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Black-Scholes Formula in Subdiffusive Regime

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Abstract In the classical approach the price of an asset is described by the celebrated Black-Scholes model. In this paper we consider a generalization of this model, which captures the subdiffusive characteristics of financial markets. We introduce a subdiffusive geometric Brownian motion as a model of asset prices exhibiting subdiffusive dynamics. We find the corresponding fractional Fokker-Planck equation governing the dynamics of the probability density function of the introduced process. We prove that the considered model is arbitrage-free and incomplete. We find the corresponding subdiffusive Black-Scholes formula for the fair prices of European options and show how these prices can be evaluated using Monte-Carlo methods. We compare the obtained results with the classical ones.

Keywords Subdiffusion · Black-Scholes formula · Fractional Fokker-Planck equation

1 Introduction

Analysis of various real-life data shows that many processes observed in economics display characteristic periods in which they stay motionless [8]. This feature is most common for emerging markets in which the number of participants, and thus the number of transactions, is rather low. Notably, similar behavior is observed in physical systems exhibiting subdiffusion. The constant periods of financial processes correspond to the trapping events in which the subdiffusive test particle gets immobilized [5]. Subdiffusion is a well known and established phenomenon in statistical physics. Its usual mathematical description is in terms of the celebrated Fractional Fokker-Planck equation (FFPE). This equation was first derived from the continuous-time random walk scheme with heavy-tailed waiting times [2, 18, 19, 27], and since then became fundamental in modelling and analysis of complex systems exhibiting slow dynamics.

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The Langevin picture corresponding to the FFPE [13, 14, 16] reveals that subdiffusion is actually a combination of two independent mechanisms: the first mechanism is the standard diffusion represented by some Itô process $X(\tau)$, the second mechanism introduces the trapping events and is represented by the so-called inverse α-stable subordinator $S_\alpha(t)$. The subordinated process $X(S_\alpha(t))$ combines both mechanisms and gives the subdiffusive dynamics. The inverse α-stable subordinator is defined in the following way

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, \tag{1}$$

where $U_\alpha(t)$ is the α-stable subordinator [25, 26] with Laplace transform given by $E(e^{-uU_\alpha(t)}) = e^{-tu^\alpha}$, $0 < \alpha < 1$. $U_\alpha(t)$ is a pure-jump process. For every jump of $U_\alpha(t)$ there is a corresponding flat period of its inverse $S_\alpha(t)$. These heavy-tailed flat periods of $S_\alpha(t)$ represent long waiting-times in which the subdiffusive particle gets immobilized in the trap. When $\alpha \nearrow 1$, $S_\alpha(t)$ reduces to the "objective time" t.

The classical and still most popular model of the market is the Black-Scholes (BS) model based on the diffusion process called geometric Brownian motion. However, the empirical studies show that many of the characteristic properties of markets cannot be captured by the BS model. One should mention here such properties as: long-range correlations, heavy-tailed and skewed marginal distributions, lack of scale invariance, periods of constant values, etc. Therefore, in recent years one observes many generalizations of the BS model based on the ideas and methods known from statistical and quantum physics [15]. In this paper we apply the subdiffusive mechanism of trapping events in order to describe properly financial data exhibiting periods of constant values. In the next section we introduce the process called subdiffusive geometric Brownian motion, and derive the FFPE governing its probability density function (PDF). In Sect. 3, we show that the considered model is arbitrage-free and incomplete. Moreover, we prove the subdiffusive Black-Scholes formula for the prices of European options and show how these prices can be evaluated using Monte-Carlo methods. Finally, we present some numerical results.

2 Subdiffusive Geometric Brownian Motion

Brownian motion and financial engineering have been tied up since 1900, when Louis Bachelier proposed his model of asset prices [1]. It was improved later in the famous BS model [3, 17, 24], in which the price of an asset $Z(t)$ follows a geometric Brownian motion (GBM)

$$Z(t) = Z_0 \exp\{\sigma B(t) + \mu t\}, \quad Z_0 > 0. \tag{2}$$

Here, $B(t)$ is the standard Brownian motion, $\sigma > 0$ is the volatility, and $\mu \in \mathbb{R}$ is the drift parameter. The process $Z(t)$ can be equivalently defined in the form of the stochastic differential equation

$$dZ(t) = \left(\mu + \frac{\sigma^2}{2}\right) Z(t)dt + \sigma Z(t)dB(t), \quad Z(0) = Z_0. \tag{3}$$

Equivalently, $Z(t)$ can be written in the form $dZ(t) = \widehat{\mu} Z(t)dt + \sigma Z(t)dB(t)$, where $\widehat{\mu} = \mu + \frac{\sigma^2}{2}$. One of the main reasons of popularity of the BS model is the advanced mathematical apparatus, which allows to investigate various properties of the model. It is particularly important in the context of derivative pricing. Computation of various option prices and hedging strategies in the BS model is straightforward. The most relevant application of the BS model is the corresponding formula for the value of European call and put options.

A European call option is a financial contract between two parties, the buyer and the seller. It gives the buyer right, but not the obligation, to buy from the seller a specified amount of an underlying asset $Z(t)$ at a specified strike price K within a specified maturity time T. The buyer pays a fee (price) for this right. The value of the call option at the maturity is equal to $\max\{Z(T)-K, 0\}$. A put option is opposite of a call option, and it gives the holder right to sell shares. Thus, the value of the put option at the maturity equals $\max\{K-Z(T), 0\}$ The relationship between the price C of the call option and the price P of the put option, both with the identical strike price and maturity time, is given by the so-called put-call parity [20]

$$C - P = Z(0) - Ke^{-rT}, \tag{4}$$

where r is the risk-free interest rate, i.e the interest rate that can be obtained by investing the money with no default risk. In the following, we assume for simplicity that $r = 0$.

The fair price $C_{BS}(Z_0, K, T, \sigma)$ of the European call option in the BS model is given by [20]

$$C_{BS}(Z_0, K, T, \sigma) = Z_0 \Phi(d_+) - K\Phi(d_-), \tag{5}$$

with

$$d_\pm = \frac{\log\frac{Z_0}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

Here, T is the maturity date, K denotes the strike price, and Φ is the cumulative distribution function of standard normal distribution. By the put-call parity, the price of the put option yields

$$P_{BS}(Z_0, K, T, \sigma) = C_{BS}(Z_0, K, T, \sigma) + K - Z_0. \tag{6}$$

In spite of many obvious advantages of the BS model, there are also some drawbacks. In the light of empirical facts, it became evident that the BS model cannot capture many of the characteristic features of prices. One of such features are the periods of constant prices, which can be observed on emerging markets in which the number of transactions is low. Notably, these constant periods of financial processes are similar in nature to the trapping events in which the subdiffusive particle gets immobilized. Therefore, to capture the above mentioned property of financial time series, we propose a generalization of the BS model. We introduce the following process

$$Z_\alpha(t) = Z(S_\alpha(t)) \tag{7}$$

as the model of asset prices. We call $Z_\alpha(t)$ the *subdiffusive GBM*. Recall that $Z(t)$ is the GBM defined in (2) and $S_\alpha(t)$, $0 < \alpha < 1$, is the inverse α-stable subordinator defined in (1). $S_\alpha(t)$ is assumed to be independent of the Brownian motion $B(t)$. The process $S_\alpha(t)$ introduces the additional mechanism of trapping events, therefore $Z_\alpha(t)$ captures the empirical property of constant price periods. In Fig. 1 we can see the typical trajectories of the GBM $Z(t)$ and its subdiffusive counterpart $Z_\alpha(t)$. Both processes Z and Z_α share the same spatial properties. The temporal properties are governed by the subordinator S_α, which is responsible for the flat periods of the trajectories of the subdiffusive GBM, cf. Fig. 1. For the market description in the context of subordination see also [7, 28].

The next theorem describes the dynamics of the PDF of $Z_\alpha(t)$.

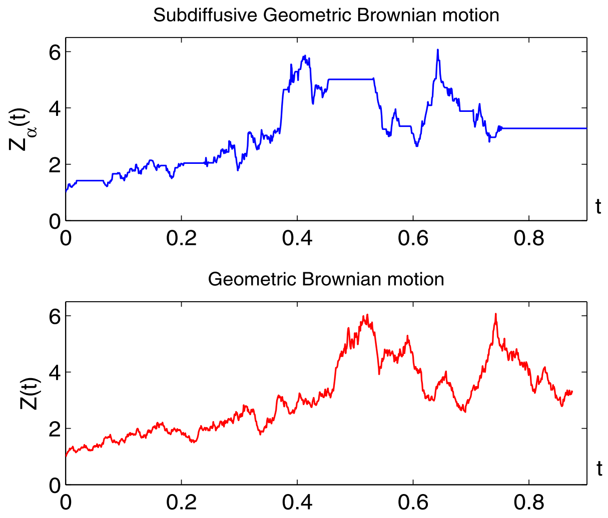

Fig. 1 Comparison of the trajectories of the classical GBM and its subdiffusive counterpart. In the subdiffusive case we observe the constant periods, which are characteristic for emerging markets and for the particles performing slow dynamics. Both processes Z and Z_α share the same spatial properties. The temporal properties are governed by the subordinator S_α, which is responsible for the flat periods of the trajectories of the subdiffusive GBM. The parameters are $\alpha = 0.8$, $\sigma = \mu = Z_0 = 1$

Theorem 1 *Let $Z_\alpha(t)$ be the subdiffusive GMB defined in (7). Then, the PDF of $Z_\alpha(t)$ is the solution of the fractional Fokker-Planck-type equation*

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[-\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}xw(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2 w(x,t)\right], \tag{8}$$

$w(x,0) = \delta_{Z_0}(x)$. *Here, the operator*

$${}_0D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

$0 < \alpha < 1$, $f \in C^1([0,\infty))$, *is the fractional derivative of the Riemann-Liouville type* [23].

Proof See Appendix. □

The above theorem can be used to investigate the dynamics of the PDF of $Z_\alpha(t)$. Alternatively, one can employ Monte Carlo methods to estimate the PDF of $Z_\alpha(t)$. Some remarks concerning methods of simulation will be given in the last part of the manuscript. In the next section, we prove that the introduced model of asset prices is arbitrage-free and incomplete. Moreover, we derive the subdiffusive Black-Scholes formula for option prices with the appropriate martingale measure.

3 Black-Scholes Formula in Subdiffusion

Let us now consider a market, whose evolution up to time horizon T is contained in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, Ω is the sample space, $\mathcal{F}$ contains all statements, which can be made about behavior of prices, and $\mathbb{P}$ is the "objective" probability measure. The underlying asset price is described by the subdiffusive GBM $(Z_\alpha(t))_{t\in[0,T]}$ defined in (7). We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the information about the history of asset prices $Z_\alpha(t)$ up to time t. $(\mathcal{F}_t)$ is also called filtration and is interpreted as the background information which is available for the investor. The more time proceeds the more information is revealed to the investor.

The essential requirement for pricing rules in a given market model is that it does not admit arbitrage opportunities. In finance and economy, an arbitrage is the simultaneous purchase and sale of an asset in order to profit from a difference in the price. In other words, an arbitrage is a "free lunch" transaction or portfolio that makes a profit without risk. Formally, an arbitrage opportunity [4] is a self-financing investing strategy ϕ, which can lead to a positive terminal gain, without any probability of intermediate loss:

$$\mathbb{P}(\forall t \in [0,T], V_t(\phi) \geq 0) = 1, \qquad \mathbb{P}(V_T(\phi) > V_0(\phi)) \neq 0,$$

where $V_t(\phi)$ denotes the value of portfolio ϕ at the moment t.

By the Fundamental theorem of asset pricing, the market model defined by $(\Omega, \mathcal{F}, \mathbb{P})$ and asset prices $(Z_\alpha(t))_{t\in[0,T]}$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is *arbitrage-free* if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the asset $(Z_\alpha(t))_{t\in[0,T]}$ is a martingale with respect to $\mathbb{Q}$. The measure $\mathbb{Q}$ is called the risk-neutral measure. Under this measure, financial assets have the same expected rate of return, regardless the variability in the price of the asset. This is in contrast to the physical probability measure (the actual probability distribution of prices), under which more risky assets have a greater expected rate of return than less risky assets.

Let us prove the following result

Theorem 2 *Let $\mathbb{Q}$ be the probability measure defined as*

$$\mathbb{Q}(A) = \int_A \exp\left\{-\gamma B(S_\alpha(T)) - \frac{\gamma^2}{2} S_\alpha(T)\right\} d\mathbb{P}, \tag{9}$$

where $\gamma = \frac{\mu+\sigma^2/2}{\sigma}$ and $A \in \mathcal{F}$. Then, the process $(Z_\alpha(t))_{t\in[0,T]}$ is a martingale with respect to $\mathbb{Q}$.

Proof See Appendix. □

Thus, by the Fundamental theorem of asset pricing [4] we get

Corollary 1 *The market model, in which the asset price follows the subdiffusive GBM $Z_\alpha(t)$, is arbitrage-free.*

Besides the idea of lack of arbitrage, another essential concept originates from the classical BS model. Namely, the concept of *market completeness*. Intuitively, a complete market is one in which the complete set of possible gambles on future states-of-the-world can be constructed with existing assets. Formally, we say that the market is complete if every $\mathcal{F}_T$-measurable random variable X (also called contingent claim) admits a replicating self-financing strategy ϕ [4]. The Second Fundamental theorem of asset pricing yields that a market defined by the asset $(Z_\alpha(t))_{t\in[0,T]}$ is complete if and only if there is a unique martingale measure equivalent to $\mathbb{P}$. The next result verifies the incompleteness of the market described by the subdiffusive GBM.

Theorem 3 *The equivalent martingale measure $\mathbb{Q}$ defined in (9) is not unique.*

Proof See Appendix. □

Therefore, by the Second Fundamental theorem of asset pricing [4]

Corollary 2 *The market model in which the asset price follows the subdiffusive GBM $Z_\alpha(t)$ is incomplete.*

Since the considered model is incomplete, different martingale measures lead to different prices of derivatives. However, the martingale measure $\mathbb{Q}$ defined in (9) has one important advantage. When $\alpha \nearrow 1$, $\mathbb{Q}$ reduces to the martingale measure of the classical BS model, which is known to be arbitrage-free and complete. Therefore, it allows to compare the obtained prices of the classical and subdiffusive BS models. Thus, in the following we concentrate on the martingale measure $\mathbb{Q}$ defined in (9). In the next theorem, we derive the subdiffusive BS formula for European option prices corresponding to $\mathbb{Q}$. Recall that for the call and put options the payoff functions take the form $\max\{Z_\alpha(T) - K, 0\}$ and $\max\{K - Z_\alpha(T), 0\}$, respectively.

Theorem 4 *Let us assume that the asset price follows the subdiffusive GBM $Z_\alpha(t)$, and that the martingale measure $\mathbb{Q}$ is given by (9). Then, the BS formula for the European call option price $C_{BS}^{Sub}(Z_0, K, T, \sigma, \alpha)$ satisfies*

$$C_{BS}^{Sub}(Z_0, K, T, \sigma, \alpha) = E(C_{BS}(Z_0, K, S_\alpha(T), \sigma))$$
$$= \int_0^\infty C_{BS}(Z_0, K, x, \sigma) T^{-\alpha} g_\alpha(x/T^\alpha) dx. \qquad (10)$$

Here, $g_\alpha(z)$ is the entire function given in terms of Fox function [23]

$$g_\alpha(z) = H_{1\,1}^{1\,0}\left(z\Big|_{(0,1)}^{(1-\alpha,\alpha)}\right),$$

whereas the function C_{BS} is given in (5).

Proof See Appendix. □

By the put-call parity (4), the put option price $P_{BS}^{Sub}(Z_0, K, T, \sigma, \alpha)$ satisfies

$$P_{BS}^{Sub}(Z_0, K, T, \sigma, \alpha) = C_{BS}^{Sub}(Z_0, K, T, \sigma, \alpha) + K - Z_0. \qquad (11)$$

It should be noted that in a similar way one can calculate prices of derivatives with different than European options payoff functions.

One can take advantage of the above theorem to calculate fair price of the European options in the subdiffusive BS model. For some values of α (i.e. for $\alpha = 1/2$) it is possible to evaluate Fox function $g_\alpha(\cdot)$ and to integrate numerically expressions in formulas (10) and (11). On the other hand it is always possible to estimate the expected values in the above expressions using Monte Carlo methods. One only needs to simulate trajectories of the inverse α-stable subordinator $S_\alpha(t)$. This can be done by the following very efficient approximation scheme introduced in [10]

$$S_{\alpha,\delta}(t) = (\min\{n \in \mathbb{N} : U_\alpha(\delta n) > t\} - 1)\,\delta. \qquad (12)$$

Here, $\delta > 0$ is the step length and $U_\alpha(\tau)$ is the α-stable subordinator. Therefore, to simulate $S_{\alpha,\delta}(t)$, one has to generate the values $U_\alpha(\delta n)$, $n = 1, 2, \ldots$. This is done by the Euler method of summing up the increments of the subordinator $U_\alpha(\tau)$, see [9]:

$$U_\alpha(0) = 0,$$
$$U_\alpha(\delta n) = U_\alpha(\delta(n-1)) + \delta^{1/\alpha}\xi_n,$$

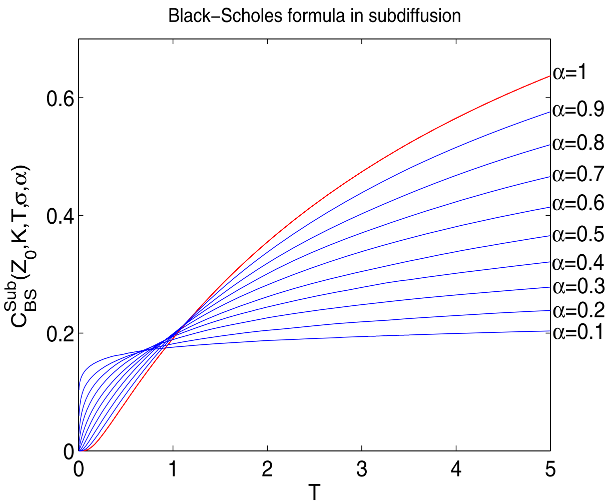

Fig. 2 Presented are prices of European call options corresponding to the subdiffusive BS model for different parameters α. The case $\alpha = 1$ is the classical BS formula. When α approaches 1, the prices converge to the classical BS result. Since $E(S_\alpha(T)) = \frac{T^\alpha}{\Gamma(\alpha+1)}$, the subdiffusive BS prices are higher than the standard one for $T < (\Gamma(\alpha+1))^{-1/(1-\alpha)}$, and lower for $T > (\Gamma(\alpha+1))^{-1/(1-\alpha)}$. The parameters are $\sigma = Z_0 = 1$, $K = 2$

where ξ_n, $n \in \mathbb{N}$, are the i.i.d. totally skewed positive α-stable random variables. The procedure of generating realizations of ξ_n is the following [30]

$$\xi_n = \frac{\sin(\alpha(V + c_1))}{(\cos(V))^{1/\alpha}} \left(\frac{\cos(V - \alpha(V + c_1))}{W} \right)^{(1-\alpha)/\alpha},$$

where $c_1 = \pi/2$, V is the uniformly distributed on $(-\pi/2, \pi/2)$ random variable and W has exponential distribution with mean one.

Simulation of the process $Z_\alpha(t)$ is just the combination of the well known method of simulating Brownian motion $B(t)$ [9] with the above mentioned algorithm of approximating $S_\alpha(t)$.

In Fig. 2 we compare the classical BS formula for European call options with the subdiffusive one. We estimated the values of C_{BS}^{Sub} using Monte Carlo methods based on the above described simulation procedure.

To fit the subdiffusive BS model to the real data, one has to estimate the parameters α, σ and μ. To estimate the first one, it is enough to extract the sample of the consecutive heavy-tailed waiting times (flat periods) from the observed trajectory. Analysis of the tail of the obtained sample will give the parameter α. Estimation of the remaining parameters can be performed in the same way as for the standard BS model. However, one has to eliminate first the effects of the inverse subordinator $S_\alpha(t)$ by removing the flat periods from the observed trajectory. Since the subdiffusive model is incomplete, different martingale measures lead to different pricing rules. Unfortunately, a straightforward criterion of choosing the true martingale measure is still missing. As already noted, the martingale measure $\mathbb{Q}$ defined in (9) has the advantage that for $\alpha \nearrow 1$ it reduces to the martingale measure of the classical BS model. This may correspond to the real-life process of increase of the market efficiency.

It should be emphasized that the reasoning presented in the manuscript can be easily generalized to the case of arbitrary inverse subordinators (cf. [12, 29]). Depending on the empirical distribution of the constant periods of asset prices, one can choose the appropriate infinitely divisible distribution and the corresponding inverse subordinator to model the dynamics of prices. Moreover, pricing of options for arbitrary inverse subordinator follows the same line as for the inverse α-stable subordinator.

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Appendix

Proof of Theorem 1 Denote by $p(x,t)$ the PDF of $Z_\alpha(t)$. From the total probability formula we get for the Laplace transform

$$\widehat{p}(x,k) = \int_0^\infty e^{-kt} p(x,t)dt = \int_0^\infty f(x,\tau)\widehat{g}(\tau,k)d\tau. \tag{13}$$

Here, $f(x,\tau)$ and $g(\tau,t)$ are the PDFs of $Z(\tau)$ and $S_\alpha(t)$, respectively, and $\widehat{g}(\tau,k) = \int_0^\infty e^{-kt} g(\tau,t)dt$. Since the process $Z(\tau)$ is given by (3), its PDF satisfies the standard Fokker-Planck equation

$$\frac{\partial f(x,\tau)}{\partial \tau} = -\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x f(x,\tau) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2 f(x,\tau),$$

or equivalently in the Laplace space

$$k\widehat{f}(x,k) - f(x,0) = -\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x\widehat{f}(x,k) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2 \widehat{f}(x,k), \tag{14}$$

where $\widehat{f}(x,k) = \int_0^\infty e^{-k\tau} f(x,\tau)d\tau$. Using the fact that $\mathbb{P}(S_\alpha(t) \le \tau) = \mathbb{P}((t/U_\alpha(1))^\alpha \le \tau)$, we obtain after some standard calculations

$$g(\tau,t) = -\frac{\partial}{\partial \tau}\int_0^t u(y,\tau)dy = \frac{t}{\alpha\tau} u(t,\tau).$$

Here, $u(t,\tau)$ is the PDF of $U_\alpha(\tau)$. Consequently,

$$\widehat{g}(\tau,k) = k^{\alpha-1} e^{-\tau k^\alpha}.$$

Using the above result in combination with (13), we get

$$\widehat{p}(x,k) = \int_0^\infty f(x,\tau) k^{\alpha-1} e^{-\tau k^\alpha} d\tau = k^{\alpha-1}\widehat{f}(x,k^\alpha).$$

The last formula applied to (14), after the change of variables $k \to k^\alpha$, gives

$$k\widehat{p}(x,k) - p(x,0) = k^{1-\alpha}\left[-\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x\widehat{p}(x,k) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2\widehat{p}(x,k)\right].$$

Finally, inverting the Laplace transform in the last equation, we obtain

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[-\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x p(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2 p(x,t)\right],$$

and the proof is completed. □

Proof of Theorem 2 Let us introduce the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ as

$$\mathcal{G}_t = \mathcal{H}_{S_\alpha(t)} \tag{15}$$

where

$$\mathcal{H}_\tau = \bigcap_{u>\tau} \{\sigma(B(y) : 0 \le y \le u) \vee \sigma(S_\alpha(y) : y \ge 0)\}. \tag{16}$$

Clearly, $\mathcal{F}_t \subseteq \mathcal{G}_t$. Moreover, from Theorem 2.1 in [11] we get that both processes $B(S_\alpha(t))$ and $\exp\{-\gamma B(S_\alpha(t)) - \frac{\gamma^2}{2} S_\alpha(t)\}$ are $(\mathcal{G}_t)$-martingales with respect to $\mathbb{P}$. Thus, $\mathbb{Q}$ is a probability measure. Clearly, it is equivalent to $\mathbb{P}$.

We will show that $Z_\alpha(t)$ is a $(\mathcal{G}_t)$-martingale with respect to $\mathbb{Q}$. Let us put

$$K_\alpha(t) = B(S_\alpha(t)) + \gamma S_\alpha(t)$$

and

$$H(t) = E\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{G}_t\right) = E\left(\exp\left\{-\gamma B(S_\alpha(T)) - \frac{\gamma^2}{2} S_\alpha(T)\right\}\middle|\mathcal{G}_t\right).$$

By the martingale property, we get that

$$H(t) = \exp\left\{-\gamma B(S_\alpha(t)) - \frac{\gamma^2}{2} S_\alpha(t)\right\}$$

or, equivalently

$$dH(t) = -\gamma H(t) dB(S_\alpha(t)), \quad H(0) = 1. \tag{17}$$

Moreover, we have that

$$Z_\alpha(t) = \exp\left\{\sigma K_\alpha(t) - \frac{\sigma^2}{2} S_\alpha(t)\right\} \tag{18}$$

and the quadratic variation of $K_\alpha(t)$ satisfies $\langle K_\alpha(t), K_\alpha(t)\rangle = S_\alpha(t)$. Now, taking advantage of formula (17) and the Girsanov-Meyer theorem [22], we get that the process

$$\begin{aligned} &B(S_\alpha(t)) - \int_0^t \frac{1}{H(s)} d\langle H(s), B(S_\alpha(s))\rangle \\ &\quad = B(S_\alpha(t)) + \gamma \int_0^t \frac{1}{H(s)} H(s) d\langle B(S_\alpha(s)), B(S_\alpha(s))\rangle \\ &\quad = B(S_\alpha(t)) + \gamma S_\alpha(t) = K_\alpha(t) \end{aligned}$$

is a local martingale with respect to $\mathbb{Q}$. By (18), also $Z_\alpha(t)$ is a local martingale with respect to $\mathbb{Q}$. Finally, since

$$E^{\mathbb{Q}}(Z_\alpha(t)) = E\left(\exp\left\{\sigma K_\alpha(t) - \frac{\sigma^2}{2} S_\alpha(t) - \gamma B(S_\alpha(T)) - \frac{\gamma^2}{2} S_\alpha(T)\right\}\right) = 1,$$

$Z_\alpha(t)$ is a $(\mathcal{G}_t)$ (and also $(\mathcal{F}_t)$)-martingale with respect to $\mathbb{Q}$. □

Proof of Theorem 3 For every $\epsilon > 0$, we define a probability measure $\mathbb{Q}_\epsilon$ in the following way

$$\mathbb{Q}_\epsilon(A) = C \int_A \exp\left\{-\gamma B(S_\alpha(T)) - \left(\epsilon + \frac{\gamma^2}{2}\right) S_\alpha(T)\right\} d\mathbb{P}, \tag{19}$$

where $C = (E(\exp\{-\gamma B(S_\alpha(T)) - (\epsilon + \frac{\gamma^2}{2})S_\alpha(T)\}))^{-1}$ is the normalizing constant, $\gamma = \frac{\mu+\sigma^2/2}{\sigma}$, and $A \in \mathcal{F}$. We will show that $Z_\alpha(t)$ is a $(\mathcal{G}_t)$-martingale with respect to $\mathbb{Q}_\epsilon$, where $\mathcal{G}_t = \mathcal{H}_{S_\alpha(t)}$ is defined in (15).

Clearly, $\mathbb{Q}_\epsilon$ is equivalent to $\mathbb{P}$. Put

$$Y(t) = \exp\left\{-\gamma B(t) - \frac{\gamma^2}{2} t\right\}, \qquad Z(t) = \exp\{\sigma B(t) + \mu t\}.$$

Then, we have

$$Y(t)Z(t) = \exp\left\{(\sigma - \gamma)B(t) - \frac{(\sigma-\gamma)^2}{2} t\right\}.$$

Thus, $Y(t)Z(t)$ is a $(\mathcal{H}_t)$-martingale with respect to $\mathbb{P}$, where $(\mathcal{H}_t)$ is defined in (16). Let us define

$$Z^{S_\alpha(T)}(t) = Z(t \wedge S_\alpha(T)).$$

Then, the stopped process $Y(t \wedge S_\alpha(T))Z^{S_\alpha(T)}(t)$ is also a $(\mathcal{H}_t, \mathbb{P})$-martingale. Since the filtration $(\mathcal{H}_t)$ is right-continuous, the bounded random variable $e^{-\epsilon S_\alpha(T)}$ is $\mathcal{H}_0$-measurable. It follows that the process

$$\left(e^{-\epsilon S_\alpha(T)} Y(t \wedge S_\alpha(T)) Z^{S_\alpha(T)}(t)\right)_{t \geq 0}$$

is also a $(\mathcal{H}_t, \mathbb{P})$-martingale. Moreover, for $A \in \mathcal{H}_t$ we have

$$\begin{aligned}
\mathbb{Q}_\epsilon(A) &= E\left(\mathbf{1}_A \exp\left\{-\gamma B(S_\alpha(T)) - \left(\epsilon + \frac{\gamma^2}{2}\right) S_\alpha(T)\right\}\right) \\
&= E\left(\mathbf{1}_A e^{-\epsilon S_\alpha(T)} E\left(\exp\left\{-\gamma B(S_\alpha(T)) - \frac{\gamma^2}{2} S_\alpha(T)\right\}\middle|\mathcal{H}_t\right)\right) \\
&= E\left(\mathbf{1}_A e^{-\epsilon S_\alpha(T)} Y(t \wedge S_\alpha(T))\right).
\end{aligned}$$

Therefore, the process $Z^{S_\alpha(T)}(t)$ is a $(\mathcal{H}_t, \mathbb{Q}_\epsilon)$-martingale. Next, we have

$$\begin{aligned}
E^{\mathbb{Q}_\epsilon}\left(\sup_{t\geq 0} Z^{S_\alpha(T)}(t)\right) &= E^{\mathbb{Q}_\epsilon}\left(\sup_{t \leq S_\alpha(T)} Z(t)\right) \\
&= E\left(\exp\left\{-\gamma B(S_\alpha(T)) - \left(\epsilon + \frac{\gamma^2}{2}\right) S_\alpha(T)\right\} \sup_{t \leq S_\alpha(T)} Z(t)\right) \\
&\leq E\left(\exp\{-\gamma B(S_\alpha(T))\} e^{|\mu| S_\alpha(T)} \sup_{t \leq T} e^{\sigma B(S_\alpha(t))}\right).
\end{aligned} \tag{20}$$

Since $E(S_\alpha^n(T)) = \frac{T^{n\alpha} n!}{\Gamma(n\alpha+1)}$, $n \in \mathbb{N}$ (see [21]), for any $\lambda > 0$ we have

$$E\left(\exp\{\lambda S_\alpha(T)\}\right) = \sum_{n=0}^{\infty} \frac{\lambda^n E(S_\alpha^n(T))}{n!} = \sum_{n=0}^{\infty} \frac{(T^\alpha \lambda)^n}{\Gamma(n\alpha+1)} < \infty.$$

Moreover, conditioning on $\sigma(S_\alpha(y) : y \geq 0)$ we obtain

$$E\left(\exp\{\lambda B(S_\alpha(T))\}\right) = E\left(\exp\left\{\frac{\lambda^2}{2} S_\alpha(T)\right\}\right) < \infty.$$

Additionally, from the Doob's maximal inequality we get

$$E\left(\sup_{t\leq T} \exp\{\lambda B(S_\alpha(t))\}\right)^2 \leq 4E\left(\exp\{2\lambda B(S_\alpha(T))\}\right) < \infty.$$

Thus, by the Hölder inequality, the above results in combination with (20) yield

$$E^{\mathbb{Q}_\epsilon}\left(\sup_{t\geq 0} Z^{S_\alpha(T)}(t)\right) < \infty.$$

Therefore, $Z^{S_\alpha(T)}(t)$ is a uniformly integrable martingale. It follows that there exists a random variable X such that $Z^{S_\alpha(T)}(t) = E^{\mathbb{Q}_\epsilon}(X|\mathcal{H}_t)$ and

$$Z_\alpha(t) = Z^{S_\alpha(T)}(S_\alpha(t)) = E^{\mathbb{Q}_\epsilon}\left(X|\mathcal{H}_{S_\alpha(t)}\right).$$

Finally, we get that $Z_\alpha(t)$ is a $(\mathcal{H}_{S_\alpha(t)}, \mathbb{Q}_\epsilon)$-martingale. Since $\mathcal{F}_t \subseteq \mathcal{H}_{S_\alpha(t)}$, it is also a $(\mathcal{F}_t, \mathbb{Q}_\epsilon)$-martingale. □

Proof of Theorem 4 The arbitrage-free pricing rule requires that

$$\begin{aligned} C_{BS}^{Sub}(Z_0, K, T, \sigma, \alpha) &= E^{\mathbb{Q}}((Z_\alpha(T) - K)^+) \\ &= E\left(\exp\left\{-\gamma B(S_\alpha(T)) - \frac{\gamma^2}{2} S_\alpha(T)\right\}(Z_\alpha(T) - K)^+\right). \end{aligned}$$

Therefore, conditioning on $S_\alpha(T)$, we obtain

$$\begin{aligned} C_{BS}^{Sub}(Z_0, K, T, \sigma, \alpha) &= E(C_{BS}(Z_0, K, S_\alpha(T), \sigma)) \\ &= \int_0^\infty C_{BS}(Z_0, K, x, \sigma) g_\alpha(x, T) dx, \end{aligned}$$

where $g_\alpha(x, T)$ is the PDF of $S_\alpha(T)$. By the α-selfsimilarity of $S_\alpha(T)$, we get that $g_\alpha(x, T) = T^{-\alpha} g_\alpha(x/T^\alpha)$, where $g_\alpha(z)$ is the PDF of $S_\alpha(1)$. Since the Laplace transform of $S_\alpha(1)$ satisfies $E(\exp\{-uS_\alpha(1)\}) = E_\alpha(-u)$, where $E_\alpha(\cdot)$ is the Mittag-Leffler function [23], one can verify [6, 21] that $g_\alpha(z) = H_{1\,1}^{1\,0}(z|_{(0,1)}^{(1-\alpha,\alpha)})$. Thus, the statement follows. □

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