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Facilitated Oriented Spin Models: Some Non Equilibrium Results

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Abstract We perform the rigorous analysis of the relaxation to equilibrium for some facilitated or kinetically constrained spin models (KCSM) when the initial distribution ν is different from the reversible one, μ. This setting has been intensively studied in the physics literature to analyze the slow dynamics which follows a sudden quench from the liquid to the glass phase. We concentrate on two basic oriented KCSM: the East model on $\mathbb{Z}$, for which the constraint requires that the East neighbor of the to-be-update vertex is vacant and the AD model on the binary tree introduced in Aldous and Diaconis (J. Stat. Phys. 107(5–6):945–975, 2002), for which the constraint requires the two children to be vacant. It is important to observe that, while the former model is ergodic at any $p \neq 1$, the latter displays an ergodicity breaking transition at $p_c = 1/2$. For the East we prove exponential convergence to equilibrium with rate depending on the spectral gap if ν is concentrated on any configuration which does not contain a forever blocked site or if ν is a Bernoulli(p') product measure for any $p' \neq 1$. For the model on the binary tree we prove similar results in the regime $p, p' < p_c$ and under the (plausible) assumption that the spectral gap is positive for $p < p_c$. By constructing a proper test function, we also prove that if $p' > p_c$ and $p \leq p_c$ convergence to equilibrium cannot occur for all local functions. Finally, in a short appendix, we present a very simple argument, different from the one given in Aldous and Diaconis (J. Stat. Phys. 107(5–6):945–975, 2002), based on a combination of some combinatorial results together with "energy barrier" considerations, which yields the sharp upper bound for the spectral gap of East when $p \uparrow 1$.

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1 Introduction

Facilitated or kinetically constrained spin models (KCSM) are interacting particle systems which have been introduced in physics literature [5, 6] to model liquid/glass transition and more generally "glassy dynamics" (see [15, 17]). They are defined on a locally finite, bounded degree, connected graph $\mathcal{G} = (V, E)$ with vertex set V and edge set E. For most KCSM the graph is the integer lattice $\mathbb{Z}^d$. A configuration is given by assigning to each site $x \in V$ its occupation variable $\eta(x) \in \{0, 1\}$ which corresponds to an empty or filled site, respectively. The evolution is given by a Markovian stochastic dynamics of Glauber type. Each site waits an independent, mean one, exponential time and then, provided the current configuration around it satisfies an a priori specified constraint, its occupation variable is refreshed to an occupied or to an empty state with probability p or $1 - p$, respectively. For each site x the corresponding constraint does not involve η_x, thus detailed balance w.r.t. Bernoulli(p) product measure μ can be easily verified and the latter is an invariant reversible measure for the process.

Here we focus on the East model [10] and on a model which was introduced in [1] which we will call AD model. The East model is one-dimensional ($\mathcal{G} = \mathbb{Z}$) and particle creation/destruction at a given site can occur only if its nearest neighbor to the right is empty. AD model is instead defined on a infinite rooted binary tree, ($\mathcal{G} = \mathcal{T}$) and the constraint requires the two children of x to be vacant in order to allow an updating of the spin at x. Note that for both models (actually for all KCSM introduced in physics literature), the constraints impose a maximal number of occupied sites in a proper neighborhood in order to allow the moves. As a consequence the dynamics becomes slower at higher density and an ergodicity breaking transition may occur at a finite critical density, $p_c < 1$. This threshold, as it has been formalized in Sect. 2.3 of [3], corresponds to the lowest density at which the origin belongs with finite probability to a cluster of particles which are mutually and forever blocked due to the constraints. Among the above models the East model has $p_c = 1$ and AD model has $p_c = 1/2$. Another key feature of both models is the existence of blocked configurations, namely configurations with all creation/destruction rates identically equal to zero. This implies the existence of several invariant measures and the occurrence of unusually long mixing times compared to high-temperature Ising models (see Sect. 7.1 of [3]). Furthermore the constrained dynamics is not attractive so that monotonicity arguments valid for e.g. ferromagnetic stochastic Ising models cannot be applied. Due to the above properties the basic issues concerning the large time behavior of the process are non-trivial. The first rigorous results for the East model were derived in [1] where it was established that the spectral gap of its generator is positive for all $p < 1$. In [3] positivity of the spectral gap inside the ergodicity region (i.e. for $p < p_c$) has been proved for a much greater class of KCSM on the integer lattice. In [1] the following bounds have also been proven for the spectral gap

$$(1/(2\log 2) - o(1))\log^2(1/(1-p)) \le \log(1/\text{gap}) \le (1/\log 2 + o(1))\log^2(1/(1-p))$$

as $p \uparrow 1$, thus establishing that the spectral gap shrinks to zero faster than any polynomial in $(1 - p)$. This result had already been obtained by non rigorous analytical tools in [16] where the asymptotics $\log(1/\text{gap}) \sim \log^2(1/(1-p))/\log 2$ was claimed to be the correct one. However, in [3], the correct asymptotics was instead proved to be $\log(1/\text{gap}) \simeq \log^2(1/(1-p))/(2\log 2)$.

A key issue both from the mathematical and the physical point of view is what happens when evolution does not start from the equilibrium measure μ. The analysis of this setting usually requires much more detailed informations than just the positivity of the spectral gap, e.g. positivity of the logarithmic Sobolev constant or of the entropy constant uniformly in the system size. Since the latter requirement does certainly not hold for KCSM (see Sect. 7.1 of [3]), even the basic question of whether convergence to μ occurs remains open in this non equilibrium setting. Of course, due to the existence of blocked configurations, convergence to μ cannot be true uniformly on the initial configuration and one could try to prove it a.e. or in mean w.r.t. an initial distribution $\nu \neq \mu$. From the point of view of physicists a particularly relevant case (see e.g. [12, 14]), is when ν is a product Bernoulli(p') measure with $p' \neq p$. If $p' > p_c$, due to the occurrence of the forever blocked clusters, one cannot hope to prove any such convergence result even if the starting point is chosen at random with distribution ν (see Sect. 4, Theorems 4.4 and 4.5). If instead both p and p' are below p_c the most natural guess is that convergence to equilibrium occurs for any local function f i.e.

$$\lim_{t\to\infty} \int d\nu(\eta)\mathbb{E}_\eta\big(f(\eta_t)\big) = \mu(f) \tag{1.1}$$

where η_t denotes the process started from η at time t and that the limit is attained exponentially fast if the spectral gap is positive.

We start by proving (1.1) plus exponential convergence for *any* one dimensional ($\mathcal{G} = \mathbb{Z}$) reversible stochastic spin model with finite range jump rates and positive spectral gap when the initial distribution ν is "not too far" from the reversible one (see Theorem 2.1). This result, based on elementary perturbation theory and therefore relatively simple, covers in particular any ergodic finite range KCSM on $\mathbb{Z}$ like the East and the FA-1f (one spin facilitated Fredrickson-Andersen) [5] models. For the latter the constraint requires at least one empty site among the two nearest neighbours of the to-be-update vertex.

Then we turn to the East model and we prove exponential convergence to equilibrium in two cases:

(i) starting from a fixed configuration η for which no site is blocked forever, i.e. a configuration without a rightmost zero (Theorem 3.1);
(ii) starting from a Bernoulli(p') measure for all $p' \in [0, 1)$ (Theorem 3.2).

The proof of these results rely on two key properties: the positivity of the spectral gap and the fact that the East constraints are oriented (or directed), i.e. for any vertex x the sites that enter in its constraint (here $x + 1$) evolve independently from the occupation variable at x (see Sect. 3 for a more precise definition).

Thanks to the fact that this is again an oriented model and with the reasonable assumption gap > 0 for any $p < p_c$, we prove exponential convergence to equilibrium when $p < p_c$ in two cases:

(i) starting from a fixed configuration η without an infinite cluster of 1's (see Theorem 4.2);
(ii) starting from a Bernoulli(p') measure with $p' < p_c$ (see Theorem 4.3).

Finally we present in the appendix an argument (Theorem 5.1) which establishes the sharp upper bound for the spectral gap of East. This result had already be obtained in [1] via a properly devised test function. We follow here a different and simpler strategy based on precise "energy/entropy" considerations.

2 One-dimensional Models Near Equilibrium

In this section we consider general KCSM on $\mathbb{Z}$ and prove an easy perturbative result. Each KCMS is characterized by its infinitesimal Markov generator $\mathcal{L}$ whose action on local (i.e. depending on finitely many variables) functions $f : \Omega \mapsto \mathbb{R}$, $\Omega = \{0, 1\}^{\mathbb{Z}}$, is given by

$$\mathcal{L} f(\omega) = \sum_{x\in\mathbb{Z}} c_x(\omega)\left[\mu_x(f) - f(\omega)\right] \tag{2.1}$$

where $\mu_x(f) \equiv \int d\mu_x(\omega_x) f(\omega)$ is a function of all the $\{\omega_y\}_{y\neq x}$ and corresponds to the local mean w.r.t. to the variable ω_x computed with the Bernoulli(p) measure μ_x while all the other variables are held fixed.

On the coefficients (or constraints) $c_x(\omega)$ we only assume that they are the indicator function of some non-empty event A_x which depends only on the random variables $\{\omega_y :\ y \neq x, |x-y| \le r\}$ where $r \ge 1$ is some preassigned constant (finite range condition). Since $c_x(\omega)$ does not depend on ω_x one immediately checks that the product measure $\mu = \bigotimes_{x\in\mathbb{Z}} \mu_x$ is a reversible stationary measure i.e. $\mathcal{L}$ can be extended to a selfadjoint, non-positive operator on $L^2(\Omega, \mu)$ which we still denote by $\mathcal{L}$. As usual we denote by $\text{gap}(\mathcal{L})$ the spectral gap of $\mathcal{L}$ (see e.g. [7]).

Finally, for any local function f we will denote by $\mathbb{E}(f(\eta_t))$ or equivalently by $P_t f(\eta)$ the expectation over the process generated by $\mathcal{L}$ at time t when the initial configuration is η.

Theorem 2.1 *Assume* $\text{gap}(\mathcal{L}) > 0$. *Then there exists* $\lambda, m > 0$ *such that, for any probability measure* ν *on* Ω *satisfying*

$$\sup_{\ell} \max_{\omega_{-\ell},\dots,\omega_\ell} e^{-\lambda\ell} \frac{\nu(\omega_{-\ell},\dots,\omega_\ell)}{\mu(\omega_{-\ell},\dots,\omega_\ell)} < \infty \tag{2.2}$$

and for any local function f *there exists* $C_f < \infty$ *s.t.*

$$\int d\nu(\eta)\Big|\mathbb{E}(f(\eta_t)) - \mu(f))\Big| \le C_f e^{-mt}.$$

Remark 2.2 The condition $\text{gap}(\mathcal{L}) > 0$ has been verified in various popular one dimensional KCSM like FA-1f and East models (see [3]).

Let $\Lambda_\ell := \{i \in \mathbb{Z} :\ |i| \le \ell\}$, f be a local function with support S_f, $\mu(f) = 0$ and $\ell_f := \inf_{\ell\in\mathbb{Z}^+}\{\ell : S_f \subset \Lambda_\ell\}$. Fix $M > 0$ to be chosen later depending on the range r and denote by $n(t) \in \mathbb{N}$ the integer part of Mt. We define the volume $\hat{\Lambda} := \Lambda_{\ell_f+n(t)} \supset \Lambda_{\ell_f} \supset S_f$ and by $\hat{\nu}$ and $\hat{\mu}$ the marginal of ν and μ to $\{0, 1\}^{\hat{\Lambda}}$, respectively. We also let $\{\hat{\eta}_s\}_{s\le t}$ be the process up to time t started from the configuration indentically equal to 0 outside $\hat{\Lambda}$ and equal to η inside $\hat{\Lambda}$.

Using standard results of "finite speed of propagation" (see e.g. [13]) one gets immediately that it is possible to choose $M = M(r)$ so large that

$$\sup_{\eta}\Big|\mathbb{E}_\eta\big(f(\eta_t)\big) - \mathbb{E}_\eta\big(f(\hat{\eta}_t)\big)\Big| \le C_f e^{-t} \tag{2.3}$$

where C_f is some constant depending on f. Therefore

$$\int d\nu(\eta)\big|\mathbb{E}(f(\eta_t)) - \mu(f))\big| \le C_f e^{-t} + \int d\hat{\nu}(\eta)\big|\mathbb{E}(f(\hat{\eta}_t)) - \mu(f))\big|$$

$$
\begin{aligned}
&\leq C_f e^{-t} + \left\| \frac{\hat{\nu}}{\hat{\mu}} \right\|_\infty \int d\hat{\mu}(\eta) \left| \mathbb{E}(f(\hat{\eta}_t)) - \mu(f) \right| \\
&\leq 2C_f e^{-t} + \left\| \frac{\hat{\nu}}{\hat{\mu}} \right\|_\infty \int d\mu(\eta) \left| \mathbb{E}(f(\eta_t)) - \mu(f) \right| \\
&\leq 2C_f e^{-t} + e^{2\lambda(\ell_f + n)} e^{-\mathrm{gap}\, t} \operatorname{Var}_\mu(f)^{1/2} \qquad (2.4)
\end{aligned}
$$

where in the last line we used (2.2) together with the standard inequality $\operatorname{Var}_\mu(P_t f) \leq e^{-2\mathrm{gap}\, t} \operatorname{Var}_\mu(f)$. If we now set $\lambda = \frac{1}{4}\mathrm{gap}/M(r)$ we get immediately that the r.h.s. of (2.4) is bounded from above by $C'_f e^{-mt}$ for some $C'_f, m > 0$.

3 Non Equilibrium Results for the East Model

The East model is defined on $\mathbb{Z}$ and its infinitesimal generator $\mathcal{L}$ takes the form (2.1) with constraints

$$
c_x(\omega) = 1 - \omega_{x+1}. \tag{3.1}
$$

In this case and thanks to the special form of the constraints we are able to improve considerably over Theorem 2.1 and get an optimal result. In this section $\mathcal{L}$ will always denote the generator (2.1) with the above special form of the constraints.

Theorem 3.1 *Let η be any configuration s.t. there is an infinite number of 0's to the right of the origin. Then there exists $m > 0$ and for any local function f there exists $C_f < \infty$ and $t_0(\eta, f)$ such that for any $t > t_0$*

$$
\left| \mathbb{E}\big(f(\eta_t)\big) - \mu(f) \right| \leq C_f e^{-mt}. \tag{3.2}
$$

Theorem 3.2 *Fix $p' \in (0, 1)$ and let ν be a Bernoulli(p') product measure on Ω. There exists $m > 0$ and for any local function f there exists $C_f < \infty$ such that*:

(a) *for any $t > 0$*

$$
\int d\nu(\eta) \left| \mathbb{E}\big(f(\eta_t)\big) - \mu(f) \right| \leq C_f e^{-mt}; \tag{3.3}
$$

(b) *for ν-almost all configurations η there exists $t_0(\eta, f)$ such that for any $t > t_0(\eta, f)$*

$$
\left| \mathbb{E}\big(f(\eta_t)\big) - \mu(f) \right| \leq C_f e^{-mt}. \tag{3.4}
$$

Remark 3.3 Theorems 3.1 and 3.2 can be extended to the version of East model on any infinite rooted tree with bounded connectivity analized in [2].

Before starting the proof of the above results it is useful to recall that an explicit construction (sometimes referred to as the *graphical construction*) of the process generated by $\mathcal{L}$ goes as follows. Choose $p \in [0, 1]$ and let $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ be a probability space on which are defined countably many independent rate-one Poisson processes and countably many independent Bernoulli(p) random variables. Assign one Poisson process to each site $x \in \mathbb{Z}$ and one Bernoulli variable to each occurrence of each Poisson process. We denote by $\{t_{x,n}\}_{n\in\mathbb{N}}$ the occurrences of the Poisson clock at site x and by $\{s_{x,n}\}_{n\in\mathbb{N}}$ the corresponding coin tosses. The variables $\{t_{x,n}\}_{n\in\mathbb{N}}$ mark the possibilities for site x to change its state. At each time $t_{x,n}$

the site x queries the state of its constraint c_x. If it is satisfied, i.e. if the spin at $x+1$ is 0, then x resets its value to the value of the corresponding Bernoulli variable $s_{x,n}$ (see e.g. [11]). For notation convenience, any occurrence of the Poisson processes such that the constraint at the site of occurrence is satisfied will be called *a legal ring*. The process obtained in this way up to time t and started from the initial configuration η will be denoted by $\{\eta_s\}_{s\le t}$. We stress that the rings and coin tosses at x for $s \le t$ have no influence whatsoever on the evolution of the configuration at the sites which enter in its constraint (here $x+1$) thus they have no influence of whether a ring at x for $s > t$ is legal or not. Any model sharing this property will be called *oriented*.

The next step is to recall the notion of *distinguished zero* introduced in [1]. This definition and the property stated in Lemma 3.5 below depend crucially on the oriented nature of the East constraints. This will be further clarified when proving a similar result (Lemma 4.8) for the AD model in Sect. 4.

Definition 3.4 Given a configuration η, suppose that $\eta(x)=0$ and call the site x *distinguished*. At a later time $s>0$ the position ξ_s of the distinguished zero obeys the following iterative rule. $\xi_s = x$ for all times s strictly smaller than the first *legal* ring of the mean one Poisson clock at x when it jumps to $x+1$. Then it waits for the next legal ring at $x+1$ and when this occurs it jumps to $x+2$ and so on.

Thus, with probability one, the path $\{\xi_s\}_{s\le t}$ is right-continuous, piecewise constant, non decreasing, with at most a finite number of discontinuities at which it increases by one. In the sequel we will adopt the standard notation $\xi_{s^-} := \lim_{\epsilon\uparrow 0}\xi_{s+\epsilon}$. By exploiting the fact that the motion of the distinguished zero for $s>t$ cannot be influenced by the clock rings and coin tosses in (x,ξ_t), Aldous and Diaconis established the following important result. In what follows, for any $V\subset\mathbb{Z}$ and any $\eta\in\Omega$, we will write μ_V, η_V for the marginal of μ on $\{0,1\}^V$ and for the restriction of η to V respectively.

Lemma 3.5 (Lemma 4 of [1]) *Fix an interval $V_0=[x_-,x_0)$. Suppose that $\eta(x_0)=0$ and that $\{\eta_x\}_{x=x_-}^{x_0-1}$ are distributed according to μ_{V_0}. Make x_0 distinguished. Then the conditional distribution of the restriction of η_t to the set $V_t=[x_-,\xi_t)$ given the path $\{\xi_s\}_{s\le t}$ is μ_{V_t}.*

Remark 3.6 Actually Aldous and Diaconis proved the above statement for the conditional distribution given only ξ_t and not the whole path $\{\xi_s\}_{s\le t}$. However, as it is easily checked, the same proof applies in our setting.

Remark 3.7 The main motivation behind the notion of the "distinguished zero" is the following. Given the path $\{\xi_s\}_{s\le t}$, for any pair (s,y) satisfying $s\le t$ and $y<\xi_s$, the variable $\{\eta_s(y)\}$ is uniquely determined by the occurrences of the Poisson processes and coin tosses $\{t_{z,n}, s_{z,n}\}_{n\ge 1}$ such that $t_{z,n}\le t$ and $z<\xi_{t_{z,n}}$ according to the following "conditional graphical construction". Without loss of generality we assume $y<x$. Until the first time (if it exists) the distinguished path $\{\xi_s\}_{s\le t}$ jumps from x to $x+1$ the variables η's in the interval $[y,x-1]$ evolve according to the graphical construction of the usual East model with a fixed zero at x. When the path jumps to $x+1$ (so that all the other variables stay fixed) a new Bernoulli(p) variable is added at the site x and the process starts again in the interval $[y,x]$.

Proof of Theorem 3.1 Let for simplicity $\mu(f)=0$ and fix an interval $[x_-,x_+]$ s.t. $S_f\subset[x_-,x_+]$. Let $x_0(\eta)$ be the position of the first zero to the right of x_+ in η and make x_0 distinguished, namely $\xi_0=x_0$ and ξ_s is the position of the corresponding distinguished zero at time $s \le t$. Given the path $\{\xi_s\}_{s\le t}$, let $0 < t_1 < t_2 < \cdots < t_{n-1} < t$ be its discontinuity points and $t_n = t$. We denote by $\{P_s^{(0)}\}_{s\le t_1}$ the Markov semigroup associated to the East model in the interval $V_0 := [x_-, x_0)$ with a fixed zero boundary condition at x_0. In other words we replace $\mathbb{Z}$ with V_0 and $\mathcal{L}$ with the finite dimensional generator $\mathcal{L}^{(0)}$ given by

$$\mathcal{L}^{(0)} f(\omega) = \sum_{x\in V_0} c_x^{(0)}(\omega)\left[\mu_x(f) - f(\omega)\right] \tag{3.5}$$

where

$$c_x^{(0)}(\omega) = \begin{cases} 1-\omega_{x+1} & \text{if } x \neq x_0 - 1 \\ 1 & \text{otherwise} \end{cases}$$

For any configuration $\sigma \in \{0,1\}^{V_0}$ we write $\sigma \otimes \sigma'$ for the configuration in $\{0,1\}^{[x_-,x_0]}$ obtained from σ by adding the variable $\sigma' \in \{0,1\}$ at the site x_0. With these notation and thanks to the fact that the time evolution inside $[x_-, x_+]$ does not depend on the initial variables $\{\eta(y)\}_{y<x_-}$, we can write

$$\begin{aligned} &\mathbb{E}\big(f(\eta_t) \mid \{\xi_s\}_{s\le t}\big) \\ &\quad = \sum_{\sigma'\in\{0,1\}} \sum_{\sigma\in\{0,1\}^{V_0}} P_{t_1}^{(0)}(\eta_{V_0}, \sigma)\mu_{x_0}(\sigma')\mathbb{E}\Big(f\big((\sigma\otimes\sigma')_{t-t_1}\big) \mid \{\xi_s\}_{t_1\le s\le t}\Big) \end{aligned} \tag{3.6}$$

where, with a slight abuse of notation, $(\sigma\otimes\sigma')_{t-t_1}$ denotes the configuration in the interval $[x_-, \xi_t)$ obtained from the configuration $\sigma\otimes\sigma'$ in the interval $[x_-, x_0]$ according to the conditional graphical construction described in Remark 3.7 applied to the time interval $(t_1, t]$. Therefore, if we let $V_1 := [x_-, x_0 + 1)$, we get

$$\begin{aligned} &\mathrm{Var}_\mu\Big(\mathbb{E}\big(f(\eta_t) \mid \{\xi_s\}_{s\le t}\big)\Big) \\ &\quad \le e^{-2\mathrm{gap} t_1}\, \mathrm{Var}_{\mu_{V_0}}\Big[\sum_{\sigma'\in\{0,1\}} \mu_{x_0}(\sigma')\mathbb{E}\Big(f\big((\sigma\otimes\sigma')_{t-t_1}\big) \mid \{\xi_s\}_{t_1\le s\le t}\Big)\Big] \\ &\quad \le e^{-2\mathrm{gap} t_1}\, \mathrm{Var}_{\mu_{V_1}}\Big[\mathbb{E}\Big(f\big((\sigma\otimes\sigma')_{t-t_1}\big) \mid \{\xi_s\}_{t_1\le s\le t}\Big)\Big] \end{aligned} \tag{3.7}$$

where $\mathrm{gap} > 0$ is the infinite volume spectral gap for East and in the first inequality we used the fact that spectral gap of $\mathcal{L}^{(0)}$ is always greater or equal than gap (see Lemma 2.11 of [3]) and in the second inequality we used convexity of the variance. If $t_1 \neq t$ we can now iterate the above inequality (3.6) for the term inside the variance by replacing t_1 with the second discontinuity point t_2 for the path $\{\xi_s\}_{\sigma\le t}$ (or by $t_2 = t$ if $\xi_t = x_1$) and by replacing $P_s^{(0)}$ with $\{P_s^{(1)}\}_{t_1<s\le t_2}$ defined as the Markov semigroup associated to the East model in V_1 with empty boundary condition on x_1. By continuing the iteration up to $t_n = t$ we get

$$\mathrm{Var}_\mu\Big(\mathbb{E}\big(f(\eta_t) \mid \{\xi_s\}_{s\le t}\big)\Big) \le e^{-2\mathrm{gap} t}\, \mathrm{Var}_\mu(f). \tag{3.8}$$

Furthermore by using $\xi_t \ge x_0 > x_+$, Lemma 3.5, the assumption $\mu(f) = 0$ and the above equality (3.6) it follows that $\mathbb{E}(f(\eta_t) \mid \{\xi_s\}_{s\le t})$ has μ-mean zero w.r.t. the initial configuration η, namely

$$\int d\mu(\eta)\mathbb{E}\big(f(\eta_t) \mid \{\xi_s\}_{s\le t}\big) = 0 \tag{3.9}$$

Finally, putting together (3.8) and (3.9) yields

$$
\begin{aligned}
\left|\mathbb{E}\Big(f(\eta_t)\Big)\right| &\le \mathbb{E}\Big(\big|\mathbb{E}\big(f(\eta_t)\mid\{\xi_s\}_{s\le t}\big)\big|\Big)\\
&\le \Big(1/(p\wedge q)\Big)^{x_0-x_-}\mathbb{E}\Big(\int d\mu(\eta)\big|\mathbb{E}\big(f(\eta_t)\mid\{\xi_s\}_{s\le t}\big)\big|\Big)\\
&\le \Big(1/(p\wedge q)\Big)^{x_0-x_-}\mathbb{E}\Big(\mathrm{Var}_\mu\Big(\mathbb{E}\big(f(\eta_t)\mid\{\xi_s\}_{s\le t}\big)\Big)^{1/2}\Big)\\
&\le \Big(1/(p\wedge q)\Big)^{x_0-x_-} e^{-\mathrm{gap}\,t}\,\mathrm{Var}_\mu(f)^{1/2}
\end{aligned}
\tag{3.10}
$$

where $a\wedge b:=\min\{a,b\}$ and to obtain the third inequality we used Cauchy-Schwartz inequality and (3.9). The claim is proved by taking $C_f=(1/(p\wedge q))^{x_+-x_-}\,\mathrm{Var}_\mu(f)^{1/2}$, $m=\mathrm{gap}/2$ and $t_0(f,\eta)=2(x_0(\eta)-x_+)|\log(p\wedge q)|1/\mathrm{gap}$. □

Proof of Theorem 3.2 Part (b) follows immediately from Theorem 3.1. In order to prove part (a) we use the same notation as above and, for a given $\delta>0$ and local f, we let $\mathcal{A}_{\delta,t}:=\{\eta:\ x_0(\eta)-x_+(f)\ge\delta t\}$. Clearly $\nu(\mathcal{A}_{\delta,t})=(p')^{\delta t}$. We can then split the average $\int d\nu(\eta)|\mathbb{E}(f(\eta_t))-\mu(f)|$ into the contribution from $\eta\in\mathcal{A}_{\delta,t}$ and $\eta\notin\mathcal{A}_{\delta,t}$. By choosing $\delta=\mathrm{gap}/(2|\log(p\wedge q)|)$ and C_f as above, we immediately get

$$
\int d\nu(\eta)\big|\mathbb{E}\Big(f(\eta_t)\Big)-\mu(f)\big|\le\|f\|_\infty e^{-c_\delta t}+C_f e^{-\frac{1}{2}\mathrm{gap}\,t}
\tag{3.11}
$$

with $c_\delta>0$ since $p'\neq 1$. □

4 Non Equilibrium Results for the AD Model

As already mentioned in the introduction the AD model is defined on the (infinite) binary rooted tree $\mathcal{T}$ with root 0. Its generator takes the form (2.1) with constraints given by

$$
c_x(\omega)=\begin{cases}1 & \text{if both children of } x \text{ are zero}\\ 0 & \text{otherwise.}\end{cases}
\tag{4.1}
$$

In this section $\mathcal{L}$ will always denotes the generator (2.1) with the above special form of the constraints. Note that, as for East, this choice is oriented: if we make the graphical construction as in Sect. 3 it is immediate to verify that the rings and coin tosses at x for $s\le t$ have no influence in the evolution of its two children, thus they do not influence the fact that a ring at x for $s>t$ is legal or not. In order to state our results we need to introduce some notation of site percolation on the tree. We call *path* any sequence $\{x^0,x^1,\dots,x^n\}$ of distinct points in $\mathcal{T}$ such that, for all i, x^i is the parent of $x^{(i+1)}$. For a given configuration η we say that $x\to y$ if there is a path of occupied sites starting in x and ending in y (thus $x\to x$ iff $\eta(x)=1$). We also define the *occupied cluster* of x as the random set

$$
\mathcal{C}_x(\eta):=\{y\in\mathcal{T}:x\to y\}.
$$

Let $\mathcal{P}_x^\ell:=\{\eta:|\mathcal{C}_x(\eta)|\ge\ell\}$, let $\mathcal{P}_x^\infty:=\{\eta:|\mathcal{C}_x(\eta)|=\infty\}$ and let $\theta(p):=\mu(\mathcal{P}_0^\infty)$ (recall that μ is the Bernoulli(p) product measure on $\{0,1\}^{\mathcal{T}}$). The corresponding site percolation critical density is defined as

$$p_{sp} := \sup\{p \in [0,1] : \theta(p) = 0\}$$

and, thanks to Proposition 2.5 of [3], it coincides with the threshold of the ergodicity regime for AD model, namely

$$p_{sp} = p_c \tag{4.2}$$

with

$$p_c := \sup\{p \in [0,1] : 0 \text{ is simple eigenvalue of } \mathcal{L}\}.$$

The following results are well known (see for example [8])

Proposition 4.1

(i) $p_{sp} = 1/2$
(ii) *If* $p < 1/2$ *there exists* $\beta(p) > 0$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} |\log \mu(\mathcal{P}_0^n)| \geq \beta \tag{4.3}$$

(iii) *If* $p = 1/2$ *there exists* $c_1, c_2 > 0$ *s.t.*

$$\frac{c_1}{n} < \mu(\mathcal{P}_0^n) < \frac{c_2}{n}. \tag{4.4}$$

As a consequence of the existence of an infinite percolation cluster above p_c it is very easy to see (simply use the test function $f(\eta) = \mathbb{1}_{\mathcal{P}_0^\infty}(\eta)$), that $\text{gap}(\mathcal{L}) = 0$ for $p > p_c$. The same holds at the critical case $p = p_c$ with a slightly subtler proof. Completely open is instead the interesting conjecture made in [1] that $\text{gap}(\mathcal{L}) > 0$ for $p < p_c$. *In all what follows we will always assume that this is the case.*

We are now ready to state our results. In what follows ν will always denotes the Bernoulli(p') product measure on $\mathcal{T}$.

Theorem 4.2 *Let* η *be a configuration s.t.* $|\mathcal{C}_x(\eta)| < \infty$ *for all* x *and assume that* $\text{gap}(\mathcal{L}) > 0$. *Then the same exponential convergence result as in Theorem 3.1 hold true.*

Theorem 4.3 *Let* $0 \leq p' < p_c$ *and assume that* $\text{gap}(\mathcal{L}) > 0$ (*so that necessarily* $p < p_c$). *Then the same exponential convergence results as in Theorem 3.2 hold true.*

We shall now explore the regime outside the validity of the hypothesis for Theorem 4.3.

Theorem 4.4 *If* $p \leq p_c < p'$ *then for any* $c > 0$ *there exists a local* f *s.t. for all* $t > 0$

$$\left| \int d\nu(\eta) \mathbb{E}(f(\eta_t) - \mu(f)) \right| \geq c.$$

If instead $p_c = p'$ and $p < p_c$ we cannot exclude convergence to equilibrium but we can set a bound on the speed of convergence which excludes exponential convergence

Theorem 4.5 *If $p < p_c = p'$ then for any $c > 0$ there exists a local f s.t. for all $t > 0$*

$$\int d\nu(\eta)\, |\mathbb{E}(f(\eta_t) - \mu(f))| \geq \frac{c}{t^2}.$$

The regime $p' < p_c \leq p$ remains to be explored. We conjecture that at least for sufficiently high p there exist local functions that do not converge to equilibrium. This conjecture is supported by the fact that we can prove this result for the following modified AD model.

Consider a non-rooted tree $\widetilde{\mathcal{T}}$ with connectivity three and let the constraint require at least two empty nearest neighbours. On this graph we can define as before the occupied clusters and the site percolation critical density which again coincides with the ergodicity threshold, p_c. Then we recall that on $\widetilde{\mathcal{T}}$ Theorem 1.6 of [9] proves that if the local density is sufficiently large there exists necessarily an infinite percolation cluster. More precisely if $\tilde{\nu}$ is a translation invariant measure on $\widetilde{\mathcal{T}}$, then there exists $1/2 < \delta < 1$ such that $\tilde{\nu}(\eta(0)\eta(1)) \geq \delta$ implies $\tilde{\nu}(\mathcal{P}_0^{\infty}) > 0$. Let ν_t denote the evoluted of time t of the initial Bernoulli(p') measure ν with $p' < p_c$. Thanks to the translation invariance of the constraints and of the initial measure, ν_t is also translation invariant. Furthermore, since the characteristic function $\mathbb{1}_{\mathcal{P}_0}$ is left invariant by the dynamics (an infinite cluster can neither be created nor disrupted), it holds $\nu_t(\mathcal{P}_0) = \nu(\mathcal{P}_0) = \theta(p') = 0$, where the latter equality follows from $p' < p_c$. Therefore if we set $f = \eta(0)\eta(1)$ we have necessarily $\int d\nu(\eta)\mathbb{E}(f(\eta_t)) = \nu_t(f) < \delta$, otherwise $\nu_t(\mathcal{P}_0) = 0$ would be in contradiction with Haggstrom theorem. This inequality together with $\mu(f) = p^2$ yields for any $t > 0$

$$\left| \int d\nu(\eta)\mathbb{E}(f(\eta_t)) - \mu(f) \right| > p^2 - \delta > 0.$$

Thus for this modified (and non oriented) model we have identified a local function which does not converge to equilibrium in the regime $p' < p_c$ and $p > \sqrt{\delta}$.

4.1 The Distinguished Set of Zeros and Its Properties

In analogy with the definition of "distinguished zero" introduced in the analysis of the East model we will begin by defining the *distinguished set of zeros*. In what follows, for any $x \in \mathcal{T}$, $\mathcal{T}_x$ will denote the binary tree rooted at x and $\mathcal{K}_x$ the two children of x. For a subset $\Lambda \subset \mathcal{T}$ the set of vertices outside Λ but such that their parent is in Λ will be denoted by $\partial_+\Lambda$.

Definition 4.6 Consider a region $\Lambda \subset \mathcal{T}$ with the property that

$$\left(\cup_{x \in \partial_+\Lambda} \mathcal{T}_x\right) \cap \Lambda = \emptyset \tag{4.5}$$

and a configuration η such that

$$\eta(x) = 0 \quad \forall x \in \partial_+\Lambda. \tag{4.6}$$

We define the *distinguished set of zeros* $B_{t=0}$ and the *distinguished volume* $V_{t=0}$ at time $t = 0$ to be the sets $\partial_+\Lambda$ and Λ respectively. At a later time $s > 0$, the distinguished set of zeros and the distinguished volume are defined as follows. $V_s = V_0$ and $B_s = B_0$ until the first time t_1 at which a legal ring occurs for one of the Poisson clocks at the sites in B_0. Call x_0 this site. Then we set $V_{t_1} = V_0 \cup x_0$ and $B_{t_1} = (B_0 \cup \mathcal{K}_{x_0}) \setminus \{x_0\} = \partial_+ V_{t_1}$ and the rule is iteratively applied to define the distinguished volume and border at any later time.

Note that, for any $t < \infty$, with probability one there are at most a finite number of times $0 < t_1 < t_2 < \cdots < t_n < t$ such that $V_{t_i+1} \neq V_{t_i}$ and $B_{t_i+1} \neq B_{t_i}$. For all $s < t$ the following properties hold:

Claim 4.7

(i) $V_t \supseteq V_s$;
(ii) $V_t \subseteq \Lambda \cup (\cup_{x \in \partial \Lambda_+} \mathcal{T}_x)$;
(iii) $B_t = \partial_+ V_t \subseteq \cup_{x \in \partial \Lambda_+} \mathcal{T}_x$;
(iv) $\eta_t(x) = 0 \ \forall x \in B_t$;
(v) *if* $x \neq y$ *and* $x, y \in B_t$, *then* $\mathcal{T}_x \cap \mathcal{T}_y = \emptyset$;
(vi) $(\cup_{x \in B_t} \mathcal{T}_x) \cap V_t = \emptyset$;
(vii) *for all* i, *given* V_{t_i} *and* t_i, *the random variable* $t_{i+1} - t_i$ *does not depend on the occurrences of the Poisson clocks at sites* $x \in V_{t_i}$ *for times* $t > t_i$ *neither on the corresponding coin tosses.*

Proof (i), (ii), (iii) and (iv) follow immediately from Definition 4.6.

(v): Let $x, y \in B_0$ and assume by contradiction that $\mathcal{T}_x \cap \mathcal{T}_y \neq \emptyset$. Then, thanks to the tree structure, either $x \in \mathcal{T}_y$ (and $\mathcal{T}_x \subset \mathcal{T}_y$) or $y \in \mathcal{T}_x$ (and $\mathcal{T}_y \subset \mathcal{T}_x$). Consider the former case (the other may be treated analogously) and call z the ancestor of x. Since $x \neq y$, z also belongs to $\mathcal{T}_y$. But this is in contradiction with hypothesis (4.5), since $x \in \partial_+ \Lambda$ implies $z \in \Lambda$. Thus property (v) holds at $t = 0$. Let us proceed by induction: suppose (v) holds up to t_i (and therefore also for $t_i < s < t_{i+1}$), we will prove that it holds at t_{i+1}. Let $x, y \in B_{t_{i+1}}$. Since $B_{t_1+1} = (B_{t_i} \setminus x_i) \cup \mathcal{K}_{x_i}$ either $x, y \in B_{t_i} \setminus x_i$ or $x, y \in \mathcal{K}_{x_i}$ or $x \in B_{t_i} \setminus x_i$, $y \in \mathcal{K}_{x_i}$ (or the converse). Property (v) follows immediately in the first case by the induction hypothesis, in the second case by the tree structure, in the third case because $y \in \mathcal{T}_{x_i}$ and by the induction hypothesis.

(vi): At time zero the property holds by Definition 4.6. Let us suppose it holds at t_i, we will now prove it holds at t_{i+1}. From Definition 4.6 it is immediate to verify that

$$\bigcup_{x \in B_{t_{i+1}}} \mathcal{T}_x \cap V_{t_{i+1}}$$
$$= \Bigg(\bigcup_{x \in B_{t_i} \setminus x_i} \mathcal{T}_x \cap V_{t_i}\Bigg) \cup \Bigg(\bigcup_{x \in \mathcal{K}_{x_i}} \mathcal{T}_x \cap V_{t_i}\Bigg) \cup \Bigg(\bigcup_{x \in B_{t_i} \setminus x_i} \mathcal{T}_x \cap x_i\Bigg) \cup \Bigg(\bigcup_{x \in \mathcal{K}_{x_i}} \mathcal{T}_x \cap x_i\Bigg)$$

The proof is then completed by noticing that all the above sets are empty: the first and second ones thanks to the recursive hypothesis ($x \in \mathcal{K}_{x_i}$ implies $\mathcal{T}_x \subset \mathcal{T}_{x_i}$), the third one thanks to property (v) (note that $x_i \in B_{t_i}$ and $x_i \in \mathcal{T}_{x_i}$), the forth one because $x \in \mathcal{K}_{x_i}$ implies $x_i \notin \mathcal{T}_x$.

(vii): the time $t_{i+1} - t_i$ is the time before the first legal ring of a site $x \in B_{t_i}$ and it clearly depends only on the Poisson clocks and coin tosses at $\bigcup_{x \in B_{t_i}} \mathcal{T}_x$. The desired independency property then follows from property (vi). □

We are now ready to state the analog of Lemma 3.5

Lemma 4.8 *Consider a region* Λ *and a configuration* η *which satisfy the hypothesis* (4.5) *and* (4.6) *of Definition* 4.6 *and make* Λ *and* $\partial_+ \Lambda$ *distinguished. If the restriction of* η *to* Λ *is distributed according to* μ_Λ, *then for each* $t > 0$ *the conditional distribution of* η_t *restricted to* V_t *given* $\{V_s\}_{s \leq t}$ *is* μ_{V_t}.

Proof Let $\tilde{\nu}_t$ be the marginal on V_t of the conditional distribution of η_t given $\{V_s\}_{s\le t}$ and distinguish two cases.

(a) $V_t = V_0$. The evolution up to time t inside V_0 is therefore the evolution of the model with empty boundary condition on $\partial_+ V_0$. This is true thanks to the property (iv) of Claim 4.7. Denoting by $P_t^{(0)}$ the corresponding Markov semigroup on $\Omega_{V_t} = \Omega_{V_0} = \{0,1\}^{|V_0|}$ and recalling that $P_t^{(0)}$ is reversible with respect to μ_{V_0} we immediately get $\tilde{\nu}_t(\sigma) = \mu_{V_t}(\eta)$.

(b) $V_t \ne V_0$. We denote by $t_1, t_2, \dots, t_n$ the subsequent times $0 < t_1 < t_2 < \cdots < t_n < t$ at which V_s changes and by $\eta_{t_i^-}$ ($\eta_{t_i^+}$) the configurations before (after) the change occurring at t_i. By the previous argument it is immediate to verify that $\tilde{\nu}_{t_1^-} = \mu_{V_{t_1^-}}$. We shall now assume inductively that $\tilde{\nu}_{t_i^-} = \mu_{V_{t_i^-}}$ and prove that $\tilde{\nu}_{t_{i+1}^-} = \mu_{V_{t_{i+1}^-}}$. If we denote by x_i the site belonging to B_{t_i-} on which the legal rings occurs at t_i, namely $x_i = V_{t_{i+1}} \setminus V_{t_i}$, the restriction of $\eta_{t_i^+}$ to $V_{t_{i+1}}$ is given by the restriction of $\eta_{t_i^-}$ to $V_{t_i^-}$ plus an independent Bernoulli(p) random variable at site x_i. Thus it follows immediately from the induction hypothesis that $\tilde{\nu}_{t_i^+} = \mu_{V_{t_i^+}}$. Then, noticing that $B_{t_i^+} = \partial_+ V_{t_i^+}$ stays empty up to time t_{i+1}^-, we can denote by $P_t^{(i+1)}$ the Markov semigroup with empty boundary conditions on $\Omega_{V_{t_i^+}}$ and apply the same argument as in (a) to conclude that $\tilde{\nu}_{t_{i+1}^-} = \mu_{V_{t_{i+1}^-}}$. □

4.2 Proof of the Theorems

Proof of Theorem 4.2 The proof follows the same pattern of the proof of Theorem 2.1. For a given local function f with support $S_f \subset \mathcal{T}$ we denote by $\mathcal{T}_f$ the smallest regular substree of $\mathcal{T}$ containing S_f. Given a configuration η such that any percolation cluster $\mathcal{C}_x(\eta)$ is finite, we denote be $\mathcal{A}_f(\eta) = \bigcup_{y\in\partial_+\mathcal{T}_f} \mathcal{C}_y(\eta)$. If we set $V_0(\eta) := \mathcal{A}_f(\eta) \cup \mathcal{T}_f$, V_0 clearly verifies property (4.5) and (4.6). Therefore we can make V_0 and $\partial_+ V_0$ distinguished at time 0 and call V_t and B_t the corresponding distinguished sets at time t. Given the path $\{V_s\}_{s\le t}$, we denote by t_1 be the first time at which $V_s \ne V_0$ and by $\{P_s^{(0)}\}_{s\le t_1}$ the Markov semigroup associated to AD model on V_0 with empty boundary conditions on $\partial_+ V_0$ (as in (3.5)). Then we get

$$\begin{aligned}&\mathbb{E}(f(\eta_t)|\{V_s\}_{s\le t})\\&\quad= \sum_{\sigma'\in\{0,1\}} \sum_{\sigma\in\{0,1\}^{V_0}} P_{t_1}^0(\eta_{V_0},\sigma)\mu_{x_0}(\sigma')\mathbb{E}(f((\sigma\times\sigma')_{t-t_1})|\{V_s\}_{t_1\le s\le t}).\end{aligned} \tag{4.7}$$

In analogy with what we did to derive (3.8) for the East model and under the hypothesis $\mathrm{gap}(\mathcal{L}) > 0$, it follows that

$$\mathrm{Var}_\mu\left(\mathbb{E}(f(\eta_t)|\{V_s\}_{s\le t})\right) \le e^{-2\mathrm{gap}t}\,\mathrm{Var}_\mu(f). \tag{4.8}$$

Lemma 4.8 yields

$$\int d\mu(\eta)\mathbb{E}(f(\eta_t)|\{V_s\}_{s\le t}) = 0 \tag{4.9}$$

and again in analogy to the procedure used to derive (3.10) we get

$$|\mathbb{E}\left(f(\eta_t)\right)| \le (1/(p\wedge q))^{|V_0(\eta)|} e^{-\mathrm{gap}t}\left(\mathrm{Var}_\mu(f)\right)^{1/2}. \tag{4.10}$$

The proof is then completed by choosing $C_f = (1/(p\wedge q))^{|\mathcal{T}^f|}\,\mathrm{Var}_\mu(f)^{1/2}$, $m = \mathrm{gap}(\mathcal{L})/2$ and $t_0(f,\eta) = -2|\mathcal{A}_f(\eta)|\log(p\wedge q)|/\mathrm{gap}(\mathcal{L})$. □

Proof of Theorem 4.3 The proof of part (b) follows immediately from Theorem 4.2 and $p < p_c$. Part (a): for a chosen δ and f we let $\mathcal{E}_{\delta,t} := \{\eta : |\mathcal{A}_f(\eta)| > \delta t\}$. Thanks to (4.3), asymptotically in t, we can bound the probability of this event as

$$\nu(\mathcal{E}_{\delta,t}) \leq p^{-2^\ell} \nu(|\mathcal{C}_0| > 2^\ell + \delta t) \leq p^{-2^\ell} \exp(-\delta t \beta). \tag{4.11}$$

Thus, if we use (4.10) together with (4.11) and choose C_f as for Theorem 3.2 and $\delta = \text{gap}/(2|\log(p \wedge q)|$ we get

$$\int d\nu(\eta) \Big| \mathbb{E}(t(\eta_t)) \Big| \leq \|f\| p^{-2^\ell} e^{-tc_\delta} + C_f e^{-t\text{gap}/2} \tag{4.12}$$

$\square$

Proof of Theorem 4.4 Fix ℓ and let f and g be the characteristic functions of the event that the occupied cluster of the origin has cardinality at least ℓ and infinite cardinality, respectively. Namely $f = \mathbb{1}_{\mathcal{P}_0^\ell}$ and $g = \mathbb{1}_{\mathcal{P}_0}$ (thus f is local and g is not local). Then for any choice of ℓ it holds $f(\eta) > g(\eta)$. Since g is left invariant by the dynamics we have

$$\int d\nu(\eta) \mathbb{E}\Big(f(\eta_t)\Big) \geq \nu(\mathcal{P}_0) = \theta(p') > 0.$$

On the other hand from Proposition 4.1 (case (ii) if $p < p_c$ or the upper bound of case (iii) if $p = p_c$), provided ℓ is chosen sufficiently large, $\ell > \bar{\ell}(c, p')$, it holds

$$\mu(f) \leq \theta(p') - c$$

and the proof is concluded. $\square$

Proof of Theorem 4.5 We start by inequality

$$\int d\nu(\eta) \int d\nu(\sigma)\,(P_t f(\eta) - P_t f(\sigma))^2 = \text{Var}_\nu(P_t f) \leq \|f\|_\infty \nu(|P_t f|).$$

Then we lower bound the left hand side by requiring that: (i) $\eta(0) = 1$, (ii) $|C_0(\eta)| \geq 2t$, (iii) $\sigma(0) = 0$, (iv) $|C_0(\sigma)| \geq 2t$ and (v) up to time t neither for the evoluted of η nor for the evoluted of σ the ordered sequence of $2t$ rings necessary to make the origin unconstrained has occurred. By using the lower bound in Proposition 4.1(iii) to bound the events (ii) and (iv) and the large deviation for the Poisson distribution for event (v) we get the desired result. $\square$

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Appendix: An Alternative Proof for the Upper Bound on the Spectral Gap of the East Model

Consider East model on $\mathbb{Z}$ and let $q := 1 - p$. We will now present an argument different from the one in Sect. 5 of [1] to prove a sharp upper bound for its spectral gap. This will help to further clarify the role played by dynamical energy barriers.

Theorem 5.1 *There exists a constant C independent of $q < 1/2$ such that* gap $< Cq^{-2}q^{\log(1/q)/(2\log 2)}$

Proof Let $\ell = 1/q$. By using Lemma 2.11 of [3] we can upper bound the spectral gap on $\mathbb{Z}$ with the one on $[0,\ell)$ with zero boundary condition on $x=\ell$ (defined by (3.5) with $V_0=[0,\ell)$), which we call gap_ℓ. With a little abuse of notation we write here μ for $\mu_{[0,\ell)}$. If we consider the East model Markov chain in $[0,\ell)$ in discrete time (at each step choose a random site in $[0,\ell)$ and try to update the current configuration with the correct East probabilities), the obvious relation

$$\mathrm{gap}_\ell = \ell\,\mathrm{gap}_{\mathrm{d},\ell}$$

holds true, where $\mathrm{gap}_{d,\ell}$ is the spectral gap in the discrete time setting. In the sequel we denote by $\vec{1}$ the completely filled configuration and by $\mathbb{P}_{\vec{1}}(A)$, where A is an event which depends on $\eta_{s\,\{s\geq 0\}}$, the probability of A under the discrete evolution started at time zero from $\vec{1}$. Finally we denote by T the first time there are $n\equiv\lfloor\log_2(\ell)\rfloor$ zeros and by T_0 the first time there is exactly one zero located at the origin.

If the process starts from $\vec{1}$, then $T_0\geq T$. In fact, on the half lattice with zero boundary condition, starting from all ones and under the condition that at most n zeros can be created, in [4] it has been proven that:

(1) the minimum distance from the origin of the zero in a configuration with only one zero is $\ell - 2^{n-1}$;
(2) the set Ω_0 of different configurations that the chain can explore has cardinality $2^{\binom{n}{2}}n!c^n$ with $c\approx 0.67$.

Thus up to time T the cardinality of the set Ω_0 of accessible configurations is at most $2^{\binom{n}{2}}n!c^n$ and necessarily (provided $1/(2q)=\ell/2>1$) $T\leq T_0$ since otherwise the configuration with exactly one single zero at the origin would have been unreachable.

Remark 5.2 The entropic factor $2^{\binom{n}{2}}n!c^n$ is much smaller (for small q) than the binomial entropic factor $\binom{\ell}{n}$.

We denote by $\Omega_n\subset\Omega_0$ those configurations with exactly n zeros. For $t=\frac{1}{2}q^{-(n/2)(1-o(1))}$ and q small enough we have:

$$\begin{aligned}\mathbb{P}_{\vec{1}}(T<t) &\leq t\sum_{\sigma\in\Omega_n}\sup_s \mathbb{P}_{\vec{1}}(\sigma_s=\sigma)\leq t2^{\binom{n}{2}}n!c^n\frac{1}{\mu(\vec{1})}q^n p^{\ell-n}\\ &\leq tq^{(n/2)(1-o(1))}\leq\frac{1}{2}.\end{aligned}$$

Next we recall that if T_A denotes the hitting time of a set A, then

$$\mathbb{P}_\mu(T_A\geq t)\leq e^{-\lambda_A t}$$

where ($\mathcal{D}_{\mathrm{d}}(f)$ is the discrete-time Dirichlet form of f)

$$\lambda_A := \inf\left\{\mathcal{D}_{\mathrm{d}}(f):\ \mu(f^2)=1,\ f\equiv 0 \text{ on } A\right\}\geq\mu(A)\mathrm{gap}_{\mathrm{d},\ell}.$$

We apply the above observation to the set A consisting of the single configuration with all ones except the origin, thus $\mu(A) = q(1-q)^{n-1}$ and $T_A = T_0$. For $t = \frac{1}{2}q^{-(n/2)(1-o(1))}$ we get

$$e^{-t\mu(A)\mathrm{gap}_{\mathrm{d},\ell}} \geq \mathbb{P}_\mu(T_0 \geq t) \geq p^\ell \mathbb{P}_1(T_0 > t) \geq p^\ell \mathbb{P}_1(T > t) \geq \frac{e^{-1}}{2}$$

which implies that

$$t\mu(A)\mathrm{gap}_{\mathrm{d},\ell} \leq 1 + \log(2).$$

In conclusion

$$\mathrm{gap} \leq \mathrm{gap}_\ell = \ell \mathrm{gap}_{\mathrm{d},\ell} \leq Cq^{-2} q^{(n/2)(1-o(1))}. \tag{5.1}$$

□

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