

Restriction Properties of Annulus SLE

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Abstract For $\kappa \in (0, 4]$, a family of annulus $\text{SLE}(\kappa; \Lambda)$ processes were introduced in (Zhan in arXiv:1004.1865v1) to prove the reversibility of whole-plane $\text{SLE}(\kappa)$. In this paper we prove that those annulus $\text{SLE}(\kappa; \Lambda)$ processes satisfy a restriction property, which is similar to that for chordal $\text{SLE}(\kappa)$. Using this property, we construct $n \geq 2$ curves crossing an annulus such that, when any $n - 1$ curves are given, the last curve is a chordal $\text{SLE}(\kappa)$ trace.

Keywords Annulus SLE · Restriction · Reversibility · Multiple SLE

1 Introduction

Oded Schramm's SLE process generates a family of random curves that grow in plane domains. The evolution is described by the classical Loewner differential equation with the driving function being $\sqrt{\kappa}B(t)$, where $B(t)$ is a standard Brownian motion and κ is a positive parameter. SLE behaves differently for different value of κ . We use $\text{SLE}(\kappa)$ to emphasize the parameter. See [5] and [9] for the fundamental properties of SLE.

There are several versions of SLE, among which chordal SLE and radial SLE are most well known. They describe random curves that grow in simply connected domains. A number of statistical physics models in simply connected domains have been proved to converge in their scaling limits to chordal or radial SLE with different parameters.

People have been working on extending SLE to general plane domains. A version of SLE in doubly connected domains, called annulus SLE, was introduced in [12]. The definition uses the annulus Loewner equation, in which the Poisson kernel function is used for the vector field, and the driving function is still $\sqrt{\kappa}B(t)$. Annulus $\text{SLE}(2)$ turns out to be the scaling limit of loop-erased random walk in doubly connected domains. In fact, loop-erased random walk in any finitely connected plane domain converges to some $\text{SLE}(2)$ -type curve (cf. [14]).

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Annulus SLE defined in [12] generates a trace in a doubly connected domain that starts from a marked boundary point and ends at a random point on the other boundary component (cf. [13]). This is different from the behavior of chordal SLE or radial SLE, whose trace ends at a fixed boundary point or interior point. The reason for this phenomena is that the definition of annulus SLE does not specify any point other than the initial point.

The annulus SLE($\kappa; \Lambda$) process was defined in [15] to describe SLE in doubly connected domains with one marked boundary point other than the initial point. Here the Λ is a function, and the marked boundary point may or may not lie on the same boundary component as the initial point. The definition uses the annulus Loewner equation with the driving function equal to $\sqrt{\kappa}B(t)$ plus some drift function, and the drift function is determined by λ . The precise relation will be given in Definition 4.3.

There is very little restriction on the function Λ in the above definition. For any $\kappa \in (0, 4]$, there is a family of particular functions $\Lambda_{\kappa;(s)}$, $s \in \mathbb{R}$, such that the annulus SLE($\kappa; \Lambda_{\kappa;(s)}$) process satisfies the remarkable reversibility properties as follows. Suppose D is a doubly connected domain, and z_0, w_0 are two boundary points that lie on different boundary components. Let β be an annulus SLE($\kappa; \Lambda_{\kappa;(s)}$) trace in D that grows from z_0 with w_0 as the marked point. Then almost surely β ends at w_0 , and the time-reversal of β is a time-change of an annulus SLE($\kappa; \Lambda_{\kappa;(-s)}$) trace in D that grows from w_0 with z_0 as the marked point. This property was used [15] to prove the reversibility of whole-plane SLE(κ) process for $\kappa \in (0, 4]$. We will state the definitions of $\Lambda_{\kappa;(s)}$ in Sect. 4.2, especially in (4.10).

In this paper we study the restriction property of the annulus SLE($\kappa; \Lambda_{\kappa;(s)}$) process. We use μ_{loop} to denote the Brownian loop measure defined in [7], which is a σ -finite infinite measure on the space of loops, and define

$$c = c(\kappa) = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}. \tag{1.1}$$

It is well known that c is the central charge for SLE(κ). Set $\mathbb{A}_p = \{e^{-p} < |z| < 1\}$, $\mathbb{T} = \{|z| = 1\}$ and $\mathbb{T}_p = \{|z| = e^{-p}\}$. We will prove the following two theorems.

Theorem 1.1 *Let $p > 0, \kappa \in (0, 4], s \in \mathbb{R}, z_0 \in \mathbb{T}$ and $w_0 \in \mathbb{T}_p$. Let ν be the distribution of an annulus SLE($\kappa; \Lambda_{\kappa;(s)}$) trace in \mathbb{A}_p started from z_0 with marked point w_0 . Let $L \subset \mathbb{A}_p$ be such that $\mathbb{A}_p \setminus L$ is a doubly connected domain and $\text{dist}(L, \{z_0, \mathbb{T}_p\}) > 0$. Define a probability measure ν_L by*

$$\frac{d\nu_L}{d\nu} = \frac{\mathbf{1}_{\{\beta \cap L = \emptyset\}}}{Z} \exp(c(\kappa)\mu_{\text{loop}}[\mathcal{L}_{L,p}]), \tag{1.2}$$

where β is the SLE trace, $\mathcal{L}_{L,p}$ is the set of all loops in \mathbb{A}_p that intersect both L and β , and $Z \in (0, \infty)$ is a normalization constant. Then ν_L is the distribution of a time-change of an annulus SLE($\kappa; \Lambda_{\kappa;(s)}$) trace in $\mathbb{A}_p \setminus L$ started from z_0 with marked point w_0 .

Theorem 1.2 *Let $p, \kappa, s, z_0, w_0, \nu$ be as in Theorem 1.1. Let $L \subset \mathbb{A}_p$ be such that $\mathbb{A}_p \setminus L$ is a simply connected domain, and $\text{dist}(L, \{z_0, w_0\}) > 0$. Define ν_L by (1.2). Then ν_L is the distribution of a time-change of a chordal SLE(κ) trace in $\mathbb{A}_p \setminus L$ from z_0 to w_0 .*

The proofs of the two theorems are similar. For the proof of Theorem 1.1, we consider two random curves $\beta(t)$ and $\gamma(t)$, $0 \leq t < p$, where β is an annulus SLE($\kappa; \Lambda_{\kappa;(s)}$) trace in \mathbb{A}_p started from z_0 with marked point w_0 , and γ is an SLE($\kappa; \Lambda_{\kappa;(s)}$) trace in $\mathbb{A}_p \setminus L$ started from z_0 with marked point w_0 , reparameterized by the capacity in \mathbb{A}_p . Decomposing β and γ according to their winding numbers, we get a family of random curves β_m and γ_m .

$m \in \mathbb{Z}$. For each m , we find a positive local martingale $M_m(t)$, $0 \leq t < p$, which serves as the “local” Radon-Nikodym derivative between β_m and γ_m , which means that, if a stopping time T satisfies that M_m is uniformly bounded on $[0, T]$, then $M_m(T)$ is the Radon-Nikodym derivative between the distributions of β_m and γ_m both stopped at T . The construction of M_m uses Itô’s calculus and Girsanov Theorem. It turns out that $M_m(p) := \lim_{t \rightarrow p^-} M_m(t)$ serves as the global Radon-Nikodym derivative between the distributions of β_m and γ_m , and is equal to the right-hand side of (1.2) with Z depending on m . This result then can be easily passed to the Radon-Nikodym derivative between β and γ . The rest of the proof focuses on studying the properties of M_m . We will derive the uniform boundedness of M_m in a countable family of events whose union supports the distribution of β_m , and calculate the limit of $M_m(t)$ as $t \rightarrow p^-$. The argument relies heavily on the analysis of the drift function $\Lambda_{\kappa;(s)}$ and several special functions related to $\Lambda_{\kappa;(s)}$. For this purpose, we cite estimations from [15] and develop some new estimations.

If $\kappa = \frac{8}{3}$, then $c = 0$. The above two theorems imply that, if we condition an annulus $\text{SLE}(\frac{8}{3}, \Lambda_{\frac{8}{3};(s)})$ trace in \mathbb{A}_p to avoid some set L , then the resulting curve is a time-change of an annulus $\text{SLE}(\frac{8}{3}, \Lambda_{\frac{8}{3};(s)})$ or chordal $\text{SLE}(\frac{8}{3})$ trace in $\mathbb{A}_p \setminus L$. This is similar to the restriction property of chordal or radial $\text{SLE}(\frac{8}{3})$ [6]. If $\kappa \in (0, \frac{8}{3})$, then $c < 0$, and the strong restriction property does not hold. But we may use the argument in [6] to construct a random shape with the restriction property as below.

Let β be an annulus $\text{SLE}_{\kappa;(s)}$ trace in \mathbb{A}_p from z_0 to w_0 . Let \mathcal{L}_{-c} denote a Brownian loop soup in \mathbb{A}_p with density $-c$ (a Poisson point process with intensity $-c\mu_{\text{loop}}$ restricted in \mathbb{A}_p) independent of β . Let $S'_{\kappa;s}(\mathbb{A}_p; z_0, w_0)$ denote the union of the image of β and the loops in \mathcal{L}_{-c} that intersect β . Let $S_{\kappa;s}(\mathbb{A}_p; z_0, w_0)$ be obtained by filling the holes in $S'_{\kappa;s}(\mathbb{A}_p; z_0, w_0)$. In other words, $S_{\kappa;s}(\mathbb{A}_p; z_0, w_0) = \mathbb{A}_p \setminus (D_0 \cup D_p)$, where D_0 (resp. D_p) is the connected component of $\mathbb{A}_p \setminus S'_{\kappa;s}(\mathbb{A}_p; z_0, w_0)$ whose boundary contains $\mathbb{T} \setminus \{z_0\}$ (resp. $\mathbb{T}_p \setminus \{w_0\}$). Then $S_{\kappa;s}(\mathbb{A}_p; z_0, w_0)$ satisfies restriction property, which means that, if we condition $S_{\kappa;s}(\mathbb{A}_p; z_0, w_0)$ to avoid a hull H in \mathbb{A}_p with $\text{dist}(H, \{z_0, w_0\}) > 0$, then we get $S_{\kappa;s}(\mathbb{A}_p \setminus H; z_0, w_0)$.

The paper is organized as follows. We introduce notation, symbols and definitions in Sects. 2, 3 and 4. The proof of Theorem 1.1 is started at Sect. 5, and finished at the end of Sect. 7. The argument introduced in [6] is used. In Sect. 8 we give a sketch of the proof of Theorem 1.2, and use Theorem 1.2 to prove Theorem 8.1, which generates $n \geq 2$ mutually disjoint random curves crossing an annulus such that conditioned on all but one trace, the remaining trace is a chordal $\text{SLE}(\kappa)$ trace. We believe that, in the case $n = 2$, if the inner circle of the annulus shrinks to a single point, then the two curves tend to the two arms of a two-sided radial $\text{SLE}(\kappa)$ (cf. [5]) in the disc. This may be used to understand the microscopic behavior of an $\text{SLE}(\kappa)$ trace near a typical point on this trace.

After finishing the first draft, the author noticed Lawler’s work in [4], which studies the same object from a different perspective. Lawler defines SLE_κ ($\kappa \leq 4$) process in a doubly connected domain connecting two opposite boundary points, which satisfies Theorems 1.1 and 1.2 here without the normalization constants, and if normalized agrees with our annulus $\text{SLE}(\kappa, \Lambda_{\kappa;(s)})$ process. The distribution of SLE in [4] may not be a probability measure, but a positive measure in general. Lawler defines his annulus SLE_κ process in the covering space by comparing it with the chordal SLE_κ in that space. He derives a PDE for the partition function and proves the existence of the solution. A nontrivial result proved in [4] is that the total mass of such annulus SLE is in fact finite, and so can be normalized to get a probability measure.

2 Preliminary

2.1 Symbols and Notation

We will frequently use functions $\cot(z/2)$, $\tan(z/2)$, $\coth(z/2)$, $\tanh(z/2)$, $\sin(z/2)$, $\cos(z/2)$, $\sinh(z/2)$, and $\cosh(z/2)$. For simplicity, we write 2 as a subscript. For example, $\cot_2(z)$ means $\cot(z/2)$, and $\cot'_2(z) = -\frac{1}{2} \sin_2^{-2}(z)$.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For $p > 0$, let $\mathbb{A}_p = \{z \in \mathbb{C} : 1 > |z| > e^{-p}\}$, $\mathbb{S}_p = \{z \in \mathbb{C} : 0 < \text{Im } z < p\}$, $\mathbb{T}_p = \{z \in \mathbb{C} : |z| = e^{-p}\}$, and $\mathbb{R}_p = \{z \in \mathbb{C} : \text{Im } z = p\}$. Then $\partial\mathbb{A}_p = \mathbb{T} \cup \mathbb{T}_p$ and $\partial\mathbb{S}_p = \mathbb{R} \cup \mathbb{R}_p$. Let e^i denote the map $z \mapsto e^{iz}$. Then e^i is a covering map from \mathbb{S}_p onto \mathbb{A}_p , maps \mathbb{R} onto \mathbb{T} and maps \mathbb{R}_p onto \mathbb{T}_p .

A subset K of a simply connected domain D is called a hull in D if $D \setminus K$ is a simply connected domain. A subset K of a doubly connected domain D is called a hull in D if $D \setminus K$ is a doubly connected domain, and K is bounded away from a boundary component of D . In this case, we define $\text{cap}_D(K) := \text{mod}(D) - \text{mod}(D \setminus K)$ to be the capacity of K in D , where $\text{mod}(\cdot)$ is the modulus of a doubly connected domain. We have $0 \leq \text{cap}_D(K) < \text{mod}(D)$, where the equality holds iff $K = \emptyset$. For example, the L in Theorem 1.1 is a hull in \mathbb{A}_p .

We say a set $K \subset \mathbb{C}$ has period $p \in \mathbb{C}$ if $p + K = K$. We say that a function f has progressive period $(p_1; p_2)$ if $f(\cdot \pm p_1) = f \pm p_2$. In this case, the definition domain of f has period p_1 , and the range of f has period p_2 .

An increasing function in this paper will always be strictly increasing. For a real interval J , we use $C(J)$ to denote the space of real continuous functions on J . The maximal solution to an ODE or SDE with initial value is the solution with the biggest definition domain.

A conformal map in this paper is an injective analytic function. We say that f maps D_1 conformally onto D_2 , and write $f : D_1 \xrightarrow{\text{Conf}} D_2$, if f is a conformal map defined on the domain D_1 and $f(D_1) = D_2$. If, in addition, for $j = 1, 2$, c_j is a point or a set in D or on ∂D , and f or its continuation maps c_1 onto c_2 , then we write $f : (D_1; c_1) \xrightarrow{\text{Conf}} (D_2; c_2)$.

Throughout this paper, a Brownian motion means a standard one-dimensional Brownian motion, and $B(t)$, $0 \leq t < \infty$, will always be used to denote a Brownian motion. This means that $B(t)$ is continuous, $B(0) = 0$, and $B(t)$ has independent increment with $B(t) - B(s) \sim \mathcal{N}(0, t - s)$ for $t \geq s \geq 0$.

Many functions in this paper depend on two variables. The first variable represents time or modulus, and the second variable does not. We use ∂_t and ∂_t'' to denote the partial derivatives w.r.t. the first variable, and use $'$, $''$, and the superscripts (h) , $h \in \mathbb{N}$, to denote the partial derivatives w.r.t. the second variable.

2.2 Special Functions

The vector fields used in the definition of the annulus Loewner equation are the following special functions $\mathbf{S}(t, z)$ and $\mathbf{H}(t, z)$. For $t > 0$, define

$$\mathbf{S}(t, z) = \lim_{M \rightarrow \infty} \sum_{k=-M}^M \frac{e^{2kt} + z}{e^{2kt} - z} = \text{P.V.} \sum_{2|n} \frac{e^{nt} + z}{e^{nt} - z},$$

$$\mathbf{H}(t, z) = -i\mathbf{S}(t, e^i(z)) = -i \text{P.V.} \sum_{2|n} \frac{e^{nt} + e^{iz}}{e^{nt} - e^{iz}} = \text{P.V.} \sum_{2|n} \cot_2(z - int).$$

Then $\mathbf{H}(t, \cdot)$ is a meromorphic function in \mathbb{C} , whose poles are $\{2m\pi + i2kt : m, k \in \mathbb{Z}\}$, which are all simple poles with residue 2. Moreover, $\mathbf{H}(t, \cdot)$ is an odd function and takes

real values on $\mathbb{R} \setminus \{\text{poles}\}$; $\text{Im } \mathbf{H}(t, \cdot) \equiv -1$ on \mathbb{R} ; $\mathbf{H}(t, \cdot)$ has period 2π and progressive period $(i2t; -2i)$. Let $\mathbf{r}(t) \in \mathbb{R}$ be such that the power series expansion of $\mathbf{H}(t, \cdot)$ near 0 is

$$\mathbf{H}(t, z) = \frac{2}{z} + \mathbf{r}(t)z + O(z^3). \tag{2.1}$$

Let $\mathbf{S}_I(t, z) = \mathbf{S}(t, e^{-t}z) - 1$ and $\mathbf{H}_I(t, z) = -i\mathbf{S}_I(t, e^{iz}) = \mathbf{H}(t, z + it) + i$. It is easy to check:

$$\mathbf{S}_I(t, z) = \text{P.V.} \sum_{2 \nmid n} \frac{e^{nt} + z}{e^{nt} - z}, \quad \mathbf{H}_I(t, z) = \text{P.V.} \sum_{2 \nmid n} \cot_2(z - int).$$

So $\mathbf{H}_I(t, \cdot)$ is a meromorphic function in \mathbb{C} with poles $\{2m\pi + i(2k + 1)t : m, k \in \mathbb{Z}\}$, which are all simple poles with residue 2; $\mathbf{H}_I(t, \cdot)$ is an odd function and takes real values on \mathbb{R} ; $\mathbf{H}_I(t, \cdot)$ has period 2π and progressive period $(i2t; -2i)$.

It is possible to express \mathbf{H} and \mathbf{H}_I using classical functions. Let $\theta(v, \tau)$ and $\theta_k(v, \tau)$, $k = 1, 2, 3$, be the Jacobi theta functions defined in [1]. Define $\Theta(t, z) = \theta(\frac{z}{2\pi}, \frac{it}{\pi})$ and $\Theta_I(t, z) = \theta_2(\frac{z}{2\pi}, \frac{it}{\pi})$. Then $\Theta(t, \cdot)$ has antiperiod 2π , $\Theta_I(t, \cdot)$ has period 2π , and

$$\mathbf{H} = 2 \frac{\Theta'}{\Theta}, \quad \mathbf{H}_I = 2 \frac{\Theta'_I}{\Theta_I}. \tag{2.2}$$

It is useful to rescale the special functions. Let

$$\widehat{\Theta}(t, z) = e^{\frac{z^2}{4t}} \left(\frac{\pi}{t}\right)^{\frac{1}{2}} \Theta\left(\frac{\pi^2}{t}, \frac{\pi}{t}z\right), \quad \widehat{\Theta}_I(t, z) = e^{\frac{z^2}{4t}} \left(\frac{\pi}{t}\right)^{\frac{1}{2}} \Theta_I\left(\frac{\pi^2}{t}, \frac{\pi}{t}z\right). \tag{2.3}$$

From the Jacobi identities, we have $\widehat{\Theta}(t, z) = \theta(i\frac{z}{2\pi}, \frac{it}{\pi}) = \Theta(t, iz)$ and $\widehat{\Theta}_I(t, z) = \theta_1(i\frac{z}{2\pi}, \frac{it}{\pi})$. From the product representations of θ_1 , we get

$$\widehat{\Theta}_I(t, z) = 2e^{-\frac{t}{4}} \cosh_2(z) \prod_{m=1}^{\infty} (1 - e^{-2mt})(1 + e^{z-2mt})(1 + e^{-z-2mt}). \tag{2.4}$$

Let $\widehat{\mathbf{H}} = 2 \frac{\widehat{\Theta}'}{\widehat{\Theta}}$ and $\widehat{\mathbf{H}}_I = 2 \frac{\widehat{\Theta}'_I}{\widehat{\Theta}_I}$. From (2.2) and (2.3) we have

$$\widehat{\mathbf{H}}(t, z) = \frac{\pi}{t} \mathbf{H}\left(\frac{\pi^2}{t}, \frac{\pi}{t}z\right) + \frac{z}{t}, \quad \widehat{\mathbf{H}}_I(t, z) = \frac{\pi}{t} \mathbf{H}_I\left(\frac{\pi^2}{t}, \frac{\pi}{t}z\right) + \frac{z}{t}. \tag{2.5}$$

Since $\widehat{\Theta}(t, z) = \Theta(t, iz)$ and $\mathbf{H}_I(t, z) = \mathbf{H}(t, z + it) + i$, we have

$$\widehat{\mathbf{H}}(t, z) = i\mathbf{H}(t, iz) = \text{P.V.} \sum_{2 \nmid n} \coth_2(z - nt); \tag{2.6}$$

$$\widehat{\mathbf{H}}_I(t, z) = \widehat{\mathbf{H}}(t, z + \pi i) = \text{P.V.} \sum_{2 \nmid n} \tanh_2(z - nt). \tag{2.7}$$

From (2.6), the power series expansion of $\widehat{\mathbf{H}}(t, \cdot)$ near 0 is

$$\widehat{\mathbf{H}}(t, z) = \frac{2}{z} + \widehat{\mathbf{r}}(t)z + O(z^3), \tag{2.8}$$

where $\widehat{\mathbf{r}}(t) := -\sum_{k=1}^{\infty} \sinh^{-2}(kt) + \frac{1}{6} = O(e^{-t}) + \frac{1}{6}$ as $t \rightarrow \infty$. Hence we may define

$$\widehat{\mathbf{R}}(t) = -\int_t^{\infty} \left(\widehat{\mathbf{r}}(s) - \frac{1}{6}\right) ds, \quad 0 < t < \infty. \tag{2.9}$$

Then $\widehat{\mathbf{R}}$ is positive and decreasing as $\widehat{\mathbf{r}} - \frac{1}{6} < 0$. From (2.1), (2.5), and (2.8), we have

$$\widehat{\mathbf{r}}(t) = \left(\frac{\pi}{t}\right)^2 \mathbf{r}\left(\frac{\pi^2}{t}\right) + \frac{1}{t}. \tag{2.10}$$

3 Loewner Equations

3.1 Annulus Loewner Equation

The annulus Loewner equations are defined in [12]. Fix $p \in (0, \infty)$ and $T \in (0, p]$. Let $\xi \in C([0, T])$. The annulus Loewner equation of modulus p driven by ξ is

$$\partial_t g(t, z) = g(t, z)\mathbf{S}(p - t, g(t, z)/e^{i\xi(t)}), \quad g(0, z) = z.$$

For $0 \leq t < T$, let $K(t)$ denote the set of $z \in \mathbb{A}_p$ such that the solution $g(s, z)$ blows up before or at time t . Then each $K(t)$ is a hull in \mathbb{A}_p , $\text{cap}_{\mathbb{A}_p}(K(t)) = t$, and $g(t, \cdot)$ maps $\mathbb{A}_p \setminus K(t)$ conformally onto \mathbb{A}_{p-t} , and maps \mathbb{T}_p onto \mathbb{T}_{p-t} . We call $K(t)$ and $g(t, \cdot)$, $0 \leq t < T$, the annulus Loewner hulls and maps of modulus p driven by ξ .

The equivalence between annulus SLE and radial SLE (Theorem 1.1 in [12]) and the existence of radial SLE trace imply the existence of annulus SLE trace. If ξ is a semi-martingale whose stochastic part is $\sqrt{\kappa}B(t)$, and whose drift part is continuously differentiable, then ξ generates an annulus Loewner trace β of modulus p , which means that

$$\beta(t) := \lim_{\mathbb{A}_{p-t} \ni z \rightarrow e^{i\xi(t)}} g(t, \cdot)^{-1}(z) \tag{3.1}$$

exists for all $0 \leq t < T$, and β is a continuous simple curve in $\mathbb{A}_p \cup \mathbb{T}$ with $\beta(0) = e^{i\xi(0)} \in \mathbb{T}$. If $\kappa \in (0, 4]$, then β is simple and $\beta((0, T)) \subset \mathbb{A}_p$. In this case, $K(t) = \beta((0, t])$ for $0 \leq t < T$, and we say that β is parameterized by its capacity in \mathbb{A}_p w.r.t. \mathbb{T}_p , i.e., $\text{cap}_{\mathbb{A}_p}(\beta((0, t])) = t$ for $0 \leq t < T$.

On the other hand, if $\beta(t)$, $0 \leq t < T$, is a simple curve with $\beta(0) \in \mathbb{T}$, $\beta((0, T)) \subset \mathbb{A}_p$, and if β is parameterized by its capacity in \mathbb{A}_p w.r.t. \mathbb{T}_p , then β is a simple annulus Loewner trace of modulus p driven by some $\xi \in C([0, T])$. If β is not parameterized by its capacity, then $\beta(v^{-1}(t))$, $0 \leq t < v(T)$, is an annulus Loewner trace of modulus p , where $v(t) := \text{cap}_{\mathbb{A}_p}(\beta((0, t]))$ is an increasing function with $v(0) = 0$. See Proposition 2.1 in [12].

3.2 Covering Annulus Loewner Equation

The covering annulus Loewner equation of modulus p driven by $\xi \in C([0, T])$ is

$$\partial_t \widetilde{g}(t, z) = \mathbf{H}(p - t, \widetilde{g}(t, z) - \xi(t)), \quad \widetilde{g}(0, z) = z. \tag{3.2}$$

For $0 \leq t < T$, let $\widetilde{K}(t)$ denote the set of $z \in \mathbb{S}_p$ such that the solution $\widetilde{g}(s, z)$ blows up before or at time t . Then for $0 \leq t < T$,

$$\widetilde{g}(t, \cdot) : (\mathbb{S}_p \setminus \widetilde{K}(t); \mathbb{R}_p) \xrightarrow{\text{Conf}} (\mathbb{S}_{p-t}; \mathbb{R}_{p-t}). \tag{3.3}$$

We call $\widetilde{K}(t)$ and $\widetilde{g}(t, \cdot)$, $0 \leq t < T$, the covering annulus Loewner hulls and maps of modulus p driven by ξ .

The relation between the covering annulus Loewner equation and the annulus Loewner equation is as follows. Let $K(t)$ and $g(t, \cdot)$ be the annulus Loewner hulls and maps of modulus p driven by ξ . Then we have $\widetilde{K}(t) = (e^i)^{-1}(K(t))$ and $e^i \circ \widetilde{g}(t, \cdot) = g(t, \cdot) \circ e^i$, $0 \leq t < T$. Thus, $\widetilde{K}(t)$ has period 2π , and $\widetilde{g}(t, \cdot)$ has progressive period $(2\pi; 2\pi)$.

If ξ generates an annulus Loewner trace β defined by (3.1), then there is a continuous simple curve $\tilde{\beta}(t), 0 \leq t < T$, which is defined by

$$\tilde{\beta}(t) = \lim_{\mathbb{S}_{p-t} \ni z \rightarrow \xi(t)} \tilde{g}(t, \cdot)^{-1}(z), \quad 0 \leq t < T. \tag{3.4}$$

Such $\tilde{\beta}$ is called the covering annulus Loewner trace of modulus p driven by ξ , and satisfies that $\beta = e^i \circ \tilde{\beta}$ and $\tilde{\beta}(0) = \xi(0)$. If β is simple with $\beta((0, T)) \subset \mathbb{A}_p$, then $\tilde{\beta}$ is also simple, $\tilde{\beta}((0, T)) \subset \mathbb{S}_p$, and $\tilde{K}(t) = \tilde{\beta}((0, t]) + 2\pi\mathbb{Z}, 0 \leq t < T$.

Since $\tilde{g}(t, \cdot)$ maps \mathbb{R}_p onto \mathbb{R}_{p-t} and $\mathbf{H}_I(t, z) = \mathbf{H}(t, z + it) + i$, we have

$$\partial_t \operatorname{Re} \tilde{g}(t, z) = \mathbf{H}_I(p - t, \operatorname{Re} \tilde{g}(t, z) - \xi(t)), \quad z \in \mathbb{R}_p. \tag{3.5}$$

Differentiating (3.5) w.r.t. z , we see that

$$\partial_t \tilde{g}'(t, z) = \tilde{g}'(t, z) \mathbf{H}'_I(p - t, \operatorname{Re} \tilde{g}(t, z) - \xi(t)), \quad z \in \mathbb{R}_p. \tag{3.6}$$

Since $\mathbf{S}(p - t, \cdot)$ and $\mathbf{H}(p - t, \cdot)$ have period 2π , for any $n \in \mathbb{Z}$, ξ and $\xi + 2n\pi$ generate the same family of annulus Loewner maps and the same family of covering annulus Loewner maps.

3.3 Strip Loewner Evolution

Strip Loewner equations will be used in Sect. 8. The strip Loewner equation [11] driven by $\xi \in C([0, T])$ is

$$\partial_t \tilde{g}(t, z) = \operatorname{coth}_2(\tilde{g}(t, z) - \xi(t)), \quad 0 \leq t < T, \quad \tilde{g}(0, z) = z.$$

For $0 \leq t < T$, let $\tilde{K}(t)$ denote the set of $z \in \mathbb{S}_\pi$ such that the solution $\tilde{g}(s, z)$ blows up before or at time t . Then $\tilde{K}(t)$ and $\tilde{g}(t, \cdot), 0 \leq t < T$, are called the strip Loewner hulls and maps driven by ξ . For each $t \in [0, T)$, $\tilde{K}(t)$ is a bounded hull in \mathbb{R}_π with $\operatorname{dist}(\tilde{K}(t), \mathbb{R}_\pi) > 0$, $\tilde{g}(t, \cdot) : (\mathbb{S}_\pi \setminus \tilde{K}(t); \mathbb{R}_\pi) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi)$, and $\tilde{g}(t, z) - z \rightarrow \pm t$ as $z \rightarrow \pm\infty$ in $\mathbb{S}_\pi \setminus \tilde{K}(t)$. If \tilde{K} is a bounded hull in \mathbb{R}_π with $\operatorname{dist}(\tilde{K}(t), \mathbb{R}_\pi) > 0$, then there exist a number $c_{\tilde{K}} \geq 0$ and a map $\tilde{g}_{\tilde{K}}$ determined by \tilde{K} such that $\tilde{g}_{\tilde{K}} : (\mathbb{S}_\pi \setminus \tilde{K}; \mathbb{R}_\pi) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi)$ and $\tilde{g}_{\tilde{K}} - z \rightarrow \pm c_{\tilde{K}}$ as $z \rightarrow \pm\infty$. We call $c_{\tilde{K}}$ the capacity of \tilde{K} in \mathbb{S}_π w.r.t. \mathbb{R}_π . Thus, the capacity of $\tilde{K}(t)$ in \mathbb{S}_π w.r.t. \mathbb{R}_π is t , and $\tilde{g}(t, \cdot) = \tilde{g}_{\tilde{K}(t)}$.

Since $\tilde{g}(t, \cdot)$ maps \mathbb{R}_π onto \mathbb{R}_π and $\operatorname{coth}_2(z + \pi i) = \tanh_2(t, z)$, we have

$$\partial_t \operatorname{Re} \tilde{g}(t, z) = \tanh_2(\operatorname{Re} \tilde{g}(t, z) - \xi(t)), \quad z \in \mathbb{R}_\pi. \tag{3.7}$$

Differentiating (3.7) w.r.t. z , we see that

$$\partial_t \tilde{g}'(t, z) = \tilde{g}'(t, z) \tanh'_2(\operatorname{Re} \tilde{g}(t, z) - \xi(t)), \quad z \in \mathbb{R}_\pi. \tag{3.8}$$

If ξ is a semi-martingale whose stochastic part is $\sqrt{\kappa}B(t)$, and whose drift part is continuously differentiable, then ξ generates a strip Loewner trace $\tilde{\beta}$, which is defined by

$$\tilde{\beta}(t) := \lim_{\mathbb{S}_\pi \ni z \rightarrow \xi(t)} \tilde{g}(t, \cdot)^{-1}(z), \quad 0 \leq t < T. \tag{3.9}$$

Such $\tilde{\beta}$ is a continuous curve in $\mathbb{S}_\pi \cup \mathbb{R}$ which satisfies that $\tilde{\beta}(0) = \xi(0) \in \mathbb{R}$. If $\kappa \in (0, 4]$, then $\tilde{\beta}$ is simple, $\tilde{\beta}((0, T)) \subset \mathbb{S}_\pi$, and $\tilde{K}(t) = \tilde{\beta}((0, t])$ for $0 \leq t < T$.

On the other hand, suppose $\tilde{\beta}(t)$ is a simple curve in $\mathbb{S}_\pi \setminus \mathbb{R}$, which intersects \mathbb{R} only at $t = 0$. Let $v(t)$ be the capacity of $\tilde{\beta}((0, t])$ in \mathbb{S}_π w.r.t. \mathbb{R}_π . Then v is a continuous increasing function, which maps $[0, T)$ onto $[0, S)$ for some $S \in (0, \infty]$, and there is $\xi \in C([0, S])$ which generates the strip Loewner trace $\tilde{\beta} \circ v^{-1}$.

The chordal SLE($\kappa; \rho$) process defined in [6] naturally extends to strip SLE($\kappa; \rho$) process. Let $\kappa > 0$ and $\rho \in \mathbb{R}$. Let $x_0, y_0 \in \mathbb{R}$. Let $\xi(t)$ and $q(t)$, $0 \leq t < \infty$, be the solution of

$$d\xi(t) = \sqrt{\kappa} dB(t) + \frac{\rho}{2} \tanh_2(\xi(t) - q(t)) dt, \quad \xi(0) = x_0;$$

$$dq(t) = \tanh_2(q(t) - \xi(t)), \quad q(0) = y_0.$$

Then the strip Loewner trace $\tilde{\beta}$ driven by ξ is called a strip SLE($\kappa; \rho$) trace in \mathbb{S}_π started from x_0 with marked point $y_0 + \pi i$. From [10] we know that, when $\rho = \kappa - 6$, $\tilde{\beta}$ is a time-change of a chordal SLE(κ) trace in \mathbb{S}_π from x_0 to $y_0 + \pi i$, stopped when it hits \mathbb{R}_π . If, in addition, $\kappa \leq 4$, since the chordal SLE(κ) trace does not hit \mathbb{R}_π before it ends, we see that $\tilde{\beta}$ is a time-change of a complete chordal SLE(κ) trace.

4 One SLE Curve Crossing an Annulus

4.1 Annulus SLE with One Marked Point

We now cite some definitions in Sect. 4.1 of [15].

Definition 4.1 A covering crossing annulus drift function is a real valued $C^{0,1}$ differentiable function defined on $(0, \infty) \times \mathbb{R}$. A covering crossing annulus drift function with period 2π in its second variable is called a crossing annulus drift function.

Definition 4.2 Suppose Λ is a covering crossing annulus drift function. Let $\kappa > 0$, $p > 0$, and $x_0, y_0 \in \mathbb{R}$. Let $\xi(t)$, $0 \leq t < p$, be the maximal solution to the SDE

$$d\xi(t) = \sqrt{\kappa} dB(t) + \Lambda(p - t, \xi(t) - \operatorname{Re} \tilde{g}(t, y_0 + \pi i)) dt, \quad \xi(0) = x_0, \tag{4.1}$$

where $\tilde{g}(t, \cdot)$, $0 \leq t < p$, are the covering annulus Loewner maps of modulus p driven by ξ . Then the covering annulus Loewner trace of modulus p driven by ξ is called the covering annulus SLE($\kappa; \Lambda$) trace in \mathbb{S}_p started from x_0 with marked point $y_0 + \pi i$.

Definition 4.3 Suppose Λ is a crossing annulus drift function. Let $\kappa \geq 0$, $p > 0$, $a \in \mathbb{T}$ and $b \in \mathbb{T}_p$. Choose $x_0, y_0 \in \mathbb{R}$ such that $a = e^{ix_0}$ and $b = e^{-p+iy_0}$. Let $\xi(t)$, $0 \leq t < p$, be the maximal solution to (4.1). The annulus Loewner trace of modulus p driven by $\xi(t)$, $0 \leq t < p$, is called the annulus SLE($\kappa; \Lambda$) trace in \mathbb{A}_p started from a with marked point b .

Remark The above definition does not depend on the choices of x_0 and y_0 because $\Lambda(p - t, \cdot)$ has period 2π , $\tilde{g}(t, \cdot)$ has progressive period $(2\pi; 2\pi)$, and for any $n \in \mathbb{Z}$, the annulus Loewner objects driven by $\xi(t) + 2n\pi$ agree with those driven by $\xi(t)$. Via conformal maps, we can define the annulus SLE($\kappa; \Lambda$) trace in any doubly connected domain.

4.2 Annulus SLE with Reversibility

A family of functions are defined in Sect. 7 of [15], which are $\widehat{\Psi}_\infty, \widehat{\Psi}_q, \widehat{\Psi}_0, \Psi_0, \Psi_m, m \in \mathbb{Z}, \Psi_{(s)}, \Lambda_0$, and $\Lambda_{(s)}, s \in \mathbb{R}$. They are all smooth functions on $(0, \infty) \times \mathbb{R}$, and depend on three parameters: $\kappa \in (0, 4]$, $\sigma \in [0, \frac{4}{\kappa}]$, and $\tau = \frac{\kappa}{4} - \sqrt{\frac{\kappa^2}{16} + \kappa\sigma} \leq 0$. Now we suppose $\kappa \in (0, 4]$ is fixed, and

$$\sigma = \frac{4}{\kappa} - 1 \geq 0, \quad \tau = \frac{\kappa}{2} - 2 \leq 0. \tag{4.2}$$

Then these function depend only on $\kappa \in (0, 4]$, $m \in \mathbb{Z}$ and $s \in \mathbb{R}$. For simplicity, we omit the symbol κ . The $\Lambda_{(s)}$ here is the $\Lambda_{\kappa;(s)}$ in Theorems 1.1 and 1.2.

The $\widehat{\Psi}_\infty$ is defined in (7.31) of [15]:

$$\widehat{\Psi}_\infty(t, x) = e^{-\frac{\tau^2 t}{2\kappa}} \cosh_2^{\frac{\tau}{2}}(x). \tag{4.3}$$

The $\widehat{\Psi}_q$ is defined by (7.33) of [15]:

$$\widehat{\Psi}_q(t, x) = \mathbf{E} \left[\exp \left(\sigma \int_0^\infty \widehat{\mathbf{H}}_{I,q}(t+s, X_x(s)) ds \right) \right], \tag{4.4}$$

where $\widehat{\mathbf{H}}_{I,q}$ is defined by (7.8) of [15]: $\widehat{\mathbf{H}}_{I,q}(t, z) = \widehat{\mathbf{H}}_I(t, z) - \tanh_2(z)$, and $X_x(t)$, $0 \leq t < \infty$, is a diffusion process which satisfies SDE (7.2) of [15]:

$$dX_x(t) = \sqrt{\kappa} dB(t) + \tau \tanh_2(X_x(t)) dt, \quad X_x(0) = x. \tag{4.5}$$

The $\widehat{\Psi}_0$ is defined in Theorem 7.2 of [15]:

$$\widehat{\Psi}_0 = \widehat{\Psi}_\infty \widehat{\Psi}_q. \tag{4.6}$$

The Ψ_0 is defined in Theorem 7.3 of [15]:

$$\Psi_0(t, x) = e^{-\frac{\sigma^2}{2\kappa t}} \left(\frac{\pi}{t} \right)^{\sigma + \frac{1}{2}} \widehat{\Psi}_0 \left(\frac{\pi^2}{t}, \frac{\pi}{t} x \right). \tag{4.7}$$

For $m \in \mathbb{Z}$ and $s \in \mathbb{R}$, the Ψ_m and $\Psi_{(s)}$ are defined in Theorem 7.4 of [15]:

$$\Psi_m(t, x) = \Psi_0(t, x - 2m\pi), \quad \Psi_{(s)} = \sum_{m \in \mathbb{Z}} e^{\frac{2\pi}{\kappa} ms} \Psi_m. \tag{4.8}$$

The functions $\widehat{\Psi}_\infty, \widehat{\Psi}_q, \widehat{\Psi}_0, \Psi_0, \Psi_m, \Psi_{(s)}$ are all positive. The functions Λ_0 and $\Lambda_{(s)}$ are defined in Proposition 7.4 and Theorem 7.4, respectively, of [15]: $\Lambda_0 = \kappa \frac{\Psi'_0}{\Psi_0} - \mathbf{H}_I$, $\Lambda_{(s)} = \kappa \frac{\Psi'_{(s)}}{\Psi_{(s)}} - \mathbf{H}_I$. For the sake of completeness, we now define $\Lambda_m = \kappa \frac{\Psi'_m}{\Psi_m} - \mathbf{H}_I = \Lambda_0(\cdot - 2m\pi)$ and

$$\Gamma_m = \Psi_m \Theta_I^{-\frac{2}{\kappa}}, \quad \Gamma_{(s)} = \Psi_{(s)} \Theta_I^{-\frac{2}{\kappa}}. \tag{4.9}$$

From (2.2), we see that Λ_m and $\Lambda_{(s)}$ have simpler expressions:

$$\Lambda_m = \kappa \frac{\Gamma'_m}{\Gamma_m}, \quad \Lambda_{(s)} = \kappa \frac{\Gamma'_{(s)}}{\Gamma_{(s)}}. \tag{4.10}$$

From Lemma 5.2 of [15], we see that Γ_m and $\Gamma_{(s)}$ solve the PDE (5.6) in [15]. Since we here set the value of σ by (4.2), this PDE becomes (5.2) in [15], i.e.,

$$\partial_t \Gamma_m = \frac{\kappa}{2} \Gamma''_m + \Gamma'_m \mathbf{H}_I + \alpha \mathbf{H}'_I \Gamma_m, \tag{4.11}$$

where

$$\alpha = \frac{6 - \kappa}{2\kappa}. \tag{4.12}$$

Define $\widehat{\Gamma}_0$ on $(0, \infty) \times \mathbb{R}$ such that

$$\widehat{\Gamma}_0(t, x) = \left(\frac{\pi}{t} \right)^\alpha \Gamma_0 \left(\frac{\pi^2}{t}, \frac{\pi}{t} x \right). \tag{4.13}$$

From (2.3), (4.7), and (4.9), we have

$$\widehat{\Gamma}_0 = \widehat{\Psi}_0 \widehat{\Theta}_I^{-\frac{2}{\kappa}}. \tag{4.14}$$

Define $\widehat{\Theta}_{I,\infty}$, $\widehat{\Theta}_{I,q}$, $\widehat{\Gamma}_\infty$, and $\widehat{\Gamma}_q$ on $(0, \infty) \times \mathbb{R}$ such that

$$\widehat{\Theta}_{I,\infty}(t, x) = 2e^{-\frac{t}{4}} \cosh_2(x); \quad \widehat{\Theta}_{I,q} = \widehat{\Theta}_I / \widehat{\Theta}_{I,\infty}; \tag{4.15}$$

$$\widehat{\Gamma}_\infty(t, x) = 2^{-\frac{2}{\kappa}} e^{-\frac{t^2-1}{2\kappa}t} \cosh_2(x)^{\frac{2}{\kappa}(\tau-1)}; \quad \widehat{\Gamma}_q = \widehat{\Gamma}_0 / \widehat{\Gamma}_\infty. \tag{4.16}$$

One may check that $\widehat{\Gamma}_\infty$ solves

$$-\partial_t \widehat{\Gamma}_\infty = \frac{\kappa}{2} \widehat{\Gamma}_\infty'' + \widehat{\Gamma}_\infty' \tanh_2 + \alpha \tanh_2' \widehat{\Gamma}_\infty. \tag{4.17}$$

From (4.3) we have $\widehat{\Gamma}_\infty = \widehat{\Psi}_\infty \widehat{\Theta}_{I,\infty}^{-\frac{2}{\kappa}}$. From (4.6) and (4.9) we have

$$\widehat{\Gamma}_q = \widehat{\Psi}_q \widehat{\Theta}_{I,q}^{-\frac{2}{\kappa}}. \tag{4.18}$$

Let $p > 0$ and $x_0, y_0 \in \mathbb{R}$. Let $y_m = y_0 + 2m\pi$, $m \in \mathbb{Z}$. Consider the following two SDEs.

$$\begin{aligned} d\xi(t) &= \sqrt{\kappa} dB(t) + \Lambda_0(p - t, \xi(t) - \operatorname{Re} \tilde{g}(t, y_m + pi)) dt, \\ 0 \leq t < p, \quad \xi(0) &= x_0, \end{aligned} \tag{4.19}$$

$$\begin{aligned} d\xi(t) &= \sqrt{\kappa} dB(t) + \Lambda_{(s)}(p - t, \xi(t) - \operatorname{Re} \tilde{g}(t, y_0 + pi)) dt, \\ 0 \leq t < p, \quad \xi(0) &= x_0, \end{aligned} \tag{4.20}$$

where $\tilde{g}(t, \cdot)$ are the covering annulus Loewner maps driven by ξ . Let μ_m or $\mu_{(s)}$ denote the distribution of $(\xi(t), 0 \leq t < p)$ if it solves (4.19) or (4.20), respectively. Then

$$\mu_{(s)} = \sum_{m \in \mathbb{Z}} e^{\frac{2\pi}{\kappa}ms} \frac{\Psi_m(p, x_0 - y_0)}{\Psi_{(s)}(p, x_0 - y_0)} \mu_m = \sum_{m \in \mathbb{Z}} e^{\frac{2\pi}{\kappa}ms} \frac{\Gamma_0(p, x_0 - y_m)}{\Gamma_{(s)}(p, x_0 - y_0)} \mu_m, \tag{4.21}$$

where the first equality follows from Proposition 7.4 in [15], and the second equality follows from (4.8), (4.9), and the fact that $\Theta_I(p, \cdot)$ has period 2π .

Let β and $\tilde{\beta}$ be the annulus Loewner trace and covering annulus Loewner trace, respectively, of modulus p , driven by ξ . If (ξ) has distribution μ_m , then $\tilde{\beta}$ is a covering annulus SLE($\kappa; \Lambda_0$) trace in \mathbb{S}_p started from x_0 with marked point $y_m + pi$. If (ξ) has distribution $\mu_{(s)}$, then β is an annulus SLE($\kappa; \Lambda_{(s)}$) trace in \mathbb{A}_p started from e^{iy_0} with marked point e^{iy_0-p} . Let \mathcal{E}_m denote the event that the covering trace ends at $y_m + pi$. Proposition 7.4, Theorem 8.3, and Theorem 9.3 in [15] together imply that $\mu_m(\mathcal{E}_m) = 1$ and $\mu_{(s)}(\bigcup_{m \in \mathbb{Z}} \mathcal{E}_m) = 1$. Since $\mathcal{E}_m, m \in \mathbb{Z}$, are mutually disjoint, the μ_m 's are singular to each other. From (4.21) we have

$$\frac{d\mu_m}{d\mu_{(s)}} = e^{\frac{2\pi}{\kappa}ms} \frac{\Gamma_0(p, x_0 - y_m)}{\Gamma_{(s)}(p, x_0 - y_0)} \mathbf{1}_{\mathcal{E}_m}. \tag{4.22}$$

4.3 Some Estimations

Lemma 4.1 *For any $t > 0$ and $0 \leq x \leq 3t$,*

$$\widehat{\mathbf{H}}'_{I,q}(t, x) < \min \left\{ \frac{1}{2}, 2e^{x-2t} \right\} + \frac{4e^{-t}}{1 - e^{-2t}}.$$

Proof Since $\widehat{\mathbf{H}}_{I,q}(t, z) = \widehat{\mathbf{H}}_I(t, z) - \tanh_2(z)$, from (2.7), we have

$$\widehat{\mathbf{H}}'_{I,q}(t, x) = \tanh'_2(x - 2t) + \sum_{n=2}^{\infty} \tanh'_2(x - 2nt) + \sum_{n=-\infty}^{-1} \tanh'_2(x - 2nt).$$

Note that $\tanh'_2(x) = \frac{2}{(e^{x/2} + e^{-x/2})^2} \leq \min\{\frac{1}{2}, 2e^x, 2e^{-x}\}$ for $x \in \mathbb{R}$. If $0 \leq x \leq 3t$, then

$$\begin{aligned} \sum_{n=2}^{\infty} \tanh'_2(x - 2nt) &\leq 2 \sum_{n=2}^{\infty} e^{x-2nt} = \frac{2e^{x-4t}}{1 - e^{-2t}} \leq \frac{2e^{-t}}{1 - e^{-2t}}, \\ \sum_{n=-\infty}^{-1} \tanh'_2(x - 2nt) &\leq 2 \sum_{n=-\infty}^{-1} e^{2nt-x} = \frac{2e^{-x-2t}}{1 - e^{-2t}} \leq \frac{2e^{-2t}}{1 - e^{-2t}}. \end{aligned}$$

The conclusion follows from the above displayed formulas. □

Proposition 4.1 *If F is one of the following functions: $\widehat{\Theta}_{I,q}$, $\widehat{\Psi}_q$, or $\widehat{\Gamma}_q$, then*

- (i) $\lim_{2t-|x| \rightarrow +\infty} \ln(F(t, x)) = 0$;
- (ii) *for every $R > 0$, $\ln(F)$ is bounded on $\{t \geq R, |x| \leq 2t + R\}$.*

Proof From (2.4) and (4.15), the conclusion is clearly true for $F = \widehat{\Theta}_{I,q}$. From (4.18), we suffice to prove this proposition for $F = \widehat{\Psi}_q$. Throughout this proof, we use $O_t(1)$ to denote a positive quantity which depends on κ, σ, t , and is uniformly bounded when t is bigger than any positive constant.

Fix $t > 0$ and $x \in \mathbb{R}$. Let $X_x(s)$ be as in (4.5), and (\mathcal{F}_s) be the filtration generated by $(X_x(s))$. Define a uniformly integrable martingale $M_{t,x}(s)$, $0 \leq s < \infty$, by

$$M_{t,x}(s) := \mathbf{E} \left[\exp \left(\sigma \int_0^{\infty} \widehat{\mathbf{H}}'_{I,q}(t+r, X_x(r)) dr \right) \middle| \mathcal{F}_s \right].$$

From (4.4) we have $M_{t,x}(s) = \widehat{\Psi}_q(t+s, X_x(s)) \exp(\sigma \int_0^s \widehat{\mathbf{H}}'_{I,q}(t+r, X_x(r)) dr)$. Suppose S is an a.s. finite (\mathcal{F}_s) -stopping time. From the Optional Stopping Theorem, $\mathbf{E}[M_{t,x}(S)] = M_{t,x}(0)$. Since $M_{t,x}(0) = \widehat{\Psi}_q(t, x)$, we have

$$\widehat{\Psi}_q(t, x) = \mathbf{E} \left[\widehat{\Psi}_q(t+S, X_x(S)) \exp \left(\sigma \int_0^S \widehat{\mathbf{H}}'_{I,q}(t+s, X_x(s)) ds \right) \right]. \tag{4.23}$$

Let $\lambda(s, x) = \min\{\frac{1}{2}, 2e^{x-2s}\}$. If $0 \leq X_x(s) \leq 3(t+s)$ for $0 \leq s \leq S$, then from Lemma 4.1, we have

$$\begin{aligned} \int_0^S \widehat{\mathbf{H}}'_{I,q}(t+s, X_x(s)) ds &\leq \int_0^S \lambda(t+s, X_x(s)) ds + \int_0^{\infty} \frac{4e^{-(t+s)}}{1 - e^{-2(t+s)}} ds \\ &= \int_0^S \lambda(t+s, X_x(s)) ds + 2 \ln \left(\frac{1+e^{-t}}{1-e^{-t}} \right), \end{aligned}$$

which together with (4.23) implies that

$$\widehat{\Psi}_q(t, x) \leq \exp(O_t(1)e^{-t}) \mathbf{E} \left[\widehat{\Psi}_q(t+S, X_x(S)) \exp \left(\sigma \int_0^S \lambda(t+s, X_x(s)) ds \right) \right]. \tag{4.24}$$

Recall that $\sigma \in [0, \frac{4}{\kappa}]$. Let $\sigma' = \frac{\kappa}{4}\sigma$. From Proposition 7.1 in [15], for any $c_0 \in (1+\sigma', 2)$, there is $C > 0$ depending only on κ, σ , and c_0 such that for any $t \in (0, \infty)$ and $x \in \mathbb{R}$,

$$1 \leq \widehat{\Psi}_q(t, x) \leq \exp(C(t^{-1} + 1)e^{(c_0-2)t}) (1 + Ce^{\frac{2}{\kappa}|x| - \frac{2}{\kappa}c_0t}). \tag{4.25}$$

This immediately implies that $\ln(\widehat{\Psi}_q)$ is bounded on $\{|x| \leq c_0 t, t \geq t_0\}$ for any $t_0 > 0$.

Choose any $c_0 \in (1 + \sigma', 2)$ such that $c_0 \geq \frac{2}{1+2/\kappa}$ and $c_0 \neq 3 - \frac{\kappa}{2}$. Let $a = 3 - c_0 \in (1, 2 - \sigma')$. Then $a \neq \frac{\kappa}{2}$. Since $c_0 - 1 > \sigma'$ and $2 > a > 1$, we have $a(2 - a) = a(c_0 - 1) > \sigma'$. Thus,

$$-\frac{2}{\kappa}a + \frac{2}{\kappa} \frac{a + \sigma'}{c_0} = -\frac{2}{\kappa c_0} (a(c_0 - 1) - \sigma') < 0; \tag{4.26}$$

$$-\frac{2}{\kappa}a + \frac{\sigma}{2(2 - a)} = -\frac{2}{\kappa(2 - a)} (a(2 - a) - \sigma') < 0. \tag{4.27}$$

For $m \in \mathbb{N} \cup \{0\}$, let \mathcal{G}_m denote the event that $\sqrt{\kappa}B(s) < as + m$ for any $s \geq 0$. Then $\emptyset = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \dots \subset \mathcal{G}_m \subset \mathcal{G}_{m+1} \subset \dots$. It is well known that $\mathbb{P}[\bigcup_{m=0}^\infty \mathcal{G}_m] = 1$ and

$$\mathbb{P}[\mathcal{G}_m^c] \leq e^{-\frac{2}{\kappa}ma}, \quad m \in \mathbb{N}. \tag{4.28}$$

Suppose $t > 0$ and $2t \leq x \leq 3t$. Let S be the first time that $X_x(s) \leq 0$ or $X_x(s) \geq 3(t + s)$. Then S is a stopping time, and $0 \leq X_x(s) \leq 3(t + s)$ for $0 \leq s \leq S$. Since $X_x(s)$ is recurrent, S is a.s. finite. Since $\tau \leq 0$ and $\tanh_2(x) \geq 0$ for $x \geq 0$, from (4.5) we have

$$X_x(s) \leq x + as + m, \quad 0 \leq s \leq S, \text{ on } \mathcal{G}_m. \tag{4.29}$$

Let \mathcal{E}_l and \mathcal{E}_r denote the event that $X_x(S) = 0$ and $X_x(S) = 3(t + S)$, respectively. From (4.25) and the facts that $0 > c_0 - 2 \geq -\frac{2}{\kappa}c_0$ and $3 - c_0 > 0$ we see that

$$\widehat{\Psi}_q(t + S, X_x(S)) \leq \exp(O_t(1)e^{(c_0-2)t}) \leq O_t(1) \quad \text{on } \mathcal{E}_l, \tag{4.30}$$

$$\widehat{\Psi}_q(t + S, X_x(S)) \leq O_t(1)e^{\frac{2}{\kappa}(3-c_0)(t+S)} \quad \text{on } \mathcal{E}_r. \tag{4.31}$$

From (4.24) we have

$$\begin{aligned} \widehat{\Psi}_q(t, x) &\leq \exp(O_t(1)e^{-t}) \sum_{m=1}^\infty \mathbf{E} \left[\mathbf{1}_{(\mathcal{G}_m \setminus \mathcal{G}_{m-1}) \cap \mathcal{E}_r} \widehat{\Psi}_q(t + S, X_x(S)) \right. \\ &\quad \times \exp\left(\sigma \int_0^S \lambda(t + s, X_x(s)) ds\right) \Big] \\ &\quad + \exp(O_t(1)e^{-t}) \sum_{m=1}^\infty \left[\mathbf{1}_{(\mathcal{G}_m \setminus \mathcal{G}_{m-1}) \cap \mathcal{E}_l} \widehat{\Psi}_q(t + S, X_x(S)) \right. \\ &\quad \times \exp\left(\sigma \int_0^S \lambda(t + s, X_x(s)) ds\right) \Big]. \end{aligned} \tag{4.32}$$

Suppose $\mathcal{G}_m \cap \mathcal{E}_r$ occurs. From (4.29) we have $3(t + S) = X_x(S) \leq x + aS + m$. Since $3 - a = c_0 > 0$, we have

$$S \leq \frac{x - 3t + m}{c_0} \quad \text{on } \mathcal{G}_m \cap \mathcal{E}_r. \tag{4.33}$$

Since $S \geq 0$, we see that $\mathcal{G}_m \cap \mathcal{E}_r = \emptyset$ when $m < 3t - x$. Let $m_0 = \lceil 3t - x \rceil$. Then from (4.28), (4.31), and the fact that $\lambda \leq \frac{1}{2}$, we find that for any $m \in \mathbb{N}$ and $m \geq m_0$,

$$\begin{aligned} &\mathbf{E} \left[\mathbf{1}_{(\mathcal{G}_m \setminus \mathcal{G}_{m-1}) \cap \mathcal{E}_r} \widehat{\Psi}_q(t + S, X_x(S)) \exp\left(\sigma \int_0^S \lambda(t + s, X_x(s)) ds\right) \right] \\ &\leq \mathbf{E} \left[e^{-\frac{2}{\kappa}(m-1)a} O_t(1) \exp\left(\frac{2}{\kappa}(3 - c_0)(t + S) + \frac{\sigma}{2}S\right) \right]. \end{aligned}$$

Since $\frac{2}{\kappa}(3 - c_0) + \frac{\sigma}{2} = \frac{2}{\kappa}(a + \sigma') > 0$, from (4.33) we find that the RHS of the above formula is

$$\leq O_t(1) \exp\left(-\frac{2}{\kappa}a(m - 1 - t) + \frac{2}{\kappa}(a + \sigma') \cdot \frac{x - 3t + m}{c_0}\right).$$

So we have

$$\begin{aligned} & \sum_{m=1}^{\infty} \mathbf{E} \left[\mathbf{1}_{(\mathcal{G}_m \setminus \mathcal{G}_{m-1}) \cap \mathcal{E}_t} \widehat{\Psi}_q(t + S, X_x(S)) \exp\left(\sigma \int_0^S \lambda(t + s, X_x(s)) ds\right) \right] \\ & \leq O_t(1) \sum_{m=m_0}^{\infty} \exp\left(-\frac{2}{\kappa}a(m - t) + \frac{2}{\kappa}(a + \sigma') \cdot \frac{x - 3t + m}{c_0}\right) \\ & = O_t(1) \exp\left(\frac{2}{\kappa}at + \frac{2}{\kappa} \frac{a + \sigma'}{c_0}(x - 3t)\right) \sum_{m=m_0}^{\infty} \exp\left(-\frac{2}{\kappa}a + \frac{2}{\kappa} \frac{a + \sigma'}{c_0}\right)^m \\ & \leq O_t(1) \exp\left(\frac{2}{\kappa}at + \frac{2}{\kappa} \frac{a + \sigma'}{c_0}(x - 3t) - \frac{2}{\kappa}am_0 + \frac{2}{\kappa} \frac{a + \sigma'}{c_0}m_0\right) \\ & \leq O_t(1)e^{\frac{2}{\kappa}a(x-2t)} \tag{4.34} \end{aligned}$$

where the second last inequality follows from (4.26), and the last inequality follows from the fact that $|m_0 - (3t - x)| < 1$.

Suppose $\mathcal{G}_m \cap \mathcal{E}_t$ occurs. From (4.29) we have $X_x(s) - 2(t + s) \leq x - 2t + m + (a - 2)s$, $0 \leq s \leq S$. Suppose that $2t \leq x \leq 3t$. Then $x - 2t + m \geq 0$ for any $m \in \mathbb{N}$. Let $p = \frac{x-2t+m}{2-a} \geq 0$. Then we have

$$\begin{aligned} & \int_0^S \lambda(t + s, X_x(s)) ds \\ & \leq \int_0^p \frac{1}{2} ds + \int_p^\infty 2e^{x-2t+m+(a-2)s} ds \\ & = \frac{p}{2} + \int_p^\infty 2e^{(a-2)(s-p)} ds = \frac{p}{2} + \frac{2}{2-a} = \frac{x - 2t + m + 4}{2(2 - a)}. \tag{4.35} \end{aligned}$$

From (4.27), (4.28) and (4.30), we have

$$\begin{aligned} & \sum_{m=1}^{\infty} \mathbf{E} \left[\mathbf{1}_{(\mathcal{G}_m \setminus \mathcal{G}_{m-1}) \cap \mathcal{E}_t} \widehat{\Psi}_q(t + S, X_x(S)) \exp\left(\sigma \int_0^S \lambda(t + s, X_x(s)) ds\right) \right] \\ & \leq O_t(1) \sum_{m=1}^{\infty} \exp\left(-\frac{2}{\kappa}(m - 1)a + \sigma \cdot \frac{x - 2t + m + 4}{2(2 - a)}\right) \\ & \leq O_t(1)e^{\frac{\sigma(x-2t)}{2(2-a)}} \sum_{m=1}^{\infty} \exp\left(-\frac{2}{\kappa}a + \frac{\sigma}{2(2-a)}\right)^m \leq O_t(1)e^{\frac{2}{\kappa}a(x-2t)}. \tag{4.36} \end{aligned}$$

From (4.32), (4.34), and (4.36), we have

$$\widehat{\Psi}_q(t, x) \leq O_t(1)e^{\frac{2}{\kappa}a(x-2t)}, \quad 2t \leq x \leq 3t. \tag{4.37}$$

Suppose $0 \leq x \leq 2t$. Let $m_1 = \lceil 2t - x \rceil$. Then $x - 2t + m \geq 0$ if and only if $m \geq m_1$. If $m \geq m_1$, then (4.35) still holds. Following the argument of (4.36), we get

$$\begin{aligned}
 & \sum_{m=m_1}^{\infty} \mathbf{E} \left[\mathbf{1}_{(\mathcal{G}_m \setminus \mathcal{G}_{m-1}) \cap \mathcal{E}_l} \widehat{\Psi}_q(t+S, X_x(S)) \exp \left(\sigma \int_0^S \lambda(t+s, X_x(s)) ds \right) \right] \\
 & \leq O_t(1) e^{\frac{\sigma(x-2t)}{2(2-a)}} \sum_{m=m_1}^{\infty} \exp \left(-\frac{2}{\kappa} a + \frac{\sigma}{2(2-a)} \right)^m \\
 & \leq O_t(1) e^{\frac{\sigma(x-2t)}{2(2-a)}} \exp \left(-\frac{2}{\kappa} a + \frac{\sigma}{2(2-a)} \right)^{m_1} \\
 & \leq O_t(1) e^{\frac{2}{\kappa} a(x-2t)}, \tag{4.38}
 \end{aligned}$$

where the last inequality holds because $|m_1 - (2t - x)| < 1$.

For $m < m_1$, we use the estimation:

$$\int_0^S \lambda(t+s, X_x(s)) ds \leq \int_0^{\infty} 2e^{x-2t+m+(a-2)s} ds = \frac{2e^{x-2t+m}}{2-a} \quad \text{on } \mathcal{G}_m.$$

From (4.30) we see that, when $m < m_1$, on the event $\mathcal{G}_m \cap \mathcal{E}_l$,

$$\begin{aligned}
 & \widehat{\Psi}_q(t+S, X_x(S)) \exp \left(\sigma \int_0^S \lambda(t+s, X_x(s)) ds \right) \\
 & = \exp \left(O_t(1) e^{(c_0-2)t} + \sigma \frac{e^{x-2t+m}}{2-a} \right) \\
 & = 1 + O_t(1) e^{(c_0-2)t} + O_t(1) e^{x-2t+m},
 \end{aligned}$$

where the last equality holds because $e^{x-2t+m} \leq O_t(1)$ for $m < m_1$. Thus,

$$\begin{aligned}
 & \sum_{m=1}^{m_1-1} \mathbf{E} \left[\mathbf{1}_{(\mathcal{G}_m \setminus \mathcal{G}_{m-1}) \cap \mathcal{E}_l} \left(\widehat{\Psi}_q(t+S, X_x(S)) \exp \left(\sigma \int_0^S \lambda(t+s, X_x(s)) ds \right) - 1 \right) \right] \\
 & \leq \sum_{m=1}^{m_1-1} e^{-\frac{2}{\kappa}(m-1)a} O_t(1) (e^{(c_0-2)t} + e^{x-2t+m}) \\
 & = O_t(1) \left(e^{(c_0-2)t} + e^{x-2t} \sum_{m=1}^{m_1-1} \exp \left(1 - \frac{2}{\kappa} a \right)^m \right) \\
 & \leq O_t(1) e^{(c_0-2)t} + O_t(1) e^{x-2t} (1 + e^{(1-\frac{2}{\kappa}a)m_1}) \\
 & \leq O_t(1) e^{(c_0-2)t} + O_t(1) (e^{x-2t} + e^{\frac{2}{\kappa}a(x-2t)}),
 \end{aligned}$$

where the second last inequality holds because $a \neq \frac{\kappa}{2}$. The above inequality together with (4.32), (4.34), and (4.38) implies that,

$$\begin{aligned}
 \widehat{\Psi}_q(t, x) - 1 & \leq O_t(1) (e^{-t} + e^{(c_0-2)t} + e^{x-2t} + e^{\frac{2}{\kappa}a(x-2t)}) \\
 & \leq O_t(1) e^{(1-\frac{c_0}{2})(x-2t)}, \quad 0 \leq x \leq 2t.
 \end{aligned}$$

Since $1 \leq \widehat{\Psi}_q$, the above inequality implies that

$$0 \leq \ln(\widehat{\Psi}_q(t, x)) \leq O_t(1) e^{(1-\frac{c_0}{2})(x-2t)}, \quad 0 \leq x \leq 2t,$$

which finishes the proof of (i) for $F = \widehat{\Psi}_q$. The above inequality together with (4.37) implies that

$$0 \leq \ln(\widehat{\Psi}_q(t, x)) \leq O_t(1) + \frac{2}{\kappa} a (0 \vee (x - 2t)), \quad 0 \leq x \leq 3t,$$

which finishes the proof of (ii) for $F = \widehat{\Psi}_q$. □

5 Annulus SLE with Domain Changed

We now start proving Theorem 1.1. The proof will be finished at the end of Sect. 7.2. Let $p > 0$, $\kappa \in (0, 4]$, $s \in \mathbb{R}$, $z_0 \in \mathbb{T}$, $w_0 \in \mathbb{T}_p$, and the hull L be as in Theorem 1.1. Choose $x_0, y_0 \in \mathbb{R}$ such that $z_0 = e^{ix_0}$ and $w_0 = e^{iy_0-p}$. Let $y_m = y_0 + 2m\pi$, $m \in \mathbb{Z}$.

Note that $\mathbb{A}_p \setminus L$ is a doubly connected domain, whose boundary contain \mathbb{T}_p and e^{ix_0} . Let $p_L = \text{mod}(\mathbb{A}_p \setminus L)$. Let $\tilde{L} = (e^i)^{-1}(L)$. Then \tilde{L} is a subset of \mathbb{S}_p with period 2π . We may find W_L and \tilde{W}_L such that $W_L : (\mathbb{A}_p \setminus L; \mathbb{T}_p) \xrightarrow{\text{Conf}} (\mathbb{A}_{p_L}; \mathbb{T}_{p_L})$, $\tilde{W}_L : (\mathbb{S}_p \setminus \tilde{L}; \mathbb{R}_p) \xrightarrow{\text{Conf}} (\mathbb{S}_{p_L}; \mathbb{R}_{p_L})$, $e^i \circ \tilde{W}_L = W_L \circ e^i$, and \tilde{W}_L has progressive period $(2\pi; 2\pi)$.

5.1 Stochastic Differential Equations

Suppose $\xi \in C([0, p])$ with $\xi(0) = x_0$. Let $g(t, \cdot)$ and $\tilde{g}(t, \cdot)$, $0 \leq t < p$, be the annulus and covering annulus Loewner maps of modulus p , respectively, driven by ξ . Let $K(t)$ and $\tilde{K}(t)$ be the corresponding hulls and covering hulls. Suppose ξ generates a simple annulus Loewner trace β of modulus p with $\beta((0, p)) \subset \mathbb{A}_p$. Then ξ also generates a simple covering annulus Loewner trace $\tilde{\beta}$ of modulus p with $\tilde{\beta}((0, p)) \subset \mathbb{S}_p$. We have $\beta = e^i \circ \tilde{\beta}$, $\tilde{\beta}(0) = \xi(0) = x_0$, $K(t) = \beta((0, t])$, and $\tilde{K}(t) = \tilde{\beta}((0, t]) + 2\pi\mathbb{Z}$, $0 \leq t < p$.

Let T be the biggest number in $(0, p]$ such that $\beta((0, T)) \cap L = \emptyset$. Let $\tilde{\beta}_L(t) = \tilde{W}_L(\tilde{\beta}(t))$ and $\beta_L(t) = W_L(\beta(t))$, $0 \leq t < T$. Then β_L and $\tilde{\beta}_L$ are simple curves, $\beta_L = e^i \circ \tilde{\beta}_L$, $\beta_L(0) \in \mathbb{T}$, and $\beta_L((0, T)) \subset \mathbb{A}_{p_L}$. Let $v(t) = \text{cap}_{\mathbb{A}_{p_L}}(\beta_L((0, t]))$. Then v is a continuous increasing function, which maps $[0, T)$ onto $[0, S)$ for some $S \in (0, p_L]$. Let $\gamma_L(t) = \beta_L(v^{-1}(t))$, $0 \leq t < S$. Then $\gamma_L(t)$, $0 \leq t < S$, is the annulus Loewner trace of modulus p_L driven by some $\eta_L \in C([0, S])$. Let $h_L(t, \cdot)$ and $\tilde{h}_L(t, \cdot)$, $0 \leq t < S$, be the annulus and covering annulus Loewner maps of modulus p_L , respectively, driven by η_L .

For $0 \leq t < T$, define $\xi_L(t) = \eta_L(v(t))$, $\tilde{g}_L(t, \cdot) = \tilde{h}_L(v(t), \cdot)$;

$$\tilde{g}_{L,W}(t, \cdot) = \tilde{g}_L(t, \cdot) \circ \tilde{W}_L; \quad \tilde{W}(t, \cdot) = \tilde{g}_{L,W}(t, \cdot) \circ g(t, \cdot)^{-1}. \tag{5.1}$$

Then both $\tilde{g}_{L,W}(t, \cdot)$ and $\tilde{W}(t, \cdot)$ have progressive period $(2\pi; 2\pi)$, and

$$\tilde{g}_{L,W}(t, \cdot) : (\mathbb{S}_p \setminus (\tilde{L} \cup (\tilde{\beta}((0, t]) + 2\pi\mathbb{Z})); \mathbb{R}_p) \xrightarrow{\text{Conf}} (\mathbb{S}_{p_L-v(t)}; \mathbb{R}_{p_L-v(t)}), \tag{5.2}$$

$$\tilde{W}(t, \cdot) : (\mathbb{S}_{p-t} \setminus \tilde{L}_t; \mathbb{R}_{p-t}) \xrightarrow{\text{Conf}} (\mathbb{S}_{p_L-v(t)}; \mathbb{R}_{p_L-v(t)}), \tag{5.3}$$

where $\tilde{L}_t := \tilde{g}(t, \tilde{L}) \subset \mathbb{S}_{p-t}$. We have $g_L(\beta_L(t)) = e^{i\xi_L(t)}$. Since $\beta_L = e^i(\tilde{\beta}_L)$, there is $n \in \mathbb{Z}$ such that $\tilde{g}_L(t, \tilde{\beta}_L(t)) = \xi_L(t) + 2n\pi$ for $0 \leq t < T$. We now add $2n\pi$ to the driving function η_L . Then the new η_L is still the driving function for γ_L , $h_L(t, \cdot)$ and $\tilde{h}_L(t, \cdot)$, and we have

$$\tilde{g}_{L,W}(t, \tilde{\beta}(t)) = \xi_L(t); \tag{5.4}$$

$$\tilde{W}(t, \xi(t)) = \xi_L(t). \tag{5.5}$$

Define $q_m(t)$, $q_{L,m}(t)$, $A_j(t)$, $A_{I,m}(t)$, $X_m(t)$, $X_{L,m}(t)$, $0 \leq t < T$, such that

$$q_m(t) + (p - t)i = \tilde{g}(t, y_m + pi); \tag{5.6}$$

$$q_{L,m}(t) + (p_L - v(t))i = \tilde{g}_{L,W}(t, y_m + pi); \tag{5.7}$$

$$A_j(t) = \tilde{W}^{(j)}(t, \xi(t)), \quad j = 1, 2, 3; \quad A_{I,m}(t) = \tilde{W}'(t, q_m(t) + (p - t)i); \tag{5.8}$$

$$X_m(t) = \xi(t) - q_m(t); \quad X_{L,m}(t) = \xi_L(t) - q_{L,m}(t). \tag{5.9}$$

A standard argument together with Lemma 2.1 in [12] shows that

$$v'(t) = \tilde{W}'(t, \xi(t))^2 = A_1(t)^2. \tag{5.10}$$

Hence,

$$\partial_t \tilde{g}_{L,W}(t, z) = \tilde{W}'(t, \xi(t))^2 \mathbf{H}(p_L - v(t), \tilde{g}_{L,W}(t, z) - \xi_L(t)). \tag{5.11}$$

Since $\mathbf{H}_I(t, \cdot)$ is odd, from (3.5), (3.6), and (5.11) we have

$$dq_m(t) = -\mathbf{H}_I(p - t, X_m(t)) dt; \tag{5.12}$$

$$\frac{d\tilde{g}'(t, y_m + pi)}{\tilde{g}'(t, y_m + pi)} = \mathbf{H}'_I(p - t, X_m(t)) dt; \tag{5.13}$$

$$d q_{L,m}(t) = -A_1(t)^2 \mathbf{H}_I(p_L - v(t), X_{L,m}(t)) dt; \tag{5.14}$$

$$\frac{d\tilde{g}'_{L,W}(t, y_m + pi)}{\tilde{g}'_{L,W}(t, y_m + pi)} = A_1(t)^2 \mathbf{H}'_I(p_L - v(t), X_{L,m}(t)) dt. \tag{5.15}$$

From (5.1), (5.13) and (5.15) we get

$$\frac{dA_{I,m}(t)}{A_{I,m}(t)} = A_1(t)^2 \mathbf{H}'_I(p_L - v(t), X_{L,m}(t)) dt - \mathbf{H}'_I(p - t, X_m(t)) dt. \tag{5.16}$$

Differentiating $\tilde{W}(t, \cdot) \circ \tilde{g}(t, z) = \tilde{g}_{L,W}(t, z)$ w.r.t. t using (3.2) and (5.11), and letting $w = \tilde{g}(t, z)$, we obtain an equality for $\partial_t \tilde{W}(t, w)$ with $w \in \mathbb{S}_{p-t} \setminus \tilde{L}_t$. Differentiating this equality w.r.t. w , we get an equality for $\partial_t \tilde{W}'(t, w)$. Letting $w \rightarrow \xi(t)$ in $\mathbb{S}_{p-t} \setminus \tilde{L}_t$ in these two equalities and using (2.1) we get

$$\partial_t \tilde{W}(t, \xi(t)) = -3A_2(t). \tag{5.17}$$

$$\frac{\partial_t A_1(t)}{A_1(t)} = \frac{1}{2} \left(\frac{A_2(t)}{A_1(t)} \right)^2 - \frac{4}{3} \frac{A_3(t)}{A_1(t)} + A_1(t)^2 \mathbf{r}(p_L - v(t)) - \mathbf{r}(p - t). \tag{5.18}$$

Let $\kappa \in (0, 4]$. Suppose now (ξ) is a semimartingale, and $d(\xi)_t = \kappa dt, 0 \leq t < p$. We will frequently apply Itô's formula (cf. [8]). From (5.5) and (5.17) we have

$$d\xi_L(t) = A_1(t) d\xi(t) + \left(\frac{\kappa}{2} - 3 \right) A_2(t) dt. \tag{5.19}$$

From (5.12) and (5.14) we see that $X_m(t)$ and $X_{L,m}(t)$ satisfy

$$dX_m(t) = d\xi(t) + \mathbf{H}_I(p - t, X_m(t)) dt; \tag{5.20}$$

$$dX_{L,m}(t) = d\xi_L(t) + A_1(t)^2 \mathbf{H}_I(p_L - v(t), X_{L,m}(t)) dt. \tag{5.21}$$

From (5.18) we see that

$$\begin{aligned} \frac{dA_1(t)}{A_1(t)} &= \frac{A_2(t)}{A_1(t)} d\xi(t) + \left[\frac{1}{2} \left(\frac{A_2(t)}{A_1(t)} \right)^2 + \left(\frac{\kappa}{2} - \frac{4}{3} \right) \frac{A_3(t)}{A_1(t)} \right. \\ &\quad \left. + A_1(t)^2 \mathbf{r}(p_L - v(t)) - \mathbf{r}(p - t) \right] dt. \end{aligned}$$

Let c and α be as in (1.1) and (4.12), respectively. Then we compute that

$$\frac{dA_1(t)^\alpha}{A_1(t)^\alpha} = \alpha \frac{A_2(t)}{A_1(t)} d\xi(t) + \left[\frac{c}{6} A_S(t) + \alpha A_1(t)^2 \mathbf{r}(p_L - v(t)) - \alpha \mathbf{r}(p - t) \right] dt, \tag{5.22}$$

where $A_S(t) := \frac{A_3(t)}{A_1(t)} - \frac{3}{2} \left(\frac{A_2(t)}{A_1(t)} \right)^2$ is the Schwarz derivative of $\tilde{W}(t, \cdot)$ at $\xi(t)$.

Let

$$Y_m(t) = \Gamma_0(p - t, X_m(t)), \quad Y_{L,m}(t) = \Gamma_0(p_L - v(t), X_{L,m}(t)), \tag{5.23}$$

From (4.10), (4.11), (5.10), (5.20) and (5.21) we find that

$$\frac{dY_m(t)}{Y_m(t)} = \frac{1}{\kappa} \Lambda_0(p - t, X_m(t)) d\xi(t) - \alpha \mathbf{H}'_I(p - t, X_m(t)) dt, \tag{5.24}$$

$$\frac{dY_{L,m}(t)}{Y_{L,m}(t)} = \frac{1}{\kappa} \Lambda_0(p_L - v(t), X_{L,m}(t)) d\xi_L(t) - \alpha A_1(t)^2 \mathbf{H}'_I(p_L - v(t), X_{L,m}(t)) dt. \tag{5.25}$$

Define M_m on $[0, T)$ by

$$M_m = A_1^\alpha A_{I,m}^\alpha \frac{Y_{L,m}}{Y_m} \exp\left(-\frac{c}{6} \int_0^\cdot A_S(s) ds + \alpha \int_{p-}^{p_L-v(\cdot)} \mathbf{r}(s) ds\right). \tag{5.26}$$

From (5.16), (5.19), (5.22), (5.24) and (5.25), we find that

$$\begin{aligned} \frac{dM_m(t)}{M_m(t)} &= \left[\alpha \frac{A_2(t)}{A_1(t)} + \frac{A_1(t)}{\kappa} \Lambda_0(p_L - v(t), X_{L,m}(t)) - \frac{1}{\kappa} \Lambda_0(p - t, X_m(t)) \right] \\ &\quad \times (d\xi(t) - \Lambda_0(p - t, X_m(t)) dt), \quad 0 \leq t < T. \end{aligned} \tag{5.27}$$

Let $C_{p,L} = \exp(-\frac{c}{2} \int_p^{p_L} (\mathbf{r}(s) + \frac{1}{s}) ds) > 0$. We have $M_m = N_m \exp(cU)$, where

$$N_m = C_{p,L} A_1^\alpha A_{I,m}^\alpha \frac{Y_{L,m}}{Y_m} \exp\left(\left(\alpha + \frac{c}{2}\right) \int_{p-}^{p_L-v(\cdot)} \mathbf{r}(s) ds + \frac{c}{2} \int_{p-}^{p_L-v(\cdot)} \frac{1}{s} ds\right). \tag{5.28}$$

$$U = -\frac{1}{6} \int_0^\cdot A_S(s) ds - \frac{1}{2} \int_{p-}^{p_L-v(\cdot)} \left(\mathbf{r}(s) + \frac{1}{s}\right) ds + \frac{1}{2} \int_p^{p_L} \left(\mathbf{r}(s) + \frac{1}{s}\right) ds. \tag{5.29}$$

5.2 Rescaling

Let $\widehat{p} = \frac{\pi^2}{p}$, $\widehat{T} = \frac{\pi^2}{p-T} - \widehat{p}$ and $\check{t} = p - \frac{\pi^2}{\widehat{p}+t}$. Then the function $t \mapsto \check{t}$ maps $[0, \infty)$ onto $[0, p)$, and maps $[0, \widehat{T})$ onto $[0, T)$. Let $\widehat{L} = \frac{\pi}{p} \widetilde{L} \subset \mathbb{S}_\pi$, $\widehat{x}_0 = \frac{\pi}{p} x_0$, and $\widehat{y}_m = \frac{\pi}{p} y_m$, $m \in \mathbb{Z}$. Since \widetilde{L} has period 2π , and $\text{dist}(\widetilde{L}, \{x_0\} \cup \mathbb{R}_p) > 0$, we see that \widehat{L} has period $2\widehat{p}$, and $\text{dist}(\widehat{L}, \{\widehat{x}_0\} \cup \mathbb{R}_\pi) > 0$. Let $\widehat{\beta}(t) = \frac{\pi}{p} \widetilde{\beta}(\check{t})$, $0 \leq t < \infty$. Then $\widehat{\beta}$ is a curve in $\mathbb{S}_\pi \cup \mathbb{R}$ started from \widehat{x}_0 , and \widehat{T} is the biggest number in $(0, \infty]$ such that $\widehat{\beta}((0, \widehat{T})) \cap \widehat{L} = \emptyset$. Furthermore, we have

$$\{T = p\} = \{\widehat{T} = \infty\} = \{\beta \cap L = \emptyset\} = \{\widetilde{\beta} \cap \widetilde{L} = \emptyset\} = \{\widehat{\beta} \cap \widehat{L} = \emptyset\}. \tag{5.30}$$

For $0 \leq t < \widehat{T}$, let $\widehat{p}_L(t) = \frac{\pi^2}{p_L-v(\check{t})}$. Since $p_L - v(t) = \text{mod}(\mathbb{A}_p \setminus L \setminus \beta((0, t]))$, while $p - t = \text{mod}(\mathbb{A}_p \setminus \beta((0, t]))$, we have $p_L - v(t) \leq p - t$, $0 \leq t < T$. Thus,

$$\widehat{p}_L(t) \geq \frac{\pi^2}{p-t} = \widehat{p} + t, \quad 0 \leq t < \widehat{T}. \tag{5.31}$$

From (2.10), for any $0 \leq t < \widehat{T}$,

$$-\int_{\widehat{p}+t}^{\widehat{p}_L(t)} \widehat{\mathbf{r}}(s) ds = \int_{p-\check{t}}^{p_L-v(\check{t})} \left(\mathbf{r}(s) + \frac{1}{s}\right) ds. \tag{5.32}$$

Let $m \in \mathbb{Z}$. For $0 \leq t < \widehat{T}$, define

$$\begin{aligned} \widehat{\xi}(t) &= \frac{\widehat{p}+t}{\pi} \cdot \xi(\check{t}); & \widehat{q}_m(t) &= \frac{\widehat{p}+t}{\pi} \cdot q_m(\check{t}); & \widehat{X}_m(t) &= \frac{\widehat{p}+t}{\pi} \cdot X_m(\check{t}); \\ \widehat{\xi}_L(t) &= \frac{\widehat{p}_L(t)}{\pi} \cdot \xi_L(\check{t}); & \widehat{q}_{L,m}(t) &= \frac{\widehat{p}_L(t)}{\pi} \cdot q_{L,m}(\check{t}); & \widehat{X}_{L,m}(t) &= \frac{\widehat{p}_L(t)}{\pi} \cdot X_{L,m}(\check{t}). \end{aligned} \tag{5.34}$$

From (5.9) we have $\widehat{X}_m = \widehat{\xi} - \widehat{q}_m$ and $\widehat{X}_{L,m} = \widehat{\xi}_L - \widehat{q}_{L,m}$. For $0 \leq t < \widehat{T}$, define

$$\widehat{g}(t, z) = \frac{\widehat{p} + t}{\pi} \check{g}\left(\check{t}, \frac{p}{\pi} z\right); \quad \widehat{g}_{L,W}(t, z) = \frac{\widehat{p}_L(t)}{\pi} \check{g}_{L,W}\left(\check{t}, \frac{p}{\pi} z\right). \tag{5.35}$$

From (3.3), (3.4), (5.4), (5.2), (5.6) and (5.7) we have

$$\widehat{g}(t, \cdot) : (\mathbb{S}_\pi \setminus (\widehat{\beta}((0, t]) + 2\widehat{p}\mathbb{Z}); \mathbb{R}_\pi, \widehat{\beta}(t), \widehat{y}_m + \pi i) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi, \widehat{\xi}(t), \widehat{q}_m(t)); \tag{5.36}$$

$$\begin{aligned} \widehat{g}_{L,W}(t, \cdot) : (\mathbb{S}_\pi \setminus ((\widehat{\beta}((0, t]) + 2\widehat{p}\mathbb{Z}) \cup \widehat{L}); \mathbb{R}_\pi, \widehat{\beta}(t), \widehat{y}_m + \pi i) \\ \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi, \widehat{\xi}_L(t), \widehat{q}_{L,m}(t) + \pi i). \end{aligned} \tag{5.37}$$

For $0 \leq t < \widehat{T}$, define

$$\widehat{W}(t, \cdot) = \widehat{g}_{L,W}(t, \cdot) \circ \widehat{g}(t, \cdot)^{-1}; \tag{5.38}$$

$$\widehat{A}_1(t) = \widehat{W}'(t, \widehat{\xi}(t)), \quad \widehat{A}_{L,m}(t) = \widehat{W}'(t, \widehat{q}_m(t) + \pi i). \tag{5.39}$$

Let $\widehat{L}_t = \frac{\widehat{p}+t}{\pi} \check{L}_t$. Since $\check{L}_t = \check{g}(t, \check{L})$, we have $\widehat{L}_t = \widehat{g}(t, \widehat{L})$. From (5.36) and (5.37) we have

$$\widehat{W}(t, \cdot) : (\mathbb{S}_\pi \setminus \widehat{L}_t; \mathbb{R}_\pi) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi); \tag{5.40}$$

$$\widehat{W}(t, \widehat{\xi}(t)) = \widehat{\xi}_L(t); \quad \widehat{W}(t, \widehat{q}_m(t) + \pi i) = \widehat{q}_{L,m}(t) + \pi i. \tag{5.41}$$

From (5.1), (5.8), (5.35) and (5.38) we have

$$\widehat{A}_1(t) = \frac{\widehat{p}_L(t)}{\widehat{p} + t} A_1(\check{t}), \quad \widehat{A}_{L,m}(t) = \frac{\widehat{p}_L(t)}{\widehat{p} + t} A_{L,m}(\check{t}). \tag{5.42}$$

For $m \in \mathbb{Z}$ and $0 \leq t < \widehat{T}$, let

$$\widehat{Y}_m(t) = \widehat{\Gamma}_0(\widehat{p} + t, \widehat{X}_m(t)), \quad \widehat{Y}_{L,m}(t) = \widehat{\Gamma}_0(\widehat{p}_L(t), \widehat{X}_{L,m}(t)). \tag{5.43}$$

From (4.13), (5.23), (5.33), and (5.34), we have

$$\widehat{Y}_m(t) = \left(\frac{\pi}{\widehat{p} + t}\right)^\alpha Y_m(\check{t}), \quad \widehat{Y}_{L,m}(t) = \left(\frac{\pi}{\widehat{p}_L(t)}\right)^\alpha Y_{L,m}(\check{t}). \tag{5.44}$$

Define \widehat{N}_m on $[0, \widehat{T})$ such that

$$\widehat{N}_m = C_{p,L} \widehat{A}_1^\alpha \widehat{A}_{L,m}^\alpha \widehat{Y}_{L,m} \widehat{Y}_m^{-1} \exp\left(-\left(\alpha + \frac{c}{2}\right) \int_{\widehat{p}_+}^{\widehat{p}_L(\cdot)} \widehat{\mathbf{r}}(s) ds\right).$$

From (5.28), (5.32), (5.42) and (5.44) we find that

$$\widehat{N}_m(t) = N_m(\check{t}), \quad 0 \leq t < \widehat{T}. \tag{5.45}$$

From (1.1), (2.9), (4.2), (4.12), (4.14), (4.16), and (5.43), we see that for $0 \leq t < \widehat{T}$,

$$\begin{aligned} \widehat{N}_m(t) = \widehat{A}_1(t)^\alpha \widehat{A}_{L,m}(t)^\alpha \widehat{\Gamma}_q(\widehat{p}_L(t), \widehat{X}_{L,m}(t)) \widehat{\Gamma}_q(\widehat{p} + t, \widehat{X}_m(t))^{-1} \\ \times \exp\left(-\alpha \int_{\widehat{X}_m(t)}^{\widehat{X}_{L,m}(t)} \tanh_2(s) ds - \left(\alpha + \frac{c}{2}\right) (\widehat{\mathbf{R}}(\widehat{p}_L(t)) - \widehat{\mathbf{R}}(\widehat{p} + t))\right). \end{aligned} \tag{5.46}$$

6 Estimations on $\widehat{N}_m(t)$

For $m \in \mathbb{Z}$, let \mathcal{P}_m denote the set of (ρ_1, ρ_2) with the following properties.

1. For $j = 1, 2$, ρ_j is a polygonal crosscut in \mathbb{S}_π that grows from a point on \mathbb{R} to a point on \mathbb{R}_π , whose line segments are parallel to either x -axis or y -axis, and whose vertices other than the end points have rational coordinates.
2. $\rho_1 + 2n\widehat{p}$, $\rho_2 + 2n\widehat{p}$, $n \in \mathbb{Z}$, and \widehat{L} are mutually disjoint; ρ_1 lies to the left of ρ_2 .
3. $\rho_1 \cup \rho_2$ disconnects \widehat{x}_0 and $\widehat{y}_m + \pi i$ from \widehat{L} in \mathbb{S}_π .

For each $(\rho_1, \rho_2) \in \mathcal{P}_m$, let $\widehat{T}_{\rho_1, \rho_2}$ denote the biggest number such that $\widehat{\beta}((0, \widehat{T}_{\rho_1, \rho_2})) \cap (\rho_1 \cup \rho_2) = \emptyset$. Since $\widehat{\beta}$ starts from \widehat{x}_0 , we have $\widehat{T}_{\rho_1, \rho_2} \leq \widehat{T}$. Let \mathcal{E}_m be as in Sect. 4.2. Then

$$\mathcal{E}_m = \left\{ \lim_{t \rightarrow p} \widetilde{\beta}(t) = y_m + \pi i \right\} = \left\{ \lim_{t \rightarrow \infty} \widehat{\beta}(t) = \widehat{y}_m + \pi i \right\}.$$

We will prove the following proposition at the end of Sect. 6.1 and Sect. 6.2.

Proposition 6.1 *Let $m \in \mathbb{Z}$.*

- (i) $\lim_{t \rightarrow \infty} \ln(\widehat{N}_m(t)/C_{p,L}) = 0$ on the event $\mathcal{E}_m \cap \{\widehat{T} = \infty\}$.
- (ii) For any $(\rho_1, \rho_2) \in \mathcal{P}_m$, $\ln(\widehat{N}_m(t))$ is uniformly bounded on $[0, \widehat{T}_{\rho_1, \rho_2})$.

For $m \in \mathbb{Z}$, define $\widetilde{\mathcal{P}}_m = \{(\frac{p}{\pi}\rho_1, \frac{p}{\pi}\rho_2) : (\rho_1, \rho_2) \in \mathcal{P}_m\}$. Then for each $(\rho_1, \rho_2) \in \widetilde{\mathcal{P}}_m$, ρ_1 and ρ_2 are simple curves that grow from a point on \mathbb{R} to a point on \mathbb{R}_π , and $\rho_1 \cup \rho_2$ disconnects x_0 and $y_m + \pi i$ from \widetilde{L} . For each $(\rho_1, \rho_2) \in \widetilde{\mathcal{P}}_m$, let T_{ρ_1, ρ_2} denote the biggest time such that $\widetilde{\beta}((0, T_{\rho_1, \rho_2})) \cap (\rho_1 \cup \rho_2) = \emptyset$. Then $T_{\rho_1, \rho_2} \leq T$, and the function $t \mapsto \check{t}$ maps $[0, \widehat{T}_{\rho_1, \rho_2})$ onto $[0, T_{\frac{p}{\pi}\rho_1, \frac{p}{\pi}\rho_2})$. From Proposition 6.1, (5.30), and (5.45) we conclude the following proposition:

Proposition 6.2 *Let $m \in \mathbb{Z}$.*

- (i) $\lim_{t \rightarrow p} N_m(t) = C_{p,L}$ on the event $\mathcal{E}_m \cap \{T = p\}$.
- (ii) For any $(\rho_1, \rho_2) \in \widetilde{\mathcal{P}}_m$, $N_m(t)$ is uniformly bounded on $[0, T_{\rho_1, \rho_2})$.

6.1 The Limit Value

We now use \mathbb{H} and \mathbb{D} to denote the upper half-plane $\{\text{Im } z > 0\}$ and the unit disk $\{|z| < 1\}$, respectively. Let \mathcal{H} denote the set of bounded hulls in \mathbb{H} . For every $H \in \mathcal{H}$, there is unique φ_H which maps $\mathbb{H} \setminus H$ conformally onto \mathbb{H} such that as $z \rightarrow \infty$, $\varphi_H(z) = z + \frac{c}{z} + O(|z|^{-2})$, where $c =: \text{hcap}(H)$ is called the half-plane capacity of H . If $H = \emptyset$, then $\varphi_H = \text{id}$ and $\text{hcap}(H) = 0$; otherwise $\text{hcap}(H) > 0$.

Suppose $H \in \mathcal{H}$ and $H \neq \emptyset$. Then $\overline{H} \cap \mathbb{R} \neq \emptyset$. Let $a_H = \inf(\overline{H} \cap \mathbb{R})$ and $b_H = \sup(\overline{H} \cap \mathbb{R})$. Let

$$\Sigma_H = \mathbb{C} \setminus (H \cup \{\bar{z} : z \in H\} \cup [a_H, b_H]).$$

By reflection principle, φ_H extends to Σ_H , and $\varphi_H : \Sigma_H \xrightarrow{\text{Conf}} \mathbb{C} \setminus [c_H, d_H]$ for some $c_H < d_H \in \mathbb{R}$. Moreover, φ_H is increasing on $(-\infty, a_H)$ and (b_H, ∞) , and maps them onto $(-\infty, c_H)$ and (d_H, ∞) , respectively. So φ_H^{-1} extends conformally to $\mathbb{C} \setminus [c_H, d_H]$.

Example 1 Suppose $r > 0$. Let $H = \{z \in \mathbb{H} : |z| \leq r\}$. Then $H \in \mathcal{H}$, $a_H = -r$ and $b_H = r$. It is clear that $\varphi_H(z) = z + \frac{r^2}{z}$. Thus $\text{hcap}(H) = r^2$, and $[c_H, d_H] = [-2r, 2r]$.

From (5.1) in [14] there is a measure μ_H supported by $[c_H, d_H]$ with $|\mu_H| = \text{hcap}(H)$ such that for any $z \in \Sigma_H$,

$$\varphi_H^{-1}(z) - z = \int_{c_H}^{d_H} \frac{-1}{z-x} d\mu_H(x). \tag{6.1}$$

Since $\varphi_\emptyset = \text{id}$, (6.1) is also true for $H = \emptyset$ if we set $\mu_\emptyset = 0$, $a_\emptyset = c_\emptyset = \infty$ and $b_\emptyset = d_\emptyset = -\infty$. The following lemma is a combination of Lemma 5.2 and Lemma 5.3 in [14].

Lemma 6.1

- (i) For any $H \in \mathcal{H}$, $\varphi_H(x) \leq x$ on $(-\infty, a_H)$ and $\varphi_H(x) \geq x$ on (b_H, ∞) .
- (ii) If $H_1, H_2 \in \mathcal{H}$ and $H_1 \subset H_2$, then $[c_{H_1}, d_{H_1}] \subset [c_{H_2}, d_{H_2}]$.

Lemma 6.2 Let $r > 0$. Suppose $H \in \mathcal{H}$ and $H \subset \{|z| \leq r\}$. Suppose $W : (\mathbb{H} \setminus H; \infty) \xrightarrow{\text{Conf}} (\mathbb{H}; \infty)$, and satisfies that $W'(\infty) = 1$, $W((-\infty, -r)) \subset (-\infty, 0)$ and $W((r, \infty)) \subset (0, \infty)$. Then for any $z \in \mathbb{H} \cup \mathbb{R}$ with $|z| \geq (1+r)^2$, $|W(z) - z| \leq r^2 + 2r$.

Proof Let $H_r = \{z \in \mathbb{H} : |z| \leq r\}$. Then $H \subset H_r \in \mathcal{H}$. From Lemma 6.1(i) and Example 1, we have $[c_H, d_H] \subset [c_{H_r}, d_{H_r}] = [-2r, 2r]$. Define $K = \varphi_H(H_r \setminus H)$. Then $K \in \mathcal{H}$ and $\varphi_{H_r} = \varphi_K \circ \varphi_H$, which implies that $\text{hcap}(H_r) = \text{hcap}(K) + \text{hcap}(H)$. Thus, $\text{hcap}(H) \leq \text{hcap}(H_r) = r^2$. Applying Lemma 6.1(ii) to K , we find that $\varphi_H(x) \leq \varphi_{H_r}(x)$ for any $x \in (b_{H_r}, \infty) = (r, \infty)$. Thus, $\inf \varphi_H((r, \infty)) \leq \inf \varphi_{H_r}((r, \infty)) = d_{H_r} = 2r$. Similarly, $\sup \varphi_H((-\infty, -r)) \geq -2r$. Since both W and φ_H map $\mathbb{H} \setminus H$ conformally onto \mathbb{H} and fix ∞ , and have derivative 1 at ∞ , there is $w \in \mathbb{R}$ such that $W(z) = \varphi_H(z) - w$ for any $z \in \mathbb{H} \setminus H$. From the assumption of W , we get $\inf W((r, \infty)) \geq 0 \geq \sup W((-\infty, -r))$. So we have $|w| \leq 2r$. We now suffice to show that for any $z \in \mathbb{H} \cup \mathbb{R}$ with $|z| \geq (1+r)^2$, $|\varphi_H(z) - z| \leq r^2$.

Let $z \in \mathbb{H} \cup \mathbb{R}$ and $|z| \geq 1 + 2r$. Since $[c_H, d_H] \subset [-2r, 2r]$ and $|\mu_H| = \text{hcap}(H) \leq r^2$, from (6.1) we have $|\varphi_H^{-1}(z) - z| \leq r^2$. Let $\gamma = \{z \in \mathbb{H} : |z| = 1 + 2r\}$. Then $\varphi_H^{-1}(\gamma)$ is a crosscut in \mathbb{H} , which divides \mathbb{H} into two components. Let D denote the unbounded component. Then φ_H^{-1} maps $\{z \in \mathbb{H} \cup \mathbb{R} : |z| \geq 1 + 2r\}$ onto \overline{D} . Since $|\varphi_H^{-1}(z) - z| \leq r^2$ for $z \in \gamma$, we have $\varphi_H^{-1}(\gamma) \subset \{|z| \leq 1 + 2r + r^2\}$, which implies that $\overline{D} \supset \{z \in \mathbb{H} \cup \mathbb{R} : |z| \geq (1+r)^2\}$. Thus, if $z \in \mathbb{H} \cup \mathbb{R}$ and $|z| \geq (1+r)^2$, then $\varphi_H(z) \in \{z \in \mathbb{H} \cup \mathbb{R} : |z| \geq 1 + 2r\}$, which implies that $|\varphi_H(z) - z| = |\varphi_H^{-1}(\varphi_H(z)) - \varphi_H(z)| \leq r^2$. □

Lemma 6.3 Let K be a hull in \mathbb{S}_π such that $\text{Re } z \leq c$ for any $z \in K$. Suppose that $V : (\mathbb{S}_\pi \setminus K; +\infty) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; +\infty)$, and satisfies $V((c, \infty)) \subset \mathbb{R}$ and $V((c, \infty) + \pi i) \subset \mathbb{R}_\pi$. Then there is $h \in \mathbb{R}$ such that if $z \in \mathbb{S}_\pi \cup \mathbb{R} \cup \mathbb{R}_\pi$ and $\text{Re } z \geq c + \ln(4)$, then $|V(z) - z - h| \leq 12e^{c-\text{Re } z}$.

Proof There is $h \in \mathbb{R}$ such that $V(z) = z + h + o(1)$ as $z \in \mathbb{S}_\pi$ and $z \rightarrow +\infty$. By considering $V - h$ instead of V , we may assume that $V(z) = z + o(1)$ as $z \in \mathbb{S}_\pi$ and $z \rightarrow +\infty$. Let $z \in \mathbb{S}_\pi \cup \mathbb{R} \cup \mathbb{R}_\pi$ with $\text{Re } z \geq c + \ln(4)$. Let $a = \text{Re } z - c$. Then $e^a \geq 4$, and there is $r \in (0, 1]$ such that $(1+r)^2/r = e^a$. Let $H = \frac{r}{c} \exp(K)$ and

$$W(z) = \exp(V(\ln z - \ln(r) + c) + \ln(r) - c). \tag{6.2}$$

Then $H \in \mathcal{H}$, $H \subset \{z \in \mathbb{H} : |z| \leq r\}$, $W : (\mathbb{H} \setminus H; \infty) \xrightarrow{\text{Conf}} (\mathbb{H}; \infty)$, and $W'(\infty) = 1$. Since $V((c, \infty)) \subset \mathbb{R}$ and $V((c, \infty) + \pi i) \subset \mathbb{R}_\pi$, we have $W((-\infty, -r)) \subset (-\infty, 0)$ and $W((r, \infty)) \subset (0, \infty)$. Since $z \in \mathbb{S}_\pi \cup \mathbb{R} \cup \mathbb{R}_\pi$ and $\text{Re } z = c + a = c + 2 \ln(1+r) -$

$\ln(r)$, we have $e^{z+\ln(r)-c} \in \mathbb{H} \cup \mathbb{R}$ and $|e^{z+\ln(r)-c}| \geq (1+r)^2$. From Lemma 6.2, we have $|W(e^{z+\ln(r)-c}) - e^{z+\ln(r)-c}| < r^2 + 2r$. So the line segment $[e^{z+\ln(r)-c}, W(e^{z+\ln(r)-c})]$ lies outside \mathbb{D} . Since $|\ln'(z)| \leq 1$ for $z \in \mathbb{C} \setminus \mathbb{D}$, we have $|\ln(W(e^{z+\ln(r)-c})) - (z + \ln(r) - c)| < r^2 + 2r \leq 3r$. From (6.2), $V(z) = \ln(W(e^{z+\ln(r)-c})) - \ln(r) + c$, so $|V(z) - z| < 3r$. Finally, since $e^a r = (r + 1)^2 = r^2 + 2r + 1 \leq 3r + 1$, we have $r \leq \frac{1}{e^a - 3}$. Since $a \geq \ln(4)$, $e^a - 3 \geq 1 \geq 4e^{-a}$. Thus, $|V(z) - z| \leq 12e^{-a} = 12e^{-\operatorname{Re} z}$. \square

Proposition 6.3 *Let $K = K_+ \cup K_- \subset \mathbb{S}_\pi$, where $\operatorname{Re} K_-$ is bounded above by $c_- \in \mathbb{R}$, $\operatorname{Re} K_+$ is bounded below by $c_+ \in \mathbb{R}$, and $c_+ - c_- \geq 2\ln(12)$. Suppose W maps $\mathbb{S}_\pi \setminus K$ conformally onto \mathbb{S}_π , and satisfies $W((c_-, c_+)) \subset \mathbb{R}$ and $W((c_-, c_+) + \pi i) \subset \mathbb{R}_\pi$. Then there exists $h \in \mathbb{R}$ such that if $z \in \mathbb{S}_\pi$ satisfies that $d := \min\{c_+ - \operatorname{Re} z, \operatorname{Re} z - c_-\} \geq \ln(12)$, then $|W(z) - z - h| \leq 48e^{-d}$.*

Proof Choose $W_- : (\mathbb{S}_\pi \setminus K_-; +\infty) \xrightarrow{\operatorname{Conf}} (\mathbb{S}_\pi; +\infty)$. By composing a suitable $U : (\mathbb{S}_\pi; +\infty) \xrightarrow{\operatorname{Conf}} (\mathbb{S}_\pi; +\infty)$ on its left, we may assume that W_- satisfies $W_- \circ W^{-1}(-\infty) = -\infty$. Let $K'_+ = W_-(K_+)$ and $W_+ = W \circ W_-^{-1}$. Then $W_+ : (\mathbb{S}_\pi \setminus K'_+; -\infty) \xrightarrow{\operatorname{Conf}} (\mathbb{S}_\pi; -\infty)$.

The assumption on W implies that there is no $x \in (c_-, \infty)$ such that $W(x) = -\infty$. Since $W_- \circ W^{-1}(-\infty) = -\infty$, there is no $x \in (c_-, \infty)$ such that $W_-(x) = -\infty$. This implies that $W_-((c_-, \infty)) \subset \mathbb{R}$. Similarly, $W_-((c_-, \infty) + \pi i) \subset \mathbb{R}_\pi$. From Lemma 6.3, there is $h_- \in \mathbb{R}$ such that

$$|W_-(z) - z - h_-| \leq 12e^{c_- - \operatorname{Re} z}, \quad \text{if } z \in \mathbb{S}_\pi \cup \mathbb{R} \cup \mathbb{R}_\pi \text{ and } \operatorname{Re} z \geq c_- + \ln(4). \quad (6.3)$$

Let $c_d = c_+ - c_- \geq 2\ln(12)$ and $c'_+ = c_+ + h_- - 12e^{-c_d}$. If $z \in \mathbb{S}_\pi \cup \mathbb{R} \cup \mathbb{R}_\pi$ and $\operatorname{Re} z \geq c_+$, then $\operatorname{Re} z \geq c_- + c_d \geq c_- + \ln(4)$, which implies that $\operatorname{Re} W_-(z) \geq c'_+$ by (6.3). Since $\operatorname{Re} K_+$ is bounded below by c_+ , we find that $\operatorname{Re} K'_+$ is bounded below by c'_+ . Suppose $W_-((c_-, c_+)) = (a_-, a_+)$. The above argument show that $a_+ \geq c'_+$. Since $W = W_+ \circ W_-$ and $W((c_-, c_+)) \subset \mathbb{R}$, we have $W_+((a_-, a_+)) \subset \mathbb{R}$. Since W_+ fixes $-\infty$, we have $W_+((-\infty, a_+)) \subset \mathbb{R}$, which implies that $W_+((-\infty, c_+)) \subset \mathbb{R}$ as $a_+ \geq c'_+$. Similarly, $W_+((-\infty, c_+) + \pi i) \subset \mathbb{R}_\pi$. From a mirror result of Lemma 6.3, we see that there exists $h_+ \in \mathbb{R}$ such that

$$|W_+(w) - w - h_+| \leq 12e^{\operatorname{Re} w - c'_+}, \quad \text{if } w \in \mathbb{S}_\pi \cup \mathbb{R} \cup \mathbb{R}_\pi \text{ and } \operatorname{Re} w \leq c'_+ - \ln(4). \quad (6.4)$$

Now suppose that $z \in \mathbb{S}_\pi$ satisfies that $d := \min\{c_+ - \operatorname{Re} z, \operatorname{Re} z - c_-\} \geq \ln(12)$. From (6.3) and that $c_+ - \operatorname{Re} z, \operatorname{Re} z - c_- \geq d \geq \ln(12)$, we obtain

$$\begin{aligned} \operatorname{Re} W_-(z) &\leq \operatorname{Re} z + h_- + 12e^{c_- - \operatorname{Re} z} \leq c_+ - d + h_- + 12e^{-d} = c'_+ - d + 12e^{-c_d} + 12e^{-d} \\ &\leq c'_+ - d + 12e^{-2\ln(12)} + 12e^{-\ln(12)} = c'_+ - d + \frac{13}{12} \\ &\leq c'_+ - d + \ln(3) \leq c'_+ - \ln(4). \end{aligned} \quad (6.5)$$

Let $h = h_+ + h_-$. Applying (6.4) to $w = W_-(z)$ and using (6.3) and (6.5), we get

$$\begin{aligned} |W(z) - z - h| &\leq |W_+(W_-(z)) - W_-(z) - h_+| + |W_-(z) - z - h_-| \\ &\leq 12e^{\operatorname{Re} W_-(z) - c'_+} + 12e^{c_- - \operatorname{Re} z} < 12e^{\ln(3) - d} + 12e^{-d} = 48e^{-d}. \end{aligned} \quad \square$$

Differentiating (6.1) w.r.t. z , we see that for $z \in \Sigma_H$,

$$\begin{aligned}
 (\varphi_H^{-1})'(z) - 1 &= \int_{c_H}^{d_H} \frac{1}{(z-x)^2} d\mu_H(x); \\
 (\varphi_H^{-1})^{(n)}(z) &= \int_{c_H}^{d_H} \frac{(-1)^{n+1}n!}{(z-x)^{n+1}} d\mu_H(x), \quad n \geq 2.
 \end{aligned}$$

The proofs of Lemma 6.2, Lemma 6.3, and Proposition 6.3 can be slightly modified to prove the following proposition.

Proposition 6.4 *There are constants $C_1, C_2 > 0$ such that the following hold. Let $K = K_+ \cup K_- \subset \mathbb{S}_\pi$, where $\text{Re } K_+$ is bounded above by $c_- \in \mathbb{R}$, $\text{Re } K_-$ is bounded below by $c_+ \in \mathbb{R}$, and $c_+ \geq c_- + 2C_2$. Suppose W maps $\mathbb{S}_\pi \setminus K$ conformally onto \mathbb{S}_π , and satisfies $W((c_-, c_+)) \subset \mathbb{R}$ and $W((c_-, c_+) + \pi i) \subset \mathbb{R}_\pi$. Then for any $z \in \mathbb{S}_\pi$ with $d := \min\{c_+ - \text{Re } z, \text{Re } z - c_-\} \geq C_2$, we have $|W'(z) - 1|, |W''(z)|, |W'''(z)| \leq C_1 e^{-d}$.*

Proof of Proposition 6.1(i) From (5.46), we suffice to show that (i) holds if $\ln(\widehat{N}_m(t)/C_{p,L})$ is replaced by $\ln(\widehat{A}_1(t))$, $\ln(\widehat{A}_{L,m}(t))$, $\ln(\widehat{\Gamma}_q(\widehat{p} + t, \widehat{X}_m(t)))$, $\ln(\widehat{\Gamma}_q(\widehat{p}_L(t), \widehat{X}_{L,m}(t)))$, $\widehat{X}_{L,m}(t) - \widehat{X}_m(t)$, or $\widehat{\mathbf{R}}(\widehat{p}_L(t)) - \widehat{\mathbf{R}}(\widehat{p} + t)$, respectively. Suppose $\mathcal{E}_m \cap \{\widehat{T} = \infty\}$ occurs. From (5.31) and (2.9) we conclude that $\widehat{\mathbf{R}}(\widehat{p}_L(t)) - \widehat{\mathbf{R}}(\widehat{p} + t) \rightarrow 0$ as $t \rightarrow \infty$.

Decompose \widehat{L} into \widehat{L}_l and \widehat{L}_r such that $\widehat{L}_l \cap \mathbb{R}$ (resp. $\widehat{L}_r \cap \mathbb{R}$) lies to the left (resp. right) of \widehat{x}_0 . Let $\widehat{L}_{l,t} = \widehat{g}(t, \widehat{L}_l)$ and $\widehat{L}_{r,t} = \widehat{g}(t, \widehat{L}_r)$, $0 \leq t < T$. Then $\widehat{L}_{l,t} \cap \mathbb{R}$ (resp. $\widehat{L}_{r,t} \cap \mathbb{R}$) lies to the left (resp. right) of $\widehat{\xi}(t)$. Let E_r be a subset of \mathbb{S}_π , which touches both \mathbb{R} and \mathbb{R}_π , disconnects $\widehat{\beta}$ from \widehat{L}_r in \mathbb{S}_π , and is disjoint from \widehat{L} and $\widehat{\beta}$. As $t \rightarrow \infty$, the diameter of $\widehat{\beta}((t, \infty))$ tends to 0, which implies that the extremal distance (cf. [2]) in $\mathbb{S}_\pi \setminus (\widehat{\beta}((0, t]) + 2\widehat{p}\mathbb{Z})$ between E_r and the set

$$S_t := (-\infty, x_0] \cup (\widehat{\beta}((0, t]) - 2\widehat{p}\mathbb{N}) \cup \{\text{the left side of } \widehat{\beta}((0, t])\} \cup \{y + \pi i : y \leq \widehat{y}_m\}$$

tends to ∞ . From (5.36) and conformal invariance, the extremal distance in \mathbb{S}_π between $\widehat{g}(t, E_r)$ and $(-\infty, \widehat{\xi}(t)] \cup \{x + \pi i : x \leq \widehat{q}_m(t)\}$ tends to ∞ as $t \rightarrow \infty$. Since E_r touches both \mathbb{R} and \mathbb{R}_π , $\widehat{g}(t, E_r)$ also has this property. Thus, $\text{dist}(\{\widehat{\xi}(t), \widehat{q}_m(t)\}, \widehat{g}(t, E_r)) \rightarrow \infty$ as $t \rightarrow \infty$. Since E_r disconnects $\widehat{\beta}$ from \widehat{L}_r in \mathbb{S}_π , we see that $\widehat{g}(t, E_r)$ disconnects $\widehat{\xi}(t)$ and $\widehat{q}_m(t) + \pi i$ from $\widehat{L}_{r,t}$. Thus, $\text{dist}(\{\widehat{\xi}(t), \widehat{q}_m(t) + \pi i\}, \widehat{L}_{r,t}) \rightarrow \infty$ as $t \rightarrow \infty$. Similarly, $\text{dist}(\{\widehat{\xi}(t), \widehat{q}_m(t) + \pi i\}, \widehat{L}_{l,t}) \rightarrow \infty$ as $t \rightarrow \infty$. Thus, $\text{dist}(\{\widehat{\xi}(t), \widehat{q}_m(t) + \pi i\}, \widehat{L}_l) \rightarrow \infty$ as $t \rightarrow \infty$. From (5.39), (5.40), and Proposition 6.4 we conclude that $\ln(\widehat{A}_1(t)) \rightarrow 0$ and $\ln(\widehat{A}_{L,m}(t)) \rightarrow 0$ as $t \rightarrow \infty$. From (5.41) and Proposition 6.3, we see that $\widehat{X}_{L,m}(t) - \widehat{X}_m(t) = (\widehat{\xi}_L(t) - \widehat{\xi}(t)) - (\widehat{q}_{L,m}(t) - \widehat{q}_m(t)) \rightarrow 0$ as $t \rightarrow \infty$.

Let $a_r(t) = \min\{\widehat{L}_{r,t} \cap \mathbb{R}\}$ and $a_l(t) = \max\{\widehat{L}_{l,t} \cap \mathbb{R}\}$. From the last paragraph, we see that $a_r(t) - \widehat{\xi}(t)$, $a_r(t) - \widehat{q}_m(t)$, $\widehat{\xi}(t) - a_l(t)$, and $\widehat{q}_m(t) - a_l(t)$ all tend to $+\infty$ as $t \rightarrow \infty$. Since $\widehat{X}_m = \widehat{\xi} - \widehat{q}_m$, we have $a_r(t) - a_l(t) \pm X_m(t) \rightarrow \infty$ as $t \rightarrow \infty$. Since \widehat{L}_t has period $2(\widehat{p} + t)$, we have $a_r(t) - a_l(t) \leq 2(\widehat{p} + t)$. Thus, $2(\widehat{p} + t) - |X_m(t)| \rightarrow \infty$ as $t \rightarrow \infty$. Let $b_1, b_2 \in \mathbb{R}$ be such that $b_1 < \widehat{y}_m < b_2 = b_1 + 2\widehat{p}$. Using (5.37) and an extremal distance argument, we conclude that, as $t \rightarrow \infty$, $\widehat{\xi}_L(t) - \text{Re } \widehat{g}_{L,W}(t, b_1 + \pi i)$, $\widehat{q}_{L,m}(t) - \text{Re } \widehat{g}_{L,W}(t, b_1 + \pi i)$, $\text{Re } \widehat{g}_{L,W}(t, b_2 + \pi i) - \widehat{\xi}_L(t)$, and $\text{Re } \widehat{g}_{L,W}(t, b_2 + \pi i) - \widehat{q}_{L,m}(t)$ all tend to $+\infty$. Since $\widehat{g}_{L,W}$ has progressive period $(2\pi; 2\pi)$, from (5.35), $\widehat{g}_{L,W}(t, \cdot)$ has progressive period $(2\widehat{p}; 2\widehat{p}_L(t))$. So $\text{Re } \widehat{g}_{L,W}(t, b_2 + \pi i) - \text{Re } \widehat{g}_{L,W}(t, b_1 + \pi i) = 2\widehat{p}_L(t)$. Since $\widehat{X}_{L,m} = \widehat{\xi}_L - \widehat{q}_{L,m}$, we conclude that $2\widehat{p}_L(t) - |\widehat{X}_{L,m}(t)| \rightarrow \infty$ as $t \rightarrow \infty$. From Proposition 4.1(i) and (5.31) we see that $\ln(\widehat{\Gamma}_q(\widehat{p} + t, \widehat{X}_m(t)))$ and $\ln(\widehat{\Gamma}_q(\widehat{p}_L(t), \widehat{X}_{L,m}(t)))$ tend to 0 as $t \rightarrow \infty$. □

6.2 Uniformly Boundedness

Now we introduce the notation of convergence of domains in [14]. We have the following definition and proposition.

Definition 6.1 Suppose D_n is a sequence of plane domains and D is a plane domain. We say that (D_n) converges to D , denoted by $D_n \xrightarrow{\text{Cara}} D$, if for every $z \in D$, $\text{dist}(z, \partial D_n) \rightarrow \text{dist}(z, \partial D)$. This is equivalent to the following:

- (i) every compact subset of D is contained in all but finitely many D_n 's;
- (ii) for every point $z_0 \in \partial D$, $\text{dist}(z_0, \partial D_n) \rightarrow 0$ as $n \rightarrow \infty$.

Suppose $D_n \xrightarrow{\text{Cara}} D$, and for each n , f_n is a complex valued function on D_n , and f is a complex valued function on D . We say that f_n converges to f locally uniformly in D , or $f_n \xrightarrow{\text{l.u.}} f$ in D , if for each compact subset F of D , f_n converges to f uniformly on F .

Proposition 6.5 Suppose $D_n \xrightarrow{\text{Cara}} D$, $f_n : D_n \xrightarrow{\text{Conf}} E_n$ for each n , and $f_n \xrightarrow{\text{l.u.}} f$ in D . Then either f is constant on D , or f maps D conformally onto some domain E . And in the latter case, $E_n \xrightarrow{\text{Cara}} E$ and $f_n^{-1} \xrightarrow{\text{l.u.}} f^{-1}$ in E .

Fix $m \in \mathbb{Z}$ and $(\rho_1, \rho_2) \in \mathcal{P}_m$. Choose $(\rho_1^*, \rho_2^*) \in \mathcal{P}_m$ such that $\rho_1^* \cup \rho_2^*$ is disjoint from $(\rho_1 \cup \rho_2) + 2\widehat{p}\mathbb{Z}$, and $\rho_1^* \cup \rho_2^*$ disconnects $\rho_1 \cup \rho_2$ from \widehat{L} in \mathbb{S}_π . Then $\rho_1^*, \rho_1, \rho_2, \rho_2^*$, and $\rho_1^* + 2\widehat{p}$ lie in the order from left to right. Suppose $\rho_j \cap \mathbb{R} = \{a_j\}$, $\rho_j^* \cap \mathbb{R} = \{a_j^*\}$, $\rho_j \cap \mathbb{R}_\pi = \{b_j + \pi i\}$, and $\rho_j^* \cap \mathbb{R}_\pi = \{b_j^* + \pi i\}$, $j = 1, 2$. Then we have $a_1^* < a_1 < \widehat{x}_0 < a_2 < a_2^* < a_1^* + 2\widehat{p}$ and $b_1^* < b_1 < \widehat{y}_m < b_2 < b_2^* < b_1^* + 2\widehat{p}$.

Let $I_\pi(z) = 2\pi - \bar{z}$ denote the reflection about \mathbb{R}_π . Let Σ_{ρ_1, ρ_2} denote the region in $\mathbb{S}_{2\pi}$ bounded by $\rho_2 \cup I_\pi(\rho_2)$ and $(\rho_1 \cup I_\pi(\rho_1)) + 2\widehat{p}$. Fix $\mathring{r} \in (b_2^*, b_1^* + 2\widehat{p})$. Then $\mathring{r} + \pi i \in \Sigma_H$. Let $\mathcal{D}_{\rho_1, \rho_2}$ denote the family of simply connected subdomains of $\mathbb{S}_{2\pi}$ which contain Σ_{ρ_1, ρ_2} and are symmetric about I_π . For each $D \in \mathcal{D}_{\rho_1, \rho_2}$, there is a unique $\mathring{f}_D : (\mathbb{S}_{2\pi}; \mathring{r} + \pi i) \xrightarrow{\text{Conf}} (D; \mathring{r} + \pi i)$ such that $\mathring{f}'_D(\mathring{r} + \pi i) > 0$. Such \mathring{f}_D commutes with I_π . Define a topology on $\mathcal{D}_{\rho_1, \rho_2}$ such that $D_n \rightarrow D_0$ iff $\mathring{f}_{D_n} \xrightarrow{\text{l.u.}} \mathring{f}_{D_0}$ in $\mathbb{S}_{2\pi}$.

Lemma 6.4 Every sequence in $\mathcal{D}_{\rho_1, \rho_2}$ contains a convergent subsequence.

Proof Choose V such that $V : (\mathbb{S}_{2\pi}; \mathring{r} + \pi i) \xrightarrow{\text{Conf}} (\mathbb{D}; 0)$. Let (D_n) be a sequence in $\mathcal{D}_{\rho_1, \rho_2}$. Then $V \circ \mathring{f}_{D_n}$, $n \in \mathbb{N}$, is a family of conformal maps from $\mathbb{S}_{2\pi}$ into \mathbb{D} . Since this family is uniformly bounded, it contains a subsequence $(V \circ \mathring{f}_{D_{n_k}})$ which converges locally uniformly in $\mathbb{S}_{2\pi}$. From Lemma 6.5, this subsequence converges to either a constant function or a conformal map defined on $\mathbb{S}_{2\pi}$. Suppose that the first case happens. Since $V \circ \mathring{f}_{D_n}(\mathring{r} + \pi i) = 0$, the constant is 0. Then we conclude that $\mathring{f}_{D_{n_k}} \xrightarrow{\text{l.u.}} \mathring{r} + \pi i$ in $\mathbb{S}_{2\pi}$, which implies that $\mathring{f}'_{D_{n_k}}(\mathring{r} + \pi i) \rightarrow 0$. Since $\text{dist}(\mathring{r} + \pi i, \partial \mathbb{S}_{2\pi}) = \pi$, from Koebe's 1/4 theorem (cf. [2]), we should have $\text{dist}(\mathring{r} + \pi i, \partial D_{n_k}) \rightarrow 0$, which contradicts that $\text{dist}(\mathring{r} + \pi i, \partial D_{n_k}) \geq \text{dist}(\mathring{r} + \pi i, \mathbb{R} \cup \mathbb{R}_{2\pi} \cup \rho_2 \cup (\rho_1 + 2\widehat{p})) > 0$. Thus, $(V \circ \mathring{f}_{D_{n_k}})$ converges locally uniformly to a conformal map, which implies that $\mathring{f}_{H_{n_k}}$ converges locally uniformly to a conformal map defined on $\mathbb{S}_{2\pi}$, say \mathring{f} . Since $\mathring{f}_{D_{n_k}}$ all map into $\mathbb{S}_{2\pi}$, fix $\mathring{r} + \pi i$, have positive derivative at $\mathring{r} + \pi i$, and commute with I_π , \mathring{f} should also satisfy these

properties. Let $D_0 = \mathring{f}(\mathbb{S}_{2\pi})$. Then D_0 is a simply connected subdomain of $\mathbb{S}_{2\pi}$, contains $\mathring{r} + \pi i$, and is symmetric about \mathbb{R}_π . We suffice to show that $D_0 \supset \Sigma_{\rho_1, \rho_2}$ because if this is true, then $D_0 \in \mathcal{D}_{\rho_1, \rho_2}$ and $\mathring{f} = \mathring{f}_{D_0}$, which implies that $D_{n_k} \rightarrow D_0$. Suppose $D_0 \not\supset \Sigma_{\rho_1, \rho_2}$. Since $\mathring{r} + \pi i \in D_0$, and Σ_{ρ_1, ρ_2} is connected, there exists $z_0 \in \Sigma_{\rho_1, \rho_2} \cap D_0$. From Lemma 6.5 we have $D_{n_k} \xrightarrow{\text{Cara}} D_0$. From Definition 6.1(ii), we see that $\text{dist}(z_0, \partial D_{n_k}) \rightarrow 0$, which contradicts that $z_0 \in \Sigma_{\rho_1, \rho_2} \subset D_{n_k}$ for each k . This finishes the proof. \square

Let $I_0(z) = \bar{z}$ denote the reflection about \mathbb{R} . For $D \in \mathcal{D}_{\rho_1, \rho_2}$, let

$$D^\pm = D \cup I_0(D) \cup (a_2, a_1 + 2\hat{p}).$$

Then D^\pm is a simply connected subdomain of $\mathbb{S}_{2\pi}^\pm := \{-2\pi < \text{Im } z < 2\pi\}$, and is symmetric about \mathbb{R} . Let $\mathring{g}_D = \mathring{f}_D^{-1} : D \xrightarrow{\text{Conf}} \mathbb{S}_{2\pi}$. From Schwarz reflection principle, \mathring{g}_D extends to a conformal map \mathring{g}_D^\pm from D^\pm into $\mathbb{S}_{2\pi}^\pm$, which commutes with I_0 .

Lemma 6.5 *If $D_n \rightarrow D_0$, then $D_n^\pm \xrightarrow{\text{Cara}} D_0^\pm$ and $\mathring{g}_{D_n}^\pm \xrightarrow{\text{l.u.}} \mathring{g}_{D_0}^\pm$ in D_0^\pm .*

Proof From Lemma 6.5 we have $D_n \xrightarrow{\text{Cara}} D_0$ and $\mathring{g}_{D_n} \xrightarrow{\text{l.u.}} \mathring{g}_{D_0}$ in D_0 . Then we easily see that $D_n^\pm \xrightarrow{\text{Cara}} D_0^\pm$. Let (D_{n_k}) be a subsequence of (D_n) . Choose $V : \mathbb{S}_{2\pi}^\pm \xrightarrow{\text{Conf}} \mathbb{D}$. Then $(V \circ \mathring{g}_{D_{n_k}}^\pm)$ is uniformly bounded family, which contains a subsequence $(V \circ \mathring{g}_{D_{n_{k_l}}}^\pm)$ that converges locally uniformly to some function G in D_0^\pm . Since $\mathring{g}_{D_{n_{k_l}}}^\pm \xrightarrow{\text{l.u.}} \mathring{g}_{D_0}^\pm$ in D_0 , we see that G is the analytic extension of $V \circ \mathring{g}_{D_0}$. Thus, $G = V \circ \mathring{g}_{D_0}^\pm$. So we conclude that $\mathring{g}_{D_{n_{k_l}}}^\pm \xrightarrow{\text{l.u.}} \mathring{g}_{D_0}^\pm$ in D_0^\pm . The proof is now finished because every subsequence of $(\mathring{g}_{D_n}^\pm)$ contains a subsequence which converges to $\mathring{g}_{D_0}^\pm$ locally uniformly in D_0^\pm . \square

For each $D \in \mathcal{D}_{\rho_1, \rho_2}$, let $D(\widehat{L})$ be the connected component of $D \setminus (\widehat{L} \cup I_\pi(\widehat{L}))$ that contains $\mathring{r} + \pi i$. Then $D(\widehat{L})$ is a simply connected subdomain of $\mathbb{S}_{2\pi}$, and is symmetric about \mathbb{R}_π . There is a unique $\mathring{g}_{D, \widehat{L}}$ such that $\mathring{g}_{D, \widehat{L}} : (D(\widehat{L}); \mathring{r} + \pi i) \xrightarrow{\text{Conf}} (\mathbb{S}_{2\pi}; \mathring{r} + \pi i)$ and $\mathring{g}'_{D, \widehat{L}}(\mathring{r} + \pi i) > 0$. Let

$$D^\pm(\widehat{L}) = D(\widehat{L}) \cup I_0(D(\widehat{L})) \cup ((a_2, a_1 + 2\hat{p}) \setminus \widehat{L}).$$

Then $\mathring{g}_{D, \widehat{L}}$ extends to a conformal map $\mathring{g}_{D, \widehat{L}}^\pm$ from $D^\pm(\widehat{L})$ into $\mathbb{S}_{2\pi}^\pm$, which commutes with I_0 . We easily see that $D_n \xrightarrow{\text{Cara}} D_0$ iff $D_n^\pm(\widehat{L}) \xrightarrow{\text{Cara}} D_0^\pm(\widehat{L})$. Using some subsequence argument we can derive the following lemma.

Lemma 6.6 *If $D_n \rightarrow D_0$, then $D_n^\pm(\widehat{L}) \xrightarrow{\text{Cara}} D_0^\pm(\widehat{L})$ and $\mathring{g}_{D_n, \widehat{L}}^\pm \xrightarrow{\text{l.u.}} \mathring{g}_{D_0, \widehat{L}}^\pm$ in $D_0^\pm(\widehat{L})$.*

For each $D \in \mathcal{D}_{\rho_1, \rho_2}$, $\overline{\rho_1^*}$ and $\overline{\rho_2^*}$ are compact subsets of D^\pm and $D^\pm(\widehat{L})$. From Lemma 6.4, Lemma 6.5, and Lemma 6.6 we conclude that, there is a constant $C > 0$ which depends only on $\rho_1, \rho_2, \rho_1^*, \rho_2^*, \widehat{L}, \mathring{r}$ such that, for any $D \in \mathcal{D}_{\rho_1, \rho_2}$ and $z \in \rho_1^* \cup \rho_2^*$, the following quantities:

$$|\mathring{g}_D(z) - z|, \quad |\mathring{g}'_D(z)|, \quad |1/\mathring{g}'_D(z)|, \quad |\mathring{g}_{D, \widehat{L}}(z) - z|, \quad |\mathring{g}'_{D, \widehat{L}}(z)|, \quad |1/\mathring{g}'_{D, \widehat{L}}(z)|,$$

are all bounded above by C .

Fix $t \in [0, \widehat{T}_{\rho_1, \rho_2}]$. Let $D = \mathbb{S}_{2\pi} \setminus (\widehat{\beta}((0, t]) + 2\widehat{p}\mathbb{Z}) \setminus I_\pi(\widehat{\beta}((0, t]) + 2\widehat{p}\mathbb{Z})$. Then $D \in \mathcal{D}_{\rho_1, \rho_2}$. We have $\widehat{g}_D : (\mathbb{S}_\pi \setminus (\widehat{\beta}((0, t]) + 2\widehat{p}\mathbb{Z}); \mathbb{R}_\pi) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi)$. Let $h_1 = \widehat{g}(t, \widehat{r} + \pi i) - (\widehat{r} + \pi i) \in \mathbb{R}$. Since \widehat{g}_D fixes $\widehat{r} + \pi i$, from (5.36) we have $\widehat{g}_D = \widehat{g}(t, \cdot) - h_1$. Similarly, using (5.37) we conclude that $\widehat{g}_{D, \widehat{L}} = \widehat{g}_{L, w}(t, \cdot) - h_2$ for some $h_2 \in \mathbb{R}$. Thus, for any $z \in \rho_1^* \cup \rho_2^*$, the following quantities:

$$\begin{aligned} &|\widehat{g}(t, z) - z - h_1|, \quad |\widehat{g}'(t, z)|, \quad |1/\widehat{g}'(t, z)|, \quad |\widehat{g}_{L, w}(t, z) - z - h_2|, \\ &|\widehat{g}'_{L, w}(t, z)|, \quad |1/\widehat{g}'_{L, w}(t, z)|, \end{aligned}$$

are all bounded above by the C in the last paragraph. Let $h = h_2 - h_1$ and $C' = \max\{2C, C^2\}$. From (5.38), we find that,

$$|\widehat{W}(t, z) - z - h| \leq C', \quad 1/C' \leq |\widehat{W}'(t, z)| \leq C', \quad z \in \widehat{g}(t, \rho_1^*) \cup \widehat{g}(t, \rho_2^*). \tag{6.6}$$

Proof of Proposition 6.1(ii) From (5.46), we suffice to show that (ii) holds if $\ln(\widehat{N}_m(t))$ is replaced by $\ln(\widehat{A}_1(t))$, $\ln(\widehat{A}_{l, m}(t))$, $\ln(\widehat{\Gamma}_q(\widehat{p} + t, \widehat{X}_m(t)))$, $\ln(\widehat{\Gamma}_q(\widehat{p}_L(t), \widehat{X}_{L, m}(t)))$, $\widehat{X}_{L, m}(t) - \widehat{X}_m(t)$, or $\widehat{\mathbf{R}}(\widehat{p}_L(t)) - \widehat{\mathbf{R}}(\widehat{p} + t)$, respectively. From (2.9) and (5.31) we see that $\widehat{\mathbf{R}}(\widehat{p}_L(t))$ and $\widehat{\mathbf{R}}(\widehat{p} + t)$ are both positive and bounded above by $\widehat{\mathbf{R}}(\widehat{p})$, which is a uniform constant. So $\widehat{\mathbf{R}}(\widehat{p}_L(t)) - \widehat{\mathbf{R}}(\widehat{p} + t)$ is uniformly bounded.

Fix $(\rho_1, \rho_2) \in \mathcal{P}_m$ and $t \in [0, \widehat{T}_{\rho_1, \rho_2}]$. Then (6.6) holds. From Schwarz reflection principle, $\widehat{W}(t, \cdot)$ extends conformally to a conformal map on $\Sigma := \mathbb{C} \setminus (\widehat{L}_t \cup I_0(\widehat{L}_t) + 2\pi i\mathbb{Z})$, and the extended map commutes with both I_0 and I_π . Thus, $\widehat{W}(t, \cdot)$ has progressive period $(2\pi i; 2\pi i)$. So $\widehat{W}'(t, \cdot)$, $1/\widehat{W}'(t, \cdot)$, and $\widehat{W}(t, \cdot) - \cdot$ are all analytic functions with period $2\pi i$. Let

$$\rho_{j, t}^* = (\widehat{g}(t, \rho_j^*) \cup I_0(\widehat{g}(t, \rho_j^*))) + 2\pi i\mathbb{Z}, \quad j = 1, 2.$$

Then $\rho_{1, t}^*$ and $\rho_{2, t}^*$ are two disjoint simple curves with period $2\pi i$, which lie inside Σ , and (6.6) holds for any $z \in \rho_{1, t}^* \cup \rho_{2, t}^*$. Since $\widehat{\xi}(t)$ and $\widehat{q}_m(t) + \pi i$ lie inside the region bounded by $\rho_{1, t}^*$ and $\rho_{2, t}^*$, from Maximum Principle, (5.39), and (5.41) we have

$$|\widehat{\xi}_L(t) - \widehat{\xi}(t) - h|, |\widehat{q}_{L, m}(t) - \widehat{q}_m(t) - h| \leq C', \quad 1/C' \leq \widehat{A}_1(t), \widehat{A}_{l, m}(t) \leq C'.$$

Since $\widehat{X}_m = \widehat{\xi} - \widehat{q}_m$ and $\widehat{X}_{L, m} = \widehat{\xi}_L - \widehat{q}_{L, m}$, we have $|\widehat{X}_{L, m}(t) - \widehat{X}_m(t)| \leq 2C'$. Thus, the lemma holds if $\ln(\widehat{N}_m(t))$ is replaced by $\ln(\widehat{A}_1(t))$, $\ln(\widehat{A}_{l, m}(t))$, or $\widehat{X}_{L, m}(t) - \widehat{X}_m(t)$.

We know that ρ_1^* , ρ_2^* , and $\rho_1^* + 2\widehat{p}$ are pairwise disjoint, and lie in the order from left to right. Since $\widehat{g}(\widehat{r}, \cdot)$ has progressive period $(2\pi; 2\pi)$, from (5.35), $\widehat{g}(t, \cdot)$ has progressive period $(2\widehat{p}; 2\widehat{p} + t)$. Thus, $\widehat{g}(t, \rho_1^*)$, $\widehat{g}(t, \rho_2^*)$, and $\widehat{g}(t, \rho_1^*) + 2(\widehat{p} + t)$ are pairwise disjoint, and lie in the order from left to right. Since $\widehat{\xi}(t)$ and $\widehat{q}_m(t) + \pi i$ are bounded by $\widehat{g}(t, \rho_1^*)$ and $\widehat{g}(t, \rho_2^*)$ in \mathbb{S}_π , they are also bounded by $\widehat{g}(t, \rho_1^*)$ and $\widehat{g}(t, \rho_1^*) + 2(\widehat{p} + t)$ in \mathbb{S}_π . Thus, $|\widehat{X}_m(t)| = |\widehat{\xi}(t) - \widehat{q}_m(t)|$ is bounded above by $2(\widehat{p} + t) + \text{diam}(\widehat{g}(t, \rho_1^*))$. Since $|\widehat{g}'(t, z)| \leq C$ on ρ_1^* , $\text{diam}(\widehat{g}(t, \rho_1^*)) \leq C \text{diam}(\rho_1^*)$. Thus, $|\widehat{X}_m(t)| - 2(\widehat{p} + t)$ is bounded above by a uniform constant. From Proposition 4.1(ii) we see that the lemma holds if $\ln(\widehat{N}_m(t))$ is replaced by $\ln(\widehat{\Gamma}_q(\widehat{p} + t, \widehat{X}_m(t)))$. Similarly, $|\widehat{X}_{L, m}(t)| - 2\widehat{p}_L(t)$ is bounded above by a uniform constant, which implies that the lemma holds if $\ln(\widehat{N}_m(t))$ is replaced by $\ln(\widehat{\Gamma}_q(\widehat{p}_L(t), \widehat{X}_{L, m}(t)))$. \square

7 Restriction

7.1 Brownian Loop Measure

Lemma 7.1 *Let $p_0 > 0$ and L_0 be a hull in \mathbb{A}_{p_0} w.r.t. \mathbb{T}_{p_0} . Let $\widetilde{L}_0 = (e^i)^{-1}(L_0)$. Suppose that $p_1 = \text{mod}(\mathbb{A}_{p_0} \setminus L_0) \in (0, p_0)$, $W_0 : (\mathbb{A}_{p_0} \setminus L_0; \mathbb{T}_{p_0}) \xrightarrow{\text{Conf}} (\mathbb{A}_{p_1}; \mathbb{R}_{p_1})$, and $\widetilde{W}_0 : (\mathbb{S}_{p_0} \setminus$*

$\tilde{L}_0; \mathbb{R}_{p_0} \xrightarrow{\text{Conf}} (\mathbb{S}_{p_1}; \mathbb{R}_{p_1})$, and $e^i \circ \tilde{W}_0 = W_0 \circ e^i$. Let $x_0 \in \mathbb{R}$ be such that $\text{dist}(e^{ix_0}, L_0) > 0$. Let $S\tilde{W}_0(x_0)$ denote the Schwarz derivative of \tilde{W}_0 at x_0 . Let $\mu_{e^{ix_0}}$ denote the Brownian bubble measure in \mathbb{A}_{p-t} rooted at e^{ix_0} . Let \mathcal{E}_{L_0} denote the set of curves that intersect L_0 . Then

$$\mu_{e^{ix_0}}[\mathcal{E}_{L_0}] = -\frac{1}{6}S\tilde{W}_0(x_0) + \frac{1}{2}\tilde{W}'_0(x_0)^2\left(\mathbf{r}(p_1) + \frac{1}{p_1}\right) - \frac{1}{2}\left(\mathbf{r}(p_0) + \frac{1}{p_0}\right).$$

Proof Let $z_0 \in \mathbb{S}_{p_0}$. The bubble measure $\mu_{e^{iz_0}}$ equals $\lim_{z_0 \rightarrow x_0} \frac{\mathbb{P}_{z_0;x_0}}{|z_0-x_0|^2}$, where $\mathbb{P}_{z_0;x_0}$ is the distribution of a planar Brownian motion started from e^{iz_0} conditioned to exit \mathbb{A}_{p_0} from e^{ix_0} . Choose $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_0 < x_2 < x_1 + 2\pi$. Then $\mathbb{P}_{z_0;x_0}$ equals the limit of $\mathbb{P}_{z_0;(x_1,x_2)}$ as $x_1, x_2 \rightarrow x_0$, where $\mathbb{P}_{z_0;(x_1,x_2)} := \mathbb{P}_{z_0}[\cdot | \mathcal{E}_{x_1,x_2}]$, \mathbb{P}_{z_0} is the distribution of a planar Brownian motion started from e^{iz_0} , and \mathcal{E}_{x_1,x_2} denotes the event that the curve ends at the arc $e^i((x_1, x_2))$.

Since the Poisson kernel function in \mathbb{A}_{p_0} with the pole at $e^{ix} \in \mathbb{T}$ is $z \mapsto \frac{1}{2\pi}(\text{Re}\mathbf{S}(p_0, z/e^{ix}) + \frac{\ln|z|}{p_0})$, we get

$$\mathbb{P}_{z_0}[\mathcal{E}_{x_1,x_2}] = -\frac{1}{2\pi} \int_{x_1}^{x_2} \text{Im}\left(\mathbf{H}(p_0, z_0 - x) + \frac{z_0}{p_0}\right) dx. \tag{7.1}$$

From conformal invariance of planar Brownian motions, $\mathbb{P}_{z_0}[\mathcal{E}_{x_1,x_2} \setminus \mathcal{E}_{L_0}]$ is equal to the probability of a planar Brownian motion started from $W_0(e^{iz_0}) = e^i(\tilde{W}_0(z_0))$ hits $\partial\mathbb{A}_{p_1}$ at the arc $W_0(e^i((x_1, x_2))) = e^i((\tilde{W}_0(x_1), \tilde{W}_0(x_2)))$. From (7.1) and change of variables, we get

$$\mathbb{P}_{z_0}[\mathcal{E}_{x_1,x_2} \setminus \mathcal{E}_{L_0}] = -\frac{1}{2\pi} \int_{x_1}^{x_2} \text{Im}\left(\mathbf{H}(p_1, \tilde{W}_0(z_0) - \tilde{W}_0(x)) + \frac{\tilde{W}_0(z_0)}{p_1}\right) \tilde{W}'_0(x) dx.$$

Then we get an expression for $\mathbb{P}_{z_0;x_1,x_2}[\mathcal{E}_{L_0}] = \mathbb{P}_{z_0}[\mathcal{E}_{L_0} | \mathcal{E}_{x_1,x_2}]$. Letting $x_1, x_2 \rightarrow x_0$, we get

$$\mathbb{P}_{z_0;x_0}[\mathcal{E}_{L_0}] = 1 - \frac{\tilde{W}'_0(x_0) \text{Im}(\mathbf{H}(p_1, \tilde{W}_0(z_0) - \tilde{W}_0(x_0)) + \frac{\tilde{W}_0(z_0) - \tilde{W}_0(x_0)}{p_1})}{\text{Im}(\mathbf{H}(p_0, z_0 - x_0) + \frac{z_0 - x_0}{p_0})}.$$

Finally we compute $\lim_{z_0 \rightarrow x_0} \frac{\mathbb{P}_{z_0;x_0}[\mathcal{E}_{L_0}]}{|z_0-x_0|^2}$. The proof is completed by some tedious but straightforward computation involving power series expansions. □

Lemma 7.2 *For the $U(t)$ defined in (5.29), we have $\mu_{\text{loop}}[\mathcal{L}_{L,t}] = U(t)$, $0 \leq t \leq T$, where $\mathcal{L}_{L,t}$ denotes the set of loops in \mathbb{A}_p that intersect both L and $\beta((0, t))$.*

Proof For $0 \leq t < T$, let μ_t denote the Brownian bubble measure in \mathbb{A}_{p-t} rooted at $e^{i\xi(t)}$. The argument in [7] shows that $\mu_{\text{loop}}[\mathcal{L}_{L,t}] = \int_0^t \mu_s[\{\cdot \cap L_s \neq \emptyset\}] ds$ for $0 \leq t \leq T$. From (5.3), (5.8) and the previous lemma, we have

$$\begin{aligned} \mu_s[\{\cdot \cap L_s \neq \emptyset\}] &= -\frac{1}{6}A_s(s) + \frac{1}{2}A_1(s)^2\left(\mathbf{r}(p_L - v(s)) + \frac{1}{p_L - v(s)}\right) \\ &\quad - \frac{1}{2}\left(\mathbf{r}(p - s) + \frac{1}{p - s}\right). \end{aligned}$$

The proof can now be completed by integrating the right-hand side of this formula from 0 to t and using (5.10) and that $v(0) = 0$. □

Lemma 7.3 *Let $m \in \mathbb{Z}$.*

- (i) *On the event $\mathcal{E}_m \cap \{\beta \cap L = \emptyset\}$, $U(p)$ is finite.*
- (ii) *For any $(\rho_1, \rho_2) \in \tilde{\mathcal{P}}_m$, $U(t)$ is uniformly bounded on $[0, \tilde{T}_{\rho_1, \rho_2}]$.*

Proof From [7], if two sets in \mathbb{C} have positive distance from each other, then the Brownian loop measure of the loops that intersect both of them is finite. (i) If \mathcal{E}_m occurs and $\beta \cap L = \emptyset$, then $\text{dist}(L, \beta((0, p))) > 0$. From Lemma 7.2 and the above observation, $U(p) = \mu_{\text{loop}}[\mathcal{L}_{L,p}]$ is finite. (ii) Let $\mathcal{L}_{L, \rho_1, \rho_2}$ denote the set of loops in \mathbb{A}_p that intersect both L and $\rho_1 \cup \rho_2$. Since $\text{dist}(L, \rho_1 \cup \rho_2) > 0$, we have $\mu_{\text{loop}}[\mathcal{L}_{L, \rho_1, \rho_2}] < \infty$. If $t < T_{\rho_1, \rho_2}$, then $\rho_1 \cup \rho_2$ disconnects $\beta((0, t])$ from L , which means that a loop in \mathbb{A}_p that intersects both L and $\beta((0, t])$ must also intersect $\rho_1 \cup \rho_2$. So $\mathcal{L}_{L,t} \subset \mathcal{L}_{L, \rho_1, \rho_2}$. Thus, $U(t), 0 \leq t < T$, is bounded above by $\mu_{\text{loop}}[\mathcal{L}_{L, \rho_1, \rho_2}]$. □

7.2 Radon-Nikodym Derivatives

Let $s \in \mathbb{R}$ and $m \in \mathbb{Z}$. Consider the following two SDEs:

$$d\xi(t) = \sqrt{\kappa} dB(t) + \left(3 - \frac{\kappa}{2}\right) \frac{A_2(t)}{A_1(t)} dt + A_1(t) \Lambda_{(s)}(p_L - v(t), X_{L,0}(t)) dt, \quad 0 \leq t < T; \tag{7.2}$$

$$d\xi(t) = \sqrt{\kappa} dB(t) + \left(3 - \frac{\kappa}{2}\right) \frac{A_2(t)}{A_1(t)} dt + A_1(t) \Lambda_0(p_L - v(t), X_{L,m}(t)) dt, \quad 0 \leq t < T. \tag{7.3}$$

Let the distribution of $(\xi(t), 0 \leq t < T)$ be denoted by $\mu_{L,(s)}$ or $\mu_{L,m}$, respectively, if $(\xi(t), 0 \leq t < T)$, is the maximal solution of (7.2) or (7.3), respectively, and $\xi(0) = x_0$.

Suppose that (ξ) has distribution $\mu_{L,m}$. From (5.1), (5.7), (5.9) and (5.19), we get

$$d\xi_L(t) = A_1(t) \sqrt{\kappa} dB(t) + A_1(t)^2 \Lambda_0(p_L - v(t), \xi_L(t) - \text{Re } \tilde{g}_L(t, \tilde{W}_L(y_m + pi))) dt, \quad 0 \leq t < T.$$

Since $\xi_L(t) = \eta_L(v(t))$ and $\tilde{g}_L(t, \cdot) = \tilde{h}_L(v(t), \cdot)$, from (5.10) and (5.8) we conclude that there is another Brownian motion $B_v(t)$ such that

$$d\eta_L(t) = \sqrt{\kappa} dB_v(t) + \Lambda_0(p_L - t, \eta_L(t) - \text{Re } \tilde{h}_L(t, \tilde{W}_L(y_m + pi))) dt, \quad 0 \leq t < v(T).$$

Recall that \tilde{h}_L and $\tilde{\gamma}_L$ are the covering annulus Loewner maps and trace of modulus p_L driven by η_L . Thus, $\tilde{\gamma}_L(t), 0 \leq t < v(T)$, is a covering annulus $\text{SLE}(\kappa; \Lambda_0)$ trace in \mathbb{S}_{p_L} started from $\tilde{W}_L(\xi(0))$ with marked point $\tilde{W}_L(y_m + pi)$, stopped at $v(T)$.

There are two possibilities. Case 1: $v(T) = p_L$. Then $\tilde{\gamma}_L(t), 0 \leq t < v(T)$ is a complete covering annulus $\text{SLE}(\kappa; \Lambda_0)$ trace. From the last paragraph of Sect. 4.2 we know that a.s. $\lim_{t \rightarrow v(T)^-} \tilde{\gamma}_L(t) = \tilde{W}_L(y_m + pi)$. Since $\tilde{\gamma}_L(t) = \tilde{\beta}_L(v(t)) = \tilde{W}_L(\beta(v(t)))$, we have $T = p$ and $\lim_{t \rightarrow T^-} \tilde{\beta}(t) = y_m + pi$, which means that the event \mathcal{E}_m occurs. Case 2: $v(T) < p_L$. Then $\lim_{t \rightarrow v(T)^-} \tilde{\gamma}_L(t)$ exists and lie in \mathbb{S}_{p_L} , which implies that $\lim_{t \rightarrow T^-} \tilde{\beta}(t)$ exists and lie in $\mathbb{S}_p \setminus \tilde{L}$. This means that the solution $\xi(t), 0 \leq t < T$, can be further extended, which is a contradiction. So only Case 1 can happen, which implies that $\mu_{L,m}(\{T = p\} \cap \mathcal{E}_m) = 1$.

Similarly, if $(\xi(t))$ has the distribution $\mu_{L,(s)}$, then a.s. $v(T) = p_L, \tilde{\gamma}_L(t), 0 \leq t < v(T)$, is a complete covering annulus $\text{SLE}(\kappa; \Lambda_{(s)})$ trace in \mathbb{S}_{p_L} started from $\tilde{W}_L(\xi(0))$ with marked point $\tilde{W}_L(y_0 + pi)$, and $\lim_{t \rightarrow v(T)^-} \tilde{\gamma}_L(t)$ exists and belongs to $y_0 + pi + 2\pi\mathbb{Z}$. Thus,

$\mu_{L,(s)}(\{T = p\}) = 1$ and $\mu_{L,(s)}(\bigcup_{m \in \mathbb{Z}} \mathcal{E}_m) = 1$. Since $X_{L,m}(0) = \tilde{W}_L(\xi(0)) - \text{Re } \tilde{W}_L(y_m + pi)$, from (4.22) we have

$$\frac{d\mu_{L,m}}{d\mu_{L,(s)}} = e^{\frac{2\pi}{\kappa}ms} \frac{\Gamma_0(p_L, X_{L,m}(0))}{\Gamma_{(s)}(p_L, X_{L,0}(0))} \mathbf{1}_{\mathcal{E}_m}. \tag{7.4}$$

Suppose $(\xi(t))$ has the distribution $\mu_{L,(s)}$. Since γ_L is the trace driven by η_L , the above argument shows that, $\gamma_L(t), 0 \leq t < v(T)$, is a complete annulus SLE($\kappa; \Lambda_{(s)}$) trace in \mathbb{A}_{p_L} started from $e^i \circ \tilde{W}_L(\xi(0)) = W_L(e^{ix_0})$ with marked point $e^i \circ \tilde{W}_L(y_0 + pi) = W_L(e^{iy_0-p})$. Since $W_L : (\mathbb{A}_p \setminus L; \beta(v^{-1}(t))) \xrightarrow{\text{Conf}} (\mathbb{A}_{p_L}; \gamma_L(t))$, we see that $\beta(v^{-1}(t)), 0 \leq t < v(T)$, is an annulus SLE($\kappa; \Lambda_{(s)}$) trace in $\mathbb{A}_p \setminus L$ started from e^{ix_0} with marked point e^{iy_0-p} .

The process $(M_m(t))$ defined earlier will be used to derive the Radon-Nikodym derivative between the $\mu_{L,m}$ defined here and the μ_m defined as the distribution of the solution of (4.19). Suppose that $(\xi(t))$ has distribution μ_m . Then $\xi(t), 0 \leq t < p$, solves the SDE:

$$d\xi(t) = \sqrt{\kappa} dB(t) + \Lambda_0(p - t, X_m(t)) dt, \quad 0 \leq t < p, \quad \xi(0) = x_0. \tag{7.5}$$

From (5.27) we see that $M_m(t), 0 \leq t < T$, is a local martingale under μ_m .

Let $(\rho_1, \rho_2) \in \tilde{\mathcal{P}}_m$. From Proposition 6.2(ii), Lemma 7.3(ii), and $M_m = N_m \exp(cU)$, we see that $M_m(t)$ is uniformly bounded on $[0, T_{\rho_1, \rho_2})$. Thus, $M_m(t \wedge T_{\rho_1, \rho_2})$ is a bounded martingale, and we have $\mathbf{E}_{\mu_m}[M_m(T_{\rho_1, \rho_2})] = M_m(0)$. If we now change the distribution of $(\xi(t))$ from μ_m to a new probability measure ν defined by $d\nu/d\mu_m = M_m(T_{\rho_1, \rho_2})/M_m(0)$, then from Girsanov's Theorem and (5.27) we see that the current $\xi(t)$ satisfies SDE (7.3) for $0 \leq t < T_{\rho_1, \rho_2}$. Thus, on the event $\{T_{\rho_1, \rho_2} = p\}$, $\mu_{L,m} \ll \mu_m$, and the Radon-Nikodym derivative between the two measures restricted to the event $\{T_{\rho_1, \rho_2} = p\}$ is $M_m(p)/M_m(0)$. From Proposition 6.2(i), Lemma 7.3(i), (5.30) and $M_m = N_m \exp(cU)$, we see that $M_m(p) = C_{p,L} \exp(cU(p))$. So

$$d\mu_{L,m}/d\mu_m = C_{p,L} \exp(cU(p))/M_m(0) \quad \text{on } \{T_{\rho_1, \rho_2} = p\}. \tag{7.6}$$

Suppose \mathcal{E}_m occurs and $T = p$. Then $\tilde{\beta} \cap \tilde{L} = \emptyset$. Since $\tilde{\beta}$ starts from x_0 , we can find $(\rho_1, \rho_2) \in \tilde{\mathcal{P}}_m$ such that $\tilde{\beta} \cap (\rho_1 \cup \rho_2) = \emptyset$, which implies that $T_{\rho_1, \rho_2} = p$. Thus,

$$\mathcal{E}_m \cap \{T = p\} \subset \bigcup_{(\rho_1, \rho_2) \in \tilde{\mathcal{P}}_m} \{T_{\rho_1, \rho_2} = p\}. \tag{7.7}$$

Since $\mu_m(\mathcal{E}_m) = 1$ and \mathcal{P}_m is countable, from (7.6) and (7.7) we see that $d\mu_{L,m}/d\mu_m = C_{p,L} \exp(cU(p))/M_m(0)$ on $\{T = p\}$. Since $\mu_{L,m}(\{T = p\}) = 1$, from (5.30) and Lemma 7.2 we know that

$$\frac{d\mu_{L,m}}{d\mu_m} = \frac{C_{p,L}}{M_m(0)} \mathbf{1}_{\{\tilde{\beta} \cap \tilde{L} = \emptyset\}} \exp(c\mu_{\text{loop}}[\mathcal{L}_{L,p}]), \tag{7.8}$$

where $\mathcal{L}_{L,p}$ is the set of loops in \mathbb{A}_p that intersect both L and $\beta((0, p))$.

Let $s \in \mathbb{R}$. Now we compare $\mu_{(s)}$ with $\mu_{L,(s)}$. Define

$$Y_{(s)}(t) = \Gamma_{(s)}(p - t, X_0(t)), \quad Y_{L,(s)}(t) = \Gamma_{(s)}(p_L - v(t), X_{L,0}(t)).$$

Define $M_{(s)}$ using (5.26) with Y_m and $Y_{L,m}$ replaced by $Y_{(s)}$ and $Y_{L,(s)}$, respectively, and $A_{I,m}$ replaced by $A_{I,0}$.

Since $\tilde{g}(t, \cdot)$ has progressive period $(2\pi; 2\pi)$, from (5.6) we have $q_m(t) = q_0(t) + 2m\pi$. Since $\tilde{W}(t, \cdot)$ has progressive period $(2\pi; 2\pi)$, from (5.8) we have $A_{I,m} = A_{I,0}$. Thus,

$$\frac{M_m(0)}{M_{(s)}(0)} = \frac{\Gamma_0(p_L, X_{L,m}(0))/\Gamma_0(p, X_m(0))}{\Gamma_{(s)}(p_L, X_{L,0}(0))/\Gamma_{(s)}(p, X_0(0))}.$$

Since $X_m(0) = x_0 - y_m$, from (4.22), (7.4), (7.8) and the above formula, we get

$$\frac{d\mu_{L,(s)}}{d\mu_{(s)}} = \frac{C_{p,L}}{M_{(s)}(0)} \mathbf{1}_{\{\beta \cap L = \emptyset\}} \exp(c\mu_{\text{loop}}[\mathcal{L}_{L,p}]).$$

Recall that when $(\xi(t))$ has distribution $\mu_{(s)}$, $\beta(t)$, $0 \leq t < p$, is an annulus SLE($\kappa; \Lambda_{(s)}$) trace in \mathbb{A}_p started from $z_0 = e^{ix_0}$ with marked point $w_0 = e^{iy_0-p}$. When $(\xi(t))$ has distribution $\mu_{L,(s)}$, a time change of β : $\beta(v^{-1}(t))$, $0 \leq t < v^{-1}(T)$, is an annulus SLE($\kappa; \Lambda_{(s)}$) trace in $\mathbb{A}_p \setminus L$ started from z_0 with marked point w_0 . This finishes the proof of Theorem 1.1 with $Z = \frac{C_{p,L}}{M_{(s)}(0)}$.

8 Other Results

8.1 Restriction in a Simply Connected Subdomain

We now give a sketch of the proof of Theorem 1.2. Let $p > 0$, $\kappa \in (0, 4]$, $s \in \mathbb{R}$ $z_0 \in \mathbb{T}$, $w_0 \in \mathbb{T}_p$, and the set L be as in Theorem 1.1. Choose $x_0, y_0 \in \mathbb{R}$ such that $z_0 = e^{ix_0}$ and $w_0 = e^{iy_0-p}$. Let $y_m = y_0 + 2m\pi$, $m \in \mathbb{Z}$. Let $\tilde{L} = (e^i)^{-1}(L)$. Then $\mathbb{S}_p \setminus \tilde{L}$ is a disjoint union of simply connected domains \tilde{D}_m , $m \in \mathbb{Z}$, such that $\tilde{D}_m = \tilde{D}_0 + 2m\pi$ for $m \in \mathbb{Z}$. We label one of the domains \tilde{D}_0 such that $x_0 \in \partial\tilde{D}_0$. There is a unique $m_0 \in \mathbb{Z}$ such that $y_{m_0} + pi \in \partial\tilde{D}_0$. We have $e^i : \tilde{D}_0 \xrightarrow{\text{Conf}} \mathbb{A}_p \setminus L$. Let J_0 be the component of $\mathbb{T}_p \setminus \tilde{L}$ that contains w_0 . We may find W_L such that $W_L : (\mathbb{A}_p \setminus L; J_0) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi)$. Let $\tilde{W}_L = W_L \circ e^i$, and \tilde{J}_0 be a component of $\mathbb{R}_p \setminus \tilde{L}$ that contains $y_{m_0} + pi$. Then $\tilde{W}_L : (\tilde{D}_0; \tilde{J}_0) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi)$.

Let $\xi(t)$, $g(t, \cdot)$, $\tilde{g}(t, \cdot)$, $\beta(t)$, $\tilde{\beta}(t)$, $0 \leq t < p$, and $T \in (0, p]$ be as in Sect. 5.1. Now we define $\tilde{\beta}_L(t) = W_L(\beta(t)) = \tilde{W}_L(\tilde{\beta}(t))$, $0 \leq t < T$. Then $\tilde{\beta}_L$ is a simple curve with $\tilde{\beta}(0) \in \mathbb{R}$ and $\tilde{\beta}((0, p)) \subset \mathbb{S}_\pi$. Let $v(t)$ be the capacity of $\tilde{\beta}_L((0, t])$ in \mathbb{S}_π w.r.t. \mathbb{R}_π for $0 \leq t < T$. Let $S = \sup v([0, T])$, and $\tilde{\gamma}_L(t) = \tilde{\beta}_L(v^{-1}(t))$, $0 \leq t < S$. Then $\tilde{\gamma}_L$ is the strip Loewner trace driven by some $\eta_L \in C([0, S])$.

Let $\tilde{h}_L(t, \cdot)$, $0 \leq t < S$, be the strip Loewner maps driven by η_L . Define $\xi_L(t) = \eta_L(v(t))$ and $\tilde{g}_L(t, \cdot) = \tilde{h}_L(v(t), \cdot)$. Define $\tilde{g}_{L,w}(t, \cdot)$ and $\tilde{W}(t, \cdot)$ using (5.1). Then (5.2) and (5.3) hold with $p_L - v(t)$ replaced by π . From (3.9) we see that (5.4) and (5.5) hold.

For $m \in \mathbb{Z}$, define $q_m(t)$ and $q_{L,m}(t)$ using (5.6) and (5.7) with $p_L - v(t)$ replaced by π . Define $A_j(t)$ and $A_{L,m}(t)$ using (5.8). Define $X_m(t)$ and $X_{L,m}(t)$ using (5.9). A standard argument shows that (5.10) holds here. So (5.11) holds with $\mathbf{H}(p_L - v(t), \cdot)$ replaced by coth_2 . Now (5.12) and (5.13) still hold here. From (3.7) and (3.8) we see that (5.14), (5.15) and (5.16) hold here with $\mathbf{H}_I(p_L - v(t), \cdot)$ replaced by \tanh_2 .

By differentiating $\tilde{W}(t, \cdot) \circ \tilde{g}(t, z) = \tilde{g}_{L,w}(t, z)$ w.r.t. t and z , and letting $w = \tilde{g}(t, z) \rightarrow \xi(t)$, we conclude that (5.17) holds here, and (5.18) holds with $\mathbf{r}(p_L - v(t))$ replaced by $\frac{1}{\delta}$, which comes from the power series expansion: $\text{coth}_2(z) = \frac{2}{z} + \frac{z}{6} + O(z^2)$ when z is near 0. Then (5.19) and (5.20) still hold here; (5.21) holds with $\mathbf{H}_I(p_L - v(t), \cdot)$ replaced by \tanh_2 ; and (5.22) should be modified with $\frac{1}{\delta}$ in place of $\mathbf{r}(p_L - v(t))$.

Define $Y_m(t)$ using (5.23), but define $Y_{L,m}(t) := \widehat{\Gamma}_\infty(v(t), X_{L,m}(t))$. Since Γ_0 solves (4.11) and $\widehat{\Gamma}_\infty$ solves (4.17), using (5.10), (5.20), and the modified (5.16) and (5.21) we find that (5.24) still holds, and (5.25) holds with $\mathbf{H}_I(p_L - v(t), \cdot)$ and $\Lambda_0(p_L - v(t), \cdot)$ replaced by \tanh_2 and $\kappa\widehat{\Gamma}_\infty/\widehat{\Gamma}_\infty = (\frac{\kappa}{2} - 3)\tanh_2$, respectively.

Define M_m using (5.26) with $\alpha \int_{p-}^{p_L-v(\cdot)} \mathbf{r}(s) ds$ replaced by $\alpha \int_{p-}^p \mathbf{r}(s) ds - \frac{\alpha}{\delta} v(\cdot)$. Using (5.10), (5.19), (5.24), and the modified (5.16), (5.21), (5.22), and (5.25), we find

that (5.27) holds here with $\Lambda_0(p_L - v(t), \cdot)$ replaced by $(\frac{\kappa}{2} - 3) \tanh_2$. We may write $M_m = N_m \exp(cU)$, where

$$N_m = A_1^\alpha A_{l,m}^\alpha \frac{Y_{L,m}}{Y_m} \exp\left(\left(\alpha + \frac{c}{2}\right) \left(\int_{p_-}^p \left(\mathbf{r}(s) + \frac{1}{s}\right) ds - \frac{v}{6}\right) - \alpha \int_{p_-}^p \frac{1}{s} ds\right);$$

$$U = -\frac{1}{6} \int_0^{\widehat{T}} A_S(s) ds + \frac{1}{12} v - \frac{1}{2} \int_{p_-}^p \left(\mathbf{r}(s) + \frac{1}{s}\right) ds.$$

To get estimations on $N_m(t)$, we do some rescaling. Let $\widehat{p}, \widehat{T}, \check{i}, \widehat{x}_0, \widehat{y}_m$, and $\widehat{\beta}$ be as defined in the first paragraph of Sect. 5.2. Then (5.30) holds here. From (2.10), we see that (5.32) holds if $\widehat{p}_L(t)$ is replaced by \widehat{p} and $p_L - v(\check{t})$ is replaced by p . Define $\widehat{\xi}(t), \widehat{q}_m(t)$, and $\widehat{X}_m(t)$ using (5.33); define $\widehat{\xi}_L, \widehat{q}_{L,m}$, and $\widehat{X}_{L,m}$ using (5.34) with the factors $\frac{\widehat{p}_L(t)}{\pi}$ removed. Define $\widehat{g}(t, \cdot)$ and $\widehat{g}_{L,W}(t, \cdot)$ using (5.35) with the factor $\frac{\widehat{p}_L(t)}{\pi}$ removed. Then (5.36) holds here and (5.37) holds if “ $\mathbb{S}_\pi \setminus ((\widehat{\beta}((0, t]) + 2\widehat{p}\mathbb{Z}) \cup \widehat{L}); \mathbb{R}_\pi$ ” is replaced by “ $\widehat{D}_0 \setminus \widehat{\beta}((0, t]); \widehat{J}_0$ ”, where $\widehat{D}_0 := \frac{\widehat{p}}{\pi} \widehat{D}_0$ and $\widehat{J}_0 := \frac{\widehat{p}}{\pi} \widehat{J}_0$. Define $\widehat{W}(t, \cdot), \widehat{A}_1(t)$, and $\widehat{A}_{l,m}(t)$ using (5.38) and (5.39). Then (5.41) still holds, (5.42) holds with $\widehat{p}_L(t)$ replaced by π , and (5.40) should be replaced by $\widehat{W}(t, \cdot) : (\widehat{D}_{0,t}; \widehat{J}_{0,t}) \xrightarrow{\text{Conf}} (\mathbb{S}_\pi; \mathbb{R}_\pi)$, where $\widehat{D}_{0,t} := \widehat{g}(t, \widehat{D}_0)$ and $\widehat{J}_{0,t} := \widehat{g}(t, \widehat{J}_0)$.

Let $\widehat{v}(t) = v(\check{t})$. Define $\widehat{Y}_m(t)$ using (5.43), but define $\widehat{Y}_{L,m}(t) = \widehat{\Gamma}_\infty(\widehat{v}(t), \widehat{X}_{L,m}(t))$. Then (5.44) holds with $\widehat{p}_L(t)$ replaced by π . Define \widehat{N}_m on $[0, \widehat{T})$ such that

$$\widehat{N}_m = \left(\frac{\widehat{p}}{\pi}\right)^\alpha \widehat{A}_1^\alpha \widehat{A}_{l,m}^\alpha \widehat{Y}_{L,m} \widehat{Y}_m^{-1} \exp\left(\left(\alpha + \frac{c}{2}\right) \left(\int_{\widehat{p}}^{\widehat{p}^+} \widehat{\mathbf{r}}(s) ds - \frac{\widehat{v}}{6}\right)\right).$$

From the modified (5.32), (5.42) and (5.44), we find that (5.45) holds here. From (1.1), (2.9), (4.2), (4.12), (4.14), (4.16), and the modified (5.43), we see that

$$\widehat{N}_m = C_p \widehat{A}_1^\alpha \widehat{A}_{l,m}^\alpha \widehat{\Gamma}_q(\widehat{p} + \cdot, \widehat{X}_m)^{-1} \exp\left(-\alpha \int_{\widehat{X}_m}^{\widehat{X}_{L,m}} \tanh_2(s) ds + \left(\alpha + \frac{c}{2}\right) \widehat{\mathbf{R}}(\widehat{p} + \cdot)\right),$$

where $C_p := (\frac{\widehat{p}}{\pi})^\alpha \exp(-(\alpha + \frac{c}{2})(\widehat{\mathbf{R}}(\widehat{p}) + \frac{\widehat{v}}{6}))$.

Let $\mathcal{E}_m, m \in \mathbb{Z}$, be as in Sect. 6. Since $\widehat{\beta}(t)$ stays inside \widehat{D}_0 before time \widehat{T} , we see that $\{\widehat{T} = \infty\} \cap \mathcal{E}_m = \emptyset$ for $m \in \mathbb{Z} \setminus \{m_0\}$. Suppose that $\{\widehat{T} = \infty\} \cap \mathcal{E}_{m_0}$ occurs. An argument using extremal length shows that $\text{dist}(\{\widehat{\xi}(t), \widehat{q}_m(t) + \pi i\}, (\mathbb{S}_\pi \cup \mathbb{R}_\pi) \setminus \widehat{D}_{0,t}) \rightarrow \infty$ as $t \rightarrow \infty$. Applying Proposition 6.3 and Proposition 6.4, we find that Proposition 6.1(i) holds here with $m = m_0$ and $C_{p,L}$ replaced by C_p .

Let \mathcal{P}_m denote the family of pairs of disjoint polygonal crosscuts (ρ_1, ρ_2) in \widehat{D}_0 such that, (i) for $j = 1, 2$, the two end points of ρ_j lie on \mathbb{R} and \mathbb{R}_π , respectively; (ii) for $j = 1, 2$, the line segments of ρ_j are parallel to x or y axes, and all vertices other than the end points have rational coordinates; and (iii) $\text{dist}(\rho_1 \cup \rho_2, \partial \widehat{D}_0) > 0$ and $\rho_1 \cup \rho_2$ disconnect \widehat{x}_0 and $\widehat{y}_m + \pi i$ from $\partial \widehat{D}_0$ in \mathbb{S}_π . For each $(\rho_1, \rho_2) \in \mathcal{P}_m$, define $\widehat{T}_{\rho_1, \rho_2}$ to be the biggest time such that $\widehat{\beta}((0, \widehat{T}_{\rho_1, \rho_2})) \cap (\rho_1 \cup \rho_2) = \emptyset$. Applying Lemma 6.4, Lemma 6.5 and Lemma 6.6 we find that Proposition 6.1(ii) holds here. We define $\widetilde{\mathcal{P}}_m$ as in Sect. 6. Then Proposition 6.2 holds here with $m = m_0$ and $C_{p,L}$ replaced by C_p .

Following the argument of Lemma 7.1 and Lemma 7.2, we can show that $U(t)$ equals the Brownian loop measure of the loops in \mathbb{A}_p that intersect both L and $\beta((0, t))$. Here we use the fact that $-\frac{1}{2\pi} \text{Im} \coth_2(\cdot - x)$ is the Poisson kernel in \mathbb{S}_π with the pole at $x \in \mathbb{R}$.

Let $\mu_{L,m}$ denote the distribution of $(\xi(t))$ if $\xi(t), 0 \leq t < T$, is the maximal solution of (7.3) with $(\frac{\kappa}{2} - 3) \tanh_2$ in place of $\Lambda_0(p_L - v(t), \cdot)$, and satisfies $\xi(0) = x_0$. Suppose $(\xi(t))$ has distribution μ_{L,m_0} . From (5.19) we conclude that $\beta_L(t) = W_L(\beta(t)), 0 \leq t < T$,

is a time-change of strip SLE(κ ; $\kappa - 6$) trace in \mathbb{S}_π started from $W_L(e^{ix_0})$ with marked point $W_L(e^{-p+iy_0})$. Thus, under this distribution, β is a time-change of a chordal SLE(κ) trace in $\mathbb{A}_p \setminus L$ from $z_0 = e^{ix_0}$ to $w_0 = e^{-p+iy_0}$. Let μ_m denote the distribution of the maximal solution of (4.19), or equivalently (7.5). Using the argument in Sect. 7.2, Girsanov’s theorem, and the modified (5.27) and Proposition 6.2 we conclude that for some constant $Z_{m_0} > 0$,

$$\frac{d\mu_{L,m_0}}{d\mu_{m_0}} = \frac{\mathbf{1}_{\beta \cap L = \emptyset}}{Z_{m_0}} \exp(c\mu_{\text{loop}}[\mathcal{L}_{L,p}]). \tag{8.1}$$

Let $s \in \mathbb{R}$. If the distribution of $(\xi(t))$ is the $\mu_{(s)}$ in Sect. 4.2, then β is an annulus SLE(κ ; $\Lambda_{(s)}$) trace in \mathbb{A}_p started from $z_0 = e^{ix_0}$ with marked point $w_0 = e^{-p+iy_0}$. Since $\{\beta \cap L = \emptyset\} \cap \mathcal{E}_m = \emptyset$ for $m \in \mathbb{Z} \setminus \{m_0\}$, from (4.22) we see that (8.1) holds with μ_{m_0} replaced by $\mu_{(s)}$ and Z_{m_0} replaced by some other $Z_{(s)} > 0$. This finishes the sketch of the proof of Theorem 1.2.

8.2 Multiple SLE Crossing an Annulus

Fix $\kappa \in (0, 4]$ and $p > 0$. Let $n \in \mathbb{N}$ and $n \geq 2$. Let z_1, \dots, z_n be n distinct points that lie on \mathbb{T} in the counterclockwise direction. Let w_1, \dots, w_n be n distinct points that lie on \mathbb{T}_p in the counterclockwise direction. Let $\vec{z} = (z_1, \dots, z_n)$ and $\vec{w} = (w_1, \dots, w_n)$. Let \mathcal{G} denote the set of $(\beta_1, \dots, \beta_n)$ such that each β_j is a crosscut in \mathbb{A}_p that connects z_j and w_j , and the n curves are mutually disjoint.

Definition 8.1 A random n -tuple $(\beta_1, \dots, \beta_n)$ with values in \mathcal{G} is called a multiple SLE(κ) in \mathbb{A}_p from \vec{z} to \vec{w} if for any $j \in \{1, \dots, n\}$, conditioned on all other $n - 1$ curves, β_j is a chordal SLE(κ) trace from z_j to w_j that grows in D_j , which is the subregion in \mathbb{A}_p bounded by β_{j-1} and β_{j+1} ($\beta_0 = \beta_n$ and $\beta_{n+1} = \beta_1$) that has z_j and w_j as its boundary points.

Theorem 8.1 Let $s_1, \dots, s_n \in \mathbb{R}$. For $j = 1, \dots, n$, let ν_j denote the distribution of the annulus SLE(κ ; $\Lambda_{\kappa;(s_j)}$) trace in \mathbb{A}_p started from z_j with marked point w_j . Define a joint distribution ν^M of $(\beta_1, \dots, \beta_n)$ by

$$\frac{d\nu^M}{\prod_{j=1}^n \nu_j} = \frac{\mathbf{1}_{\mathcal{E}_{\text{disj}}}}{Z} \exp\left(c \sum_{s=2}^n \mu_{\text{loop}}(\mathcal{L}_{\geq s})\right), \tag{8.2}$$

where $\mathcal{E}_{\text{disj}}$ is the event that β_j , $1 \leq j \leq n$, are mutually disjoint; $\mathcal{L}_{\geq s}$ is the set of loops in \mathbb{A}_p that intersect at least s curves among β_j , $1 \leq j \leq n$; and $Z > 0$ is a constant. Then ν^M is the distribution of a multiple SLE(κ) in \mathbb{A}_p from \vec{z} to \vec{w} .

Proof Suppose for $1 \leq j \leq n$, β_j is a crosscut in \mathbb{A}_p connecting z_j with w_j . Fix $j \in \{1, \dots, n\}$. Let $\mathcal{L}_{\geq s}^{j,1}$ (resp. $\mathcal{L}_{\geq s}^{j,0}$) denotes the set of loops in \mathbb{A}_p that intersect at least s curves among β_k , $k \neq j$, and intersect (resp. do not intersect) β_j . Then $\mathcal{L}_{\geq s} = \mathcal{L}_{\geq s}^{j,0} \cup \mathcal{L}_{\geq s-1}^{j,1}$. Let $\mathcal{L}_{\geq s}^j = \mathcal{L}_{\geq s}^{j,0} \cup \mathcal{L}_{\geq s}^{j,1}$. Then $\mathcal{L}_{\geq s}^j$ depends only on β_k , $k \neq j$. Since $\mathcal{L}_{\geq n}^{j,0} = \emptyset$, we have

$$\sum_{s=2}^n \mu_{\text{loop}}(\mathcal{L}_{\geq s}) = \sum_{s=2}^n \mu_{\text{loop}}(\mathcal{L}_{\geq s}^{j,0}) + \sum_{s=1}^{n-1} \mu_{\text{loop}}(\mathcal{L}_{\geq s}^{j,1}) = \mu_{\text{loop}}(\mathcal{L}_{\geq 1}^{j,1}) + \sum_{s=2}^{n-1} \mu_{\text{loop}}(\mathcal{L}_{\geq s}^j).$$

Let $\mathcal{E}_{\text{disj}}^j$ denote the event that $\beta_k, k \neq j$, are mutually disjoint. When $\mathcal{E}_{\text{disj}}^j$ occurs, let D_j be the simply connected subdomain of \mathbb{A}_p as in Definition 8.1. Let $L_j = \mathbb{A}_p \setminus D_j$. Then $\mathcal{E}_{\text{disj}} = \mathcal{E}_{\text{disj}}^j \cap \{\beta_j \cap L_j = \emptyset\}$. Thus, we may rewrite the right-hand side of (8.2) as

$$\frac{\mathbf{1}_{\mathcal{E}_{\text{disj}}^j} \mathbf{1}_{\{\beta_j \cap L_j = \emptyset\}}}{Z} \exp\left(c \sum_{s=2}^{n-1} \mu_{\text{loop}}(\mathcal{L}_{\geq s}^j) + c \mu_{\text{loop}}(\mathcal{L}_{\geq 1}^{j,1})\right) = C_* \mathbf{1}_{\{\beta_j \cap L_j = \emptyset\}} \exp(c \mu_{\text{loop}}(\mathcal{L}_{\geq 1}^{j,1})),$$

where $C_* = \frac{1}{Z} \mathbf{1}_{\mathcal{E}_{\text{disj}}^j} \exp(c \sum_{s=2}^{n-1} \mu_{\text{loop}}(\mathcal{L}_{\geq s}^j))$ is measurable w.r.t. the σ -algebra generated by $\beta_k, k \neq j$. Let ν_j^M denote the conditional distribution of β_j when $(\beta_1, \dots, \beta_n) \sim \nu^M$ and all β_k other than β_j are given. The above argument shows that the conditional Radon-Nikodym derivative between ν_j^M and ν_j is $C_* \mathbf{1}_{\{\beta_j \cap L_j = \emptyset\}} \exp(c \mu_{\text{loop}}(\mathcal{L}_{\geq 1}^{j,1}))$. Note that $\mathcal{L}_{\geq 1}^{j,1}$ is the set of all loops in \mathbb{A}_p that intersect both β_j and L_j . From Theorem 1.2 we conclude that ν_j^M is the distribution of a time-change of a chordal SLE(κ) trace in $\mathbb{A}_p \setminus L_j = D_j$ from z_j to w_j . □

Choose $x_j, y_j \in \mathbb{R}$ such that $z_j = e^{ix_j}, w_j = e^{iy_j - p}, 1 \leq j \leq n, z_1 < z_2 < \dots < z_n < z_1 + 2\pi$, and $w_1 < w_2 < \dots < w_n < w_1 + 2\pi$. For each $m \in \mathbb{Z}$, let \mathcal{G}_m denote the set of $(\beta_1, \dots, \beta_n) \in \mathcal{G}$ such that for each $j, (e^i)^{-1}(\beta_j)$ has a component that connects x_j with $y_j + 2m\pi + pi$. Then \mathcal{G} is the disjoint union of \mathcal{G}_m 's. Let ν^M be given by Theorem 8.1, and let $\nu_m^M = \nu^M[\cdot | \mathcal{G}_m], m \in \mathbb{Z}$. Then each ν_m^M is also the distribution of a multiple SLE(κ) in \mathbb{A}_p from \vec{z} to \vec{w} , and the same is true for any convex combination of ν_m^M 's. In fact, the converse is also true.

Proposition 8.1 *If ν is the distribution of a multiple SLE(κ) in \mathbb{A}_p from \vec{z} to \vec{w} , then ν is some convex combination of $\nu_m^M, m \in \mathbb{Z}$.*

Proof Define another probability measure ν^* by $\frac{d\nu^*}{d\nu} = \frac{1}{Z} \exp(-c \sum_{s=2}^n \mu_{\text{loop}}(\mathcal{L}_{\geq s}))$, where $Z \in (0, \infty)$ is a normalization constant. From the proof of Theorem 8.1, we see that, if $(\beta_1, \dots, \beta_n) \sim \nu^*$, then for any j , conditioning on the other $n - 1$ curves, β_j has the distribution of an annulus SLE($\kappa; \Lambda_{(s_j)}$) trace in \mathbb{A}_p from z_j to w_j conditioned to avoid other curves.

Let \mathcal{A} denote the set of $(\Omega_1, \dots, \Omega_n)$ such that each Ω_j is a subdomain of \mathbb{A}_p bounded by two crosscuts crossing \mathbb{A}_p , and the Ω_j 's are mutually disjoint. Let \mathcal{S}_{Ω_j} denote the event that the curve stays within Ω_j . Let $\mu = \nu^*[\cdot | \prod_{j=1}^n \mathcal{S}_{\Omega_j}]$. From the property of ν^* , we see that, if $(\beta_1, \dots, \beta_n) \sim \mu$, then for any j , conditioning on the other $n - 1$ curves, β_j has the distribution of an annulus SLE($\kappa; \Lambda_{(s_j)}$) trace in \mathbb{A}_p from z_j to w_j conditioned to stay inside Ω_j . Thus, $\mu = \prod_{j=1}^n \nu_j[\cdot | \mathcal{S}_{\Omega_j}]$. This implies that $\nu^* = C(\Omega_1, \dots, \Omega_n) \prod_{j=1}^n \nu_j$ on $\prod_{j=1}^n \mathcal{S}_{\Omega_j}$ for some positive constant $C(\Omega_1, \dots, \Omega_n)$.

Decompose \mathcal{A} into $\mathcal{A}_m, m \in \mathbb{Z}$, such that \mathcal{A}_m is the set of all $(\Omega_1, \dots, \Omega_n) \in \mathcal{A}$ such that there exists $(\beta_1, \dots, \beta_n) \in \mathcal{G}_m$ with $\beta_j \in \Omega_j, 1 \leq j \leq n$. Fix $m \in \mathbb{Z}$ and $(\Omega_1, \dots, \Omega_n), (\Omega'_1, \dots, \Omega'_n) \in \mathcal{A}_m$. Then $\nu_j(\mathcal{S}_{\Omega_j} \cap \mathcal{S}_{\Omega'_j}) > 0$ for each j . Thus, $\prod_{j=1}^n \mathcal{S}_{\Omega_j} \cap \prod_{j=1}^n \mathcal{S}_{\Omega'_j}$ is a positive event under $\prod \nu_j$. So we must have $C(\Omega_1, \dots, \Omega_n) = C(\Omega'_1, \dots, \Omega'_n)$. This means that the function $C(\Omega_1, \dots, \Omega_n)$ is constant, say C_m , on each \mathcal{A}_m . For $m \in \mathbb{Z}$, we may find countably many $(\Omega_1, \dots, \Omega_n) \in \mathcal{A}_m$ such that the events $\prod_{j=1}^n \mathcal{S}_{\Omega_j}$ cover \mathcal{G}_m . Thus, $\nu^* = C_m \prod_{j=1}^n \nu_j$ on \mathcal{G}_m for each $m \in \mathbb{Z}$, which implies that $\nu[\cdot | \mathcal{G}_m] = \nu_m^M$ for each $m \in \mathbb{Z}$. Since ν is supported by $\mathcal{G} = \bigcup_{m \in \mathbb{Z}} \mathcal{G}_m$, the proof is finished. □

Remarks

1. Theorem 8.1 extends the main result in [3] which states that, if \mathbb{A}_p is replaced by a simply connected domain D , if $z_1, \dots, z_n, w_n, \dots, w_1$ are $2n$ distinct points that lie on ∂D in the counterclockwise direction, if v_j is the distribution of a chordal SLE(κ) trace in D from z_j to w_j , and if $(\beta_1, \dots, \beta_n)$ has joint distribution v^M which is defined by (8.2), then for any $1 \leq j \leq n$, conditioning on the other $n - 1$ curves, β_j is a time-change of a chordal SLE(κ) trace from z_j to w_j that grows in the component of $D \setminus \bigcup_{k \neq j} \beta_k$ whose boundary contains z_j and w_j . In fact, for the $(\beta_1, \dots, \beta_n)$ in Theorem 8.1, if we condition on one of the curves, say β_n , then the conditional joint distribution of the rest of the curves $\beta_1, \dots, \beta_{n-1}$ agrees with the joint distribution given by [3] with $D = \mathbb{A}_p \setminus \beta_1$.
2. Since \mathcal{G}_m 's are mutually disjoint, Proposition 8.1 implies that for each $m \in \mathbb{Z}$, v_m^M does not depend on the choice of s_1, \dots, s_n . In fact, if we define another multiple SLE(κ) distribution $v^{M'}$ using $s'_1, \dots, s'_n \in \mathbb{R}$, then there is a constant $Z > 0$ such that for each $m \in \mathbb{Z}$, $\frac{dv^{M'}}{dv^M} = e^{\frac{2\pi m}{\kappa}(\sum s'_j - \sum s_j)}$ on \mathcal{G}_m . Moreover, since each μ_j satisfies reversibility, we see that v^M and v_m^M should also satisfy reversibility.
3. In the case $n = 2$, if we let the inner circle shrink to 0, it is expected that the two curves tend to the two arms of a two-sided radial SLE(κ). The two-sided radial SLE(κ) ($\kappa \leq 4$) generates two simple curves in \mathbb{D} , which connect 0 with two different points on \mathbb{T} , and intersect only at 0. The union of the two arms can be understood as a chordal SLE(κ) trace connecting the two boundary points, conditioned to pass through 0. Thus, the knowledge on multiple SLE(κ) with $n = 2$ can be used to study the microscopic behavior of an SLE trace near a typical point on the trace.

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