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Clustering Bounds on n-Point Correlations for Unbounded Spin Systems

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Abstract We prove clustering estimates for the truncated correlations, i.e., cumulants of an unbounded spin system on the lattice. We provide a unified treatment, based on cluster expansion techniques, of four different regimes: large mass, small interaction between sites, large self-interaction, as well as the more delicate small self-interaction or near massive Gaussian regime. A clustering estimate in the latter regime is needed for the Bosonic case of the recent result obtained by Lukkarinen and Spohn on the rigorous control on kinetic scales of quantum fluids.

Keywords Cluster estimates · Cluster expansion · Mayer expansion · Unbounded spin systems · Correlation decay

1 Introduction

In this paper we consider the following example of unbounded lattice spin system. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of lattice sites which contains the origin $\mathbf{0}$. For each site $\mathbf{x} \in \Lambda$ we associate a complex-valued random variable $\psi(\mathbf{x})$ called a spin or a field. Note that we will use "$*$" to denote complex conjugation instead of a bar. The collection

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$\psi_\Lambda = (\psi(\mathbf{x}))_{\mathbf{x}\in\Lambda} \in \mathbb{C}^\Lambda$ of these variables is sampled according to the finite volume Gibbs measure

$$\langle\cdot\rangle_\Lambda = \frac{1}{Z_\Lambda}\int_{\mathbb{C}^\Lambda} D\psi^* D\psi\ e^{-H_\Lambda(\psi_\Lambda)}(\cdot) \tag{1}$$

where

$$Z_\Lambda = \int_{\mathbb{C}^\Lambda} D\psi^* D\psi\ e^{-H_\Lambda(\psi_\Lambda)} > 0 \tag{2}$$

and

$$D\psi^* D\psi = \prod_{\mathbf{x}\in\Lambda} \frac{d\Re\psi(\mathbf{x})d\Im\psi(\mathbf{x})}{\pi}$$

is proportional to the Lebesgue measure in $\mathbb{C}^\Lambda$, and the Hamiltonian with free boundary conditions is given by

$$H_\Lambda(\psi_\Lambda) = \sum_{(\mathbf{x},\mathbf{y})\in\Lambda^2} J(\mathbf{x}-\mathbf{y})\psi^*(\mathbf{x})\psi(\mathbf{y}) + \frac{\lambda}{4}\sum_{\mathbf{x}\in\Lambda}|\psi(\mathbf{x})|^4. \tag{3}$$

The assumptions on the parameters appearing in the Hamiltonian are the following:

- $\lambda > 0$.
- The pair potential J is a function $\mathbb{Z}^d \to \mathbb{C}$ of compact support such that

$$J(\mathbf{x})^* = J(-\mathbf{x}).$$

- $J_{\neq} = \sum_{\mathbf{x}\neq\mathbf{0}} |J(\mathbf{x})|$ satisfies $J_{\neq} \leq J(\mathbf{0})$.

Let $\psi^\sharp$ stand for either ψ or ψ^*. The main goal of this article is to study the so called *truncated correlation functions*, defined by

$$\begin{aligned}&\langle\psi^\sharp(\mathbf{x}_1),\ldots,\psi^\sharp(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}\\ &\quad= \frac{\partial}{\partial\alpha_1}\cdots\frac{\partial}{\partial\alpha_n}\mathrm{Log}\big\langle(1+\alpha_1\psi^\sharp(\mathbf{x}_1))\cdots(1+\alpha_n\psi^\sharp(\mathbf{x}_n))\big\rangle_\Lambda\big|_{\alpha_1,\ldots,\alpha_n=0}\end{aligned}$$

where Log denotes the principal logarithm of a complex number, i.e. $\mathrm{Log}(re^{i\theta}) = \ln(r) + i\theta$, with $r > 0$ and $\theta \in (-\pi, \pi]$. It is a standard task to show that the series in $\alpha_1, \ldots, \alpha_n$ is analytic in a small polydisc D_Λ around $\alpha_1, \ldots, \alpha_n = 0$. These truncated correlations are also known in the literature as cumulants, semi-invariants or connected correlation functions.

We will find uniformly in Λ and the assignments for the $\sharp$ symbols, explicit bounds of the form

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n\in\Lambda} \left|\langle\psi^{\sharp_1}(\mathbf{0}),\psi^{\sharp_2}(\mathbf{x}_2),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}\right| \leq c_n(J,\lambda) \tag{4}$$

also called l^1-clustering estimates. We will derive such bounds in four different regimes:

- $J(\mathbf{0})$ big, i.e., the large-mass regime;
- $J_{\neq}$ small, i.e., the regime of small interaction between lattice sites;
- λ big, or the 'high temperature' regime;
- λ small and $J(\mathbf{0}) > J_{\neq}$, i.e., the near massive Gaussian regime or near-Gaussian regime for short.

Remark 1 Note that one could introduce a temperature dependence by replacing the Hamiltonian $H_\Lambda(\psi_\Lambda)$ by $\beta H_\Lambda(\psi_\Lambda)$ where β as usual denotes the inverse of the temperature times Boltzmann's constant. However, one can do the change of variable $\psi = \beta^{-\frac{1}{2}}\psi'$ and absorb β into a modification of the coupling $\lambda \to \beta^{-1}\lambda$. In this setting, λ big would coincide with 'high temperature', and λ small with 'low temperature'. Some authors (see e.g. [8, 9]) use the expression 'low temperature' for what we called the near-Gaussian regime. We prefer instead to reserve the expression 'low temperature' for the situation where there is a broken symmetry, which is not the case here.

Remark 2 We excluded from our working hypotheses the trivial situation $\lambda = 0$ since the higher truncated functions vanish identically in that case and there is nothing to prove.

In the statement of the theorems below we will denote by the same symbol $O(1)$ the various constants which appear. These are absolute constants such as $\sqrt{\pi}$ or quantities which only depend on the dimension d which is fixed throughout. In the following sections we will express such constants in details, to prove that they are effectively computable. However we are not interested in this paper in optimal bounds so the values that we give may be improved with more refined estimates.

Our results for the various regimes above consist of the following theorems.

Theorem 1 *In the event that $J_{\neq}$ and $\lambda > 0$ are fixed, there exists a $K(J_{\neq}, \lambda) > J_{\neq} \geq 0$ which is independent of the volume Λ such that the l^1-clustering estimate*

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n \in \Lambda} |\langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}| \leq n! \times \left[\frac{O(1)}{\sqrt{J(\mathbf{0}) - J_{\neq}}}\right]^n$$

holds as soon as $J(\mathbf{0}) \geq K(J_{\neq}, \lambda)$.

Theorem 2 *In the event that $J(\mathbf{0}) > 0$ and $\lambda > 0$ are fixed, there exists an $\epsilon(J(\mathbf{0}), \lambda) \in (0, J(\mathbf{0}))$ which is independent of the volume Λ such that the l^1-clustering estimate*

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n \in \Lambda} |\langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}| \leq n! \times \left[\frac{O(1)(1 + \frac{1}{J(\mathbf{0})}\sqrt{\frac{\lambda}{2}})}{\sqrt{J(\mathbf{0}) - J_{\neq}}}\right]^n$$

holds as soon as $0 \leq J_{\neq} \leq \epsilon(J(\mathbf{0}), \lambda)$.

Theorem 3 *In the event that $J_{\neq}$ and $J(\mathbf{0}) \geq J_{\neq}$ are fixed, there exists a $K(J(\mathbf{0}), J_{\neq}) > 0$ which is independent of the volume Λ such that the l^1-clustering estimate*

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n \in \Lambda} |\langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}| \leq n! \times \left[O(1) \times \lambda^{-\frac{1}{4}}\right]^n$$

holds as soon as $\lambda \geq K(J(\mathbf{0}), J_{\neq})$.

Theorem 4 *Let $N \geq 1$ be an integer and suppose that J is fixed. Then there exist quantities $\epsilon(J) > 0$, $c_1(N, J) > 0$ and $c_2(J) > 0$ such that, for any λ with $0 < \lambda \leq \epsilon(J)$, for any* *nonempty finite volume* $\Lambda \subset \mathbb{Z}^d$, *and any even integer* n, $n \geq 2(N+1)$, *we have the* l^1-*clustering bound*

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n \in \Lambda} |\langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle_{\Lambda}^{\mathrm{T}}| \leq c_1(N,J) \times c_2(J)^n \times \lambda^N \times n!.$$

Theorem 5 *There exists a quantity* $c_3(J) > 0$ *such that for the same* $\epsilon(J)$ *as in the previous theorem, for any* λ *with* $0 < \lambda \leq \epsilon(J)$, *and for any nonempty finite volume* $\Lambda \subset \mathbb{Z}^d$, *one has*

$$\sum_{\mathbf{x}_2 \in \Lambda} |\langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2)\rangle_{\Lambda,\lambda}^{\mathrm{T}} - \langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2)\rangle_{\Lambda,0}^{\mathrm{T}}| \leq c_3(J)\lambda$$

where we have restored the λ-*dependence in the notation for truncated correlation functions*.

Such l^1-clustering estimates in statistical mechanics have a long history. See [56, 57] for the case of classical gases at low activity. The importance of such estimates in relation to analyticity of thermodynamic functions was stressed in [23]. However, to the best of our knowledge, clustering estimates which include the results of the previous theorems have not previously appeared in the literature. The cluster expansion methods we used to derive these results also provide the exponential tree decay of the truncated correlations as a function of the locations $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of the sources. The reader can easily extract such decay estimates from the proofs provided here. However, in this article we focus instead on l^1 estimates where the $\mathbf{x}_i$ are summed over the lattice. Indeed, our primary motivation comes from the recent work of Lukkarinen and Spohn [44, 45] where they obtained the rigorous control of the kinetic regime for the time-evolution of a class of interacting quantum fluids. Their result is conditional on establishing l^1-clustering estimates such as the ones provided in Theorems 4 and 5. Similar estimates in the Fermionic case, and in the precise form needed for the work of Lukkarinen and Spohn, were recently obtained by Salmhofer [59]. Bounds on Fermionic truncated correlation functions were previously derived by many authors, see for example [13, 50, 52]. This kind of Fermionic estimates may be derived using a variety of techniques: Hadamard bounds on determinants [20], Gram bounds together with the free propagator decay [25], the Pauli exclusion principle in momentum space [28], and in direct space [38], the Brydges-Battle-Federbush tree identity [42], the Brydges-Kennedy-Abdesselam-Rivasseau (BKAR) forest formula [4], and the ring expansion of [26]. A good introductory reference on these Fermionic estimates is [48, Chap. 2]. Here we will exclusively deal with the Bosonic situation. Our methods are robust enough to handle much more general unbounded spin systems, but for better readability we refrained from stating our results with maximal generality, and restricted our attention to the complex Bosonic model presented above which is the one needed for [44, 45].

Another motivation for the present work comes from the recent interest in the decay of correlations for unbounded spin systems, especially in relation to Log-Sobolev inequalities (see e.g. [7, 35, 36, 53, 68, 70] and references therein). For the equivalence between the exponential decay of the truncated 2-point function, the Log-Sobolev inequality and the spectral gap property, in the case of unbounded spins, see [69]. The study of the decay properties for the 2-point function for this class of models has a long history [21, 33, 39, 41, 63]. In the near-Gaussian case, i.e., in a regime similar to the setting of Theorems 4 and 5, recent proofs of exponential decay for the truncated 2-point function were given in [8, 9, 49, 61, 62]. These works use the methods introduced by Helffer and Sjöstrand in [37]. However, to the best of our knowledge, one has not been able to treat higher truncated n-point functions with Helffer-Sjöstrand and Witten laplacian techniques. The only such results [37, 43] for higher correlations that we are aware of concern centered moments, i.e., expectations of the form $\langle (X_1 - \langle X_1\rangle)\cdots(X_n - \langle X_n\rangle)\rangle$ which are different from fully truncated correlations $\langle X_1, \ldots, X_n\rangle^{\mathrm{T}}$.

It is quite well known that the first three regimes mentioned above are amenable to cluster expansion techniques of the kind that is standard in the statistical mechanics literature (see e.g. [47, 57, 60]). Much less known, in the mathematical analysis and probability theory communities, is that the near-Gaussian can also be treated using a special kind of cluster expansion technique. We refer to the latter as the field theoretic cluster expansion. It also has a long history, and originates in the work of Glimm, Jaffe and Spencer in constructive quantum field theory [30, 31]. It has then been simplified and improved by many authors (see in particular [15, 29, 46, 51, 55] and references therein). Our approach for the proof of Theorems 4 and 5 owes much to the pioneering work [24] in the context of $P(\phi)_2$ quantum field theories. This was adapted to the lattice setting in [22, 66]. These use the original Glimm-Jaffe-Spencer cluster expansion, which involves a decoupling procedure for the Gaussian measure. A simpler way to do this was introduced by Brydges, Battle and Federbush [12, 16] and it is based on a combinatorial tree expansion identity. The third generation of such decoupling procedures, which is the one used in this article, is based on the BKAR forest formula [2, 17]. We give an account of this basic tool and of the general results of cluster expansion in Sects. 2.4 and 2.5. We will use it many times, in the proof of the estimates, first for the 'high-temperature' regimes and then for the near-Gaussian regime. One of the difficulties which we have to address here and which does not seem to have received a great deal of attention in the previous literature, is the aim for bounds which are uniform in n, growing as $n!$, and where one *simultaneously* extracts as many powers of λ as possible. Such an extraction of perturbation theory in the context of constructive field theory can be done using additional Taylor expansions (see e.g. [10, Sect. 5.14]). However, we have been unable to extract the optimal bound $C^n\lambda^{\frac{n}{2}-1}n!$ predicted by tree level perturbation theory, in a way which is uniform in n. The precise statement of this problem is given as Conjecture 1 from Sect. 2.2. Of course, one can ask the same question for the real-valued scalar field model with φ^4 interaction, on the lattice. Note that a related bound with a factor $\lambda^{\frac{n}{4}}$ was obtained by Brydges, Dimock and Hurd [19, Theorem 9] in the context of the UV limit of the ϕ_3^4 model. It is quite likely that one can obtain estimates similar to ours using the methods of [11] together with an additional large versus small field analysis. It would be interesting to see if this alternate approach would produce $C^n\lambda^{\frac{n}{2}-1}n!$ bounds.

Let us conclude this introduction by indicating some of the notation used throughout this article. We use $|\cdot|$ for the cardinality of finite sets. If n is a nonnegative integer, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. We will denote by $\mathbb{1}\{\cdots\}$ the characteristic function of the condition between braces.

2 Preliminaries

2.1 Basic Properties of the Model

By hypothesis on the function J, the matrix $\tilde{J} = (J(\mathbf{x}-\mathbf{y}))_{\mathbf{x},\mathbf{y}\in\Lambda}$ is Hermitian positive definite. This can be proved as follows. Let us denote the inner product on $l^2(\Lambda)$ by $\langle \psi_1, \psi_2\rangle = \sum_{\mathbf{x}\in\Lambda} \psi_1^*(\mathbf{x})\psi_2(\mathbf{x})$, and the norm of a vector by $\|\cdot\|$. The hypothesis $J(\mathbf{x})^* = J(-\mathbf{x})$ trivially implies that the matrix $\tilde{J} = (J(\mathbf{x}-\mathbf{y}))_{\mathbf{x},\mathbf{y}\in\Lambda}$ is Hermitian $\tilde{J}^\dagger = \tilde{J}$. Besides for any field

ψ on $l^2(\Lambda)$, on has

$$\langle\psi,\tilde{J}\psi\rangle=\sum_{\mathbf{x},\mathbf{y}\in\Lambda}\psi^*(\mathbf{x})J(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})=J(\mathbf{0})\|\psi\|^2+\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}\psi^*(\mathbf{x})J(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})$$

and

$$\begin{aligned}\left|\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}\psi^*(\mathbf{x})J(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\right| &\leq \sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}\left(|\psi(\mathbf{x})|\sqrt{|J(\mathbf{x}-\mathbf{y})|}\right)\left(|\psi(\mathbf{y})|\sqrt{|J(\mathbf{x}-\mathbf{y})|}\right)\\ &\leq \frac{1}{2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}|\psi(\mathbf{x})|^2|J(\mathbf{x}-\mathbf{y})|+\frac{1}{2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}|\psi(\mathbf{y})|^2|J(\mathbf{x}-\mathbf{y})|\end{aligned}$$

where we used the inequality $ab\leq\frac{1}{2}(a^2+b^2)$. Therefore the last expression is bounded by $J_{\neq}\|\psi\|^2$ and

$$\langle\psi,\tilde{J}\psi\rangle\geq(J(\mathbf{0})-J_{\neq})\|\psi\|^2$$

so that $\tilde{J}$ is positive definite when $J(\mathbf{0})>J_{\neq}$. In this case, the covariance matrix $C=\tilde{J}^{-1}=(C(\mathbf{x},\mathbf{y}))_{\mathbf{x},\mathbf{y}\in\Lambda}$ is well defined, and there exists a unique mean zero normalized Gaussian probability measure denoted by $d\mu_C(\psi^*,\psi)$ on $\mathbb{C}^\Lambda$ with covariance matrix C, i.e., such that

$$\int_{\mathbb{C}^\Lambda}d\mu_C(\psi^*,\psi)\psi(\mathbf{x})\psi^*(\mathbf{y})=C(\mathbf{x},\mathbf{y}),$$

$$\int_{\mathbb{C}^\Lambda}d\mu_C(\psi^*,\psi)\psi^*(\mathbf{x})\psi^*(\mathbf{y})=0,$$

$$\int_{\mathbb{C}^\Lambda}d\mu_C(\psi^*,\psi)\psi(\mathbf{x})\psi(\mathbf{y})=0.$$

The measure can be written

$$d\mu_C(\psi^*,\psi)=(\det\tilde{J})e^{-\sum_{(\mathbf{x},\mathbf{y})\in\Lambda^2}J(\mathbf{x}-\mathbf{y})\psi^*(\mathbf{x})\psi(\mathbf{y})}D\psi^*D\psi.$$

The moments of this measure can be expressed via the Isserlis-Wick Theorem [40, 67] as follows:

- If $p\neq q$ then

$$\int_{\mathbb{C}^\Lambda}d\mu_C(\psi^*,\psi)\psi(\mathbf{x}_1)\dots\psi(\mathbf{x}_p)\psi^*(\mathbf{y}_1)\dots\psi^*(\mathbf{y}_q)=0.$$

- In the $p=q$ case, one has

$$\begin{aligned}&\int_{\mathbb{C}^\Lambda}d\mu_C(\psi^*,\psi)\psi(\mathbf{x}_1)\dots\psi(\mathbf{x}_p)\psi^*(\mathbf{y}_1)\dots\psi^*(\mathbf{y}_p)\\ &\quad=\sum_{\gamma\in\mathfrak{S}_p}C(\mathbf{x}_{\gamma(1)},\mathbf{y}_1)\dots C(\mathbf{x}_{\gamma(p)},\mathbf{y}_p),\end{aligned}\tag{5}$$

  i.e., one sums over all possible pairwise Wick contractions of the ψ's with the ψ^*'s, as indicated by the permutation γ.

2.2 Feynman Diagrams and Tree Level Analysis

The relation (5) may be expressed in terms of oriented graphs (Feynman diagrams) in which for each $\psi(\mathbf{x})$ we draw an ingoing half edge from the vertex $\mathbf{x}$, for each $\psi^*(\mathbf{y})$ we draw an outgoing half edge from the vertex $\mathbf{y}$, and the expectation is the sum over all the possible oriented graphs obtained contracting only outgoing half edges with ingoing half edges and associating a free propagator $C(\mathbf{x},\mathbf{y})$ to each edge.

In this section we will focus on the near-Gaussian regime, i.e., λ small, and we will analyze the truncated n-point function $\langle \psi^{\sharp_1}(\mathbf{x}_1),\dots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}$. A trivial change of variable $\psi \to e^{i\theta}\psi$ shows that truncated correlation functions vanish unless the number of arguments n is even, and there are equal numbers of ψ's and ψ^*'s involved. We will therefore always assume this to be the case.

An easy but crucial lemma we will later need is the following.

Lemma 1 *For n even, $n \geq 4$, and for all $k < \frac{n}{2} - 1$ we have*

$$\left(\frac{\mathrm{d}}{d\lambda}\right)^k \langle \psi^{\sharp_1}(\mathbf{x}_1),\dots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}\Big|_{\lambda=0} = 0.$$

We assume the reader is familiar with the rigorous formalism of Feynman diagrams used to express formal perturbation theory. A precise mathematical treatment can be found in [1, 58]. A well-known fact from this formalism is that the function $\langle \psi^{\sharp}(\mathbf{x}_1),\dots,\psi^{\sharp}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}$ is C^∞ in λ in the interval $[0,+\infty)$, and that its Taylor series at the origin, seen as a formal power series in λ, is the sum of the contributions of all connected Feynman diagrams with external legs $\mathbf{x}_1,\dots,\mathbf{x}_n$. It is a trivial exercise in organic chemistry to see that the minimal number N of internal 4-valent vertices needed to build such a connected graph is $\frac{n}{2}-1$. This corresponds to tree graphs. The lemma is an immediate consequence of this fact.

The remainder of this section will be devoted to some heuristic considerations which we hope will shed some light on the near-Gaussian regime. It is part of constructive field theory folklore that one should expect the correct bound in the l^1-clustering estimate to be dictated by the contribution of tree graphs. For a given tree graph, the l^1 sum over the sites $\mathbf{x}_2,\dots,\mathbf{x}_n$ in $\mathbb{Z}^d$ can be easily bounded, using a standard pin and sum argument, by Cst^n. This is because the hypotheses on J imply the exponential decay of the free propagator $C(\mathbf{x},\mathbf{y})$, as is recalled in Sect. 2.3. The issue is the number of trees. Because of the constraints due to the orientations of the edges, the counting is not an immediate consequence of Cayley's formula. One can nevertheless obtain an exact formula.

For $n \geq 2$ and even, let $\kappa_{\frac{n}{2}}$ denote the number of Wick contractions γ, with n external legs and $N = \frac{n}{2} - 1$ internal vertices, which produce connecting trees. One can easily check by inspection that $\kappa_1 = 1$, $\kappa_2 = 4$, $\kappa_3 = 288$, and $\kappa_4 = 82944$. In general one has the following result.

Lemma 2 *For any $k \geq 1$ one has*

$$\kappa_k = \frac{2^{k-1}k!(k-1)!(3k-3)!}{(2k-1)!}.$$

Proof We find a quadratic induction formula for the κ_k, namely, for any $k \geq 3$ one has

$$(k-2)\kappa_k = \sum_{i=1}^{k-2}\binom{k-1}{i}\binom{k}{i+1}\binom{k}{k-i}\kappa_{i+1}\kappa_{k-i}.$$

Indeed, the left-hand side counts Wick contraction schemes together with the choice of an internal edge of the corresponding trees. If one cuts that distinguished edge, the tree falls apart into two trees T_1, T_2. The numbering is unambiguous, if one decides that the cut edge goes from T_2 to T_1. Let i, $1 \le i \le k-2$, be the number of internal vertices of T_1. Choosing them accounts for the first binomial coefficient. The second binomial coefficient is the number of ways one can choose the $i+1$ external ψ vertices of T_1 among the initial $k = \frac{n}{2}$. The third coefficient is for the choices of the $k-i$ external ψ^* vertices in T_2. Note that the cut edge introduces an extra distinguished ψ^* leaf for T_1 and an extra ψ leaf for T_2. Now the given formula for κ_k satisfies the quadratic induction because of the identity

$$\sum_{j=0}^{k-1} \frac{(3j)!(3k-3j-3)!}{j!(2j+1)!(2k-2j-1)!(k-j-1)!} = \frac{(3k-2)!}{k!(2k-1)!} \tag{6}$$

which holds for any $k \ge 1$, and which is an easy consequence of [32, (5.62)]. □

Remark 3 One can also prove (6) using Clausen's ${}_4F_3$ hypergeometric summation formula. The κ_k's are related to the Fuss-Catalan numbers of order 3 (see [32, p. 347]).

Now the rough estimate for the l^1-clustering bound which comes from the aforementioned analysis of the tree level contribution, for even $n \ge 2$, is

$$c_n(\lambda) \sim \mathrm{Cst}^n \lambda^{\frac{n}{2}-1} \frac{\kappa_{\frac{n}{2}}}{N!} \sim \mathrm{Cst}^n \lambda^{\frac{n}{2}-1} n!$$

by Stirling's formula and Lemma 2.

It is therefore natural to make the following conjecture.

Conjecture 1 In the massive case $J(\mathbf{0}) > J_{\neq}$, there exists a constant $c(J) > 0$, such that for $\lambda > 0$ small enough, for any even integer $n \ge 2$, for any finite volume $\Lambda \in \mathbb{Z}^d$, and for any assignment of the $\sharp$ symbols, one has

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n \in \Lambda} |\langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}| \le c(J)^n \lambda^{\frac{n}{2}-1} n!.$$

Although we do not see a fundamental reason against this conjecture, due to technical difficulties inherent to the cluster expansion method we follow in this paper, we have been unable to prove so much. Related bounds of the $n!$ type have been obtained in the literature [19, 24, 51], but without extracting the optimal power of λ. Using the method of this article, it is possible to extract this power, but at the cost of a higher power of $n!$. The difficulty is in obtaining optimal bounds which are *uniform* in n. Theorem 4 is a weakening of this conjecture.

2.3 Free Propagator Decay

Picking up the thread from Sect. 2.1, we let $\tilde{J}_{\neq}$ denote the off-diagonal part of $\tilde{J}$, so that $\tilde{J} = J(\mathbf{0})\mathrm{I} + \tilde{J}_{\neq}$. We also assume in this section that $0 < J_{\neq} < J(\mathbf{0})$. The matrix $\tilde{J}_{\neq}$ is also Hermitian and, because of the previous inequalities, has its operator norm bounded by $\|\tilde{J}_{\neq}\| \le J_{\neq} < J(\mathbf{0})$. Thus the Neumann series

$$C = \frac{1}{J(\mathbf{0})} \sum_{p \ge 0} \left(\frac{-1}{J(\mathbf{0})} \tilde{J}_{\neq} \right)^p$$

converges, and provides the following random path representation for the free propagator

$$C(\mathbf{x},\mathbf{y}) = \frac{1}{J(\mathbf{0})}\delta_{\mathbf{x},\mathbf{y}} - \frac{1}{J(\mathbf{0})^2}\mathbb{1}_{\{\mathbf{x}\neq\mathbf{y}\}}J(\mathbf{x}-\mathbf{y}) + \sum_{p\geq 2}\frac{(-1)^p}{J(\mathbf{0})^{p+1}} \times \sum_{\mathbf{z}_1,\ldots,\mathbf{z}_{p-1}} J(\mathbf{x}-\mathbf{z}_1)J(\mathbf{z}_1-\mathbf{z}_2)\cdots J(\mathbf{z}_{p-2}-\mathbf{z}_{p-1})J(\mathbf{z}_{p-1}-\mathbf{y}) \tag{7}$$

where the last sum is over sequences of sites in Λ such that, $\mathbf{z}_1 \neq \mathbf{x}$, $\mathbf{z}_{p-1} \neq \mathbf{y}$, and $\mathbf{z}_i \neq \mathbf{z}_{i-1}$ for $2 \leq i \leq p-1$. We assumed the function $J : \mathbb{Z}^d \to \mathbb{C}$ to be compactly supported. Let therefore $r_0 > 0$ be such that $J(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^d$ satisfying $|\mathbf{x}| \geq r_0$. We now have the following elementary lemma.

Lemma 3 *For any $\Lambda \subset \mathbb{Z}^d$, and for any $\mathbf{x},\mathbf{y} \in \Lambda$ one has the uniform exponential decay bound*

$$|C(\mathbf{x},\mathbf{y})| \leq K_0 e^{-\mu_0|\mathbf{x}-\mathbf{y}|}$$

where

$$K_0 = \frac{J(\mathbf{0})}{J_{\neq}(J(\mathbf{0}) - J_{\neq})} > 0 \quad \textit{and}$$

$$\mu_0 = \frac{1}{r_0}\log\left(\frac{J(\mathbf{0})}{J_{\neq}}\right).$$

Proof Given a nonempty Λ in $\mathbb{Z}^d$, and $\mathbf{x}, \mathbf{y}$ in Λ, let $p_0 = \lfloor \frac{|\mathbf{x}-\mathbf{y}|}{r_0} \rfloor$. First suppose that $p_0 \geq 1$, so that $|\mathbf{x}-\mathbf{y}| \geq r_0$. Then obviously the first two terms on the right-hand side of (7) vanish. Besides, if a term in the last sum over $p \geq 2$ is nonzero, then there exists a sequence of sites $\mathbf{z}_1,\ldots,\mathbf{z}_{p-1}$ such that $|\mathbf{x}-\mathbf{z}_1| < r_0$, $|\mathbf{z}_{p-1}-\mathbf{y}| < r_0$, and $|\mathbf{z}_i - \mathbf{z}_{i-1}| < r_0$ for $2 \leq i \leq p-1$. Thus $|\mathbf{x}-\mathbf{y}| < pr_0$ and $p_0 < p$, i.e., $p \geq p_0 + 1$. Therefore at the level of matrix elements on has

$$C(\mathbf{x},\mathbf{y}) = \left[\frac{1}{J(\mathbf{0})}\sum_{p\geq p_0+1}\left(\frac{-1}{J(\mathbf{0})}\tilde{J}_{\neq}\right)^p\right]_{\mathbf{x},\mathbf{y}}$$

and bounding matrix elements by operator norms, one has

$$|C(\mathbf{x},\mathbf{y})| \leq \frac{1}{J(\mathbf{0})}\left(\frac{J_{\neq}}{J(\mathbf{0})}\right)^{p_0+1} \times \frac{1}{1-\frac{J_{\neq}}{J(\mathbf{0})}}$$

from which the estimate easily follows, since $p_0 + 1 > \frac{|\mathbf{x}-\mathbf{y}|}{r_0}$. Now if $p_0 = 0$, one has $|\mathbf{x}-\mathbf{y}| < r_0$ and therefore using the full Neumann series and the coarse bound by the operator norm

$$|C(\mathbf{x},\mathbf{y})| \leq \|C\| \leq \frac{1}{J(\mathbf{0}) - J_{\neq}} = \frac{1}{J(\mathbf{0}) - J_{\neq}}e^{-\mu_0|\mathbf{x}-\mathbf{y}|} \times e^{\mu_0|\mathbf{x}-\mathbf{y}|}$$

and the last factor is obviously bounded by $e^{\mu_0 r_0} = \frac{J(\mathbf{0})}{J_{\neq}}$, so the lemma follows. □

This exponential decay trivially implies the l^1 property of the free propagator thanks to

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} e^{-\mu|\mathbf{z}|} \le K_1(d,\mu) \tag{8}$$

for any $\mu > 0$ and dimension $d \ge 1$, where $K_1(d,\mu)$ is some constant.

2.4 The Brydges-Kennedy-Abdesselam-Rivasseau Formula

We recall in this section and the next one a list of well known combinatorial identities and inequalities and some basic results of cluster expansions that we will use to establish our bounds.

The BKAR formula [2, 17] is a simpler and more symmetric version of the earlier Brydges-Battle-Federbush tree formula [12, 15, 16] in constructive field theory. This earlier formula itself is an improvement on the pioneering approach of Glimm-Jaffe-Spencer [30, 31].

Let us consider a finite set $E \neq \emptyset$, and let us denote by $E^{(2)}$ the set of unordered pairs $\{a,b\}$, where a and b are any distinct elements in E. Of course $|E^{(2)}| = \binom{|E|}{2}$. We will consider the space $\mathbb{R}^{E^{(2)}}$ of multiplets $s = (s_l)_{l\in E^{(2)}}$ indexed by pairs $l \in E^{(2)}$, and functions defined on a particular compact convex set $\mathcal{K}_E$ in this space. Let Π_E denote the set of partitions of E. For any partition $\pi = \{X_1,\dots X_q\}$ in Π_E we associate a vector $v_\pi = (v_{\pi,l})_{l\in E^{(2)}}$ defined as

$$v_{\pi,l} = \mathbb{1}_{\{\exists i, 1\le i\le q, l\subset X_i\}}.$$

Now $\mathcal{K}_E$ is by definition the convex hull of the vectors v_π, for $\pi \in \Pi_E$. It is easy to see that $\mathcal{K}_E$ affinely generates $\mathbb{R}^{E^{(2)}}$. Indeed, let $\hat{0}$ be the partition entirely made of singletons, and for any pair $l \in E^{(2)}$ let $\hat{l}$ denote the partition made of the two element set l and the singletons $\{a\}$, for $a \in E\backslash l$. Then, the vectors $v_{\hat{l}} - v_{\hat{0}}$, for $l \in E^{(2)}$ form a basis of the vector space $\mathbb{R}^{E^{(2)}}$. As a result, the open domain $\Omega_E = \mathring{\mathcal{K}}_E$ is nonempty, and $\mathcal{K}_E$ is equal to the closure $\bar{\Omega}_E$. Let $C^k(\bar{\Omega}_E)$ denote the usual space of functions of class C^k on the domain Ω_E which, together with their derivatives up to order k, admit uniformly continuous extensions to the closure $\mathcal{K}_E = \bar{\Omega}_E$ (see, e.g., [5]).

Now a simple graph with vertex set E can be thought of as a subset of the complete graph $E^{(2)}$. A forest $\mathfrak{F}$ is a graph with no circuits, and it is made of a vertex-disjoint collection of trees. Let $\mathfrak{F}$ be a forest, and let $\vec{h} = (h_l)_{l\in\mathfrak{F}}$ be a vector of real parameters indexed by the edges l in the forest $\mathfrak{F}$. To such data we canonically associate a multiplet $s(\mathfrak{F},\vec{h}) = (s(\mathfrak{F},\vec{h})_l)_{l\in E^{(2)}}$ in $\mathbb{R}^{E^{(2)}}$ as follows. Let a and b be two distinct elements in E. If a and b belong to two distinct connected components of the forest $\mathfrak{F}$, then $s(\mathfrak{F},\vec{h})_{\{a,b\}} = 0$. Otherwise let, by definition, $s(\mathfrak{F},\vec{h})_{\{a,b\}} = \min_l h_l$ where l belongs to the unique path in the forest $\mathfrak{F}$ joining a to b. We are now ready to state the BKAR formula.

Theorem 6 [2, 17] *Let $f \in C^{|E|-1}(\bar{\Omega}_E)$, and let $1 \in \mathbb{R}^{E^{(2)}}$ denote the multiplet with all entries equal to one. This is also the same as $v_{\hat{1}}$ where $\hat{1}$ is the single block partition $\{E\}$. We then have*

$$f(1) = \sum_{\mathfrak{F}\text{ forest}} \int_{[0,1]^{\mathfrak{F}}} d\vec{h}\, \frac{\partial^{|\mathfrak{F}|} f}{\prod_{l\in\mathfrak{F}} \partial s_l}\big(s(\mathfrak{F},\vec{h})\big)$$

where the sum is over all forests $\mathfrak{F}$ with vertex set E, the notation $d\vec{h}$ is for the Lebesgue measure on the set of parameters $[0,1]^{\mathfrak{F}}$, the partial derivatives of f are with respect to the *entries indexed by the pairs belonging to $\mathfrak{F}$, and the evaluation of these derivatives is at the $\vec{h}$ dependent point $s(\mathfrak{F}, \vec{h})$. Such points belong to $\mathcal{K}_E$.*

Note that the empty forest always occurs and its contribution is $f(0) = f(v_{\hat{0}})$. There are several proofs of this identity [2, 17, 18], but we believe the most natural and most easily generalizable is the one given in [3, Sect. 2]. This proof also most clearly shows the $s(\mathfrak{F}, \vec{h})$ belong to $\mathcal{K}_E$. This point is important for the positivity of the interpolated covariance matrices in Sect. 4.1, and also when proving Lemma 5 via Lemma 4.

We will now recall a lemma which, via the uniqueness of the Möbius inverse in the partition lattice, is a corollary of the BKAR forest formula.

Lemma 4 *Again let us consider a finite set E and let us denote by $E^{(2)}$ the set of unordered pairs $l = \{a, b\}$ in E. Let $V_{\{a,b\}}$ be a collection of complex numbers indexed by $E^{(2)}$. Then*

$$\sum_{\mathsf{g} \rightsquigarrow E} \prod_{l \in \mathsf{g}} (e^{-V_l} - 1) = \sum_{\substack{\mathfrak{T} \rightsquigarrow E \\ \mathfrak{T} \text{ tree}}} \int_{[0,1]^{\mathfrak{T}}} d\vec{h} \left\{ \prod_{l \in \mathfrak{T}} (-V_l) \right\} e^{-\sum_{l \in E^{(2)}} s(\mathfrak{T}, \vec{h})_l V_l}. \tag{9}$$

Here g is summed over all simple graphs (i.e. subsets of $E^{(2)}$) which connect E. We abbreviate this property by the notation $\mathsf{g} \rightsquigarrow E$. On the right-hand side the sum is on spanning trees $\mathfrak{T}$ which connect E. The notation $s(\mathfrak{T}, \vec{h})$ is as in Theorem 6.

The following tree graph inequality, initially due to Brydges, Battle and Federbush (see [12, 15, 54]) is the basic tool we will need for the estimates in the first three regimes related to the 'high temperature' scenario. It is an easy consequence of Lemma 4.

Lemma 5 *Under the same hypotheses as in Lemma 4, let us assume that the numbers V_l satisfy, in addition, the following stability hypothesis: there are nonnegative numbers U_a, for $a \in E$, such that for any subset $S \subset E$ one has*

$$\left| \sum_{l \in S^{(2)}} V_l \right| \leq \sum_{a \in S} U_a.$$

Then the following inequality holds

$$\left| \sum_{\mathsf{g} \rightsquigarrow E} \prod_{l \in \mathsf{g}} (e^{-V_l} - 1) \right| \leq e^{\sum_{a \in E} U_a} \sum_{\substack{\mathfrak{T} \rightsquigarrow E \\ \mathfrak{T} \text{ tree}}} \prod_{l \in \mathfrak{T}} |V_l|.$$

2.5 The Cluster Expansion for the Polymer Gas

We now recall the basics of polymer gas cluster expansions. Any nonempty finite subsets $R \subset \Lambda$ is called a polymer. We denote by $\mathbf{P}(\Lambda)$ the set of all such polymers. We associate to each $R \in \mathbf{P}(\Lambda)$ a variable $\rho(R) \in \mathbb{C}$ called the activity of the polymer R. They can be collected in a vector $\rho = (\rho(R))_{R \in \mathbf{P}(\Lambda)} \in \mathbb{C}^{\mathbf{P}(\Lambda)}$.

On the complex space $\mathbb{C}^{\mathbf{P}(\Lambda)}$ we consider the polynomial function $\mathcal{Z}$ defined by

$$\mathcal{Z}(\rho) = \sum_{p \geq 0} \frac{1}{p!} \sum_{R_1, \ldots, R_p \in \mathbf{P}(\Lambda)} \mathbb{1}\{\text{the } R_i \text{ are disjoint}\} \rho(R_1) \cdots \rho(R_p)$$

for any $\rho \in \mathbb{C}^{\mathbf{P}(\Lambda)}$. This function is usually called the *grand canonical partition function of the polymer gas* at finite volume Λ. It is well known (see e.g. [15, 21]) that the logarithm of $\mathcal{Z}_\Lambda$ can be written in terms of the following series

$$\log \mathcal{Z}(\rho) = \sum_{p \geq 1} \frac{1}{p!} \sum_{R_1, \ldots, R_p \in \mathbf{P}(\Lambda)} \phi^{\mathrm{T}}(R_1, \ldots, R_p) \rho(R_1) \ldots \rho(R_p)$$

with

$$\phi^{\mathrm{T}}(R_1, \ldots, R_p) = \sum_{\substack{H \rightsquigarrow [p] \\ H \subset G}} (-1)^{|H|}$$

where G is the graph with vertex set $[p] = \{1, \ldots, p\}$ and edges corresponding to the pairs $\{i, j\}$, $i \neq j$, such that $R_i \cap R_j \neq \emptyset$. The sum is over all spanning connecting subgraphs H which are identified with their edge sets.

Note that by the so-called Whitney-Tutte-Fortuin-Kasteleyn representation (see, e.g., [64]), one can write the chromatic polynomial of G as

$$P(G, x) = \sum_{H \subset G} (-1)^{|H|} x^{c(H)} \tag{10}$$

where $c(H)$ is the number of connected components of H. For nonnegative integer values of x, the quantity $P(G, x)$ is by definition the number of proper vertex colorings of G with x colors. A good way to see the Ursell function $\phi^{\mathrm{T}}(R_1, \ldots, R_p)$ is as the coefficient of x in the chromatic polynomial $P(G, x)$.

The condition for the convergence of the series above is a well studied subject. It can be expressed in terms of the following norm, depending on a parameter $a > 0$, defined on the space $\mathbb{C}^{\mathbf{P}(\Lambda)}$ of polymer activities

$$\|\rho\|_a = \sup_{\mathbf{x} \in \Lambda} \sum_{R \in \mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{x} \in R\} |\rho(R)|\, e^{a|R|}.$$

The best result on this subject, essentially proven almost four decades ago by Gruber and Kunz [34] but largely forgotten, and then rediscovered very recently by Fernández and Procacci [27] with a new proof, is the following theorem.

Theorem 7 *Let $a > 0$ and let ρ denote an element of $\mathbb{C}^{\mathbf{P}(\Lambda)}$. Then the series*

$$f(\rho) = \sum_{p \geq 1} \frac{1}{p!} \sum_{R_1, \ldots, R_p \in \mathbf{P}(\Lambda)} \phi^{\mathrm{T}}(R_1, \ldots, R_p) \rho(R_1) \cdots \rho(R_p)$$

is absolutely convergent in the closed ball $\|\rho\|_a \leq e^a - 1$. The function f is analytic on the open ball $\|\rho\|_a < e^a - 1$ and satisfies

$$\exp f(\rho) = \mathcal{Z}$$

for any ρ with $\|\rho\|_a \leq e^a - 1$.

In fact, one can extract a more precise result (see [27, p. 132]), when $\rho \geq 0$, i.e., when the polymer activities $\rho(R)$ are real and nonnegative.

Theorem 8 *If $\rho \geq 0$ and $\|\rho\|_a \leq e^a - 1$, then for any $R_0 \in \mathbf{P}(\Lambda)$, we have the estimate*

$$\sum_{p\geq 1} \frac{1}{p!} \sum_{R_1,\dots,R_p \in \mathbf{P}(\Lambda)} |\phi^{\mathrm{T}}(R_0, R_1, \dots, R_p)| \rho(R_1) \cdots \rho(R_p) \leq e^{a|R_0|} - 1.$$

We will use this theorem when R_0 is a singleton, namely, when $R_0 = \{\mathbf{z}\}$ for some $\mathbf{z} \in \Lambda$. In this case one has the identity

$$\phi^{\mathrm{T}}(R_0, R_1, \dots, R_p) = (-r)\phi^{\mathrm{T}}(R_1, \dots, R_p) \tag{11}$$

where r is the number of indices q, $1 \leq q \leq p$, such that $\mathbf{z} \in R_q$. Indeed, if G is the intersection graph on $\{0, 1, \dots, p\}$ defined by the collection of polymers $R_0, R_1, \dots, R_p$, then the restriction of G to the set formed by the vertex 0 and its neighbors is the complete graph on $r + 1$ elements. This property and the representation (10) gives a very easy proof of the reduction formula (11). A trivial consequence of (11) and Theorem 8 is the following lemma which will be used repeatedly in the sequel.

Lemma 6 *For nonnegative polymer activities $\rho(R)$, $R \in \mathbf{P}(\Lambda)$, such that $\|\rho\|_a \leq e^a - 1$ we have the bound*

$$\sup_{\mathbf{z}\in\Lambda} \sum_{p\geq 1} \frac{1}{p!} \sum_{R_1,\dots,R_p \in \mathbf{P}(\Lambda)} \mathbb{1}\left\{\mathbf{z} \in \bigcup_{q=1}^{p} R_q\right\} \\ \times |\phi^{\mathrm{T}}(R_1, \dots, R_p)| \rho(R_1) \cdots \rho(R_p) \leq e^a - 1. \tag{12}$$

Remark 4 The important point to note here is that the bound is uniform in $\Lambda \in \mathbb{Z}^d$.

Since we are not interested in optimal bounds we will choose hereafter $a = \log 2$ and we will denote the norm $\|\rho\|_{\log 2}$ simply by $\|\rho\|$. So in what follows we will use the norm

$$\|\rho\| = \sup_{\mathbf{x}\in\Lambda} \sum_{R\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{x} \in R\} |\rho(R)| 2^{|R|} \tag{13}$$

with the condition ensuring absolute convergence of $f(\rho)$ being

$$\|\rho\| \leq 1. \tag{14}$$

3 The Large Mass, Small Interaction, and Large Self-interaction Regimes

3.1 The Mayer Series Representation for the Truncated Correlations

Given a source specification $(\mathbf{x}_i, \sharp_i)_{i\in I}$, we will consider the perturbed partition function

$$Z_\Lambda(\boldsymbol{\alpha}) = \int_{\mathbb{C}^\Lambda} D\psi^* D\psi\, e^{-H_\Lambda(\psi_\Lambda)} \prod_{i\in I} (1 + \alpha_i \psi^{\sharp_i}(\mathbf{x}_i))$$

and the truncated correlation function given by

$$\langle (\psi^{\sharp_i}(\mathbf{x}_i))_{i\in I} \rangle_\Lambda^{\mathrm{T}} = \frac{\partial^{|I|}}{\prod_{i\in I} \partial\alpha_i} \operatorname{Log} Z_\Lambda(\boldsymbol{\alpha}) \bigg|_{\boldsymbol{\alpha}=\mathbf{0}}.$$

As mentioned before, $\operatorname{Log} Z_\Lambda(\boldsymbol{\alpha})$ is analytic in a small polydisc D_Λ around $\boldsymbol{\alpha} = \mathbf{0}$.

Now let us introduce the normalized single site measure

$$d\nu(z^*,z)=\frac{1}{\mathcal{N}}e^{-J(\mathbf{0})|z|^2-\frac{\lambda}{4}|z|^4}d\Re z\,d\Im z$$

on $\mathbb{C}$, where

$$\mathcal{N}=\int_{\mathbb{C}}e^{-J(\mathbf{0})|z|^2-\frac{\lambda}{4}|z|^4}d\Re z\,d\Im z.$$

Clearly $Z_\Lambda(\boldsymbol{\alpha})=\mathcal{N}^{|\Lambda|}\check{Z}_\Lambda(\boldsymbol{\alpha})$ where

$$\check{Z}_\Lambda(\boldsymbol{\alpha})=\int_{\mathbb{C}^\Lambda}\prod_{\mathbf{x}\in\Lambda}d\nu(\psi^*(\mathbf{x}),\psi(\mathbf{x}))e^{-\sum_{\{\mathbf{x},\mathbf{y}\}\in\Lambda^{(2)}}I_{\{\mathbf{x},\mathbf{y}\}}(\psi)}\prod_{i\in I}(1+\alpha_i\psi^{\sharp_i}(\mathbf{x}_i))$$

with

$$I_{\{\mathbf{x},\mathbf{y}\}}(\psi)=J(\mathbf{x}-\mathbf{y})\psi^*(\mathbf{x})\psi(\mathbf{y})+J(\mathbf{y}-\mathbf{x})\psi^*(\mathbf{y})\psi(\mathbf{x}).$$

Since $Z_\Lambda(\boldsymbol{\alpha})=\mathcal{N}^{|\Lambda|}\check{Z}_\Lambda(\boldsymbol{\alpha})$ with $\mathcal{N}>0$, one has the analyticity in the polydisc D_Λ of $\mathrm{Log}\,\check{Z}_\Lambda(\boldsymbol{\alpha})=-|\Lambda|\ln\mathcal{N}+\mathrm{Log}\,Z_\Lambda(\boldsymbol{\alpha})$ as well as the identity

$$\langle(\psi^{\sharp_i}(\mathbf{x}_i))_{i\in I}\rangle_\Lambda^{\mathrm{T}}=\frac{\partial^{|I|}}{\prod_{i\in I}\partial\alpha_i}\mathrm{Log}\,\check{Z}_\Lambda(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\mathbf{0}}.$$

On the other hand, we can also write

$$\begin{aligned}e^{-\sum_{\{\mathbf{x},\mathbf{y}\}\in\Lambda^{(2)}}I_{\{\mathbf{x},\mathbf{y}\}}(\psi)}&=\prod_{\{\mathbf{x},\mathbf{y}\}\in\Lambda^{(2)}}\left[1+(e^{-I_{\{\mathbf{x},\mathbf{y}\}}(\psi)}-1)\right]\\&=\sum_{\mathsf{g}\subset\Lambda^{(2)}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathsf{g}}\left(e^{-I_{\{\mathbf{x},\mathbf{y}\}}(\psi)}-1\right)\end{aligned}$$

where the sum is over all simple graphs g on the vertex set Λ. Using this equation and also expanding the product of the $(1+\alpha_i\psi^{\sharp_i}(\mathbf{x}_i))$ one easily obtains, after reorganization according to the connected components of the graph g, the following polymer representation for $\check{Z}_\Lambda(\boldsymbol{\alpha})$. Namely, one has

$$\check{Z}_\Lambda(\boldsymbol{\alpha})=\sum_{p\ge0}\frac{1}{p!}\sum_{R_1,\dots,R_p\in\mathbf{P}(\Lambda)}\mathbb{1}\{\text{the }R_i\text{ are disjoint}\}\rho(R_1,\boldsymbol{\alpha})\cdots\rho(R_p,\boldsymbol{\alpha})\tag{15}$$

where the polymer activity of a polymer R is defined as

$$\begin{aligned}\rho(R,\boldsymbol{\alpha})=&\sum_{\mathsf{g}\rightsquigarrow R}\sum_{J\subset I_R}\mathbb{1}\{|R|\ge2\text{ or }J\ne\emptyset\}\int_{\mathbb{C}^R}\prod_{\mathbf{x}\in R}d\nu(\psi^*(\mathbf{x}),\psi(\mathbf{x}))\\&\times\prod_{i\in J}(\alpha_i\psi^{\sharp_i}(\mathbf{x}_i))\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathsf{g}}\left(e^{-I_{\{\mathbf{x},\mathbf{y}\}}(\psi)}-1\right)\end{aligned}\tag{16}$$

with the notation $I_R=\{i\in I|\mathbf{x}_i\in R\}$. Let $\rho(\boldsymbol{\alpha})$ denote the activity specification $(\rho(R,\boldsymbol{\alpha}))_{R\in\mathbf{P}(\Lambda)}$. Using the notation and definitions of Sect. 2.5 one obviously has $\check{Z}_\Lambda(\boldsymbol{\alpha})=\mathcal{Z}(\rho(\boldsymbol{\alpha}))$.

Now suppose the condition $\|\rho(\mathbf{0})\| < 1$ holds. Then on a small polydisc D'_Λ the hypothesis $\|\rho(\boldsymbol{\alpha})\| < 1$ will also hold. On the polydisc $D_\Lambda \cap D'_\Lambda$ both the functions $f(\rho(\boldsymbol{\alpha}))$ and $\operatorname{Log} \check{Z}_\Lambda(\boldsymbol{\alpha})$ are analytic and exponentiate to $\check{Z}_\Lambda(\boldsymbol{\alpha})$. By connectedness the difference $f(\rho(\boldsymbol{\alpha})) - \operatorname{Log} \check{Z}_\Lambda(\boldsymbol{\alpha})$ is a constant in $2i\pi\mathbb{Z}$. However this difference takes a real value at $\boldsymbol{\alpha} = \mathbf{0}$, and therefore it must vanish. Indeed, the hypothesis on the $J(\mathbf{x})$ function implies that $I_{\{\mathbf{x},\mathbf{y}\}}(\psi)$ is real and therefore, by (16), the activities $\rho(R, \mathbf{0})$ are also real. Finally, by the definition in Theorem 7 it follows that $f(\rho(\mathbf{0})) \in \mathbb{R}$. As a result of these considerations, on a small polydisc around $\boldsymbol{\alpha} = \mathbf{0}$ one has the equality

$$f(\rho(\boldsymbol{\alpha})) = \operatorname{Log} \check{Z}_\Lambda(\boldsymbol{\alpha})$$

thus

$$\begin{aligned}
\langle(\psi^{\sharp_i}(\mathbf{x}_i))_{i\in I}\rangle_\Lambda^{\mathrm{T}} &= \frac{\partial^{|I|}}{\prod_{i\in I}\partial\alpha_i} f(\rho(\boldsymbol{\alpha}))\bigg|_{\boldsymbol{\alpha}=\mathbf{0}} \\
&= \sum_{p\geq 0}\frac{1}{p!}\sum_{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)} \phi^{\mathrm{T}}(R_1,\ldots,R_p)\frac{\partial^{|I|}}{\prod_{i\in I}\partial\alpha_i}\rho(R_1,\boldsymbol{\alpha})\ldots\rho(R_p,\boldsymbol{\alpha})\bigg|_{\boldsymbol{\alpha}=\mathbf{0}} \\
&= \sum_{p\geq 0}\frac{1}{p!}\sum_{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)} \phi^{\mathrm{T}}(R_1,\ldots,R_p) \\
&\quad\times\sum_{I_1,\ldots,I_p\subset I}\mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q = I\right\}\prod_{q=1}^{p}\left[\frac{\partial^{|I_q|}}{\prod_{i\in I_q}\partial\alpha_i}\rho(R_q,\boldsymbol{\alpha})\bigg|_{\boldsymbol{\alpha}=\mathbf{0}}\right].
\end{aligned}$$

Comparing with (16) it is apparent that the effect of the α derivatives is to force the summed over J to be equal to I_q. Hence, as an immediate consequence of Theorem 7, we have the following statement.

Proposition 1 *Define for any subset J of the source label set I, and any polymer $R \in \mathbf{P}(\Lambda)$,*

$$\begin{aligned}
\tilde{\rho}(R, J) &= \mathbb{1}\{\forall i \in J, \mathbf{x}_i \in R\}\mathbb{1}\{|R| \geq 2, \text{ or } J \neq \emptyset\} \\
&\quad\times\sum_{\mathsf{g}\rightsquigarrow R}\int_{\mathbb{C}^R}\prod_{\mathbf{x}\in R} d\nu(\psi^*(\mathbf{x}),\psi(\mathbf{x}))\prod_{i\in J}\psi^{\sharp_i}(\mathbf{x}_i)\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathsf{g}}\left(e^{-I_{\{\mathbf{x},\mathbf{y}\}}(\psi)} - 1\right).
\end{aligned}$$

Provided the source-free condition

$$\sup_{\mathbf{x}\in\Lambda}\sum_{R\in\mathbf{P}(\Lambda)}\mathbb{1}\{\mathbf{x}\in R\}\,|\tilde{\rho}(R,\emptyset)|\,2^{|R|} < 1$$

holds, one has the absolutely convergent series representation for all truncated correlation functions

$$\begin{aligned}
&\langle(\psi^{\sharp_i}(\mathbf{x}_i))_{i\in I}\rangle_{\Lambda,\lambda}^{\mathrm{T}} \\
&\quad= \sum_{p\geq 1}\frac{1}{p!}\sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)\\ I_1,\ldots,I_p\subset I}}\phi^{\mathrm{T}}(R_1,\ldots,R_p)\mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q = I\right\}\prod_{q=1}^{p}\tilde{\rho}(R_q, I_q).
\end{aligned}$$

Note that with this new definition, polymer activities $\tilde{\rho}(R, J)$ do not have to contain all the sources localized in sites belonging to R.

3.2 Estimates for a Single Polymer Activity

Consider a polymer activity $\tilde{\rho}(R, J)$ as defined in Proposition 1. By moving the sum over connecting graphs inside the integral, one has the estimate

$$|\tilde{\rho}(R, J)| \le \mathbb{1}\{\forall i \in J, \mathbf{x}_i \in R\}\mathbb{1}\{|R| \ge 2, \text{ or } J \ne \emptyset\} \int_{\mathbb{C}^R} \prod_{\mathbf{x}\in R} d\nu(\psi^*(\mathbf{x}), \psi(\mathbf{x}))$$
$$\times \left(\prod_{i\in J} |\psi^{\sharp_i}(\mathbf{x}_i)|\right) \left| \sum_{\mathsf{g}\rightsquigarrow R} \prod_{\{\mathbf{x},\mathbf{y}\}\in \mathsf{g}} \left(e^{-I_{\{\mathbf{x},\mathbf{y}\}}(\psi)} - 1\right)\right|.$$

Now for any subset $S \subset R$ and for any fixed field ψ, by the argument given in Sect. 2.1, one has

$$\left| \sum_{\{\mathbf{x},\mathbf{y}\}\in S^{(2)}} I_{\{\mathbf{x},\mathbf{y}\}}(\psi)\right| \le J_{\ne} \sum_{\mathbf{x}\in S} |\psi(\mathbf{x})|^2.$$

As a result, Lemma 5 implies, for any given field configuration ψ, the inequality

$$\left| \sum_{\mathsf{g}\rightsquigarrow R} \prod_{\{\mathbf{x},\mathbf{y}\}\in \mathsf{g}} \left(e^{-I_{\{\mathbf{x},\mathbf{y}\}}(\psi)} - 1\right)\right| \le e^{J_{\ne}\sum_{\mathbf{x}\in R}|\psi(\mathbf{x})|^2} \sum_{\substack{\mathfrak{T}\rightsquigarrow R \\ \mathfrak{T}\text{ tree}}} \prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} (2|J(\mathbf{x}-\mathbf{y})||\psi(\mathbf{x})||\psi(\mathbf{y})|).$$

Therefore

$$|\tilde{\rho}(R, J)| \le \mathbb{1}\{\forall i \in J, \mathbf{x}_i \in R\}\mathbb{1}\{|R| \ge 2, \text{ or } J \ne \emptyset\} 2^{|R|-1} \sum_{\substack{\mathfrak{T}\rightsquigarrow R \\ \mathfrak{T}\text{ tree}}} \prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |J(\mathbf{x}-\mathbf{y})|$$
$$\times \prod_{\mathbf{x}\in R} \int_{\mathbb{C}} d\nu(\psi^*(\mathbf{x}), \psi(\mathbf{x})) e^{J_{\ne}|\psi(\mathbf{x})|^2} |\psi(\mathbf{x})|^{m(\mathbf{x})} \tag{17}$$

with $m(\mathbf{x}) = c_J(\mathbf{x}) + d_{\mathfrak{T}}(\mathbf{x})$ where $c_J(\mathbf{x})$ is number of source labels $i \in J$ such that $\mathbf{x}_i = \mathbf{x}$, while $d_{\mathfrak{T}}(\mathbf{x})$ is the degree of the vertex $\mathbf{x}$ in the tree $\mathfrak{T}$. The integrals in (17) are now estimated thanks to the following lemma.

Lemma 7 *For any integer $m \ge 0$, we have*

$$\int_{\mathbb{C}} d\nu(z^*, z) e^{J_{\ne}|z|^2} |z|^m \le 2\left(J(\mathbf{0}) + \sqrt{\frac{\lambda}{2}}\right) (J(\mathbf{0}) - J_{\ne})^{-\frac{m+2}{2}} m!^{\frac{1}{2}} \tag{18}$$

when $J(\mathbf{0}) > J_{\ne}$, as well as

$$\int_{\mathbb{C}} d\nu(z^*, z) e^{J_{\ne}|z|^2} |z|^m \le 4\left(J(\mathbf{0}) + \sqrt{\frac{\lambda}{2}}\right) \left(\frac{\lambda}{4}\right)^{-\frac{m+2}{4}} m!^{\frac{1}{4}}. \tag{19}$$

Proof We have

$$\int_{\mathbb{C}} d\nu(z^*, z) e^{J_{\ne}|z|^2} |z|^m = \frac{1}{\mathcal{N}} \int_{\mathbb{C}} e^{-(J(\mathbf{0})-J_{\ne})|z|^2 - \frac{\lambda}{4}|z|^4} |z|^m \, d\Re z \, d\Im z.$$

In order to obtain a lower bound on the denominator $\mathcal{N}$, we write

$$\begin{aligned}
\mathcal{N} &= \int_{\mathbb{C}} e^{-J(\mathbf{0})|z|^2-\frac{\lambda}{4}|z|^4} d\Re z\, d\Im z = 2\pi \int_0^\infty e^{-J(\mathbf{0})r^2-\frac{\lambda}{4}r^4} r\,dr \\
&= \pi \int_0^\infty e^{-\frac{\lambda}{4}t^2-J(\mathbf{0})t} dt \\
&= \pi e^{\frac{J(\mathbf{0})^2}{\lambda}} \int_0^\infty e^{-\frac{1}{2}(\sqrt{\frac{\lambda}{2}}t+J(\mathbf{0})\sqrt{\frac{2}{\lambda}})^2} dt = \pi\sqrt{\frac{2}{\lambda}} e^{\frac{J(\mathbf{0})^2}{\lambda}} \int_{J(\mathbf{0})\sqrt{\frac{2}{\lambda}}}^\infty e^{-\frac{q^2}{2}} dq.
\end{aligned}$$

Now recall Birnbaum's inequality [14] for Mill's ratio, i.e., essentially the erfc function:

$$e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{q^2}{2}} dq \geq \frac{\sqrt{4+x^2}-x}{2}$$

for $x \geq 0$. One can simplify this to

$$e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{q^2}{2}} dq \geq \frac{2}{x+\sqrt{4+x^2}} \geq \frac{2}{x+\sqrt{4+4x+x^2}} = \frac{1}{x+1}.$$

The latter applied to $x = J(\mathbf{0})\sqrt{\frac{2}{\lambda}}$ readily provides the needed bound

$$\mathcal{N} \geq \frac{\pi}{J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}}. \tag{20}$$

Now for the upper bound on the numerator, we have two possibilities corresponding to the two estimates in the lemma.

1st Option—the Bound Using the Gaussian Part We write

$$\begin{aligned}
&\int_{\mathbb{C}} e^{-(J(\mathbf{0})-J_{\neq})|z|^2-\frac{\lambda}{4}|z|^4} |z|^m\, d\Re z\, d\Im z \\
&\quad \leq \int_{\mathbb{C}} e^{-(J(\mathbf{0})-J_{\neq})|z|^2} |z|^m\, d\Re z\, d\Im z \\
&\quad = 2\pi \int_0^\infty e^{-(J(\mathbf{0})-J_{\neq})r^2} r^{m+2} \frac{dr}{r} = 2\pi (J(\mathbf{0})-J_{\neq})^{-\frac{m+2}{2}} \Gamma\left(\frac{m+2}{2}\right).
\end{aligned}$$

We therefore only need to show the elementary inequality $\frac{1}{\sqrt{m!}}\Gamma(\frac{m+2}{2}) \leq 1$ in order to complete the proof of (18). This can be done easily by induction using the well-known properties of Euler's gamma function (see e.g. [6]). The quantity $\frac{1}{\sqrt{m!}}\Gamma(\frac{m+2}{2})$ is equal to 1 and $\frac{\sqrt{\pi}}{2} < 1$ for $m = 0$, and $m = 1$ respectively. Besides, when going from m to $m+2$, this quantity changes by a factor of

$$\frac{1}{\sqrt{(m+1)(m+2)}} \cdot \frac{m+2}{2} = \frac{1}{2}\sqrt{1+\frac{1}{m+1}} < 1$$

and the desired inequality propagates.

2nd Option—the Bound Using the Quartic Self-interaction We write

$$\int_{\mathbb{C}} e^{-(J(\mathbf{0})-J_{\neq})|z|^2-\frac{\lambda}{4}|z|^4}\,|z|^m d\Re z\, d\Im z \le \int_{\mathbb{C}} e^{-\frac{\lambda}{4}|z|^4}|z|^m d\Re z\, d\Im z$$
$$= 2\pi\int_0^\infty e^{-\frac{\lambda}{4}r^4} r^{m+2}\frac{dr}{r} = 2\pi\left(\frac{\lambda}{4}\right)^{-\frac{m+2}{4}}\Gamma\left(\frac{m+2}{4}\right). \tag{21}$$

We are left with showing $\Gamma(\frac{m+2}{4}) \le 2m!^{\frac{1}{4}}$ for any nonnegative integer m. By Dirichlet's multidimensional extension of the formula for the beta integral [6, Theorem 1.8.6], we have

$$\Gamma\left(\frac{m+2}{4}\right)^4 = (m+1)!\int_{t_i>0,\sum t_i=1}(t_1t_2t_3t_4)^{\frac{m-2}{4}}dt_1dt_2dt_3dt_4$$
$$\le (m+1)!\int_{t_i>0,\sum t_i=1}\left(\frac{1}{4^4}\right)^{\frac{m-2}{4}}dt_1dt_2dt_3dt_4$$

by the arithmetic vs. geometric mean inequality. Thus

$$\Gamma\left(\frac{m+2}{4}\right)^4 \le (m+1)!\times\frac{1}{3!}\times 4^{-m+2}\le 2^4 m!$$

for $m \ge 0$. Indeed,

$$4^m = (1+3)^m \ge 1+3m \ge \frac{(m+1)}{6}.$$ □

From the raw estimate (17), the previous lemma, the inequality

$$m(\mathbf{x})! \le 2^{c_J(\mathbf{x})+d_{\mathfrak{T}}(\mathbf{x})}c_J(\mathbf{x})!d_{\mathfrak{T}}(\mathbf{x})!$$

and the trivial relations

$$\sum_{\mathbf{x}\in R}c_J(\mathbf{x}) = |J|, \qquad \sum_{\mathbf{x}\in R}d_{\mathfrak{T}}(\mathbf{x}) = 2|R|-2$$

one easily derives the following basic bounds on single polymer activities.

Lemma 8 (Gaussian estimate) *When $J(\mathbf{0}) > J_{\neq}$, one has the bound*

$$|\tilde{\rho}(R,J)| \le \mathbb{1}\{\forall i\in J, \mathbf{x}_i\in R\}\mathbb{1}\{|R|\ge 2,\ \text{or}\ |J|\ge 1\}2^{3|R|+\frac{|J|}{2}-2}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R|}$$
$$\times (J(\mathbf{0})-J_{\neq})^{-2|R|-\frac{|J|}{2}+1}\prod_{\mathbf{x}\in R}c_J(\mathbf{x})!^{\frac{1}{2}}\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\ \text{tree}}}\prod_{\mathbf{x}\in R}d_{\mathfrak{T}}(\mathbf{x})!^{\frac{1}{2}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|J(\mathbf{x}-\mathbf{y})|.$$

Lemma 9 (Quartic estimate (a.k.a. domination)) *One has the bound*

$$|\tilde{\rho}(R,J)| \le \mathbb{1}\{\forall i\in J, \mathbf{x}_i\in R\}\mathbb{1}\{|R|\ge 2,\ \text{or}\ |J|\ge 1\}2^{\frac{11|R|}{2}+\frac{3|J|}{4}-\frac{5}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R|}$$
$$\times \lambda^{-|R|-\frac{|J|}{4}+\frac{1}{2}}\prod_{\mathbf{x}\in R}c_J(\mathbf{x})!^{\frac{1}{4}}\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\ \text{tree}}}\prod_{\mathbf{x}\in R}d_{\mathfrak{T}}(\mathbf{x})!^{\frac{1}{4}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|J(\mathbf{x}-\mathbf{y})|.$$

3.3 The Large Mass and Small Interaction Regimes

In this section we will use the estimate of Lemma 8 which allows us to treat at once both the cases when $J(\mathbf{0})$ is large or when $J_{\neq}$ is small. Our first task is to check the source-free (i.e. $J=\emptyset$) condition for the applicability of Proposition 1. Let $\mathbf{z}$ be a site in Λ. We need to bound

$$\begin{aligned} A &= \sum_{R\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z}\in R\}|\tilde{\rho}(R,\emptyset)|2^{|R|} \\ &\le \sum_{R\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z}\in R\}\mathbb{1}\{|R|\ge 2\}2^{4|R|-2}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R|} \\ &\quad \times (J(\mathbf{0})-J_{\neq})^{-2|R|+1}\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\text{ tree}}}\prod_{\mathbf{x}\in R} d_{\mathfrak{T}}(\mathbf{x})!^{\frac{1}{2}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|J(\mathbf{x}-\mathbf{y})|. \end{aligned}$$

We now condition this sum on the size $m=|R|\ge 2$ of the polymer, and introduce a sum over labellings of the sites in R by the fixed set of indices $[m]$. Namely we introduce in the sum the identity

$$1=\frac{1}{(m-1)!}\sum_{\mathbf{x}_1,\dots,\mathbf{x}_m\in\Lambda}\mathbb{1}\{\mathbf{x}_1=\mathbf{z}\}\mathbb{1}\{\mathbf{x}_i\text{ distinct}\}\mathbb{1}\{R=\{\mathbf{x}_1,\dots,\mathbf{x}_m\}\}. \tag{22}$$

The next step is to use this artifice to transport the summation over trees $\mathfrak{T}$ on the variable set R to a sum over trees $\mathfrak{t}$ on the fixed set $[m]$. Then, partly releasing the second condition in (22), and eliminating R we obtain

$$\begin{aligned} A &\le \sum_{m\ge 2}\frac{1}{(m-1)!}\sum_{\mathbf{x}_1,\dots,\mathbf{x}_m\in\Lambda}\mathbb{1}\{\mathbf{x}_1=\mathbf{z}\}2^{4m-2}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{m} \\ &\quad \times (J(\mathbf{0})-J_{\neq})^{-2m+1}\sum_{\substack{\mathfrak{t}\rightsquigarrow[m]\\ \mathfrak{t}\text{ tree}}}\prod_{i\in[m]} d_{\mathfrak{t}}(i)!^{\frac{1}{2}}\prod_{\{i,j\}\in\mathfrak{t}}\left(\mathbb{1}\{\mathbf{x}_i\neq\mathbf{x}_j\}|J(\mathbf{x}_i-\mathbf{x}_j)|\right) \end{aligned}$$

with the obvious notation $d_{\mathfrak{t}}(i)$ for the degree of $i\in[m]$ in the tree $\mathfrak{t}$. Now the sum over the locations $\mathbf{x}_i$, starting with the leafs and then progressing towards the root $1\in[m]$, is easy and gives a factor $J_{\neq}^{m-1}$. The sum over the tree is done using the following lemma whose proof is given further below.

Lemma 10 *We have the bound*

$$\sum_{\substack{\mathfrak{t}\rightsquigarrow[m]\\ \mathfrak{t}\text{ tree}}}\prod_{i\in[m]} d_{\mathfrak{t}}(i)!\le 2^{3m-3}(m-2)!.$$

Thanks to the coarse bound $d_{\mathfrak{t}}(i)!^{\frac{1}{2}}\le d_{\mathfrak{t}}(i)!$ and the last lemma, we now have

$$A\le\sum_{m\ge 2}2^{7m-5}J_{\neq}^{m-1}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{m}(J(\mathbf{0})-J_{\neq})^{-2m+1}.$$

Therefore, as soon as the condition

$$\frac{2^7 J_{\neq}.(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}})}{(J(\mathbf{0})-J_{\neq})^2}\le\frac{1}{2} \tag{23}$$

holds, one will have

$$\|\rho(\cdot,\emptyset)\| \le A \le \frac{2^{10} J_{\neq}.\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^2}{(J(\mathbf{0})-J_{\neq})^3}. \tag{24}$$

Clearly, if we take either one of the limits $J(\mathbf{0}) \to \infty$ or $J_{\neq} \to 0$, the condition (23) will hold and $\|\rho(\cdot,\emptyset)\|$ can be made arbitrarily small. Note that crucial to this last fact is $m \ge 2$, i.e., the absence of single-site polymers, also called monomers.

Proof of Lemma 10 We have by Cayley's Theorem (see, e.g, [65, Theorem 5.3.4]) which counts labelled spanning trees with fixed vertex degrees

$$\begin{aligned}
\sum_{\substack{\mathfrak{t}\leadsto[m]\\ \mathfrak{t}\ \text{tree}}} \prod_{i=1}^{m} d_{\mathfrak{t}}(i)! &= \sum_{\substack{d_1,\ldots,d_m\ge 1\\ \Sigma d_i=2m-2}} \sum_{\substack{\mathfrak{t}\leadsto[m]\\ \mathfrak{t}\ \text{tree}}} \mathbb{1}\{\forall i,\ \text{degree of } i \text{ in } \mathfrak{t} \text{ is } d_i\} \prod_{i=1}^{m} d_{\mathfrak{t}}(i)! \\
&= \sum_{\substack{d_1,\ldots,d_m\ge 1\\ \Sigma d_i=2m-2}} \frac{(m-2)!}{\prod_{i=1}^{m}(d_i-1)!} \prod_{i=1}^{m} d_i! = (m-2)! \sum_{\substack{d_1,\ldots,d_m\ge 1\\ \Sigma d_i=2m-2}} d_1\cdots d_m.
\end{aligned}$$

Using the arithmetic versus geometric mean inequality

$$d_1\cdots d_m \le \left[\frac{2m-2}{m}\right]^m \le 2^m$$

as well as

$$\sum_{\substack{d_1,\ldots,d_m\ge 1\\ \Sigma d_i=2m-2}} 1 = \binom{2m-3}{m-1} \le 2^{2m-3}$$

the lemma follows. □

We are now in a position to tackle the l^1-clustering estimate (4), where the source label set is $I=[n]$. Assuming either $J(\mathbf{0})$ is large enough or $J_{\neq}$ is small enough to guarantee conditions (23) and $\|\rho(\cdot,\emptyset)\|<1$ hold, we can use Proposition 1 as well as Lemma 8 to write

$$\begin{aligned}
&\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n\in\Lambda} |\langle \psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle_{\Lambda}^{\mathrm{T}}| \\
&\le \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_n\in\Lambda} \mathbb{1}\{\mathbf{x}_1=\mathbf{0}\} \sum_{p\ge 1}\frac{1}{p!} \sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)\\ I_1,\ldots,I_p\subset[n]}} |\phi^T(R_1,\ldots,R_p)| \\
&\quad\times \mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q=[n]\right\} \prod_{q=1}^{p}\Bigg[\mathbb{1}\{\forall i\in I_q, \mathbf{x}_i\in R_q\}\mathbb{1}\{|R_q|\ge 2, \text{ or } |I_q|\ge 1\} \\
&\quad\times 2^{3|R_q|+\frac{|I_q|}{2}-2}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R_q|} (J(\mathbf{0})-J_{\neq})^{-2|R_q|-\frac{|I_q|}{2}+1}
\end{aligned}$$

$$\times \prod_{\mathbf{x}\in R_q} c_{I_q}(\mathbf{x})!^{\frac{1}{2}} \sum_{\substack{\mathfrak{T}_q \rightsquigarrow R_q \\ \mathfrak{T}_q \text{ tree}}} \prod_{\mathbf{x}\in R_q} d_{\mathfrak{T}_q}(\mathbf{x})!^{\frac{1}{2}} \prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}_q} |J(\mathbf{x}-\mathbf{y})| \Bigg]. \tag{25}$$

The first step is to push $\mathbb{1}\{\mathbf{x}_1=\mathbf{0}\}$ through the sums over p, the R_q's and the I_q's. We then bound it by the coarser condition $\mathbb{1}\{\mathbf{0}\in\bigcup_{q=1}^p R_q\}$. The second step is to push the sums over the $\mathbf{x}_i$'s inside the appropriate (i.e., as dictated by the choice of the I_q's) bracket factor. The sums over the source localizations $\mathbf{x}_i$ are therefore bounded with the knowledge of which polymer they belong to. This rests on the following lemma.

Lemma 11 *Using the notation $c_{\mathbf{y}}(\mathbf{x})=|\{j, 1\le j\le k|\mathbf{y}_j=\mathbf{x}\}|$, for any polymer R, and for any power β, we have*

$$\sum_{\mathbf{y}_1,\ldots,\mathbf{y}_k\in R}\prod_{\mathbf{x}\in R} c_{\mathbf{y}}(\mathbf{x})!^{\beta} \le 2^{|R|+k-1}\cdot k!^{\max(\beta,1)}.$$

Proof Summing first over multiindices $c=c(\mathbf{x})_{\mathbf{x}\in R}$ and then over the sequences $(\mathbf{y}_1,\ldots,\mathbf{y}_k)$ for which $c_{\mathbf{y}}=c$, we have

$$\sum_{\mathbf{y}_1,\ldots,\mathbf{y}_k\in R}\prod_{\mathbf{x}\in R} c_{\mathbf{y}}(\mathbf{x})!^{\beta} = k!\sum_{c,|c|=k}\prod_{\mathbf{x}\in R} c(\mathbf{x})!^{\beta-1}$$

where we denoted by $|c|$ the length $\sum_{\mathbf{x}\in R} c(\mathbf{x})$ of the multiindex c. We then bound the product by $k!^{\beta-1}$ if $\beta\ge 1$ and by 1 otherwise. We also use the trivial bound

$$\sum_{c,|c|=k} 1 = \binom{|R|-1+k}{|R|-1} \le 2^{|R|+k-1}$$

and the result follows. $\square$

As a result of this lemma we have for every q, $1\le q\le p$,

$$\sum_{(\mathbf{x}_i)_{i\in I_q}\in\Lambda^{I_q}} \mathbb{1}\{\forall i\in I_q, \mathbf{x}_i\in R_q\}\prod_{\mathbf{x}\in R_q} c_{I_q}(\mathbf{x})!^{\frac{1}{2}} \le 2^{|R_q|+|I_q|-1}\times|I_q|!. \tag{26}$$

Calling $\mathfrak{W}$ the sum to be estimated on the left-hand side of (25), and using the relation $\sum_{q=1}^p |I_q|=n$ in order to pull out some factors from the sums, we now have

$$\begin{aligned}
\mathfrak{W} \le{}& 2^{\frac{3n}{2}}(J(\mathbf{0})-J_{\neq})^{-\frac{n}{2}}\sum_{p\ge 1}\frac{1}{p!}\sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)\\ I_1,\ldots,I_p\subset[n]}} \mathbb{1}\left\{\mathbf{0}\in\bigcup_{q=1}^p R_q\right\} \\
&\times|\phi^{\mathrm{T}}(R_1,\ldots,R_p)|\mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q=[n]\right\}\times|I_1|!\times\cdots\times|I_p|! \\
&\times\prod_{q=1}^p\left[\mathbb{1}\{|R_q|\ge 2, \text{ or } |I_q|\ge 1\}2^{4|R_q|-3}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R_q|}\right. \\
&\left.\times(J(\mathbf{0})-J_{\neq})^{-2|R_q|+1}\sum_{\substack{\mathfrak{T}_q\rightsquigarrow R_q\\ \mathfrak{T}_q \text{ tree}}}\prod_{\mathbf{x}\in R_q} d_{\mathfrak{T}_q}(\mathbf{x})!^{\frac{1}{2}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}_q}|J(\mathbf{x}-\mathbf{y})|\right].
\end{aligned}$$

We now insert the decomposition

$$\mathbb{1}\{|R_q| \geq 2, \text{ or } |I_q| \geq 1\} = \mathbb{1}\{|R_q| \geq 2\} + \mathbb{1}\{|R_q| = 1 \text{ and } |I_q| \geq 1\} \tag{27}$$

so each bracket factor takes the form $A_q + B_q$ where the nonnegative numbers A_q and B_q correspond to the first and second conditions of (27) respectively. Note that since the I_q's form a disjoint decomposition with possibly empty subsets of the set $[n]$, the number of q's for which $B_q \neq 0$ is bounded by n. Hence, for any number γ such that $0 < \gamma \leq 1$, we can write

$$\begin{aligned}
\prod_{q=1}^{p}(A_q + B_q) &= \prod_{\substack{q=1 \\ B_q \neq 0}}^{p} (A_q + B_q) \prod_{\substack{q=1 \\ B_q = 0}}^{p} A_q \\
&\leq \prod_{\substack{q=1 \\ B_q \neq 0}}^{p} (\gamma^{-1} A_q + B_q) \prod_{\substack{q=1 \\ B_q = 0}}^{p} A_q \\
&\leq \gamma^{-|\{q, B_q \neq 0\}|} \prod_{\substack{q=1 \\ B_q \neq 0}}^{p} (A_q + \gamma B_q) \prod_{\substack{q=1 \\ B_q = 0}}^{p} A_q \\
&\leq \gamma^{-n} \prod_{q=1}^{p}(A_q + \gamma B_q) \\
&\leq \gamma^{-n} \prod_{q=1}^{p}(A_q + \gamma \tilde{B}_q)
\end{aligned}$$

where $\tilde{B}_q$ means we now forget about the condition $|I_q| \geq 1$. Leaving the appropriate choice of γ for later, we now have

$$\begin{aligned}
\mathfrak{W} \leq \gamma^{-n} 2^{\frac{3n}{2}} (J(\mathbf{0}) - J_{\neq})^{-\frac{n}{2}} \sum_{p \geq 1} \frac{1}{p!} & \sum_{\substack{R_1, \ldots, R_p \in \mathbf{P}(\Lambda) \\ I_1, \ldots, I_p \subset [n]}} \mathbb{1}\left\{ \mathbf{0} \in \bigcup_{q=1}^{p} R_q \right\} \\
& \times |\phi^{\mathrm{T}}(R_1, \ldots, R_p)| \mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q = [n] \right\} \times |I_1|! \times \cdots \times |I_p|! \\
& \times \prod_{q=1}^{p} \Bigg[\mathbb{1}\{|R_q| \geq 2\} 2^{4|R_q|-3} \left(J(\mathbf{0}) + \sqrt{\frac{\lambda}{2}} \right)^{|R_q|} \\
& \times (J(\mathbf{0}) - J_{\neq})^{-2|R_q|+1} \sum_{\substack{\mathfrak{T}_q \rightsquigarrow R_q \\ \mathfrak{T}_q \text{ tree}}} \prod_{\mathbf{x} \in R_q} d_{\mathfrak{T}_q}(\mathbf{x})!^{\frac{1}{2}} \prod_{\{\mathbf{x}, \mathbf{y}\} \in \mathfrak{T}_q} |J(\mathbf{x} - \mathbf{y})| \\
& + \gamma \mathbb{1}\{|R_q| = 1\} 2 \left(J(\mathbf{0}) + \sqrt{\frac{\lambda}{2}} \right) (J(\mathbf{0}) - J_{\neq})^{-1} \Bigg].
\end{aligned}$$

We now perform the sum over the I_q's

$$\sum_{I_1,\ldots,I_p\subset[n]} \mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q=[n]\right\} \times |I_1|!\times\cdots\times|I_p|!$$
$$= \sum_{\substack{k_1,\ldots,k_p\geq 0\\ k_1+\cdots+k_p=n}} \binom{n}{k_1\cdots k_p}\cdot k_1!\cdots k_p! = n!\binom{n+p-1}{p-1} \leq n!2^{n+p-1}. \tag{28}$$

Thus

$$\mathfrak{W} \leq n!\gamma^{-n}2^{\frac{5n}{2}-1}(J(\mathbf{0})-J_{\neq})^{-\frac{n}{2}}\sum_{p\geq 1}\frac{1}{p!}$$
$$\times \sum_{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)} \mathbb{1}\left\{\mathbf{0}\in\bigcup_{q=1}^{p}R_q\right\}|\phi^{\mathrm{T}}(R_1,\ldots,R_p)|\varrho(R_1)\ldots\varrho(R_p) \tag{29}$$

where the nonnegative polymer activities $\varrho(\cdot)$ are defined by

$$\varrho(R) = 4\gamma\mathbb{1}\{|R|=1\}\frac{J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}}{J(\mathbf{0})-J_{\neq}}$$
$$+\mathbb{1}\{|R|\geq 2\}2^{4|R|-2}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R|}(J(\mathbf{0})-J_{\neq})^{-2|R|+1}$$
$$\times\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\text{ tree}}}\prod_{\mathbf{x}\in R}d_{\mathfrak{T}}(\mathbf{x})!^{\frac{1}{2}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|J(\mathbf{x}-\mathbf{y})|.$$

The norm $\|\varrho\|$ is estimated in the same manner as we did for $\|\rho(\cdot,\emptyset)\|$ at the beginning of this section. Namely, we find

$$\|\varrho\|\leq 8\gamma\frac{J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}}{J(\mathbf{0})-J_{\neq}}+\sum_{m\geq 2}2^{8m-5}J_{\neq}^{m-1}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{m}(J(\mathbf{0})-J_{\neq})^{-2m+1}.$$

By taking $J(\mathbf{0})$ large or $J_{\neq}$ small we will ensure that

$$\frac{2^8J_{\neq}.(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}})}{(J(\mathbf{0})-J_{\neq})^2}\leq\frac{1}{2}.$$

and also, summing the geometric series, that

$$\|\varrho\|\leq 8\gamma\times\frac{J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}}{J(\mathbf{0})-J_{\neq}}+\frac{2^{12}J_{\neq}.(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}})^2}{(J(\mathbf{0})-J_{\neq})^3} \tag{30}$$

holds.

1st Case. When $J(\mathbf{0})$ becomes large, the right-hand side of (30) approaches 8γ. So we simply choose $\gamma=\frac{1}{16}<1$ and Lemma 6 together with the previous estimate (29) complete the proof of Theorem 1.

2nd Case. When $J_{\neq}$ becomes small, the right-hand side of (30) approaches $8\gamma(1+\frac{1}{J(\mathbf{0})}\sqrt{\frac{\lambda}{2}})$. Therefore, we choose

$$\gamma = \frac{1}{16(1+\frac{1}{J(\mathbf{0})}\sqrt{\frac{\lambda}{2}})} < 1$$

and now Theorem 2 follows.

3.4 The Large Self-interaction Regime

This essentially is an encore presentation of Sect. 3.3. The difference is that we now use the estimate of Lemma 9. Following the same line of argument as in Sect. 3.3, we therefore successively have

$$\begin{aligned}
A &= \sum_{R\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z}\in R\}\, |\tilde{\rho}(R,\emptyset)|\, 2^{|R|} \\
&\le \sum_{R\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z}\in R\}\mathbb{1}\{|R|\ge 2\}2^{\frac{13|R|}{2}-\frac{5}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R|} \\
&\quad \times \lambda^{-|R|+\frac{1}{2}} \times \sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\text{ tree}}} \prod_{\mathbf{x}\in R} d_{\mathfrak{T}}(\mathbf{x})!^{\frac{1}{4}} \prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |J(\mathbf{x}-\mathbf{y})|
\end{aligned}$$

then, introducing the labeling and using Lemma 10,

$$A \le \sum_{m\ge 2} 2^{\frac{19}{2}m-\frac{11}{2}} \cdot \lambda^{-m+\frac{1}{2}} \cdot J_{\neq}^{m-1}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{m}.$$

For λ large we will have

$$\frac{2^{\frac{19}{2}} J_{\neq}.(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}})}{\lambda} \le \frac{1}{2}$$

and

$$\|\rho(\cdot,\emptyset)\| \le 2^{\frac{29}{2}} J_{\neq}\lambda^{-\frac{3}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^2 < 1.$$

So we now have,

$$\begin{aligned}
\mathfrak{W} &= \sum_{\mathbf{x}_2,\dots,\mathbf{x}_n\in\Lambda} |\langle\psi^{\sharp_1}(\mathbf{0}),\psi^{\sharp_2}(\mathbf{x}_2),\dots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}| \\
&\le \sum_{\mathbf{x}_1,\dots,\mathbf{x}_n\in\Lambda} \mathbb{1}\{\mathbf{x}_1=\mathbf{0}\}\sum_{p\ge 1}\frac{1}{p!}\sum_{\substack{R_1,\dots,R_p\in\mathbf{P}(\Lambda)\\ I_1,\dots,I_p\subset[n]}} |\phi^T(R_1,\dots,R_p)| \\
&\quad \times \mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q=[n]\right\}\prod_{q=1}^{p}\Bigg[\mathbb{1}\{\forall i\in I_q, \mathbf{x}_i\in R_q\}\mathbb{1}\{|R_q|\ge 2, \text{or } |I_q|\ge 1\}
\end{aligned}$$

$$\times 2^{\frac{11}{2}|R_q|+\frac{3}{4}|I_q|-\frac{5}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R_q|}\lambda^{-|R_q|-\frac{|I_q|}{4}+\frac{1}{2}}$$

$$\times\prod_{\mathbf{x}\in R_q}c_{I_q}(\mathbf{x})!^{\frac{1}{4}}\times\sum_{\substack{\mathfrak{T}_q\rightsquigarrow R_q\\ \mathfrak{T}_q\ \text{tree}}}\prod_{\mathbf{x}\in R_q}d_{\mathfrak{T}_q}(\mathbf{x})!^{\frac{1}{4}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}_q}|J(\mathbf{x}-\mathbf{y})|\Bigg]. \tag{31}$$

Following the same steps as before, including the introduction of $\gamma\in(0,1]$, we arrive at

$$\mathfrak{W}\le n!\gamma^{-n}2^{\frac{11n}{4}-1}\lambda^{-\frac{n}{4}}\times\sum_{p\ge 1}\frac{1}{p!}\sum_{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)}\mathbb{1}\left\{\mathbf{0}\in\bigcup_{q=1}^{p}R_q\right\}$$
$$\times|\phi^{\mathrm{T}}(R_1,\ldots,R_p)|\varsigma(R_1)\cdots\varsigma(R_p)$$

with the nonnegative polymer activities

$$\varsigma(R)=2^4\gamma.\mathbb{1}\{|R|=1\}\times\lambda^{-\frac{1}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)$$
$$+\mathbb{1}\{|R|\ge 2\}2^{\frac{13}{2}|R|-\frac{5}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{|R|}$$
$$\times\lambda^{-|R|+\frac{1}{2}}\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\ \text{tree}}}\prod_{\mathbf{x}\in R}d_{\mathfrak{T}}(\mathbf{x})!^{\frac{1}{4}}\prod_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|J(\mathbf{x}-\mathbf{y})|.$$

Their norm is similarly bounded by

$$\|\varsigma\|\le 2^{\frac{9}{2}}\gamma\cdot\left(1+J(\mathbf{0})\sqrt{\frac{2}{\lambda}}\right)+\sum_{m\ge 2}2^{\frac{21}{2}m-\frac{11}{2}}J_{\neq}^{m-1}\lambda^{-m+\frac{1}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{m}.$$

For λ large enough we will have

$$\frac{2^{\frac{21}{2}}J_{\neq}(1+J(\mathbf{0})\sqrt{\frac{2}{\lambda}})}{\lambda}\le\frac{1}{2}$$

and therefore also

$$\|\varsigma\|\le 2^{\frac{9}{2}}\gamma\times\left(1+J(\mathbf{0})\sqrt{\frac{2}{\lambda}}\right)+2^{\frac{33}{2}}J_{\neq}\lambda^{-\frac{3}{2}}\left(J(\mathbf{0})+\sqrt{\frac{\lambda}{2}}\right)^{2}.$$

The latter expression approaches $2^{\frac{9}{2}}\gamma$ as λ becomes large. Therefore, choosing $\gamma=\frac{1}{2^{11/2}}$ and applying Lemma 6 completes the proof of Theorem 3.

4 The Small Self-interaction or Near-Gaussian Regime

4.1 The Cluster and Mayer Expansions for the Truncated Correlation Functions

We will follow a line of argument similar to Sect. 3.1 in order to obtain a convergent series representation of the truncated correlations which is adapted to the small λ regime to be considered in the remainder of this article. Using the notation of Sects. 2.1 and 3.1 we have $Z_\Lambda(\boldsymbol{\alpha}) = (\det \tilde{J})^{-1}\tilde{Z}_\Lambda(\boldsymbol{\alpha})$ where

$$\tilde{Z}_\Lambda(\boldsymbol{\alpha}) = \int_{\mathbb{C}^\Lambda} d\mu_C(\psi^*, \psi)\, e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Lambda}|\psi(\mathbf{x})|^4} \prod_{i\in I}(1 + \alpha_i \psi^{\sharp_i}(\mathbf{x}_i)).$$

Since $\tilde{J}$ is positive definite, we again have that $\operatorname{Log}\tilde{Z}_\Lambda(\boldsymbol{\alpha})$ is analytic in a small polydisc D_Λ around $\boldsymbol{\alpha} = \mathbf{0}$ and

$$\langle(\psi^{\sharp_i}(\mathbf{x}_i))_{i\in I}\rangle_\Lambda^{\mathrm{T}} = \frac{\partial^{|I|}}{\prod_{i\in I}\partial\alpha_i} \operatorname{Log}\tilde{Z}_\Lambda(\boldsymbol{\alpha})\bigg|_{\boldsymbol{\alpha}=\mathbf{0}}.$$

The next step is to write an expansion for $\tilde{Z}_\Lambda(\boldsymbol{\alpha})$ similar to (15). This step is usually called the cluster expansion in the constructive quantum field theory literature and was introduced by Glimm, Jaffe and Spencer [30, 31]. These expansions which have been simplified and improved over the years by a small group of experts, are not so well known in the wider mathematical community. In what follows we will try to explain the method in detail, on the simple case of the lattice field theory considered in this article. We also use one of the more recent technical implementations based on the BKAR forest interpolation formula of Sect. 2.4. This is a kind of combinatorial Taylor expansion with integral reminder which interpolates between a 'complex' fully coupled situation and a 'simpler' fully decoupled one. In the present case this decoupling expansion will be applied to the Gaussian measure, since it is the only feature preventing the random variables of different lattice sites from being independent.

Before introducing the decoupling expansion for the Gaussian measure, we need a preliminary expansion of the self-interaction term $e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Lambda}|\psi(\mathbf{x})|^4}$. This is a matter of convenience and spares us the division by the amplitude of trivial polymers when deriving the polymer gas representation. This is especially useful in view of the forthcoming derivatives with respect to the coupling constant λ which one would rather have act on products instead of ratios. This preliminary expansion consists in writing

$$\begin{aligned} e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Lambda}|\psi(\mathbf{x})|^4} &= \prod_{\mathbf{x}\in\Lambda}\left[1 + \left(e^{-\frac{\lambda}{4}|\psi(\mathbf{x})|^4} - 1\right)\right] \\ &= \sum_{\Upsilon\subset\Lambda}\prod_{\mathbf{x}\in\Upsilon}\left(e^{-\frac{\lambda}{4}|\psi(\mathbf{x})|^4} - 1\right). \end{aligned}$$

Then for each $\mathbf{x}\in\Upsilon$ we write

$$e^{-\frac{\lambda}{4}|\psi(\mathbf{x})|^4} - 1 = \int_0^1 dt_{\mathbf{x}}\left(-\frac{\lambda}{4}|\psi(\mathbf{x})|^4\right)e^{-\frac{\lambda}{4}t_{\mathbf{x}}|\psi(\mathbf{x})|^4}.$$

As a result, one has

$$\begin{aligned} \tilde{Z}_\Lambda(\boldsymbol{\alpha}) = \sum_{\Upsilon\subset\Lambda}\left(-\frac{\lambda}{4}\right)^{|\Upsilon|}\int_{[0,1]^\Upsilon} d\vec{t}\int_{\mathbb{C}^\Lambda} d\mu_C(\psi^*,\psi) \\ \times\left(\prod_{i\in I}(1+\alpha_i\psi^{\sharp_i}(\mathbf{x}_i))\right)\left(\prod_{\mathbf{x}\in\Upsilon}|\psi(\mathbf{x})|^4\right)e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Upsilon}t_{\mathbf{x}}|\psi(\mathbf{x})|^4} \end{aligned} \tag{32}$$

where $d\vec{t}$ denotes the Lebesgue measure $\prod_{\mathbf{x}\in\Upsilon} dt_{\mathbf{x}}$.

We are now ready to apply Theorem 6, using $E=\Lambda$, as follows. For any multiplet $s=(s_l)_{l\in\Lambda^{(2)}}$ in the closed convex set $\mathcal{K}_\Lambda$, we replace the covariance C by a modified covariance $C[s]$ defined by

$$C[s](\mathbf{x},\mathbf{x})=C(\mathbf{x},\mathbf{x})\quad \text{for any } \mathbf{x}\in\Lambda,$$

and

$$C[s](\mathbf{x},\mathbf{y})=s_{\{\mathbf{x},\mathbf{y}\}}C(\mathbf{x},\mathbf{y})\quad \text{for } \mathbf{x}\neq\mathbf{y} \text{ in } \Lambda.$$

Clearly this new matrix is also Hermitian. Moreover, the following key positivity property follows from the previous definitions.

Lemma 12 *If $s\in\mathcal{K}_\Lambda$ then $C[s]$ is positive definite.*

Proof For $s\in\mathcal{K}_\Lambda$ one can find an expression $s=\sum_{j=1}^k w_j v_{\pi_j}$ with the w_j's nonnegative and summing up to 1, and the π_j some suitable partitions of Λ. Using notations similar to Sect. 2.1, for any vector $\psi\in\mathbb{C}^\Lambda$ we have

$$\begin{aligned}\langle\psi,C[s]\psi\rangle&=\sum_{\mathbf{x},\mathbf{y}\in\Lambda}\psi^*(\mathbf{x})C[s](\mathbf{x},\mathbf{y})\psi(\mathbf{y})\\&=\sum_{\mathbf{x}\in\Lambda}\psi^*(\mathbf{x})C(\mathbf{x},\mathbf{x})\psi(\mathbf{x})+\sum_{j=1}^k w_j\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}\psi^*(\mathbf{x})(v_{\pi_j})_{\{\mathbf{x},\mathbf{y}\}}C(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\\&=\sum_{j=1}^k w_j\Bigg[\sum_{\mathbf{x}\in\Lambda}\psi^*(\mathbf{x})C(\mathbf{x},\mathbf{x})\psi(\mathbf{x})\\&\quad+\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}\psi^*(\mathbf{x})\Bigg(\sum_{X\in\pi_j}\mathbb{1}\{\mathbf{x}\in X\}\mathbb{1}\{\mathbf{y}\in X\}\Bigg)C(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\Bigg]\\&=\sum_{j=1}^k w_j\sum_{X\in\pi_j}\langle\psi_X,C\psi_X\rangle\end{aligned}$$

where $\psi_X(\mathbf{x})=\mathbb{1}\{\mathbf{x}\in X\}\psi(\mathbf{x})$. Therefore the result is nonnegative since C is positive definite. Now suppose $\langle\psi,C[s]\psi\rangle$ vanishes. Since $\sum_{j=1}^k w_j=1$, one can choose j_0 such that $w_{j_0}>0$, and then have for every block $X\in\pi_{j_0}$ that $\langle\psi_X,C\psi_X\rangle=0$, i.e., that $\psi_X=0$. Thus $\psi=\sum_{X\in\pi_{j_0}}\psi_X=0$. □

As a result of this lemma the corresponding Gaussian measures $d\mu_{C[s]}$ are well defined. The last Gaussian integral in (32) therefore becomes a function

$$f(s)=\int_{\mathbb{C}^\Lambda}d\mu_{C[s]}(\psi^*,\psi)\Bigg(\prod_{i\in I}(1+\alpha_i\psi^{\sharp_i}(\mathbf{x}_i))\Bigg)\Bigg(\prod_{\mathbf{x}\in\Upsilon}|\psi(\mathbf{x})|^4\Bigg)e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Upsilon}t_\mathbf{x}|\psi(\mathbf{x})|^4}$$

of $s\in\mathcal{K}_\Lambda$ to which one can apply Theorem 6. The well known rule for computing derivatives of Gaussian integrals with respect to the covariance matrix, namely as a Laplace type operator acting on the integrand (see e.g. [29, Sect. 9.2]), immediately implies the following representation:

$$\tilde{Z}_\Lambda(\boldsymbol{\alpha}) = \sum_{\Upsilon\subset\Lambda}\left(-\frac{\lambda}{4}\right)^{|\Upsilon|}\int_{[0,1]^\Upsilon} d\vec{t}\sum_{\substack{\mathfrak{F}\text{ forest}\\ \text{on }\Lambda}}\int_{[0,1]^{\mathfrak{F}}} d\vec{h}\int_{\mathbb{C}^\Lambda} d\mu_{C[s(\mathfrak{F},\vec{h})]}(\psi^*,\psi)$$
$$\times\left(\prod_{l\in\mathfrak{F}}\Delta_l\right)\left(\prod_{i\in I}(1+\alpha_i\psi^{\sharp_i}(\mathbf{x}_i))\right)\left(\prod_{\mathbf{x}\in\Upsilon}|\psi(\mathbf{x})|^4\right)e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Upsilon}t_{\mathbf{x}}|\psi(\mathbf{x})|^4}.$$

Here we denoted by Δ_l the operator of Laplace type given, for any unordered pair of distinct sites $\mathbf{x}$ and $\mathbf{y}$ by

$$\Delta_{\{\mathbf{x},\mathbf{y}\}} = C(\mathbf{x},\mathbf{y})\frac{\partial}{\partial\psi(\mathbf{x})}\frac{\partial}{\partial\psi^*(\mathbf{y})} + C(\mathbf{y},\mathbf{x})\frac{\partial}{\partial\psi(\mathbf{y})}\frac{\partial}{\partial\psi^*(\mathbf{x})} \tag{33}$$

using the standard $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial\bar{z}}$ vector fields of multivariate complex analysis. The differential operators act on everything to their right.

Now, let π be the partition of Λ into connected components of the forest $\mathfrak{F}$. An important feature of the definition of $s(\mathfrak{F},\vec{h})$ is that it vanishes between components. This implies the componentwise factorization of the Gaussian integral. One can also factorize the different combinatorial sums involved, i.e., those over the sets $\Upsilon_R = \Upsilon\cap R$, as well as the ones over the trees $\mathfrak{T}_R$ connecting each $R\in\pi$ and which together make up the forest $\mathfrak{F}$. In sum, one has

$$\tilde{Z}_\Lambda(\boldsymbol{\alpha}) = \sum_{\pi\in\Pi_\Lambda}\prod_{R\in\pi}\zeta_0(R,\boldsymbol{\alpha},\lambda) \tag{34}$$

where for each polymer R we defined the polymer activity

$$\zeta_0(R,\boldsymbol{\alpha},\lambda) = \sum_{\Upsilon\subset R}\sum_{J\subset I_R}\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\text{ tree}}}\left(-\frac{\lambda}{4}\right)^{|\Upsilon|}\left(\prod_{i\in J}\alpha_i\right)\int_{[0,1]^\Upsilon} d\vec{t}\int_{[0,1]^{\mathfrak{T}}} d\vec{h}$$
$$\times\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*,\psi)\left(\prod_{l\in\mathfrak{T}}\Delta_l\right)\left(\prod_{i\in J}\psi^{\sharp_i}(\mathbf{x}_i)\right)$$
$$\times\left(\prod_{\mathbf{x}\in\Upsilon}|\psi(\mathbf{x})|^4\right)e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Upsilon}t_{\mathbf{x}}|\psi(\mathbf{x})|^4} \tag{35}$$

again with $I_R = \{i\in I|\mathbf{x}_i\in R\}$. Note that the covariance C which is used here is the restriction to R of the one defined on Λ as the inverse of $\tilde{J}$. It therefore retains a slight dependence on the volume Λ which contains the polymer R. We nevertheless suppressed it in the notation for better readability.

We now introduce in (35) the decomposition

$$1 = \mathbb{1}\{|R|=1,\ \Upsilon=\emptyset\text{ and }J=\emptyset\} + \mathbb{1}\{|R|\geq 2,\text{ or }|\Upsilon|\geq 1,\text{ or }|J|\geq 1\}$$

and break $\zeta_0(R,\boldsymbol{\alpha},\lambda)$ accordingly as a sum of two contributions. It is easy to see, because all sums and integrals become trivial and also because even for a single site the Gaussian measures are normalized, that the first contribution reduces to $\mathbb{1}\{|R|=1\}$. Therefore, by only keeping track of polymers for which the second contribution is selected, one can rewrite (34) as a polymer gas representation similar to (15), namely

$$\tilde{Z}_\Lambda(\boldsymbol{\alpha}) = \sum_{p\geq 0} \frac{1}{p!} \sum_{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)} \mathbb{1}\{\text{the } R_i \text{ are disjoint}\}\zeta(R_1,\boldsymbol{\alpha},\lambda)\cdots\zeta(R_p,\boldsymbol{\alpha},\lambda) \tag{36}$$

with the polymer activities defined as

$$\begin{aligned}\zeta(R,\boldsymbol{\alpha},\lambda) = &\sum_{\Upsilon\subset R}\sum_{J\subset I_R}\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\text{ tree}}} \mathbb{1}\{|R|\geq 2 \text{ or } |\Upsilon|\geq 1 \text{ or } |J|\geq 1\}\\ &\times\left(-\frac{\lambda}{4}\right)^{|\Upsilon|}\left(\prod_{i\in J}\alpha_i\right)\int_{[0,1]^\Upsilon} d\vec{t}\int_{[0,1]^{\mathfrak{T}}} d\vec{h}\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*,\psi)\\ &\times\left(\prod_{l\in\mathfrak{T}}\Delta_l\right)\left(\prod_{i\in J}\psi^{\sharp_i}(\mathbf{x}_i)\right)\left(\prod_{\mathbf{x}\in\Upsilon}|\psi(\mathbf{x})|^4\right)e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Upsilon}t_{\mathbf{x}}|\psi(\mathbf{x})|^4}.\end{aligned} \tag{37}$$

Now the same line of argument leading up to Proposition 1 shows the following.

Proposition 2 *Define for any subset J of the source label set I, and any polymer $R\in\mathbf{P}(\Lambda)$,*

$$\begin{aligned}\tilde{\zeta}(R,J,\lambda) = &\mathbb{1}\{\forall i\in J, \mathbf{x}_i\in R\}\sum_{\Upsilon\subset R}\left(-\frac{\lambda}{4}\right)^{|\Upsilon|}\mathbb{1}\{|R|\geq 2 \text{ or } |\Upsilon|\geq 1,\ \text{ or } |J|\geq 1\}\\ &\times\sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T}\text{ tree}}}\int_{[0,1]^\Upsilon} d\vec{t}\int_{[0,1]^{\mathfrak{T}}} d\vec{h}\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*,\psi)\\ &\times\left(\prod_{l\in\mathfrak{T}}\Delta_l\right)\left(\prod_{i\in J}\psi^{\sharp_i}(\mathbf{x}_i)\right)\left(\prod_{\mathbf{x}\in\Upsilon}|\psi(\mathbf{x})|^4\right)e^{-\frac{\lambda}{4}\sum_{\mathbf{x}\in\Upsilon}t_{\mathbf{x}}|\psi(\mathbf{x})|^4}.\end{aligned} \tag{38}$$

Provided the source-free condition

$$\sup_{\mathbf{x}\in\Lambda}\sum_{R\in\mathbf{P}(\Lambda)}\mathbb{1}\{\mathbf{x}\in R\}|\tilde{\zeta}(R,\emptyset,\lambda)|2^{|R|}<1 \tag{39}$$

holds, one has the absolutely convergent series representation for all truncated correlation functions

$$\begin{aligned}\langle(\psi^{\sharp_i}(\mathbf{x}_i))_{i\in I}\rangle_\Lambda^{\mathrm{T}} = &\sum_{p\geq 1}\frac{1}{p!}\sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)\\ I_1,\ldots,I_p\subset I}}\phi^{\mathrm{T}}(R_1,\ldots,R_p)\\ &\times\mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q = I\right\}\prod_{q=1}^p\tilde{\zeta}(R_q,I_q,\lambda).\end{aligned} \tag{40}$$

Remark 5 Although this is not really necessary for the proof of Proposition 2, note that the source-free activities $\tilde{\zeta}(R,\emptyset,\lambda)$ are real-valued. This is because of the hypothesis $J(-\mathbf{x}) = J(\mathbf{x})^*$ which implies that the differential operators $\Delta_{\{\mathbf{x},\mathbf{y}\}}$ preserve the real-valuedness of functions.

Remark 6 In the constructive quantum field theory literature, an equation such as (36) would be called a cluster expansion for the partition function $\tilde{Z}(\boldsymbol{\alpha})$ which is written as a sum over collections of disjoint polymers. An equation such as (40) involving sums over collections of polymers with the coefficient ϕ^{T} would be called a Mayer expansion for the truncated correlation functions.

We now go back to the setting of Theorem 4 and also restore the λ dependence in the notation for correlation functions. In order to extract the factor λ^N in the clustering estimate, we will perform an additional Taylor expansion of the connected correlation function. We write

$$\begin{aligned}\langle\psi^{\sharp_1}(\mathbf{x}_1),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda} &= \sum_{k=0}^{N-1}\frac{\lambda^k}{k!}\left(\frac{d}{d\lambda}\right)^k\langle\psi^{\sharp_1}(\mathbf{x}_1),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}\big|_{\lambda=0}\\ &\quad+\int_0^1 du\,\frac{(1-u)^{N-1}}{(N-1)!}\left(\frac{d}{du}\right)^N\langle\psi^{\sharp_1}(\mathbf{x}_1),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,u\lambda}.\end{aligned}$$

However, by Lemma 1 and the hypothesis $n\geq 2(N+1)$, the Taylor polynomial vanishes and the right-hand side reduces to the integral remainder, i.e.,

$$\langle\psi^{\sharp_1}(\mathbf{x}_1),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}=\int_0^1 du\,\frac{(1-u)^{N-1}}{(N-1)!}\left(\frac{d}{du}\right)^N\langle\psi^{\sharp_1}(\mathbf{x}_1),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,u\lambda}.$$

So far we only used the fact that correlations as C^∞ functions of λ on $[0,+\infty)$. However, the next step is to use the representation (40), and also to differentiate it, term by term, N times. This will require some estimates, in order to justify the convergence criterion (39) as well as the following outcome of term by term differentiation

$$\begin{aligned}&\langle\psi^{\sharp_1}(\mathbf{x}_1),\ldots,\psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}\\ &\quad=\int_0^1 du\,\frac{(1-u)^{N-1}}{(N-1)!}\sum_{p\geq 1}\frac{1}{p!}\sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)\\ I_1,\ldots,I_p\subset[n]}}\phi^{\mathrm{T}}(R_1,\ldots,R_p)\\ &\qquad\times\mathbb{1}\left\{I_q\text{ disjoint},\ \bigcup I_q=[n]\right\}\sum_{N_1+\cdots+N_p=N}\frac{N!}{N_1!\ldots N_p!}\prod_{q=1}^{p}\left(\frac{d}{du}\right)^{N_q}\tilde{\zeta}(R_q,I_q,u\lambda)\end{aligned}\tag{41}$$

for $n\geq 2(N+1)$. Finally we will use this last representation as input for the l^1-clustering property.

4.2 The Estimates

This section will provide the necessary estimates for the proof of Theorem 4. Many of the ideas used in these estimates originated in the work Glimm-Jaffe-Spencer (see e.g. [29, Chap. 18], [55, Sect. III.1], and [3]). We will first suppose that the J function governing the Gaussian measure is so chosen that the constant K_0 appearing in Lemma 3 is equal to 1. Later in Sect. 4.2.4 we will get rid of this restriction by a simple scaling transformation on the field variable ψ.

4.2.1 The Bound on a Single Polymer Activity and Its Derivatives

The basic quantity we need to bound is $(\frac{d}{du})^M \tilde{\zeta}(R, I, u\lambda)$, for any integer $M \geq 0$, polymer R, index set $I \in [n]$, and $u \in [0, 1]$. From (38) one easily gets

$$
\begin{aligned}
&\left(\frac{d}{du}\right)^M \tilde{\zeta}(R, I, u\lambda) \\
&\quad = \mathbb{1}\{\forall i \in I, \mathbf{x}_i \in R\} \sum_{\Upsilon \subset R} \mathbb{1}\{|R| \geq 2 \text{ or } |\Upsilon| \geq 1, \text{ or } |I| \geq 1\} \\
&\qquad \times \sum_{k=0}^{M} \frac{M!}{k!(M-k)!} \mathbb{1}\{|\Upsilon| \geq M-k\} \frac{|\Upsilon|!}{(|\Upsilon|-M+k)!} u^{|\Upsilon|-M+k} \left(-\frac{\lambda}{4}\right)^{|\Upsilon|+k} \\
&\qquad \times \sum_{\substack{\mathfrak{T} \rightsquigarrow R \\ \mathfrak{T} \text{ tree}}} \sum_{\mathbf{y}_1, \ldots, \mathbf{y}_k \in \Upsilon} \int_{[0,1]^\Upsilon} d\vec{t} \int_{[0,1]^{\mathfrak{T}}} d\vec{h} \left(\prod_{j=1}^{k} t_{\mathbf{y}_j}\right) \mathcal{I}
\end{aligned} \tag{42}
$$

where $\mathcal{I}$ refers to the remaining Gaussian integral, namely,

$$
\begin{aligned}
\mathcal{I} = &\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*, \psi) \left(\prod_{l \in \mathfrak{T}} \Delta_l\right) \left(\prod_{i \in I} \psi^{\sharp_i}(\mathbf{x}_i)\right) \left(\prod_{\mathbf{x} \in \Upsilon} |\psi(\mathbf{x})|^4\right) \\
&\times \left(\prod_{j=1}^{k} |\psi(\mathbf{y}_j)|^4\right) e^{-\frac{\lambda u}{4} \sum_{\mathbf{x} \in \Upsilon} t_{\mathbf{x}} |\psi(\mathbf{x})|^4}.
\end{aligned} \tag{43}
$$

When one performs the derivatives coming from the Δ_l operators, this further splits into

$$
\mathcal{I} = \sum_{\mathfrak{p}} \mathcal{I}_{\mathfrak{p}}
$$

where, avoiding excessive formalization, we denoted by $\mathfrak{p}$ any derivation procedure including the detailed information as to which specific ψ or ψ^* factor has been destroyed, and by which $\frac{\partial}{\partial \psi}$ or $\frac{\partial}{\partial \psi^*}$ operator. We will use a bound of the form

$$
|\mathcal{I}| \leq \left(\max_{\mathfrak{p}} |\mathcal{I}_{\mathfrak{p}}|\right) \sum_{\mathfrak{p}} 1.
$$

Note that a term $\mathcal{I}_{\mathfrak{p}}$ has the form

$$
\mathcal{I}_{\mathfrak{p}} = C_{\mathfrak{p}}.L_{\mathfrak{p}}. \int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*, \psi) \left(\prod_{\mathbf{x} \in R} \{\psi(\mathbf{x})^{m(\mathbf{x})} \psi^*(\mathbf{x})^{m^*(\mathbf{x})}\}\right) e^{-\frac{\lambda u}{4} \sum_{\mathbf{x} \in \Upsilon} t_{\mathbf{x}} |\psi(\mathbf{x})|^4}
$$

where the $m(\mathbf{x})$ and $m^*(\mathbf{x})$ are some local multiplicities and $L_{\mathfrak{p}}$ is a product of factors $-\frac{\lambda u t_{\mathbf{x}}}{4}$ produced by the derivatives which acted on the exponential. Finally, $C_{\mathfrak{p}}$ is a product of propagators corresponding to the edges in the tree $\mathfrak{T}$. It depends on whether, for each Δ_l, one choses the $C(\mathbf{x}, \mathbf{y})$ or the $C(\mathbf{y}, \mathbf{x})$ term. In any case, Lemma 3 together with the assumption $K_0 = 1$ implies

$$
|C_{\mathfrak{p}}| \leq e^{-\mu_0 \sum_{\{\mathbf{x},\mathbf{y}\} \in \mathfrak{T}} |\mathbf{x}-\mathbf{y}|}.
$$

We will use the notation $|m| = \sum_{\mathbf{x}\in R} m(\mathbf{x})$ for the total multiplicity of a multiindex such as $(m(\mathbf{x}))_{\mathbf{x}\in R}$. Using the positivity of the interaction, and in particular the positivity of λ, one has

$$0 < e^{-\frac{\lambda u}{4}\sum_{\mathbf{x}\in\Upsilon} t_{\mathbf{x}}|\psi(\mathbf{x})|^4} \leq 1.$$

Therefore, by the Cauchy-Schwarz inequality

$$\left|\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*,\psi)\prod_{\mathbf{x}\in R}\{\psi(\mathbf{x})^{m(\mathbf{x})}\psi^*(\mathbf{x})^{m^*(\mathbf{x})}\}e^{-\frac{\lambda u}{4}\sum_{\mathbf{x}\in\Upsilon} t_{\mathbf{x}}|\psi(\mathbf{x})|^4}\right|$$

$$\leq \left[\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*,\psi)\prod_{\mathbf{x}\in R}\{\psi(\mathbf{x})^{\hat{m}(\mathbf{x})}\psi^*(\mathbf{x})^{\hat{m}(\mathbf{x})}\}\right]^{\frac{1}{2}}$$

where $\hat{m}(\mathbf{x}) = m(\mathbf{x}) + m^*(\mathbf{x})$.

From now on we also impose the hypothesis $0 < \lambda \leq 4$, which implies $|L_{\mathfrak{p}}| \leq 1$. We now invoke the following classical lemma of constructive field theory [24, 30] also called the principle of local factorials, in order to bound the last Gaussian integral.

Lemma 13 *For any q, and collection of sites $\mathbf{z}_1,\ldots,\mathbf{z}_q$ and $\mathbf{w}_1,\ldots,\mathbf{w}_q$ in R, one has*

$$\left|\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*,\psi)\psi(\mathbf{z}_1)\cdots\psi(\mathbf{z}_q)\psi^*(\mathbf{w}_1)\cdots\psi^*(\mathbf{w}_q)\right| \leq K_1^q\prod_{\mathbf{x}\in R} n^*(\mathbf{x})!$$

where $n^*(\mathbf{x})$ denotes the number of indices i, $1\leq i\leq q$, such that $\mathbf{w}_i = \mathbf{x}$ and $K_1 = K_1(d,\mu_0)$ of (8).

Proof By the Isserlis-Wick Theorem we have

$$\left|\int_{\mathbb{C}^R} d\mu_{C[s(\mathfrak{T},\vec{h})]}(\psi^*,\psi)\psi(\mathbf{z}_1)\cdots\psi(\mathbf{z}_q)\psi^*(\mathbf{w}_1)\cdots\psi^*(\mathbf{w}_q)\right|$$

$$\leq \sum_{\sigma\in\mathfrak{S}_q}\prod_{i=1}^{q}|C(\mathbf{z}_i,\mathbf{w}_{\sigma(i)})| \leq \sum_{\sigma\in\mathfrak{S}_q} e^{-\mu_0\sum_{i=1}^q|\mathbf{z}_i-\mathbf{w}_{\sigma(i)}|}$$

because the decoupling parameters in $s(\mathfrak{T},\vec{h})$ are between 0 and 1, and also because of Lemma 3. Now

$$\sum_{\sigma\in\mathfrak{S}_q} e^{-\mu_0\sum_{i=1}^q|\mathbf{z}_i-\mathbf{w}_{\sigma(i)}|} = \sum_{\mathbf{u}_1,\ldots,\mathbf{u}_q\in R} e^{-\mu_0\sum_{i=1}^q|\mathbf{z}_i-\mathbf{u}_i|}\sum_{\sigma\in\mathfrak{S}_q}\mathbb{1}\{\forall i,\mathbf{w}_{\sigma(i)} = \mathbf{u}_i\}$$

and the last sum over σ is either equal to $\prod_{\mathbf{x}\in R} n^*(\mathbf{x})!$ or vanishes, depending on whether or not the $\mathbf{u}$ sequence is a permutation of the $\mathbf{w}$ sequence. Now the claim follows using (8). □

We now apply Lemma 13 to get the following bound

$$|\mathcal{I}_{\mathfrak{p}}| \leq e^{-\mu_0\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|\mathbf{x}-\mathbf{y}|} K_1^{|m|}\prod_{\mathbf{x}\in R}(m(\mathbf{x})+m^*(\mathbf{x}))!^{\frac{1}{2}}$$

where we used the easily verifiable fact $|m| = |m^*| = \frac{|\hat{m}|}{2}$, i.e., in any $\mathcal{I}_{\mathfrak{p}}$ one always has an equal number of ψ and ψ^* factors remaining. Let us define the following initial multiplicities, i.e., before applying the derivatives, for the ψ and ψ^* fields respectively:

$$m_0(\mathbf{x}) = |\{i \in I \mid \mathbf{x}_i = \mathbf{x} \text{ and } \sharp_i = \emptyset\}| + 2\mathbb{1}_{\{\mathbf{x}\in\Upsilon\}} + 2|\{j, 1 \leq j \leq k \mid \mathbf{y}_j = \mathbf{x}\}|$$

and

$$m_0^*(\mathbf{x}) = |\{i \in I \mid \mathbf{x}_i = \mathbf{x} \text{ and } \sharp_i = *\}| + 2\mathbb{1}_{\{\mathbf{x}\in\Upsilon\}} + 2|\{j, 1 \leq j \leq k \mid \mathbf{y}_j = \mathbf{x}\}|.$$

Once again we have $|m_0| = |m_0^*|$. When expanding the sums (33), each site will receive a number $\delta(\mathbf{x})$ of $\frac{\partial}{\partial\psi(\mathbf{x})}$ derivatives, as well as a number $\delta^*(\mathbf{x})$ of $\frac{\partial}{\partial\psi^*(\mathbf{x})}$ derivatives. Besides, one trivially has

$$\delta(\mathbf{x}) + \delta^*(\mathbf{x}) = d(\mathbf{x})$$

where $d(\mathbf{x})$ denotes the degree of the vertex $\mathbf{x} \in R$ in the tree $\mathfrak{T}$.

Now it is easy to see that the maximal number of field factors occurs when all the derivatives pull new $|\psi|^4$ vertices from the exponential. Namely, for any $\mathbf{x} \in R$,

$$\begin{aligned} m(\mathbf{x}) &\leq m_0(\mathbf{x}) + 2\delta^*(\mathbf{x}) + \delta(\mathbf{x}), \\ m^*(\mathbf{x}) &\leq m_0^*(\mathbf{x}) + \delta^*(\mathbf{x}) + 2\delta(\mathbf{x}), \\ m(\mathbf{x}) + m^*(\mathbf{x}) &\leq m_0(\mathbf{x}) + m_0^*(\mathbf{x}) + 3d(\mathbf{x}). \end{aligned}$$

We will also need the identities

$$\sum_{\mathbf{x}\in R} (m_0(\mathbf{x}) + m_0^*(\mathbf{x})) = |I| + 4|\Upsilon| + 4k \tag{44}$$

and

$$\sum_{\mathbf{x}\in R} d(\mathbf{x}) = 2(|R| - 1) \tag{45}$$

which imply

$$|m_0| + |m_0^*| + 3|d| = |I| + 4|\Upsilon| + 4k + 6|R| - 6 \tag{46}$$

and

$$|m| \leq \frac{|I|}{2} + 2|\Upsilon| + 2k + 3|R| - 3.$$

We are now ready to bound the number of derivation procedures $\mathfrak{p}$.

Lemma 14 *The number of derivation procedures is bounded by*

$$\sum_{\mathfrak{p}} 1 \leq 2^{|R|-1} e^{|I|+4|\Upsilon|+4k+6|R|-3} \prod_{\mathbf{x}\in R} d(\mathbf{x})!.$$

Proof One pays a factor $2^{|R|-1}$ for the sums (33). For $\mathbf{x} \in R\backslash\Upsilon$ the derivatives can only act on the fields already present, and we therefore have a number of choices limited by

$$\mathbb{1}_{\{m_0(\mathbf{x})\geq\delta(\mathbf{x})\}} \mathbb{1}_{\{m_0^*(\mathbf{x})\geq\delta^*(\mathbf{x})\}} \frac{m_0(\mathbf{x})! m_0^*(\mathbf{x})!}{(m_0(\mathbf{x}) - \delta(\mathbf{x}))!(m_0^*(\mathbf{x}) - \delta^*(\mathbf{x}))!}.$$

For $\mathbf{x} \in \Upsilon$, the number of terms produced by applying the derivatives is bounded by

$$d(\mathbf{x})! \times \exp\left\{m_0(\mathbf{x}) + m_0^*(\mathbf{x}) + \frac{3}{2}d(\mathbf{x}) + 3\right\}. \tag{47}$$

Indeed, this is the number of terms produced when computing

$$\left(\frac{\partial}{\partial\psi(\mathbf{x})}\right)^{\delta(\mathbf{x})}\left(\frac{\partial}{\partial\psi^*(\mathbf{x})}\right)^{\delta^*(\mathbf{x})} \psi(\mathbf{x})^{m_0(\mathbf{x})}\psi^*(\mathbf{x})^{m_0^*(\mathbf{x})}\, e^{-\frac{\lambda u}{4}\psi(\mathbf{x})^2\psi^*(\mathbf{x})^2}.$$

Let us first perform the $\frac{\partial}{\partial\psi^*(\mathbf{x})}$ derivatives and then the $\frac{\partial}{\partial\psi(\mathbf{x})}$ derivatives. When evaluating the very last derivative $\frac{\partial}{\partial\psi(\mathbf{x})}$ one has to choose between deriving a new vertex from the exponential which gives a factor 2 for the choice of ψ in the $\psi(\mathbf{x})^2\psi^*(\mathbf{x})^2$ vertex, or deriving a field factor which was already there which at most gives $m_0(\mathbf{x}) + 2\delta^*(\mathbf{x}) + \delta(\mathbf{x}) - 1$ possibilities. Indeed, either the factor was present initially which corresponds to $m_0(\mathbf{x})$ choices, or it was in a vertex first derived from the exponential by a $\frac{\partial}{\partial\psi^*(\mathbf{x})}$ derivative, in which case one has to pay a factor $\delta^*(\mathbf{x})$ to identify that derivative and a factor 2 for the choice of field ψ within the vertex $-\frac{\lambda u}{4}\psi(\mathbf{x})^2\psi^*(\mathbf{x})^2$. The last possibility is when the derived factor was in a vertex first produced by one of the previous $\delta(\mathbf{x}) - 1$ derivatives of type $\frac{\partial}{\partial\psi(\mathbf{x})}$. Such a derivative already consumes one of the two ψ's in the vertex, so we only have to pay a factor of $\delta(\mathbf{x}) - 1$. In sum, the last $\frac{\partial}{\partial\psi(\mathbf{x})}$ derivative at most gives $(m_0(\mathbf{x}) + 2\delta^*(\mathbf{x}) + \delta(\mathbf{x}) + 1)$ possibilities. Likewise the before last $\frac{\partial}{\partial\psi(\mathbf{x})}$ derivatives has at most $(m_0(\mathbf{x}) + 2\delta^*(\mathbf{x}) + \delta(\mathbf{x}))$ options, etc. Therefore, the number of possibilities for the $\frac{\partial}{\partial\psi(\mathbf{x})}$ derivatives is bounded by $\frac{(m_0(\mathbf{x})+2\delta^*(\mathbf{x})+\delta(\mathbf{x})+1)!}{(m_0(\mathbf{x})+2\delta^*(\mathbf{x})+1)!}$. By a similar reasoning, that of the $\frac{\partial}{\partial\psi^*(\mathbf{x})}$ derivatives, which are performed first, is bounded by $\frac{(m_0^*(\mathbf{x})+\delta^*(\mathbf{x})+1)!}{(m_0^*(\mathbf{x})+1)!}$. As a result, the total number of possibilities is at most

$$\begin{aligned}
&\frac{(m_0(\mathbf{x}) + 2\delta^*(\mathbf{x}) + \delta(\mathbf{x}) + 1)!}{(m_0(\mathbf{x}) + 2\delta^*(\mathbf{x}) + 1)!} \cdot \frac{(m_0^*(\mathbf{x}) + \delta^*(\mathbf{x}) + 1)!}{(m_0^*(\mathbf{x}) + 1)!} \\
&\quad \le \left[m_0(\mathbf{x}) + 2\delta^*(\mathbf{x}) + \frac{\delta(\mathbf{x})}{2} + \frac{3}{2}\right]^{\delta(\mathbf{x})} \left[m_0^*(\mathbf{x}) + \frac{\delta^*(\mathbf{x})}{2} + \frac{3}{2}\right]^{\delta^*(\mathbf{x})}
\end{aligned} \tag{48}$$

where we used the arithmetic versus geometric mean inequality

$$\begin{aligned}
\frac{s!}{(s-q)!} &= s(s-1)\cdots(s-q+1) \\
&\le \left[\frac{s + (s-1) + \cdots + (s-q+1)}{q}\right]^q \\
&\le \left(s - \frac{q-1}{2}\right)^q.
\end{aligned}$$

Finally using the inequality $x^n \le n! e^x$ for each of the two factors on the right-hand side of (48), as well as the trivial inequality $\delta(\mathbf{x})!\delta^*(\mathbf{x})! \le d(\mathbf{x})!$, we find that the number of terms produced by the derivatives at the site $\mathbf{x}$ is bounded by

$$d(\mathbf{x})! \times \exp\left\{m_0(\mathbf{x}) + m_0^*(\mathbf{x}) + \frac{5}{2}\delta^*(\mathbf{x}) + \frac{1}{2}\delta(\mathbf{x}) + 3\right\}.$$

Now redo the same reasoning, but this time first applying the $\frac{\partial}{\partial\psi(\mathbf{x})}$ and then the $\frac{\partial}{\partial\psi^*(\mathbf{x})}$ derivatives. One will get the same bound but with $\delta(\mathbf{x})$ and $\delta^*(\mathbf{x})$ exchanged. Taking the geometric mean of the two bounds gives the desired estimate (47). Using the same inequalities, the bound for $\mathbf{x} \in R\backslash\Upsilon$ is easily seen to be no greater than the one for the $\mathbf{x} \in \Upsilon$ case. The lemma now follows from (44) and (45). $\square$

Putting the previous considerations together we now have a bound on $\mathcal{I}$ from (43):

$$|\mathcal{I}| \leq 2^{|R|-1} e^{|I|+4|\Upsilon|+4k+6|R|-3} K_1^{\frac{1}{2}|I|+2|\Upsilon|+2k+3|R|-3} \\ \times e^{-\mu_0 \sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |\mathbf{x}-\mathbf{y}|} \cdot \prod_{\mathbf{x}\in R} d(\mathbf{x})! \prod_{\mathbf{x}\in R} (m_0(\mathbf{x}) + m_0^*(\mathbf{x}) + 3d(\mathbf{x}))!^{\frac{1}{2}}.$$

Note that we used $K_1 \geq 1$ which is clear from (8) since $\sum_{\mathbf{z}\in\mathbb{Z}^d} e^{-\mu|\mathbf{z}|} \geq 1$. We now use the estimate

$$(m_0(\mathbf{x}) + m_0^*(\mathbf{x}) + 3d(\mathbf{x}))! \leq (m_0(\mathbf{x}) + m_0^*(\mathbf{x}))! d(\mathbf{x})!^3 4^{m_0(\mathbf{x})+m_0^*(\mathbf{x})+3d(\mathbf{x})}$$

and identity (46) in order to write

$$\prod_{\mathbf{x}\in R} (m_0(\mathbf{x}) + m_0^*(\mathbf{x}) + 3d(\mathbf{x}))!^{\frac{1}{2}} \leq 2^{|I|+4|\Upsilon|+4k+6|R|-6} \prod_{\mathbf{x}\in R} d(\mathbf{x})!^{\frac{3}{2}} \prod_{\mathbf{x}\in R} (m_0(\mathbf{x}) + m_0^*(\mathbf{x}))!^{\frac{1}{2}}.$$

We also need the following important remark: if $|R| \geq 2$, then the integral $\mathcal{I}$ of (43) is zero unless $|R| \leq |\Upsilon| + |I| + k$. Indeed, if $|R| \geq 2$ then the connecting tree $\mathfrak{T}$ is nonempty and therefore each site $\mathbf{x} \in R$ receives at least one field derivative. If such a site $\mathbf{x}$ contains no source, i.e., $\mathbf{x} \neq \mathbf{x}_i$, for any $i \in I$, and if the site is not in Υ, and if $\mathbf{x} \neq \mathbf{y}_j$ for any j, $1 \leq j \leq k$, then the derivative has nothing to act on and the integral $\mathcal{I}$ is zero. Note that if $|R| = 1$, then we already have a characteristic function in (42) enforcing $|\Upsilon| \geq 1$ or $|I| \geq 1$. As a result, we always have $|R| \leq |\Upsilon| + |I| + k$, which can be exploited by introducing the corresponding characteristic function. This condition, together with $|\Upsilon| \geq M - k$ and the assumption $0 < \lambda \leq 4$ implies

$$\left(\frac{\lambda}{4}\right)^{|\Upsilon|+k} \leq \left(\frac{\lambda}{4}\right)^{\max\{|R|-|I|,M\}}. \tag{49}$$

The previous considerations allow us to write

$$\left|\left(\frac{d}{du}\right)^M \tilde{\xi}(R, I, u\lambda)\right| \\ \leq \mathbb{1}\{\forall i \in I, \mathbf{x}_i \in R\} \sum_{\Upsilon\subset R} \mathbb{1}\{|R| \geq 2 \text{ or } |\Upsilon| \geq 1, \text{ or } |I| \geq 1\} \\ \times \sum_{k=0}^{M} \mathbb{1}\{|\Upsilon| \geq M-k\} \frac{M!}{k!} 2^{|\Upsilon|} \left(\frac{\lambda}{4}\right)^{\max\{|R|-|I|,M\}} \mathbb{1}\{|R| \leq |\Upsilon| + |I| + k\} \\ \times \sum_{\substack{\mathfrak{T}\rightsquigarrow R \\ \mathfrak{T} \text{ tree}}} \sum_{\mathbf{y}_1,\ldots,\mathbf{y}_k\in\Upsilon} 2^{|R|-1} (2e\sqrt{K_1})^{|I|+4|\Upsilon|+4k+6|R|-3} \cdot e^{-\mu_0 \sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |\mathbf{x}-\mathbf{y}|} \\ \times \prod_{\mathbf{x}\in R} d(\mathbf{x})!^{\frac{5}{2}} \prod_{\mathbf{x}\in R} (m_0(\mathbf{x}) + m_0^*(\mathbf{x}))!^{\frac{1}{2}} \tag{50}$$

where we used $K_1 \geq 1$ as well as the inequality $\frac{|\Upsilon|!}{(M-k)!(|\Upsilon|-M+k)!} \leq 2^{|\Upsilon|}$.

We now need a lemma which bounds local factorials of the degrees in the tree by a portion of the tree decay. This is a volume effect due to the finite dimensionality of the host lattice $\mathbb{Z}^d$.

Lemma 15 *For any $\alpha > 0$ we have*

$$\prod_{\mathbf{x}\in R} d(\mathbf{x})! e^{-\alpha \sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |\mathbf{x}-\mathbf{y}|} \le K_2^{|R|}$$

where $K_2 = \max\{K_{2,1}, K_{2,2}\}$ with

$$K_{2,1} = \left(\left\lfloor \frac{2\pi^{\frac{d}{2}} d^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \right\rfloor\right)!$$

and

$$K_{2,2} = \exp\left(\sup_{x\in[1,+\infty[} \left\{ x\log x - \frac{\alpha\Gamma(\frac{d}{2}+1)^{\frac{1}{d}}}{2^{2+\frac{1}{d}}\sqrt{\pi}} x^{1+\frac{1}{d}} + \frac{\alpha\sqrt{d}}{4} x \right\}\right).$$

Proof We write the quantity to be estimated as the product over $\mathbf{x}\in R$ of

$$d(\mathbf{x})! e^{-\frac{\alpha}{2}\sum_{\mathbf{y}|\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |\mathbf{x}-\mathbf{y}|} \tag{51}$$

and we will bound the last expression using the fact that the $d(\mathbf{x})$ sites $\mathbf{y}$ which are neighbors of $\mathbf{x}$ in the tree $\mathfrak{T}$ are distinct, and the more they are the further away from $\mathbf{x}$ they have to be. Indeed, for any $r \ge 0$, the number B_r of lattice points at distance at most r from $\mathbf{x}$ satisfies

$$\begin{aligned}
B_r &\le 2^d |\{\mathbf{z}\in\mathbb{N}^d, |\mathbf{z}| \le r\}| \\
&\le 2^d \operatorname{Vol}\left(\bigcup_{\mathbf{z}\in\mathbb{N}^d, |\mathbf{z}|\le r} \mathbf{z} + [0,1[^d\right) \\
&\le 2^d \operatorname{Vol}(\{\mathbf{z}\in\mathbb{R}_+^d, |\mathbf{z}| \le r+\sqrt{d}\}) \\
&\le \operatorname{Vol}(S^{d-1}) \int_0^{r+\sqrt{d}} d\rho\rho^{d-1} \\
&\le \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} (r+\sqrt{d})^d.
\end{aligned}$$

Now if $d(\mathbf{x}) \ge 2B_r$, at least half of the $d(\mathbf{x})$ neighbors of $\mathbf{x}$ are at a distance greater than r from $\mathbf{x}$. This would imply

$$e^{-\frac{\alpha}{2}\sum_{\mathbf{y}|\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |\mathbf{x}-\mathbf{y}|} \le e^{-\frac{\alpha r}{4} d(\mathbf{x})}.$$

Let us first suppose that

$$d(\mathbf{x}) \ge \frac{2\pi^{\frac{d}{2}} d^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}.$$

Letting

$$r_{\max} = \left\{ \frac{\Gamma(\frac{d}{2}+1)}{2\pi^{\frac{d}{2}}} d(\mathbf{x}) \right\}^{\frac{1}{d}} - \sqrt{d},$$

we see that $r_{\max} \geq 0$ and $r = r_{\max}$ satisfies the hypothesis $d(\mathbf{x}) \geq 2B_r$. Consequently, it follows that

$$e^{-\frac{\alpha}{2}\sum_{\mathbf{y}|\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|\mathbf{x}-\mathbf{y}|} \leq \exp\left[-\frac{\alpha d(\mathbf{x})}{4}\left(\left(\frac{\Gamma(\frac{d}{2}+1)}{2\pi^{\frac{d}{2}}}d(\mathbf{x})\right)^{\frac{1}{d}} - \sqrt{d}\right)\right].$$

Using the trivial inequality $d(\mathbf{x})! \leq d(\mathbf{x})^{d(\mathbf{x})}$ and the definition of $K_{2,2}$ which clearly is finite when $\alpha > 0$, we have

$$d(\mathbf{x})!e^{-\frac{\alpha}{2}\sum_{\mathbf{y}|\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|\mathbf{x}-\mathbf{y}|} \leq K_{2,2}.$$

In the second case where

$$d(\mathbf{x}) < \frac{2\pi^{\frac{d}{2}}d^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$$

then (51) is trivially bounded by $K_{2,1}$. $\square$

Since the $d(\mathbf{x})!$ to be bounded in (50) appear at the power $\frac{5}{2}$ and we want to use only a fraction, say half, of the tree decay; we will use the previous lemma with $\alpha = \frac{\mu_0}{5}$ so that $\frac{5}{2}\alpha = \frac{\mu_0}{2}$.

We now turn our attention to the $(m_0(\mathbf{x}) + m_0^*(\mathbf{x}))!^{\frac{1}{2}}$ in (50). Let

$$\begin{aligned} c_I(\mathbf{x}) &= |\{i \in I|\mathbf{x}_i = \mathbf{x}\}|, \\ b(\mathbf{x}) &= 4\mathbb{1}_{\{\mathbf{x}\in\Upsilon\}}, \\ v_{\mathbf{y}}(\mathbf{x}) &= |\{j, 1 \leq j \leq k|\mathbf{y}_j = \mathbf{x}\}|. \end{aligned}$$

Then

$$\begin{aligned} \prod_{\mathbf{x}\in R}(m_0(\mathbf{x}) + m_0^*(\mathbf{x}))!^{\frac{1}{2}} &= \prod_{\mathbf{x}\in R}(c_I(\mathbf{x}) + b(\mathbf{x}) + 4v_{\mathbf{y}}(\mathbf{x}))!^{\frac{1}{2}} \\ &\leq \prod_{\mathbf{x}\in R}\{c_I(\mathbf{x})!b(\mathbf{x})!v_{\mathbf{y}}(\mathbf{x})!^4 \cdot 6^{c_I(\mathbf{x})+b(\mathbf{x})+4v_{\mathbf{y}}(\mathbf{x})}\}^{\frac{1}{2}}. \end{aligned}$$

That is

$$\prod_{\mathbf{x}\in R}(m_0(\mathbf{x}) + m_0^*(\mathbf{x}))!^{\frac{1}{2}} \leq 6^{\frac{1}{2}|I|+2|R|+2k} \cdot 24^{\frac{1}{2}|R|}\prod_{\mathbf{x}\in R}c_I(\mathbf{x})!^{\frac{1}{2}}\prod_{\mathbf{x}\in R}v_{\mathbf{y}}(\mathbf{x})!^2 \tag{52}$$

where we used $|\Upsilon| \leq |R|$.

We now simplify the bound (50) by using Lemma 15 with $\alpha = \frac{\mu_0}{5}$, as well as (52). We also bound the sum over the no longer needed Υ by $2^{|R|}$, and we bound k by M in the various exponents where it appears. After some cleaning up, we therefore get

$$\begin{aligned} &\left|\left(\frac{d}{du}\right)^M \tilde{\zeta}(R, I, u\lambda)\right| \\ &\quad\leq \left(\frac{\lambda}{4}\right)^{\max\{|R|-|I|,M\}} K_3^{|R|}K_4^{|I|}K_5^M\mathbb{1}\{\forall i \in I, \mathbf{x}_i \in R\} \\ &\qquad\times\sum_{k=0}^M\frac{M!}{k!}\prod_{\mathbf{x}\in R}c_I(\mathbf{x})!^{\frac{1}{2}}\left[\sum_{\mathbf{y}_1,\ldots,\mathbf{y}_k\in R}\prod_{\mathbf{x}\in R}v_{\mathbf{y}}(\mathbf{x})!^2\right]\left[\sum_{\substack{\mathfrak{T}\rightsquigarrow R \\ \mathfrak{T}\text{ tree}}}e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|\mathbf{x}-\mathbf{y}|}\right] \end{aligned}$$

where

$$K_3 = 2^{16}3^2e^{10}\sqrt{6}K_1^5K_2^{\frac{5}{2}},$$
$$K_4 = 2e\sqrt{6K_1},$$
$$K_5 = 2^6 3^2 e^4 K_1^2.$$

Note that K_2 in Lemma 15 is such that $K_2 \geq K_{2,1} \geq 1$ and therefore K_3, K_4, K_5 all are ≥ 1.

The sums over the positions $\mathbf{y}_1, \dots, \mathbf{y}_k$ of the vertices created by the additional Taylor expansion, as well as over the positions $\mathbf{x}_2, \dots, \mathbf{x}_n$ in the l^1-clustering bound will be done thanks to Lemma 11.

Remark 7 We will use this lemma with both $\beta = 2$ and $\beta = \frac{1}{2}$ in order to sum over the $\mathbf{y}_j$ and $\mathbf{x}_i$ respectively. However the β versus 1 dichotomy prevents us from compensating the factorials at the power 2 by the ones at the power $\frac{1}{2}$. This is the main bottleneck we found on the way to Conjecture 1.

Thanks to Lemma 11, the sum over the $\mathbf{y}$'s is bounded by $2^{|R|+k-1}k!^2$. We then use the coarse bounds

$$\sum_{k=0}^{M} k!\, M!\, 2^{|R|+k-1} \leq M!^2 2^{|R|+M}(M+1) \leq M!^2 2^{|R|+2M}.$$

As a result we have the desired bound which is summarized in the following proposition.

Proposition 3 *Suppose the function $J : \mathbb{Z}^d \to \mathbb{C}$ satisfies $K_0 = 1$, where K_0 is the quantity defined in Lemma 3. Suppose M is a nonnegative integer, R is a polymer in Λ, u belongs to $[0, 1]$ and $0 < \lambda \leq 4$. Then, we have the estimate*

$$\left|\left(\frac{d}{du}\right)^M \tilde{\zeta}(R, I, u\lambda)\right| \leq \left(\frac{\lambda}{4}\right)^{\max\{|R|-|I|, M\}} (2K_3)^{|R|} K_4^{|I|} (4K_5)^M M!^2$$
$$\times \mathbb{1}\{\forall i \in I, \mathbf{x}_i \in R\} \times \prod_{\mathbf{x}\in R} c_I(\mathbf{x})!^{\frac{1}{2}} \times \left[\sum_{\substack{\mathfrak{T} \rightsquigarrow R \\ \mathfrak{T} \text{ tree}}} e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|\mathbf{x}-\mathbf{y}|}\right] \tag{53}$$

where right derivatives are meant when $u = 0$, and left derivatives when $u = 1$.

4.2.2 The Convergence Criterion

We now need to address the convergence criterion (39) of Proposition 2. Take any fixed site $\mathbf{z} \in \Lambda$, we then need to bound the quantity

$$Q = \sum_{R\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z} \in R\}|\tilde{\zeta}(R, \emptyset, \lambda)|2^{|R|}.$$

The simple case $I = \emptyset$, $u = 1$, $M = 0$, of the basic raw estimate (53) gives

$$Q \leq \sum_{R\in\mathbf{P}(\Lambda)} \mathbb{1}\{\mathbf{z} \in R\}(\lambda K_3)^{|R|} \sum_{\substack{\mathfrak{T} \rightsquigarrow R \\ \mathfrak{T} \text{ tree}}} e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|\mathbf{x}-\mathbf{y}|}.$$

We now proceed as in Sect. 3.3 and condition the sum on $m = |R|$ and also introduce the identity (22), in order to eliminate R. The sum over the locations of the labeled sites gives a factor K_6^{m-1} using (8) and letting $K_6 = K_1(d, \frac{\mu_0}{2})$. The sum over trees $\mathfrak{t}$ on the set $[m]$, this time without the $d_{\mathfrak{t}}(i)!$'s is simply bounded by $m^{m-2} \le (m-2)!e^m$ by the coarser version of Cayley's Theorem [65, Proposition 5.3.2]. Note that one has to treat separately $m \ge 2$ and $m = 1$. The result is easily seen to again be a geometric series bound, namely

$$Q \le \sum_{m \ge 1} (\lambda e K_3 K_6)^m.$$

Therefore if $\lambda \le \frac{1}{3eK_3K_6} < 4$, which we assume from now on, then we have

$$\|\tilde{\zeta}(\cdot, \emptyset, \lambda)\| \le \frac{1}{2}$$

and the criterion is satisfied.

4.2.3 Justification of the Term by Term Differentiation

We now address the issue of term by term differentiation leading to the series expression (41). Recall that as a corollary of the mean value theorem and Lebesgue dominated convergence, the equation

$$\begin{aligned}
&\left(\frac{d}{du}\right)^N \langle \psi^{\sharp_1}(\mathbf{x}_1), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda, u\lambda} \\
&\quad = \sum_{p \ge 1} \frac{1}{p!} \sum_{\substack{R_1, \ldots, R_p \in \mathbf{P}(\Lambda) \\ I_1, \ldots, I_p \subset [n]}} \phi^{\mathrm{T}}(R_1, \ldots, R_p) \mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q = [n] \right\} \\
&\quad \times \sum_{N_1 + \cdots + N_p = N} \frac{N!}{N_1! \ldots N_p!} \prod_{q=1}^{p} \left(\frac{d}{du}\right)^{N_q} \tilde{\zeta}(R_q, I_q, u\lambda)
\end{aligned} \tag{54}$$

will be established for $u \in [0, 1]$ provided we can find majorants $G(R, I, \lambda, k)$ which are uniform in u such that

$$\left| \left(\frac{d}{du}\right)^k \tilde{\zeta}(R, I, u\lambda) \right| \le G(R, I, \lambda, k)$$

for any $u \in [0, 1]$, and such that for any integer M, $1 \le M \le N$, one has

$$\begin{aligned}
&\sum_{p \ge 1} \frac{1}{p!} \sum_{\substack{R_1, \ldots, R_p \in \mathbf{P}(\Lambda) \\ I_1, \ldots, I_p \subset [n]}} |\phi^{\mathrm{T}}(R_1, \ldots, R_p)| \mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q = [n] \right\} \\
&\quad \times \sum_{M_1 + \cdots + M_p = M} \frac{M!}{M_1! \cdots M_p!} \prod_{q=1}^{p} G(R_q, I_q, \lambda, M_q) < +\infty.
\end{aligned}$$

We will take the majorants $G(R, I, \lambda, k)$ provided by the right-hand side of (53). We therefore have to show the finiteness of

$$\mathcal{Q} = \sum_{p\geq 1} \frac{1}{p!} \sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)\\ I_1,\ldots,I_p\subset[n]}} |\phi^{\mathrm{T}}(R_1,\ldots,R_p)| \mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q = [n] \right\}$$
$$\times \sum_{M_1+\cdots+M_p=M} \frac{M!}{M_1!\ldots M_p!} \prod_{q=1}^{p} \left[\left(\frac{\lambda}{4}\right)^{\max\{|R_q|-|I_q|, M_q\}} (2K_3)^{|R_q|} K_4^{|I_q|} (4K_5)^{M_q} M_q!^2 \right.$$
$$\left. \times\, \mathbb{1}\{\forall i \in I_q, \mathbf{x}_i \in R_q\} \times \prod_{\mathbf{x}\in R_q} c_{I_q}(\mathbf{x})!^{\frac{1}{2}} \left[\sum_{\substack{\mathfrak{T}_q \rightsquigarrow R_q\\ \mathfrak{T}_q \text{ tree}}} e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}_q} |\mathbf{x}-\mathbf{y}|} \right] \right].$$

Since the only issue is finiteness, we will use very coarse bounds for $\mathcal{Q}$. We write

$$\left(\frac{\lambda}{4}\right)^{\max\{|R_q|-|I_q|, M_q\}} \leq \left(\frac{\lambda}{4}\right)^{|R_q|-|I_q|}$$

as well as

$$\prod_{q=1}^{p} \prod_{\mathbf{x}\in R_q} c_{I_q}(\mathbf{x})!^{\frac{1}{2}} \leq n!^{\frac{1}{2}}.$$

We also bound the characteristic functions by

$$\mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q = [n] \right\} \prod_{q=1}^{p} \mathbb{1}\{\forall i \in I_q, \mathbf{x}_i \in R_q\}$$
$$\leq \mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q = [n] \right\} \mathbb{1}\left\{ \mathbf{x}_1 \in \bigcup_{q=1}^{p} R_q \right\}.$$

We finally use

$$\sum_{M_1+\cdots+M_p=M} M_1!\cdots M_p! \leq M!\, 2^{M+p} \tag{55}$$

and

$$\sum_{I_1,\ldots,I_p\subset[n]} \mathbb{1}\left\{ I_q \text{ disjoint}, \bigcup I_q = [n] \right\} = p^n \leq n! e^p$$

and therefore get

$$\mathcal{Q} \leq n!^{\frac{3}{2}} \left(\frac{4K_4}{\lambda}\right)^n M!^2 (8K_5)^M \sum_{p\geq 1} \frac{1}{p!}$$
$$\times \sum_{R_1,\ldots,R_p\in\mathbf{P}(\Lambda)} \mathbb{1}\left\{ \mathbf{x}_1 \in \bigcup_{q=1}^{p} R_q \right\} |\phi^{\mathrm{T}}(R_1,\ldots,R_p)| \rho(R_1)\cdots\rho(R_p)$$

with

$$\rho(R) = 2e\left(\frac{\lambda K_3}{2}\right)^{|R|} \sum_{\substack{\mathfrak{T} \rightsquigarrow R \\ \mathfrak{T} \text{ tree}}} e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}} |\mathbf{x}-\mathbf{y}|}.$$

Now using the same argument as in Sect. 4.2.2 we see that

$$\|\rho\| \le 2e \sum_{m\ge 1} (\lambda e K_3 K_6)^m.$$

As a result, we will have $\|\rho\| \le 1$ provided

$$\lambda \le \frac{1}{(1+2e)eK_3K_6} < \frac{1}{3eK_3K_6}$$

which we now assume. Finally, Lemma 6 shows that $\mathcal{Q}$ is finite, and therefore (54) as well as (41) are justified.

4.2.4 The Clustering Estimate

We now, under the hypotheses $n \ge 2(N+1)$, $K_0 = 1$ and $0 < \lambda \le \frac{1}{(1+2e)eK_3K_6}$, come to the clustering estimate proper, i.e., the bound on

$$\mathfrak{C} = \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_n\in\Lambda} \mathbb{1}\{\mathbf{x}_1 = \mathbf{0}\}\, |\langle \psi^{\sharp_1}(\mathbf{x}_1), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle^{\mathrm{T}}_{\Lambda,\lambda}|.$$

Using (41), we can write

$$\mathfrak{C} \le \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_n\in\Lambda} \mathbb{1}\{\mathbf{x}_1 = \mathbf{0}\} \int_0^1 du \frac{(1-u)^{N-1}}{(N-1)!} \sum_{p\ge 1} \frac{1}{p!} \sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda) \\ I_1,\ldots,I_p\subset[n]}} |\phi^{\mathrm{T}}(R_1,\ldots,R_p)|$$
$$\times\, \mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q = [n]\right\} \sum_{N_1+\cdots+N_p=N} \frac{N!}{N_1!\ldots N_p!} \prod_{q=1}^{p} \left|\left(\frac{d}{du}\right)^{N_q} \tilde{\zeta}(R_q, I_q, u\lambda)\right|.$$

Inserting the bound (53), performing the u integral, and tidying the resulting inequality, one obtains

$$\mathfrak{C} \le (\lambda K_5)^N K_4^n \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_n\in\Lambda} \mathbb{1}\{\mathbf{x}_1 = \mathbf{0}\} \sum_{p\ge 1} \frac{1}{p!} \sum_{\substack{R_1,\ldots,R_p\in\mathbf{P}(\Lambda) \\ I_1,\ldots,I_p\subset[n]}} |\phi^{\mathrm{T}}(R_1,\ldots,R_p)|$$
$$\times\, \mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q = [n]\right\} \sum_{N_1+\cdots+N_p=N} N_1!\ldots N_p! \prod_{q=1}^{p} \left[\left(\frac{\lambda}{4}\right)^{\max\{|R_q|-|I_q|-N_q,0\}}\right.$$
$$\left.\times\, (2K_3)^{|R_q|} \mathbb{1}\{\forall i\in I_q, \mathbf{x}_i \in R_q\} \prod_{\mathbf{x}\in R_q} c_{I_q}(\mathbf{x})!^{\frac{1}{2}} \left\{ \sum_{\substack{\mathfrak{T}_q \rightsquigarrow R_q \\ \mathfrak{T}_q \text{ tree}}} e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}_q} |\mathbf{x}-\mathbf{y}|} \right\}\right].$$

We now proceed as in Sect. 3.3 and push $\mathbb{1}\{\mathbf{x}_1 = \mathbf{0}\}$ through the sums over p, the R_q's and the I_q's, before bounding it by the coarser condition $\mathbb{1}\{\mathbf{0} \in \bigcup_{q=1}^{p} R_q\}$. We likewise push the sums over the $\mathbf{x}_i$'s inside the appropriate bracket factor. The sums over the source localizations $\mathbf{x}_i$ are then performed using Lemma 11 and yield the same bound as (26). Hence,

$$\mathfrak{C} \le (\lambda K_5)^N (2K_4)^n \sum_{p\ge 1} \frac{1}{p!} \sum_{\substack{R_1,\dots,R_p \in \mathbf{P}(\Lambda)\\ I_1,\dots,I_p \subset [n]}} \mathbb{1}\left\{\mathbf{0} \in \bigcup_{q=1}^{p} R_q\right\}$$
$$\times |\phi^{\mathrm{T}}(R_1,\dots,R_p)| \mathbb{1}\left\{I_q \text{ disjoint}, \bigcup I_q = [n]\right\} |I_1|!\dots|I_p|!$$
$$\times \sum_{N_1+\dots+N_p=N} N_1!\dots N_p! \prod_{q=1}^{p}\left[\left(\frac{\lambda}{4}\right)^{\max\{|R_q|-|I_q|-N_q,0\}} (4K_3)^{|R_q|}\right.$$
$$\left.\times \left\{\sum_{\substack{\mathfrak{T}_q \rightsquigarrow R_q\\ \mathfrak{T}_q \text{ tree}}} e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}_q}|\mathbf{x}-\mathbf{y}|}\right\}\right].$$

We now introduce a constant γ, $0<\gamma\le 1$, to be fine-tuned shortly; and we suppose that λ satisfies the extra hypothesis $0<\frac{\lambda}{4}\le\gamma$. We can now use the estimate

$$\left(\frac{\lambda}{4}\right)^{\max\{|R_q|-|I_q|-N_q,0\}} \le \gamma^{\max\{|R_q|-|I_q|-N_q,0\}} \le \gamma^{|R_q|-|I_q|-N_q}.$$

We also use the previously derived inequalities (28) and (55) which yield

$$\mathfrak{C} \le \left(\frac{2\lambda K_5}{\gamma}\right)^N \left(\frac{4K_4}{\gamma}\right)^n N!n!$$
$$\times \sum_{p\ge 1}\frac{1}{p!} \sum_{\substack{R_1,\dots,R_p \in \mathbf{P}(\Lambda)\\ I_1,\dots,I_p \subset [n]}} \mathbb{1}\left\{\mathbf{0} \in \bigcup_{q=1}^{p} R_q\right\} |\phi^{\mathrm{T}}(R_1,\dots,R_p)| \prod_{q=1}^{p} \varpi(R_q)$$

with

$$\varpi(R) = 4.(4\gamma K_3)^{|R|} \sum_{\substack{\mathfrak{T}\rightsquigarrow R\\ \mathfrak{T} \text{ tree}}} e^{-\frac{\mu_0}{2}\sum_{\{\mathbf{x},\mathbf{y}\}\in\mathfrak{T}}|\mathbf{x}-\mathbf{y}|}.$$

Now, again as in Sect. 4.2.2, we have

$$\|\varpi\| \le 4\sum_{m\ge 1}(8e\gamma K_3 K_6)^m.$$

Thus we will have $\|\varpi\|\le 1$ as soon as

$$\gamma \le \frac{1}{40eK_3K_6}.$$

We therefore choose $\gamma = \frac{1}{40eK_3K_6}$. Since we have by hypothesis that

$$0<\lambda\le 4\gamma = \frac{1}{10eK_3K_6} < \frac{1}{(1+2e)eK_3K_6} < \frac{1}{3eK_3K_6} < 4$$

we checked the validity of every condition we needed to check and we can use Lemma 6 and conclude

$$\mathfrak{C} \le \left(\frac{2\lambda K_5}{\gamma}\right)^N \left(\frac{4K_4}{\gamma}\right)^n N!n!.$$

In other words we have the desired clustering bound

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n\in\Lambda} |\langle\psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}| \le \lambda^N\ n!\ N!\ (80eK_3K_5K_6)^N\ (160eK_3K_4K_6)^n.$$

We will now get rid of the restriction to $K_0 = 1$ by a simple scaling transformation on the field variables ψ. Indeed if one does the change of variable $\psi = \eta\psi'$, for some $\eta > 0$, in the original model, one easily sees that the n-point truncated correlations of the ψ fields becomes η^n times the analogous correlation for the ψ' fields. The latter are sampled according to the measure corresponding to the input parameters $\eta^2 J$, $\eta^4\lambda$ instead of the original function J and coupling λ respectively. Now one can go back through the definitions of the various constants μ_0, α, K_1, K_2, K_3, K_4, K_5 and K_6, and easily check that they are invariant if one multiplies the J function by a nonzero scalar η^2. On the other hand $K_0 = \frac{J(\mathbf{0})}{J_{\neq}(J(\mathbf{0})-J_{\neq})}$ gets multiplied by η^{-2}. Therefore one can make the new $K_0 = 1$ by choosing $\eta = \sqrt{\frac{J(\mathbf{0})}{J_{\neq}(J(\mathbf{0})-J_{\neq})}}$. Thus we have proved that if

$$0 < \lambda \le \frac{J_{\neq}^2(J(\mathbf{0}) - J_{\neq})^2}{10eK_3K_6J(\mathbf{0})^2};$$

then, uniformly in $\Lambda \subset \mathbb{Z}^d$, and n even satisfying $n \ge 2(N+1)$, we have

$$\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_n\in\Lambda} |\langle\psi^{\sharp_1}(\mathbf{0}), \psi^{\sharp_2}(\mathbf{x}_2), \ldots, \psi^{\sharp_n}(\mathbf{x}_n)\rangle_\Lambda^{\mathrm{T}}| \le c_1(N, J)c_2(J)^n\lambda^N \times n!$$

where

$$c_1(N, J) = N!\left(\frac{80eK_3K_5K_6J(\mathbf{0})^2}{J_{\neq}^2(J(\mathbf{0}) - J_{\neq})^2}\right)^N$$

and

$$c_2(J) = 160eK_3K_4K_6\sqrt{\frac{J(\mathbf{0})}{J_{\neq}(J(\mathbf{0}) - J_{\neq})}}.$$

This completes the proof of Theorem 4, i.e., the clustering estimate for n-point functions with $n \ge 4$.

The proof of Theorem 5 is exactly the same as the one given above for Theorem 4 with the choices $n = 2$ and $N = 1$. The only difference is that the right-hand side of the starting point equation (41) now expresses the Taylor remainder

$$\langle\psi^{\sharp_1}(\mathbf{x}_1), \psi^{\sharp_2}(\mathbf{x}_2)\rangle_{\Lambda,\lambda}^{\mathrm{T}} - \langle\psi^{\sharp_1}(\mathbf{x}_1), \psi^{\sharp_2}(\mathbf{x}_2)\rangle_{\Lambda,0}^{\mathrm{T}}$$

instead of the full 2-point function.

5 Additional Remarks

As mentioned in the introduction our main motivation for this work is to provide a proof for some key assumptions needed in [45], namely (2.18) and (2.19) therein. We therefore limited our analysis to the precise model studied in the latter reference. It should be clear, however, that our methods are quite robust and apply in much more general situations. Since this article is technical enough as it is, we refrained from stating our results and proofs with maximal generality, which would most likely be at the expense of better readability. In the following we will briefly indicate how one can generalize our results.

In Theorem 3, the hypothesis $J(\mathbf{0}) \geq J_{\neq}$ can be relaxed. Indeed, in (21) one could have borrowed a portion $\frac{\lambda}{8}|z|^4$ of the quartic term in order to control a quadratic part with the wrong sign. This would then bring an extra $O(1)$ factor per site in the polymers, which is easily taken care of.

For the near-Gaussian regime, all that is required from the Gaussian measure is the exponential decay of $C(\mathbf{x}, \mathbf{y})$. For the sake of completeness we provided a proof in Lemma 3 when J is of compact support. One can easily obtain the exponential decay of C when J is of infinite range but itself is exponentially decaying with large enough diagonal part. In fact, one can relax the hypothesis of exponential decay of C to simply a power law decay $(1 + |\mathbf{x} - \mathbf{y}|)^{-\alpha}$ with α large enough. The main lower bound on this exponent comes from the needs of Lemma 15. See e.g. [3, Lemma 22] for how to accommodate the power law case.

Translation invariance is not needed. For instance Theorems 1, 2, 3 and their proofs remain valid if the Hermitian matrix $\tilde{J}(\mathbf{x}, \mathbf{y})$ is not of the form $J(\mathbf{x} - \mathbf{y})$. One simply needs to replace throughout $J(\mathbf{0})$ by $\min_{\mathbf{x}} \tilde{J}(\mathbf{x}, \mathbf{x})$ and $J_{\neq}$ by $\max_{\mathbf{x}} \sum_{\mathbf{y}} |\tilde{J}(\mathbf{x}, \mathbf{y})|$. One can also adapt our methods to the situation where the coupling λ is site dependent.

Note that our methods do not require a ferromagnetic type hypothesis $J(\mathbf{x} - \mathbf{y}) \leq 0$ for $\mathbf{x} \neq \mathbf{y}$, nor the convexity of the interaction potential. One can easily treat more general potentials $\lambda P(|\psi|^2)$ where P is a polynomial with strictly positive leading coefficient.

Our results and in particular Theorem 4 can be transposed to the case of a real valued massive scalar field with ϕ^4 interaction. One could then apply Theorem 2.1 in [69] to reduce the proof of a log-Sobolev inequality to that of the exponential decay of the two-point function. Note that in Sect. 4.2.4 we consumed all the remaining tree decay

$$\prod_{q=1}^{p} \sum_{\substack{\mathfrak{T}_q \rightsquigarrow R_q \\ \mathfrak{T}_q \text{ tree}}} e^{-\frac{\mu_0}{2} \sum_{\{\mathbf{x},\mathbf{y}\} \in \mathfrak{T}_q} |\mathbf{x}-\mathbf{y}|}$$

for the clustering estimate. However one could have only used a fraction $\frac{\mu_0}{4}$. The remaining decay could then be used in order to obtain clustering estimates where the absolute value of the truncated function gets multiplied by a penalizing factor

$$\exp\left(\frac{\mu_0}{4} D(\mathbf{x}_1, \ldots, \mathbf{x}_N)\right)$$

where $D(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is the infimum over trees connecting the sources (possibly through additional internal vertices) of the sum of edge lengths. In particular for the two-point function one easily gets exponential decay. The remaining technicality towards the log-Sobolev inequality is to adapt our estimates to make them uniform in the boundary conditions (see e.g. [53]).

The most interesting extension of our results would be to the case of a near-Gaussian regime in the presence of nonlocal non-Gaussian interactions, namely where instead of $\sum_{\mathbf{x}} |\psi(\mathbf{x})|^4$ one would have more general terms of the form

$$\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_p} W(\mathbf{x}_1, \sharp_1; \ldots; \mathbf{x}_p, \sharp_p)\psi^{\sharp_1}(\mathbf{x}_1)\cdots\psi^{\sharp_p}(\mathbf{x}_p)$$

for some nonlocal kernels W. We expect that one would need a sufficiently strong tree decay hypothesis for these kernels, for instance an exponential one. We believe our methods, with much more work, can be used to prove such results. Let us briefly outline a strategy for instance in the case of an interaction of the form

$$\sum_{\mathbf{x},\mathbf{y}\in\Lambda} |\psi(\mathbf{x})|^2 W(\mathbf{x}-\mathbf{y})|\psi(y)|^2$$

where W is real and nonnegative, obeying the symmetry $W(-\mathbf{x}) = W(\mathbf{x})$, and satisfying an exponential decay hypothesis. One first needs to separate the diagonal terms form the off-diagonal terms in the latter sum. One then performs the preliminary t-expansion for the diagonal terms as in Sect. 4.1. Then one needs to expand the off-diagonal terms using the BKAR formula, with parameters $s_{\{\mathbf{x},\mathbf{y}\}}$ multiplying the corresponding $2|\psi(\mathbf{x})|^2 W(\mathbf{x}-\mathbf{y})|\psi(y)|^2$ term. This will produce a sum over forests with W edges. Only then, one uses the BKAR formula again in order to decouple the Gaussian measure, however the underlying set E should not be Λ but rather the set of connected components created by the W forest. Namely, the definition of the interpolated matrix $C[s]$ is done blockwise, with blocks corresponding to these connected components. The outcome will be a jungle formula in the sense of [2], i.e., the activity of a polymer R will involve a sum over spanning trees made of two types of edges: the W edges produced by the first expansion and propagator edges produced by the second one. The main technical difficulty is that a derivative $\frac{\partial}{\partial\psi(\mathbf{x})}$ or $\frac{\partial}{\partial\psi^*(\mathbf{x})}$ can produce fields at a far away location $\mathbf{y}$. These fields produced by derivatives at various locations in the polymer can conspire to fetch the same $\mathbf{y}$. This could cause a problem when trying to use Lemma 15 because the local factorials due to the Gaussian bound would not be immediately related to the degrees in the spanning tree. Fortunately, one can move these local factorials by Lemma 17 in [3] and control them using the local multiplicities of the $\mathbf{x}$ derivatives which created the $\mathbf{y}$ fields, thanks to the decay of the involved $W(\mathbf{x}-\mathbf{y})$ kernels. In the situation of general multi-body kernels W with $p > 2$, one would presumably need a hypergraph expansion as in [3, Sect. 2].

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References

1. Abdesselam, A.: Feynman diagrams in algebraic combinatorics. Sém. Lothar. Comb. **49** (2002/04), Art. B49c, 45 p. (electronic)

2. Abdesselam, A., Rivasseau, V.: Trees, forests and jungles: a botanical garden for cluster expansions. In: Constructive Physics, Palaiseau, 1994. Lecture Notes in Physics, vol. 446, pp. 7–36. Springer, Berlin (1995)
3. Abdesselam, A., Rivasseau, V.: An explicit large versus small field multiscale cluster expansion. Rev. Math. Phys. **9**(2), 123–199 (1997)
4. Abdesselam, A., Rivasseau, V.: Explicit fermionic tree expansions. Lett. Math. Phys. **44**(1), 77–88 (1998)
5. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, 2nd edn. Academic Press, New York (2003)
6. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge University Press, Cambridge (1999)
7. Ané, C., Blachère, S., Chafaï, D., Fougères, P., Gentil, I., Malrieu, F., Roberto, C., Scheffer, G.: Sur les Inégalités de Sobolev Logarithmiques. With a Preface by Dominique Bakry and Michel Ledoux. Panoramas et Synthèses, vol. 10. Soc. Math. France, Paris (2000)
8. Bach, V., Møller, J.S.: Correlation at low temperature I. Exponential decay. J. Funct. Anal. **203**(1), 93–148 (2003)
9. Bach, V., Jecko, T., Sjöstrand, J.: Correlation asymptotics of classical lattice spin systems with nonconvex Hamilton function at low temperature. Ann. H. Poincaré **1**(1), 59–100 (2000)
10. Balaban, T., Imbrie, J.Z., Jaffe, A.: Effective action and cluster properties of the Abelian Higgs model. Commun. Math. Phys. **114**, 257–315 (1988)
11. Balaban, T., Feldman, J., Knörrer, H., Trubowitz, E.: Power series representations for bosonic effective actions. J. Stat. Phys. **134**, 839–857 (2009)
12. Battle, G.A., Federbush, P.: A note on cluster expansions, tree graph identities, extra $1/N!$ factors! Lett. Math. Phys. **8**(1), 55–57 (1984)
13. Benfatto, G., Gallavotti, G., Procacci, A., Scoppola, B.: Beta function and Schwinger functions for a many fermions system in one dimension. Anomaly of the Fermi surface. Commun. Math. Phys. **160**(1), 93–171 (1994)
14. Birnbaum, Z.W.: An inequality for Mill's ratio. Ann. Math. Stat. **13**(2), 245–246 (1942)
15. Brydges, D.C.: A short course on cluster expansions. In: Phénomènes Critiques, Systèmes Aléatoires, Théories de Jauge, Part I, II, Les Houches, 1984, pp. 129–183. North-Holland, Amsterdam (1986)
16. Brydges, D.C., Federbush, P.: A new form of the Mayer expansion in classical statistical mechanics. J. Math. Phys. **19**(10), 2064–2067 (1978)
17. Brydges, D., Kennedy, T.: Mayer expansions and the Hamilton-Jacobi equation. J. Stat. Phys. **48**, 19 (1987)
18. Brydges, D., Martin, P.: Coulomb systems at low density: a review. J. Stat. Phys. **96**, 1163–1330 (1999)
19. Brydges, D., Dimock, J., Hurd, T.R.: The short distance behavior of $(\phi^4)_3$. Commun. Math. Phys. **172**, 143–186 (1995)
20. Caianiello, E.R.: Number of Feynman graphs and convergence. Nuovo Cimento **10**(3), 223–225 (1956)
21. Cammarota, C.: Decay of correlations for infinite range interactions in unbounded spin systems. Commun. Math. Phys. **85**(4), 517–528 (1982)
22. Constantinescu, F.: Analyticity in the coupling constant of the $\lambda P(\varphi)$ lattice theory. J. Math. Phys. **21**(8), 2278–2281 (1980)
23. Duneau, M., Iagolnitzer, D., Souillard, B.: Decrease properties of truncated correlation functions and analyticity properties for classical lattices and continuous systems. Commun. Math. Phys. **31**, 191–208 (1973)
24. Eckmann, J.-P., Magnen, J., Sénéor, R.: Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories. Commun. Math. Phys. **39**, 251–271 (1974/75)
25. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: A renormalizable field theory: the massive Gross-Neveu model in two dimensions. Commun. Math. Phys. **103**(1), 67–103 (1986)
26. Feldman, J., Knörrer, H., Trubowitz, E.: A representation for Fermionic correlation functions. Commun. Math. Phys. **195**(2), 465–493 (1998)
27. Fernández, R., Procacci, A.: Cluster expansion for abstract polymer models. New bounds from an old approach. Commun. Math. Phys. **274**(1), 123–140 (2007)
28. Gawędzki, K., Kupiainen, A.: Gross-Neveu model through convergent perturbation expansions. Commun. Math. Phys. **102**(1), 1–30 (1985)
29. Glimm, J., Jaffe, A.: Quantum Physics. A Functional Integral Point of View, 2nd edn. Springer, New York (1987)
30. Glimm, J., Jaffe, A., Spencer, T.: The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions, part II: The cluster expansion. In: Velo, G., Wightman, A. (eds.) Constructive Quantum Field Theory, Erice, 1973. Lecture Notes in Physics, vol. 25. Springer, New York (1973)

31. Glimm, J., Jaffe, A., Spencer, T.: The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model. Ann. Math. **100**, 585–632 (1974)
32. Graham, R.L., Knuth, D.E., Patashnik, O.: Concrete Mathematics. A Foundation for Computer Science. Advanced Book Program. Addison-Wesley, Reading (1989)
33. Gross, L.: Decay of correlations in classical lattice models at high temperature. Commun. Math. Phys. **68**(1), 9–27 (1979)
34. Gruber, C., Kunz, H.: General properties of polymer systems. Commun. Math. Phys. **22**, 133–161 (1971)
35. Guionnet, A., Zegarlinski, B.: Lectures on logarithmic Sobolev inequalities. In: Séminaire de Probabilités, XXXVI. Lecture Notes in Mathematics, vol. 1801, pp. 1–134. Springer, Berlin (2003)
36. Helffer, B.: Semiclassical Analysis, Witten Laplacians, and Statistical Mechanics. Series in Partial Differential Equations and Applications, vol. 1. World Scientific, River Edge (2002)
37. Helffer, B., Sjöstrand, J.: On the correlation for Kac-like models in the convex case. J. Stat. Phys. **74**(1–2), 349–409 (1994)
38. Iagolnitzer, D., Magnen, J.: Asymptotic completeness and multiparticle structure in field theories II. Theories with renormalization: the Gross-Neveu model. Commun. Math. Phys. **111**(1), 81–100 (1987)
39. Israel, R.B., Nappi, C.R.: Exponential clustering for long-range integer-spin systems. Commun. Math. Phys. **68**(1), 29–37 (1979)
40. Isserlis, L.: On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. Biometrika **12**, 134–139 (1918)
41. Kunz, H.: Analyticity and clustering properties of unbounded spin systems. Commun. Math. Phys. **59**(1), 53–69 (1978)
42. Lesniewski, A.: Effective action for the Yukawa$_2$ quantum field theory. Commun. Math. Phys. **108**(3), 437–467 (1987)
43. Lo, A.: On the exponential decay of the n-point correlation functions and the analyticity of the pressure. J. Math. Phys. **48**(12), 123506 (2007)
44. Lukkarinen, J., Spohn, H.: Not to normal order—notes on the kinetic limit for weakly interacting quantum fluids. J. Stat. Phys. **134**(5–6), 1133–1172 (2009)
45. Lukkarinen, J., Spohn, H.: Weakly nonlinear Schrödinger equation with random initial data. Preprint arXiv:0901.3283v1 [math-ph] (2009)
46. Mack, G., Pordt, A.: Convergent perturbation expansions for Euclidean quantum field theory. Commun. Math. Phys. **97**(1–2), 267–298 (1985)
47. Malyshev, V.A., Minlos, R.A.: Gibbs Random Fields. Cluster Expansions. Translated from the Russian by R. Kotecký and P. Holický. Mathematics and Its Applications (Soviet Series), vol. 44. Kluwer Academic, Dordrecht (1991)
48. Mastropietro, V.: Non-perturbative Renormalization. World Scientific, Hackensack (2008)
49. Matte, O.: Supersymmetric Dirichlet operators, spectral gaps, and correlations. Ann. H. Poincaré **7**(4), 731–780 (2006)
50. Pereira, E., Procacci, A., O'Carroll, M.: Multiscale formalism for correlation functions of fermions. Infrared analysis of the tridimensional Gross-Neveu model. J. Stat. Phys. **95**(3–4), 665–692 (1999)
51. Pordt, A.: Mayer expansions for Euclidean lattice field theory: convergence properties and relation with perturbation theory. Desy preprint 85-103, unpublished. Available at http://www-lib.kek.jp/top-e.html (1985)
52. Procacci, A., Pereira, E.: Infrared analysis of the tridimensional Gross-Neveu model: pointwise bounds for the effective potential. Ann. Inst. H. Poincaré Phys. Théor. **71**(2), 129–198 (1999)
53. Procacci, A., Scoppola, B.: On decay of correlations for unbounded spin systems with arbitrary boundary conditions. J. Stat. Phys. **105**(3–4), 453–482 (2001)
54. Procacci, A., de Lima, B.N.B., Scoppola, B.: A remark on high temperature polymer expansion for lattice systems with infinite range pair interactions. Lett. Math. Phys. **45**(4), 303–322 (1998)
55. Rivasseau, V.: From Perturbative to Constructive Renormalization. Princeton Series in Physics. Princeton University Press, Princeton (1991)
56. Ruelle, D.: Cluster property of the correlation functions of classical gases. Rev. Mod. Phys. **36**, 580–584 (1964)
57. Ruelle, D.: Statistical Mechanics: Rigorous Results. Benjamin, New York/Amsterdam (1969)
58. Salmhofer, M.: Renormalization. An Introduction. Texts and Monographs in Physics. Springer, Berlin (1999)
59. Salmhofer, M.: Clustering of fermionic truncated expectation values via functional integration. J. Stat. Phys. **134**(5–6), 941–952 (2009)
60. Simon, B.: The Statistical Mechanics of Lattice Gases, vol. I. Princeton Series in Physics. Princeton University Press, Princeton (1993)
61. Sjöstrand, J.: Correlation asymptotics and Witten Laplacians. Algebra Anal. **8**(1), 160–191 (1996) (translation in St. Petersburg Math. J. **8**(1), 123–147 (1997))

62. Sjöstrand, J.: Complete asymptotics for correlations of Laplace integrals in the semi-classical limit. Mém. Soc. Math. France (N.S.) **83** (2000)
63. Sokal, A.: Mean-field bounds and correlation inequalities. J. Stat. Phys. **28**(3), 431–439 (1982)
64. Sokal, A.D.: Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions. Comb. Probab. Comput. **10**(1), 41–77 (2001)
65. Stanley, R.P.: Enumerative Combinatorics, vol. 2. With a foreword by Gian-Carlo Rota and Appendix 1 by Sergey Fomin. Cambridge Studies in Advanced Mathematics, vol. 62. Cambridge University Press, Cambridge (1999)
66. Wagner, W.: Analyticity and Borel-summability of the perturbation expansion for correlation functions of continuous spin systems. Helv. Phys. Acta **54**(3), 341–363 (1981/1982)
67. Wick, G.C.: The evaluation of the collision matrix. Phys. Rev. (2) **80**, 268–272 (1950)
68. Yoshida, N.: The log-Sobolev inequality for weakly coupled lattice fields. Probab. Theory Relat. Fields **115**(1), 1–40 (1999)
69. Yoshida, N.: The equivalence of the log-Sobolev inequality and a mixing condition for unbounded spin systems on the lattice. Ann. Inst. H. Poincaré Probab. Stat. **37**(2), 223–243 (2001)
70. Zegarlinski, B.: The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice. Commun. Math. Phys. **175**(2), 401–432 (1996)