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Some Universal Properties for Restricted Trace Gaussian Orthogonal, Unitary and Symplectic Ensembles

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Abstract Consider fixed and bounded trace Gaussian orthogonal, unitary and symplectic ensembles, closely related to Gaussian ensembles without any constraint. For three restricted trace Gaussian ensembles, we prove universal limits of correlation functions at zero and at the edge of the spectrum edge. Our argument also applies to restricted trace ensembles with monomial potentials. In addition, by using the universal result in the bulk for fixed trace Gaussian unitary ensemble, which has been obtained by Götze and Gordin, we also prove the universal limits of correlation functions everywhere in the bulk for bounded trace Gaussian unitary ensemble.

Keywords Fixed trace ensembles · Bounded trace ensembles · Universality

1 Introduction and Main Results

One important class of random matrix models are Gaussian orthogonal, unitary and symplectic ensembles, denoted respectively by GOE, GUE and GSE. The joint probability density function (p.d.f.) for the eigenvalues $x_1, \ldots, x_N$ of these three canonical ensembles is given [24] by

$$P_{N\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N\beta}} \prod_{1 \le j < k \le N} |x_j - x_k|^\beta \prod_{i=1}^{N} e^{-\beta N x_i^2} \tag{1.1}$$

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where the partition function is

$$Z_{N\beta} = (2\pi)^{\frac{N}{2}} (2\beta N)^{-\frac{N_\beta}{2}} \prod_{j=1}^{N} \frac{\Gamma(1+\frac{\beta j}{2})}{\Gamma(1+\frac{\beta}{2})}. \tag{1.2}$$

Here $N_\beta = N + \beta N(N-1)/2$, and $\beta = 1, 2$ or 4 correspond to GOE, GUE or GSE respectively. The n-point correlation functions for the Gaussian ensembles are defined [12, 24] by

$$R_{n\beta}(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} P_{N\beta}(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N, \tag{1.3}$$

which measures the probability density of finding an eigenvalue (regardless of labeling) around each of the points $x_1, \ldots, x_n$, the positions of the remaining levels being unobserved. In particular, $R_{1\beta}$ gives the overall level density. A classic result shows that the asymptotic normalized eigenvalue density $\frac{1}{N} R_{1\beta}$ (density of states) as $N \to \infty$ is given by the Wigner semicircle law $\omega(x) = \frac{2}{\pi}\sqrt{(1-x^2)_+}$ for $\beta = 1, 2, 4$. However, when $\beta = 1, 2, 4$, for $n \geq 2$, the study of a finer asymptotics presents a pattern (called universality) in the bulk $(-1, 1)$ and at the edge ± 1 of the spectrum.

To present this universal pattern, let us introduce an integral kernel [27, 28, 32]

$$K(x, y) = \frac{\varphi(x)\, \varphi'(y) - \varphi'(x)\, \varphi(y)}{x - y} \tag{1.4}$$

where $\varphi(x)$ is a real-valued function. When one takes $\varphi(x) = \frac{\sin \pi x}{\pi}$ or $\mathrm{Ai}(x)$, $K(x, y)$ in (1.4) will be rewritten as $K_{\mathrm{sine}}(x, y)$ and $K_{\mathrm{Airy}}(x, y)$, respectively. Here $\mathrm{Ai}(x)$ stands for the Airy function satisfying the differential equation

$$\mathrm{Ai}''(x) = x\, \mathrm{Ai}(x),$$

with the initial condition $\mathrm{Ai}(\infty) = 0$. Sine-kernel in the bulk for GOE, GUE and GSE [6, 24, 33] says: for every $u \in (-1, 1)$ and $t_1, \ldots, t_n \in \mathbb{R}^1$,

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{(N\omega(u))^n} R_{n\beta}\left(u + \frac{t_1}{N\omega(u)}, \ldots, u + \frac{t_n}{N\omega(u)}\right) \\ &= \begin{cases} \det[K_{\mathrm{sine}}(t_j, t_k)]_{j,k=1}^n, & \beta = 2 \\ \left(\det[K_{\mathrm{sine}}^{(\beta)}(t_j, t_k)]_{j,k=1}^n\right)^{\frac{1}{2}}, & \beta = 1, 4, \end{cases} \end{aligned} \tag{1.5}$$

where

$$K_{\mathrm{sine}}^{(1)}(x, y) = \begin{pmatrix} K_{\mathrm{sine}}(x-y) & \frac{\partial}{\partial x} K_{\mathrm{sine}}(x-y) \\ \int_0^{x-y} K_{\mathrm{sine}}(t)dt - \frac{1}{2}\mathrm{sgn}(x-y) & K_{\mathrm{sine}}(x-y) \end{pmatrix}, \tag{1.6}$$

and

$$K_{\mathrm{sine}}^{(4)}(x, y) = \begin{pmatrix} K_{\mathrm{sine}}(2(x-y)) & \frac{\partial}{\partial x} K_{\mathrm{sine}}(2(x-y)) \\ \int_0^{x-y} K_{\mathrm{sine}}(2t)dt - \frac{1}{2}\mathrm{sgn}(2(x-y)) & K_{\mathrm{sine}}(2(x-y)) \end{pmatrix}. \tag{1.7}$$

It has been proved that this is also true in other invariant ensembles under the orthogonal, unitary or symplectic groups [5, 8, 10, 25] and in certain ensembles of Hermitian

Wigner matrices [21]. Similarly, Airy-kernel at the edge of spectrum for GOE, GUE and GSE [16, 24, 31] says: for $t_1, \ldots, t_n \in \mathbb{R}^1$,

$$\lim_{N\to\infty} \frac{1}{(2N^{\frac{2}{3}})^n} R_{n\beta}\left(1+\frac{t_1}{2N^{\frac{2}{3}}}, \ldots, 1+\frac{t_n}{2N^{\frac{2}{3}}}\right)$$
$$= \begin{cases} \det[K_{\mathrm{Airy}}(t_j, t_k)]_{j,k=1}^n, & \beta = 2 \\ \left(\det[K_{\mathrm{Airy}}^{(\beta)}(t_j, t_k)]_{j,k=1}^n\right)^{\frac{1}{2}}, & \beta = 1, 4. \end{cases} \tag{1.8}$$

where

$$K_{\mathrm{Airy}}^{(1)} = \begin{pmatrix} (K_{\mathrm{Airy}}^{(1)})_{11} & (K_{\mathrm{Airy}}^{(1)})_{12} \\ (K_{\mathrm{Airy}}^{(1)})_{21} & (K_{\mathrm{Airy}}^{(1)})_{22} \end{pmatrix},$$

$$(K_{\mathrm{Airy}}^{(1)})_{11} = (K_{\mathrm{Airy}}^{(1)})_{22} = K_{\mathrm{Airy}}(x,\ y) + \frac{1}{2}\mathrm{Ai}(x)\left(1 - \int_y^\infty \mathrm{Ai}(z)\,dz\right),$$

$$(K_{\mathrm{Airy}}^{(1)})_{12} = -\partial_y K_{\mathrm{Airy}}(x,\ y) - \frac{1}{2}\mathrm{Ai}(x)\mathrm{Ai}(y), \tag{1.9}$$

$$(K_{\mathrm{Airy}}^{(1)})_{21} = -\int_x^\infty K_{\mathrm{Airy}}(z, y)\,dz + \frac{1}{2}\left(\int_y^x \mathrm{Ai}(z)\,dz + \int_x^\infty \mathrm{Ai}(z)\,dz \cdot \int_y^\infty \mathrm{Ai}(z)\,dz\right)$$
$$- \frac{1}{2}\mathrm{sgn}(x-y),$$

and in the case $\beta = 4$,

$$K_{\mathrm{Airy}}^{(4)} = \frac{1}{2}\begin{pmatrix} (K_{\mathrm{Airy}}^{(4)})_{11} & (K_{\mathrm{Airy}}^{(4)})_{12} \\ (K_{\mathrm{Airy}}^{(4)})_{21} & (K_{\mathrm{Airy}}^{(4)})_{22} \end{pmatrix}, \tag{1.10}$$

$$(K_{\mathrm{Airy}}^{(4)})_{11} = (K_{\mathrm{Airy}}^{(4)})_{22} = K_{\mathrm{Airy}}(x,\ y) - \frac{1}{2}\mathrm{Ai}(x)\int_y^\infty \mathrm{Ai}(z)\,dz,$$

$$(K_{\mathrm{Airy}}^{(4)})_{12} = -\partial_y K_{\mathrm{Airy}}(x,\ y) - \frac{1}{2}\mathrm{Ai}(x)\mathrm{Ai}(y),$$

$$(K_{\mathrm{Airy}}^{(4)})_{21} = -\int_x^\infty K_{\mathrm{Airy}}(z, y)\,dz + \frac{1}{2}\int_x^\infty \mathrm{Ai}(z)dz \int_y^\infty \mathrm{Ai}(z)dz.$$

This Airy-kernel has been proved true in other invariant ensembles [5, 9], and in real symmetric and Hermitian Wigner random matrices [27]. Notice that we have used a quaternion determinant [24] defined by Dyson, after $K_{\mathrm{sine}}^{(1)}$, $K_{\mathrm{sine}}^{(4)}$, $K_{\mathrm{Airy}}^{(1)}$, $K_{\mathrm{Airy}}^{(4)}$, these 2×2 matrices are thought of as quaternions. It is expected that the sine and Airy kernels are rather common for very general ensembles. This is known as the *universality conjecture*. Recently, very relevant results have been obtained for Wigner matrix ensembles. T. Tao and V. Vu [30] showed that local statistical properties of the eigenvalues for Wigner matrices only depend on the first four moments of the elements. Erdős, L., Péché, S., Ramírez, J.A. Schlein, B., Yau, H.-T., [13, 15] proved the bulk universality under subexponential decay. Combining [30] and [13], Erdős-Ramírez-Schlein-Tao-Vu-Yau established universality of the gap distribution and averaged k-point correlations for all Wigner matrices (with subexponentially decaying entries), with no extra assumptions [14].

In the present paper, we will deal with three restricted trace Gaussian ensembles: fixed trace Gaussian ensembles and bounded trace Gaussian ensembles. The aim is to extend the properties (1.8) and (1.5) when $u = 0$ to the fixed and bounded trace GOE, GUE and GSE. First, let us give a review [24, 26] for restricted trace ensembles. Proceeding from the analogy of a fixed energy in classical statistical mechanics, Rosenzweig defines [26] his "fixed trace" ensemble for a Gaussian real symmetric, Hermitian or self-dual matrix H by the requirement that the trace of H^2 be fixed to a number r^2 with no other constraint. The number r is called the strength of the ensemble. The joint probability density function for the matrix elements of H is therefore given by

$$P_r(H) = K_r^{-1}\,\delta\left(\frac{1}{r^2}\operatorname{tr} H^2 - 1\right)$$

where K_r is the normalization constant. Note that this probability density function is invariant under a conjugate action by orthogonal, unitary or symplectic groups, because of the invariance of the quantity $\operatorname{tr} H^2$. Its eigenvalue joint p.d.f. has the form

$$P_{N\beta}^{FT,r}(x_1,\ldots,x_N) = \frac{1}{Z_{N\beta}^{FT,r}}\delta\left(\sum_{i=1}^{N} x_i^2 - r^2\right)\prod_{1\le j<k\le N}|x_j - x_k|^{\beta}, \tag{1.11}$$

where the normalization constant $Z_{N\beta}^{FT,r}$ can be computed by virtue of variable substitution for the partition function $Z_{N\beta}$ [22]:

$$Z_{N\beta}^{FT,r} = r^{N_\beta - 1}\frac{(2\pi)^{\frac{N}{2}}2^{-\frac{N_\beta}{2}+1}}{\Gamma(\frac{N_\beta}{2})}\prod_{j=1}^{N}\frac{\Gamma(1+\frac{\beta j}{2})}{\Gamma(1+\frac{\beta}{2})}. \tag{1.12}$$

Notice the analogy: fixed trace ensemble bears the same relationship to the unconstrained ensemble that the microcanonical ensemble to the canonical ensemble in statistical physics [4]. Instead of keeping the trace constant we might require it to be bounded [7, 24], the eigenvalue joint p.d.f for the bounded trace Gaussian ensembles is given by

$$P_{N\beta}^{BT,r}(x_1,\ldots,x_N) = \frac{1}{Z_{N\beta}^{BT,r}}\theta\left(r^2 - \sum_{i=1}^{N} x_i^2\right)\prod_{1\le j<k\le N}|x_j - x_k|^{\beta}, \tag{1.13}$$

where $\theta(x)$ denotes the Heaviside step function, i.e., $\theta(x) = 1$ for $x \ge 0$, otherwise $\theta(x) = 0$. Proceeding further, G. Akemann et al [1, 2] have considered a generalization of a fixed or bounded Gaussian ensemble up to an arbitrary polynomial potential, and described further interesting physical features of restricted trace Gaussian ensembles due to the interaction among eigenvalues introduced through a constraint. In addition, we also notice that fixed and bounded trace Gaussian ensembles are norm-dependent ensembles. Recently, T. Guhr [19, 20] has extended the supersymmetric integral representations of the more general correlation function from norm dependent random matrix ensembles to arbitrary unitarily invariant matrix ensembles by generalizing the Hubbard–Stratonovich transformation.

The n-point correlation function for fixed trace Gaussian ensembles is defined by

$$R_{n\beta}^{FT,r}(x_1,\ldots,x_n) = \frac{N!}{(N-n)!}\int_{\mathbb{R}^{N-n}} P_{N\beta}^{FT,r}(x_1,\ldots,x_N)\,dx_{n+1}\cdots dx_N. \tag{1.14}$$

More precisely,

$$R_{n\beta}^{FT,r}(x_1,\dots,x_n)=\frac{N!}{(N-n)!}\int_{\Omega_{N-n}} P_{N\beta}^{FT,r}(x_1,\dots,x_N)d\sigma_{N-n} \tag{1.15}$$

where Ω_{N-n} denotes the sphere $\sum_{j=n+1}^{N}x_j^2=r^2-\sum_{j=1}^{n}x_j^2$, and $d\sigma_{N-n}$ denotes the spherical measure. Similarly, the n-point correlation function for the bounded trace Gaussian ensembles is given by

$$R_{n\beta}^{BT,r}(x_1,\dots,x_n)=\frac{N!}{(N-n)!}\int_{\mathbb{R}^{N-n}} P_{N\beta}^{BT,r}(x_1,\dots,x_N)dx_{n+1}\cdots dx_N. \tag{1.16}$$

The expectation of $\sum_{j=1}^{N}x_j^2$ is $\frac{N}{4}+(\frac{1}{2\beta}-\frac{1}{4})\approx N/4$ as $N\to\infty$ (see [24]). Thus we choose the strength $r=\sqrt{N}/2$ for restricted trace Gaussian ensembles and abbreviate $R_{n\beta}^{FT,r}$ and $R_{n\beta}^{BT,r}$ to $R_{n\beta}^{FT}$ and $R_{n\beta}^{BT}$. It is easy to see that the following relation:

$$r^n R_{n\beta}^{FT,r}(rx_1,\dots,rx_n)=R_{n\beta}^{FT,1}(x_1,\dots,x_n). \tag{1.17}$$

The important thing to be noted about fixed trace GOE, GUE and GSE is their moment equivalence with the associated Gaussian ensembles of large dimensions (implying the semicircle law), see Mehta's book [24], Sect. 27.1, p. 488. At the end of this section, p. 490, he writes:

> It is not very clear whether this moment equivalence implies that all local statistical properties of the eigenvalues in two sets of ensembles are identical. This is so because these local properties of eigenvalues may not be expressible only in terms of finite moments of the matrix elements.

In addition, in Sect. 27.3 Mehta also speculated that working out the eigenvalues spacing distribution for bounded trace Gaussian ensembles (the semicircle law is also applicable) is much more difficult. Compared with the classical problem of "equivalence of ensembles" in statistical physics, the problem of equivalence of local statistical properties in random matrix ensembles refers to one rather delicate manifestation of the phenomenon of the equivalence of ensembles. So to prove the equivalence of local properties we need to study a finer "concentration phenomenon" in two sets of ensembles, see [17] for detailed explanation.

However, to our knowledge, only very few results are obtained on the local limit behavior of the correlation functions for fixed and bounded trace Gaussian ensembles. An important breakthrough about local scaling limits of the correlation functions for fixed trace Gaussian ensembles comes from [1, 17, 18]. In [1], universality at zero was proved for a class of fixed trace GUE with monomial potentials (including Gaussian case). In [18], universality at zero is shown for the correlation measure for fixed trace GUE and then extended to the bulk [17] for the correlation functions using a different method from that in [18]. Recently, we have obtained local result at the edge of the density for Gaussian ensembles in [34] and all kinds of universal properties of correlation functions for fixed and bounded trace Laguerre unitary ensemble in [23].

With universality of the since-kernel in the bulk for fixed trace GUE it is meant the following: for every $u\in(-1,1)$ and $t_1,\dots,t_n\in\mathbb{R}^1$,

$$\lim_{N\to\infty}\frac{1}{(N\omega(u))^n}R_{n2}^{FT}\left(u+\frac{t_1}{N\omega(u)},\dots,u+\frac{t_n}{N\omega(u)}\right)=\det[K_{\text{sine}}(t_j,t_k)]_{j,k=1}^{n}. \tag{1.18}$$

In the present paper, we prove the universality of sine-kernel at zero (in the bulk for bounded trace GUE) and Airy-kernel at the edge for fixed and bounded trace GOE, GUE and GSE. All these known results, to some extent, answer the problem: "equivalence of ensembles" posed by Mehta. However, we must emphasize that some (macroscopic) universal property from the unconstrained ensembles might be destroyed when referring to restricted ensembles, for example, G. Akemann et al [3] have discovered non-universality of n-point resolvent ($n \geq 2$) for restricted ensembles.

Let $C_c(\mathbb{R}^n)$ be the set of all continuous functions on $\mathbb{R}^n$ with compact support, now we can state our main results as follows.

Theorem 1 *Let $R_{n\beta}^{FT}$ ($\beta = 1, 2, 4$) be the n-point correlation functions of fixed trace GOE, GUE and GSE, defined by (1.14). For any $f \in C_c(\mathbb{R}^n)$, the following asymptotic properties hold.*

(i) *At zero of the spectrum*:

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \left(\frac{\pi}{2N}\right)^n R_{n\beta}^{FT}\left(\frac{\pi t_1}{2N}, \ldots, \frac{\pi t_n}{2N}\right) d^n t$$
$$= \begin{cases} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{\mathrm{sine}}(t_j, t_k)]_{j,k=1}^n \, d^n t, & \beta = 2, \\ \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \big(\det[K_{\mathrm{sine}}^{(\beta)}(t_j, t_k)]_{j,k=1}^n\big)^{\frac{1}{2}} \, d^n t, & \beta = 1, 4. \end{cases} \tag{1.19}$$

(ii) *The soft edge of the spectrum*:

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \frac{1}{(2N^{2/3})^n} R_{n\beta}^{FT}\left(1 + \frac{t_1}{2N^{2/3}}, \ldots, 1 + \frac{t_n}{2N^{2/3}}\right) d^n t$$
$$= \begin{cases} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{\mathrm{Airy}}(t_j, t_k)]_{j,k=1}^n \, d^n t, & \beta = 2, \\ \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \big(\det[K_{\mathrm{Airy}}^{(\beta)}(t_j, t_k)]_{j,k=1}^n\big)^{\frac{1}{2}} \, d^n t, & \beta = 1, 4. \end{cases} \tag{1.20}$$

Theorem 2 *Let $R_{n\beta}^{BT}$ ($\beta = 1, 2, 4$) be the n-point correlation functions of bounded trace GOE, GUE and GSE, defined by (1.16). For any $f \in C_c(\mathbb{R}^n)$, the following asymptotic properties hold.*

(i) *At zero of the spectrum*:

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \left(\frac{\pi}{2N}\right)^n R_{n\beta}^{BT}\left(\frac{\pi t_1}{2N}, \ldots, \frac{\pi t_n}{2N}\right) d^n t$$
$$= \begin{cases} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{\mathrm{sine}}(t_j, t_k)]_{j,k=1}^n \, d^n t, & \beta = 2, \\ \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \big(\det[K_{\mathrm{sine}}^{(\beta)}(t_j, t_k)]_{j,k=1}^n\big)^{\frac{1}{2}} \, d^n t, & \beta = 1, 4. \end{cases} \tag{1.21}$$

(ii) *The soft edge of the spectrum*:

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \frac{1}{(2N^{2/3})^n} R_{n\beta}^{FT}\left(1 + \frac{t_1}{2N^{2/3}}, \ldots, 1 + \frac{t_n}{2N^{2/3}}\right) d^n t$$
$$= \begin{cases} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{\mathrm{Airy}}(t_j, t_k)]_{j,k=1}^n \, d^n t, & \beta = 2, \\ \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \big(\det[K_{\mathrm{Airy}}^{(\beta)}(t_j, t_k)]_{j,k=1}^n\big)^{\frac{1}{2}} \, d^n t, & \beta = 1, 4. \end{cases} \tag{1.22}$$

(iii) *The bulk of the spectrum for the bounded trace GUE*: *for every* $u \in (-1,1)$ *and* $t_1, \dots, t_n \in \mathbb{R}^1$,

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} f(t_1,\dots,t_n) \frac{1}{(N\omega(u))^n} R_{n2}^{BT}\left(u + \frac{t_1}{N\omega(u)}, \dots, u + \frac{t_n}{N\omega(u)}\right) d^n t$$
$$= \int_{\mathbb{R}^n} f(t_1,\dots,t_n) \det[K_{\text{sine}}(t_j,t_k)]_{j,k=1}^n d^n t. \tag{1.23}$$

The rest of this paper is organized as follows: in Sect. 2, a relation of correlation functions between fixed trace and unconstrained ensembles is formulated, which is the starting point of our arguments. Then a useful lemma (see Lemma 3) is given, which plays a vital role on the proof of Theorem 1. Theorems 1 and 2 will be proved in Sects. 3 and 4, respectively.

Notation We will use the notation $\vec{v}_n$ for the n-dimensional row vector $(v_1,\dots,v_n)$ without further explanation. The abbreviation $d\vec{v}_n$ denotes the Lebesgue measure $dv_1 \cdots dv_n$ on $\mathbb{R}^n$. The symbol $\|\vec{v}_n\|$ is expressed as the l^2 (Euclidean) norm of the vector $\vec{v}_n$. We will use the convention that the C's denote generically bounded constants independent on N, whose values may depend on β and change from line to line.

2 Relation Between Fixed Trace and Unconstrained Ensembles

For the n-point correlation function of the Gaussian ensembles, given by (1.3), and that of the fixed trace Gaussian ensembles defined by (1.14), references [2, 11, 22] give an integral equation

$$R_{n\beta}(x_1,\dots,x_n) = \frac{1}{C_{N\beta}}\left(\frac{N}{\sqrt{2}}\right)^{N_\beta} \int_{\frac{2}{\sqrt{N}}\|\vec{x}_n\|}^{+\infty} e^{-\frac{\beta u^2 N^2}{4}} u^{N_\beta - n - 1} R_{n\beta}^{FT}\left(\frac{x_1}{u},\dots,\frac{x_n}{u}\right) du, \tag{2.1}$$

where

$$C_{N\beta} = \Gamma\left(\frac{N_\beta}{2}\right) 2^{\frac{N_\beta}{2}-1} \beta^{-\frac{N_\beta}{2}}. \tag{2.2}$$

Now we give a new argument by a direct calculation. $\forall f \in C_c(\mathbb{R}^n)$, a transformation to polar coordinates yields

$$\int_{\mathbb{R}^n} f(x_1,\dots,x_n) R_{n\beta}(x_1,\dots,x_n)\, dx_1 \cdots dx_n$$
$$= \frac{1}{Z_{N\beta}} \int_{\mathbb{R}^N} f(x_1,\dots,x_n) \prod_{1\le j<k\le N} |x_j - x_k|^\beta \prod_{i=1}^N e^{-\beta N x_i^2} dx_1 \cdots dx_N$$
$$= \frac{1}{Z_{N\beta}} \int_0^{+\infty} \int_{S^{N-1}} f(uy_1,\cdots,uy_n) \prod_{1\le j<k\le N} |y_j - y_k|^\beta e^{-\beta N u^2} u^{N_\beta - 1} dy_1 \cdots dy_N\, du$$
$$= \frac{Z_{N\beta}^{FT,1}}{Z_{N\beta}} \int_0^\infty \int_{\|\vec{y}_n\|\le 1} f(uy_1,\dots,uy_n) R_{n\beta}^{FT,1}(y_1,\dots,y_n) e^{-\beta N u^2} u^{N_\beta - 1} dy_1 \cdots dy_n\, du$$
$$= \frac{Z_{N\beta}^{FT,1}}{Z_{N\beta}} \int_{\mathbb{R}^n} \int_{\|\vec{x}_n\|}^\infty f(x_1,\dots,x_n) R_{n\beta}^{FT,1}\left(\frac{x_1}{u},\dots,\frac{x_n}{u}\right) e^{-\beta N u^2} u^{N_\beta - n - 1} du\, dx_1 \cdots dx_n. \tag{2.3}$$

Combining (1.2), (1.12) and (1.17), we thus conclude (2.1).

If we replace the square potential x^2 of (1.1) by a monomial potential x^{2p}, then the corresponding fixed trace ensembles should be (1.11) where $\delta(\sum_{i=1}^N x_i^2 - r^2)$ is replaced by $\delta(\sum_{i=1}^N x_i^{2p} - r^2)$. Thus we can get a similar relation between correlation functions:

$$R_{n\beta}(x_1,\dots,x_n) = \int_0^\infty \Phi_{N\beta}(u)\left(\frac{r}{u}\right)^{n/p} R_{n\beta}^{FT}\left(\left(\frac{r}{u}\right)^{\frac{1}{p}}x_1,\dots,\left(\frac{r}{u}\right)^{\frac{1}{p}}x_n\right)du,$$

where

$$\Phi_{N\beta}(u) = u^{N_\beta/p-1}e^{-\beta N u^2}\Big/\int_0^\infty u^{N_\beta/p-1}e^{-\beta N u^2}\,du.$$

Our arguments also apply to the fixed trace and bounded trace ensembles with the monomial potentials, since we mainly exploit some homogeneity of the monomial potentials, see Remark 9. But for simplicity we only deal with the square potential x^2 case. The integral equation (2.1) as a bridge between the two ensembles, is the starting point of our arguments. Now we state a lemma which plays a vital role in the proof of Theorem 1.

Lemma 3 *Let $\{\alpha_N\}$ be a sequence such that $\alpha_N \to 0$ but $\alpha_N N/\sqrt{\ln N} \to \infty$ as $N \to \infty$. Then for any given $\beta > 0$ and nonnegative integer n, as $N \to \infty$, we have*

$$\frac{1}{C_{N\beta}}\int_0^{\frac{N}{\sqrt{2}}(1-\alpha_N)} e^{-\frac{\beta u^2}{2}}u^{N_\beta-n-1}\,du = O(e^{-0.5\beta(N\alpha_N)^2(1+o(1))}) \tag{2.4}$$

and

$$\frac{1}{C_{N\beta}}\int_{\frac{N}{\sqrt{2}}(1+\alpha_N)}^{+\infty} e^{-\frac{\beta u^2}{2}}u^{N_\beta-n-1}\,du = O(e^{-0.5\beta(N\alpha_N)^2(1+o(1))}) \tag{2.5}$$

where $C_{N\beta}$ is given by (2.2).

Proof We will first consider the left hand side of (2.4). It is not difficult to observe that the function $e^{-\beta u^2/2}u^{N_\beta-n-1}$ attains its maximum at $\sqrt{(N_\beta-n-1)/\beta}$ which satisfies

$$\sqrt{\frac{N_\beta-n-1}{\beta}} > \frac{N}{\sqrt{2}}(1-\alpha_N) \tag{2.6}$$

for sufficiently large N. Therefore, using Stirling's formula for the gamma function:

$$\Gamma\left(\frac{N_\beta}{2}\right) = (2\pi)^{1/2}e^{-N_\beta/2}\left(\frac{N_\beta}{2}\right)^{\frac{N_\beta-1}{2}}\left(1+O\left(\frac{1}{N}\right)\right), \tag{2.7}$$

one obtains

$$\begin{aligned}
&\frac{1}{C_{N\beta}}\int_0^{\frac{N}{\sqrt{2}}(1-\alpha_N)} e^{-\frac{\beta u^2}{2}}u^{N_\beta-n-1}\,du \\
&\le \frac{e^{-\frac{\beta N^2}{4}(1-\alpha_N)^2}(\frac{N}{\sqrt{2}}(1-\alpha_N))^{N_\beta-n-1}}{\Gamma(N_\beta/2)2^{N_\beta/2-1}\beta^{-N_\beta/2}}\,\frac{N}{\sqrt{2}}(1-\alpha_N) \\
&\le C\,\frac{e^{-\frac{\beta N^2}{4}(1-\alpha_N)^2}2^{-\frac{N_\beta-n-1}{2}}(1-\alpha_N)^{N_\beta-n-1}}{e^{-N_\beta/2}}\,\frac{(\sqrt{\beta}N)^{N_\beta}}{N_\beta^{N_\beta/2}}N^{1-n}.
\end{aligned} \tag{2.8}$$

Expanding

$$\ln(N_\beta^{N_\beta/2}) = \frac{N_\beta}{2}\left(\ln\left(\frac{\beta}{2}N^2\right) + \ln\left(1 + \left(\frac{2}{\beta} - 1\right)\frac{1}{N}\right)\right)$$

and by a direct calculation, one obtains

$$\frac{(\sqrt{\beta}N)^{N_\beta}}{N_\beta^{N_\beta/2}} = \exp\left(\frac{N_\beta}{2}\ln 2 - \frac{N}{2} + \frac{\beta N}{4} + O(1)\right).$$

Again, expanding

$$\ln(1-\alpha_N) = -\alpha_N - \alpha_N^2/2 + O(\alpha_N^3),$$

thus we obtain the estimate

$$\begin{aligned}(2.8) \le\; & C\exp\left(-\frac{\beta N^2}{4}(1-\alpha_N)^2 - \frac{N_\beta - n - 1}{2}\ln 2 + (N_\beta - n - 1)\ln(1-\alpha_N) + \frac{N_\beta}{2}\right. \\ & \left. + \frac{N_\beta}{2}\ln 2 - \frac{N}{2} + \frac{\beta N}{4} + (1-n)\ln N\right) \\ =\; & C\exp\left(-0.5\beta(N\alpha_N)^2 + \frac{n+1}{2}\ln 2 + (1-n)\ln N\right. \\ & \left. + \left(N\left(1 - \frac{\beta}{2}\right) - n - 1\right)\ln(1-\alpha_N) + O(\alpha_N^3 N^2)\right) \\ =\; & C\exp\big(-0.5\beta(N\alpha_N)^2(1+o(1))\big). \end{aligned} \tag{2.9}$$

The assumption on the sequence α_N leads to (2.4). Following the similar argument, (2.5) can be obtained. Indeed, for sufficiently large N, the function $e^{-u^2\beta/2}u^{N_\beta+1}$ attains its maximum at $\sqrt{(N_\beta+1)/\beta}$ because the inequality $\sqrt{(N_\beta+1)/\beta} < (1+\alpha_N)N/\sqrt{2}$ holds. Thus we have

$$\begin{aligned} & \frac{1}{C_{N\beta}}\int_{\frac{N}{\sqrt{2}}(1+\alpha_N)}^{\infty} e^{-\frac{\beta u^2}{2}} u^{N_\beta+1} u^{-n-2}\,du \\ & \le \frac{e^{-\frac{\beta N^2}{4}(1+\alpha_N)^2}(\frac{N}{\sqrt{2}}(1+\alpha_N))^{N_\beta+1}}{\Gamma(N_\beta/2)2^{N_\beta/2-1}\beta^{-N_\beta/2}} \int_{\frac{N}{\sqrt{2}}(1+\alpha_N)}^{\infty} u^{-n-2}\,du \\ & \le C\exp\left(-0.5\beta(N\alpha_N)^2 - \frac{1}{2}\ln 2 + (1-n)\ln N\right. \\ & \quad \left. + \left(N\left(1-\frac{\beta}{2}\right) + 1\right)\ln(1+\alpha_N) + O(\alpha_N^2 N^3)\right) \\ & = C\exp\big(-0.5\beta(N\alpha_N)^2(1+o(1))\big). \end{aligned} \tag{2.10}$$

This completes the proof of this lemma. □

Remark 4 For the convenience of our arguments, set

$$\Psi_{N\beta}(u) \triangleq \frac{1}{C_{N\beta}}\left(\frac{N}{\sqrt{2}}\right)^{N_\beta} e^{-\frac{\beta u^2 N^2}{4}} u^{N_\beta - 1}. \tag{2.11}$$

Thus

$$\int_0^{1-\alpha_N} \Psi_{N\beta}(u)du = \frac{1}{C_{N\beta}} \int_0^{\frac{N}{\sqrt{2}}(1-\alpha_N)} e^{-\frac{\beta u^2}{2}} u^{N\beta-1}\, du = O(e^{-0.5\beta(N\alpha_N)^2(1+o(1))}), \quad (2.12)$$

$$\int_{1+\alpha_N}^{\infty} \Psi_{N\beta}(u)du = \frac{1}{C_{N\beta}} \int_{\frac{N}{\sqrt{2}}(1+\alpha_N)}^{+\infty} e^{-\frac{\beta u^2}{2}} u^{N\beta-1}\, du = O(e^{-0.5\beta(N\alpha_N)^2(1+o(1))}). \quad (2.13)$$

By Lemma 3, it follows

$$\int_{1-\alpha_N}^{1+\alpha_N} \Psi_{N\beta}(u)du = 1 + O(e^{-0.5\beta(N\alpha_N)^2(1+o(1))}). \quad (2.14)$$

By (2.11), the integral equation (2.1) can be rewritten as follows:

$$R_{n\beta}(\vec{x}_n) = \int_{\frac{2}{\sqrt{N}}\|\vec{x}_n\|}^{+\infty} \Psi_{N\beta}(u)\frac{1}{u^n} R_{n\beta}^{FT}\left(\frac{1}{u}\vec{x}_n\right) du. \quad (2.15)$$

At the end of this section, we give a simple sketch of the proof of Theorem 1. Notice the fact that the right hand side of the above equation can be divided into three parts, i.e.,

$$\left(\int_{\frac{2}{\sqrt{N}}\|\vec{x}_n\|}^{1-\alpha_N} + \int_{1-\alpha_N}^{1+\alpha_N} + \int_{1+\alpha_N}^{\infty}\right) \Psi_{N\beta}(u)\frac{1}{u^n} R_{n\beta}^{FT}\left(\frac{1}{u}\vec{x}_n\right) du. \quad (2.16)$$

Using Lemma 3, the first and third parts will rapidly disappear for the large N. By the integral intermediate value theorem, the middle part of the above identity equals

$$\int_{1-\alpha_N}^{1+\alpha_N} \Psi_{N\beta}(u)\, du\ \frac{1}{(\xi_N(u))^n} R_{n\beta}^{FT}\left(\frac{1}{\xi_N(u)}\vec{x}_n\right) \quad (2.17)$$

where $1-\alpha_N \le \xi_N(u) \le 1+\alpha_N$. It follows from $1/\xi_N(u) = 1 + O(\alpha_N)$ that

$$R_{n\beta}(\vec{x}_n) \sim R_{n\beta}^{FT}((1+O(\alpha_N))\vec{x}_n). \quad (2.18)$$

Let $x_i = \pi t_i/2N,\ 1 \le i \le n$, then the above relation can be rewritten as

$$R_{n\beta}(\pi t_1/2N, \ldots, \pi t_1/2N) \sim R_{n\beta}^{FT}\Big((1+O(\alpha_N))\pi t_1/2N, \ldots, (1+O(\alpha_N))\pi t_1/2N\Big). \quad (2.19)$$

It is not difficult to see that $(1+O(\alpha_N))\pi t_1/2N = \pi t_1/2N + O(\alpha_N/N)$. The fact that $\alpha_N \to 0$ as N goes to infinity means that $\alpha_N/N = o(1/N)$. Therefore the term $O(\alpha_N/N)$ is a small perturbation, comparing with $\pi t_1/2N$, and the relation

$$R_{n\beta}(\pi t_1/2N, \ldots, \pi t_1/2N) \sim R_{n\beta}^{FT}(\pi t_1/2N, \ldots, \pi t_1/2N) \quad (2.20)$$

can be expected. Similarly, at the edge of the spectrum, we assume that $x_i = 1 + t_i/2N^{2/3},\ 1 \le i \le n$, then

$$\begin{aligned} &R_{n\beta}(1+t_1/2N^{2/3}, \ldots, 1+t_i/2N^{2/3}) \\ &\quad \sim R_{n\beta}^{FT}\Big((1+O(\alpha_N))(1+t_1/2N^{2/3}), \ldots, (1+O(\alpha_N))(1+t_1/2N^{2/3})\Big). \end{aligned} \quad (2.21)$$

Notice

$$(1+O(\alpha_{_N}))(1+t_i/2N^{2/3}) = 1+t_i/2N^{2/3} + O(\alpha_{_N})(1+t_i/2N^{2/3}),$$

and if one takes $\alpha_{_N} = N^{-\theta},\ 2/3 < \theta < 1$, then the relation

$$R_{n\beta}(1+t_1/2N^{2/3},\dots,1+t_i/2N^{2/3}) \sim R^{FT}_{n\beta}\left(1+t_1/2N^{2/3},\dots,1+t_i/2N^{2/3}\right) \tag{2.22}$$

can also be expected.

In the present paper, we mainly use the case $\alpha_{_N} = N^{-\theta}$ for any fixed $\frac{2}{3} < \theta < 1$, which obviously satisfies the assumption of Lemma 3 about $\alpha_{_N}$.

3 Proof of Theorem 1

In principle, universality of sine-kernel at zero can be proved by one of the arguments [34]: giving an upper bound estimation of the correlation functions with the help of the maximum of Vandermonde determinant on the sphere by Stieltjes [29]. However, in this paper we will deal with it in a slightly different way. From Lemma 3, let us take $\alpha_{_N} = N^{-\theta}, \theta \in (0,1)$. It is not clear that the scaling at the soft edge of the spectrum is proportional to $N^{-2/3}$ (see [16]). Thus we can choose $\theta > 2/3$ and give a very close approximation of correlation functions near the radial sharp cutoff point. Then using known results about the unconstrained ensembles, we can obtain Airy-kernel for the fixed trace ensembles. However, this argument seems to be insufficient for proving universality in the whole bulk. The main difficulty is that the "rate" index θ has been rather sharp, in the sense that it cannot be replaced with a larger number than 1.

Proof of Theorem 1(i) By (2.15), we obtain

$$\begin{aligned}
&\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} f(\vec{t}_n) R_{n\beta}\left(\frac{\pi}{2N}\vec{t}_n\right) d\,\vec{t}_n \\
&= \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} f(\vec{t}_n) \int_{\frac{\pi}{\sqrt{N^3}}\|\vec{t}_n\|}^{+\infty} \Psi_{N\beta}(u)\,\frac{1}{u^n}\, R^{FT}_{n\beta}\left(\frac{\pi}{2Nu}\vec{t}_n\right) du\,\vec{t}_n \\
&= \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} f(\vec{t}_n) \Big(\int_{\frac{\pi}{\sqrt{N^3}}\|\vec{t}_n\|}^{1-\alpha_N} + \int_{1-\alpha_N}^{1+\alpha_N} + \int_{1+\alpha_N}^{\infty} \Big) \Psi_{N\beta}(u)\,\frac{1}{u^n}\, R^{FT}_{n\beta}\left(\frac{\pi}{2Nu}\vec{t}_n\right) du\,\vec{t}_n \\
&= I + II + III.
\end{aligned} \tag{3.1}$$

We will first estimate I. Making the change of variables

$$t_i = u\,y_i, \quad 1 \le i \le n, \tag{3.2}$$

then I can be reduced to

$$\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} \int_0^{1-\alpha_N} f(u\vec{y}_n) 1_{\{\frac{\pi}{\sqrt{N^3}}\|\vec{y}_n\| \le u \le 1-\alpha_{_N}\}}(u)\, \Psi_{N\beta}(u) R^{FT}_{n\beta}\left(\frac{\pi\,\vec{y}_n}{2N}\right) du\, d\vec{y}_n. \tag{3.3}$$

Since $f \in C_c(\mathbb{R}^n)$, there exists some positive constant M such that $|f(x)| \le M$. It follows from (2.12) that I can be dominated by

$$\begin{aligned} I &\le M\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n}\int_0^{1-\alpha_N} \Psi_{N\beta}(u)\, R_{n\beta}^{FT}\left(\frac{\pi \vec{y}_n}{2N}\right) du\, d\vec{y}_n \\ &= M\int_0^{1-\alpha_N} \Psi_{N\beta}(u) du \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} R_{n\beta}^{FT}\left(\frac{\pi \vec{y}_n}{2N}\right) d\vec{y}_n \\ &= M\int_0^{1-\alpha_N} \Psi_{N\beta}(u) du \int_{\mathbb{R}^n} R_{n\beta}^{FT}(\vec{z}_n) d\vec{z}_n \\ &= O(N^n e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}). \end{aligned} \tag{3.4}$$

Here we make the change of variables: $z_i = (2N)/\pi y_i$, $1 \le i \le n$ for the integral

$$\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} R_{n\beta}^{FT}\left(\frac{\pi \vec{y}_n}{2N}\right) d\vec{y}_n.$$

On the other hand, the fact that $\int_{\mathbb{R}^n} R_{n\beta}^{FT}(\vec{z}_n) d\vec{z}_n = N!/(N-n)!$ has been used. According to (2.13), a similar argument shows that III can be dominated by

$$III \le M\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n}\int_{1+\alpha_N}^{\infty} \Psi_{N\beta}(u) R_{n\beta}^{FT}\left(\frac{\pi \vec{y}_n}{2N}\right) du d\vec{y}_n = O\left(N^n e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}\right). \tag{3.5}$$

Next, we will consider II. Under the transform (3.2), we find

$$II = \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} f(u\,\vec{y}_n)\Psi_{N\beta}(u) R_{n\beta}^{FT}\left(\frac{\pi}{2N}\vec{y}_n\right) du d\vec{y}_n. \tag{3.6}$$

Combining (3.1), (3.4) and (3.5), it is not difficult to see that for any given $f(x) \in C_c(\mathbb{R})$,

$$\begin{aligned} &\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} f(\vec{t}_n) R_{n\beta}\left(\frac{\pi}{2N}\vec{t}_n\right) d\vec{t}_n \\ &\quad = \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} f(u\,\vec{y}_n)\Psi_{N\beta}(u) R_{n\beta}^{FT}\left(\frac{\pi}{2N}\vec{y}_n\right) du d\vec{y}_n \\ &\quad\quad + O\left(N^n e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}\right). \end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned} &\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} f(\vec{y}_n)\Psi_{N\beta}(u) R_{n\beta}^{FT}\left(\frac{\pi}{2N}\vec{y}_n\right) du d\vec{y}_n \\ &= \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} f(\vec{y}_n) R_{n\beta}^{FT}\left(\frac{\pi \vec{y}_n}{2N}\right) du\, d\vec{y}_n \left(1 + O(N^n e^{-0.5\beta N^{2(1-\theta)}(1+o(1))})\right). \end{aligned} \tag{3.8}$$

Next we will prove that the difference between (3.8) and (3.6) is zero as N goes to infinity, i.e.

$$\lim_{N\to\infty}\left|\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} \left(f(u\vec{y}_n) - f(\vec{y}_n)\right)\Psi_{N\beta}(u) R_{n\beta}^{FT}\left(\frac{\pi \vec{y}_n}{2N}\right) du\, d\vec{y}_n\right| = 0. \tag{3.9}$$

Note that $f(\vec{x}) \in C_c(\mathbb{R}^n)$. For any $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that

$$|f(\vec{x}) - f(\vec{y})| < \epsilon.$$

whenever $\|\vec{x} - \vec{y}\| < \delta$. We remind that $\mathbf{B}_R$ denotes the ball of the radius R in $\mathbb{R}^n$ centered at zero. Choose a ball $\mathbf{B}_R$ such that $supp(f) \subset \mathbf{B}_R$ and $\{u\vec{y}_n | \vec{y}_n \in supp(f), 1 - \alpha_{_N} \le u \le 1 + \alpha_{_N}\} \subset \mathbf{B}_R$. $\forall \vec{y}_n \in supp(f)$, there exists N_0 not depending on $\vec{y}_n$ such that

$$\|u\vec{y}_n - \vec{y}_n\| \le |u-1| \|\vec{y}_n\| < N^{-\theta} R < \delta(\epsilon)$$

for $N > N_0$. Therefore, we have

$$|f(u\vec{y}_n) - f(\vec{y}_n)| < \epsilon. \tag{3.10}$$

For large N, it follows from (2.14) that

$$\begin{aligned} |(3.8) - (3.6)| &\le \left(\frac{\pi}{2N}\right)^n \int_{\mathbf{B}_R} \int_{1-\alpha_{_N}}^{1+\alpha_{_N}} |f(u\,\vec{y}_n) - f(\vec{y}_n)| \Psi_{N\beta}(u)\, R_{n\beta}^{FT}\left(\frac{\pi\,\vec{y}_n}{2N}\right) du\, d\vec{y}_n \\ &= \epsilon \left(\frac{\pi}{2N}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(\frac{\pi\,\vec{y}_n}{2N}\right) d\vec{y}_n \int_{1-\alpha_{_N}}^{1+\alpha_{_N}} \Psi_{N\beta}(u) du \le 2\epsilon \left(\frac{\pi}{2N}\right)^n \\ &\quad \times \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(\frac{\pi\,\vec{y}_n}{2N}\right) d\vec{y}_n \\ &\le 2\epsilon\, M_{_R}. \end{aligned} \tag{3.11}$$

Here we have used Lemma 5 below. Thus, (3.9) can be obtained. According to (3.7) and (3.8), it follows that

$$\begin{aligned} &\left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} f(\vec{t}_n) R_{n\beta}\left(\frac{\pi}{2N}\vec{t}_n\right) d\vec{t}_n \\ &\quad = \left(\frac{\pi}{2N}\right)^n \int f(\vec{y}_n) R_{n\beta}^{FT}\left(\frac{\pi\,\vec{y}_n}{2N}\right) du\, d\vec{y}_n (1 + o(1)) + o(1). \end{aligned} \tag{3.12}$$

Hence, from the known results (1.5) for the Gaussian ensembles, we complete the proof of Theorem 1(i). □

Now we will prove the following lemma, which is inspired by Lemma 4 in [18].

Lemma 5 *Let* $\beta = 1, 2, 4,$ *for any fixed* $R > 0$, *there exists a constant* $M_{_R}$ *such that*

$$\left(\frac{\pi}{2N}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(\frac{\pi}{2N} y_1, \ldots, \frac{\pi}{2N} y_n\right) dy_1 \cdots dy_n \le M_{_R}. \tag{3.13}$$

Proof Choose any positive $\delta < R$, there exists N_o depending only on R and δ such that $R\,N^{-\theta} < \delta$ for $N > N_0$. That is, for any $\vec{y}_n \in \mathbf{B}_R$, when $u \in [1 - N^{-\theta}, 1 + N^{-\theta}]$, we have

$$u\|\vec{y}_n\| \le R + N^{-\theta} R < R + \delta.$$

Let $\eta \in (0,1)$ be a real number and let $\phi(t)$ be a smooth decreasing function on $[0, R+\delta)$ such that $\phi(t)=1$ for $t\in[0,R+\delta)$ and $\phi(t)=0$ for $t \geq (1+\eta)(R+\delta)$. Set $\varphi(\vec{x}_n) = \phi(\|\vec{x}_n\|)$ for $\vec{x}_n=(x_1,\ldots,x_n)\in\mathbb{R}^n$. For $N>N_0$, we have

$$\left(\frac{\pi}{2N}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(\frac{\pi\vec{y}_n}{2N}\right)d\vec{y}_n \leq \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} \varphi(u\vec{y}_n) R_{n\beta}^{FT}\left(\frac{\pi\vec{y}_n}{2N}\right)d\vec{y}_n. \tag{3.14}$$

Multiplying by $\Psi_{N\beta}(u)$ then integrating (3.14) with respect to u on $[1-\alpha_N, 1+\alpha_N]$, one obtains

$$\begin{aligned}
&\int_{1-\alpha_N}^{1+\alpha_N} \Psi_{N\beta}(u)du \left(\frac{\pi}{2N}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(\frac{\pi\vec{y}_n}{2N}\right)d\vec{y}_n \\
&\quad\leq \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} \varphi(u\vec{y}_n)\Psi_{N\beta}(u) R_{n\beta}^{FT}\left(\frac{\pi\vec{y}_n}{2N}\right)dud\vec{y}_n \\
&\quad= \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} \varphi(\vec{y}_n) R_{n\beta}\left(\frac{\pi}{2N}\vec{y}_n\right)d\vec{y}_n + O\left(N^n\, e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}\right).
\end{aligned} \tag{3.15}$$

Here we have used (3.7) and (2.14). From (1.5), we conclude (3.13). □

Next, we will prove the part (ii) of Theorem 1.

Proof of Theorem 1(ii) Set $\vec{w}_n = (1+\frac{t_1}{2N^{2/3}},\ldots,1+\frac{t_n}{2N^{2/3}})$. By (2.15), we find

$$\begin{aligned}
&\frac{1}{(2N^{2/3})^n}\int_{\mathbb{R}^n} f(t_1,\ldots,t_n) R_{n\beta}\left(1+\frac{t_1}{2N^{2/3}},\ldots,1+\frac{t_n}{2N^{2/3}}\right)dt_1\cdots dt_n \\
&\quad= \frac{1}{(2N^{2/3})^n}\int_{\mathbb{R}^n} f(\vec{t}_n)\int_{\frac{2}{\sqrt{N}}\|\vec{w}_n\|}^{+\infty} \Psi_{N\beta}(u)\frac{1}{u^n} R_{n\beta}^{FT}\left(\frac{\vec{w}_n}{u}\right)dud\vec{t}_n \\
&\quad= \frac{1}{(2N^{2/3})^n}\int_{\mathbb{R}^n} f(\vec{t}_n)\Big(\int_{\frac{2}{\sqrt{N}}\|\vec{w}_n\|}^{1-\alpha_N} + \int_{1-\alpha_N}^{1+\alpha_N} + \int_{1+\alpha_N}^{\infty}\Big)\Psi_{N\beta}(u)\frac{1}{u^n} R_{n\beta}^{FT}\left(\frac{\vec{w}_n}{u}\right)dud\vec{t}_n \\
&\quad= I' + II' + III'.
\end{aligned} \tag{3.16}$$

We will follow an analogous argument for the proof of Theorem 1(i). Making the change of variables

$$t_i = uy_i + 2N^{2/3}(u-1), \quad 1\leq i\leq n, \tag{3.17}$$

then I' can be reduced to

$$\begin{aligned}
&\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n}\int_0^{1-\alpha_N} f\Big(u\,y_1 + 2N^{2/3}(u-1),\ldots,uy_n+2N^{2/3}(u-1)\Big) \\
&\quad\times 1_{\{\frac{2}{\sqrt{N}}\|\vec{y}_n\|\leq u\leq 1-\alpha_N\}}(u)\,\Psi_{N\beta}(u) R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right)dud\vec{y}_n.
\end{aligned} \tag{3.18}$$

Assume that $|f(x)| \le M$. By (2.12), I' can be dominated by

$$I' \le M\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n}\int_0^{1-\alpha_N} \Psi_{N\beta}(u) R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) du\, d\vec{y}_n$$
$$= M\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n} R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) d\vec{y}_n \int_0^{1-\alpha_N} \Psi_{N\beta}(u) du$$
$$= M\frac{N!}{(N-n)!} O(e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}) = O(N^n\, e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}). \tag{3.19}$$

Here we use the fact that

$$\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n} R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) d\vec{y}_n = \frac{N!}{(N-n)!}. \tag{3.20}$$

Similarly, by (2.13), one gets

$$III' \le M\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n}\int_{1+\alpha_N}^{\infty} \Psi_{N\beta}(u) R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) du\, d\vec{y}_n$$
$$= O(N^n\, e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}). \tag{3.21}$$

It follows from (3.16), (3.19) and (3.21) that for any $f(\vec{x}_n) \in C_c(\mathbb{R}^n)$,

$$\frac{1}{(2N^{2/3})^n} \int_{\mathbb{R}^n} f(\vec{t}_n) R_{n\beta}\left(1+\frac{t_1}{2N^{2/3}},\ldots,1+\frac{t_n}{2N^{2/3}}\right) d\vec{t}_n$$
$$= II' + O\left(N^n\, e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}\right) \tag{3.22}$$

Under the transform (3.17), we have

$$II' = \left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} f\left(u\, y_1 + 2N^{2/3}(u-1),\ldots,uy_n + 2N^{2/3}(u-1)\right)$$
$$\times \Psi_{N\beta}(u) R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) du\, dy_1\cdots dy_n. \tag{3.23}$$

We also notice that

$$\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} f(\vec{y}_n)\Psi_{N\beta}(u) R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) du\, d\vec{y}_n$$
$$= \left(\frac{\pi}{2N}\right)^n \int_{\mathbb{R}^n} f(y_1,\ldots,y_n) R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) d\vec{y}_n$$
$$\times \left(1 + O(e^{-0.5\beta N^{2(1-\theta)}(1+o(1))})\right). \tag{3.24}$$

We need to prove that the difference between the right hand side of (3.23) and (3.24) is zero when N tends to infinity, i.e.,

$$\lim_{N\to\infty}\left|\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n}\int_{1-\alpha_N}^{1+\alpha_N} [f(u\, y_1 + 2N^{2/3}(u-1),\ldots,uy_n + 2N^{2/3}(u-1)) - f(\vec{y}_n)]\right.$$
$$\left.\times \Psi_{N\beta}(u) R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}},\ldots,1+\frac{y_n}{2N^{2/3}}\right) du\, d\vec{y}_n\right| = 0. \tag{3.25}$$

For any $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that $|f(\vec{x}) - f(\vec{y})| < \epsilon$, whenever $\|\vec{x} - \vec{y}\| < \delta$. Since $1 - \alpha_N \le u \le 1 + \alpha_N$ and $2/3 < \theta < 1$, we can choose a ball $\mathbf{B}_R$ such that $supp(f) \subset \mathbf{B}_R$ and

$$\{(uy_1 + 2N^{2/3}(u-1), \ldots, uy_n + 2N^{2/3}(u-1)) | \vec{y}_n \in supp(f), 1 - \alpha_N \le u \le 1 + \alpha_N\} \subset \mathbf{B}_R.$$

For $\vec{y}_n \in supp(f)$, there exist N_0 independent on $\vec{y}_n$ such that

$$\begin{aligned} &\|(u\,y_1 + 2N^{2/3}(u-1), \ldots, uy_n + 2N^{2/3}(u-1) - (y_1, \ldots, y_n)\| \\ &\le \sqrt{\sum_{i=1}^{n} \left(2N^{2/3-\theta} + |y_i|N^{-\theta}\right)^2} \le \sqrt{n}(RN^{-\theta} + 2N^{2/3-\theta}) \le \delta(\epsilon) \end{aligned} \tag{3.26}$$

for $N > N_0$. Therefore, $\forall \vec{y}_n \in supp(f)$

$$|f(u\,y_1 + 2N^{2/3}(u-1), \ldots, uy_n + 2N^{2/3}(u-1)) - f(y_n, \ldots, y_n)| < \epsilon. \tag{3.27}$$

Furthermore, we get

$$\begin{aligned} &|II' - (3.24)| \\ &\le \epsilon \left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbf{B}_R} \int_{1-\alpha_N}^{1+\alpha_N} \Psi_{N\beta}(u)\, R_{n\beta}^{FT}\left(1 + \frac{y_1}{2N^{2/3}}, \ldots, 1 + \frac{y_n}{2N^{2/3}}\right) du\, d\vec{y}_n \\ &= \epsilon \int_{1-\alpha_N}^{1+\alpha_N} \Psi_{N\beta}(u) du \left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(1 + \frac{y_1}{2N^{2/3}}, \ldots, 1 + \frac{y_n}{2N^{2/3}}\right) d\vec{y}_n \\ &\le 2\epsilon \left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(1 + \frac{y_1}{2N^{2/3}}, \ldots, 1 + \frac{y_n}{2N^{2/3}}\right) d\vec{y}_n \\ &\le 2\epsilon\, M_R. \end{aligned} \tag{3.28}$$

Here we have used Lemma 6 below. Thus, (3.25) can be obtained. Combining (3.22), (3.23), (3.24) and (3.25), we have

$$\begin{aligned} &\int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \frac{1}{(2N^{2/3})^n} R_{n\beta}\left(1 + \frac{t_1}{2N^{2/3}}, \ldots, 1 + \frac{t_n}{2N^{2/3}}\right) dt_1 \cdots dt_n \\ &= \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) \left(\frac{1}{2N^{2/3}}\right)^n R_{n\beta}^{FT}\left(1 + \frac{y_1}{2N^{2/3}}, \ldots, 1 + \frac{y_n}{2N^{2/3}}\right) d\vec{y}_n (1 + o(1)) + o(1), \end{aligned}$$

for large N. From (1.8), we complete the proof of Theorem 1(ii). □

Lemma 6 *Let* $\beta = 1, 2, 4$, *for any fixed* $R > 0$, *there exists a constant* M_R *such that*

$$\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(1 + \frac{y_1}{2N^{2/3}}, \ldots, 1 + \frac{y_n}{2N^{2/3}}\right) dy_1 \cdots dy_n \le M_R. \tag{3.29}$$

Proof Choose any positive $\delta < R$, there exists $N_o(R, \delta)$ such that for $N > N_0$, $\vec{y}_n \in \mathbf{B}_R$, we have

$$\sqrt{\sum_{i=1}^{n} (uy_i + 2N^{2/3}(u-1))^2} \le \sqrt{\sum_{i=1}^{n} (u|y_i| + 2N^{2/3-\theta})^2} < R + \delta \tag{3.30}$$

where $u \in [1-N^{-\theta}, 1+N^{-\theta}]$, $2/3 < \theta < 1$. Let $\eta \in (0,1)$ be a real number and let $\phi(t)$ be a smooth decreasing function on $[0, R+\delta)$ such that $\phi(t) = 1$ for $t \in [0, R+\delta)$ and $\phi(t) = 0$ for $t \geq (1+\eta)(R+\delta)$. Set $\varphi(\vec{x}_n) = \phi(\|\vec{x}_n\|)$ for $\vec{x}_n = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $N > N_0$, we have

$$\begin{aligned}&\left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}}, \ldots, 1+\frac{y_n}{2N^{2/3}}\right) dy_1 \cdots dy_n \\ &\quad \leq \left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n} \varphi\Big(u\, y_1 + 2N^{2/3}(u-1), \ldots, u y_n + 2N^{2/3}(u-1)\Big) \\ &\qquad \times R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}}, \ldots, 1+\frac{y_n}{2N^{2/3}}\right) dy_1 \cdots dy_n. \end{aligned} \tag{3.31}$$

Multiplying by $\Psi_{N\beta}(u)$ and then integrating (3.31) with respect to u on $[1-\alpha_N, 1+\alpha_N]$, one obtains

$$\begin{aligned}&\int_{1-\alpha_N}^{1+\alpha_N} \Psi_{N\beta}(u) du \left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbf{B}_R} R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}}, \ldots, 1+\frac{y_n}{2N^{2/3}}\right) dy_1 \cdots dy_n \\ &\quad \leq \left(\frac{1}{2N^{2/3}}\right)^n \int_{\mathbb{R}^n} \int_{1-\alpha_N}^{1+\alpha_N} \varphi\Big(u\, y_1 + 2N^{2/3}(u-1), \ldots, u y_n + 2N^{2/3}(u-1)\Big) \\ &\qquad \times \Psi_{N\beta}(u)\, R_{n\beta}^{FT}\left(1+\frac{y_1}{2N^{2/3}}, \ldots, 1+\frac{y_n}{2N^{2/3}}\right) du dy_1 \cdots dy_n \\ &\quad = \frac{1}{(2N^{2/3})^n} \int_{\mathbb{R}^n} \varphi(\vec{t}_n) R_{n\beta}\left(1+\frac{t_1}{2N^{2/3}}, \ldots, 1+\frac{t_n}{2N^{2/3}}\right) d\vec{t}_n \\ &\qquad + O\left(N^n\, e^{-0.5\beta N^{2(1-\theta)}(1+o(1))}\right). \end{aligned} \tag{3.32}$$

Here we make use of (3.22) and (3.23). Thus, (1.8) implies (3.29).

Hence we complete the proof of this lemma. □

4 Bounded Trace Gaussian Ensembles

We first give a representation of correlation functions for the bounded trace Gaussian ensembles in terms of these for the fixed trace Gaussian ensembles, just as we dealt with bounded trace Laguerre unitary ensemble in [23]. Actually, the correlation function of fixed trace Gaussian ensembles can be given in terms of that of bounded trace ensembles [2, 11].

Proposition 7 *Let $R_{n\beta}^{BT,r}$ and $R_{n\beta}^{FT,r}$ be the n-point correlation functions for the bounded trace and fixed trace Gaussian ensembles respectively, then we have the following relation*

$$R_{n\beta}^{BT,r}(x_1, \ldots, x_n) = \int_0^1 N_\beta\, u^{N_\beta - 1} \frac{1}{u^n} R_{n\beta}^{FT,r}\left(\frac{x_1}{u}, \ldots, \frac{x_n}{u}\right) du, \tag{4.1}$$

where $N_\beta = N + \beta N(N-1)/2$.

Proof It suffices to prove

$$R_{n\beta}^{BT,r}(x_1, \ldots, x_n) = \int_0^r \frac{N_\beta}{r^{N_\beta}} u^{N_\beta - 1} \left(\frac{r}{u}\right)^n R_{n\beta}^{FT,r}\left(\frac{r}{u}x_1, \ldots, \frac{r}{u}x_n\right) du. \tag{4.2}$$

For every $u > 0$, let

$$\Omega_N(u) = \left\{(x_1, \ldots, x_N) | \sum_{j=1}^{N} x_j^2 = u^2\right\} \tag{4.3}$$

be the sphere in $\mathbb{R}^N$, which carries the volume element induced by the standard Euclidean metric on $\mathbb{R}^N$, denoted by $u^{N-1} d\sigma_N$. For $h \in L^\infty(\mathbb{R}^N)$, let $\langle h(\cdot)\rangle_\theta$ and $\langle h(\cdot)\rangle_\delta$ denote the ensemble average taken in the bounded trace and fixed trace ensembles, respectively. From (1.11) and (1.13), we have

$$\begin{aligned}\langle h(\cdot)\rangle_\theta &= \frac{1}{Z_{N\beta}^{BT,r}} \int_0^r u^{N-1} du \int_{\Omega_N(u)} h(x_1, \ldots, x_N) \prod_{1\le i<j\le N} |x_i - x_j|^\beta d\sigma_N \\ &= \frac{1}{Z_{N\beta}^{BT,r}} \int_0^r u^{N_\beta-1} du \int_{\Omega_N(1)} h(ux_1, \ldots, ux_N) \prod_{1\le i<j\le N} |x_i - x_j|^\beta d\sigma_N,\end{aligned}$$

and

$$\begin{aligned}\langle h(a\,\cdot)\rangle_\delta &= \frac{1}{Z_{N\beta}^{FT,r}} \int_{\Omega_N(r)} r^{N-1} h(ax_1, \ldots, ax_N) \prod_{1\le i<j\le N} |x_i - x_j|^\beta d\sigma_N \\ &= \frac{1}{Z_{N\beta}^{FT,r}} \int_{\Omega_N(1)} r^{N_\beta-1} h(arx_1, \ldots, arx_N) \prod_{1\le i<j\le N} |x_i - x_j|^\beta d\sigma_N.\end{aligned}$$

Choose $a = \frac{u}{r}$, we get

$$\langle h(\cdot)\rangle_\theta = \frac{Z_{N\beta}^{FT,r}}{Z_{N\beta}^{BT,r}} \int_0^r \left(\frac{u}{r}\right)^{N_\beta-1} \left\langle h\left(\frac{u}{r}\,\cdot\right)\right\rangle_\delta du. \tag{4.4}$$

Setting $h \equiv 1$, we get the ratio of the partition functions $Z_{N\beta}^{FT,r}$ and $Z_{N\beta}^{BT,r}$. Substituting this ratio, we then obtain

$$\langle h(\cdot)\rangle_\theta = \int_0^r \frac{N_\beta}{r^{N_\beta}} u^{N_\beta-1} \left\langle h\left(\frac{u}{r}\,\cdot\right)\right\rangle_\delta du. \tag{4.5}$$

In particular, taking

$$h(x_1, \ldots, x_N) = \sum_{1\le i_1<\cdots<i_n\le N} f(x_{i_1}, \ldots, x_{i_n}), \tag{4.6}$$

we have

$$\begin{aligned}&\int_{\mathbb{R}^n} f(x_1, \ldots, x_n) R_{n\beta}^{BT,r}(x_1, \ldots, x_n)\, d^n x \\ &= \int_0^r \frac{N_\beta}{r^{N_\beta}} u^{N_\beta-1} du \int_{\mathbb{R}^n} f\left(\frac{u}{r}x_1, \ldots, \frac{u}{r}x_n\right) R_{n\beta}^{FT,r}(x_1, \ldots, x_n)\, d^n x \\ &= \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\, d^n x \int_0^r \frac{N_\beta}{r^{N_\beta}} u^{N_\beta-1} \left(\frac{r}{u}\right)^n R_{n\beta}^{FT,r}\left(\frac{r}{u}x_1, \ldots, \frac{r}{u}x_n\right) du.\end{aligned}$$

Since $R_{n\beta}^{BT,r}$ and $R_{n\beta}^{FT,r}$ are both continuous, we complete the proof. □

Next, we notice a "sharp" concentration phenomenon along the radial coordinate between correlation functions of the bounded trace and fixed trace Gaussian ensembles. Although its proof is simple, the following lemma plays a crucial role in dealing with local statistical properties of the eigenvalues between the fixed and bounded ensembles.

Lemma 8 *Let $\{b_N\}$ be a sequence such that $b_N \to 0$ but $N^2 b_N \to \infty$ as $N \to \infty$, then we have*

$$\int_0^1 N_\beta\, u^{N_\beta - 1}\, du = \int_{u_-}^1 N_\beta\, u^{N_\beta - 1}\, du + e^{-0.5\beta N^2 b_N (1+o(1))}, \tag{4.7}$$

where $u_- = 1 - b_N$.

Proof

$$\int_0^{u_-} N_\beta\, u^{N_\beta - 1}\, du = (1 - b_N)^{N_\beta} = e^{N_\beta \ln(1-b_N)}$$
$$= e^{N_\beta(-b_N + O(b_N^2))} = e^{-0.5\beta N^2 b_N(1+o(1))}.$$

This completes the proof. □

Remark 9 In Lemma 8, let us take $b_N = N^{-\kappa}, \kappa \in (0,2)$. Since the "rate" index κ can be chosen larger than 1 while the scaling in the bulk is proportional to N^{-1} and at the soft edge of the spectrum is proportional to $N^{-2/3}$, in principle we can prove all local statistical properties of the eigenvalues between the fixed and bounded trace Gaussian ensembles are identical in the limit. Such arguments also apply to the equivalence of ensembles between the fixed trace and bounded trace ensembles with monomial potentials, where we exploit some homogeneity of the monomial potentials. In addition, we notice that equivalence of n-point resolvents of the fixed and bounded trace ensembles with monomial potentials has turned out to be identical in the limit in [3].

Now we turn to the proof of Theorem 2, by using the associated results about the fixed trace ensembles.

Proof The proof is very similar to that of Theorem 1, we only point out some different places in the bulk case for bounded trace GUE.

In Lemma 8, choose $b_N = N^{-\kappa}$, $\kappa \in (1,2)$. The change of variables corresponding to (3.17) reads:

$$t_i = (u-1)N\, x\omega(x) + u y_i, \quad i = 1, \ldots, n \tag{4.8}$$

where fixed $x \in (-1,1)$. The condition that $b_N = N^{-\kappa}$, $\kappa \in (1,2)$ ensures $(1-u)N \le N^{-\kappa+1} \to 0$ as $N \to \infty$ for $u \in [u_-, 1]$. On the other hand, by (1.18), the following fact similar to Lemma 6 is obvious: for any fixed $R > 0$,

$$\frac{1}{(N\omega(x))^n} \int_{B_R} R_{n\beta}^{FT}\left(x + \frac{y_1}{N\omega(x)}, \ldots, x + \frac{y_n}{N\omega(x)}\right) d^n y \le C_R. \tag{4.9}$$

Here B_R is the ball of the radius R in $\mathbb{R}^n$ centered at zero, and C_R is a constant. Using Proposition 7 and universality of sine-kernel (1.18) for fixed trace GUE, we complete the proof after a similar procedure. □

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