J Stat Phys (2008) 131: 575–595
DOI 10.1007/s10955-008-9514-2

On the Study of Jamming Percolation

M. Jeng · J.M. Schwarz

Received: 31 August 2007 / Accepted: 27 February 2008 / Published online: 26 March 2008

Abstract We investigate kinetically constrained models of glassy transitions, and determine which model characteristics are crucial in allowing a rigorous proof that such models have discontinuous transitions with faster than power law diverging length and time scales. The models we investigate have constraints similar to that of the knights model, introduced by Toninelli, Biroli, and Fisher (TBF), but differing neighbor relations. We find that such knights-like models, otherwise known as models of jamming percolation, need a "No Parallel Crossing" rule for the TBF proof of a glassy transition to be valid. Furthermore, most knights-like models fail a "No Perpendicular Crossing" requirement, and thus need modification to be made rigorous. We also show how the "No Parallel Crossing" requirement can be used to evaluate the provable glassiness of other correlated percolation models, by looking at models with more stable directions than the knights model. Finally, we show that the TBF proof does not generalize in any straightforward fashion for three-dimensional versions of the knights-like models.

Keywords Percolation · Jamming · Kinetically constrained models · Glassy transition

1 Introduction

The puzzle of glass-forming systems has remained sufficiently elusive over the years such that even the puzzle pieces themselves have changed shape. For example, the puzzle piece of the lack of a growing lengthscale may now have to be modified since a growing lengthscale can perhaps be extracted from a higher-order correlation function [1]. A less recent change in puzzle pieces is the distinction by Angell between fragile and strong glasses, where the excess of low-frequency vibrational models is more pronounced in strong glasses than in fragile ones [2]. One piece of the puzzle that has remained constant over the years, however,

M. Jeng (✉) · J.M. Schwarz
Physics Department, Syracuse University, Syracuse, NY 13244, USA
e-mail: mjeng@physics.syr.edu

J.M. Schwarz
e-mail: jschwarz@physics.syr.edu

is the dramatic slowing down of the dynamics of the particles near the glass transition. More precisely, a supercooled liquid's viscosity can increase by fourteen orders of magnitude as the temperature is decreased near a "working" definition of the glass temperature [2].

There have been many beautiful ideas developed over the years to explain this dynamical slowing down. An excellent survey of these many ideas can be found in [3] and [4]. One particular idea relevant for this work that has emerged over the years is to understand whether glassy dynamics can be understood as arising from steric constraints on the particles alone [5]. One of the simplest such examples is the Kob-Andersen model [6]. It is motivated by the caging of particles, ultimately observed in larger scale systems such as colloidal glasses [7, 8]. The Kob-Andersen model is a hard-core lattice gas model, but with the constraint that a particle can only hop to an adjacent site if and only if it has less than a certain number of neighbors, m, both before and after the move. Early simulations of the Kob-Andersen model on the hypercubic lattice for relevant values of m appeared to find a dynamical phase transition at a nontrivial critical density [6]. However, subsequent mathematically rigorous results found that the phase transition does not occur until the fully packed state [9, 10]. This corresponds to a zero-temperature glass transition. TBF proved this by showing that at any monomer density, there were mobile cores that could diffuse at sufficiently long time scales.

While the hypercubic version of the Kob-Andersen model does not exhibit a finite-temperature glass transition, the mean field version does [9, 10]. Since it is still up for debate whether or not mean field is relevant for physical systems, one can ask whether or not there exists another finite-dimensional, kinetically constrained model that exhibits a finite temperature glass transition. While the Kob-Andersen (and the Fredrickson-Andersen [11]) models, are elegant in their simplicity, there are indeed two more involved kinetically constrained models in two dimensions that can be proven to exhibit a finite-temperature glass transition. These two models have been dubbed the spiral model [12, 13] and the sandwich model [14]. Both models exhibit an unusual phase transition in that the fraction of frozen particles jumps discontinuously at the transition, typical of a first-order phase transition. However, as the transition is approached from below, there exists a crossover lengthscale that diverges faster than a power law in $T - T_g$. The crossover length, Γ, distinguishes between squares of size $L \ll \Gamma$, which are likely to contain a frozen cluster, and squares of size $L \gg \Gamma$, for which the probability of containing a frozen cluster is exponentially unlikely. Given this combination of a discontinuity in the fraction of frozen particles, and a faster than power law diverging length scale, the transition has unique characteristics.[1]

Models such as the spiral and sandwich models are proof in principle that further exploration of kinetically constrained models in finite-dimensions may be fruitful, in particular, because should an ideal glass transition exist, it may indeed be of unusual character. More specifically, the Edwards-Anderson order parameter should be discontinuous at the transition, yet accompanied by rapidly diverging time scales [15, 16].

To analyze these kinetically constrained models, one can map a kinetically constrained model onto a percolation problem. More specifically, the immobile particles of the kinetically constrained model are mapped to occupied sites in the percolation model. When immobile particles/occupied sites percolate through the system, the system is frozen into a glassy

[1] The Kosterlitz-Thouless transition also exhibits unusual characteristics. There is a discontinuity in the spin wave stiffness, or the free energy cost to applying a gradient, accompanied with an exponentially diverging correlation length. However, there is no magnetization, or local order parameter, that goes to zero at the transition. The jamming percolation models have a simple order parameter, the fraction of sites in the infinite cluster.

phase. Therefore, a glass transition is mapped to a percolation transition. However, the percolation problem is not the usual one where each site is initially occupied with an independent probability ρ. Instead, the steric constraints in the kinetically constrained models map to constraints on the occupation of sites in the percolation models, so that the occupation of sites is now correlated, as opposed to being independent of one another. The simplest model of correlated percolation is k-core/bootstrap percolation, in which an occupied site must have at least k occupied neighbors [17, 18]. Occupied sites that do not satisfy this stability requirement are removed, and this condition is applied repeatedly, until all remaining sites are stable. This correlated percolation model can be mapped to the Fredrickson-Andersen model, which is a lattice spin model for the glass transition where a spin can flip only if its number of nearest neighbor down spins (mobile particles) is equal to or larger than some integer, f, with $f = Z - k + 1$, where Z is the coordination number of the lattice [5, 19].

One might think that kinetically constrained models/correlated percolation systems would be easy to numerically simulate, and that construction of rigorous proofs would be nothing more than an interesting problem for mathematicians. However, while simulating these systems is easy, extracting their properties in the infinite system size limit is much harder. Just as with the Kob-Andersen model, initial numerical simulations of k-core percolation for certain values of k found evidence of first-order phase transitions at nontrivial critical densities, and of second-order phase transitions in a different universality class than normal percolation [20–22]. However, subsequent mathematically rigorous analyses found that the critical point is $\rho_c = 1$ for those k [23–25]. Because the critical point approaches unity very slowly in the limit of infinite system size, the simulations on finite-size systems were misled as to the location of the true critical point, thereby highlighting the importance of rigorous results for these models as well. For a review of k-core percolation, see [26].

There is another finite-dimensional system presumably exhibiting an unusual transition. It is the jamming transition in repulsive soft spheres. Numerical simulations of soft spheres show a critical point at which the average coordination number jumps discontinuously to a universal, isostatic value. But quantities such as the shear modulus and the deviation of the average coordination number from its isostatic value show a nontrivial power law behavior in the vicinity of the critical point [27–29]. Recent experiments on two-dimensional photoelastic beads support this notion of a mixed transition [30].

Interestingly, it has been conjectured that the physics of granular systems, colloidal systems, and glassy systems are of a similar character [31]. The mean field behaviour of glassy models also points towards this unification [32]. Furthermore, experimental evidence of caging in another two-dimensional granular system also supports this notion [33]. The question of finite-dimensional glassy models being quantitatively similar to the repulsive soft sphere system is still being investigated. Certainly the spiral and sandwich models show that, qualitatively, one can have a glassy system exhibiting an unusual finite-dimensional transition. However, they do not appear to be in the same class as the jamming system, since the order parameter exponent is unity just above the transition in the jamming percolation models, but is one-half in the jamming system.

To explore the possible link between jamming and glassy systems in terms of a finite-temperature glass transition, TBF initially introduced the knights model, a model of correlated percolation similar to the spiral model [15], and called it a model of jamming percolation. In fact, the spiral, knights, and sandwich models are all models of jamming percolation. In this paper, we expand on our earlier work [14], in which we introduced the sandwich model, by presenting the details of the proof of an unusual transition in this model. This proof is based on modifying the proof developed by TBF in [15], where it was originally misapplied to the knights model. The proof was later discussed in more detail in [34], and more recently applied to the spiral model in [13, 35]. We also introduce further generalizations of models of jamming percolation to demonstrate that the phenomenon of a finite-dimensional transition is indeed somewhat generic. In doing so, we show that the TBF proof only gives a rigorous derivation of these novel properties if a "No parallel crossing" rule holds. This rule says that two similarly-oriented directed percolation chains cannot cross without having sites in common. The effect of this rule is that one directed percolation-like process cannot be used to locally stabilize the other. For models such as the sandwich model, which satisfy this "No Parallel Crossing" rule, but fail a "No perpendicular crossing" rule, the TBF proof works only with some modification.

In this paper we address for which models the TBF proof can be used to show a glassy transition, and for which they cannot. The question of for which models a glassy transition can be rigorously proven is important, because numerical and qualitative arguments are often misleading, as with the already-mentioned numerical simulations of k-core percolation and the Kob-Andersen model. Of course, for models in which the TBF proof fails, it is possible that the system is still glassy, and that this can be shown by some other, as yet undiscovered, rigorous methods. Or, one can resort to numerical methods to shed some light on the situation. For example, while the TBF proof does not apply to TBF's original model, the knights model, they presented analytical/numerical arguments in [12] suggesting that the knights model nevertheless has a glassy transition that is in the same universality class as the spiral model. However, here, given past misinterpretations of numerical data, we focus on extending mathematical methods initially developed by TBF, with the ultimate goal of understanding the physics of glassy systems. We should point out that given that there are several detailed papers on the TBF proof [13, 34], and that our work is based on modifying that proof, we will refer to them quite often, as opposed to making this paper self-contained. Finally, we will discuss connections between jamming percolation and force-balance percolation, another correlated percolation model inspired by granular systems, where analytical/numerical evidence points toward an unusual transition in finite dimensions [36, 37].

2 The Class of Models

We consider a class of models that generalizes the knights model, the earliest of the jamming percolation models. The class of models is defined on the two-dimensional square lattice. Initially, each site is occupied with an independent probability ρ. Each site has four neighboring sets, and each set contains two sites. The four sets are labelled as the northeast, northwest, southeast, and southwest neighboring sets. (The sites in those sets only lie in precisely those compass directions for the knights model, but we continue to label the four sets in this manner for all our correlated percolation models until Sect. 5.) To be stable, an occupied site must either have (1) at least one northeast neighbor and at least one southwest neighbor, or (2) at least one northwest neighbor and at least one southeast neighbor. All other occupied sites are unstable, and are vacated. This culling process is then repeatedly applied—sites that were previously stable may become unstable by earlier cullings—until all remaining sites are stable. The neighboring sets of the original knights model introduced by TBF [15] are shown in Fig. 1. Figure 2 shows the neighboring sets in the sandwich model, which we introduced in [14], and Fig. 3 shows the neighboring sets in the spiral model, which TBF introduced in [12].

In these models, stable sites must either be part of a chain running from the northeast to the southwest, or part of a chain running from the northwest to the southeast. In the final configuration, all sites must be stable, so any chain must either continue forever (to the

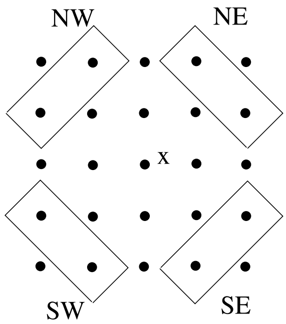


Fig. 1 The knights model neighbors

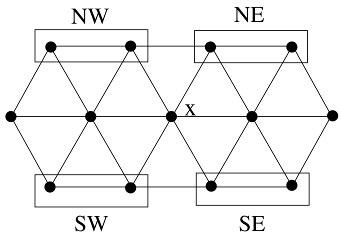


Fig. 2 The sandwich model neighbors

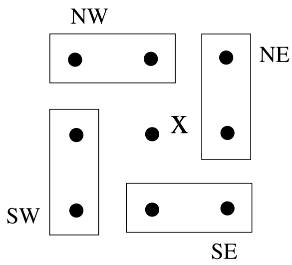


Fig. 3 The spiral model neighbors

boundary), or terminate in a chain of the other type (this latter case is called a "T-junction"). Any sites left after the culling procedure must thus be connected by an infinite series of chains (or, in a finite system, connected by a series of chains to the distant boundary), so asking for the critical probability at which an infinite cluster first appears is the same as asking for the minimum probability at which some sites remain unculled in the infinite size limit.

If sites could only be stable by having northeast and southwest neighbors, then every stable site would be part of a northeast-southwest chain on a particular sublattice—the sublattice extending to the northeast of the site for the sandwich model is shown in Fig. 4. Chains on this sublattice are isomorphic to infinite chains in directed percolation, which has a well-studied second-order phase transition [38]. However, in these models there is an additional mechanism by which sites can be made stable—that is, by having northwest and southeast neighbors. Adding an extra way for a sites to be stable can only possibly depress the critical probability, so we immediately see that for these models $\rho_c \leq \rho_c^{\mathrm{DP}}$.

It will turn out to be useful to divide these models into classes based on two properties. We define a model as having "No Parallel Crossing" property if whenever two northeast-southwest chains (or two northwest-southeast chains) intersect, they must have sites in common—we abbreviate this a "property A." And we define a model as having a "No

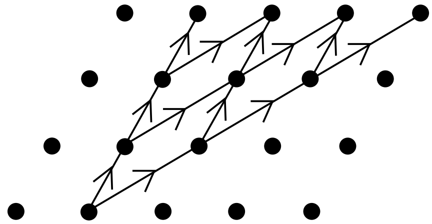


Fig. 4 A sublattice in the sandwich model

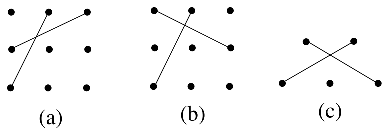


Fig. 5 (**a**) Failure of property A ("No Parallel Crossings") in the knights model. (**b**) Failure of property B ("No Perpendicular Crossings") in the knights model. (**c**) Failure of property B in the sandwich model

Table 1 Properties satisfied for the knights, sandwich, and spiral models

Model	Property A	Property B
Knights	No	No
Sandwich	Yes	No
Spiral	Yes	Yes

Perpendicular Crossing" property, or property B, if whenever a northeast-southwest and northwest-southeast chain intersect, they must have sites in common. Table 1 shows which properties each of the three models possesses. Examples of where the properties fail for each model are shown in Fig. 5.

Our analysis here is based to a large extent on the claimed TBF proof of a glassy transition for the knights model [15, 34]. Their proof consisted of two parts. First, they claimed to show that the critical point of the knights model is exactly the same as that for directed percolation. Second, once this was done, they were able to use well-known results on directed percolation (assuming a well-tested conjecture about anisotropic rescaling in directed percolation) to show that this model has a glassy transition—specifically, they were able to find structures with a finite density at the critical point of directed percolation, and to show that just below this critical point, the crossover length and culling times diverged.

Our analysis of this more general class of models shows that the TBF proof that $\rho_c = \rho_c^{\mathrm{DP}}$ is only valid for models satisfying property A; so it works for the sandwich and spiral models, but fails for the knights model. The second part of their proof, showing a glassy transition, implicitly assumes property B. The spiral model exhibits property B and hence the TBF method of proof carries through.[2] However, we show that the proof can be modified to work for models that fail to have property B. The spiral *and* the sandwich models thus have provably glassy transitions, while the knights model does not.

[2]This observation was kindly pointed by C. Toninelli and G. Biroli in [35] after an initial draft of our paper was made available making the opposite claim.

3 Identifying the Critical Point

We sketch the TBF proof that $\rho_c^{\text{knights}} = \rho_c^{\text{DP}}$, which will let us understand why property A is sufficient, and most likely necessary to the result. The key to the TBF proof is to show that voids (clusters of empty sites) of particular shapes have a finite probability of growing forever. For example, for the diamond-shaped void in the sandwich model, shown in Fig. 6, if the key site labelled y is vacant, it will trigger the removal of all the sites marked with stars, increasing the void size by one. (The corresponding void for the knights model appears as Fig. 1c of [15].) If this process repeats forever, with such key sites repeatedly removed, this void will grow to infinity.

The TBF proof for the knights model is based on the claim that the key sites of the knights model, located at the corners of octagonal voids of size L, can only be stable if part of a directed percolation chain of $\mathcal{O}(L)$. Once this claim is granted, the rest of the proof is straightforward. For $\rho < \rho_c^{\text{DP}}$, such long chains are exponentially suppressed, and thus for a large void, the vertices at the corners of the void are exponentially likely to be culled. Summing up the relevant probabilities, this results in a finite probability that the void will grow to infinity. For an infinite lattice, it is thus certain that there will be at least one void that grows to infinity, showing that all sites are culled below ρ_c^{DP}. Since $\rho_c^{\text{knights}} \le \rho_c^{\text{DP}}$, this is supposed to show that $\rho_c^{\text{knights}} = \rho_c^{\text{DP}}$.

The claim that sites at the corners of voids can only be stable if part of a long chain of $\mathcal{O}(L)$ is only valid for models with property A. A counterexample to this claim for the knights model can be seen in Fig. 7. This counterexample also shows why property A is necessary and sufficient for this claim to be valid. To be stable, the site must be part of a northeast-directed chain; pick the lowest northeast-directed chain coming out of the corner site y. If that chain stops before reaching length $\mathcal{O}(L)$, it must terminate in a northwest-southeast chain. Since the northeast chain is not as long as a wall of the void, the new southeast-directed chain will eventually hit the void (thus resulting in the culling of all chains, and the corner site y), unless it hits a T-junction. That new T-junction results a second northeast-southwest chain, which will eventually reach the first northeast chain, as in Fig. 7. For models with property A, the two chains will intersect, contradicting the original assumption that we chose the lowest northeast-directed chain out of y. Thus, by contradiction, for models with property A, the first northeast chain must be $\mathcal{O}(L)$ for y to be stable, and it indeed follows that $\rho_c = \rho_c^{\text{DP}}$.

What about for models such as the knights model, that lack property A? Is it possible that despite this counterexample to the claim, ρ_c^{knights} is actually equal to ρ_c^{DP}, for some other reason? While we do not have a mathematically rigorous proof that $\rho_c^{\text{knights}} \neq \rho_c^{\text{DP}}$, we present an argument that the two are almost certainly unequal. We present our arguments in the context of the knights model, but they generalize to other models that lack property A.

Consider the substructure in Fig. 8. All sites in it are stable under the knights model culling rules, except for the two sites at its ends, and those sites will become stable if the

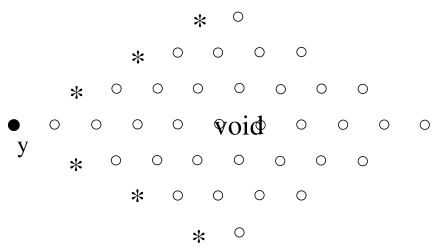


Fig. 6 A void in the sandwich model. If the site y is unstable, its removal will cause the culling of all the sites marked with *stars*. The corresponding void for the spiral model is a diamond, regular in shape

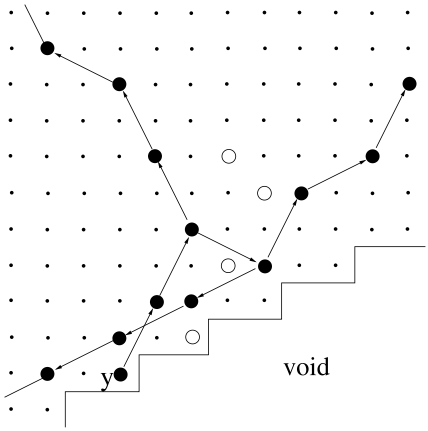


Fig. 7 A counterexample to the claim in [15, 34] that for y to be stable it must be part of a long, uninterrupted chain to the northeast. *Large solid circles* are occupied sites, and *large empty circles* are vacant sites. All sites in the void are vacant. All other sites can be either occupied or unoccupied. *Arrows* are drawn from each occupied site to indicate the neighboring sites that make it stable

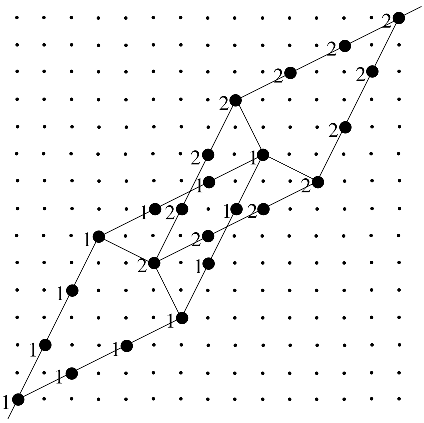


Fig. 8 A substructure that depresses the critical point of the knights model. Numbers indicate sites in the same sublattice

substructure is attached between two northeast-southwest chains. Furthermore, there is no northeast-southwest chain internal to the substructure connecting the two ends. The substructure is internally stabilized by northwest-southeast links.

This means that the substructure can act as a "rest stop." Northeast-southwest chains can have breaks in their paths, connected by this substructure. Just below ρ_c^{DP}, long northeast-southwest chains almost form an infinite structure. They are almost linked, so a few extra connections, through these substructures, should create an infinite cluster even below ρ_c^{DP}. So we expect that $\rho_c^{\mathrm{knights}} < \rho_c^{\mathrm{DP}}$.

We can make the argument more formal by considering the following modification of the directed percolation problem, which we call jumping directed percolation. As with normal directed percolation, we occupy sites on the square lattice with probability ρ, and connect each site with directed bonds to its neighbors to the north and east. However, now we define an additional way for sites to be connected. We divide the lattice into blocks of size 9×9, and for each block, if the two hollow squares of sites shown in Fig. 9 are occupied, we with probability s connect the two hollow squares with a directed bond from the southwest square to the northeast square. The critical point of this model is a function of s: $\rho_c^{\mathrm{Jump}}(s)$.

By repeating Fig. 8 three times, to create diamonds connecting sublattice #1 to #2 to #3 and then back to #1, we obtain a structure that links two separated diamonds on sublattice

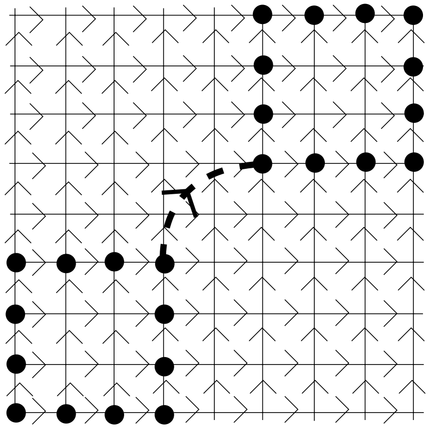


Fig. 9 A configuration in "jumping directed percolation" that receives an additional connection with probability s

#1, through sites in the other two sublattices. So if we restrict ourselves to looking at sites in sublattice #1, and the directed percolation structures on that sublattice, sites that appear disconnected may be connected by these sites in sublattices #2 and #3. The structure obtained by repeating Fig. 8 three times has 24 sites in sublattices #2 and #3, each of which is occupied with probability ρ, and has 24 sites in sublattice #1. The sites in sublattice #1 in this repeated structure map onto the occupied sites in Fig. 9. So with $s=\rho^{24}$, s gives the probability of having appropriate "hidden" occupied sites in sublattices #2 and #3 that connect and make stable the two hollow squares. Infinite chains in the jumping directed percolation model are infinite stable clusters in the knights model, and thus $\rho_c^{\text{knights}} \le \rho_c^{\text{Jump}}(\rho^{24}) \le \rho_c^{\text{DP}}$.

However, since jumping directed percolation is just directed percolation with an extra connection process, it is reasonable to expect that $\rho_c^{\text{Jump}}(s) < \rho_c^{\text{DP}}$ for all $s>0$, implying $\rho_c^{\text{knights}} < \rho_c^{\text{DP}}$. While this argument is not mathematically rigorous, it is strongly suggestive, particularly when we recall previous results on enhancements in percolation by Aizenman and Grimmett [39]. Their work showed that if percolation on a lattice was "enhanced" by adding, for specified subconfigurations, extra connections or occupations with probability s, this would *strictly* decrease the critical probability, for any $s>0$, so long as the enhancement was essential. Essential enhancements were defined as those such that a single enhancement could create a doubly-infinite path where none existed before. See [39] for a more rigorous and precise statement of the results on enhancements, and [40] for a general discussion of enhancements. The results of [39] were obtained for undirected percolation, so are not directly relevant for the jumping directed percolation model considered here, but they are analogous enough to strongly suggest that $\rho_c^{\text{Jump}}(s) < \rho_c^{\text{DP}}$ for all $s>0$. It is difficult to see how such adding such a new route for paths to infinity could leave the critical probability completely unchanged.

4 Property B

For models satisfying property A, we have $\rho_c = \rho_c^{\text{DP}}$, but we still need to check that the TBF proof that the transition is glassy (discontinuous with a diverging crossover length) is valid.

The TBF proof of a glassy transition implicitly assumes that the knights model has property B. For example, to show discontinuity, they construct a configuration that has a finite density at ρ_c^{DP}—see Fig. 2b of [15]. This figure, and others like it, implicitly assume property B, because they are based on drawing overlapping rectangles in independent directions,

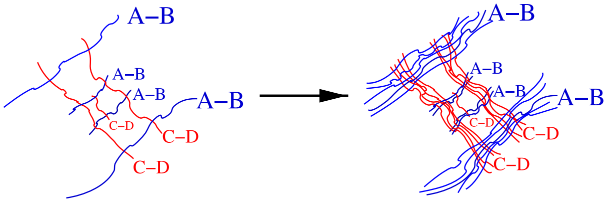


Fig. 10 The modification in the TBF discontinuity structure for models failing the "No Perpendicular Crossing" rule (property B)

and assuming that if paths in these intersecting rectangles cross, they must stabilize each other (form a T-junction). The resulting frozen structure is shown on the left side of Fig. 10. However, if a model lacks property B, the paths can cross without stabilizing each other.

The knights model does not satisfy property A, so whether or not it satisfies property B is a moot point. But what about the sandwich model, which satisfies property A, but not property B? The TBF proof as it stands is not immediately valid in these cases. Nevertheless, it turns out that for such models, the TBF proof can be made to work by a modification of their structures.

The basic idea of the modification is as follows. The TBF proof of a glassy transition is based on drawing structures consisting of sets of overlapping rectangles, showing that there is a sufficiently high probability (finite for the proof of discontinuity, and approaching 1 for the proof of diverging crossover length) that each rectangle has a spanning path in the desired direction, and then using property B to conclude that the intersecting paths form a frozen cluster. For models that lack property B, we use the same figures as in the TBF proof (e.g. Figs. 2a and 2b of [15]), but pick the rectangle sizes large enough that there is a high probability that each rectangle has *multiple* spanning paths ($\mathcal{O}(L^{1-z})$ for a rectangle of $\mathcal{O}(L)$), each occurring in a disjoint parallel subrectangle. Then in each place where the TBF proof assumes a T-junction based on property B, we will have a northeast (northwest) path crossing many northwest (northeast) paths. The probability that no T-junction occurs turns out to decay exponentially with the number of northwest (northeast) paths. In Fig. 10 we show how the discontinuity structure of the TBF proof (from Fig. 2a of [15]) is modified by this procedure.

To implement these ideas, we need the following proposition, which says that sufficiently large rectangles of size $aL \times L$ are very likely at the critical point to have many chains connecting the sides of length aL:

Proposition 1 *For $L \to \infty$ there exists $c > 0$ and $r > 0$ s.t.*

$$\begin{aligned}\mu^{\rho_c}_{L,aL}(\neg\exists \lfloor rL^{1-z}\rfloor \textit{ disjoint northeast occupied clusters},\\ \textit{occurring in disjoint parallel subrectangles},\\ \textit{connecting the sides of length } aL)\\ \le \exp(-cL^{1-z}).\end{aligned}$$

Here z is the dynamical exponent, which has been numerically found to be approximately 0.63 in two dimensions [41]. In [15], TBF used the fact that a single large rectangle of size $aL \times L$ was very likely to have a single crossing path. This proposition uses similar arguments to find $\mathcal{O}(L^{1-z})$ paths.

Proof We divide the box of size L by aL into aL^{1-z} parallel disjoint subrectangles, each of size $L \times L^z$. Assuming the conjecture of anisotropic scaling in directed percolation, each subrectangle has a probability $q > 0$ of having a path connecting the two sizes of length L^z, contained within that subrectangle. The expected number of crossings is qaL^{1-z}, and for any $r < qa$ the probability of having less than rL^{1-z} crossings decays exponentially in L^{1-z}. □

The TBF proof of discontinuity shows that for certain structures of rectangles, there is a nonzero probability that each rectangle has a suitable "event," and that, assuming property B, the existence of each event (a rectangle crossing) results in a stable structure at the critical point. Now, with our modified proposition, each "event" is the presence of multiple crossings (in disjoint parallel subrectangles) in each rectangle, rather than single crossings.

Without property B, this does not guarantee a frozen cluster. However, we see in Appendix A that when a northeast (northwest) path crosses n northwest (northeast) paths, in n disjoint subrectangles, the probability of not forming a single T-junction decays exponentially in n. This is physically obvious, since for large subrectangles, the probability of each T-junction in each subrectangle is essentially independent; however, since the probabilities are not truly independent, more work is needed to make this rigorous. The details of the proof are relegated to Appendix A. More generally, the arguments in Appendix A show that we can treat the probabilities of T-junctions in different subrectangles as independent, when establishing an upper bound on the probability. We will use this throughout this section to multiply such probabilities as if they were independent.

Assuming property B, the TBF proof shows that the probability of having suitable events in each rectangle is nonzero at the critical point. We now need to show that, even without property B, this results in a finite probability of an appropriate set of T-junctions. In the TBF discontinuity structure, shown in Fig. 2b of [15], there are two infinite sequences of rectangles, $\mathcal{R}_i$, $i \geq 1$. Each pair of rectangles $\mathcal{R}_i$ is twice as large as the rectangles $\mathcal{R}_{i-1}$. In [15], the "suitable events" needed for a stable spanning structure are crossing paths in each rectangle. We modify this so that the "suitable events" are those of Proposition 1—each $\mathcal{R}_i$ should have at least $k2^{i(1-z)}$ crossings parallel to its long direction, where k is some positive constant.

The arguments in Appendix A show that this results in a T-junction with probability $(1-(1-r)^{k2^{i(1-z)}})$, for some positive r and k.

Starting at the origin, the probability of forming appropriate T-junctions off to infinity can then be seen to be

$$\big(1-(1-r_1)^{k2^{1-z}}\big)^2 \prod_{i=2}^{\infty}\big(1-(1-r_2)^{k2^{i(1-z)}}\big)^2, \tag{1}$$

for some positive r_1, r_2, and k. This product converges to a positive number, so the transition is proven to be discontinuous (subject to assumption of the well-tested conjecture of an anisotropic critical exponent in directed percolation).

The proof of the diverging crossover length can be made to avoid the assumption of property B by a similar modification of the TBF structures. Again, we begin by repeating the TBF structures, with the set of parallel rectangles in Fig. 2a of [15]. In that picture, if every rectangle has a spanning path, *and* the paths all intersect, there will be a spanning frozen cluster. TBF consider the case where each rectangle has sides of order the directed percolation parallel correlation length, $\xi_{||}$. They then show that c_3 and c_4 can be chosen such that if the system size is $L < c_4\xi_{||}\exp[c_3\xi_\ell(p)^{1-z}]$, the probability that each rectangle is occupied by a spanning cluster approaches 1 as $L \to \infty$, $\rho \to \rho_c^-$. If property B were to hold, this would result in T-junctions that would create a frozen structure, and show that the crossover length diverges as $\rho \to \rho_c^-$.

We no longer have property B; but instead, by using Proposition 1, we can choose the rectangle sizes such that each rectangle is occupied by "many" spanning clusters (with "many" defined by Proposition 1). Given this, we can start at an arbitrary rectangle, and then work our way out, looking for T-junctions to create a spanning frozen structure. We will only fail to create a frozen structure if at some point we reach intersecting rectangles where one spanning path in one rectangle crosses many spanning paths in the other rectangle, but without creating a T-junction. By the arguments in Appendix A, the probability of this occurring decays exponentially in cL^{1-z} for some $c > 0$. So even with $\mathcal{O}(L/\xi_{||})^2$ intersections, the probability that we ever fail in this process goes to 0 as $L \to \infty$ and $\rho \to \rho_c^-$, and we are essentially guaranteed a frozen structure. This shows that the crossover length diverges as we approach the critical point, with the same lower bound that TBF found.

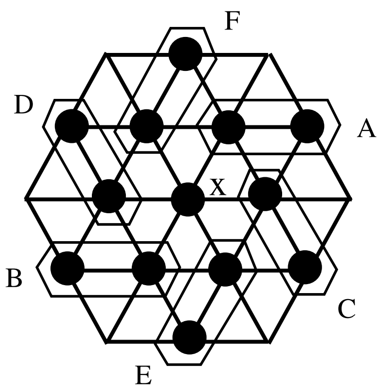
5 Pinwheel Model and 8-Spiral Model

The models we have discussed so far have two possible ways in which a site can be stable, and varying neighboring relations for the culling relations. However, we can also define generalizations in which there are three or more ways in which a site can be stabilized. For example, for the Pinwheel model, shown in Fig. 11, the condition for a site to be stable is that it (1) have neighbors in both the sets A and B, or (2) have neighbors in both the sets C and D, or (3) have neighbors in both the sets E and F. This gives a site three possible directions for stabilizing chains. Similarly, Fig. 12 shows a model in which there are four possible directions for a site to be stable, such that there is an extra "or": for the sets G and H. We denote this model the 8-spiral model.

Despite the extra ways in which sites can be made stable, the TBF proof of a glassy transition is still valid, because property A holds for both the pinwheel model and the 8-spiral model. That is, in both of these models if two A-B chains (or two C-D chains, or two E-F chains, or two G-H chains) cross, they must have sites in common. This turns out to be sufficient to show that there will be a void that grows forever in the infinite system limit.

For the sandwich and spiral models, we needed to show that a stable site at the corner of a diamond-shaped void has to be part of a directed percolation-like chain (DP-like chain) of order the size of the void. For the pinwheel model, we consider hexagonal voids, and show that stable sites at the corners of these voids must be "associated" with a long DP-like chain, where "associated" will be defined by the construction below. Then, just as for the sandwich and spiral models, for $\rho < \rho_c$, long DP-like chains are exponentially suppressed,

Fig. 11 Neighboring relations for the pinwheel model



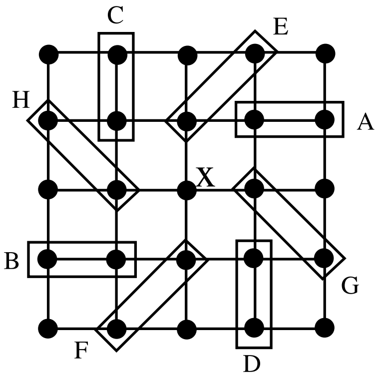


Fig. 12 Neighboring relations for the 8-spiral model in which there are four processes by which a site can be made stable. For a site to be stable it must have neighbors in A and B, or C and D, or E and F, or G and H

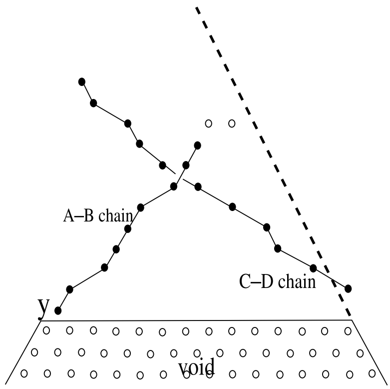


Fig. 13 The top of a hexagonal void in the Pinwheel model, and examples of the A-B and C-D chains discussed in the text

giving voids a finite probability to grow forever, showing that the infinite system is empty for $\rho < \rho_c$.

Consider the site y at the corner of the hexagonal void in Fig. 13. Look at all A-B chains coming out of y, and pick the lowest possible chain; in other words, look for successive A neighbors, and if a site has two possible A neighbors, pick the lower one. If that chain reaches the dashed line in Fig. 13, we have a DP-like chain of order the size of the void, and are done. Otherwise, this A-B chain must terminate either in a C-D chain, or an E-F chain. If it terminates in an E-F chain, that chain must terminate in a C-D chain, which must then cross the original A-B chain. (Note that the E-F chain cannot terminate in an A-B chain, since by the "No Parallel Crossing" rule, the new A-B chain would intersect the first A-B chain, and contradict our assumption that we chose the lowest A-B chain coming out of the site y.) So if the A-B chain coming out of y does not reach the dashed line, it must either terminate in a C-D chain or cross a C-D chain (the latter case is shown in Fig. 13). Pick the lowest of all the C-D chains that cross or intersect our A-B chain. This C-D chain must reach the dashed line, using the same logic as before (if it terminated in an A-B chain, that would intersect the first A-B chain, and contradict the assumption that we chose the lowest A-B chain coming out of y; while if it terminated in an E-F chain, that E-F chain would have to turn into either an A-B or C-D chain before reaching the void, again resulting in a contradiction). Since the A-B and C-D chains that we have constructed cross, and together include both y and the dashed line, at least one of the chains must be of order the size of the void. A similar construction can also be used for the 8-spiral model using a diamond-shaped void.

Having established where the critical point is, we follow TBF, and as before consider an infinite sequence of two types of rectangular regions, $\mathcal{R}_{1,2}$ (see Fig. 10), one of which contains AB paths, and the other of which contains EF paths. (Any pair of types of paths—AB/CD, AB/EF, or CD/EF—are permissible, as are any of the 6 possible pairs for the 8-spiral model.) Again, the sequence is constructed such that the AB paths and the EF paths are mutually intersecting with a frozen backbone that contains the origin, and the TBF proof simply carries through with the additional modification we have introduced for models that do not obey property B. All in all, since having more neighboring relations gives more ways for an occupied site to be stable (without depressing the DP critical point), the TBF constructions of discontinuous percolation structures at the critical point simply carry through. Just as for the sandwich model, the TBF proof needs to be modified to deal with the lack of property B. One needs at least two intersecting rectangular regions in which the probability for two "transverse" blocking directions each undergoing a directed percolation transition independently is nonzero.

6 Models in Higher Dimensions

Consideration of the "No Parallel Crossing" rule shows that for higher-dimensional generalizations of the knights model, the TBF proof cannot be generalized in a straightforward manner to show provably glassy transitions. We will show that the "No Parallel Crossing" rule never holds, so the critical point is always depressed below that of directed percolation.

Specializing to three dimensions for convenience, let $\mathbf{A} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ consist of three linearly independent 3-vectors, and $\mathbf{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ consist of three linearly independent 3-vectors with $\mathbf{A} \neq \mathbf{C}$. Also, define $\mathbf{B} = \{-\vec{a}_1, -\vec{a}_2, -\vec{a}_3\}$ and $\mathbf{D} = \{-\vec{c}_1, -\vec{c}_2, -\vec{c}_3\}$. For an occupied site $\vec{r}$ to be stable, it must have either (1) occupied neighbors from both $\vec{r} + \mathbf{A}$ and $\vec{r} + \mathbf{B}$ or (2) occupied neighbors from both $\vec{r} + \mathbf{C}$ and $\vec{r} + \mathbf{D}$. The first condition we denote the A-B condition, the second, the C-D condition. Then, just as for the sandwich and spiral models, there are two directed percolation processes by which a site can be made stable (A-B chains and C-D chains), and one might think that for an appropriate set of neighboring relations the TBF proof could be used to show a glassy transition.

However, it turns out that because these models never satisfy the "No Parallel Crossing" rule, the critical point is always depressed below that of three-dimensional directed percolation, and the TBF proof cannot be directly generalized for these models.

Property A says that two chains running in similar directions cannot cross without having sites in common. For two-dimensional models, this can be required by the topology and neighboring relations. However, in three dimensions, the topology always makes it easy for two directed chains to miss each other, and so no three-dimensional generalization of property A can be satisfied, regardless of the neighboring relations. Furthermore, if the two chains miss each other, then the buttressing of each type of chain does not occur and the resulting transition may be continuous.

If we only enforced the A-B condition, then we would just have three dimensional directed percolation (modulo finite clusters). However, for the model defined above, there is an extra way to be stable, resulting in $\rho_c < \rho_c^{\mathrm{DP}}$. Consider a *finite* structure with the following properties: (1) all occupied sites are stable under the culling rules except occupied sites $\vec{r}_i$ and $\vec{r}_f$, (2) $\vec{r}_i$ has an occupied neighbor in $\vec{r} + \mathbf{B}$ and $\vec{r}_f$ has an occupied neighbor in $\vec{r} + \mathbf{A}$, and (3) there is no AB path connecting $\vec{r}_i$ and $\vec{r}_f$. This structure is analogous to Fig. 8 for the knights model. We relegate to Appendix B the proof of the existence of such a finite structure.

By the arguments in Sect. 3, we expect these structures to depress the critical point below that of three-dimensional directed percolation. They will make stable some occupied sites that were unstable under just the A-B condition. Slightly below ρ_c^{DP}, the system is about to percolate using the A-B condition alone. The substructures act as extra, local bonds, joining up long A-B paths and pushing the system above the critical point, just as in the knights model. Again, arguments similar to these have been made rigorous by Aizenmann and Lebowitz in the case of undirected percolation [25].

While a rigorous proof along the lines of the TBF proof cannot be done for three-dimensional models, this does not mean that such three-dimensional models fail to have a glassy transition. Further numerical and analytical studies of three-dimensional models, similar to the one conducted in [12] for the knights model, would be useful.

7 Discussion

The discovery of a two-dimensional percolation transition, where the sudden emergence of a discontinuous backbone coincides with a crossover length diverging faster than a power law, is recent, and of great interest for glassy systems, jamming systems, and phase transitions in general. Unusual transitions have been found previously in mean field systems of a slightly different nature, but not in finite dimensions. And while the finite-dimensional transition is discontinuous, it is not driven by nucleation, as with ordinary discontinuous transitions, but instead by a scaffolding of many tenuous directed percolation paths occurring simultaneously to form a bulky structure. We have shown that property A is required for the proof that $\rho_c = \rho_c^{\mathrm{DP}}$, but property B is not. All that is needed to prove that the transition is discontinuous (once $\rho_c = \rho_c^{\mathrm{DP}}$ is established) is a finite probability for two transverse percolating structures to intersect, to prevent each other from being culled. Therefore, one can construct other models, such as the pinwheel and 8-spiral models, that exhibit a similar transition in two dimensions. The phenomena is not as specific as might seem at first glance. However, such a buttressing mechanism in dimensions higher than two is more difficult because it is more difficult for percolating *paths* to intersect and form a buttressing, bulky structure.

Models like the knights model, where ρ_c is most likely less than ρ_c^{DP}, provide physicists, mathematicians, and computer scientists with a motivation to study new models of correlated percolation—models that are not isomorphic to directed percolation, but quite possibly in the same universality class. Once this avenue is pursued further, one can then easily extend the class of models for which a finite-temperature transition can be rigorously shown. To begin, it would be interesting to consider a directed percolation model in two dimensions where the number of nearest neighbors is greater than two. For example, if the number of nearest neighbors was increased to four, would the percolation transition still be in the same universality class as directed percolation? If so, as is presumably the case, then one could construct a jamming percolation model with sets larger than two sites. These jamming percolation models would then be isomorphic to the next-neighbor directed percolation models and then one could use results from directed percolation to prove a percolation transition. The results on the knights model in [12] suggest that even though the TBF proof cannot be directly applied, further development of mathematical tools should be able to enlarge this new universality class even beyond the scope of this paper.

There exists another class of correlated percolation models called force-balance percolation. The first model in this class was defined in [36]. Other force-balance percolation models are currently being constructed and studied [37]. The force-balance percolation models differ from the jamming percolation models in that (1) the sets, such as A, B, etc., are overlapping and (2) the "or" between pairs of sets is changed to "and". Given these differences, the methods of proof used here cannot be easily applied. Numerical results indicate that the transition is discontinuous with a nontrivial "correlation length" exponent, indicating that the transition may not the garden-variety discontinuous transition. The force-balance models are perhaps less artificial in that they mimic force-balance by requiring that an occupied site (i.e. a particle) have occupied neighbors to its left and right, as well as its top and bottom, in order to be stable. However, little has been rigorously proven about them. To make progress along these lines would be useful.

What about the lack of finite stable clusters in models of jamming percolation? Recent numerical work on another correlated percolation model, k-core percolation, with $k=4$ on the four-dimensional hypercubic lattice, appears to exhibit an *ordinary*, discontinuous percolation transition driven by nucleation [42]. Finite clusters exist in this model, unlike in the jamming percolation models. Therefore, in the jamming percolation models, there can be no surface tension between the percolating and nonpercolating phases, which is typical of an ordinary discontinuous transition. If finite clusters are allowed in a correlated percolation model, one might guess that the unusual nature of the transition would be destroyed. However, finite clusters, other than individually floating particles, do not appear in the jamming transition of granular particles. Otherwise, the packing would not be static. So it is unclear whether the existence of finite clusters pertains to the jamming of granular particles. This is also the case for the glass transition.

If more of an analogy between jamming and models of jamming percolation is to be made in finite dimensions (setting aside the matter of the critical dimension of jamming), a model where the fraction of sites participating in the infinite cluster increases smaller than linearly just above the transition must be found. Furthermore, the existence of a jamming percolation model with a universal jump in the number of occupied sites at the transition that "naturally" emerges as opposed to being externally imposed [43], is yet another necessary quest if a jamming percolation model of jamming is to be found.

Finally, models of correlated percolation, such as the sandwich and spiral models, tell us that there do indeed exist kinetically constrained models of glassy dynamics that exhibit unusual phase transitions in finite dimensions. Therefore, this avenue of exploration for understanding possible finite-temperature glass transitions in finite-dimensions remains open. Since our work helps to clarify which jamming/correlated percolation models can be rigorously shown to have an unusual finite-temperature glass transition with a particular set of properties, other models exhibiting possibly other unusual behaviours can hopefully be more easily developed in the near future.

Appendix A: Proof that Multiple Crossings Are Exponentially Unlikely to Avoid Creating T-junctions

In this appendix we justify the claim made in Sect. 4 that given n crossings, the probability that no crossing results in a T-junction decays at least exponentially in n. This would be immediately true if each crossing resulted in an independent probability of a T-junction. So what we show is that these crossings, by occurring in disjoint subrectangles, can be effectively treated as independent (in establishing an upper bound on the probability).

Fig. 14 Multiple crossings, resulting in at least one T-junction with high probability. Crossings are drawn with *open circles* to emphasize that these crossings may or may not result in sites common to the crossings paths

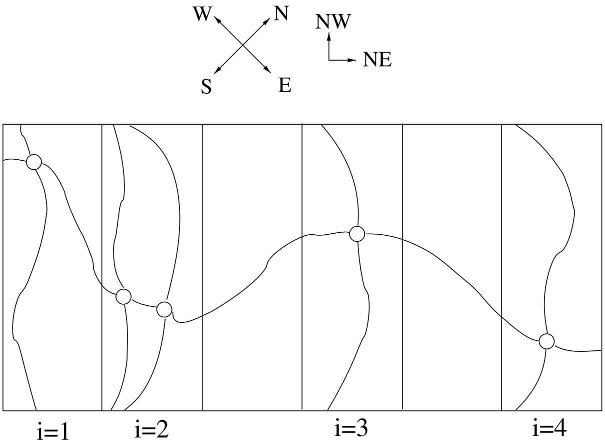
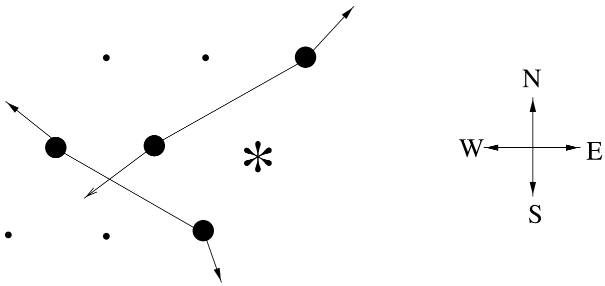


Fig. 15 The only configuration in the sandwich model where northeast and northwest chains cross, but the northeast part of the northeast chain is not stabilized by a T-junction. If the site marked with a *star* was occupied, this would create a T-junction



The relevant picture is shown for $n = 4$ in Fig. 14. There are n disjoint rectangles, labeled by i, $1 \leq i \leq n$, each of which has at least one northwest spanning path. We call this event H, and conditionalize upon the occurrence of H. Each northwest path must cross the northeast spanning path, but without property B, these crossings do not necessarily result in T-junctions, where by a T-junction we mean specifically a site in common between the paths that stabilizes the northeast part of the northeast path. Let G_i be the event that at least one crossing in rectangle i forms a T-junction.

Now for a configuration with $\neg G_i$, look at the sites in the vicinity of the crossing (if there is more than one crossing, we choose one by an arbitrary ordering of possible crossing locations). Restricting ourselves first to the sandwich model, Fig. 15 shows the only way that G_i can fail to happen. The site labeled by a star must be vacant, and if that site is made occupied, the new configuration is in G_i. So a local change in the vicinity of the crossing can always create a T-junction. This local change induces a mapping f_i from the set of states with $\neg G_i$ to a subset of the set of states with G_i. The mapping is one-to-one onto this subset, and for any state S with $\neg G_i$, the probability of the configuration $f_i(S)$ is $\rho/(1-\rho)$ times the probability of the configuration S, where ρ is the site occupation probability. Thus

$$\mu^{\rho}(\neg G_i \mid H) \leq \frac{1-\rho}{\rho}\mu^{\rho}(G_i \mid H). \tag{A1}$$

More generally, if we want to consider other variations of the knights model that lack property B, we need only that for any crossing without a T-junction, some local configuration of changes in a bounded region around the crossing can create a T-junction. The induced mapping can be many-to-one, so long as the "many" is bounded (which follows automatically from the restriction that the configuration changes occur in a bounded region around the crossing). This will more generally give

$$\mu^{\rho}(\neg G_i \mid H) \leq c\mu^{\rho}(G_i \mid H), \tag{A2}$$

for some $0 < c < 1$.

Since the configuration changes only take place within a rectangle i, they do not affect whether or not we have G_j for $j \neq i$, and we can write the above inequality for a state where we specify whether these other G_j occur. For example, for $i = 3$ we might write

$$\begin{aligned} &\mu^{\rho}(G_1 \cap \neg G_2 \cap \neg G_3 \cap G_4 \cdots \cap G_n \mid H) \\ &\quad \leq c\mu^{\rho}(G_1 \cap \neg G_2 \cap G_3 \cap G_4 \cdots \cap G_n \mid H), \end{aligned} \tag{A3}$$

with the same c as above. This can be intuitively thought of as treating the different probabilities of forming T-junctions as independent. Repeatedly using (A3) in different subrectangles, we find that

$$\mu^{\rho}\left(\bigcap_{i=1}^{n} \neg G_i \mid H\right) \leq \frac{1}{(1+c)^n}. \tag{A4}$$

We are thus exponentially unlikely to have no T-junctions.

Appendix B: Existence of Finite Structures in Three-Dimensional Models

In this appendix we prove the existence of the finite structures discussed in Sect. 6. Before giving the formal proof, we sketch the qualitative idea behind the construction of these structures. In the structure for the knights model, shown in Fig. 8, there are two parallelograms consisting of A-B (northeast-southwest) chains. The two parallelograms are parallel to each other, and are connected and stabilized by C-D (northwest-southeast) chains. In two dimensions, such a figure can only be constructed by having some chains cross each other, and this results in a long A-B chain connecting the two ends, unless the model violates property A. However, in three dimensions, the C-D chains can always be run in a direction independent of the plane of the A-B parallelograms, so such a substructure can always be formed, regardless of the neighboring relations.

We now formalize this argument. Recalling that $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, and $\vec{c}_1$ are four three-dimensional vectors, with the first three being linearly independent and $\vec{c}_1$ not equal to any of the first three, there must exist $n \neq 0$, and m_1, m_2, m_3, with at least two nonzero m_i, such that

$$\sum_{i=1}^{3} m_i \vec{a}_i = n\vec{c}_1. \tag{B1}$$

If all of the m_i are positive, then it is easy to make the desired structure. Let $\vec{F} \equiv \sum_{i=1}^{3} m_i \vec{a}_i = n\vec{c}_1$. Then make the structure in Fig. 16, where each of the vectors $\vec{F}$ represents an A-B chain of length $\sum_{i=1}^{3} m_i$, and nc_i represents a C-D chain of length n. This structure has the desired properties. If all of the m_i are negative, we simply replace **C** with **D** and use the same argument.

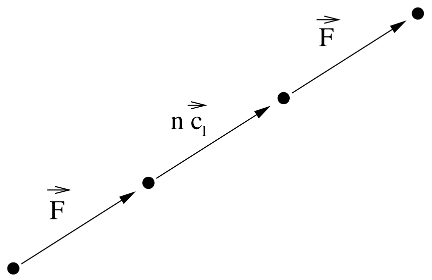


Fig. 16 The structure in the case where $\vec{c}_1 = \sum_{i=1}^{3} m_i \vec{a}_i$ with all $m_i > 0$

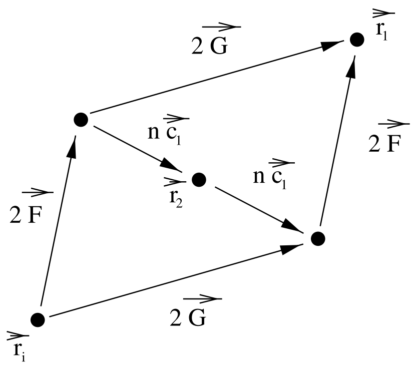


Fig. 17 A substructure

If some m_i are positive, and some m_i are negative, we redefine the m_i, and rewrite (B1) as

$$\sum_{i=1}^{3} m_i \vec{a}_i = \sum_{i=1}^{3} \tilde{m}_i \vec{a}_i + n\vec{c}_1, \tag{B2}$$

where n, all m_i, and all $\tilde{m}_i$, are positive, and for any i, either m_i or $\tilde{m}_i$ is zero. Then define $\vec{F} \equiv \sum_{i=1}^{3} m_i \vec{a}_i$ and $\vec{G} \equiv \sum_{i=1}^{3} \tilde{m}_i \vec{a}_i$. $\vec{F}$ and $\vec{G}$ are both nonzero and linearly independent.

We can now make the structure shown in Fig. 17. The vectors $2\vec{F}$ and $2\vec{G}$ represent A-B chains of length $2\sum_{i=1}^{3} m_i$ and $2\sum_{i=1}^{3} \tilde{m}_i$. In these chains every site is stable by the A-B condition except for $\vec{r}_i$ and $\vec{r}_1$. The vectors $n\vec{c}_1$ are chains of length n in which every site is stable by the C-D condition.

Next, since $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ form a complete basis for three-dimensional space, and $\{\vec{F}, \vec{G}\}$ are only two vectors, there exists a vector $\vec{H} = \sum_{i=1}^{3} p_i \vec{a}_i$, with all $p_i > 0$, such that $\vec{H}$ is linearly independent of $\{\vec{F}, \vec{G}\}$. (Note: this is where the three-dimensional case differs from the two-dimensional case. For models such as the spiral and sandwich models, the two vectors $\vec{F}$ and $\vec{G}$ already span the space.) We can then make a second copy of Fig. 17, displaced from the original by $\vec{H}$, as shown in Fig. 18.

We can now check that in Fig. 18 all sites except the start and end sites, $\vec{r}_i$ and $\vec{r}_f$, are stable under the culling rules, and that $\vec{r}_i$ and $\vec{r}_f$ have neighbors from **B** and **A**, respectively. It remains to check that there is no A-B chain connecting $\vec{r}_i$ to $\vec{r}_f$ in this structure. There is no obvious such chain, but depending on the vectors $\vec{a}_i$ and $\vec{c}_1$, it is possible that there are some sites in the $\vec{F}$, $\vec{G}$, and $\vec{H}$ chains by chance are separated by a vector $\vec{a}_i$, inadvertently forming an A-B chain between $\vec{r}_i$ and $\vec{r}_f$. However, if this is the case, we can create new larger structure, simply by multiplying n, all m_i, all $\tilde{m}_i$, and all p_i by the same multiplicative constant. The structure thus grows larger, while the vectors $\vec{a}_i$ stay the same, so for a suffi-

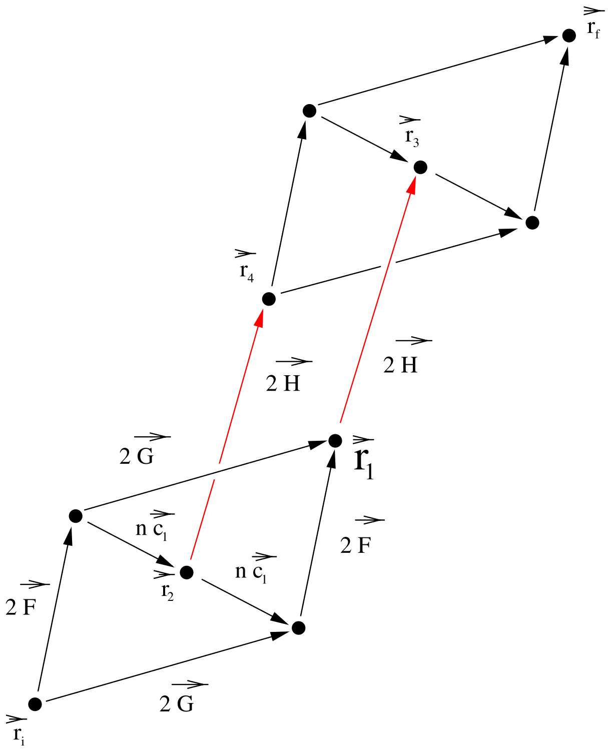


Fig. 18 The full structure with the desired properties

ciently large multiplicative constant, it is impossible for the different chains in the structure to be adjacent by $\vec{a}_i$ connections. We thus have a structure with the desired properties.

References

1. Berthier, L., Biroli, G., Bouchaud, J.P., Cipelletti, L., et al.: Science **310**, 1797 (2005). For earlier hints of a diverging length scale, see Menon, N., Nagel, S.R.: Phys. Rev. Lett. **74**, 1230 (1995)
2. Angell, C.A.: Science **267**, 1924 (1995)
3. Debenedetti, P.G.: Metastable Liquids: Concepts and Principles. Princeton University Press, Princeton (1996)
4. Kob, W. In: Barrat, J.-L., Feigelman, M., Kurchan, J. (eds.) Slow Relaxations and Nonequilibrium Dynamics in Condensed Matter. Springer, New York (2003)
5. Ritort, F., Sollich, P.: Adv. Phys. **52**, 219 (2003)
6. Kob, W., Andersen, H.C.: Phys. Rev. E **48**, 4364 (1993)
7. Weeks, E.R., Weitz, D.A.: Phys. Rev. Lett. **89**, 095704 (2002)
8. Weeks, E.R., Weitz, D.A.: Chem. Phys. **284**, 361 (2002)
9. Toninelli, C., Biroli, G., Fisher, D.S.: Phys. Rev. Lett. **92**, 185504 (2004)
10. Toninelli, C., Biroli, G., Fisher, D.S.: J. Stat. Phys. **120**, 167 (2005)
11. Fredrickson, G.H., Andersen, C.H.: Phys. Rev. Lett. **53**, 1244 (1984)
12. Toninelli, C., Biroli, G., Fisher, D.S.: Phys. Rev. Lett. **98**, 129602 (2007)
13. Toninelli, C., Biroli, G.: arXiv:0709.0378
14. Jeng, M., Schwarz, J.M.: Phys. Rev. Lett. **98**, 129601 (2007)
15. Toninelli, C., Biroli, G., Fisher, D.S.: Phys. Rev. Lett. **96**, 035702 (2006)
16. Leonard, S., et al.: J. Stat. Mech.: Theory and Expt. P07107 (2007)
17. Chalupa, J., Leath, P.L., Reich, G.R.: J. Phys. C **12**, L31 (1979)
18. Pittel, B., Spencer, J., Wormald, N.: J. Comb. Theory Ser. B **67**, 111 (1996)

19. Sellitto, M., Biroli, G., Toninelli, C.: Europhys. Lett. **69**, 496 (2005)
20. Kogut, P.M.: J. Phys. C Solid State Phys. **14**, 3187 (1981)
21. Branco, N.S., de Queiroz, S.L.A., dos Santos, R.R.: J. Phys. C Solid State Phys. **19**, 1909 (1986)
22. Adler, J., Stauffer, D.: J. Phys. A.: Math. Gen. **23**, L1119 (1990)
23. van Enter, A.C.D.: J. Stat. Phys. **48**, 943 (1987)
24. Schonmann, R.H.: Ann. Probab. **20**, 174 (1992)
25. Aizenman, M., Lebowitz, J.L.: J. Phys. A **21**, 3801 (1988)
26. Adler, J.: Physica A **171**, 453 (1991)
27. O'Hern, C.S., Langer, S.A., Liu, A.J., Nagel, S.R.: Phys. Rev. Lett. **88**, 075507 (2002)
28. O'Hern, C.S., Silbert, L.E., Liu, A.J., Nagel, S.R.: Phys. Rev. E **68**, 011306 (2003)
29. Silbert, L.E., Liu, A.J., Nagel, S.R.: Phys. Rev. Lett. **95**, 098301 (2005)
30. Majumdar, T.S., et al.: Phys. Rev. Lett. **98**, 058001 (2007)
31. Liu, A.J., Nagel, S.R.: Nature **396**, 21–22 (1998)
32. Kurchan, J.: Adv. Complex Systems **4**, 363 (2001)
33. Marty, G., Dauchot, O.: Phys. Rev. Lett. **94**, 015701 (2005)
34. Toninelli, C., Biroli, G.: J. Stat. Phys. **126**, 731 (2007)
35. Biroli, G., Toninelli, C.: arXiv:0709.0583
36. Schwarz, J.M., Liu, A.J., Chayes, L.Q.: Europhys. Lett. **73**, 560 (2006)
37. Jeng, M., Liu, A.J., Schwarz, J.M.: in preparation
38. Hinrichsen, H.: Adv. Phys. **49**, 815 (2000)
39. Aizenman, M., Grimmett, G.R.: J. Stat. Phys. **63**, 817 (1991)
40. Grimmett, G.: Percolation, 2nd edn. Springer, Berlin (1999)
41. Buldyrev, S.V., et al.: Phys. Rev. A **45**, R8313 (1992)
42. Rizzo, T., Parisi, G.: arXiv:cond-mat/0609777
43. Duxbury, P.M., et al.: Phys. Rev. E **59**, 2084 (1999)