J Stat Phys (2009) 134: 941–952
DOI 10.1007/s10955-009-9698-0

Clustering of Fermionic Truncated Expectation Values Via Functional Integration

Manfred Salmhofer

Received: 19 September 2008 / Accepted: 20 February 2009 / Published online: 5 March 2009

Abstract I give a simple proof that the correlation functions of many-fermion systems have a convergent functional Grassmann integral representation, and use this representation to show that the cumulants of fermionic quantum statistical mechanics satisfy ℓ^1-clustering estimates.

Keywords Quantum statistical mechanics · Fermions · Functional integrals · Truncated correlations · Decay properties

1 Introduction

Formal coherent-state functional integrals of the type

$$\begin{aligned} Z &= \operatorname{Tr} e^{-\beta(H-\mu N)} \\ &= \int \prod_{\substack{\tau\in[0,\beta)\\ x\in\Lambda}} d\bar\phi(\tau,x)\, d\phi(\tau,x)\, e^{-\int_0^\beta d\tau (\bar\phi(\tau,x)(\frac{\partial}{\partial\tau}+\mu)\phi(\tau,x)-H(\bar\phi(\tau),\phi(\tau)))} \end{aligned} \tag{1}$$

have become ubiquitous in the physics literature. These formulas, and their generalizations to correlation functions, have become powerful heuristic and calculational tools in theoretical physics. However, even when space Λ is replaced by a *finite* set, their mathematical meaning is not obvious because the product measure over a continuum of τ does not exist. The natural way to obtain a mathematically well-defined identity is to use the Lie–Trotter product formula to introduce a discrete Euclidian time τ and recover a rigorous variant of the above formal equation in the limit where the time discretization $\Delta\tau$ vanishes. This involves the solution of a mild, but not completely trivial, ultraviolet problem, which arises because only a first-order derivative with respect to τ appears in the exponent, so that the covariance

M. Salmhofer (✉)
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany
e-mail: m.salmhofer@thphys.uni-heidelberg.de

of the Gaussian integral on the right hand side is not absolutely summable as a function of the frequency ω dual to τ. This problem has been solved both for fermions and bosons. The solution for bosons is very recent [1, 2]; for fermions, a solution that does not require rather obscure multiscale arguments was first given in [5] at zero temperature ($\beta \to \infty$) and then in [7] for all $\beta > 0$ (the basic problems and the background are discussed in [7]).

The main technical difficulty in the study of condensed matter systems is, of course, not this ultraviolet problem, but the infrared problem associated to perturbations of Hamiltonians that do not have a gap above the ground state, entailing a slow, non–summable, decay of the above–mentioned Gaussian covariance at long times and distances, and much effort has gone into understanding its physical and mathematical consequences. On the other hand, proving simple statements about the reduced density matrices of equilibrium states requires a solution of the UV problem as well. This may be the reason why no simple functional integral proof of decay properties of truncated correlations (a.k.a cumulants) seems to have been published.

In this paper I prove the convergence of the above functional integral representation for fermions on a finite lattice. The bounds for the generating function for the reduced density matrices will be uniform in the lattice size, so that the thermodynamic limit can be taken.

This paper is partly motivated by the work of Lukkarinen and Spohn, who control the many-body time evolution on the short kinetic timescale [6] under the hypothesis of an ℓ^1 clustering property of the truncated expectation values. The result of [6] only requires estimates at some fixed inverse temperature β, but not sharp bounds on the dependence on β, so that, using the results of [7], multiscale techniques can be avoided altogether. Inspection of the hypothesis of [6] reveals that the clustering property needed there is a special case of finiteness of the natural norms used in the analysis of fermionic systems in [5] and [7, 9]. Thus the results given here are easy applications of the results of [9] and [7], and the main purpose of this paper is therefore to make them more widely accessible. To this end, I also include in the Appendix a largely streamlined derivation of the functional integral representation along the lines of [8], Appendix B, and a new proof of its convergence using the bounds of [7].

2 Setup and Main Results

Let X be a finite set. In typical applications, $X = \Lambda \times \{1, \dots, n\}$, where Λ is a spatial lattice and the second index labels all internal degrees of freedom, e.g. spin. This choice is not necessary; one could also think of X as indexing any orthonormal basis of a finite-dimensional Hilbert space $\mathcal{H}$, such as the energy eigenbasis of a system in a trapping potential (with an ultraviolet cutoff). Let $\mathcal{F} = \bigwedge \mathcal{H}$ be the antisymmetric Fock space over $\mathcal{H}$; it is also finite-dimensional. Let a^+ and a^- denote the standard fermionic creation and annihilation operators satisfying canonical anticommutation relations.

Let ρ be a density operator of the form $\rho = Z^{-1} \mathrm{e}^{-\beta(H-\mu N)}$ where $H = H_0 + V$ and N is the number operator (we also call $H_0 - \mu N = K_0$). On the finite lattice, all these operators are bounded, and the partition function $Z = Z_X(\beta, \mu) = \mathrm{Tr}_{\mathcal{F}} \mathrm{e}^{-\beta(H-\mu N)}$ is well-defined.

$$\langle A \rangle_{\beta,\mu,X} = \frac{1}{Z_X(\beta,\mu)} \mathrm{Tr}\big(\mathrm{e}^{-\beta(H-\mu N)} A\big) \tag{2}$$

(provided that $Z_X \neq 0$, which will be the case in our applications, where $H = H^*$).

2.1 Normal-Ordered Monomials

The reduced density matrices in the grand canonical state associated to H, β and μ are

$$\gamma_{m,n}(x_1,\dots,x_m;y_1,\dots,y_n)=\left\langle \prod_{i=1}^{m} a^+_{x_i}\prod_{j=n}^{1} a^-_{y_j}\right\rangle_{\beta,\mu,X}. \tag{3}$$

If the Hamiltonian is $U(1)$-charge-invariant, $\gamma_{m,n}\neq 0$ only if $m=n$, in which case I denote it by γ_m.

The reduced density matrices of the state can be obtained by taking derivatives of the following generating functional. Let $(c^+_x)_{x\in X}$ and $(c^-_x)_{x\in X}$ be Grassmann source fields satisfying for all $x,y\in X$ and all $u,v\in\{-1,1\}$

$$c^u_x c^v_y + c^v_y c^u_x = 0,\qquad c^u_x a^v_y + a^v_y c^u_x = 0 \tag{4}$$

for all $x,y\in X$, and define

$$Z(c^-,c^+)=\mathrm{Tr}\big[\mathrm{e}^{-\beta(H-\mu N)}\mathrm{e}^{(c^+,a^+)_X}\mathrm{e}^{(c^-,a^-)_X}\big] \tag{5}$$

where $(c^+,a^+)_X=\int_X \mathrm{d}x\, c^+_x a^+_x$ (here we use a continuum notation $\int_X \mathrm{d}x=\varepsilon\sum_{x\in X}$ where ε can be a suitable power of a lattice spacing). Then

$$\gamma_m(x_1,\dots,x_m;y_1,\dots,y_m)=\frac{\delta^m}{\delta c^+_{x_1}\cdots\delta c^+_{x_m}}\frac{\delta^m}{\delta c^-_{y_m}\cdots\delta c^-_{y_1}}\frac{Z(c^-,c^+)}{Z(0,0)}\bigg|_{c^+=0,c^-=0} \tag{6}$$

with $\frac{\delta}{\delta c^\pm_x}=\varepsilon^{-1}\frac{\partial}{\partial c^\pm_x}$. As will become clear in the following, it is natural to take Grassmann-valued sources.

2.2 General Monomials

In applications, one often wants to get information about the expectation values of monomials where annihilation operators can be to the left of creation operators. This can be included in the present setup: although the two exponentials under the trace in (5) do not commute, they are easily combined into a single one, as follows. Because the c^+ and c^- are Grassmann variables,

$$[c^+_x a^+_x, c^-_y a^-_y]=c^+_x c^-_y\{a^+_x,a^-_y\}=c^+_x c^-_y\delta_X(x,y) \tag{7}$$

which commutes both with $(c^+,a^+)_X$ and $(c^-,a^-)_X$. The Baker–Campbell–Hausdorff formula then implies that

$$\begin{aligned}\mathrm{e}^{(c^+,a^+)_X}\mathrm{e}^{(c^-,a^-)_X}&=\mathrm{e}^{(c^+,a^+)_X+(c^-,a^-)_X+\frac12(c^+,c^-)_X}\\&=\mathrm{e}^{(c^-,a^-)_X}\mathrm{e}^{(c^+,a^+)_X}\mathrm{e}^{(c^+,c^-)_X}.\end{aligned} \tag{8}$$

Thus commuting two such exponentials of source terms produces an additional factor $\mathrm{e}^{-(c^-,c^+)_X}$.

Consider now the expectation value of a monomial $\prod_{i=1}^{2m}a^{\sigma_i}(x_i)$, where $\sigma=(\sigma_1,\dots,\sigma_{2m})\in\{-1,1\}^{2m}$ is an arbitrary sequence of creation/annihilation indices. For

$i \in \{1, \ldots, 2m\}$ with $\sigma_i = +$ let $L_i(\sigma) = \{j < i : \sigma_j = -1\}$ be the set of labels of annihilation operators to the left of i; if $\sigma_i = -1$, set $L_i = \emptyset$. Then by (8)

$$\prod_{i=1}^{2m} \mathrm{e}^{c_i^{\sigma_i} a^{\sigma_i}(x_i)} = \mathrm{e}^{-\sum_i \sum_{j \in L_i(\sigma)} (c_j^-, c_i^+)_X} : \prod_{i=1}^{2m} \mathrm{e}^{c_i^{\sigma_i} a^{\sigma_i}(x_i)} : \tag{9}$$

It is an elementary exercise to show that for quasifree states, the prefactor gives the familiar f_β, $1 - f_\beta$ structure for the expectation values of unordered monomials in quasifree states (used in the context of deriving the quantum Boltzmann equation in [4] and [6]). Here f_β is the Fermi function (see below).

2.3 Truncated Expectations

The truncated expectations (also called cumulants, or connected parts, of the reduced density matrices), are the derivatives of the logarithm of $Z(c^-, c^+)$: let

$$F(c^-, c^+) = \log Z(c^-, c^+) \tag{10}$$

then

$$\gamma_m^T(x_1, \ldots, x_m; x_{m+1}, \ldots, x_{2m}) = \frac{\delta^m}{\delta c_{x_1}^+ \cdots \delta c_{x_m}^+} \frac{\delta^m}{\delta c_{x_{2m}}^- \cdots \delta c_{x_{m+1}}^-} F(c^-, c^+)\Big|_{c^+=0, c^-=0}. \tag{11}$$

By straightforward expansion of the exponential in $Z = \mathrm{e}^F$ and evaluation of the Grassmann derivatives at zero sources c^- and c^+, one can verify that the γ_m^T indeed coincide with the truncated expectations, as defined in [3].

Equation (9) makes evident that for $m \geq 2$, the thus defined truncated expectations γ_m^T are indeed totally antisymmetric under permutations of $x_1, \ldots, x_{2m}$ (as stated in Theorem A.1 of [6]): when taking the expectation value of (9) and then the logarithm, the σ-dependent prefactor gives a quadratic contribution which drops out in the derivatives with respect to the sources c^- and c^+ for $m > 1$. Antisymmetry follows because the product of exponentials is invariant under permutations, and because Grassmann derivatives anticommute. Moreover, (9) implies that the truncated expectation values of order $m \geq 2$ of the sequence of monomials $(\prod_{i=1}^{2m} a^{\sigma_i}(x_i))_{m \in \mathbb{N}}$ are in absolute value independent of the ordering, therefore bounds on the truncated reduced m-particle density matrices imply the same bounds for all truncated expectations (for $m \geq 2$).

2.4 ℓ^1-clustering

Let H_0 be selfadjoint so that $K_0 = H_0 - \mu N = (a^+, \mathcal{E} a^-)_X$ with a hermitian matrix $\mathcal{E}$. Let

$$\mathbb{C}(\tau, E) = -1_{\tau > 0} f_\beta(-E) \mathrm{e}^{-\tau E} + 1_{\tau \leq 0} f_\beta(E) \mathrm{e}^{-\tau E} \tag{12}$$

where $f_\beta(E) = (1 + \mathrm{e}^{\beta E})^{-1}$ is the Fermi function, and let

$$\mathcal{C}_{\tau, \tau'}(\mathcal{E}) = \mathbb{C}(\tau - \tau', \mathcal{E}) \tag{13}$$

be the standard Euclidian-time-ordered free fermion two-point function. $\mathcal{C}$ is an $\mathbb{X} \times \mathbb{X}$-matrix with $\mathbb{X} = [0, \beta) \times X$. Let $\delta > 0$ be a determinant bound for $\mathcal{C}$, as defined in Definition 1.2 of [7], namely δ is such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$

$$\sup_{p_1, \ldots, p_n, q_1, \ldots, q_n \in B} \left| \det\bigl(\langle p_i, q_j \rangle C_{x_i y_j}\bigr)_{1 \leq i, j \leq n} \right| \leq \delta^{2n} \tag{14}$$

(here $B = \{\xi \in \mathbb{C}^n : \|\xi\|_2 \le 1\}$), and let

$$\alpha = \sup_{y \in X} \int_0^\beta \mathrm{d}\tau \int_X \mathrm{d}x \, |\mathcal{C}(\tau, \mathcal{E})|_{x,y} \tag{15}$$

be the decay constant of $\mathcal{C}$ (see Sect. 4.2 of [7]). Both of these constants are finite for any $\beta > 0$ and finite index set X; δ is uniform in β and in X if $f_\beta(\pm\mathcal{E})$ is trace class uniformly in X (see below), and α is uniform in β if $\mathcal{C}$ decays fast enough (for sufficient conditions for this in terms of $\mathcal{E}$, see [7] and below).

The interaction V can be a generic charge-invariant multibody interaction, assumed to be given in normal ordered form

$$V = \sum_{m \in \mathbb{N}} \int \prod_{i=1}^{2m} \mathrm{d}x_i \, v_m(x_1, \ldots, x_{2m}) a^+_{x_1} \cdots a^+_{x_m} a^-_{x_{m+1}} \cdots a^-_{x_{2m}}. \tag{16}$$

For $f_m : X^{2m} \to \mathbb{C}$, let

$$|f_m|_{1,\infty} = \max_j \sup_{x_j} \int \prod_{\substack{i=1 \\ i \ne j}}^{2m} \mathrm{d}x_i \, |f_m(x_1, \ldots, x_{2m})|. \tag{17}$$

For $h > 0$ and a sequence $f = (f_m)_{m \in \mathbb{N}}$ let

$$\|f\|_h = \sum_{m \ge 1} |f_m|_{1,\infty} h^{2m}. \tag{18}$$

Theorem 1 *Let $\omega = 2\alpha\delta^{-2}$, and assume that the interaction V has the properties $V = V^*$ and $\omega\|V\|_{3\delta} \le 1/2$. Then for all $m \ge 2$, γ_m^T obeys ℓ^1-clustering, i.e.*

$$|\gamma_m^T|_{1,\infty} \le 2(m!)^2 \alpha^{2m} \delta^{-2m} \|V\|_{3\delta}, \tag{19}$$

and the same bound holds for truncated expectation values of unordered monomials of degree $2m \ge 4$. For $m = 1$ (and denoting γ_1^T at $V = 0$ by $\gamma_{1,0}^T$)

$$\left|\gamma_1^T - \gamma_{1,0}^T\right|_{1,\infty} \le 2\alpha^2\delta^{-2} \|V\|_{3\delta}. \tag{20}$$

Proof Equations (19) and (20) are proven in Sect. 3. That the truncated expectation values of unordered monomials of degree $2m \ge 4$ satisfy the same bounds as γ_m^T was shown in Sect. 2.3. □

Remarks

1. For a two-body interaction $V = \int_X \mathrm{d}x \int_X \mathrm{d}y : n_x v(x,y) n_y :$, $\|V\|_h = h^4 |v|_{1,\infty}$ is small for small (and summable) v. Thus, when measuring the strength of the interaction, one can regard the coefficient of h^4 as the coupling constant, $\lambda = |v|_{1,\infty}$. For general V, I take the convention that, up to the lowest power of h that appears in it, $\|V\|_h$ is the "coupling constant". With this convention,
   - Equation (19) implies (6.19) of [6] because $(m!)^2 \le (2m)!$
   - Equation (20) implies (6.20) of [6].

The factorial in (19) reflects the property that the radius of analyticity of $F(c^-, c^+)$ as a function of (c^-, c^+) is finite, while the partition function $Z(c^-, c^+)$ itself is an exponentially bounded entire function of (c^-, c^+). The advantage of using truncated functions is, of course, that the radius of convergence for γ_m^T is uniform in $|X|$ and in N, and that the expansion coefficients converge in the limits $|X| \to \infty$ and $N \to \infty$, while the expansion coefficients in Z diverge for $|X| \to \infty$.

2. Similar bounds hold for states that are not charge invariant; the restriction was imposed here mainly for notational simplicity.
3. Estimates for the determinant bound and the decay constant are given in [7] for the translation-invariant case where Λ is a discrete torus of sidelength L, so that the propagator $\mathcal{C}$ has a standard Fourier representation. In particular, in this case, δ is uniform in L. The decay constant α will in general depend on β, but there are a few cases where it does not, in particular if there is an energy gap, i.e. the distance of 0 to the spectrum of $\mathcal{E}$ is bounded below by a fixed positive number.
4. More generally, let $(\varphi_x)_{x\in X}$ be an orthonormal basis of $\mathcal{H}$, and let

$$C_{(\tau,x),(\tau',x')} = \langle \varphi_x | \mathcal{C}(\tau - \tau', \mathcal{E}) \varphi_{x'} \rangle. \tag{21}$$

The spectral representation $\mathcal{E} = \sum_l \epsilon_l |e_l\rangle\langle e_l|$ then gives a natural Gram representation (with $\mathrm{e}^{-\mathrm{i}p\cdot x}$ replaced by $\langle e_l \mid \varphi_x \rangle$ and integration over p replaced by summation over l) that replaces the Fourier representation in Lemma 4.1 of [7]. Proceeding as in the proof of this Lemma, one obtains

$$\delta \le 2 \max_{\sigma=\pm} \sum_l f_\beta(\sigma \epsilon_l) \tag{22}$$

which is bounded uniformly in X if $f_\beta(\pm\mathcal{E})$ is trace class uniformly in X. This effectively requires an ultraviolet cutoff because $f_\beta(-E) \to 1$ as $E \to \infty$. The decay constant depends on the properties of the e_l and, again, on the presence or absence of spectrum of $\mathcal{E}$ near $E = 0$.

5. More detailed bounds can be obtained by the methods described in Sect. 3: by using weighted norms, one can obtain precise decay estimates; by using more details of tree expansions [9] one can also obtain pointwise estimates. All these estimates depend on the decay constant and the determinant bound of the covariance. In multiscale approaches to the low-temperature infrared problem, the bounds given in Sect. 3 recover the full power counting, as already noted in [7].

3 Functional Integral Representation

To analyse connected functions, it is convenient to have a graded algebra in which all even elements commute. This is never the case in the CAR algebra of fermionic operators on Fock space. This is the main motivation for using a Grassmann integral representation of the grand canonical traces.

$Z(c^-, c^+)$ has the following Grassmann integral representation.

Theorem 2 *Let H_0 be selfadjoint so that $K_0 = H_0 - \mu N = (a^+, \mathcal{E} a^-)_X$ with a hermitian matrix $\mathcal{E}$. Let V be any linear map from $\mathcal{F}$ to $\mathcal{F}$.*

For $m \in \{1, \dots, N\}$, let $\tau_m = m\frac{\beta}{N}$ and $R^{(N)}$ be the operator on $\mathbb{C}^N \otimes \mathbb{C}^{|X|}$ given by

$$R^{(N)}_{m,n} = \mathcal{C}_{\tau_n,\tau_m}(\mathcal{E}) \tag{23}$$

with $\mathcal{C}$ given by (13). *Then*

$$Z(c^-, c^+) = Z_0 e^{(c^-, f_\beta(\mathcal{E}) c^+)_X} \lim_{N\to\infty} \left(\mu_{R^{(N)}} * e^{-\mathcal{V}}\right)(\bar{\eta}^{(N)}, \eta^{(N)}) \tag{24}$$

where for $k \in \{1, \ldots, N\}$

$$\bar{\eta}_k^{(N)} = \mathcal{C}^{\mathrm{T}}_{\beta,\tau_k} c^-, \qquad \eta_k^{(N)} = \mathcal{C}_{\tau_k,\tau_1} c^+, \tag{25}$$

(*the* T *denotes the transpose as an operator on* $\mathbb{C}^{|X|}$), $Z_0 = \det(1 + e^{-\beta\mathcal{E}}) = Tr\, e^{-\beta(H_0 - \mu N)}$ *is the partition function for $V = 0$, and*

$$\mathcal{V} = \frac{\beta}{N} \sum_{i=1}^{N} \mathcal{N}(V)(\bar{\psi}_i, \psi_i). \tag{26}$$

Here $\mathcal{N}(V)$ is the normal ordered form of V, and $*$ denotes the standard Grassmann Gaussian convolution integral (*see* [8])

$$(\mu_C * F)(\bar{\phi}, \phi) = \int d\mu_C(\bar{\psi}, \psi)\, F(\bar{\psi} + \bar{\phi}, \psi + \phi) \tag{27}$$

where the integration is over the Grassmann variables $(\bar{\psi}_{k,x}, \psi_{k,x})_{(k,x)\in\{1,\ldots,N\}\times X}$.

Theorem 2 is proven in the appendix.

In the Theorem, we recognize $e^{-\mathcal{W}^{(N)}(R^{(N)},\mathcal{V})} = \mu_{R^{(N)}} * e^{-\mathcal{V}}$ as *Wilson's effective action*, which generates the connected amputated Green function of the time-discretized theory. In [9] and [7], this function was shown to converge uniformly in X, provided that δ and α are uniform in X. The variables $\eta^{(N)}$ and $\bar{\eta}^{(N)}$ on which $\mathcal{W}$ depends contain a $\mathcal{C}$ because Z generates non-amputated functions, where a propagator is associated to each external leg. The sources c appear in the time slices τ_1 and $\tau_N = \beta$, as expected since the monomial "sits" at the fixed time $\tau = 0$.

I briefly recall some of the properties of $\mathcal{W}$ proven in [7, 9]. By Theorem 4.5 of [7], the limit $\mathcal{W}(\mathcal{C}, \mathcal{V}) = \lim_{N\to\infty} \mathcal{W}^{(N)}(R^{(N)}, \mathcal{V})$ exists in $\|\cdot\|_h$ for small enough h. Let $h = \delta$ in that theorem and denote the part of $\mathcal{W}$ that is homogeneous of degree p in $\mathcal{V}$ by $\mathcal{W}(\mathcal{C}, \mathcal{V}; p)$, then

$$\left\| \mathcal{W}(\mathcal{C}, \mathcal{V}) - \sum_{p=1}^{P} \frac{1}{p!} \mathcal{W}(\mathcal{C}, \mathcal{V}; p) \right\|_{\delta,\mathbb{X}} \le \frac{(\omega \|\mathcal{V}\|_{3\delta,\mathbb{X}})^P}{1 - \omega\|\mathcal{V}\|_{3\delta,\mathbb{X}}} \|\mathcal{V}\|_{3\delta,\mathbb{X}} \tag{28}$$

holds for all $P \ge 0$, provided $\omega\|\mathcal{V}\|_{3\delta,\mathbb{X}} < 1$ (for $P = 0$ the empty sum in (28) is zero; this is the only case needed to prove Theorem 1).

The norm $\|\cdot\|_{h,\mathbb{X}}$ is similar to the norm $\|\cdot\|_h$ defined in (18), but with $|\cdot|_{1,\infty}$ in (17) replaced by a $1,\infty$-norm $|\cdot|_{1,\infty,\mathbb{X}}$ in which integrals and suprema run over $\mathbb{X} = [0, \beta) \times X$. Because the interaction V gives rise to a $\mathcal{V}$ that is local in the time $\tau \in [0, \beta)$, $\|\mathcal{V}\|_{h,\mathbb{X}} = \|V\|_h$.

In summary, for small enough $\omega\|V\|_{3\delta}$, $\mathcal{W}$ is analytic in $\mathcal{V}$ and in the fields. In particular, $\mathcal{W}$ is continuous in the fields, and

$$\lim_{N\to\infty} \mathcal{W}^{(N)}(R^{(N)}, \mathcal{V})(\bar{\eta}^{(N)}, \eta^{(N)}) = \mathcal{W}(\mathcal{C}, \mathcal{V})(\bar{\eta}, \eta) \tag{29}$$

with $\bar\eta(\tau) = \mathcal{C}^{\mathrm{T}}_{\beta,\tau} c^-$ and $\eta(\tau) = \mathcal{C}_{\tau,0} c^+$. Let $\mathcal{W}^{(N)}_{2m}$ be the homogeneous part of $\mathcal{W}^{(N)}$ of degree $2m$ in $\bar\eta, \eta$. It generates the connected amputated $2m$-point functions $W^{(N)}_{2m} = \frac{1}{(m!)^2} \frac{\delta^m}{\delta\eta^m} \frac{\delta^m}{\delta\bar\eta^m} \mathcal{W}^{(N)}_{2m}$. Then $W^{(N)}_{2m}$ is given by a tree expansion [7, 9], i.e. an explicit, absolutely and uniformly in N convergent expansion in which the limit $N \to \infty$ can be taken termwise. Denote the limit by W_{2m}. Then, if $\omega \|V\|_{3\delta} \le 1/2$,

$$|W_{2m}|_{1,\infty,\mathbb{X}} \le 2\delta^{-2m} \|V\|_{3\delta} \tag{30}$$

for all $m \ge 1$, and

$$Z(c^-, c^+) = Z_0 \mathrm{e}^{(c^-, f_\beta(\mathcal{E}) c^+)_X} \mathrm{e}^{-\mathcal{W}(\bar\eta,\eta)} \tag{31}$$

so $F = \log Z$ becomes

$$\begin{aligned} F(c^-, c^+) &= \log Z_0 + (c^-, f_\beta(\mathcal{E}) c^+)_X - \mathcal{W}(\bar\eta, \eta) \\ &= \log Z_0 + (c^-, \left[f_\beta(\mathcal{E}) - \mathcal{K}_\beta \right] c^+)_X - \sum_{m \ge 2} \mathcal{W}_{2m}(\mathcal{C}_{\beta,\cdot}\, c^-, \mathcal{C}_{\cdot,0} c^+) \end{aligned} \tag{32}$$

with $\mathcal{K}_\beta = \int \mathrm{d}\tau \int \mathrm{d}\tau' \mathcal{C}_{\beta,\tau} \mathcal{W}_2(\tau, \tau') \mathcal{C}_{\tau',0}$. In (32), the full two-point function is explicit in the quadratic part of the generating function. The analogue of the free fermion formula, obtained by discarding the $\mathcal{W}_{2m}$ for all $m \ge 2$, changes correspondingly. Under the present assumptions, $\mathcal{W}$ is small when $\|\mathcal{V}\|_h$ is small, and hence it is clear that all higher truncated functions are small and $\mathcal{K}_\beta$ is small in $|\cdot|_{1,\infty}$, i.e. the two-point function ($m = 1$) is close to the free one.

Theorem 1 now follows immediately from the above properties of $\mathcal{W}$: Because the arguments of $\mathcal{W}$ in (32) are $\bar\eta = \mathcal{C}_{\beta,\cdot}\, c^-$ and $\eta = \mathcal{C}_{\cdot,0} c^+$,

$$\begin{aligned} \gamma^T_m(x_1, \ldots, x_{2m}) = \int \prod_{k=1}^{2m} \mathrm{d}\tau_k\, \mathrm{d}y_k\, (m!)^2 W_{2m}\left((\tau_1, y_1), \ldots, (\tau_{2m}, y_{2m})\right) \\ \times \prod_{k=1}^{m} \left(\mathcal{C}(\beta, \tau_k)\right)_{x_k, y_k} \left(\mathcal{C}(\tau_{k+m}, 0)\right)_{y_{k+m}, x_{k+m}} \end{aligned} \tag{33}$$

for all $m \ge 2$, hence

$$\frac{1}{(m!)^2} \left|\gamma^T_m\right|_{1,\infty} \le |W_{2m}|_{1,\infty,\mathbb{X}} |\mathcal{C}|^{2m}_{1,\infty,\mathbb{X}} = |W_{2m}|_{1,\infty,\mathbb{X}} \alpha^{2m} \tag{34}$$

by (15). By (30),

$$\left|\gamma^T_m\right| \le 2(m!)^2 \delta^{-2m} \|V\|_{3\delta} \alpha^{2m} \tag{35}$$

which proves (19). For $m = 1$, recall that $\gamma^T_{1,0} = f_\beta(\mathcal{E})$, so $\gamma^T_1 - \gamma^T_{1,0} = \mathcal{K}_\beta$, as defined after (32). Thus, again by (30) and (15)

$$\left|\gamma^T_1 - \gamma^T_{1,0}\right|_{1,\infty} = |\mathcal{K}_\beta|_{1,\infty} \le \alpha^2 |W_2|_{1,\infty,\mathbb{X}} \le 2\alpha^2 \delta^{-2} \|V\|_{3\delta}, \tag{36}$$

which proves (20).

Acknowledgements This work was supported by DFG grants FOR 718 and FOR 723. Part of it was done during a very inspiring summer school at the *Erwin–Schrödinger–Institut* in Vienna. I would like to thank Christian Hainzl, Robert Seiringer, and Jakob Yngvason for the invitation to this summer school, and Herbert Spohn for discussions.

Appendix: Traces and Grassmann Integrals

Lemma 3 *Let $\mathcal{F}$ be a Hilbert space and A and B be bounded linear operators on $\mathcal{F}$. Then*

$$e^{A+B} = \lim_{N\to\infty}\left[e^{\frac{A}{N}}\left(1+\frac{B}{N}\right)\right]^N. \tag{37}$$

Proof This is an easy variant of the standard proof of the Lie product formula, as given, e.g. in [8]. Let $C = e^{(A+B)/N}$ and $D = e^{A/N}(1+B/N)$. Then

$$\max\{\|C\|, \|D\|\} \le e^{\frac{\|A\|+\|B\|}{N}} \tag{38}$$

and by following the proof given in [8], one obtains the estimate

$$\|C^N - D^N\| \le \frac{4}{N}(\|A\|+\|B\|)^2 e^{\|A\|+\|B\|}. \tag{39}$$

□

By assumption, the one-particle Hilbert space is finite-dimensional, i.e. has an ONB indexed by a finite set X. Thus the Fock space $\mathcal{F}$ is finite-dimensional, too, hence all linear operators on $\mathcal{F}$ are bounded and the trace is a continuous linear functional. Therefore by Lemma 3,

$$Z(c^-, c^+) = \lim_{N\to\infty} \tilde{Z}_N(c^-, c^+) \tag{40}$$

where

$$\tilde{Z}_N(c^-, c^+) = \mathrm{Tr}\big[\rho_N e^{(c^+, a_+)} e^{(c^-, a)}\big] \tag{41}$$

and $\rho_N = (C_N D_N)^N$ with $C_N = e^{-\frac{\beta}{N}K_0}$ and $D_N = 1 - \frac{\beta}{N}V$. If V is normal ordered, the same is true for D_N (but not for $e^{-\beta V/N}$). This is the main reason that makes Lemma 3 convenient in the transition to the Grassmann integral done in the following.

By a purely algebraic transformation, $\tilde{Z}_N(c^-, c^+)$ is represented as a finite-dimensional Grassmann integral. Some basic facts needed for this are in Appendix B of [8]. Here I collect only the most important definitions and identities. The *Grassmann symbol* $\mathcal{G}(A)$ of an operator A with normal ordered form $\mathcal{N}(A)(a^+, a^-)$ is defined as

$$\mathcal{G}(A)(\psi_+, \psi_-) = e^{(\psi_+, \psi_-)_X}\mathcal{N}(A)(\psi_+, \psi_-). \tag{42}$$

The product of two operators has Grassmann symbol

$$\mathcal{G}(AB)(\bar{\psi}, \psi) = \int \mathcal{D}_X(\bar{\psi}', \psi')\mathcal{G}(A)(\bar{\psi}, \psi')e^{-(\bar{\psi}', \psi')_X}\mathcal{G}(B)(\bar{\psi}', \psi). \tag{43}$$

The trace is represented by

$$\mathrm{Tr}\, A = \int \mathcal{D}_X(\psi_+, \psi_-)\mathcal{G}(A)(-\psi_+, \psi_-)e^{-(\psi_+, \psi_-)_X}. \tag{44}$$

By these identities, Z can be written as the convolution

$$\tilde{Z}_N(c^-, c^+) = \int \mathcal{D}_X(\bar{\theta}, \theta)\mathcal{G}(\rho_N)(\bar{\theta}, \theta)e^{(\bar{\theta}+c^-, \theta+c^+)}. \tag{45}$$

By the rule for products of Grassmann symbols, (43),

$$\tilde{Z}_N(c^-, c^+) = \int \prod_{k=1}^{N} \mathcal{D}_X(\bar\psi_k, \psi_k) \mathrm{e}^{(\bar\psi_1 + c^-, \psi_N + c^+)_X} \times \prod_{k=1}^{N} \mathcal{G}(C_N D_N)(\bar\psi_k, \psi_k) \prod_{k=1}^{N-1} \mathrm{e}^{-(\bar\psi_{k+1}, \psi_k)_X}. \tag{46}$$

The decomposition $H = H_0 + V$ is chosen such that H_0 is quadratic, so that $K_0 = (a^+, \mathcal{E} a^-)_X$ with a hermitian matrix $\mathcal{E}$. For this special case, the Grassmann symbol is easily obtained by Grassmann Gaussian integration, with the result

$$\mathcal{G}(C_N D_N)(\bar\psi_k, \psi_k) = \mathcal{G}(D_N)({u_N}^T \bar\psi_k, \psi_k) \tag{47}$$

where $u_N = \mathrm{e}^{-\frac{\beta}{N}\mathcal{E}}$ (and T denotes the transpose). The interaction V is assumed to be in normal ordered form, so its Grassmann symbol is straightforwardly obtained by replacing a^+ by $\bar\psi$ and a^- by ψ. Inserting this and changing variables from $\bar\psi_k$ to ${u_N}^T \bar\psi_k$ gives

$$\begin{aligned} \tilde{Z}_N(c^-, \sigma) &= \det \mathrm{e}^{-\mathcal{E}} \int \prod_{k=1}^{N} \mathcal{D}_X(\bar\psi_k, \psi_k) \mathrm{e}^{({u_N}^T \bar\psi_1 + c^-, \psi_N + c^+)_X} \\ &\quad \times \prod_{k=1}^{n} \left(1 - \frac{\beta}{N} V(\bar\psi_k, \psi_k)\right) \prod_{k=1}^{N-1} \mathrm{e}^{-({u_N}^T \bar\psi_{k+1}, \psi_k)_X} \\ &= \mathrm{e}^{(c^-, c^+)_X} \det \mathrm{e}^{-\mathcal{E}} \int \prod_{k=1}^{N} \mathcal{D}_X(\bar\psi_k, \psi_k) \mathrm{e}^{(\bar\psi, Q^{(N)}\psi)} \\ &\quad \times \prod_{k=1}^{n} \left(1 - \frac{\beta}{N} V(\bar\psi_k, \psi_k)\right) \mathrm{e}^{(c^-, \psi_N)_X + (\bar\psi_1, u_N c^+)_X} \end{aligned} \tag{48}$$

with

$$(\bar\psi, Q^{(N)}\psi) = \sum_{k=1}^{N} (\bar\psi_k, \psi_k)_X - \sum_{k=1}^{N-1} (\bar\psi_{k+1}, u_N \psi_k)_X + (\bar\psi_1, u_N \psi_N)_X. \tag{49}$$

That $\bar\psi_1$ couples to ψ_N with the opposite sign means that there is an antiperiodic boundary condition for the fields $\bar\psi$ and ψ. $Q^{(N)}$ acts as a matrix on the "time" indices k and as an operator on the $x \in X$:

$$Q^{(N)}_{m,n} = \delta_{m,n} - u_N \delta_{m-1,n} + u_N \delta_{m,1}\delta_{N,n} = Q^{(0,N)}_{m,n} + u_N \delta_{m,1}\delta_{N,n} \tag{50}$$

$Q^{(0,N)}$ is lower triangular and its inverse is easily found to be $R^{(0,N)}$ where

$$R^{(0,N)}_{kl} = {u_N}^{k-l} \mathbb{1}(k \ge l). \tag{51}$$

The additional term in Q is similar to a rank one perturbation. Using the standard formulas, one obtains $(Q^{(N)})^{-1} = R^{(N)}$ as

$$R^{(N)}_{m,n} = {u_N}^{m-n} \big(\mathbb{1}(m \ge n) - (1 + {u_N}^{-N})^{-1}\big)$$

$$= {u_N}^{m-n} \begin{cases} -(1+{u_N}^{-N})^{-1} & \text{for } m < n \\ (1+{u_N}^{N})^{-1} & \text{for } m \geq n \end{cases} \tag{52}$$

we obtain the formula (23) for $R^{(N)}$. Thus, in summary

$$\tilde{Z}_N(c^-, c^+) = \mathrm{e}^{(c^-,c^+)_X} Z_0^{(N)} \int \mathrm{d}\mu_{R^{(N)}}(\bar{\psi}, \psi)\mathrm{e}^{(c^-,\psi_N)_X + (\bar{\psi}_1, \mathrm{e}^{-\frac{\beta}{N}\mathcal{E}} c^+)_X} \times \prod_{k=1}^{n}\left(1 - \frac{\beta}{N} V(\bar{\psi}_k, \psi_k)\right) \tag{53}$$

where $Z_0^{(N)} = \det \mathrm{e}^{-\mathcal{E}} \det Q^{(N)}$.

It remains to reexponentiate V. To this end, recall from Definition B.11 of [8] the ℓ^1 seminorm $|||\cdot|||_q$ on the Grassmann algebra, namely $|||A|||_q = \sum_{m\geq 1} |A_m|_1 q^m$ with $|A_m|_1 = \int \prod_{i=1}^m \mathrm{d}x_i |A_m(x_1,\dots,x_m)|$. It satisfies the product inequality $|||AB|||_q \leq |||A|||_q |||B|||_q$, and if the covariance C has determinant bound δ, then

$$\left|\int \mathrm{d}\mu_C(\Psi) F(\Psi)\right| \leq |||F|||_\delta \tag{54}$$

(see Theorem B.14 in [8]). Now let

$$\Delta(\Psi) = \prod_{k=1}^{N} \mathrm{e}^{-\frac{\beta}{N}V(\Psi_k)} - \prod_{k=1}^{N}\left(1 - \frac{\beta}{N}V(\Psi_k)\right). \tag{55}$$

By the discrete product rule

$$\prod_{k=1}^{N} A_k - \prod_{k=1}^{N} B_k = \sum_{l=1}^{N} \prod_{k<l} A_k (A_l - B_l) \prod_{k>l} B_k, \tag{56}$$

and by (54) and the triangle and product inequalities,

$$\left|\int \mathrm{d}\mu_C(\Psi)\Delta(\Psi)\right| \leq \|\Delta\|_\delta \leq N \mathrm{e}^{\frac{(N-1)}{N}\beta |||V|||_\delta} |||\mathrm{e}^{-\frac{\beta}{N}V} - 1 + \frac{\beta}{N} V|||_\delta$$
$$\leq N \mathrm{e}^{\frac{(N-1)}{N}\beta |||V|||_\delta} \frac{\beta^2 |||V|||_q^2}{N^2} \mathrm{e}^{\frac{\beta}{N}|||V|||_\delta} = \frac{\beta^2 |||V|||_q^2}{N} \mathrm{e}^{\beta |||V|||_\delta} \to 0 \tag{57}$$

as $N \to \infty$. Note that if X is a lattice of volume Ω with a translation group, $|||G|||_q$ is typically of order Ω for translation invariant elements G of the Grassmann algebra, but this does not matter here because Ω remains fixed as $N \to \infty$.

Thus

$$Z(c^-, c^+) = \lim_{N\to\infty} Z_N(c^-, c^+) \tag{58}$$

where

$$Z_N(c^-, c^+) = \mathrm{e}^{(c^-,c^+)_X} \det(1 + \mathrm{e}^{-\mathcal{E}}) \int \mathrm{d}\mu_{R^{(N)}}(\bar{\psi}, \psi)\mathrm{e}^{(c^-,\psi_N)_X + (\bar{\psi}_1, c^+)_X} \times \mathrm{e}^{-\frac{\beta}{N}\sum_{k=1}^{n} V(\bar{\psi}_k, \psi_k)} \tag{59}$$

with $R^{(N)}$ given in (23). Equation (24) now follows by a standard completion of the square in the integrand.

References

1. Balaban, T., Feldman, J., Knörrer, H., Trubowitz, E.: A functional integral representation for many boson systems I: The partition function. Ann. Henri Poincaré **9**, 1229–1273 (2008)
2. Balaban, T., Feldman, J., Knörrer, H., Trubowitz, E.: A functional integral representation for many boson systems II: Correlation functions. Ann. Henri Poincaré **9**, 1275–1307 (2008)
3. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics I and II. Springer, Berlin (2002)
4. Erdős, L., Salmhofer, M., Yau, H.-T.: On the quantum Boltzmann equation. J. Stat. Phys. **116**, 367–380 (2004)
5. Feldman, J., Knörrer, H., Trubowitz, E.: Convergence of perturbation expansions in fermionic models, part 1: nonperturbative bounds. Commun. Math. Phys. **247**, 195–242 (2004)
6. Lukkarinen, J., Spohn, H.: Not to normal order—notes on the kinetic limit for weakly interacting quantum fluids. arXiv:0807.5072
7. Pedra, W., Salmhofer, M.: Determinant bounds and the Matsubara UV problem of many-fermion systems. Commun. Math. Phys. **282**, 797–818 (2008)
8. Salmhofer, M.: Renormalization. Springer, Berlin (1998)
9. Salmhofer, M., Wieczerkowski, C.: Positivity and convergence in fermionic quantum field theory. J. Stat. Phys. **99**, 557–586 (2000)