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Current Moments of 1D ASEP by Duality

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Abstract We consider the exponential moments of integrated currents of 1D asymmetric simple exclusion process using the duality found by Schütz. For the ASEP on the infinite lattice we show that the nth moment is reduced to the problem of the ASEP with less than or equal to n particles.

Keywords Asymmetric exclusion process · Bethe ansatz · Duality

1 Introduction

The one dimensional asymmetric simple exclusion process (ASEP) is a many-particle stochastic process in which each particle is an asymmetric random walker but with exclusion interaction among particles [12, 13]. We consider the ASEP with hopping rate p to the right and q to the left with $p+q=1$ and $p \neq q$. The ratio is denoted by $\tau = p/q$.

The ASEP can be defined either on a finite lattice or on an infinite lattice. It is of much current interest to study fluctuation properties of the ASEP on $\mathbb{Z}$ because for this case one can perform detailed analysis using the connection to random matrix theory and other techniques [6, 10, 17, 24]. Until a few years ago the analysis had been restricted to the totally asymmetric case, $p=0$ or $q=0$, i.e., particles hop only in one direction.

In [26], Tracy and Widom succeeded in computing the distribution of the particle position for the ASEP with general parameter values using the transition probability derived from the Bethe ansatz. For recent developments see [11, 25, 27–33].

Recently their formula was utilized to study the height fluctuations of the KPZ equation [1, 3, 18–20]. By the Cole-Hopf transformation, the KPZ equation is mapped to a problem

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of directed polymer. As noted long time ago, the n replica partition function of this directed polymer is mapped to the problem of attractive δ-Bose gas with n particles [8, 9]. This connection has been utilized recently in [2, 4], see also [16]. By considering a generating function of the n replicas, one could reproduce the results for height fluctuations of the KPZ equation. See also a related work [14].

In this note we point out that similar consideration is possible for the ASEP. We show that the nth exponential moments of current in the ASEP can be written as a summation of transition probabilities for k particles with $0 \le k \le n$ of the ASEP, see Proposition 3. From this one could find an expression for current fluctuations of the ASEP. This is the main result of this paper, which is summarized as Theorem 1. Our argument is based on the duality relation of the ASEP found by Schütz [21].

2 Duality

First we consider the ASEP on a finite lattice $[L] = \{1, 2, \dots, L\}$ with reflective boundaries. We employ the formulation using the quantum spin chain language, see e.g. [23]. We set

$$\begin{aligned} s^+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad s^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ s^z &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad n = \frac{1}{2} - s^z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{2.1}$$

Let us introduce a vector $|0\rangle$ which corresponds to the empty system. One can construct an n particle state with particle positions at $x_1, \dots, x_n$ by

$$|x_1, \dots, x_n\rangle = s^-_{x_1} \cdots s^-_{x_n} |0\rangle. \tag{2.2}$$

Here s^-_x means it acts nontrivially only on the space of site x as a 2×2 matrix s^- in (2.1). The state of the system can also be specified by a set of particle numbers η_x on each site $x \in [L]$. Here $\eta_x = 1$ (resp. $\eta_x = 0$) means the site x is occupied (resp. empty). We sometimes abbreviate as $\eta = \{\eta_1, \dots, \eta_L\}$ and denote the corresponding state by $|\eta\rangle$.

Let $P(\eta, t)$ be the probability that the configuration of the system is η at time t and set

$$|P\rangle = \sum_\eta P(\eta, t)|\eta\rangle \tag{2.3}$$

where $\sum_\eta$ means the summation over all particle configuration. The time evolution of this is given by the master equation,

$$\frac{d}{dt}|P\rangle = -H|P\rangle, \tag{2.4}$$

where the transition rate matrix is given by

$$H = -\sum_{j=1}^{L-1} \Big[p s_j^+ s_{j+1}^- + q s_j^- s_{j+1}^+ - p n_j (1 - n_{j+1}) - q(1 - n_j) n_{j+1} \Big]. \tag{2.5}$$

The transition probability, i.e., the probability that n particles starting from $y_1, \dots, y_n$ at time 0 are on sites $x_1, \dots, x_n$ at time t is written as

$$G(x_1, \dots, x_n; t | y_1, \dots, y_n; 0) = \langle x_1, \dots, x_n | e^{-tH} | y_1, \dots, y_n\rangle \tag{2.6}$$

with $\langle x_1,\dots,x_n| = \langle 0|s^+_{x_1}\cdots s^+_{x_n}$. We sometimes abbreviate this as $G(\{x\}_n;t|\{y\}_n;0)$ with the understanding $\{x\}_n=(x_1,\dots,x_n)$.

We recall the duality of the ASEP based on a quantum group symmetry of the process[21]. Our notation in this article is slightly different from [21]. Let us set

$$X^+ = \tau^{-L/4+\frac{1}{2}}\sum_{k=1}^{L}\tau^{\sum_{j=1}^{k-1}(1-n_j)}s_k^-, \tag{2.7}$$

$$X^- = \tau^{-L/4}\sum_{k=1}^{L}s_k^+\tau^{\sum_{j=k+1}^{L}n_j}, \tag{2.8}$$

$$K = \tau^{-\sum_{k=1}^{L}s_k^z}. \tag{2.9}$$

They satisfy the $U_q(sl_2)$ algebra [5, 7, 15],

$$KX^+K^{-1}=\tau X^+, \qquad KX^-K^{-1}=\tau^{-1}X^-, \tag{2.10}$$

$$[X^+,X^-]=\frac{K-K^{-1}}{\tau^{1/2}-\tau^{-1/2}}. \tag{2.11}$$

They commute with H, i.e.,

$$[H,X^{\pm}]=[H,K]=0. \tag{2.12}$$

The last relation reflects the fact that the number of particles is conserved in the dynamics. Since the state space is finite there is the unique stationary state for each particle number $N, 0\le N\le L$. Let us denote it by $|N\rangle$. It satisfies $H|N\rangle=0$. When $N=0$ this is nothing but the state $|0\rangle$ with no particle. For $N\ge 1$, one can construct $|N\rangle$ by applying X^+ for N times to $|0\rangle$ as

$$|N\rangle=(X^+)^N|0\rangle. \tag{2.13}$$

It is easy to check $H|N\rangle=0$ using (2.12) and $H|0\rangle=0$. We also introduce

$$\langle N| = C_N\langle 0|(X^-)^N = \sum_{\eta:\sum_{k=1}^{L}\eta_k=N}\langle\eta| \tag{2.14}$$

where

$$C_N=\tau^{\frac{LN}{4}}\frac{(1-\tau)^N}{(1-\tau)\cdots(1-\tau^N)}. \tag{2.15}$$

Since $\langle N|$ is written as a summation over all possible states with N particles, the normalization of a state $|P\rangle$ is written as $\langle N|P\rangle=1$. Using (2.7), one sees that the normalized version of (2.13) is given by

$$|N\rangle_{\mathrm{norm}}=\tau^{-\frac{N(N+1)}{2}}\frac{(1-\tau)\cdots(1-\tau^N)}{(1-\tau^{L-N+1})\cdots(1-\tau^L)}\sum_{\eta:\sum_{k=1}^{L}\eta_k=N}\tau^{\sum_{k=1}^{L}kn_k}|\eta\rangle. \tag{2.16}$$

For an initial state $|I_N\rangle$ with N particles, the solution to (2.4) is $e^{-Ht}|I_N\rangle$ and the average of a quantity A which depends on η_i's at time t is

$$\langle A\rangle_t=\langle N|Ae^{-Ht}|I_N\rangle. \tag{2.17}$$

Now let us define

$$N_x = \sum_{j=1}^{x} n_j \tag{2.18}$$

and set

$$Q_x = \tau^{N_x}, \tag{2.19}$$

$$\tilde{Q}_x = \frac{Q_x - Q_{x-1}}{\tau - 1} = \tau^{N_{x-1}} n_x. \tag{2.20}$$

One can verify

Lemma 1

$$[(X^-)^N, Q_x] = (\tau^N - 1) Q_x X_x^- (X^-)^{N-1} \tag{2.21}$$

where

$$X_x^- = \tau^{-L/4} \sum_{k=1}^{x} s_k^+ \tau^{\sum_{j=k+1}^{L} n_j}. \tag{2.22}$$

Proof The $N = 1$ case is easily checked by using the following relations

$$\tau^n s^+ = s^+, \qquad s^+ \tau^n = \tau s^+, \qquad \tau^n s^- = \tau s^-, \qquad s^- \tau^n = s^-. \tag{2.23}$$

Next assume (2.21) is true for N. We want to see (2.21) holds for $N + 1$. We start from

$$[(X^-)^{N+1}, Q_x] = [(X^-)^N, Q_x] X^- + (X^-)^N [X^-, Q_x]. \tag{2.24}$$

One uses (2.21) for $N = 1$ and N to get

$$\begin{aligned}
[(X^-)^{N+1}, Q_x] &= (\tau^N - 1) Q_x X_x^- (X^-)^N + (\tau^N - 1)(\tau - 1) Q_x X_x^- (X^-)^{N-1} X_x^- \\
&\quad + (\tau - 1) Q_x (X^-)^N X_x^-.
\end{aligned} \tag{2.25}$$

Comparing this with RHS of (2.21) for $N + 1$, it is enough to show

$$(X^- - X_x^-)(X^-)^{N-1} X_x^- = \tau^N X_x^- (X^-)^{N-1} (X^- - X_x^-). \tag{2.26}$$

To verify this, one can check the $N = 1$ case by using (2.8), (2.23) and then use mathematical induction. □

Applying this lemma, we see, when x_i's are distinct,

$$\langle N | \tilde{Q}_{x_1} \cdots \tilde{Q}_{x_n} = C_{N,n} \langle x_1, \ldots, x_n | (X^-)^{N-n} \tag{2.27}$$

where

$$C_{N,n} = \frac{\tau^{\frac{1}{4}(N-n)L} (1 - \tau)^{N-n}}{(1 - \tau) \cdots (1 - \tau^{N-n})}. \tag{2.28}$$

Notice $C_N = C_{N,0}$. Using this one can show

Proposition 2 [21] *When x_i's are distinct, it holds*

$$\langle \tilde{Q}_{x_1} \cdots \tilde{Q}_{x_n} \rangle_t = \sum_{1 \le y_1 < \cdots < y_n \le L} G(x_1, \ldots, x_n; t | y_1, \ldots, y_n; 0) \langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_n} \rangle_0. \tag{2.29}$$

Proof This is seen as follows:

$$\begin{aligned} \text{LHS} &= \langle N | \tilde{Q}_{x_1} \cdots \tilde{Q}_{x_n} e^{-Ht} | I_N \rangle \\ &= C_{N,n} \langle x_1, \ldots, x_n | (X^-)^{N-n} e^{-Ht} | I_N \rangle \\ &= C_{N,n} \langle x_1, \ldots, x_n | e^{-Ht} (X^-)^{N-n} | I_N \rangle \\ &= \sum_{1 \le y_1 < \cdots < y_n \le L} \langle x_1, \ldots, x_n | e^{-Ht} | y_1, \ldots, y_n \rangle \cdot C_{N,n} \langle y_1, \ldots, y_n | (X^-)^{N-n} | I_N \rangle \\ &= \sum_{1 \le y_1 < \cdots < y_n \le L} \langle x_1, \ldots, x_n | e^{-Ht} | y_1, \ldots, y_n \rangle \langle N | \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_n} | I_N \rangle = \text{RHS}. \end{aligned} \tag{2.30}$$

Here $C_{N,n}$ is the constant appearing in (2.27). In the third equality we used (2.12) and in the forth equality we used the fact that $\sum_{1 \le y_1 < \cdots < y_n \le L} | y_1, \ldots, y_n \rangle \langle y_1, \ldots, y_n |$ acts as an identity in the subspace with n particles. □

This is a generalization of the well known duality for the symmetric simple exclusion process [12]. The computation of k point correlation functions of (2.20) is reduced to the k particle problem.

To study the exponential moments of currents, we need a formula when x_i's are equal. It turns out that the quantity can not be written as a summation of only n-particle transition probability but as a sum of k particle ones for all $k (\le n)$. The result is

Proposition 3

$$\begin{aligned} \langle Q_x^n \rangle_t = \sum_{k=0}^{n} (\tau^n - 1) \cdots (\tau^{n-k+1} - 1) \\ \times \sum_{1 \le x_1 < \cdots < x_k \le x} \sum_{1 \le y_1 < \cdots < y_k \le L} G(\{x\}_k; t | \{y\}_k; 0) \langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0. \end{aligned} \tag{2.31}$$

To derive this we need a few lemmas. One first shows

Lemma 4

$$\langle N | \tilde{Q}_x^2 = (\tau - 1) \sum_{j=1}^{x-1} \langle N | \tilde{Q}_j \tilde{Q}_x + \langle N | \tilde{Q}_x \tag{2.32}$$

Proof First one computes

$$\text{LHS} = C_N \langle 0 | (X^-)^N \tilde{Q}_x^2 = C_N \langle 0 | [(X^-)^N, \tilde{Q}_x^2] = C_N \langle 0 | [(X^-)^N, \tilde{Q}_x] \tilde{Q}_x. \tag{2.33}$$

Using Lemma 1, this becomes

$$C_N \frac{\tau^N - 1}{\tau - 1} \langle 0 | (Q_x X_x^- - Q_{x-1} X_{x-1}^-) (X^-)^{N-1} \tilde{Q}_x. \tag{2.34}$$

Rewriting $(X^-)^{N-1}\tilde{Q}_x = [(X^-)^{N-1}, \tilde{Q}_x] + \tilde{Q}_x(X^-)^{N-1}$, applying Lemma 1 again and using (2.8), it is

$$\tau^{-L/2}\frac{\tau^N-1}{\tau-1}\left\{\frac{\tau^{N-1}-1}{\tau-1}\langle 0|\left((\tau-1)\sum_{j=1}^{x-1}s_j^+s_x^+\right)(X^-)^{N-2}+\langle 0|s_x^+\tilde{Q}_x(X^-)^{N-1}\right\}$$

$$=(\tau-1)C_{N,2}\sum_{j=1}^{x-1}\langle j,x|(X^-)^{N-2}+C_{N,1}\langle x|(X^-)^{N-1}. \tag{2.35}$$

By (2.27) this is the RHS of (2.32). □

Let us define for a fixed x

$$B_n = \langle N|\sum_{1\le x_1<\cdots<x_n\le x}\tilde{Q}_{x_1}\cdots\tilde{Q}_{x_n}. \tag{2.36}$$

Then we have

Lemma 5

$$B_n\sum_{j=1}^{x}\tilde{Q}_j = \frac{1-\tau^{n+1}}{1-\tau}B_{n+1}+\frac{1-\tau^n}{1-\tau}B_n. \tag{2.37}$$

Proof First we see

$$B_n\sum_{j=1}^{x}\tilde{Q}_j = (n+1)B_{n+1}+\sum_{i=1}^{n}B_{n,i} \tag{2.38}$$

where

$$B_{n,i} = \sum_{1\le x_1<\cdots<x_l\le x}\langle N|\tilde{Q}_{x_1}\cdots\tilde{Q}_{x_i}^2\cdots\tilde{Q}_{x_n}. \tag{2.39}$$

This can be rewritten in terms of B_n, B_{n+1} only as

$$B_{n,i} = (\tau^i-1)B_{n+1}+\tau^{i-1}B_n. \tag{2.40}$$

This is seen as follows. Using (2.32), one has

$$B_{n,1} = (\tau-1)B_{n+1}+B_n, \tag{2.41}$$

$$B_{n,i} = (\tau-1)(B_{n,1}+\cdots+B_{n,i-1})+i(\tau-1)B_{n+1}+B_n. \tag{2.42}$$

Suppose (2.40) is correct for $1,2,\ldots,i-1$. Then one gets (2.40) for i by mathematical induction. Plugging (2.40) into (2.38), we get (2.37). □

Now using lemmas 4,5, it is not difficult to show

$$\langle N|Q_x^n = \sum_{k=0}^{n}(\tau^n-1)\cdots(\tau^{n-k+1}-1)\sum_{1\le x_1<\cdots<x_k\le x}\langle N|\tilde{Q}_{x_1}\cdots\tilde{Q}_{x_k} \tag{2.43}$$

by mathematical induction. From this and Proposition 2, we arrive at Proposition 3.

3 Step Markov Initial Condition for the ASEP on $\mathbb{Z}$

To consider the ASEP on $\mathbb{Z}$, we first put the ASEP with reflective boundaries of size $2L+1$ on $\{-L, -L+1, \dots, L-1, L\}$. Then by taking the $L \to \infty$ limit in (2.31), we have, for the ASEP on $\mathbb{Z}$,

$$\langle Q_x^n \rangle_t = \sum_{k=0}^{n} (\tau^n - 1) \cdots (\tau^{n-k+1} - 1) \times \sum_{-\infty < x_1 < \cdots < x_k \le x} \sum_{-\infty < y_1 < \cdots < y_k < \infty} G(\{x\}_k; t | \{y\}_k; 0) \langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0. \tag{3.1}$$

Here $G(\{x\}_k; t|\{y\}_k; 0)$ is the transition probability of the ASEP on $\mathbb{Z}$ and $\tilde{Q}_x$ is defined by (2.19), (2.20) with

$$N_x = \sum_{j=-\infty}^{x} n_j. \tag{3.2}$$

Let us assume $q > p$ from now on. For the summation in (3.1) to converge, we assume in the sequel of the paper that there are no particles far to the left and there are enough many particles far to the right. With this in mind we state the formula as

Proposition 6 *The nth moment of Q_x is written as*

$$\langle Q_x^n \rangle_t = \sum_{k=0}^{n} (\tau^n - 1) \cdots (\tau^{n-k+1} - 1) c_k \tag{3.3}$$

with

$$c_k = \sum_{-\infty < x_1 < \cdots < x_k \le x} \sum_{-\infty < y_1 < \cdots < y_k < \infty} G(\{x\}_k; t|\{y\}_k; 0) \langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0. \tag{3.4}$$

Let $N_t(x)$ be the integrated current at bond between sites x and $x+1$, i.e., the number of particles which hop from $x+1$ to x minus the number of particles which hop from x to $x+1$ up to time t. Notice that when we consider the initial condition such that $\eta_x = 0, x \le 0$, then $N_t(x)$ is the number of particles on sites $\le x$ and hence

$$\langle Q_x^n \rangle_t = \langle \tau^{n N_t(x)} \rangle \tag{3.5}$$

for $x \le 0$. Hence the quantity in Proposition 6 is the same as the exponential moment of the current of the ASEP. For $x > 0$ and more general initial conditions, one has to modify the relation between Q_x and $N_t(x)$ to incorporate the initial configuration of particles.

For the ASEP on $\mathbb{Z}$, the transition probability is written as [22, 26]

$$G(\{x\}_k; t|\{y\}_k; 0) = \sum_{\sigma \in S_k} \int_{C_R} \cdots \int_{C_R} d\xi_1 \cdots d\xi_k A_\sigma \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_i \epsilon(\xi_i) t} \tag{3.6}$$

where S_k is a set of all permutations of order k, $\epsilon(\xi) = p/\xi + q\xi - 1$ and

$$A_\sigma = \operatorname{sgn} \sigma \frac{\prod_{i<j} (p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j} (p + q\xi_i \xi_j - \xi_i)}. \tag{3.7}$$

C_R is a contour enclosing the origin anticlockwise with a radius large enough that all the poles in A_σ are included in C_R. In [26], the contour was taken to be a small one, but one can simply use the transformation $\xi \to 1/\xi$ to switch to a large contour.

Suppose we substitute this representation into (3.3). Taking $R > 1$, one has $|\xi_i| > 1, 1 \le i \le k$ so that the summation over x can be performed as

$$\sum_{-\infty<x_1<\cdots<x_k\le x} \xi_{\sigma(1)}^{x_1} \cdots \xi_{\sigma(k)}^{x_k} = \frac{(\xi_1 \cdots \xi_k)^{x+1}}{(\xi_{\sigma(1)}-1)\cdots(\xi_{\sigma(1)}\cdots\xi_{\sigma(k)}-1)}. \tag{3.8}$$

By using (3.6),(3.8) and a formula given in [26],

$$\sum_{\sigma\in S_k} \mathrm{sgn}\sigma \frac{\prod_{i<j}(p+q\xi_{\sigma(i)}\xi_{\sigma(j)}-\xi_{\sigma(i)})}{(\xi_{\sigma(1)}-1)\cdots(\xi_{\sigma(1)}\cdots\xi_{\sigma(k)}-1)} = (-1)^k q^{\frac{1}{2}k(k-1)} \frac{\prod_{i<j}(\xi_j-\xi_i)}{\prod_i(1-\xi_i)}, \tag{3.9}$$

one arrives at

Theorem 1 *The nth moment of Q_x is written as* (3.3) *with*

$$c_k = (-1)^k q^{\frac{1}{2}k(k-1)} \int_{C_R} \cdots \int_{C_R} d\xi_1 \dots d\xi_k \prod_{i<j} \frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i} \prod_i \frac{\xi_i^x e^{\epsilon(\xi_i)t}}{1-\xi_i}$$
$$\times \sum_{-\infty<y_1<\cdots<y_k<\infty} \frac{\langle \tilde{Q}_{y_1}\cdots\tilde{Q}_{y_k}\rangle_0}{\xi_1^{y_1}\cdots\xi_k^{y_k}}. \tag{3.10}$$

In this expression, the dependence on the initial condition is clearly separated. It is straightforward to check whether the summation over y can be taken for a given initial condition.

From (3.10) one guesses that the distribution function of the current is given by (again for a special case where $\eta_x = 0, x \le 0$ initially)

$$\mathbb{P}[N_t(x) \ge m] = (-1)^m \tau^{\frac{1}{2}m(m-1)} \sum_{k\ge 0} \tau^{(1-m)k} \binom{k-1}{k-m}_\tau c_k, \quad x \le 0 \tag{3.11}$$

where $\binom{N}{n}_\tau$ is τ-binomial coefficient defined as $\binom{N}{n}_\tau = \frac{(1-\tau^N)\cdots(1-\tau^{N-n+1})}{(1-\tau)\cdots(1-\tau^n)}$. In fact following the argument in [31], one sees that (3.11) leads to the expression,

$$\langle e^{\lambda N_t(x)}\rangle = \sum_{k=0}^{\infty} \tau^{-\frac{1}{2}k(k-1)} e^{\lambda k} \prod_{j=0}^{k-1}(1-e^{-\lambda}\tau^j)\cdot c_k. \tag{3.12}$$

When $\lambda = n\log\tau, n \in \{1,2,\dots\}$, the summation over k is terminated at $k = n$ and this reduces to (3.3). Of course to go in the opposite way from (3.3) to (3.12), there is a question of analytic continuation, an infamous problem in replica theory. We do not discuss it here but just mention that it looks natural to expect (3.12) from (3.3).

To further proceed, we need to take the summation in (3.10). As explained in [33], there are not many examples for which this has been done. Here we will give a generalization by observing

Lemma 7 *Suppose* $\langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0$ *has the form,*

$$\langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0 = a_k 1_{y_1 \geq 1} \prod_{i=1}^{k} g_i (y_i - y_{i-1} - 1) \tag{3.13}$$

with the convention $y_0 = 0$. *Here* a_k *does not depend on* y_i *and the function* $g_i, 1 \leq i \leq k$ *is assumed to be such that* $\sum_{y=0}^{\infty} \frac{g_i(y)}{\xi^{y+1}}$ *converges for* $|\xi|$ *large enough. Then*

$$\sum_{1 \leq y_1 < \cdots < y_k < \infty} \frac{\langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0}{\xi_1^{y_1} \cdots \xi_k^{y_k}} = a_k \prod_{i=1}^{k} \sum_{y_i=0}^{\infty} \frac{g_i(y_i)}{(\xi_i \cdots \xi_k)^{y_i+1}}. \tag{3.14}$$

Proof Due to $1_{y_1 \geq 1}$ in (3.13), the summation in (3.10) can be replaced by $\sum_{1 \leq y_1 < y_2 < \cdots < y_k}$. By shifting $y_i \to y_i - 1$, the LHS reads

$$a_k \sum_{y_1=0}^{\infty} \sum_{y_2=y_1+1}^{\infty} \cdots \sum_{y_k=y_{k-1}+1}^{\infty} \frac{g_1(y_1)}{\xi_1^{y_1+1}} \frac{g_2(y_2 - y_1 - 1)}{\xi_2^{y_2+1}} \frac{g_k(y_k - y_{k-1} - 1)}{\xi_k^{y_k+1}}. \tag{3.15}$$

One further makes a change of variable, $y_i \to y_1 + (y_2 + 1) + \cdots (y_i + 1)$, to get the RHS. □

We illustrate Lemma 7, for a non-trivial case. Let us consider the initial condition in which there is a particle at the origin and the particle occupation on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ is described as a Markov process with the 2×2 transition matrix,

$$A = \begin{bmatrix} 1-\rho & 1-\mu \\ \rho & \mu \end{bmatrix} \tag{3.16}$$

where $0 < \rho, \mu < 1$. This means that if the site $x (\geq 1)$ is empty then the site $x+1$ is empty (resp. occupied) with probability $1-\rho$ (resp. ρ) and if the site $x (\geq 1)$ is occupied then the site $x+1$ is empty (resp. occupied) with probability $1-\mu$ (resp. μ). We call this the step Markov initial condition. We remark that the transition matrix A in (3.16) can be interpreted as a transfer matrix of the one-dimensional Ising model with certain coupling constant and magnetic field so that our initial condition can be regarded as the equilibrium measure of the Ising model with infinite negative boundary field on site 0. When $\rho = \mu$, each site $x \geq 1$ is independent and this becomes the step Bernoulli initial conditions [30] except that there is a particle at the origin. When the site x is occupied, let us denote by $p_L(m)$ the probability that there is a particle at site $x + L + 1$ and that there are m particles on sites from $x+1$ up to $x + L$. We set

$$w_L(\zeta) = \sum_{m=0}^{L} p_L(m) \zeta^m. \tag{3.17}$$

Notice, if one defines

$$Z_L(\zeta) = \langle 1| \left(A \begin{bmatrix} 1 & 0 \\ 0 & \zeta \end{bmatrix} \right)^L A|1\rangle, \quad |1\rangle = {}^t(0, 1), \tag{3.18}$$

this is written as $w_L(\zeta) = Z_L(\zeta)/Z_L(1)$.

Let $y_0 = 0, 1 \le y_1 < y_2 < \cdots < y_k$. Suppose there are particles on $y_0 = 0$ and y_i, $1 \le i \le k$ and that there are m_i particles on sites between $y_{i-1} + 1$ and $y_i - 1$, $1 \le i \le k$. This happens with probability

$$\prod_{i=1}^{k} p_{y_i - y_{i-1} - 1}(m_i) \tag{3.19}$$

due to the Markov properties of the measure. We also have

$$\tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} = \prod_{i=1}^{k} \tau^{m_1 + \cdots + m_i + i} = \tau^{\frac{1}{2}k(k+1) + \sum_{i=1}^{k}(k-i+1)m_i}. \tag{3.20}$$

Hence

$$\begin{aligned}\langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0 &= \tau^{\frac{1}{2}k(k+1)} \prod_{i=1}^{k} \sum_{m_i=0}^{y_i - y_{i-1} - 1} p_{y_i - y_{i-1} - 1}(m_i) \tau^{(k-i+1)m_i} \\ &= \tau^{\frac{1}{2}k(k+1)} \prod_{i=1}^{k} w_{y_i - y_{i-1} - 1}(\tau^{k-i+1}).\end{aligned} \tag{3.21}$$

This is exactly the form in Lemma 7 with $g_i(y) = w_y(\tau^{k-i+1})$ and hence

$$\sum_{-\infty < y_1 < \cdots < y_k < \infty} \frac{\langle \tilde{Q}_{y_1} \cdots \tilde{Q}_{y_k} \rangle_0}{\xi_1^{y_1} \cdots \xi_k^{y_k}} = \tau^{\frac{1}{2}k(k+1)} \prod_{i=1}^{k} \sum_{y_i=0}^{\infty} \frac{w_{y_i}(\tau^{k-i+1})}{(\xi_i \cdots \xi_k)^{y_i+1}}. \tag{3.22}$$

Combining Theorem 1 and Lemma 7, we summarize our result as

Theorem 2 *For the step Markov initial condition described around* (3.16), *the nth moment of Q_x is given by*

$$\langle Q_x^n \rangle_t = \sum_{k=0}^{n} (\tau^n - 1) \cdots (\tau^{n-k+1} - 1) c_k \tag{3.23}$$

with

$$\begin{aligned}c_k = (-1)^k q^{\frac{1}{2}k(k-1)} \tau^{\frac{1}{2}k(k+1)} \int_{C_R} \cdots \int_{C_R} d\xi_1 \ldots d\xi_k \prod_{i<j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{\xi_i^x e^{\epsilon(\xi_i)t}}{1 - \xi_i} \\ \times \prod_{i=1}^{k} \sum_{y_i=0}^{\infty} \frac{w_{y_i}(\tau^{k-i+1})}{(\xi_i \cdots \xi_k)^{y_i+1}}.\end{aligned} \tag{3.24}$$

It may still be in general nontrivial to take the summation over y_i in (3.24) explicitly but the point here is that the summations over y_i's are now separated and hence one may be able to do asymptotics using this expression with (3.17), (3.18).

In the parallel way, one can also treat the initial condition in which there is no particle at the origin and the particle occupation on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ is described as a Markov process with the transition matrix (3.16). The only difference is that the first product $p_{y_1-1}(m_1)$ is replaced by $p_{y_1-1}^{(0)}(m_1)$ in (3.19) where $p_L^{(0)}(m)$ is the probability that, when the site x is empty, there is a particle at site $x+L+1$ and that there are m particles on sites from $x+1$ up to $x+L$. We also define $w_L^{(0)}(\zeta)$ accordingly. Then the net change in (3.24) is that one replaces $\tau^{\frac{1}{2}k(k+1)}$ by $\tau^{\frac{1}{2}k(k-1)}$ and $w_{y_1}(\tau^k)$ by $w_{y_1}^{(0)}(\tau^k)$ respectively.

When $\mu=\rho$, the measure (assuming there is no particle at the origin) becomes the step Bernoulli initial conditions. For this case, one has

$$p_L(m)=\rho\binom{L}{m}\rho^m(1-\rho)^{L-m}, \tag{3.25}$$

$$w_L(\zeta)=\sum_{m=0}^{L}p_L(m)\zeta^m=\rho(1-\rho(1-\zeta))^L. \tag{3.26}$$

Applying Lemma 7 with $a_k=\rho^k$, $g_i(y)=(1-\rho(1-\zeta))^y$, one finds

$$\sum_{y_i=0}^{\infty}\frac{g_i(y_i)}{(\xi_i\cdots\xi_k)^{y_i+1}}=\frac{1}{\xi_i\cdots\xi_k-1+\rho(1-\tau^{k-i+1})}. \tag{3.27}$$

The final result is given by

$$\sum_{-\infty<y_1<\cdots<y_k<\infty}\frac{\langle\tilde{Q}_{y_1}\cdots\tilde{Q}_{y_k}\rangle_0}{\xi_1^{y_1}\cdots\xi_k^{y_k}}=\tau^{\frac{1}{2}k(k-1)}\rho^k\prod_{i=1}^{k}\frac{1}{\xi_i\cdots\xi_k-1+\rho(1-\tau^{k-i+1})}. \tag{3.28}$$

This agrees with the expression given in [28].

Next we consider the m-periodic initial conditions in which particles start from the sites $1, m+1, 2m+1, \ldots$ [11]. In this case one has

$$\tilde{Q}_y=1_{\substack{y\geq 1\\ y\equiv 1}}\tau^{\frac{y-1}{m}} \tag{3.29}$$

where $\equiv$ means the modulo m. Hence

$$\sum_{-\infty<y_1<\cdots<y_k<\infty}\frac{\langle\tilde{Q}_{y_1}\cdots\tilde{Q}_{y_k}\rangle_0}{\xi_1^{y_1}\cdots\xi_k^{y_k}}=\sum_{\substack{y_1=1\\ y_1\equiv 1}}^{\infty}\frac{\tau^{\frac{y_1-1}{m}}}{\xi_1^{y_1}}\sum_{\substack{y_2=y_1+m\\ y_2\equiv 1}}^{\infty}\frac{\tau^{\frac{y_2-1}{m}}}{\xi_2^{y_2}}\cdots\sum_{\substack{y_k=y_{k-1}+m\\ y_k\equiv 1}}^{\infty}\frac{\tau^{\frac{y_k-1}{m}}}{\xi_k^{y_k}}. \tag{3.30}$$

This is not exactly the form treated by Lemma 7. But if we make a change of variable, $y_i=m\tilde{y}_i-m+1,\ 1\leq i\leq k$, then

$$\frac{\langle\tilde{Q}_{y_1}\cdots\tilde{Q}_{y_k}\rangle_0}{\xi_1^{y_1}\cdots\xi_k^{y_k}}=\tau^{\frac{1}{2}k(k-1)}1_{\tilde{y}_1\geq 1}\prod_{i=1}^{k}\tau^{(k-i+1)(\tilde{y}_i-\tilde{y}_{i-1}-1)}. \tag{3.31}$$

Now one can apply Lemma 7 with $g_i(y)=\tau^{(k+i-1)y}$ to obtain

$$\sum_{-\infty<y_1<\cdots<y_k<\infty}\frac{\langle\tilde{Q}_{y_1}\cdots\tilde{Q}_{y_k}\rangle_0}{\xi_1^{y_1}\cdots\xi_k^{y_k}}=\tau^{\frac{1}{2}k(k-1)}(\xi_1\cdots\xi_k)^{m-1}\prod_{i=1}^{k}\frac{1}{\xi_i^m\cdots\xi_k^m-\tau^{k-i+1}}. \tag{3.32}$$

This agrees with the expression given in [11, 31].

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