m)\left(1-m+\frac{2}{N}\right)} f\left(m - \frac{2}{N}\right)\end{aligned} \tag{5.19}$$

Pick $0 < \delta < \frac{1}{4}\min_l |m_{l+1} - m_l|$. Let $1 \equiv \sum_0^k \chi_l$ be a partition of unity satisfying: For $l = 1, \ldots, k$

$$\chi_l \equiv 1 \quad \text{on } I_\delta(m_l) \quad \text{and} \quad \chi_l \equiv 0 \quad \text{on } I_{2\delta}^c(m_l).$$

Here $I_\eta(m)$ denotes the interval $[m-\eta, m+\eta]$. By Lemma 7 there exists $\epsilon > 0$ such that for $l = 1, \ldots, k$ and all N large enough

$$\frac{1}{N}\log\max_m \left|\big(\mathcal{S}_N + R_N^1\big)\chi_l h_N(m)\right| \leq -\epsilon. \tag{5.20}$$

Let $\mathcal{S}_N^l$ be a Dirichlet restriction of $\mathcal{S}$ to $I_\delta(m_l)$. Let $-R_{N,l}^1$ be the leading eigenvalue of $\mathcal{S}_N^l$.

We are entitled to conclude: There exists $\epsilon' > 0$ such that

$$\frac{1}{N}\log\left|R_N^1 - \min_l R_{N,l}^1\right| \leq -\epsilon'. \tag{5.21}$$

Furthermore,

Proposition 2 *If $l = 0, \ldots, k$ and $\psi = \lim_{j\to\infty}\psi_{N_j}$ is a subsequential limit such that m_l is a local minimum of ψ, then there exists $\epsilon' > 0$ such that*:

$$\frac{1}{N_j}\log\left|R_{N_j}^1 - R_{N_j,l}^1\right| \leq -\epsilon'. \tag{5.22}$$

Proof In view of Lemma 4 the claim readily follows from the general theory of exponentially low lying spectra for metastable Markov chains [6]. For a direct proof note that under the assumptions of the proposition, one (possibly after further shrinking the value of δ) can upgrade (5.20) as

$$\frac{1}{N_j}\log\max_m \left|\big(\mathcal{S}_{N_j} + R_{N_j}^1\big)\frac{\chi_l h_{N_j}(m)}{h_{N_j}(m_l)}\right| \leq -\epsilon, \tag{5.23}$$

and (5.22) follows from the spectral theorem. □

Asymptotics of Dirichlet Eigenvalues $R_{N,l}^1$. Define $h(m) = \lambda\sqrt{1-m^2}$. The asymptotics of $R_{N,l}^1$ up to zero order terms is given [29] by

$$-R^1_{N,l} = -N\mathsf{r}_1 - \sqrt{\frac{V''(m_l)}{h(m_l)}} + \mathsf{O}\left(\frac{1}{N}\right) = -N\mathsf{r}_1 - \sqrt{\frac{\chi_0(m_l)}{\lambda}} + \mathsf{O}\left(\frac{1}{N}\right), \tag{5.24}$$

where we used the explicit expression (5.4) for V in the second equality. χ_0 was defined in (5.16). The claim of Corollary 1 follows now from Proposition 2. □

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Appendix: The Variational Problem

The Lagrangian $\mathcal{L}_0$ was defined in (2.12).

Lower Bounds on $\mathcal{L}_0$ Fix $\alpha \in \mathcal{A}$ and consider $\theta^t_\alpha = \frac{n-1}{n}t$ and, for $\beta \neq \alpha$, $\theta^t_\beta = -\frac{1}{n}t$. Recall that $\underline{v} \in \mathbb{R}^n_0$, that is $v_\alpha = -\sum_{\beta\neq\alpha} v_\beta$. Therefore, for any α

$$\mathcal{L}_0(\underline{m}, \underline{v}) \geq \sup_t \left\{ t v_\alpha - \sum_{\beta\neq\alpha} \sqrt{m_\alpha m_\beta}\lambda_{\alpha\beta}\big(\cosh(t) - 1\big)\right\}$$

Define $\lambda_\alpha(\underline{m}) = \sum_{\beta\neq\alpha}\sqrt{m_\alpha m_\beta}\,\lambda_{\alpha\beta}$. For $|v_\alpha| \geq \lambda_\alpha(\underline{m})$ one may choose $t^* = \mathrm{sign}(v_\alpha)\cdot \log\frac{|v_\alpha|}{\lambda_\alpha(\underline{m})}$. We infer: If $|v_\alpha| \geq \lambda_\alpha(\underline{m})$, then

$$\mathcal{L}_0(\underline{m}, \underline{v}) \geq |v_\alpha|\left(\log\frac{|v_\alpha|}{\lambda_\alpha(\underline{m})} - 1\right). \tag{A.1}$$

Upper Bounds on the Lagrangian $\mathcal{L}_0$ Consider

$$\mathcal{R}_0(\underline{m}, \underline{v}) \stackrel{\Delta}{=} \sup_{\underline{\theta}}\left\{\sum v_\alpha\theta_\alpha - \sum_{\alpha,\beta}\sqrt{m_\alpha m_\beta}\,\lambda_{\alpha,\beta}\cosh(\theta_\beta - \theta_\alpha)\right\}.$$

Since $\mathcal{L}_0(\underline{m}, \underline{v}) = \mathcal{R}_0(\underline{m}, \underline{v}) + \sum_\alpha \lambda_\alpha(\underline{m})$, it would be enough to control the dependence of $\mathcal{R}_0$ on $\underline{v}$.

Let us say that a flow $\underline{f} = \{f_{\alpha\beta}\}$ is compatible with $\underline{v} \in \mathbb{R}^n_0$; $\underline{f} \sim \underline{v}$ if:

(a) It is a flow: $f_{\alpha\beta} = -f_{\beta\alpha}$.
(b) For any $\alpha \in \mathcal{A}$, $\sum_\beta f_{\beta\alpha} = v_\alpha$.

Then $\sum v_\alpha\theta_\alpha = \frac{1}{2}\sum_{\alpha,\beta}(\theta_\beta - \theta_\alpha)f_{\alpha\beta}$. Hence, for any $\underline{f} \sim \underline{v}$,

$$\mathcal{R}_0 = \sup_{\underline{\theta}}\left\{\frac{1}{2}\sum_{\alpha,\beta}(\theta_\beta - \theta_\alpha)f_{\alpha\beta} - \sqrt{m_\alpha m_\beta}\,\lambda_{\alpha\beta}\cosh(\theta_\beta - \theta_\alpha)\right\}. \tag{A.2}$$

We shall rely on the following upper bound on each term in (A.2): For any f and $a > 0$

$$\sup_t\big\{ft - a\cosh(t)\big\} \leq |f|\log\left(1 + \frac{2|f|}{a}\right).$$

Consequently, we derive the following upper bound on $\mathcal{R}_0$:

$$\mathcal{R}_0(\underline{m}, \underline{v}) \leq \inf_{\underline{f}\sim\underline{v}} \sum_{\alpha,\beta}\frac{|f_{\alpha\beta}|}{2}\log\left(1 + \frac{|f_{\alpha\beta}|}{\sqrt{m_\alpha m_\beta}\lambda_{\alpha\beta}}\right). \tag{A.3}$$

Regularity of Minimizers Let $\underline{m} \in \mathrm{int}(\Delta_d)$. We claim that there exists $\delta_0 > 0$ and $t_0 > 0$ such that for any $\underline{m}'$ in the δ_0-neighbourhood of $\underline{m}$ the minimizer γ^* of

$$\inf_{\gamma(0)=\underline{m}', \gamma(t_0)=\underline{m}} \int_0^{t_0} \mathcal{L}\big(\gamma(s), \gamma'(s)\big) \mathrm{d}s$$

exists and is, actually, C^∞. Indeed, an absolutely continuous minimizer exists by the classical Tonelli's theorem. By lower (A.1) and upper (A.3) bounds on the Lagrangian, it is easy to understand that minimizers stay inside $\mathrm{int}(\Delta_d)$ once t_0 and δ_0 are chosen to be appropriately small. But then the regularity theory of either [11] or [3] applies and yields Lipschitz regularity on $[0, t_0]$. Since, the Lagrangian $\mathcal{L}$ is strictly convex in the second argument, and, in the interior of Δ_d, it is C^∞ in both arguments, the C^∞ of the minimizer follows from the implicit function theorem, see e.g. [7].

References

1. Aizenman, M., Nachtergaele, B.: Geometric aspects of quantum spin states. Commun. Math. Phys. **164**, 17–63 (1994)
2. Aizenman, M., Klein, A., Newman, C.: Percolation methods for disordered quantum Ising models. In: Kotecky, R. (ed.) Phase Transitions: Mathematics, Physics, Biology, pp. 1–26. World Scientific, Singapore (1993)
3. Ambrosio, L., Ascenzi, O., Buttazzo, G.: Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands. J. Math. Anal. Appl. **142**, 301–316 (1989)
4. Anantharaman, N.: On the zero-temperature or vanishing viscosity limit for certain Markov processes arising from Lagrangian dynamics. J. Eur. Math. Soc. **6**(2), 207–276 (2004)
5. Bapst, V., Semerjian, G.: On quantum mean-field models and their quantum annealing. arXiv:1203.6003v1 (2012)
6. Bovier, A., Eckhoff, M., Gayrard, V., Klein, M.: Metastability and low lying spectra in reversible Markov chains. Commun. Math. Phys. **228**(2), 219–255 (2002)
7. Buttazzo, G., Giaquinta, M., Hildebrandt, S.: One-Dimensional Variational Problems. An Introduction. Oxford Lecture Series in Mathematics and Its Applications (1998)
8. Campanino, M., Klein, A., Perez, J.F.: Localization in the ground state of the Ising model with a random transverse field. Commun. Math. Phys. **135**, 499–515 (1991)
9. Capuzzo-Dolcetta, I., Lions, P.-L.: Hamilton-Jacobi equations with state constraints. Trans. Am. Math. Soc. **318**(2), 643–683 (1990)
10. Chayes, L., Crawford, N., Ioffe, D., Levit, A.: The phase diagram of the quantum Curie-Weiss model. J. Stat. Phys. **133**(1), 131–149 (2008)
11. Clarke, F.H., Vinter, R.B.: Regularity properties of solutions to the basic problem in the calculus of variations. Trans. Am. Math. Soc. **289**(1), 73–98 (1985)
12. Crawford, N., Ioffe, D.: Random current representation for transverse field Ising model. Commun. Math. Phys. **296**(2), 447–474 (2010)
13. Dawson, D.A., Gartner, J.: Large deviations from the McKean-Vlasov limit for weakly interacting diffusions. Stochastics **20**(4), 247–308 (1987)
14. Dembo, A., Zetouni, O.: Large Deviations Techniques and Applications. Stochastic Modeling and Applied Probability, vol. 38. Springer, Berlin (2010).
15. Dimassi, M., Sjöstrand, J.: Spectral Asymptotics in the Semi-classical Limit. Mathematical Society Lecture Note Series, vol. 268. Cambridge University Press, London (1999)
16. Dorlas, T.C.: Probabilistic derivation of a noncommutative version of Varadhan's theorem. Math. Proc. R. Ir. Acad. **109**(1), 1–18 (2009).
17. Fannes, M., Spohn, H., Verbeure, A.: Equilibrium states for mean field models. J. Math. Phys. **21**, 355–358 (1980).
18. Fathi, A.: Théoreme KAM faible et théorie de Mather sur les systemes lagrangiens. C. R. Acad. Sci., Ser. I Math. **324**(9), 1043–1046 (1997).
19. Fathi, A.: The Weak KAM Theorem in Lagrangian Dynamics 10th Preliminary Version (2008). Chap. 4 (pp. 109–183), Chap. 7 (pp. 213–235). Book preprint. http://www.math.ist.utl.pt/~dgomes/austin/summer08/WeakKAMVersion10-15June2008.pdf.
20. Fathi, A., Ezequiel, M.: Weak KAM theorem on non compact manifolds. Nonlinear Differ. Equ. Appl. **14**(1), 1–27 (2007).

21. Ginibre, J.: Existence of phase transitions for quantum lattice systems. Commun. Math. Phys. **14**, 205–234 (1969)
22. Goldschmidt, Ch., Ueltschi, D., Windridge, P.: Quantum Heisenberg models and their probabilistic representations. In: Entropy and the Quantum II. Contemp. Math., vol. 552, pp. 177–224. Am. Math. Soc., Providence (2011)
23. Helffer, B.: Semi-classical Analysis for the Schrödinger Operator and Applications. LNM, vol. 1336. Springer, Heidelberg (1988)
24. Helffer, B.: Semiclassical Analysis, Witten Laplacians, and Statistical Mechanics. Partial Differential Equations and Applications, vol. 1. World Scientific, Singapore (2002)
25. Ioffe, D.: On some applicable versions of abstract large deviations theorems. Ann. Probab. **19**(4), 1629–1639 (1991)
26. Ioffe, D.: Stochastic geometry of classical and quantum Ising models. In: Lecture Notes in Mathematics, vol. 1970, pp. 87–126. Springer, Berlin (2009)
27. Klein, M., Rosenberger, E.: Agmon-type estimates for a class of difference operators. Ann. Henri Poincaré **9**(6), 1177–1215 (2008)
28. Klein, M., Rosenberger, E.: Harmonic approximation of difference operators. J. Funct. Anal. **257**(11), 3409–3453 (2009)
29. Klein, M., Rosenberger, E.: Asymptotic eigenfunctions for a class of difference operators. Asymptot. Anal. **73**(1–2), 1–36 (2011).
30. Shwartz, A., Weiss, A.: Large deviations with diminishing rates. Math. Oper. Res. **30**, 281–310 (2005)
31. Ueltschi, D.: Geometric and probabilistic aspects of boson lattice models. In: In and out of Equilibrium. Progr. Probab, vol. 51, pp. 363–391. Birkhäuser, Boston (2002)