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Classical Motion in Force Fields with Short Range Correlations

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Abstract We study the long time motion of fast particles moving through time-dependent random force fields with correlations that decay rapidly in space, but not necessarily in time. The time dependence of the averaged kinetic energy $\langle p^2(t)\rangle/2$ and mean-squared displacement $\langle q^2(t)\rangle$ is shown to exhibit a large degree of universality; it depends only on whether the force is, or is not, a gradient vector field. When it is, $\langle p^2(t)\rangle \sim t^{2/5}$ independently of the details of the potential and of the space dimension. The stochastically accelerated particle motion is then superballistic in one dimension, with $\langle q^2(t)\rangle \sim t^{12/5}$, and ballistic in higher dimensions, with $\langle q^2(t)\rangle \sim t^2$. These predictions are supported by numerical results in one and two dimensions. For force fields not obtained from a potential field, the power laws are different: $\langle p^2(t)\rangle \sim t^{2/3}$ and $\langle q^2(t)\rangle \sim t^{8/3}$ in all dimensions $d \geq 1$.

Keywords Stochastic acceleration · Diffusion · Random potential

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1 Introduction

We study in this paper the motion

$$\ddot{q}(t) = F(q(t), t) \tag{1.1}$$

of fast particles in random force fields with correlations that are short-range in space, but not necessarily in time. We consider models of two different general classes. In the first, upon which we focus most of our attention, the force is assumed to be of the form

$$F(q,t) = \sum_N f_N\left(\frac{q - q_N}{\ell}, \frac{t}{\sigma}\right), \tag{1.2}$$

where the f_N are smooth functions of compact support in a ball of radius $1/2$ centered at 0, with additional characteristics detailed in Sect. 2; $\ell, \sigma > 0$ are a length and a time scale. The f_N model a random or periodic array of identical, randomly-oriented scatterers, centered at points q_N, that evolve periodically or quasi-periodically in time. We assume $\inf_{N \neq M} \|q_N - q_M\| \geq \ell$ so that the local forces $f_N(\frac{q-q_N}{\ell}, \frac{t}{\sigma})$ do not overlap. As a result, the particle interacts with at most one scatterer at a time, and otherwise travels freely between collisions, through a random potential. The model therefore describes an *inelastic and non-dissipative soft Lorentz gas*, i.e., a distribution of "soft" scatterers centered at the points q_N, off which the particle bounces inelastically. We introduce randomness in the initial data, and assume the system to have finite horizon, so any trajectory of a free particle intersects the support of F at some future time t.

To explain our terminology and for comparison's sake, we recall that in the standard Lorentz gas, scattering is *elastic*, and scatterers are identical hard unchanging obstacles centered at fixed points q_N, with a spatial distribution chosen either randomly, periodically, or quasi-periodically. Unlike in the current model, the particle's energy in the Lorentz gas is conserved and the particle's mean squared displacement has been proven to be diffusive as a result of the strong chaotic properties of the local dynamics [5]. A hard but *inelastic* Lorentz gas is studied in [14, 15], where the obstacles' radii oscillate in time, periodically or randomly. The particle's kinetic energy is argued to grow in time, with an exponent close to one in both cases. The behaviour of the mean squared displacement of the particle is however not analyzed in these works. Finally, a *dissipative* model, related both to the Lorentz gas and to the models considered here, was studied in [12, 20]; there the scattering mechanism was provided by a one-dimensional periodic array of oscillators representing environmental degrees of freedom of the medium in thermal equilibrium. The full system consisting of the particle in interaction with the oscillator bath was treated with a Hamiltonian dynamics, with conservation of the total energy. The Hamiltonian interaction of the particle with the oscillator bath then provides, in addition to a random force, an effective friction force that allows the particle to *dissipate* any excess energy, and to thus equilibrate with its environment. As a result, it is shown the particle's averaged kinetic energy is asymptotically constant and the particle's motion is diffusive with a temperature dependent diffusion constant.

The force (1.2) considered in the present paper can be thought of as being obtained from those of [12, 20] by switching off the friction component of the force provided by the particle's back reaction with the medium. The stochastic acceleration of the particle induced by the random force field leads, then, to an unbounded acceleration of the particle. In this paper we compute the power laws associated, e.g., with the growth in time of the particle's average kinetic energy $\langle p^2(t)\rangle/2$ and mean-squared displacement $\langle q^2(t)\rangle$, as well as the time scales on which these phenomena occur.

In the other class of models that we consider, the force $F(q,t)$ is modeled as a space and time homogeneous random field satisfying

$$\langle F(q,t)\rangle = 0, \qquad \left\langle F(q,t)\, F\left(q',t'\right)\right\rangle = \frac{\ell^2}{\sigma^4} C\left(\frac{q-q'}{\ell}, \frac{t-t'}{\sigma}\right), \tag{1.3}$$

where C is a matrix function of rapid decay in the spatial variable, but need not decay in the time variable. For these models, as with (1.2), we are interested in characterizing the asymptotic growth of $\langle p^2(t)\rangle$ and $\langle q^2(t)\rangle$.

There has been a fair amount of work reported in the physics and mathematical physics literature on problems of this type, partially motivated by questions in plasma physics, astronomy, and solid state physics (see for example [4, 19, 21, 22]). Previous mathematically rigorous work has mostly dealt with deriving, under suitable scalings, Fokker-Planck equations for the particle density (as in [8, 16]). Unfortunately, analyses of this type do not directly give information about the asymptotic behavior of the particles' kinetic energy or mean squared displacement. The theoretical physics literature is mostly concerned with Gaussian random potentials and contradictory claims have been made regarding the power law growth of $\langle p^2(t)\rangle$ and $\langle q^2(t)\rangle$. For potential fields that are delta correlated in time, but not in space, it is generally agreed (see for example [10]) that in the weak coupling limit $\langle p^2(t)\rangle \sim t$ and $\langle q^2(t)\rangle \sim t^3$, but there is some controversy on what happens when the Gaussian potential field has temporal correlations of nonzero and finite duration. For this case it is argued in [7, 13, 18] that in $d=1$, $\langle p^2(t)\rangle \sim t^{2/5}$ and that $\langle q^2(t)\rangle \sim t^{12/5}$ (compatible with numerical and theoretical results presented here). In [9], on the other hand, it is claimed that for $d=1$, $\langle q^2(t)\rangle \sim t^3$, as in the case when the random potentials are delta correlated in time. For $d>1$ it is found in [7] that $\langle p^2(t)\rangle \sim t^{1/2}$, and that $\langle q^2(t)\rangle \sim t^{9/4}$. In [18], the conclusions of [7] for $d>1$ are contested and it is argued that for Gaussian random potentials with fast decaying spatial and temporal correlations $\langle p^2(t)\rangle \sim t^{2/5}$ in all dimensions, and $\langle q^2(t)\rangle \sim t^2$ for $d>1$.

Although there is some numerical work [13] that supports the predictions for $d=1$ of [7, 13, 18], to the best of our knowledge no numerical simulations have been performed in higher dimensions. To help resolve the existing controversy on this subject we present in this paper numerical results in one and two dimensions on a particularly simple (non-Gaussian) model whose random force can be expressed as in (1.2), and which allows for an efficient numerical integration of the equations of motion out to very long times. Full details of the numerical calculations are presented in Sect. 7, but our essential results for the case in which the force F is derived from a potential field are presented in Figs. 1 and 2, where we plot the quantities $\langle v^2\rangle = \langle (p\sigma/\ell)^2\rangle$ and $\langle y^2\rangle = \langle (q/\ell)^2\rangle$, as functions both of the dimensionless time $\tau = t/\sigma$ and of the collision number n, which labels the number of scattering centers visited by the particle.

Our numerical results indicate that in both one and two dimensions

$$\left\langle v^2(\tau)\right\rangle \sim \tau^{2/5}, \qquad \left\langle v_n^2\right\rangle \sim n^{1/3}, \tag{1.4}$$

which is in agreement with [7, 13, 18]. In one dimension the particle's mean-squared displacement is superballistic, with

$$\left\langle y^2(\tau)\right\rangle \sim \tau^{12/5}, \qquad \left\langle y_n^2\right\rangle \sim n^2. \tag{1.5}$$

In two dimensions, however, $\langle y^2(\tau)\rangle$ becomes ballistic, i.e.,

$$\left\langle y^2(\tau)\right\rangle \sim \tau^2, \qquad \left\langle y_n^2\right\rangle \sim n^{5/3}. \tag{1.6}$$

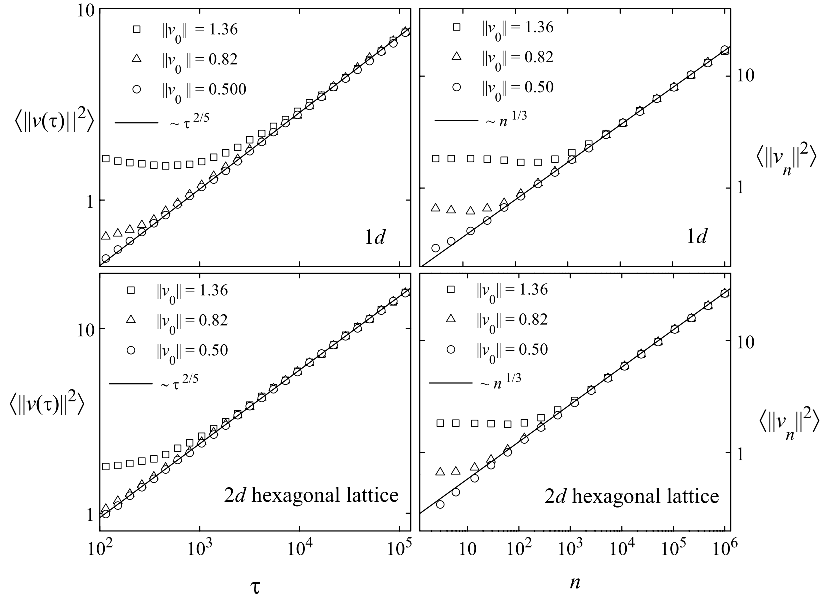

Fig. 1 Numerically determined values of $\langle v^2(\tau)\rangle$ and $\langle v_n^2\rangle$ in one dimension (*top*) and for a two-dimensional hexagonal lattice (*bottom*), for the model described in Sect. 7. In each plot, the different symbols correspond to different initial conditions, as indicated, the straight lines to the power laws in (1.4)

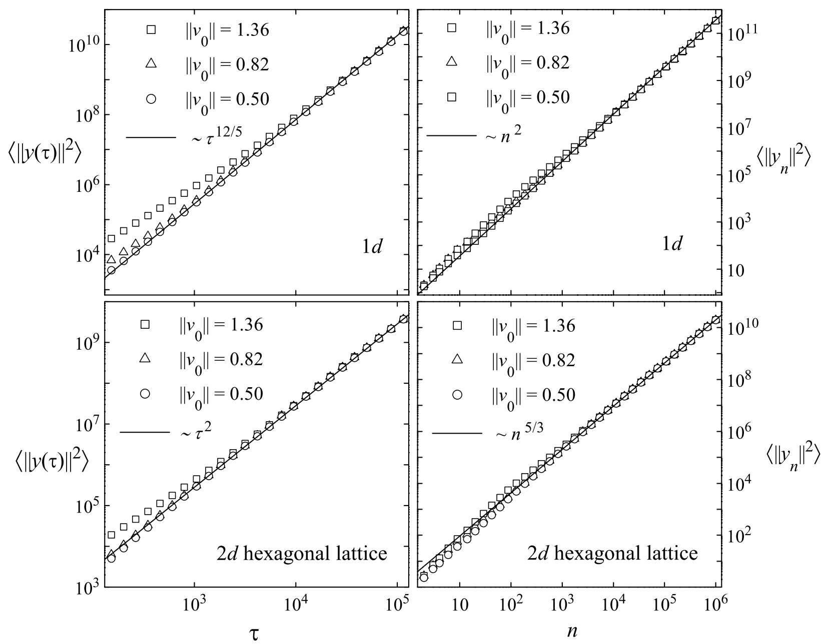

Fig. 2 Numerically determined values of $\langle y^2(\tau)\rangle$ and $\langle y_n^2\rangle$, in one dimension (*top*), and for a two-dimensional hexagonal lattice (*bottom*), for the model described in Sect. 7. In each plot, different symbols correspond to different initial conditions, as indicated, the straight lines to the power laws in (1.5) and (1.6)

This is different from what was predicted in [7] for Gaussian potentials, but in agreement with predictions made for this case in [18].

To understand our numerical results in one and two dimensions, and to more firmly establish what happens for the models of the type (1.2) and (1.3) in higher dimensions, we present in the bulk of this paper a unified mathematical analysis that captures the essential physics of the problem. It provides in particular a means for calculating the power law growth of the mean kinetic energy and the mean-squared displacement associated with an ensemble of particles moving in time-dependent random force fields of the types described above.

The analysis is based on consideration of the typical trajectory of a particle moving in a fluctuating force field described by (1.2) or (1.3), which can be viewed as a sequence of isolated scattering events. We argue, in fact, that the motion is well approximated by a coupled discrete-time random walk for the particle's momentum and position. Each time step corresponds to one collision of the particle with a single scatterer, or to one traversal by the particle of a distance of the order of the correlation length of the potential. Momentum increments are treated as independent random events whose magnitude depends upon the particle speed. Theoretical analysis of the resulting random walk reveals that the high velocity behavior of the momentum change of the particle during one such scattering event completely determines the asymptotic properties of the motion. As we show, this high energy behavior is insensitive to the details of the force field, notably to its statistical properties or to the precise geometry of the scattering centers; the asymptotic behavior of the motion is therefore quite universal, and in particular not a result that arises only with Gaussian potential fields.

Indeed, for general force fields obtainable as the gradient of a potential field, we find (Theorem 4.1) that the energy change incurred by a particle of velocity v satisfies $\Delta E \sim \|v\|^{-1}$. This fact, combined with our analysis of the resulting random walk in momentum and position space leads to an increase of $\langle p^2(t)\rangle$ that is in all dimensions of the form observed in Fig. 1, and as described by (1.4). In one dimension, $\langle q^2(t)\rangle$ is predicted by our analysis to grow in time as in (1.5), and as observed in the top left panel of Fig. 2. In all higher dimensions it is predicted to grow as observed in the bottom left panel in Fig. 2, and as described by (1.6). This slower growth of $\langle q^2(t)\rangle$ in higher dimensions arises from the fact that the particle can now turn while traveling, as its velocity vector performs an orientational random walk resulting from small random deflections.

Our analysis can also be applied to the case where $F(q,t)$ does not derive from a potential field, a situation which has attracted some attention in the mathematics literature. We find for a non-gradient force field that the energy change in a single scattering is considerably larger than in the gradient case: $\Delta E \sim 1$ (Theorem 5.1). Consequently, we predict a larger rate of acceleration $\langle p^2(t)\rangle \sim t^{2/3}$ (see (5.6)), that confirms rigorous results that have been obtained for $d \geq 4$ in [6, 11] under suitable technical conditions on the forces f_N in (1.2). Our analysis then leads to the prediction that in all dimensions particle motion in the presence of a non-gradient random force field is superballistic with $\langle q^2(t)\rangle \sim t^{8/3}$ (see (5.7)). The fundamental reason for the difference with the gradient field case is that the particle turns more slowly while traveling, because it accelerates more quickly, and so is less easily deflected. The difference between the two situations can be traced to the fact that time-dependent gradient force fields produce smaller changes in the particle's energy than non-gradient force fields do. This is a remnant of the energy conservation that is a characteristic feature of time-independent gradient fields.

The problem addressed here is related to that of the energy growth of a confined particle in a potential with (quasi-)periodic time dependence, as occurs in pulsed or kicked rotors and in so-called Fermi accelerators. It will be shown in a forthcoming paper how the techniques developed here can be applied to those problems as well [1].

The rest of the paper is organized as follows. In Sect. 2 we introduce a random walk description of the motion of a particle moving in a field of scatterers. General features of the walk that pertain to both gradient and non-gradient force fields are derived in Sect. 3. Section 4 is devoted to a derivation of the above power laws for the case of a gradient force field, and Sect. 5 analyzes the non-gradient case. In Sect. 6, we adapt our analysis of Sects. 3–5 to random force fields as described by (1.3), obtaining results for the gradient and nongradient case identical to those found, respectively, in Sects. 4 and 5. Details of our numerical calculations, the results of which are presented in figures distributed throughout the paper, are given in Sect. 7. Proofs of mathematical results used for the analysis in Sects. 3–6 comprise the Appendix.

2 Particle in a Field of Scatters: A Random Walk Description

We first describe precise conditions on the functions f_N in (1.2) under which we work. We systematically use rescaled variables ($\ell > 0, \sigma > 0$)

$$\tau = \frac{t}{\sigma} \in \mathbb{R}, \qquad y(\tau) = \frac{q(t)}{\ell} \in \mathbb{R}^d, \qquad v(\tau) = \dot{y}(\tau) = \frac{\sigma}{\ell} p(t), \qquad x_N = \frac{q_N}{\ell}$$

and suppose f_N to be of the form

$$f_N(y, \tau) = \frac{\ell}{\sigma^2} c_N M_N g\left(M_N^{-1} y, \omega\tau + \phi_N^0\right). \tag{2.1}$$

The locations $x_N, N \in \mathbb{Z}^d$ of the scattering centers can be chosen either randomly (with uniform density) or lying on a regular lattice. The coupling constants c_N are independent random variables taking values in $[-1, 1]$ and distributed according to a common probability measure ν not concentrated on 0. The M_N are rotations belonging to $\mathrm{SO}(d, \mathbb{R})$ and are also i.i.d., according to the left-invariant Haar measure on $\mathrm{SO}(d, \mathbb{R})$. Thus, the scatterers are identical objects randomly oriented in space, all described by the same function $g : \mathbb{R}^d \times \mathbb{T}^m \to \mathbb{R}^d$ which is smooth and supported in the ball of radius $1/2$ in its first variable; $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ is the m-torus and $\omega \in \mathbb{R}^m$, $\|\omega\| = 1$. When ω has components that are independent over the rationals, the force is quasi-periodic in time, otherwise it is periodic. The parameters $\phi_N^0 \in \mathbb{T}^m$ are i.i.d. random initial phases, uniformly distributed on the torus. We write $\mathrm{d}\mu(M, \phi, c)$ for the above described probability measure on $\mathrm{SO}(d, \mathbb{R}) \times \mathbb{T}^m \times [-1, 1]$. The force may or may not derive from a potential. The above class of models is sufficiently rich to allow for the description of pulsing, vibrating, and rotating scattering centers; for an explicit example, see Sect. 7.

In the rescaled variables, the equations of motion (1.1)–(1.2) become

$$\ddot{y}(\tau) = G(y(\tau), \tau), \quad G(y, \tau) = \sum_N c_N M_N g\left(M_N^{-1}(y - x_N), \omega\tau + \phi_N^0\right). \tag{2.2}$$

One should think of $g(y, \omega\tau + \phi)$ as the force produced by a soft, time-dependent scatterer centered at the origin; G then describes a *field* of *identical* scatterers, randomly oriented, and centered at the points x_N. We assume the system has a finite horizon, so that the distance over which a particle can freely travel is less than some fixed distance $L > 0$, uniformly in time and space and independently of the direction in which it moves. Thus, with probability one, for all $(y, v, \tau) \in \mathbb{R}^{2d} \times \mathbb{R}$ such that $G(y, \tau) = 0$,

$$\sup\left\{\tau' > 0 \mid \forall 0 \le \tau'' \le \tau', G\left(y + v\tau'', \tau + \tau''\right) = 0\right\} \le \frac{L}{\|v\|}.$$

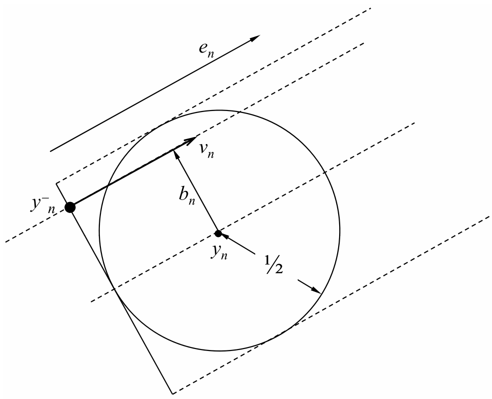

Fig. 3 A particle at time τ_n impinging with velocity v_n and impact parameter b_n on the nth scatterer, centered at the point y_n

We consider a particle which at time $\tau_0 = 0$ is close to a scatterer at $x_0 = 0$ and moving toward it with initial velocity v_0 along an initial direction that, if followed without deflection, would find the particle at its closest approach to the force center located at a point defined by the impact parameter $b_0 \in \mathbb{R}^d$ (see Fig. 3). After inelastically scattering from the center at x_0 the particle moves freely with a new velocity v_1 until it encounters a second scatterer, and in this way it undergoes a random succession of scattering events. The nth scattering event begins, by definition, at time τ_n when the particle arrives with incoming velocity v_n at the point (see Fig. 3)

$$y_n^- = y_n - \frac{1}{2}e_n + b_n, \qquad b_n \cdot e_n = 0, \qquad \|b_n\| \leq \frac{1}{2}$$

near the scattering center at $y_n = x_{N_n}$, where $e_n = v_n/\|v_n\|$, and the impact parameter b_n is a vector perpendicular to the incoming velocity vector. The nth scatterer itself is characterized by its orientation $M_n := M_{N_n}$, its phase $\phi_n := \omega\tau_n + \phi^0_{N_n}$ at the time that the particle encounters it, and the coupling strength $c_n := c_{N_n}$.

The change in velocity experienced by a sufficiently fast particle at the nth scattering center can be written (Proposition A.2)

$$v_{n+1} = v_n + R\,(v_n, b_n, M_n, \phi_n, c_n) \tag{2.3}$$

where, for all $v \in \mathbb{R}^d$, $b \in \mathbb{R}^d$ with $v \cdot b = 0$, and $(M, \phi, c) \in \mathrm{SO}(d, \mathbb{R}) \times \mathbb{T}^m \times \mathbb{R}$,

$$R\,(v, b, M, \phi, c) = c\int_0^{+\infty} \mathrm{d}\tau' M g\left(M^{-1}y\left(\tau'\right), \omega\tau' + \phi\right) \tag{2.4}$$

in which $y(\tau)$ is the unique solution of

$$\ddot{y}\,(\tau) = cMg\left(M^{-1}y\,(\tau)\,, \omega\tau + \phi\right), \qquad y\,(0) = b - \frac{1}{2}\frac{v}{\|v\|}, \qquad \dot{y}\,(0) = v.$$

After leaving the influence of the nth scatterer, the particle then travels a distance η_n with velocity v_{n+1} to scatterer $n+1$, which it encounters after a time $\Delta\tau_n = \eta_n/\|v_{n+1}\|$.

Based upon this description of the dynamics, and ignoring the role of recollisions, we now argue that the motion of an ensemble of particles moving in a force field described by (1.2) is well approximated by a coupled discrete-time random walk in momentum and position space. Each step of the walk is associated with one scattering event, where the variables M_n, ϕ_n, c_n that characterize the scatterer, and the variables η_n, b_n that characterize the approach of the particle onto the scatterer, are drawn from distributions that characterize them in the actual system of interest. Thus, starting from a given initial condition (y_0, v_0), we iteratively determine the velocity, the location, and the time of the particle immediately before the nth scattering event through the relations:

$$\left.\begin{aligned} v_{n+1} &= v_n + R\left(v_n, \kappa_n\right) \\ \tau_{n+1} &= \tau_n + \tfrac{\eta_*}{\|v_{n+1}\|} \\ y_{n+1} &= y_n + \eta_* e_{n+1} \end{aligned}\right\} \tag{2.5}$$

where $\kappa_n = (b_n, M_n, \phi_n, c_n)$. The parameters (M_n, ϕ_n, c_n) are independently chosen from the distributions already described (the distribution for ϕ_n being the same as for ϕ_n^0). Without the loss of any essential physics, we have in (2.5) replaced the random variable η_n at each time step with the average distance $\eta_* = \langle \eta_n \rangle < L$ between scattering events. The b_n are independently chosen at each step uniformly from the $d-1$ dimensional ball of radius $1/2$ perpendicular to v_n. To summarize, this random walk describes a particle that moves freely over a distance η_*, then meets, with random impact parameter, a randomly oriented scatterer at a random moment of its (quasi-)periodic evolution. After scattering, the process repeats itself. Our basic assumption, therefore, is that this gives a good description of a typical trajectory in the real system.

In what follows we write $\langle \cdot \rangle$ for averages over all realizations of the random process κ_n. In Sects. 3–6 we study the asymptotic behavior of this random walk, under conditions expressed in the following hypothesis:

Hypothesis 1 $g \in C^3(\mathbb{R}^d \times \mathbb{T}^m)$ is compactly supported in the ball of radius $1/2$ centered at the origin in the y variable. The function g and its partial derivatives up to order three are all bounded, and we write

$$0 < g_{\max} := \|g\|_\infty < +\infty.$$

If $g(y, \phi) = -\nabla_y W(y, \phi)$, we suppose $W \in C^4(\mathbb{R}^d \times \mathbb{T}^m)$ also is supported in the ball of radius $1/2$ centered at the origin in the y-variable. Moreover, $(\omega \cdot \nabla_\phi) W \neq 0$ and, if $d = 1$, we require that, for some $\phi \in \mathbb{T}^m$,

$$\int_{\mathbb{R}} \mathrm{d}y \, \left(\omega \cdot \nabla_\phi\right) W\left(y, \phi\right) \neq 0. \tag{2.6}$$

The meaning of (2.6) is explained in Remark 4.2 below.

3 Analysis of the Random Walk: General Considerations

We now turn to the analysis of the large n behavior of the first equation of (2.5)

$$v_{n+1} = v_n + R\left(v_n, \kappa_n\right) \tag{3.1}$$

which is independent of the others. We assume that the particles are fast, meaning $\|v_0\|^2 \gg c g_{\max}$ (Lemma A.1). For that purpose we need to understand the high momentum behavior of the momentum transfer $R(v_n, \kappa_n)$, as well as of the energy transfer

$$\Delta E\left(v_n, \kappa_n\right) = \frac{1}{2}\left(\left(v_n + R\left(v_n, \kappa_n\right)\right)^2 - v_n^2\right). \tag{3.2}$$

First order perturbation theory allows one to write (see Proposition A.2 for details)

$$R(v_n, \kappa_n) = \frac{c_n}{\|v_n\|} \int_{-\infty}^{+\infty} \mathrm{d}\lambda\ M_n g\left(M_n^{-1}\left(b_n + \left(\lambda - \frac{1}{2}\right) e_n\right), \frac{\omega\lambda}{\|v_n\|} + \phi_n\right) + \mathrm{O}\left(\|v_n\|^{-3}\right).$$

More generally, if g is sufficiently smooth, one can write, for $K \in \mathbb{N}$, and $(v, \kappa) \in \mathbb{R}^{2d} \times \mathrm{SO}(d, \mathbb{R}) \times \mathbb{T}^m \times \mathbb{R}$, $b \cdot v = 0$,

$$R(v, \kappa) = \sum_{k=1}^{K} \frac{\alpha^{(k)}(e, \kappa)}{\|v\|^k} + \mathrm{O}\left(\|v\|^{-K-1}\right), \quad e = \frac{v}{\|v\|}. \tag{3.3}$$

Note that

$$\alpha^{(1)}(e, \kappa) = c \int_{-\infty}^{+\infty} \mathrm{d}\lambda\ M g\left(M^{-1}\left(b + \left(\lambda - \frac{1}{2}\right) e\right), \phi\right) \tag{3.4}$$

and

$$\alpha^{(2)}(e, \kappa) = c \int_{-\infty}^{+\infty} \mathrm{d}\lambda\ \lambda \partial_\tau M g\left(M^{-1}\left(b + \left(\lambda - \frac{1}{2}\right) e\right), \phi\right), \tag{3.5}$$

in which we have introduced the suggestive notation

$$\partial_\tau := \omega \cdot \nabla_\phi. \tag{3.6}$$

Hence

$$\Delta E(v, \kappa) = \sum_{\ell=0}^{L} \frac{\beta^{(\ell)}(e, \kappa)}{\|v\|^\ell} + \mathrm{O}\left(\|v\|^{-L-1}\right), \tag{3.7}$$

where

$$\left.\begin{aligned} \beta^{(0)} &= e \cdot \alpha^{(1)} \\ \beta^{(1)} &= e \cdot \alpha^{(2)} \\ \beta^{(2)} &= \left(\tfrac{1}{2}\alpha^{(1)} \cdot \alpha^{(1)} + e \cdot \alpha^{(3)}\right) \\ \beta^{(3)} &= \left(\alpha^{(1)} \cdot \alpha^{(2)} + e \cdot \alpha^{(4)}\right) \\ \beta^{(4)} &= \left(\tfrac{1}{2}\alpha^{(2)} \cdot \alpha^{(2)} + \alpha^{(1)} \cdot \alpha^{(3)} + e \cdot \alpha^{(5)}\right). \end{aligned}\right\} \tag{3.8}$$

It is easy to see that expansion (3.3) has rather different features when g is a gradient vector field than when it is not. Indeed, when $g = -\nabla W$, the first order term in the momentum transfer (3.3) is perpendicular to the incoming momentum v, so that

$$\beta^{(0)}(e, \kappa) = e \cdot \alpha^{(1)}(e, \kappa) = 0. \tag{3.9}$$

As a result $\Delta E \sim \|v\|^{-1}$ in that case. Moreover, one then has

$$\beta^{(1)}(e, \kappa) = c \int_{-\infty}^{+\infty} \mathrm{d}\lambda\ \partial_\tau W\left(M^{-1}(b + \lambda e), \phi\right). \tag{3.10}$$

On the other hand, when g is not a gradient vector field, $\beta^{(0)}$ does not vanish and, as a consequence, $\Delta E \sim 1$. This is the source of the different asymptotics for $\langle v_n^2 \rangle$ and $\langle y_n^2 \rangle$ in those two cases, as we will see below.

For later purposes, starting from (3.1)–(3.3), a simple computation yields

$$e_{n+1} = \left(1 - \frac{\Delta E_n}{\|v_n\|^2}\right)\left[e_n + \frac{R_n}{\|v_n\|}\right] + \mathrm{O}\left(\frac{(\Delta E_n)^2}{\|v_n\|^4}\right) = e_n + \delta_n, \tag{3.11}$$

where $R_n = R(v_n, \kappa_n)$, and $\Delta E_n = \Delta E(v_n, \kappa_n)$. Hence, from (3.8),

$$\begin{aligned}\delta_n &= \left(\alpha_n^{(1)} - (\alpha_n^{(1)} \cdot e_n)e_n\right)\frac{1}{\|v_n\|^2} + \left(\alpha_n^{(2)} - (\alpha_n^{(2)} \cdot e_n)e_n\right)\frac{1}{\|v_n\|^3} \\ &\quad + \left(\alpha_n^{(3)} - (\alpha_n^{(3)} \cdot e_n)e_n\right)\frac{1}{\|v_n\|^4} - \frac{1}{2}\left(\alpha_n^{(1)} \cdot \alpha_n^{(1)}\right)\frac{e_n}{\|v_n\|^4} \\ &\quad - \left(\alpha_n^{(1)} \cdot e_n\right)\frac{\alpha_n^{(1)}}{\|v_n\|^4} + \mathrm{O}(\|v_n\|^{-5}) \\ &= \delta_n^{(4)} + \mathrm{O}(\|v_n\|^{-5}).\end{aligned}$$

Here $\alpha_n^{(k)} = \alpha^{(k)}(e_n, \kappa_n)$. We can write $\delta_n = \delta_n^{\perp} + \mu_n e_n,\ \delta_n^{\perp} \cdot e_n = 0,$ with (since $\|e_{n+1}\| = 1 = \|e_n\|$)

$$\begin{aligned}\mu_n &= -1 + \sqrt{1 - \delta_n^{\perp} \cdot \delta_n^{\perp}} \leq 0 \\ &= -\frac{1}{2}\left(\alpha_n^{(1)} \cdot \alpha_n^{(1)}\right)\frac{1}{\|v_n\|^4} - \left(\alpha_n^{(1)} \cdot e_n\right)^2\frac{1}{\|v_n\|^4} + \mathrm{O}(\|v_n\|^{-5}).\end{aligned}$$

For a function f depending on v and $\kappa = (b, M, \phi, c)$, $b \cdot v = 0$, $\|b\| \leq 1/2$ we shall denote the average over the parameters associated with a single scattering event as

$$\overline{f(v)} = \int \frac{\mathrm{d}b}{C_d} \int \mathrm{d}\mu\,(M, \phi, c)\, f(v, b, M, \phi, c)\,, \tag{3.12}$$

where C_d is the volume of the ball of radius $1/2$ in $\mathbb{R}^{d-1}$.

4 Analysis of the Random Walk: Gradient Fields

In this section we consider the more interesting case where $g = -\nabla_y W$. The following theorem, the proof of which appears in the Appendix, will be essential to our results.

Theorem 4.1 *Suppose Hypothesis 1 holds and that $g = -\nabla_y W$.*
(i) *For all unit vectors $e \in \mathbb{R}^d$,*

$$\overline{\alpha^{(1)}(e)} = 0 = \overline{\alpha^{(2)}(e)}. \tag{4.1}$$

Moreover, for all $v \in \mathbb{R}^d$

$$\overline{\Delta E(v)} = \frac{B}{\|v\|^4} + \mathrm{O}\left(\|v\|^{-5}\right), \qquad \overline{(\Delta E(v))^2} = \frac{D^2}{\|v\|^2} + \mathrm{O}\left(\|v\|^{-3}\right), \tag{4.2}$$

where

$$B = \frac{d-3}{2}D^2 \tag{4.3}$$

with

$$D^2 = \frac{\overline{c^2}}{C_d} \int_{\mathbb{T}^m} \mathrm{d}\phi \int_{\mathbb{R}^{2d}} \mathrm{d}y_0 \mathrm{d}y_0' \parallel y_0 - y_0' \parallel^{1-d} \partial_\tau W\left(y_0, \phi\right) \partial_\tau W\left(y_0', \phi\right) > 0. \tag{4.4}$$

In particular, for all unit vectors $e \in \mathbb{R}^d$ and for $\ell = 1, 2, 3$,

$$\overline{\beta^{(\ell)}\left(e\right)} = 0, \qquad B = \overline{\beta^{(4)}\left(e\right)} \quad \textit{and} \quad D^2 = \overline{\left(\beta^{(1)}\left(e\right)\right)^2} > 0. \tag{4.5}$$

(ii) *Let v_n be the random process defined by* (3.1) *and $e_n = v_n/\|v_n\|$. Let, for $\ell \in \mathbb{N}$, $\beta_n^{(\ell)} = \beta^{(\ell)}(e_n, \kappa_n)$. Then one has, for all $n \neq n' \in \mathbb{N}$, for all $0 \le \ell \le \ell' \le 3$,*

$$\left.\begin{aligned} &\langle \beta_n^{(4)} \rangle - B = 0 = \langle \beta_n^{(\ell)} \rangle \\ &\langle \beta_n^{(\ell)} \beta_{n'}^{(\ell')} \rangle = 0 = \langle \beta_n^{(\ell)} (\beta_{n'}^{(4)} - B) \rangle = \langle (\beta_n^{(4)} - B)(\beta_{n'}^{(4)} - B) \rangle. \end{aligned}\right\} \tag{4.6}$$

Moreover, $\langle (\beta_n^{(4)})^2 \rangle$ and $\langle \beta_n^{(\ell)} \beta_n^{(4)} \rangle$ are independent of n.

Remark 4.2 (i) Note that part (i) of the Theorem does not involve the random walk (3.1). It is a statement about the functions $\alpha^{(\ell)}(e, \kappa)$, $\beta^{(\ell)}(e, \kappa)$, viewed as random variables in κ.

(ii) The strict positivity of D^2 is equivalent to the requirement that $\beta^{(1)}$ does not vanish identically. This follows from Hypothesis 1, and notably from the nonvanishing of the time derivative of the potential. This is as expected, since in a time-independent potential, energy is conserved to all orders, so certainly $\beta^{(1)} = 0$. In one dimension, the extra assumption (2.6) is needed to ensure $\beta^{(1)} \neq 0$: indeed, when $d = 1$, $\beta^{(1)} = 0$ as soon as the potential has a vanishing spatial average. In that case, some lower order term $\beta^{(\ell)}$ will not vanish and, as will be clear from the discussion which follows, this would alter the power laws of the stochastic acceleration. Such situations, which are easily treated using the methods of this paper, will not be considered further.

(iii) From (4.2), one sees the typical energy change in one collision is of order $D/\|v\|$, for large $\|v\|$, whereas its average value $B/\|v\|^4$ is much smaller. Also it is a small energy loss for $d \le 2$ and a gain for $d \ge 4$. We will see below that even in low dimensions and asymptotically in time the energy grows on average, despite this loss term.

We first establish the asymptotic behavior of $\langle \|v_n\|^2 \rangle$, where v_n is the stochastic process defined by (3.1). We start from the expansion (3.7) which yields, respectively

$$\frac{\|v_{n+1}\|^2}{\|v_n\|^2} = 1 + \sum_{i=1}^{4} \frac{2\beta_n^{(i)}}{\|v_n\|^{i+2}} + \mathrm{O}\left(\|v_n\|^{-7}\right)$$

$$\frac{\|v_{n+1}\|}{\|v_n\|} = 1 + \sum_{i=1}^{3} \frac{\beta_n^{(i)}}{\|v_n\|^{i+2}} + \mathrm{O}\left(\|v_n\|^{-6}\right)$$

$$\|v_{n+1}\| - \|v_n\| = \sum_{i=1}^{3} \frac{\beta_n^{(i)}}{\|v_n\|^{i+1}} + \frac{\beta_n^{(4)} - \frac{1}{2}(\beta_n^{(1)})^2}{\|v_n\|^5} + \mathrm{O}\left(\|v_n\|^{-6}\right)$$

and consequently

$$\Delta \|v_n\|^3 = \|v_n\|^2 \Delta \|v_n\| \left[1 + \frac{\|v_{n+1}\|}{\|v_n\|} + \frac{\|v_{n+1}\|^2}{\|v_n\|^2}\right]$$

$$= \sum_{i=1}^{3} \frac{3\beta_n^{(i)}}{\|v_n\|^{i-1}} + \frac{3\left(\beta_n^{(4)} + \frac{1}{2}(\beta_n^{(1)})^2\right)}{\|v_n\|^3} + \mathrm{O}\left(\|v_n\|^{-4}\right)$$

$$= 3\beta_n^{(1)} + \frac{3\left(\beta_n^{(4)} + \frac{1}{2}(\beta_n^{(1)})^2\right)}{\|v_n\|^3} + \mathrm{O}_0\left(\|v_n\|^{-1}\right) + \mathrm{O}\left(\|v_n\|^{-4}\right). \tag{4.7}$$

Here the notation $\mathrm{O}_0(\|v_n\|^{-1})$ means the term is $\mathrm{O}(\|v_n\|^{-1})$ and of zero average. Introducing

$$\xi_n = \frac{\|v_n\|^3}{3D}, \qquad \epsilon_n = \frac{\beta_n^{(1)}}{D} \quad \text{and} \quad \gamma = \frac{1}{3}\left(\frac{B}{D^2} + \frac{1}{2}\right) = \frac{1}{6}(d-2) \geq -\frac{1}{6}, \tag{4.8}$$

we drop the error term in (4.7) to obtain the one-dimensional random walk

$$\Delta\xi_n = \epsilon_n + \frac{\gamma}{\xi_n} \quad \text{with } \langle\epsilon_n\rangle = 0, \ \left\langle\epsilon_n^2\right\rangle = 1 \tag{4.9}$$

in the variable ξ_n. Here the first term on the right hand side is the dominant term of zero average in (4.7), whereas the second term is a systematic drift term, and is its dominant term of non-zero average (when $\gamma \neq 0$).

From this simple random walk we can easily deduce the short time behavior of the dynamics. Suppose $\xi_0 \gg |\gamma|$. Then,

$$\xi_n = \xi_0 + n\frac{\gamma}{\xi_0} + \sum_{k=0}^{n-1} \epsilon_k,$$

where this approximation remains valid as long as $|\xi_n - \xi_0| \ll \xi_0$. A short calculation shows this is guaranteed provided[1]

$$n \ll N_*(\xi_0) \sim \xi_0^2 \sim \|v_0\|^6. \tag{4.10}$$

This last relation gives an estimate of the number of collisions needed before the asymptotic long time behavior, as derived below, sets in. This dependence on the initial speed can be seen in the numerical results for the model described in Sect. 7, as shown in Fig. 4. We now turn to the asymptotic behavior of ξ_n, $n \gg N_*(\xi_0)$. We will show that, for $d \geq 2$, and for $k > -3$,

$$\left\langle\|v_n\|^k\right\rangle \sim n^{\frac{k}{6}}. \tag{4.11}$$

Note that this is indeed the behavior observed numerically for the full dynamics of the numerical models described in Sect. 7, as illustrated in Fig. 1 for $k = 2$, and which we present in Fig. 5 for $k = -1$ and -2.

From a theoretical point of view, the result (4.11) is obvious for $d = 2$, since then $\gamma = 0$ and (4.9) then just describes a simple random walk on the half line. More generally, looking at (4.9), because $\gamma \geq 0$ for $d \geq 2$, one certainly expects $\langle\xi_n\rangle \to +\infty$, as a result of the combined drift-diffusion implied by (4.9). In $d = 1$, $\gamma < 0$ and the second term then acts as a friction term. We will nevertheless show that for all $\gamma > -1/2$ the friction is too small to alter the asymptotic behavior of $\langle\xi_n\rangle$. To this end we note that

$$\Delta\xi_n^2 = (2\xi_n + \Delta\xi_n)(\Delta\xi_n).$$

[1] From this point onward we use the notation $f(x) \sim g(x)$ to mean that there exist $0 < c \leq C < +\infty$ so that $cf(x) \leq g(x) \leq Cf(x)$.

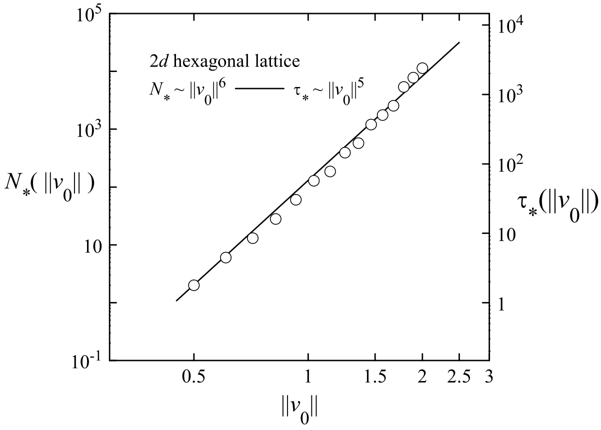

Fig. 4 The quantities $N_*(\|v_0\|)$ (*on the left vertical axis*), and $\tau_*(\|v_0\|)$ (*on the right vertical axis*) as a function of $\|v_0\|$ for a particle moving in a hexagonal lattice ($d=2$) as described in Sect. 7

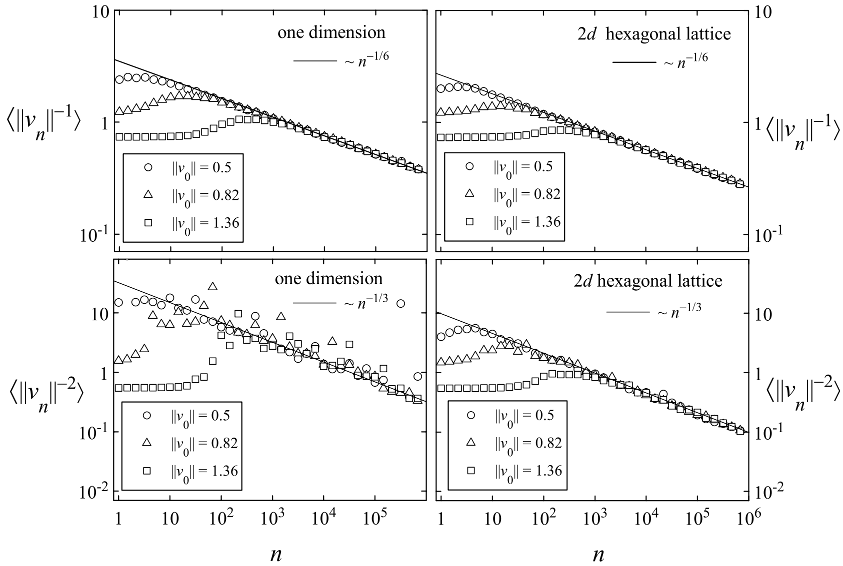

Fig. 5 Numerical results showing the asymptotic behavior of $\langle\|v_n\|^k\rangle$, for the model described in Sect. 7, in one and two dimensions, with initial conditions and k values as indicated

Again keeping only the dominant terms yields

$$\Delta\xi_n^2 = 2\xi_n\epsilon_n + 2\gamma + 1. \tag{4.12}$$

For $\gamma > -1/2$, we will now show that after rescaling the process ξ_n^2 by n, it has a well-defined limit, which is a squared Bessel process of dimension $\delta = 2\gamma + 1$. This will establish (4.11) for those values of γ and for all $k > -3$. To see this, define, for $s \geq 0$, $n \in \mathbb{N}$, and $0 \leq \sigma \leq s$,

$$Y_\sigma^{(n)} = \frac{s}{n}\xi_k^2, \quad \text{if } \sigma_k = k\frac{s}{n} \leq \sigma < (k+1)\frac{s}{n} = \sigma_{k+1}. \tag{4.13}$$

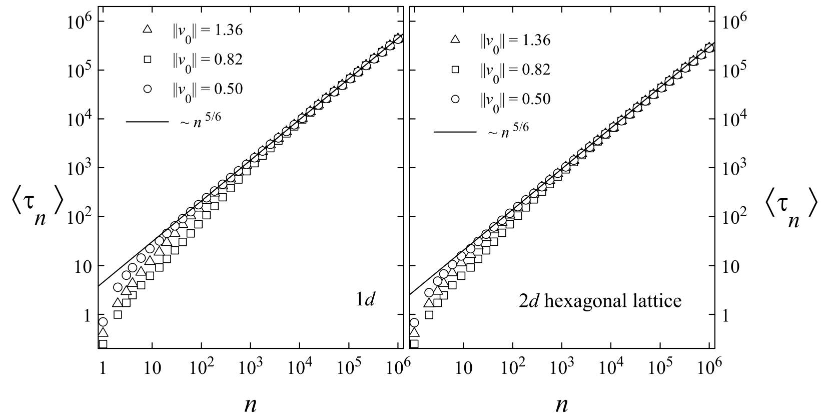

Fig. 6 Behavior of the collision time $\langle \tau_n \rangle$ for the model of Sect. 7, both in one and two dimensions, with initial speeds as indicated

Multiplying (4.12) with s/n one finds that

$$Y_s^{(n)} = Y_0^{(n)} + 2\sum_{k=0}^{n-1} \sqrt{Y_{\sigma_k}^{(n)}}\,\Delta B_{\sigma_k}^{(n)} + (2\gamma+1)\,s,$$

where

$$B_{\sigma_k}^{(n)} = \sqrt{\frac{s}{n}} \sum_{\ell=0}^{k-1} \epsilon_\ell.$$

Taking the limit $n \to +\infty$ and writing $Y_s = \lim_{n\to+\infty} Y_s^{(n)}$, one finds

$$Y_s = Y_0 + 2\int_0^s \sqrt{Y_s}\mathrm{d}B_s + (2\gamma+1)s,$$

where B_s is a one-dimensional Brownian motion since the ϵ_n are i.i.d. In other words, the limiting process Y_s satisfies the stochastic differential equation

$$\mathrm{d}Y_s = 2\sqrt{Y_s}\mathrm{d}B_s + (2\gamma+1)\mathrm{d}s, \tag{4.14}$$

of the squared Bessel process of dimension $\delta = 2\gamma + 1$ (see [17], Chap. 11), and is therefore a squared Bessel process.

Thus, since ξ_n^2/n converges, we can approximate its distribution by that of Y_1 and conclude that, for all $\ell > -1$, $\langle \xi_n^\ell \rangle \sim n^{\frac{\ell}{2}}$, which is (4.11). A more rigorous version of these arguments will be provided in [2]. Equation (4.11) in particular yields, via (2.5), (see Fig. 6)

$$\langle \tau_n \rangle \sim n^{5/6} \tag{4.15}$$

and, finally

$$\left\langle v^2(\tau) \right\rangle \sim \tau^{2/5}, \quad \tau \gg \tau_*(\|v_0\|) := \frac{N_*(\xi_0)}{\|v_0\|} \sim \|v_0\|^5. \tag{4.16}$$

We note that the asymptotic behavior does not depend on the initial speed $\|v_0\|$; the time scale $\tau_*(\|v_0\|) = \tau_{N_*(\|v_0\|)}$ on which it sets in, on the other hand, is predicted by this analysis to grow quickly, as $\|v_0\|^5$, a result that is verified in the $2d$ numerical results presented in Fig. 4, which shows the number of collisions N_* and the mean time τ_* before the asymptotic regime is observed, as a function of $\|v_0\|$.

For $d = 1$, the same power law was found for (Gaussian) random fields in [7, 13], and [3], using very different methods. For $d > 1$, the only studies we are aware of are [7] and [18], who deal with Gaussian random fields and who respectively find $\langle v^2(\tau)\rangle \sim \tau^{1/2}$, which disagrees with (4.16) and $\langle v^2(\tau)\rangle \sim \tau^{2/5}$, which agrees with it. As mentioned in the Introduction, we have corroborated our predictions (4.11) and (4.16), including the onset of the asymptotic regime at $\tau_*(\|v_0\|)$, with numerical calculations in $1d$ and $2d$, the results of which are presented in Fig. 1 and described more fully in Sect. 7.

Before turning back to an analysis of the full random walk (2.5) in order to determine the asymptotic behavior of the mean squared displacement $\langle y^2(\tau)\rangle$, we now first briefly discuss the validity of the assumption we implicitly made in passing from (4.7) to (4.9), namely that the lower order terms of (4.7) won't alter the behavior of $\langle v^2(\tau)\rangle$ that we obtained by ignoring them. To get an estimate of the error made neglecting these terms, we will evaluate them along a typical trajectory of the random walk (3.1), along which we showed $\|v_n\| \sim n^{1/6}$, and will thereby demonstrate that the contribution of each of the neglected terms to ξ_n is smaller than $n^{1/2}$, the contribution of the two dominant terms retained above. Note first that for $i = 2, 3$

$$\left\langle \sum_{k=1}^{n} \frac{\beta_k^{(i)}}{k^{(i-1)/6}} \right\rangle = 0, \qquad \left\langle \left(\sum_{k=1}^{n} \frac{\beta_k^{(i)}}{k^{(i-1)/6}} \right)^2 \right\rangle \sim n^{1-2(i-1)/6},$$

because $\langle \beta_k^{(i)}\rangle = 0$, and $\langle \beta_k^{(i)}\beta_{k'}^{(i)}\rangle = 0$ for $k \neq k'$, by Theorem 4.1(ii). Hence, since for $i = 2, 3$, $n^{1-2(i-1)/6} \ll n \sim \langle \xi_n^2\rangle \sim \langle \|v_n\|^6\rangle$, we conclude that, with the above condition on $\beta_k^{(i)}$, these neglected terms do indeed contribute a lower order correction to (4.11). The neglected term of order $\|v_n\|^{-3}$ is also of zero average, and therefore treated in the same way. Unlike the first three terms, the error in (4.7) that is of order $\|v_n\|^{-4} = \mathrm{O}(n^{-2/3})$, need not be of zero average; however after summation over n it yields a contribution of order $n^{1/3} \ll n^{1/2} \sim \xi_n$, and can therefore also be neglected. Theorem 4.1 therefore implies that all neglected terms in (4.7) provide lower order contributions to the asymptotics of $\|v_n\|$. Note the crucial role of the retained term in (4.9) involving γ, which contributes a term of exactly the same order as the dominant diffusive term $\epsilon_n = \beta_n^{(1)}/D$.

We now derive the asymptotic behavior of $\|y_n\|$ and $\|y(\tau)\|$ (see (4.20) and (4.21) below). We first consider the case with $d > 1$, which clearly depends on how much the particle's path deviates from a straight line, i.e., on how much and how quickly it turns. In particular, we need to analyze the third equation in (2.5). For that purpose, we will first study the evolution of the unit vectors e_n, which execute a random walk on the unit $(d-1)$-sphere. Note that, for all n, as a result of Theorem 4.1(i) and the observation that e_n is independent of κ_n, the step δ_n of the walk in e_n, defined in (3.11), has a mean that satisfies $\langle \delta_n\rangle = \mathrm{O}(\|v_n\|^{-4})$. On the other hand, the magnitude of the step is

$$\|\delta_n\| = \frac{\|\alpha_n^{(1)}\|}{\|v_n\|^2} + \mathrm{O}(\|v_n\|^{-3}) = \|\delta_n^{\perp}\|.$$

Given that the particle has high speed $\|v_n\|$ at the nth collision, we now wish to compute how many collisions m it takes for the particle's direction to change by a macroscopic amount.

For that purpose, we compute the conditional expectation

$$\langle \|e_{n+m} - e_n\|^2 \rangle = \sum_{k=0}^{m-1} \sum_{k'=0}^{m-1} \langle \delta_{n+k} \cdot \delta_{n+k'} \rangle.$$

We will suppose m satisfies $m \ll N_*(\xi_n) \sim \|v_n\|^6 \sim n$, so that (4.10) implies $\|v_{n+m}\| \sim \|v_n\|$; we will therefore approximate $\|v_{n+k}\|$ by $\|v_n\|$. It then follows from Theorem 4.1(i) that, for all k,

$$\langle \delta_{n+k} \cdot \delta_{n+k} \rangle = \frac{\langle \|\alpha_{n+k}^{(1)}\|^2 \rangle}{\|v_n\|^4} + \mathrm{O}\left(\|v_n\|^{-5}\right).$$

For the off-diagonal terms, we note that for $k > k'$,

$$\begin{aligned}\langle \delta_{n+k} \cdot \delta_{n+k'} \rangle = \left\langle \delta_{n+k}^{\perp} \cdot \delta_{n+k'}^{\perp} \right\rangle + \left\langle \mu_{n+k} e_{n+k} \cdot \delta_{n+k'}^{\perp} \right\rangle \\ + \left\langle \delta_{n+k}^{\perp} \cdot \mu_{n+k'} e_{n+k'} \right\rangle + \mathrm{O}\left(\|v_n\|^{-8}\right).\end{aligned}$$

In addition, the rotational invariance of the system implies that for a given e_{n+k}, the vector $\overline{\delta_{n+k}^{\perp}}$ vanishes (see (3.12) for the definition of the barred average). Hence, if $k > k'$

$$\left\langle \delta_{n+k}^{\perp} \cdot \delta_{n+k'}^{\perp} \right\rangle = 0 = \left\langle \delta_{n+k}^{\perp} \cdot \mu_{n+k'} e_{n+k'} \right\rangle.$$

On the other hand, writing that $e_{n+k} = e_{n+k'+1} + \Delta_k$, rotational invariance also implies that the conditional expectation of Δ_k given $e_{n+k'+1}$ is a vector $\nu_k e_{n+k'+1}$ of length $|\nu_k| \le 2$. Hence,

$$\left\langle \delta_{n+k'}^{\perp} \cdot \mu_{n+k} e_{n+k} \right\rangle = \left\langle \delta_{n+k'}^{\perp} \cdot \mu_{n+k} e_{n+k'+1} \right\rangle + \left\langle \delta_{n+k'}^{\perp} \cdot \mu_{n+k} \nu_k e_{n+k'+1} \right\rangle,$$

and

$$\begin{aligned}\left|\left\langle \delta_{n+k'}^{\perp} \cdot \mu_{n+k} e_{n+k} \right\rangle\right| &\le 3\left\langle |\delta_{n+k'}^{\perp} \cdot e_{n+k'+1}||\mu_{n+k}| \right\rangle \\ &\le 3\left\langle \left|\delta_{n+k'}^{\perp} \cdot \left[e_{n+k'} + \delta_{n+k'}\right]\right| |\mu_{n+k}| \right\rangle \\ &\le \frac{3}{\|v_n\|^4} \|\delta_{n+k'}^{\perp}\|^2 = \mathrm{O}\left(\|v_n\|^{-8}\right).\end{aligned}$$

Consequently,

$$\left\langle \|e_{n+m} - e_n\|^2 \right\rangle = m \frac{\langle \|\alpha_0^{(1)}\|^2 \rangle}{\|v_n\|^4} + m\mathrm{O}\left(\|v_n\|^{-5}\right) + m^2\mathrm{O}\left(\|v_n\|^{-8}\right).$$

Consequently, provided

$$m = M_*(\|v_n\|) \sim \|v_n\|^4 \sim n^{2/3} \ll n \tag{4.17}$$

we find $\langle \|e_{n+m} - e_n\|^2 \rangle \sim 1$. This shows that after $M_*(\|v_n\|)$ collisions, and aside from accidental cancellations between the diagonal and off-diagonal terms, the particle turns through a macroscopic angle with the unit vectors e_{n+m} covering the unit sphere. In Fig. 7 we display values of $M_*(\|v_0\|)$ obtained from a numerical study of the decay of the correlation function $\langle e_n \cdot e_0 \rangle$ in the $2d$ numerical model described in Sect. 7. The observed power law behavior agrees with the one predicted by the random walk analysis above.

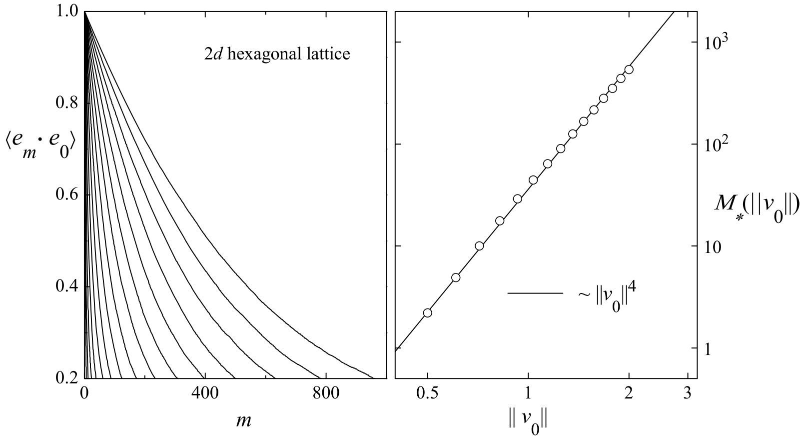

Fig. 7 *On the left*, the correlation function $\langle e_m \cdot e_0 \rangle$ is plotted as a function of m for a set of fifteen initial speeds $\|v_0\|$ lying in the range 0.5 to 2.0. *On the right*, a numerical estimate of $M_*(\|v_0\|)$ obtained from the initial slopes of the data in the left panel are plotted as a function of $\|v_0\|$

We now analyze the asymptotic behavior of $\|y_n\|$. For particles that start off with an initial speed $\|v_0\|$, it takes typically $M_1 = \|v_0\|^4$ collisions to acquire a random direction of motion. We can then define recursively

$$M_{k+1} = M_k + M_k^{\frac{2}{3}}, \tag{4.18}$$

from which one readily finds that $M_k \sim k^3$. The M_k can be interpreted by remarking that, when $n = M_k$, the particle's velocity has "turned", i.e., changed direction by a macroscopic amount, on average k times, while the trajectory along the sequence of $m \ll M_{k+1} - M_k$ collisions between M_k and M_{k+1} largely follows a more or less straight path. We use this picture to approximately compute y_{M_k} by writing

$$y_{M_{k+1}} = y_{M_k} + \eta_* (M_{k+1} - M_k) e_{M_k}. \tag{4.19}$$

This is a rough estimate, but the idea is that, on average, the particles go straight for about $M_{k+1} - M_k$ steps in the direction e_{M_k} without turning. In view of (4.17) and (4.18) we can now think of these successive directions e_{M_k} as randomly and independently chosen on the sphere, so that (4.19) describes a random walk on a larger length scale, having independent steps of order $\eta_*(M_{k+1} - M_k) \sim k^2$. This yields

$$\left\langle \|y_{M_k}\|^2 \right\rangle \sim \sum_{\ell=1}^{k} \ell^4 \sim k^5 \sim M_k^{5/3}.$$

Interpolating between the M_k then allows one to write

$$\left\langle \|y_n\|^2 \right\rangle \sim n^{5/3}. \tag{4.20}$$

Note that, together with (4.15), this finally gives the result

$$\langle \|y(\tau)\| \rangle \sim \tau. \tag{4.21}$$

The motion of the particles is therefore ballistic in the sense that $\|y(\tau)\|/\tau$, which describes the rate at which the particle's distance from the origin grows, is finite on average. Note, however, that the averaged instantaneous speed grows as $\tau^{1/5}$, as shown above. The particles therefore speed up, but turn while traveling, which decreases the rate at which they move away from the origin. The results of our numerical calculations for $d = 2$, presented in Fig. 2, and described in detail in Sect. 7 are in agreement with the results of the random walk analysis outlined above.

Finally, we briefly treat the situation for $d = 1$. For this case, the particle cannot progressively change its direction, and so the analysis presented above does not apply. Indeed, in $1d$ a direction change implies a complete reversal in its direction of motion; but this can happen only if the particle encounters a stretch of scatterers that causes it to completely decelerate first. Computations similar to the previous ones show this cannot occur on a time scale shorter than $M_*(\|v_0\|) \sim \|v_0\|^6$, which is the same scale on which, as we have shown above, the particle accelerates. On the other hand, for all times, we have the obvious upper bound

$$\langle\|y(\tau)\|\rangle \leq \int_0^\tau \mathrm{d}s\ \langle\|v(\tau)\|\rangle \sim \tau^{6/5}. \tag{4.22}$$

It is clear, then, that for time scales over which most of the particles in the ensemble have not reversed direction

$$\langle\|y_n\|\rangle \sim n, \tag{4.23}$$

which along with (4.15) implies $\langle\|y(\tau)\|\rangle \sim \tau^{6/5}$, i.e., it saturates the upper bound (4.22). At longer times, the distribution of times for the random walk (4.9) to return to the origin at $\xi = 0$, which governs events at which the velocity reverses, may alter the asymptotics. If this happens, it does so at times longer than we have been able to investigate numerically. Indeed, up to the times investigated in our numerical calculations the bound (4.22) appears to accurately describe the asymptotic properties of the growth of $y(\tau)$.

5 Non-gradient Force Fields

When the force field g does not derive from a potential W, we suppose the distribution ν of the coupling constant c is centered

$$\int c\ \mathrm{d}\nu(c) = 0, \tag{5.1}$$

so that the mean force vanishes at each point $y \in \mathbb{R}^d$. From (3.7), we have

$$\|v_{n+1}\|^2 = \|v_n\|^2 + 2\beta_n^{(0)} + \frac{2\beta_n^{(1)}}{\|v_n\|} + \frac{2\beta_n^{(2)}}{\|v_n\|^2} + \mathrm{O}\left(\|v_n\|^{-3}\right). \tag{5.2}$$

Similar to Theorem 4.1, we now have the following Theorem, the proof of which also appears in the Appendix:

Theorem 5.1 *Suppose Hypothesis 1 and (5.1) hold. Then:*

(i) *For all unit vectors $e \in \mathbb{R}^d$, $\overline{\alpha^{(1)}(e)} = 0 = \overline{\alpha^{(2)}(e)}$. Moreover, for all $v \in \mathbb{R}^d$,*

$$\overline{\Delta E(v)} = \frac{B'}{\|v\|^2} + \mathrm{O}\left(\|v\|^{-3}\right), \qquad \overline{\Delta E(v)^2} = D'^2 + \mathrm{O}\left(\|v\|^{-1}\right), \tag{5.3}$$

where $B' = \frac{(d-1)}{2} D'^2$, *with*

$$D'^2 = \frac{\overline{c^2}}{C_d} \int_{T^m} \mathrm{d}\phi \int_{\mathbb{R}^{2d}} \mathrm{d}y_0 \mathrm{d}y_0' \|y_0 - y_0'\|^{-(1+d)} \\ \times \left(y_0 - y_0'\right) \cdot g\left(y_0, \phi\right) \left(y_0 - y_0'\right) \cdot g\left(y_0', \phi\right) \geq 0.$$

In particular, for all unit vectors $e \in \mathbb{R}^d$ *and for* $\ell = 0, 1$,

$$\overline{\beta^{(\ell)}(e)} = 0, \qquad B' = \overline{\beta^{(2)}(e)} \quad \text{and} \quad D'^2 = \overline{\left(\beta^{(0)}(e)\right)^2} \geq 0. \tag{5.4}$$

$D' > 0$ *if and only if* $\beta^{(0)}(e, \kappa)$ *does not vanish identically, which implies* g *is not a gradient vector field.*

(ii) *Let* v_n *be the random process defined by* (3.1) *and* $e_n = v_n/\|v_n\|$. *Let, for* $\ell \in \mathbb{N}$, $\beta_n^{(\ell)} = \beta^{(\ell)}(e_n, \kappa_n)$. *Then one has, for all* $n \neq n' \in \mathbb{N}$, *for all* $0 \leq \ell \leq \ell' \leq 1$,

$$\left.\begin{aligned} &\langle \beta_n^{(2)} \rangle - B' = 0 = \langle \beta_n^{(\ell)} \rangle \\ &\langle \beta_n^{(\ell)} \beta_{n'}^{(\ell')} \rangle = 0 = \langle \beta_n^{(\ell)} (\beta_{n'}^{(2)} - B') \rangle = \langle (\beta_n^{(2)} - B')(\beta_{n'}^{(2)} - B') \rangle. \end{aligned}\right\} \tag{5.5}$$

Moreover, $\langle (\beta_n^{(2)})^2 \rangle$ *and* $\langle \beta_n^{(\ell)} \beta_n^{(2)} \rangle$ *are independent of* n.

Remark 5.2 Whether the force does or does not depend on time plays no role in this result, contrary to what happens in Theorem 4.1. In other words, when a force field is not a gradient field, the dominant behavior of the energy transfer to a particle is not affected by whether it depends on time or not. In particular, the coefficients B' and D' do not involve time derivatives of the force, as do B and D.

We now analyze the asymptotic behavior of the velocity and the position of the particle, as in Sect. 4. From (5.2), neglecting the subdominant terms as in (4.7)–(4.8), we find

$$\Delta \xi_n' = \epsilon_n' + \frac{\gamma'}{\xi_n'}, \quad \text{where } \xi_n' = \frac{\|v_n\|^2}{2D'}, \ \epsilon_n' = \frac{\beta_n^{(0)}}{D'}, \text{ and } \gamma' = \frac{1}{4}(d-1).$$

Note that $\gamma' \geq 0$ in all dimensions, so that from the analysis of (4.9) it follows that $\langle \|v_n\|^k \rangle \sim n^{k/4}$. Using this in (2.5) yields

$$\langle \tau_n \rangle \sim \sum_{\ell=0}^{n} \frac{1}{\ell^{1/4}} \sim n^{3/4}, \quad \text{and} \quad \langle \|v(\tau)\| \rangle \sim \tau^{1/3}. \tag{5.6}$$

This is proven rigorously in [6] for a time-independent, non-gradient force field of the type (1.2) and (2.1), in $d \geq 4$, under suitable additional technical conditions on g and the distribution of the scattering centers.

We now show what this implies for the asymptotic behavior of $\|y(\tau)\|$. First, the short time scale $N_*'(\xi_0')$ is now $N_*(\xi_0') \sim \xi_0'^2 \sim n \sim \|v_0\|^4$. Then, from (3.11) we find

$$e_{n+1} = e_n + \frac{\alpha_n^{(1)} - (\alpha_n^{(1)} \cdot e_n) e_n}{\|v_n\|^2} + \mathrm{O}\left(\|v_n\|^{-3}\right).$$

Consequently, $\langle \|e_{n+m} - e_n\|^2 \rangle \sim m/\|v_n\|^4 \sim m/n$. Thus, the particle now turns over a macroscopic angle after $M_*(\|v_n\|) \sim \|v_n\|^4 \sim n$ collisions, many more than for force fields deriving from a potential (see (4.17)) and of the same order as the number $N_*(\|v_n\|) \sim n$ of collisions it needs to accelerate significantly. This is simply due to the fact that the particle is much faster, since $\|v_n\| \sim n^{1/4}$ rather than $\|v_n\| \sim n^{1/6}$, and harder to deflect. This reflects itself in the asymptotic behavior of $\|y(\tau)\|$ as follows. We define as before $M_1 = \|v_0\|^4$, $M_{k+1} = M_k + M_k$, so that $M_k \sim 2^k$, and $y_{M_{k+1}} = y_{M_k} + \eta_*(M_{k+1} - M_k)e_{M_k}$, which integrates to $\langle \|y_{M_k}\|\rangle \sim M_k$, yielding

$$\langle \| y\,(\tau)\, \| \rangle \sim \tau^{4/3}, \tag{5.7}$$

independent of the dimension d of the ambient space.

6 Homogeneous Random Fields

As we now briefly indicate, the analysis of the previous sections can be adapted to the case where the force field is not of the form (2.2), but is a time and space homogeneous random vector field satisfying

$$\langle G\,(y,\tau)\rangle = 0, \qquad \left\langle G\,(y,\tau)\, G\left(y',\tau'\right)\right\rangle = C\left(y-y',\tau-\tau'\right).$$

Note that C is a matrix-valued function, which we assume decays quickly in its spatial variable, but not necessarily in its temporal variable.

In this situation, also, we expect the asymptotic motion of the particle to be well described by a random walk similar to the one in (2.5), where now the time step $\Delta\tau_n$ is determined by the time the particle needs to travel through a distance η_* equal to several times the correlation length (which equals 1 in the rescaled units used here) of the force field:

$$\left.\begin{aligned} v_{n+1} &= v_n + R\,(y_n, v_n, \tau_n, \Delta\tau_n) \\ \tau_{n+1} &= \tau_n + \tfrac{\eta_*}{\|v_n\|},\ \ \eta_* \geq 1 \\ y_{n+1} &= y_n + \eta_* e_n. \end{aligned}\right\} \tag{6.1}$$

Here, $R(y_n, v_n, \tau_n, \Delta\tau_n)$ is the momentum change experienced by a particle that, after arriving at y_n at time τ_n with momentum v_n, travels for a time $\Delta\tau_n$.

We consider first the case in which $G = -\nabla W$ is a random gradient field such that

$$\langle W\,(y,\tau)\rangle = 0, \quad \left\langle W\,(y,\tau)\, W\left(y',\tau'\right)\right\rangle = K\left(y-y',\tau-\tau'\right), \tag{6.2}$$

where K is a function of compact support in $\mathbb{B}(0,1)$ belonging to $\mathcal{C}^5(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, that is rotationally invariant and even in its temporal variable.

To study the asymptotic behavior of v_n in (6.1) we first need, as in the previous sections, to understand the asymptotic behavior of

$$\|v_n\|^2 = \|v_0\|^2 + \sum_{k=0}^{n-1} \Delta\|v_k\|^2 = \|v_0\|^2 + \sum_{k=0}^{n-1} 2\Delta H_k - 2\,(W_n - W_0)\,, \tag{6.3}$$

where $H_k = H(y_k, v_k, \tau_k) = \|v_k\|^2/2 + W_k$, and $W_k = W(y_k, \tau_k)$. Introducing

$$\Delta H\,(y, v, \tau, \Delta\tau) = H\,(y\,(\tau+\Delta\tau)\,, v\,(\tau+\Delta\tau)\,, \tau+\Delta\tau) - H\,(y, v, \tau)$$

we find

$$\Delta H\left(y, v, \tau, \frac{\eta_*}{\|v\|}\right) = \Delta H_I\left(y, v, \tau, \frac{\eta_*}{\|v\|}\right) + \Delta H_{II}\left(y, v, \tau, \frac{\eta_*}{\|v\|}\right) + \mathrm{O}\left(\|v\|^{-5}\right),$$

where

$$\Delta H_I\left(y, v, \tau, \frac{\eta_*}{\|v\|}\right) = \frac{\eta_*}{\|v\|}\int_0^1 \mathrm{d}\lambda\, \partial_\tau W\left(y + \eta_*\lambda e, \tau + \frac{\eta_*\lambda}{\|v\|}\right)$$

and

$$\begin{aligned}\Delta H_{II}\left(y, v, \tau, \frac{\eta_*}{\|v\|}\right) = -\frac{\eta_*^3}{\|v\|^3}\int_0^1 \mathrm{d}\lambda\, \nabla\partial_\tau W\left(y + \eta_*\lambda e, \tau + \frac{\eta_*\lambda}{\|v\|}\right)\\ \cdot\int_0^\lambda \mathrm{d}\lambda' \int_0^{\lambda'} \mathrm{d}\lambda''\, \nabla W\left(y + \eta_*\lambda'' e, \tau + \frac{\eta_*\lambda''}{\|v\|}\right).\end{aligned}$$

We then have the same kind of result as in Theorem 4.1, the proof of which is immediate:

Proposition 6.1 *Under the above conditions,* $\langle\alpha^{(1)}\rangle = 0 = \langle\alpha^{(2)}\rangle$ *and for all* $v \in \mathbb{R}^d$,

$$\langle\Delta H(v)\rangle = \frac{\tilde{B}}{\|v\|^4} + \mathrm{O}\left(\|v\|^{-5}\right), \qquad \left\langle(\Delta H(v))^2\right\rangle = \frac{\tilde{D}^2}{\|v\|^2} + \mathrm{O}\left(\|v\|^{-3}\right),$$

where

$$\tilde{B} = (d-3)\,\eta_* K^{(0)} - 2\,(d-4)\,K^{(1)}, \qquad \tilde{D}^2 = 2\left(\eta_* K^{(0)} - K^{(1)}\right),$$

and

$$K^{(0)} = \int_0^1 \mathrm{d}\mu \left(-\partial_t^2 K(\mu e, 0)\right), \qquad K^{(1)} = \int_0^1 \mathrm{d}\mu \left(-\mu\partial_t^2 K(\mu e, 0)\right).$$

Proof Noting that $\langle\Delta H_I(v)\rangle = 0$, we find

$$\begin{aligned}\langle\Delta H(v)\rangle &= \langle\Delta H_{II}(v)\rangle\\ &= \frac{\eta_*^3}{\|v\|^3}\int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\lambda''\left(\lambda - \lambda''\right)(\Delta\partial_t K)\left(\eta_*\left(\lambda - \lambda''\right)e, \frac{\eta_*(\lambda - \lambda'')}{\|v\|}\right)\\ &= \frac{\eta_*^4}{\|v\|^4}\int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\lambda''\left(\lambda - \lambda''\right)^2\left(\Delta\partial_t^2 K\right)\left(\eta_*\left(\lambda - \lambda''\right)e, 0\right) + \mathrm{O}\left(\|v\|^{-5}\right)\\ &= \frac{\eta_*^4}{\|v\|^4}\int_0^1 \mathrm{d}\lambda\,(1-\lambda)\,\lambda^2\left(\Delta\partial_t^2 K\right)(\eta_*\lambda e, 0) + \mathrm{O}\left(\|v\|^{-5}\right).\end{aligned}$$

Using the rotational invariance of $\Delta\partial_t^2 K(\cdot, 0)$, and integrating by parts, we obtain the above expression for $\tilde{B}$.

Moreover, since

$$(\Delta H(y, v, \tau, \Delta\tau))^2 = \frac{\eta_*^2}{\|v\|^2}\left(\int_0^1 \mathrm{d}\lambda\, \partial_\tau W(y + \eta_*\lambda e, \tau)\right)^2 + \mathrm{O}(\|v\|^{-3}),$$

we find

$$\langle (\Delta H(v))^2 \rangle = \frac{\widetilde{D}^2}{\|v\|^2} + \mathrm{O}(\|v\|^{-3}),$$

where

$$\begin{aligned}\widetilde{D}^2 &= \eta_*^2 \int_0^1 \mathrm{d}\lambda \int_0^1 \mathrm{d}\lambda' \left(-\partial_\tau^2 K\right)(\eta_*(\lambda - \lambda')e, 0)\\ &= 2\eta_*^2 \int_0^1 \mathrm{d}\lambda (1-\lambda) \left(-\partial_\tau^2 K\right)(\eta_* \lambda e, 0).\end{aligned}$$

A change of variables then gives the result. □

Scaling $\|v_n\|^2$ by $(s/n)^{1/3}$ in (6.3) and taking n to infinity, one finds that the limiting process Z_σ satisfies the stochastic differential equation

$$\mathrm{d}Z_\sigma = \frac{2}{3}\frac{\mathrm{d}B_\sigma}{\sqrt{Z_\sigma}} + \frac{2}{3}\left(\gamma - \frac{1}{6}\right)\frac{\mathrm{d}\sigma}{Z_\sigma^2}, \quad \gamma = \frac{1}{3}\left(\frac{\tilde{B}}{\tilde{D}^2} + \frac{1}{2}\right).$$

It then follows from the Itô formula that $Y_\sigma = Z_\sigma^3$ satisfies the stochastic differential equation of the square of the Bessel process [17] of dimension $\delta = 2\gamma + 1$. By taking η_* sufficiently large we can make γ arbitrarily close to $(d-2)/6$. The analysis of the random walk is therefore entirely analogous to the one in Sect. 4, yielding in particular the same power laws for the growth of $\langle v^2(\tau)\rangle$ and $\langle y^2(\tau)\rangle$ as in (4.16) and (4.21).

In the case that G is not a gradient field we still assume it to be rotationally invariant and reflection symmetric. This implies that there exist functions Λ_1 and Λ_2 such that the correlation function is of the form

$$C(y,\tau) = \Lambda_1(\|y\|,\tau)\,\mathbb{P}_y + \Lambda_2(\|y\|,\tau)\,\mathbb{P}_y^\perp,$$

where $\mathbb{P}_y$ is the orthogonal projector along the direction of the vector y and $\mathbb{P}_y^\perp + \mathbb{P}_y = I_d$. We in addition assume that Λ_1 and Λ_2 are $\mathcal{C}^2$ functions that decay fast in their spatial variable, and that for all $\tau \in \mathbb{R}$, $\Lambda_1(\cdot,\tau)$ and $\Lambda_2(\cdot,\tau)$ are compactly supported in $[0,1]$. Under these assumption, we then prove an analogue of Theorem 5.1:

Proposition 6.2 *Under the conditions stated above $\langle\alpha^{(1)}\rangle = 0 = \langle\alpha^{(2)}\rangle$, and for all $v \in \mathbb{R}^d$,*

$$\langle \Delta E(v)\rangle = \frac{\tilde{B}'}{\|v\|^2} + \mathrm{O}\left(\|v\|^{-3}\right), \qquad \left\langle (\Delta E(v))^2\right\rangle = \tilde{D}'^2 + \mathrm{O}\left(\|v\|^{-1}\right),$$

where

$$\tilde{B}' = \eta_*(d-1)\,K'^{(0)} - (d-2)\,K'^{(1)}, \qquad \tilde{D}'^2 = 2\left(\eta_* K'^{(0)} - K'^{(1)}\right) > 0$$

and

$$K'^{(0)} = \int_0^1 \mathrm{d}\mu \Lambda_1(\mu,0), \qquad K'^{(1)} = \int_0^1 \mathrm{d}\mu\mu\Lambda_1(\mu,0).$$

Proof A computation of $R(y,v,\tau,\eta_*/\|v\|)=\int_\tau^{\tau+\eta_*/\|v\|}G(y(\tau'),\tau')\mathrm{d}\tau'$ to second order in perturbation theory gives

$$R\left(y,v,\tau,\eta_*/\left\|v\right\|\right)=R_I\left(y,v,\tau,\eta_*/\left\|v\right\|\right)+R_{II}\left(y,v,\tau,\eta_*/\left\|v\right\|\right)$$

with

$$R_I\left(y,v,\tau,\eta_*/\left\|v\right\|\right)=\frac{\eta_*}{\|v\|}\int_0^1\mathrm{d}\lambda G\left(y+\eta_*\lambda e,\tau+\frac{\eta_*\lambda}{\|v\|}\right)$$

and

$$\begin{aligned}R_{II}\left(y,v,\tau,\eta_*/\left\|v\right\|\right)=\frac{\eta_*^3}{\|v\|^3}&\int_0^1\mathrm{d}\lambda\int_0^\lambda\mathrm{d}\lambda''\left(\lambda-\lambda''\right)\\&\times\left(G\left(y+\eta_*\lambda''e,\tau\right)\cdot\nabla\right)G\left(y+\eta_*\lambda e,\tau\right)+\mathrm{O}\left(\|v\|^{-4}\right).\end{aligned}$$

Hence, $\langle\alpha^{(1)}\rangle$ and $\langle\alpha^{(2)}\rangle$, and consequently $\langle\beta^{(0)}\rangle$ and $\langle\beta^{(1)}\rangle$, vanish. Using $w\cdot\mathbb{P}_y(v)=(v\cdot y)(w\cdot y)/y^2$ then yields

$$\begin{aligned}&\frac{1}{2}\left\langle\alpha^{(1)}\cdot\alpha^{(1)}\right\rangle\\&\quad=\frac{\eta_*^2}{2}\int_0^1\mathrm{d}\lambda\int_0^1\mathrm{d}\lambda''\left(\Lambda_1\left(L\left|\lambda-\lambda''\right|,0\right)+(d-1)\Lambda_2\left(\eta_*\left|\lambda-\lambda''\right|,0\right)\right)\\&\quad=\eta_*^2\int_0^1\mathrm{d}\lambda\left(1-\lambda\right)\left(\Lambda_1\left(\eta_*\lambda,0\right)+(d-1)\Lambda_2\left(\eta_*\lambda,0\right)\right),\quad\text{and}\\&\left\langle\alpha^{(3)}\cdot e\right\rangle\\&\quad=\eta_*^2\int_0^1\mathrm{d}\lambda\left(1-\lambda\right)\left(\eta_*\lambda\Lambda_1'\left(\lambda,0\right)+(d-1)\left(\Lambda_1\left(\lambda,0\right)-\Lambda_2\left(\lambda,0\right)\right)\right)\\&\quad=\eta_*^2\int_0^1\mathrm{d}\lambda\left(\left((d-2)-\lambda\left(d-3\right)\right)\Lambda_1\left(\eta_*\lambda,0\right)-(d-1)\left(1-\lambda\right)\Lambda_2\left(\eta_*\lambda,0\right)\right).\end{aligned}$$

Adding the last two equations and making the change of variables $\mu=\eta_*\lambda$ yields the above expression for $\tilde{B}'$. □

Analysis of the random walk now proceeds along the lines of Sect. 5, yielding the same power laws as obtained therein.

7 Numerical Results

To illustrate our theoretical analysis of the motion of a particle in random force fields presented in the previous sections, we performed numerical calculations for a periodic array of soft scatterers in one and two dimensions. For the two dimensional case we employed a hexagonal lattice, with, for $N=(N_1,N_2)\in\mathbb{Z}^2$, $x_N=N_1u+N_2v$, where $u=(1,0)$, $v=\frac{1}{2}(1,\sqrt{3})$. We focused on the case in which the force fields associated with the scatterers were derived from a potential, taking W to be of the form of a time-dependent, flat circular potential,

$$W(y,\phi) = f(\phi)\,\chi\left(\frac{\|y\|}{y_*}\right), \quad y \in \mathbb{R}^d, d = 1, 2,$$

where $\chi(x) = 1$ if $0 \leq x \leq 1$ and $\chi(x) = 0$ otherwise. Here the parameter y_* satisfies $\frac{\sqrt{3}}{4} < y_* < 1/2$ to ensure the system has a finite horizon. Three different choices were explored for the function f, namely,

$$f_1(\phi) = \cos(2\pi\phi), \qquad f_2(\phi) = 1 + \cos^2(2\pi\phi), \quad \phi \in [0, 1[,$$

each of which leads to a time-periodic potential, and

$$f_3(\phi) = f_3(\phi_1, \phi_2) = \cos(2\pi\phi_1) + \cos(2\pi\phi_2).$$

In the latter case, the frequency vector ω was chosen to be $\omega = (1, \sqrt{2})$ so that the resulting potential is quasi-periodic in time. The phases ϕ_N were chosen uniformly on the torus, independently for each scatterer. Coupling constants c_N were either drawn independently from a uniform distribution on [0, 1/2], or set to a fixed value $c_N = 1$, or $c_N = -1$, for all N.

Depending on the phase and the choice of coupling constants, each such potential describes a centrally symmetric potential barrier or well, whose maximum/minimum oscillates in time. For the choice $f = f_1$ or f_3, any given scatterer will sometimes act as a potential well, and at other times as a barrier, depending on the sign of $c_N f_1(\phi_N + \tau)$ or of $c_N f_3(\phi_N + \omega\tau)$ at the time τ of arrival of the particle; on average the force at a given point in space always vanishes. When $f = f_2$ on the other hand, and $c_N = 1$ for all N, $c_N f_2(\phi + \tau)$ is always positive, yielding a lattice of oscillating potential barriers, for which the average force at a given point in space does not vanish. Similarly, when $f = f_2$ and $c_N = -1$ for all N one obtains a lattice of oscillating potential wells. In all cases that we studied numerically, the system had finite horizon.

Motion of a particle through an array of such scatterers can be computed iteratively, by using energy and angular momentum conservation at the entry and exit of the particle from the support of the potential, and without a numerical integration of a second order differential equation. This allows one to compute the motion of the particle numerically for very long times, as required to properly study the asymptotic regime.

In our calculations, each particle was initially placed at a point randomly chosen on the boundary of the scatterer at the origin, with an initial velocity drawn with equal probability from all possible outward directions. For each ensemble of initial conditions, the initial speed $\|v_0\|$ of the particle was kept fixed (with values indicated in the figure captions, or in the figures themselves). Displayed results represent averages over, typically, 10^4 trajectories for each initial speed. For convenience of presentation, data appearing as a function of time τ or collision number n in the figures presented throughout the paper represent a subset of the data generated, evaluated at values of the independent variable that are equally spaced on a logarithmic axis.

A general finding of both our numerical calculations and of our theoretical analysis is that the power law behavior associated with stochastic acceleration is independent of the precise form of the potential employed; in particular it does not depend on whether the average force vanishes or not. Thus, in the figures that appear in the paper we have chosen to present numerical results only for the case where $f = f_1$ and c_N is uniformly distributed in [0, 1/2].

For this specific model, Fig. 1 shows the evolution of the particle's mean kinetic energy, both as a function of time τ and as a function of collision number n. As noted in the text, one observes excellent agreement with the power law behavior predicted by our analysis (see (1.4), (4.11), (4.16)), independent of dimension. One also notices in this figure that the asymptotic regime is reached after an initial period which ends after a number $N_*(\|v_0\|)$ of collisions that grows with $\|v_0\|$. The value of $N_*(\|v_0\|)$ was computed numerically for fifteen values of $\|v_0\|$ between 0.5 and 2, and the result is presented in Fig. 4. The observed power law $N_*(\|v_0\|) \sim \|v_0\|^6$, is as predicted in Sect. 4 (see (4.10)).

Similarly, Fig. 2 shows for the specific numerical model described above, the evolution of the particle's mean squared displacement as a function of τ and n. We find that the power laws obtained in one dimension (see (1.5) and (4.23)) and in two dimensions (see (1.6) and (4.20)–(4.21)) are indeed different, and precisely as predicted by the analysis of Sect. 4.

In order to obtain analytical results for a sufficiently general class of potentials, the theoretical analysis of Sect. 4 assumed scattering potentials that are smooth, which the potentials used in the numerical calculations are clearly not. Indeed, running the numerics for sufficiently long times with a smooth potential would involve repeatedly solving a second order differential equation; this would lack precision and be too time consuming. Explicit computations specific to the square potential, however, show that formulas (3.7) and (3.3) remain valid, and that their dominant terms have the same behavior as in the analysis presented, so that our arguments go through unaltered. This lends additional support to our claim that it is the high energy behavior of the energy and momentum transfer in a single scattering event that determines the asymptotic behavior of the particle, and suggests that the results are even more universal than is implied by our analysis.

As a closing comment we note also that in our numerical models the potentials are rotationally invariant, and the lattices are ordered. Thus, when c_N is constant, the only randomness left in the problem is in the initial phases ϕ_N of the scatterers and the initial directions e_0 of the particles. Thus, the essential randomness necessary for the validity of our analysis arises from the dispersive nature of the scattering event itself, which leads to a random sequence of scattering events when evaluated along the trajectory that the particle follows.

Appendix: Proof of Main Theorems

In this appendix we provide proofs of Theorems 4.1 and 5.1. We begin with some preparatory material. It is convenient to write $\hat{g}(y,\tau) = Mg(M^{-1}y, \omega\tau + \phi)$, suppressing the variables ϕ and M from the notation.

The estimates below are all uniform in ϕ and M. Note that when $g = -\nabla W$, then $\hat{g} = -\nabla \hat{W}$, with $\hat{W}(y,\tau) = W(M^{-1}y, \omega\tau + \phi)$. We need to study the solutions of

$$\ddot{y}\left(\tau'\right) = c\hat{g}\left(y\left(\tau'\right), \tau'\right) = -c\nabla \hat{W}\left(y\left(\tau'\right), \tau'\right), \qquad y\left(\tau_0\right) = y_0, \qquad \dot{y}\left(\tau_0\right) = v_0. \tag{A.1}$$

For any initial condition $y(\tau_0) = y_0$, $v(\tau_0) = v_0$, we define

$$v_\pm = \lim_{\tau \to \pm\infty} \dot{y}\left(\tau\right). \tag{A.2}$$

In particular, when $y_0 = b - \frac{1}{2}\frac{v_0}{\|v_0\|}$ and $\dot{y}_0 = v_0$, we have (see (2.4))

$$R(v_0, \kappa) = v_+ - v_- = -c\int_{\mathbb{R}} d\tau' \nabla \hat{W}(y(\tau'), \tau'). \tag{A.3}$$

That these limits exist if $\|v_0\|$ is large enough is a consequence of the following lemma.

Lemma A.1 *Suppose Hypothesis 1 holds. Let $\tau_0 \in \mathbb{R}$ and suppose $(y_0, v_0) \in \mathbb{R}^{2d}$ satisfies $\|y_0\| \leq 1/2$, $\|v_0\|^2 \geq 12cg_{\max}$. Then there exist unique $\tau_{\text{in}} \leq \tau_0 \leq \tau_{\text{out}}$ so that $\|y(\tau_{\text{in}})\| = \frac{5}{2} = \|y(\tau_{\text{out}})\|$. Moreover*

$$\frac{\sqrt{3}}{\|v_0\|} \leq \min\{\tau_0 - \tau_{\text{in}}, \tau_{\text{out}} - \tau_0\} \leq \max\{\tau_0 - \tau_{\text{in}}, \tau_{\text{out}} - \tau_0\} \leq \frac{3\sqrt{2}}{\|v_0\|}. \tag{A.4}$$

The lemma roughly says that any particle that is at some instant τ_0 inside the region where the potential does not vanish and that has enough kinetic energy at that moment, has entered it in the past and will leave again in the future, spending a time of order $\frac{1}{\|v_0\|}$ to cross it: both the upper and lower bounds in (A.4) will be used in the proof of Proposition A.3 below. Note that the Lemma does indeed imply the existence of the limits in (A.2).

Proof From (A.1),

$$y(\tau) = y_0 + v_0(\tau - \tau_0) + c\int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau'} d\tau'' \hat{g}\left(y\left(\tau''\right), \tau''\right), \tag{A.5}$$

so that $Q(\tau - \tau_0) \leq \|y(\tau)\| \leq P(\tau - \tau_0)$, where

$$Q(\tau - \tau_0) = -cg_{\max}\frac{(\tau - \tau_0)^2}{2} + \|v_0\||\tau - \tau_0| - \frac{1}{2}$$

and

$$P(\tau - \tau_0) = cg_{\max}\frac{(\tau - \tau_0)^2}{2} + \|v_0\||\tau - \tau_0| + \frac{1}{2}.$$

One checks that $Q(\sigma_+) = \frac{5}{2} = P(\sigma_-)$, with

$$\sigma_- = \frac{\|v_0\|}{cg_{\max}}\left(\sqrt{1 + \frac{4cg_{\max}}{\|v_0\|^2}} - 1\right), \quad \text{and} \quad \sigma_+ = \frac{\|v_0\|}{cg_{\max}}\left(1 - \sqrt{1 - \frac{6cg_{\max}}{\|v_0\|^2}}\right).$$

Note that $\sigma_- \leq \sigma_+$. Since $\|y(\tau_0 \pm \sigma_-)\| \leq \frac{5}{2} \leq \|y(\tau_0 \pm \sigma_+)\|$, it is clear there exist $\tau_{\text{in}}, \tau_{\text{out}}$ satisfying

$$\tau_0 - \sigma_+ \leq \tau_{\text{in}} \leq \tau_0 - \sigma_- \quad \text{and} \quad \tau_0 + \sigma_- \leq \tau_{\text{out}} \leq \tau_0 + \sigma_+.$$

Uniqueness follows from the observation that $\hat{g}$ vanishes outside the ball of radius $1/2$ so that the trajectory can enter and leave the ball of radius $5/2$ only once. Equation (A.4) now follows from the observation that, if $0 \leq x \leq A < 1$, then

$$\sqrt{1+x} - 1 \geq \frac{1}{2}\frac{1}{\sqrt{1+A}}x, \qquad 1 - \sqrt{1-x} \leq \frac{1}{2}\frac{1}{\sqrt{1-A}}x.$$

It is enough to choose $A = 1/3$ in the first inequality and $A = 1/2$ in the second. □

With $v_\pm$ from (A.2), we define, for all (y_0, v_0, τ_0) so that $\|v_0\|^2 \geq 12cg_{\max}$,

$$\Delta v(v_0, y_0, \tau_0) = v_+ - v_-, \qquad \Delta K(v_0, y_0, \tau_0) = \frac{1}{2}\left(v_+^2 - v_-^2\right). \tag{A.6}$$

Note that both Δv and ΔK are constant along trajectories:

$$\Delta v\,(v_0, y_0, \tau_0) = \Delta v(v(\tau'), y(\tau'), \tau'), \qquad \Delta K(v_0, y_0, \tau_0) = \Delta K(v(\tau'), y(\tau'), \tau'). \quad \text{(A.7)}$$

We therefore think of them as functions on the space of all trajectories with sufficient kinetic energy. We are interested in understanding the high velocity behavior of ΔK and of its average over all those trajectories that enter the support of the potential. We will see that, when $\hat{g} = -\nabla\hat{W}$, $\Delta K \sim \|v_0\|^{-1}$ (Proposition A.3) but that the average of ΔK vanishes up to terms of order $\|v_0\|^{-4}$ (Proposition A.4). In other words, the dominant terms in ΔK vanish on average. This observation is the essence of Theorem 4.1, as we will explain below.

Proposition A.2 *Suppose Hypothesis 1 is satisfied. Let* $y_0, v_0 \in \mathbb{R}^d$ *and* $\tau_0 \in \mathbb{R}$, *with* $\|v_0\|^2 \geq 12cg_{\max}$. *Then*

$$\Delta v\,(v_0, y_0, \tau_0) = \frac{c}{\|v_0\|}\int_{-\infty}^{+\infty} \hat{g}\,(y_0 + \lambda e_0, \tau_0)\;\mathrm{d}\lambda + \frac{c}{\|v_0\|^2}\int_{-\infty}^{+\infty} \partial_\tau \hat{g}\,(y_0 + \lambda e_0, \tau_0)\,\lambda \mathrm{d}\lambda + \mathrm{O}\left(\|v_0\|^{-3}\right). \quad \text{(A.8)}$$

The error term is uniform in $y_0, \tau_0, c \in [-1, 1]$ *and in* $e_0 = v_0/\|v_0\|$.

Proof Suppose first $\|y_0\| \leq \frac{1}{2}$. It follows from Lemma A.1 that, under the stated condition on $\|v_0\|$, there exist unique entrance and exit times τ_{in} and τ_{out} to the ball of radius $5/2$, with $\tau_{\text{out}} - \tau_0$ and $\tau_0 - \tau_{\text{in}}$ of order $\|v_0\|^{-1}$. As a result, for $\tau < \tau_{\text{in}}$ and $\tau > \tau_{\text{out}}$ the particle executes a free motion with speeds v_- and v_+, in the region where the force $\hat{g}$ vanishes identically. From (A.1), one now readily concludes

$$\begin{aligned}\Delta v\,(v_0, y_0, \tau_0) &= c\int_{\tau_{\text{in}}}^{\tau_{\text{out}}} \hat{g}\,(y\,(\tau)\,, \tau)\,\mathrm{d}\tau \\ &= c\int_{\tau_{\text{in}}}^{\tau_{\text{out}}} \hat{g}\,(y_0 + v_0\,(\tau - \tau_0)\,, \tau)\,\mathrm{d}\tau + \mathrm{O}\left(\|v_0\|^{-3}\right),\end{aligned}$$

where we used $\|y(\tau) - (y_0 + v_0(\tau - \tau_0))\| \leq \frac{c}{2}g_{\max}(\tau - \tau_0)^2$, which follows easily from (A.5). Let us now remark that Lemma A.1 implies that

$$\|y_0 + v_0(\tau_{\text{out/in}} - \tau_0)\| \geq \|v_0\||\tau_{\text{out/in}} - \tau_0| - \frac{1}{2} \geq 1/2.$$

As a result, we can extend the τ integration to the full real axis; indeed, the integrand vanishes for $\tau \leq \tau_{\text{in}}$ and for $\tau_{\text{out}} \leq \tau$. The change of variables $\lambda = \|v_0\|(\tau - \tau_0)$ yields

$$\Delta v\,(v_0, y_0, \tau_0) = \frac{c}{\|v_0\|}\int_{-\infty}^{+\infty} \hat{g}\left(y_0 + \lambda e_0, \tau_0 + \frac{\lambda}{\|v_0\|}\right)\mathrm{d}\lambda + \mathrm{O}\left(\|v_0\|^{-3}\right), \quad \text{(A.9)}$$

so that a first order Taylor expansion yields the result.

We now consider the case where $\|y_0\| > 1/2$. We may assume the particle trajectory intersects the ball of radius $1/2$ centered at the origin: otherwise $\Delta v(v_0, y_0, \tau_0) = 0$, and then the result stated certainly holds. Suppose therefore the trajectory intersects that ball and that $y_0 \cdot v_0 \leq 0$. Then there exists a unique time $\tau_* > \tau_0$ when the trajectory enters the above ball: so $y(\tau) = y_0 + v_0(\tau - \tau_0)$ for all $\tau \le \tau_*$, $\|y(\tau_*)\| = 1/2$ and $y(\tau_*) \cdot v_0 \le 0$. Clearly

$$\Delta v\left(v_0, y_0, \tau_0\right) = \Delta v\left(v_0, y\left(\tau_*\right), \tau_*\right),$$

and we can apply the result of the first part of the proof to write

$$\Delta v\left(v_0, y_0, \tau_0\right) = \frac{c}{\|v_0\|} \int_{-\infty}^{+\infty} \hat{g}\left(y\left(\tau_*\right) + \lambda e_0, \tau_* + \frac{\lambda}{\|v_0\|}\right) \mathrm{d}\lambda + \mathrm{O}\left(\|v_0\|^{-3}\right). \tag{A.10}$$

The change of variables

$$\tilde{\lambda} = \lambda + \|v_0\| \left(\tau_* - \tau_0\right)$$

transforms (A.10) into (A.9), which concludes the proof. The case where $y_0 \cdot v_0 \ge 0$ is treated analogously. $\square$

When $\hat{g} = -\nabla \hat{W}$, we need the high $\|v_0\|$ expansion of ΔK up to order $\|v_0\|^{-4}$ obtained in the following proposition.

Proposition A.3 *Suppose Hypothesis 1 is satisfied and suppose $\hat{g} = -\nabla \hat{W}$. Then, for all $v_0 \in \mathbb{R}^d$ such that $\|v_0\|^2 \ge 12 c g_{\max}$ and for all $y_0 \in \mathbb{R}^d$,*

$$\Delta K\left(v_0, y_0, \tau_0\right) = \Delta K_I\left(v_0, y_0, \tau_0\right) + \Delta K_{II}\left(v_0, y_0, \tau_0\right) + \mathrm{O}\left(\|v_0\|^{-5}\right) \tag{A.11}$$

where

$$\Delta K_I\left(v_0, y_0, \tau_0\right) = \frac{c}{\|v_0\|} \int_{\mathbb{R}} \mathrm{d}\lambda\ \partial_\tau \hat{W}\left(y_0 + \lambda e_0, \tau_0 + \frac{\lambda}{\|v_0\|}\right), \tag{A.12}$$

and

$$\begin{aligned}\Delta K_{II}\left(v_0, y_0, \tau_0\right) = -\frac{c^2}{\|v_0\|^3} &\int_{\mathbb{R}} \mathrm{d}\lambda \nabla \partial_\tau \hat{W}\left(y_0 + \lambda e_0, \tau_0 + \frac{\lambda}{\|v_0\|}\right) \\ &\cdot \int_0^{\lambda} \mathrm{d}\lambda' \int_0^{\lambda'} \mathrm{d}\lambda'' \nabla \hat{W}\left(y_0 + \lambda'' e_0, \tau_0 + \frac{\lambda''}{\|v_0\|}\right).\end{aligned} \tag{A.13}$$

The error term is uniform in y_0, τ_0, and in $e_0 = v_0/\|v_0\|$.

The index "I" or "II" refers to first and second order in $\hat{W}$, but note that each of the corresponding contributions has an expansion in $\|v_0\|^{-1}$.

Proof We first deal with the case where $\|y_0\| \le 1/2$. As in the proof of Proposition A.2, one can integrate the equation of motion to obtain

$$\Delta K\left(v_0, y_0, \tau_0\right) = -c \int_{\tau_{\text{in}}}^{\tau_{\text{out}}} \dot{y}\left(\tau\right) \cdot \nabla \hat{W}\left(y\left(\tau\right), \tau\right) \mathrm{d}\tau = c \int_{\tau_{\text{in}}}^{\tau_{\text{out}}} \partial_\tau \hat{W}\left(y\left(\tau\right), \tau\right) \mathrm{d}\tau.$$

From (A.5) one easily finds, for $\tau \in [\tau_{\text{in}}, \tau_{\text{out}}]$, that

$$\left.\begin{array}{ll}\|\dot{y}\left(\tau\right) - v_0\| & \le c g_{\max} |\tau - \tau_0| \\ \|y\left(\tau\right) - \left(y_0 + v_0\left(\tau - \tau_0\right)\right)\| & \le c g_{\max}\left(\tau - \tau_0\right)^2 \\ y\left(\tau\right) & = y_I\left(\tau\right) + \mathrm{O}\left(\|v_0\|^{-4}\right)\end{array}\right\} \tag{A.14}$$

where we used (A.4) in the last line and where

$$y_I(\tau) = y_0 + v_0(\tau - \tau_0) - c\int_{\tau_0}^{\tau} \mathrm{d}\tau' \int_{\tau_0}^{\tau'} \mathrm{d}\tau''\, \nabla \hat{W}\left(y_0 + v_0\left(\tau'' - \tau_0\right), \tau''\right).$$

Hence

$$\Delta K(v_0, y_0, \tau_0) = c\int_{\tau_{\mathrm{in}}}^{\tau_{\mathrm{out}}} \partial_\tau \hat{W}(y_I(\tau), \tau)\, \mathrm{d}\tau + \mathrm{O}\left(\|v_0\|^{-5}\right).$$

Expanding $\partial_\tau \hat{W}(y_I(\tau), \tau)$ around $y_0 + v_0(\tau - \tau_0)$, the result follows. The case $\|y_0\| > 1/2$ is handled as in the proof of Proposition A.2. □

For the purpose of proving Theorem 4.1, we now turn to the computation of the average energy change of all trajectories with a given, sufficiently high, incoming momentum or energy, and that enter the ball of radius $1/2$ centered at the origin. Recalling that $\hat{g}(y, \tau) = Mg(M^{-1}y, \omega\tau + \phi)$, we have, for $v_0 \in \mathbb{R}^d$, $b \cdot v_0 = 0$ and $\kappa = (b, M, \phi + \omega\tau_0, c)$,

$$\Delta E(v_0, \kappa) = \Delta K\left(v_0, b - \frac{1}{2}e_0, \tau_0\right). \tag{A.15}$$

We first compute the average of $\Delta E(v_0, b, M, \phi, c)$ over ϕ:

$$\int_{\mathbb{T}^m} \mathrm{d}\phi\; \Delta E(v_0, b, M, \phi, c). \tag{A.16}$$

Proposition A.4 *Suppose Hypothesis 1 is satisfied. Then, for all $v_0 \in \mathbb{R}^d$ and for all $b \in \mathbb{R}^d$, $b \cdot v_0 = 0$, $M \in \mathrm{SO}(d, \mathbb{R})$, $c \in [-1, 1]$,*

$$\begin{aligned}\int_{\mathbb{T}^m} \mathrm{d}\phi\; &\left(\Delta E(v_0, b, M, \phi, c) + \Delta E(-v_0, b, M, \phi, c)\right)\\ &= \frac{2\widehat{\beta_{II}^{(4)}}(e_0, b, M, c)}{\|v_0\|^4} + \mathrm{O}\left(\|v_0\|^{-5}\right).\end{aligned} \tag{A.17}$$

Here

$$\begin{aligned}\widehat{\beta_{II}^{(4)}}(e_0, b, M, c) = \frac{c^2}{2}\int_{\mathbb{T}^m} \mathrm{d}\phi \int_0^1 \mathrm{d}\lambda \int_0^1 \mathrm{d}\lambda'\; \left(\lambda - \lambda'\right)^2\\ \times\, \partial_\tau S\left(\lambda - \frac{1}{2}, M, \phi\right) \partial_\tau S\left(\lambda' - \frac{1}{2}, M, \phi\right),\end{aligned} \tag{A.18}$$

with $S(\mu, M, \phi) = \nabla W(M^{-1}(b + \mu e_0), \phi)$.

We remark that $\Delta E(v_0, b, M, \phi, c)$ and $\Delta E(-v_0, b, M, \phi, c)$ are the energy changes undergone by two distinct particles, both impinging at the same time on the same obstacle with the *same* impact parameter, but with opposite velocities. According to Proposition A.3, each of the two terms $\Delta E(\pm v_0, b, M, \phi, c)$ is of order $\|v_0\|^{-1}$ so that Proposition A.4 shows that combining a time average with a "time reversal" $v_0 \to -v_0$ diminishes the energy change undergone by the particle in a scattering event drastically.

Proof Using (A.15), we write, as in (A.12)–(A.13)

$$\Delta E = \Delta E_I + \Delta E_{II} + \mathrm{O}\left(\|v_0\|^{-5}\right). \tag{A.19}$$

We will accordingly write $\beta^{(\ell)} = \beta_I^{(\ell)} + \beta_{II}^{(\ell)}$, where $\beta^{(\ell)}$ is defined in (3.7). It is then immediately clear from (A.12) that

$$\begin{aligned}\int_{\mathbb{T}^m} \mathrm{d}\phi\ \Delta E_I\,(v_0, b, M, \phi, c)\\ = \frac{c}{\|v_0\|} \int \mathrm{d}\lambda \int \mathrm{d}\phi \partial_\tau W\left(M^{-1}\left(b + \left(\lambda - \frac{1}{2}\right) e_0\right), \omega\tau_0 + \frac{\omega\lambda}{\|v_0\|} + \phi\right) = 0,\end{aligned}$$

since $\partial_\tau = \omega \cdot \nabla_\phi$ and W is ϕ-periodic.

We now turn to ΔE_{II} which is of order $\|v_0\|^{-3}$ in view of (A.13) and write

$$\Delta E_{II} = \frac{\beta_{II}^{(3)}}{\|v_0\|^3} + \frac{\beta_{II}^{(4)}}{\|v_0\|^4} + \mathrm{O}\left(\|v_0\|^{-5}\right). \tag{A.20}$$

One then readily finds

$$\begin{aligned}\beta_{II}^{(3)}\,(e_0, b, M, \phi, c) = -c^2 \int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\lambda' \int_0^{\lambda'} \mathrm{d}\lambda''\\ \times \partial_\tau S\left(\lambda - \frac{1}{2}, M, \phi\right) \cdot S\left(\lambda'' - \frac{1}{2}, M, \phi\right).\end{aligned} \tag{A.21}$$

and, immediately performing the ϕ-average,

$$\begin{aligned}\int_{\mathbb{T}^m} \mathrm{d}\phi\ \beta_{II}^{(4)}\,(e, b, M, \phi, c) = \frac{c^2}{2} \int_{\mathbb{T}^m} \mathrm{d}\phi \int_0^1 \mathrm{d}\lambda \int_0^1 \mathrm{d}\lambda'\ \left(\lambda - \lambda'\right)^2\\ \times \partial_\tau S\left(\lambda - \frac{1}{2}, M, \phi\right) \cdot \partial_\tau S\left(\lambda' - \frac{1}{2}, M, \phi\right).\end{aligned}$$

Note that in (A.21), the integrand is in general no longer a gradient in the ϕ-variables, except in the special case where $W(y, \phi) = w(y) f(\phi)$. So there is no reason why the ϕ-average of $\beta_{II}^{(3)}(e_0, b, M, \phi, c)$ should vanish. But now remark, using (A.21) and the definition of S, that

$$\begin{aligned}\beta_{II}^{(3)}\,(-e_0, b, M, \phi, c) = -c^2 \int_0^1 \mathrm{d}\mu\ \partial_\tau S\left(\frac{1}{2} - \mu, M, \phi\right)\\ \cdot \int_0^\mu \mathrm{d}\mu' \int_0^{\mu'} \mathrm{d}\mu'' S\left(\frac{1}{2} - \mu'', M, \phi\right).\end{aligned}$$

When performing in this last expression the succession of changes of variable defined by $\frac{1}{2} - \mu'' = \lambda - \frac{1}{2}$, $\mu' = 1 - \lambda'$, $-\mu + 1 = \lambda''$, one finds

$$\begin{aligned}\beta_{II}^{(3)}\,(-e_0, b, M, \phi, c)\\ = -c^2 \int_0^1 \mathrm{d}\lambda'' \int_{\lambda''}^1 \mathrm{d}\lambda' \int_{\lambda'}^1 \mathrm{d}\lambda \partial_\tau S\left(\lambda'' - \frac{1}{2}, M, \phi\right) \cdot S\left(\lambda - \frac{1}{2}, M, \phi\right).\end{aligned} \tag{A.22}$$

Note that the domain of integration is the same as in (A.21), just the order of integration is different. So adding (A.21) and (A.22) the integrand becomes

$$S\left(\lambda-\frac{1}{2},M,\phi\right)\cdot\partial_\tau S\left(\lambda''-\frac{1}{2},M,\phi\right)+\partial_\tau S\left(\lambda-\frac{1}{2},M,\phi\right)\cdot S\left(\lambda''-\frac{1}{2},M,\phi\right)$$

which is a total time derivative. Computing the ϕ-average of the sum therefore yields

$$\int_{\mathbb{T}^m}\mathrm{d}\phi\,\left(\beta_{II}^{(3)}(e_0,b,M,\phi,c)+\beta_{II}^{(3)}(-e_0,b,M,\phi,c)\right)=0. \tag{A.23}$$

A similar computation shows that

$$\int_{\mathbb{T}^m}\mathrm{d}\phi\;\beta_{II}^{(4)}(-e_0,b,M,\phi,c)=\int_{\mathbb{T}^m}\mathrm{d}\phi\;\beta_{II}^{(4)}(e_0,b,M,\phi,c)\,. \tag{A.24}$$

Adding the various contributions, the proposition now follows from (A.19). □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1 (i) Noting that $\nabla\hat{W}(y,\tau)=M\nabla W(M^{-1}y,\omega\tau+\phi)$ one finds

$$\int_{-\infty}^{+\infty}\mathrm{d}\lambda\int_{e\cdot b=0}\mathrm{d}b\;\nabla\hat{W}\left(b+\left(\lambda-\frac{1}{2}\right)e,\tau\right)=0$$

since $\hat{W}$ has compact support in its first variable. So $\overline{\alpha^{(1)}}=0$. Similarly, integration over the ϕ variable leads to the vanishing of $\overline{\alpha^{(2)}(e)}$. To prove (4.2), we first point out that, in view of the rotational invariance of the system, $\Delta E(M'v_0,M'b,M'M,\phi,c)=\Delta E(v_0,b,M,\phi,c)$, for all $M'\in\mathrm{SO}(d,\mathbb{R})$. Consequently, $\overline{\Delta E(M'v_0)}=\overline{\Delta E(v_0)}$, where $\overline{\cdot}$ is defined in (3.12). As a result, $\overline{\Delta E(v_0)}$ depends only on $\|v_0\|$ and not on e_0. In particular $\overline{\Delta E(-v_0)}=\overline{\Delta E(v_0)}$. It therefore follows from (A.17) that

$$\overline{\Delta E\left(v_0\right)}=\frac{\overline{\beta_{II}^{(4)}}}{\|v_0\|^4}+\mathrm{O}\left(\|v_0\|^{-5}\right).$$

This proves the first equation in (4.2). Using

$$\Delta E(v,\kappa)=\frac{\beta^{(1)}(e,\kappa)}{\|v\|}+\mathrm{O}(\|v_0\|^{-2}),$$

the second equation in (4.2) and (4.5) follow immediately.

It remains to show (4.3) and (4.4). For that purpose, we compute B:

$$\begin{aligned}B&=\overline{\beta_{II}^{(4)}}\\&=\frac{\overline{c^2}}{2}\int_{\mathbb{T}^m}\mathrm{d}\phi\int_{\mathbb{S}^d}\mathrm{d}\Omega(e_0)\int_{b\cdot e_0=0}\frac{\mathrm{d}b}{C_d}\int_0^1\mathrm{d}\lambda\int_0^1\mathrm{d}\lambda'(\lambda-\lambda')^2\\&\quad\times\tilde{S}\left(b+\left(\lambda-\frac{1}{2}\right)e_0,\phi\right)\cdot\tilde{S}\left(b+\left(\lambda'-\frac{1}{2}\right)e_0,\phi\right)\end{aligned}$$

where $\tilde{S}(y,\phi) = \nabla\partial_\tau W(y,\phi)$. Using the change of variables formula (A.25), proven below, we find

$$B = \frac{\overline{c^2}}{2C_d}\int_{\mathbb{T}^m} \mathrm{d}\phi \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}y' \left\|y-y'\right\|^{3-d} \tilde{S}(y,\phi)\cdot\tilde{S}(y',\phi).$$

Using the definition of $\tilde{S}$ and integrating by parts twice, this yields

$$B = \frac{\overline{c^2}}{2C_d}\int_{\mathbb{T}^m} \mathrm{d}\phi \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}y' \sum_{i=1}^d \partial_{y_i}\partial_{y_i'}\|y-y'\|^{3-d}\partial_\tau W(y,\phi)\partial_\tau W(y',\phi).$$

Since

$$\partial_{y_i}\partial_{y_i'}\|y-y'\|^{3-d} = (d-3)\left((1-d)(y_i-y_i')^2+\|y-y'\|^2\right)\|y-y'\|^{-1-d},$$

we conclude

$$B = (d-3)\frac{\overline{c^2}}{2C_d}\int_{\mathbb{T}^m} \mathrm{d}\phi \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}y'\|y-y'\|^{1-d}\partial_\tau W(y,\phi)\partial_\tau W(y',\phi).$$

Using (3.10) and (4.5), one finds

$$\begin{aligned} D^2 = \overline{c^2}\int_{\mathbb{T}^m} \mathrm{d}\phi \int_{\mathbb{S}^d} \mathrm{d}\Omega(e_0)\int_{b\cdot e_0=0}\frac{\mathrm{d}b}{C_d}\int_{\mathbb{R}}\mathrm{d}\lambda\int_{\mathbb{R}}\mathrm{d}\lambda'\ \partial_\tau W\left(M^{-1}\left(b+\left(\lambda-\frac{1}{2}\right)e_0\right),\phi\right) \\ \times\, \partial_\tau W\left(M^{-1}\left(b+\left(\lambda'-\frac{1}{2}\right)e_0\right),\phi\right). \end{aligned}$$

In view of (A.25) this becomes

$$D^2 = \frac{\overline{c^2}}{C_d}\int_{\mathbb{T}^m} \mathrm{d}\phi \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}y'\|y-y'\|^{1-d}\partial_\tau W(y,\phi)\partial_\tau W(y',\phi),$$

which proves (4.3) and (4.4).

(ii) Note that when v_n is defined by (3.1), κ_n is independent of v_n since the latter only depends on κ_k for $k<n$. It follows therefore from (4.5) that $\langle\beta_n^{(\ell)}\rangle = 0 = \langle\beta_n^{(4)}\rangle - B$. The same remark applies to the computation of the correlations. For example, when computing $\langle\beta_n^{(\ell)}\beta_{n+k}^{(\ell')}\rangle$, for some positive k, one can integrate first over κ_{n+k}, which yields the result because of (i). □

Lemma A.5 *In dimension $d\geq 2$, for all $f:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^+\to\mathbb{R}$, such that $\|y_0-y_0'\|^{1-d}f(y_0,y_0',\|y_0-y_0'\|)\in L^1(\mathbb{R}^{2d})$, we have*

$$\begin{aligned} &\int_{\mathbb{S}^d} d\Omega(e_0)\int_{b\cdot e_0=0}\mathrm{d}b\int_{\mathbb{R}}\mathrm{d}\lambda\int_{\mathbb{R}}\mathrm{d}\lambda' f\left(b+\lambda e_0, b+\lambda' e_0, |\lambda-\lambda'|\right) \\ &\quad = \int_{\mathbb{R}^d} dy_0\int_{\mathbb{R}^d} dy_0'\left\|y_0-y_0'\right\|^{1-d} f\left(y_0,y_0',\left\|y_0-y_0'\right\|\right). \qquad \text{(A.25)} \end{aligned}$$

Proof Let y_0 and y_0' be in $\mathbb{R}^d$, $y_0\neq y_0'$. Then there exists a unique $(\lambda,\lambda',e_0,b)\in\mathbb{R}\times\mathbb{R}\times\mathbb{S}^d\times\mathbb{R}^d$ with $(y_0-y_0')\cdot e_0>0$ such that

$$y_0 = b+\lambda e_0, \qquad y_0' = b+\lambda' e_0 \quad \text{and} \quad b\cdot e_0 = 0.$$

Since $e_0 \in \mathbb{S}^d$, there exists unique angles $(\theta_1, \ldots, \theta_{d-1}) \in [0, \pi]^{d-2} \times [0, 2\pi]$ such that

$$e_0 = Ru_1, \quad R = R_{d-1}(\theta_{d-1}) \cdots R_1(\theta_1),$$

where $(u_1, \ldots, u_d)$ is the canonical basis of $\mathbb{R}^d$ and $R_i(\theta)$ is the rotation of angle θ in the plane defined by u_i and u_{i+1}. Since $R^{-1}b$ is orthogonal to u_1, there exists also a unique $(\rho, \tilde{\theta}_2, \ldots, \tilde{\theta}_{d-1}) \in \mathbb{R}^+ \times [0, \pi]^{d-3} \times [0, 2\pi]$ such that

$$b = \rho R\tilde{R}u_2, \quad \tilde{R} = R_{d-1}\left(\tilde{\theta}_{d-1}\right) \cdots R_2\left(\tilde{\theta}_2\right).$$

This gives the following equality:

$$\mathrm{d}y_0\mathrm{d}y_0' = |J|\,\mathrm{d}\lambda\mathrm{d}\lambda'\mathrm{d}\rho \prod_{i=1}^{d-1}\mathrm{d}\theta_i \prod_{j=2}^{d-1}\mathrm{d}\tilde{\theta}_j,$$

where

$$|J| = \begin{vmatrix} Ru_1 & 0_{d\times 1} & R\tilde{R}u_2 & N & M \\ 0_{d\times 1} & Ru_1 & R\tilde{R}u_2 & N & M' \end{vmatrix},$$

with $N = \rho R \nabla_{\tilde{\theta}}(\tilde{R}u_2)$,

$$M = \nabla_\theta\left(R\left(\rho\tilde{R}u_2 + \lambda u_1\right)\right) \quad \text{and} \quad M' = \nabla_\theta\left(R\left(\rho\tilde{R}u_2 + \lambda' u_1\right)\right).$$

Simple manipulations on the rows and columns yield

$$|J| = \left|\begin{array}{ccc|cc} Ru_1 & R\tilde{R}u_2 & N & 0_{d\times 1} & M \\ \hline & 0_{d\times d} & & Ru_1 & M' - M \end{array}\right| = \left|\lambda' - \lambda\right|^{d-1}\rho^{d-2}J_1J_2,$$

with $J_1 = |Ru_1; \nabla_\theta Ru_1|$, $J_2 = |u_1; \tilde{R}u_2; \nabla_{\tilde{\theta}}\tilde{R}u_2|$, the result follows upon noticing that

$$\mathrm{d}\Omega(e_0) = J_1 \prod_{i=1}^{d-1}\mathrm{d}\theta_i \quad \text{and} \quad \mathrm{d}b = \rho^{d-2}J_2\mathrm{d}\rho\prod_{j=2}^{d-1}\mathrm{d}\tilde{\theta}_j.$$

□

Proof of Theorem 5.1 Computing $R(v, \kappa)$ to second order in perturbation theory as in the proof of Proposition A.3, one finds

$$R(v, \kappa) = R_I(v, \kappa) + R_{II}(v, \kappa) + \mathrm{O}\left(\|v\|^{-4}\right), \tag{A.26}$$

where

$$R_I(v, \kappa) = \frac{c}{\|v\|}\int_{\mathbb{R}}\mathrm{d}\mu\ \hat{g}\left(b + \mu e, \tau_0 + \frac{\mu + \frac{1}{2}}{\|v\|}\right) \quad \text{and}$$

$$R_{II}(v, \kappa) = \frac{c^2}{\|v\|^3}\int_{\mathbb{R}}\mathrm{d}\mu\ \left[K(e, \kappa, \mu)\cdot\nabla\right]\hat{g}(b + \mu e, \tau_0),$$

with

$$K(e, \kappa, \mu) = \int_{-\infty}^{\mu}\mathrm{d}\mu'\int_{-\infty}^{\mu'}\mathrm{d}\mu''\hat{g}\left(b + \mu'' e, \tau_0\right).$$

As a result of (5.1), $\overline{R_I(v,\kappa)} = 0$, immediately implying $\overline{\alpha^{(\ell)}} = 0$ for $\ell = 1, 2$ and hence $\overline{\beta^{(\ell)}} = 0$ for $\ell = 0, 1$ (see (3.8)). To compute $\overline{\beta^{(2)}}$, we need $\overline{e \cdot \alpha^{(3)}}$. From (A.26) we find $\overline{e \cdot \alpha^{(3)}(e,\kappa)} = \overline{T(e,\kappa)}$ with

$$\begin{aligned} T(e,\kappa) &= c^2 \int_{\mathbb{R}} \mathrm{d}\mu \int_{-\infty}^{\mu} \mathrm{d}\mu' \int_{-\infty}^{\mu'} \mathrm{d}\mu'' \left[\hat{g}\left(b+\mu'' e, \tau_0\right) \cdot \nabla\right]\left(e \cdot \hat{g}\right)(b+\mu e, \tau_0) \\ &= c^2 \int_{\mathbb{R}} \mathrm{d}\mu \int_{-\infty}^{\mu} \mathrm{d}\mu'' \left(\mu - \mu''\right)\left[\hat{g}\left(b+\mu'' e, \tau_0\right) \cdot \nabla\right]\left(e \cdot \hat{g}\right)(b+\mu e, \tau_0). \end{aligned}$$

Noting that the integrand is unchanged under the change of variable $\tilde{e} = -e$, $\tilde{\mu} = -\mu$, $\tilde{\mu}'' = -\mu''$, one finds

$$\begin{aligned} \int \mathrm{d}\Omega(e) \int_{b\cdot e=0} \mathrm{d}b\, T(e,\kappa) = c^2 \int \mathrm{d}\Omega(\tilde{e}) \int_{b\cdot\tilde{e}=0} \mathrm{d}b \int_{\mathbb{R}} \mathrm{d}\tilde{\mu} \int_{\tilde{\mu}}^{+\infty} \mathrm{d}\tilde{\mu}'' \left(\tilde{\mu} - \tilde{\mu}''\right) \\ \times \left[\hat{g}\left(b+\tilde{\mu}''\tilde{e}, \tau_0\right) \cdot \nabla\right]\left(\tilde{e} \cdot \hat{g}\right)(b+\tilde{\mu}\tilde{e}, \tau_0). \end{aligned}$$

Averaging the last two formulas and using the change of variables formula (A.25), we conclude

$$\begin{aligned} \overline{e \cdot \alpha^{(3)}} &= \frac{\overline{c^2}}{2} \int_{\mathbb{T}^m} d\phi \int_{\mathbb{S}^d} \mathrm{d}\Omega(e_0) \int_{b\cdot e_0=0} \frac{\mathrm{d}b}{C_d} \int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}} \mathrm{d}\mu \\ &\quad \times \left(g(b+\mu'' e, \phi) \cdot \nabla\right)\left(((b+\mu e) - (b+\mu'' e)) \cdot g\right)(b+\mu e, \phi) \\ &= \frac{\overline{c^2}}{2C_d} \int_{\mathbb{T}^m} \mathrm{d}\phi \int \mathrm{d}y \mathrm{d}y'' \|y-y''\|^{1-d} \sum_j (y-y'')_j \left[g(y'',\phi) \cdot \nabla\right] g_j(y,\phi). \end{aligned}$$

A partial integration then yields

$$\begin{aligned} \overline{e \cdot \alpha^{(3)}} = &-\frac{\overline{c^2}(1-d)}{2C_d} \int_{\mathbb{T}^m} \mathrm{d}\phi \int \mathrm{d}y \mathrm{d}y'' \|y-y''\|^{-1-d} \\ &\times \left(\left(y-y''\right) \cdot g(y,\phi)\right)\left(\left(y-y''\right) \cdot g\left(y'',\phi\right)\right) \\ &- \frac{\overline{c^2}}{2C_d} \int_{\mathbb{T}^m} \mathrm{d}\phi \int \mathrm{d}y \mathrm{d}y'' \|y-y''\|^{1-d} g(y,\phi) \cdot g\left(y'',\phi\right). \end{aligned}$$

From (3.4) one easily sees the second term equals $-\frac{1}{2}\overline{\alpha^{(1)} \cdot \alpha^{(1)}(e)}$ so that, using (3.8), we find

$$\begin{aligned} \overline{\beta^{(2)}(e)} &= \frac{\overline{c^2}(d-1)}{2C_d} \int_{\mathbb{T}^m} \mathrm{d}\phi \int \mathrm{d}y \mathrm{d}y'' \|y-y''\|^{-1-d} \\ &\quad \times \left(\left(y-y''\right) \cdot g(y,\phi)\right)\left(\left(y-y''\right) \cdot g\left(y'',\phi\right)\right) \\ &= \frac{d-1}{2}\overline{\beta^{(0)}(e)^2} \geq 0. \end{aligned}$$

□

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