On Steady Distributions of Kinetic Models of Conservative Economies

Daniel Matthes · Giuseppe Toscani

Abstract We analyze the large-time behavior of various kinetic models for the redistribution of wealth in simple market economies introduced in the pertinent literature in recent years. As specific examples, we study models with fixed saving propensity introduced by Chakraborti and Chakrabarti (Eur. Phys. J. B 17:167–170, 2000), as well as models involving both exchange between agents and speculative trading as considered by Cordier et al. (J. Stat. Phys. 120:253–277, 2005) We derive a sufficient criterion under which a unique non-trivial stationary state exists, and provide criteria under which these steady states do or do not possess a Pareto tail. In particular, we prove the absence of Pareto tails in pointwise conservative models, like the one in (Eur. Phys. J. B 17:167–170, 2000), while models with speculative trades introduced in (J. Stat. Phys. 120:253–277, 2005) develop fat tails if the market is "risky enough". The results are derived by a Fourier-based technique first developed for the Maxwell-Boltzmann equation (Gabetta et al. in J. Stat. Phys. 81:901–934, 1995; Bisi et al. in J. Stat. Phys. 118(1–2):301–331, 2005; Pareschi and Toscani in J. Stat. Phys. 124(2–4):747–779, 2006) and from a recursive relation which allows to calculate arbitrary moments of the stationary state.

Keywords Econophysics · Boltzmann equation · Wealth and income distributions · Pareto distribution

1 Introduction

In recent years, various kinetic models of Boltzmann type have been introduced to study the evolution of the distribution of wealth in a simple market economy [9, 10, 13, 17, 25, 26,

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41]. In most of these models, the market is represented as an ideal gas, agents correspond to molecules, and each trading event between two agents is considered to be an elastic—or rather *money conserving*—collision of two molecules. The founding idea is that a trading market composed of a sufficiently large number of agents can essentially be described using the same laws of statistical mechanics which govern the behavior of a physical system composed of many interacting particles.

In these models, the knowledge of the large-wealth behavior of the steady state density is of primary importance, since it determines *a posteriori* if the model fits data of real economies. More than a hundred years ago, the Italian economist Vilfredo Pareto [35] first quantified the large-wealth behavior of the income distribution in a society and found it to follow a power-law distribution. More precisely, if $f(w)$ denotes the density of agents with wealth w, and w is sufficiently large, then

$$\int_w^\infty f(w_*)\,dw_* \sim w^{-\mu}. \tag{1}$$

Pareto mistakenly believed that power laws apply to the whole distribution with a universal exponent μ approximatively equal to 1.5. Later, Mandelbrot proposed a weak Pareto law that applies only to high incomes [31]. The exponent μ, usually called the Pareto index, was initially found to have a value between 1 and 3. Numerous investigations with real data during the last ten years revealed that the tails of the income distribution indeed behave as indicated above, and the value of the Pareto index varies between 1 and 2.5 (USA $\sim$ 1.6, Japan $\sim$ 1.8–2.2 [17, 18, 29]). Typically, the weak Pareto law applies to the richest 10% of the population—which possesses about 40% of the total wealth of that country. The mid- and low-income population, in fact the majority (90% or more), follow a different distribution which is debated to be either Gibbs [17, 26, 28] or log-normal [16].

In any kinetic description, the dynamics is defined through the microscopic interactions. In the econophysical context, the model is specified by the rules according to which agents perform their trades. A seemingly important and very sensible ingredient is the *saving factor*. This means that each of the interacting individuals keeps a certain fraction of his current wealth, and only puts the remaining part into the trade—at the risk of losing it. The first extensive studies of this class of models are probably due to Chakraborti and Chakrabarti [10], and to Chatterjee, Chakrabarti and Manna [11]. Indeed, some of their numerical experiments reproduce a wealth distribution similar to that of a real economy. Mainly two variants to introduce saving propensities have been studied: In the first, all agents have the same fixed saving factor [10], while in the second, the saving propensity has a quenched random distribution [11]. Numerous simulations have been carried out for both types of model, but few mathematical rigorous results are available up to now: The fixed-rate model has been understood to a certain extent (see e.g. [14, 36]); its stationary state resembles a gamma distribution [36]. On the other hand, there exists no complete analysis for the model with randomly distributed saving factors (see however, [12, 32, 33, 38]).

A somewhat different approach to trades was made by Cordier, Pareschi, and the second author in [13]. There, a very simple model of an open market economy involving both assets exchanges between individuals and speculative trading was introduced. The model is such that the average wealth is not conserved due to a random dynamics in the binary trade interactions. The latter describes the spontaneous growth or decrease of wealth related to investments in the stock market, housing, and other factors. In the absence of conservation of the total wealth, the best way to extract information on the large-time behavior is to scale the solution in a way which keeps the total wealth constant. Nevertheless, analytical studies of the resulting kinetic model remained difficult (although a thorough analysis of the

continuous trading limit has been performed). For a determination of the shape of the limit distribution, the authors made use of suitable numerical methods.

Concerning the use of numerical methods in the study of steady states to kinetic equations (and in particular the tails of these solutions), we recall that one is faced with delicate problems. In fact, the numerical reconstruction of tails in self-similar profiles of non-linear evolution equations has a long history, especially in the field of diffusion processes. The main numerical difficulty originates from the so-called retention property, which means that the support of the solution is growing in time. Indeed, because of its spreading dynamics, diffusion equations will ask for repetitive re-gridding. For some recent developments in this area we refer to [24], where a new tool has been proposed, which allows to achieve precise numerical results in the case of fast diffusion equations. An analogous problem exists in the kinetic context, and has been recently apparent in the studies of the long-time behavior of a dissipative Maxwell gas [20, 21]. Despite the fact that the self-similar profiles are know to exhibit overpopulated tails, a numerical reconstruction of the latter is difficult to obtain. The available methods, principally Monte Carlo ones, are oscillating exactly in the interesting large velocities part of the density profile [23, 39].

In conclusion of the preceding remarks, one should treat the output of numerical experiments with care, when it comes to determine the structure of tails. Rigorous analytical information is of high importance in this discussion. The intention of the present paper is to bridge the gap between the numerical observations and their analytical understanding, at least for some of the available models. First of all, we investigate under which assumptions on the model one may rightfully speak of a stationary wealth distribution. More precisely, we prove the existence and uniqueness of a steady state, and we obtain results about the convergence of the dynamics towards it in the long-time limit. Subsequently, we investigate the regularity of the steady state. We confirm the expectation that a sensible wealth distribution curve is smooth. Finally—and most importantly—we provide a method to calculate the Pareto index directly from the model parameters.

In order to extract information on the large-time behavior of the economic system, we adopt ideas from [34]. The two main ingredients are the use of Fourier metrics and estimates on the evolution of moments. In fact, this strategy is well-known in dissipative kinetic theory, where the large-time behavior of the solution to the dissipative Boltzmann equation for Maxwell molecules [4] has been investigated by exactly the same means [1, 20, 21]. We would like to stress, however, that our investigations here go beyond an application of the mentioned results. In fact, the models of interest here do not fit into the "classical picture" of kinetic equations. As we do not allow debts, the wealth of our agents is always positive; this corresponds to a kinetic model which forbids negative velocities. In particular, we *cannot* assume that the distribution function is isotropic. Moreover, the role played by the second moment, i.e. the energy, in the classical picture is now played the first moment, which is the mean wealth of the society.

For the sake of completeness, we mention that alternative strategies have been used before in order to obtain analytical information about the tails of the steady wealth distribution. For instance, in [13], the kinetic equation was replaced by a (simpler) Fokker–Planck-type equation. This was justified by proving that the latter constitutes the grazing collisions limit of the first. The equilibrium state of the Fokker–Planck equation can be computed explicitly and is of Pareto type, i.e. it exhibits the power-law tail for the richest individuals. The same Fokker–Planck equation was obtained in [6] as the mean field limit of a stochastic dynamical equation, as well as in [30, 42] in the context of a generalized Lotka–Volterra dynamic. The mathematical methods used in [13] are close to those used in the context of kinetic theory for granular flows. Analogous methods have been used for a kinetic model of a simple market economy with a constant growth mechanism by Slanina [41], and generalized

to a one-dimensional kinetic model with mixing interaction in [34]. Finally, the hydrodynamic limit of a market model with time-dependent saving has been derived and studied numerically in [19].

Below, we provide more specific description of the setup and summary of our results.

In Sect. 2, we define the class of models to study. The time evolution of the wealth distribution $f(v, t)$ is based on the assumption of collision-like trade events. In a suitable scaling, the effect of collisions is quantitatively described by a Boltzmann-like equation [8],

$$\frac{\partial f}{\partial t} = Q(f, f). \tag{2}$$

Here $f = f(v, t)$ is the distribution of wealth $v \in \mathbb{R}_+$ at time $t \geq 0$, and Q is a bilinear operator which describes the change of f due to trades among agents. The binary trade is determined by the linear exchange rules

$$v^* = p_1 v + q_1 w; \qquad w^* = p_2 v + q_2 w. \tag{3}$$

Here (v, w) denote the (positive) money of two arbitrary individuals before the trade, and (v^*, w^*) the money after the trade. The interaction coefficients p_i, q_i, $i = 1, 2$ can be either given constant or random quantities, with the obvious constraint to be nonnegative. In the present paper—except for Sect. 4.2—the random distribution (or collision kernel, in the kinetic language) is assumed to be independent of the wealth variables v and w, and independent of time. Moreover, we restrict ourselves to *conservative* models, characterized by the further property

$$\langle p_1 + p_2 \rangle = 1, \qquad \langle q_1 + q_2 \rangle = 1,$$

where $\langle \cdot \rangle$ denotes the expectation value. Such interactions define a dynamic for the distribution functions $f(v, t)$, i.e. for appropriate initial conditions, there exists a unique solution. This solution is continuous in a suitable Fourier metric for probability measures, see Theorem 2.3.

In Sect. 3, we study the large-time behavior of solutions to (2), in particular their convergence towards a steady state, and the qualitative properties of the latter. The main result of this paper is given in Theorem 3.3: generically, the distribution functions $f(v, t)$ *do* converge towards a steady state $f_\infty(v)$ as $t \to \infty$, and this steady state is unique. Moreover, we prove that the *Pareto index* μ of f_∞ in (1) is given by the non-trivial root $\bar{s} > 1$ of the quantity

$$\mathfrak{S}(s) := \frac{1}{2} \left\langle \sum_{i=1}^{2} (p_i^s + q_i^s) \right\rangle - 1. \tag{4}$$

This quantity is directly accessible once the interaction coefficients in (3) are made explicit. If no root $\bar{s} > 1$ exists, then the limiting distribution either possesses a *slim tail*, which decays super-algebraically as $v \to \infty$, or is *trivial*, i.e. all agents have wealth zero. Finally, Theorem 3.5 provides C^∞-smoothness of the steady state under very general assumptions. In fact, the steady state belongs to a Gevrey class.

In addition to Fourier metrics, the other key tool in the proof of Theorem 3.3 is an investigation of the evolution of moments. The results are summarized in Theorem 3.2: if $\mathfrak{S}(s)$ is negative, then the sth moment of $f(v, t)$ remains uniformly bounded in time; if $\mathfrak{S}(s)$ is positive, then the sth moment diverges exponentially fast as $t \to \infty$.

In Sect. 4, the behavior of the function $\mathfrak{S}$ in various special cases is analyzed, and additional information about the shape of the tails is derived. For instance, in Example 2, we choose the collision kernel in a way that allows to control its degree of randomness by a parameter μ. The typical behavior we find (see Fig. 1) is that for small randomness, the tails of the steady state decay super-algebraically; in fact, they decay exponentially by Theorem 4.4. On the other hand, for large randomness, the steady state is trivial. Pareto tails form exactly in the intermediate regime. Likewise, a small saving propensity λ results in slim tails, while for large values $\lambda \approx 1$, the stationary state collapses to into the trivial solution. We conclude that a good adjustment of both randomness and saving play a pivotal role in the definition of suitable models.

Another observation concerns pointwise conservative trades, where the total of money is conserved not only in the mean, but in *each* interaction. Then either the tail of f_∞ is slim, or is possibly of Pareto type but unstable in the sense that it is reached at a slow algebraic time scale, but not exponentially fast; see Theorem 4.1 for details. Hence, in order to obtain a natural and robust wealth structure in the long-time limit, the respective model *must* allow the total money to increase during an individual trade with a certain probability; likewise, it must allow the total to decrease. This behavior is most easily interpreted as a combination of trades with *risky investments*, as discussed above in the context of the model of Cordier et al.

Finally, we mention how our results are useful to understand the outcome of numerical experiments. By Theorem 3.3, a variety of models allows only for trivial steady distribution of wealth, in which all agents are truly pauper. However, the convergence of the money distribution $f(v, t)$ towards this pathological limit $f_\infty(v)$ goes via the blow-up of all moments above the first, i.e. while an increasing majority of agents accumulates in the pauper region, fewer and fewer individuals become extremely rich, because the total wealth of the society is preserved. On the other hand, in the long-time limit, the total wealth is zero (the convergence is really weak in this case). In a simple numerical experiment, in which finite-size effects play a role, one could mistakenly interpret the blow-up of all moments above the first as convergence to a steady state with Pareto-index $\mu \approx 1$. This gives a possible explanation of why so many numerical studies report a critical index of approximately one, corresponding to a state with infinite mean wealth.

In summary, our results improve the understanding of the trade-mechanisms which are responsible for the creation of Pareto tails. In particular, we prove that the large-wealth behavior of the steady distribution heavily depends on the degree of randomness in the microscopic trade, and on the propensity of agents to save.

2 The Kinetic Equation

2.1 Collision Rules in Conservative Economies

A kinetic model of a simple market economy describes the redistribution of wealth by means of *microscopic* binary interactions among agents or individuals which perform exchange of money. These interactions are interpreted as trades. One generally assumes that the outcome of each trade is non-negative for both agents, imposing that no debts are allowed. This rule emphasizes the difference between economic interactions and the classical interactions between molecules, where no positivity restriction applies.

On the microscopic level, we consider binary interactions described by the rules (3), i.e.,

$$v^* = p_1 v + q_1 w; \qquad w^* = p_2 v + q_2 w, \tag{5}$$

where (v, w) and (v^*, w^*) denote the money of two arbitrary individuals before and after the trade, respectively. The interaction coefficients p_i, q_i for $i = 1, 2$ can be either given constant or random quantities. Here random means with respect to an a priori given probability distribution for the p_i and q_i, which does not depend on time t, or on v and w. As debts are not allowed, the interaction coefficients are constrained to be nonnegative. The total amount of money involved in the trade changes from $v + w$ into

$$v^* + w^* = (p_1 + p_2)v + (q_1 + q_2)w, \tag{6}$$

so that, unless a.s. both

$$p_1 + p_2 = 1, \qquad q_1 + q_2 = 1, \tag{7}$$

money is generally *not* conserved. Condition (7) can be relaxed in presence of random coefficients: we require that the total money is conserved *in the mean*, that is the random coefficients $p_i, q_i, i = 1, 2$ satisfy

$$\langle p_1 + p_2 \rangle = 1, \qquad \langle q_1 + q_2 \rangle = 1. \tag{8}$$

Both conditions (7) and (8) define *conservative* trades, and we refer to an interaction with (7) as *pointwise conservative*. These interactions are such that the mean wealth

$$M = \int v f(v) \, dv$$

is conserved in time.

Remark 2.1 We shall frequently use the abbreviation

$$\langle \Psi(p_i, q_i) \rangle_+ := \frac{1}{2} \langle \Psi(p_1, q_1) + \Psi(p_2, q_2) \rangle \tag{9}$$

for expressions Ψ involving p_i and q_i. Note that (8) implies the weaker condition

$$\langle p_i + q_i \rangle_+ = 1, \tag{10}$$

which will be sufficient for most of our considerations.

Both types of conservative interaction allow for a similar (and almost complete) study. Various models in the literature define conservative trades. In their pioneering paper A. Chakraborti and B.K. Chakrabarti [10] imposed that the agents taking part in a trade change money according to the rule

$$v^* = v + \Delta(v, w); \qquad w^* = w - \Delta(v, w), \tag{11}$$

where $\Delta(v, w)$ represents the amount of money to be exchanged,

$$\Delta(v, w) = (1 - \lambda)[(\epsilon - 1)v + \epsilon w], \tag{12}$$

with a constant *saving propensity* $\lambda \in [0, 1]$, and a random fraction ϵ with given distribution on $[0, 1]$. The introduction of λ in the definition of $\Delta(v, w)$ takes into account the basic fact

that rational agents always retain some money when trading; the saving rate λ (or sometimes s) denotes the ratio of the withheld to all the money possessed. This trade corresponds to

$$
\begin{aligned}
p_1 &= 1 - (1 - \lambda)(1 - \epsilon), & q_1 &= (1 - \lambda)\epsilon, \\
p_2 &= (1 - \lambda)(1 - \epsilon), & q_2 &= 1 - (1 - \lambda)\epsilon,
\end{aligned}
\tag{13}
$$

and is pointwise conservative.

We remark that the model has been further developed [11]: instead of prescribing a "global" saving rate for all agents, an individual saving propensity $\lambda_i \in [0, 1]$ is randomly assigned to each individual agent. This rate is then held fix during the time evolution. The rules (13) change accordingly into

$$
\begin{aligned}
p_1 &= 1 - (1 - \lambda_A)(1 - \epsilon), & q_1 &= (1 - \lambda_B)\epsilon, \\
p_2 &= (1 - \lambda_A)(1 - \epsilon), & q_2 &= 1 - (1 - \lambda_B)\epsilon,
\end{aligned}
\tag{14}
$$

where the quantities λ_A and λ_B denote the saving rates of the two interacting individuals A and B, possessing the wealth v and w, respectively. Despite its pointwise conservative character, this model does *not* fit into the current framework. Note that the interaction coefficients are not distributed according to an *a priori* law; rather, finding the (time- and wealth-depending) law for λ becomes part of the problem here. Some remarks are given in Sect. 4.2. However, a thorough analysis of this model will be subject of a separate paper [32].

Another kind of trade was considered in [13]. Its rules read

$$
v^* = \lambda v + (1 - \lambda)w + \eta v; \qquad w^* = (1 - \lambda)v + \lambda w + \tilde{\eta}w.
\tag{15}
$$

The saving rate λ is a global constant, as in the rules (13), while η and $\tilde{\eta}$ are independent random variables with the same distribution and zero mean. This random term introduces the effects of an open economy describing the market returns. Provided that $\eta, \tilde{\eta} \geq -\lambda$ almost surely, the random coefficients $p_i, q_i, i = 1, 2$ in (15) are nonnegative, since

$$
\begin{aligned}
p_1 &= \lambda + \eta, & q_1 &= 1 - \lambda, \\
p_2 &= 1 - \lambda, & q_2 &= \lambda + \tilde{\eta}.
\end{aligned}
\tag{16}
$$

However, this trade is conservative only in the mean, since $\langle p_1 + p_2 \rangle = 1$, but $p_1 + p_2 = 1 + \eta \neq 1$ with probability one.

For the sake of completeness, we mention some non-conservative models. F. Slanina recently introduced an increasing wealth model [41] where

$$
p_1 = \lambda, \qquad q_1 = 1 - \lambda + \epsilon; \qquad p_2 = 1 - \lambda + \epsilon, \qquad q_2 = \lambda.
\tag{17}
$$

In the rules (17), the growth parameter ϵ is a fixed positive constant, hence the total money has increased after the trade,

$$
v^* + w^* = (1 + \epsilon)(v + w).
\tag{18}
$$

These rules introduce (artificially) the effects of a strong economy into the game, making the total wealth grow with time. The same effect can be obtained in a more natural way: allow the random variables in (15) to attain arbitrary real values, but discard all trades for which the post-trade money are negative (see the discussion in [13]). The same idea was used in

the original transfer model proposed by A. Drăgulescu and V. Yakovenko [17], where the amount of money exchanged is

$$\Delta(v, w) = \frac{\epsilon}{2}(v + w) \tag{19}$$

with a random number $0 \leq \epsilon \leq 1$. If the "loser" cannot afford the money to be exchanged, the trade does not take place.

Non-conservative kinetic models for trades of the type considered by Slanina [41] have been recently studied in [34] by means of mathematical methods first applied to the Boltzmann equation for Maxwell molecules. In the forthcoming section, we will show how these techniques generalize to treat the conservative models introduced so far.

2.2 Weak Form of the Boltzmann Equation

Let $f(v, t)$ denote the distribution of agents with wealth $v \in \mathbb{R}_+$ at time $t \geq 0$. A Boltzmann-like equation for the time evolution of $f(v, t)$ can be easily derived by standard methods of kinetic theory, considering that the temporal change of $f(v, t)$ depends on a balance between the gain and loss of wealth among agents due to trades. A useful way of writing this equation is to use the so-called weak form: for all smooth functions $\Phi(v)$, consider

$$\frac{d}{dt} \int_{\mathbb{R}_+} f(v)\Phi(v)\, dv = \int_{\mathbb{R}_+} Q(f, f)(v)\Phi(v)\, dv, \tag{20}$$

where the *collision* operator $Q(f, f)$ is defined by

$$\int_{\mathbb{R}_+} Q(f, f)(v)\Phi(v)\, dv$$
$$= \frac{1}{2} \left\langle \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (\Phi(v^*) + \Phi(w^*) - \Phi(v) - \Phi(w)) f(v) f(w)\, dv\, dw \right\rangle. \tag{21}$$

The kinetic equation (20) is the analogue of the Boltzmann equation for Maxwell molecules [3, 8], where the collision frequency is assumed to be constant. It is well-known in kinetic theory that this simplification allows for a better understanding of the qualitative behavior of the solutions. Among other properties, it can be shown that higher order moments can be evaluated recursively, since the integrals $\int v^n f(v, t)\, dv$ obey a closed hierarchy of equations.

Without loss of generality, we can fix the initial density to satisfy

$$\int_{\mathbb{R}_+} f_0(v)\, dv = 1; \qquad \int_{\mathbb{R}_+} v f_0(v)\, dv = M. \tag{22}$$

Choosing $\Phi(v) = 1$ (respectively $\Phi(v) = v$) into (20) shows that conditions (22) are propagated in time, so that, for all $t > 0$

$$\int_{\mathbb{R}_+} f(v, t)\, dv = 1; \qquad \int_{\mathbb{R}_+} v f(v, t)\, dv = M. \tag{23}$$

The initial value problem can be easily studied in the weak form (20). Denote by $\mathcal{M}_0$ the space of all probability measures in $\mathbb{R}_+$, and for $\alpha \geq 0$ by

$$\mathcal{M}_\alpha = \left\{ \mu \in \mathcal{M}_0 : \int_{\mathbb{R}_+} |v|^\alpha \mu(dv) < +\infty \right\}, \tag{24}$$

the space of all Borel probability measures of finite moment of order α, equipped with the topology of the weak convergence of measures.

A weak solution to the initial value problem for (20), with initial probability density $f_0(v) \in \mathcal{M}_\alpha, \alpha > 0$ is a probability density $f \in C^1(\mathbb{R}_+, \mathcal{M}_\alpha)$ which satisfies (20) for $t > 0$ and all smooth functions Φ, and the initial condition is realized as

$$\lim_{t \to 0} \int_{\mathbb{R}_+} \Phi(v) f(v, t) \, dv = \int_{\mathbb{R}_+} \Phi(v) f_0(v) \, dv \tag{25}$$

for all Φ.

2.3 The Boltzmann Equation in Fourier Representation

From the kinetic theory of rarefied gases it is well-known that Boltzmann-like equations are often conveniently studied in Fourier space, see e.g. [3]. This is particularly true for the model under consideration, since its collision kernel (the probability measure) is independent of v. In fact, the weak formulation (20) is equivalent to the Fourier transformed equation:

$$\frac{\partial \widehat{f}(\xi, t)}{\partial t} = \widehat{Q}\left(\widehat{f}, \widehat{f}\right)(\xi, t), \tag{26}$$

where $\widehat{f}(\xi, t)$ is the Fourier transform of $f(x, t)$,

$$\widehat{f}(\xi, t) = \int_{\mathbb{R}_+} e^{-i\xi v} f(v, t) \, dv.$$

According to the collision rule (5), the transformed kernel reads

$$\widehat{Q}\left(\widehat{f}, \widehat{f}\right)(\xi) = \frac{1}{2}\left\langle \widehat{f}(p_1\xi)\widehat{f}(q_1\xi) + \widehat{f}(p_2\xi)\widehat{f}(q_2\xi)\right\rangle - \widehat{f}(\xi)\widehat{f}(0). \tag{27}$$

The initial conditions (22) turn into

$$\widehat{f_0}(0) = 1 \quad \text{and} \quad \widehat{f_0}'(0) = iM.$$

Hence (26) can be rewritten as

$$\frac{\partial \widehat{f}(\xi, t)}{\partial t} + \widehat{f}(\xi, t) = \left\langle \widehat{f}(p_i\xi)\widehat{f}(q_i\xi)\right\rangle_+. \tag{28}$$

Equation (28) is a generalization of (43) in [34], as we allow random variables for the mixing parameters p_i and q_i. Further, (28) is connected to (4.8) considered by Bobylev and Cercignani in [5]. The difference is that in our situation v is restricted to the positive real axis, which makes it impossible to apply most of the techniques from [5]. In particular, we *cannot* rescale our model in such a way that both the first and the second moment are preserved.

To prove existence of solutions to (28) is rather classical; e.g., the same methods available for the elastic Kac model [27] apply, expressing the solution as a Wild sum [3]. In order to show uniqueness of the solution, we use the idea from [22]: it is proven that the collision operator Q is Lipschitz continuous in a suitable metric.

Introduce for arbitrary $s > 0$ the *Fourier metric* on $\mathcal{M}_\alpha$ by

$$d_s(f, g) = \sup_{\xi \in \mathbb{R}} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^s}. \tag{29}$$

Write $s = m + \beta$, where m is an integer and $0 < \beta \le 1$. Two functions f and g have a finite distance, $d_s(f, g) < \infty$, if and only if their moments up to order m agree.

The metric (29) has been introduced in [22] to investigate the trend to equilibrium of the solutions to the Boltzmann equation for Maxwell molecules. There, the case $s = 2 + \beta$ with $\beta > 0$ was considered. Further applications of d_s to various kinetic models can be found in [1, 34, 37].

In order to show Lipschitz-continuity, let f_1 and f_2 be two solutions of the kinetic equation (28), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (22), and denote by $\widehat{f_1}$, $\widehat{f_2}$ their Fourier transforms. Let $s \ge 1$ be such that $d_s(f_{1,0}, f_{2,0})$ is finite. Then

$$\frac{\partial}{\partial t} \frac{(\widehat{f_1}(\xi) - \widehat{f_2}(\xi))}{|\xi|^s} + \frac{\widehat{f_1}(\xi) - \widehat{f_2}(\xi)}{|\xi|^s} = \frac{\langle \widehat{f_1}(p_i\xi)\widehat{f_1}(q_i\xi) - \widehat{f_2}(p_i\xi)\widehat{f_2}(q_i\xi)\rangle_+}{|\xi|^s}. \quad (30)$$

Now, since $|\widehat{f_1}(\xi, t)| \le 1$ and $|\widehat{f_2}(\xi, t)| \le 1$ for all $\xi \in \mathbb{R}$, we obtain

$$\left| \frac{\langle \widehat{f_1}(p_i\xi)\widehat{f_1}(q_i\xi) - \widehat{f_2}(p_i\xi)\widehat{f_2}(q_i\xi)\rangle_+}{|\xi|^s} \right|$$

$$\le \left\langle |\widehat{f_1}(p_i\xi)| \left| \frac{\widehat{f_1}(q_i\xi) - \widehat{f_2}(q_i\xi)}{|q_i\xi|^s} \right| q_i^s \right\rangle_+ + \left\langle |\widehat{f_2}(q_i\xi)| \left| \frac{\widehat{f_1}(p_i\xi) - \widehat{f_2}(p_i\xi)}{|p_i\xi|^s} \right| p_i^s \right\rangle_+$$

$$\le \sup \left| \frac{\widehat{f_1}(\xi) - \widehat{f_2}(\xi)}{|\xi|^s} \right| \langle p_i^s + q_i^s \rangle_+.$$

In terms of the auxiliary quantity

$$h(t, \xi) = \frac{\widehat{f_1}(\xi) - \widehat{f_2}(\xi)}{|\xi|^s},$$

the preceding computation shows that

$$\left| \frac{\partial h}{\partial t} + h \right| \le \langle p_i^s + q_i^s \rangle_+ \|h\|_\infty.$$

Gronwall's lemma yields at once that

$$\|h(t)\|_\infty \le \exp\left\{(\langle p_i^s + q_i^s \rangle_+ - 1)t\right\} \|h_0\|_\infty. \quad (31)$$

Below and the following chapters, the quantity in round brackets,

$$\mathfrak{S}(s) = \frac{1}{2} \sum_{i=1}^{2} \langle p_i^s + q_i^s \rangle - 1, \quad (32)$$

plays a major role. Since $\|h(t)\|_\infty = d_s(f_1(t), f_2(t))$, we obtain from (31)

Theorem 2.2 *Let $f_1(t)$ and $f_2(t)$ be two solutions of the Boltzmann equation (20), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (22). Let $s \ge 1$ be such that $d_s(f_{1,0}, f_{2,0})$ is finite. Then, for all times $t \ge 0$,*

$$d_s(f_1(t), f_2(t)) \le \exp\{\mathfrak{S}(s) \cdot t\} d_s(f_{1,0}, f_{2,0}). \quad (33)$$

In particular, if $\mathfrak{S}(s)$ is negative, then the d_s-distance of f_1 and f_2 decays exponentially in time.

Putting $f_{1,0} = f_{2,0} = f_0$ in (33), and using $s = 1$ yields

Corollary 2.3 *If f_0 is a nonnegative density satisfying conditions* (22), *then there exists a unique weak solution $f(t)$ of the Boltzmann equation with $f(0) = f_0$.*

3 Existence, Uniqueness and Smoothness of the Steady State

In order to discuss the shape of the tail in the stationary distribution, we first need to establish the existence of the latter. We prove—under suitable hypotheses—convergence of the distribution function $f(v, t)$ to a long-time limit $f_\infty(v)$, and show that this constitutes the unique stationary state. We point out that the existence of the long-time limit *cannot* be directly concluded from the contractivity of the kinetic equation in some d_s-norm. Unfortunately, one needs additional information on the boundedness of higher moments; for further discussion on the lack of completeness w.r.t. Fourier distance, see [7]. Therefore, the temporal evolution of moments of $f(v, t)$ is investigated first. The second approach follows the idea of Bobylev and Cercignani [5] to approximate the steady state by using the stationary equation as an iteration scheme.

3.1 Evolution of Moments

In Theorem 2.2 about the large-time behavior of solutions to (20), the essential quantity $\mathfrak{S}$ has been introduced. Below, we prove that if $\mathfrak{S}(s)$ is negative for some $s > 0$, then the sth moment of the solution,

$$M_s(t) = \int_{\mathbb{R}_+} v^s f(v; t) \, dv,$$

remains bounded for all times. On the other hand, if $\mathfrak{S}(s)$ is positive for some $s \geq 2$, then $M_s(t)$ diverges exponentially fast as $t \to \infty$. We exploit this information to prove existence and decay properties of the steady state.

To start with, we note that conservation of the total wealth allows to conclude that at least all moments of order $s \leq 1$ remain uniformly bounded. In fact, by Hölder's inequality,

$$\int_{\mathbb{R}_+} v^s f(v) \, dv \leq \left(\int_{\mathbb{R}_+} v f(v) \, dv \right)^s \cdot \left(\int_{\mathbb{R}_+} f(v) \, dv \right)^{1-s} = M^s \quad (0 < s < 1).$$

Now, let $s > 1$ and suppose that the initial density $f_0(v)$ satisfies

$$M_s(0) = \int_{\mathbb{R}_+} v^s f_0(v) \, dv < \infty. \tag{34}$$

Then, putting $\Phi(v) = v^s$ in the weak form (20), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}_+} v^s f(v, t) \, dv = \frac{1}{2} \left\langle \int_{\mathbb{R}_+^2} dv \, dw \left(\sum_{i=1}^{2} (p_i v + q_i w)^s - v^s - w^s \right) f(v) f(w) \right\rangle_+. \tag{35}$$

In the following, we establish upper and lower bounds for the right-hand side of (35). These bounds rely on the following elementary inequality:

Lemma 3.1 *For arbitrary non-negative real numbers a and b, and s > 1,*

$$a^s + b^s + \theta_s(a^{s-1}b + ab^{s-1}) \leq (a+b)^s \leq a^s + b^s + \Theta_s(a^{s-1}b + ab^{s-1}), \quad \text{with}$$

$$
\theta_s = \begin{cases} s & (s > 3), \\ 2^{s-3}s & (2 \leq s \leq 3), \\ 0 & (1 < s < 2) \end{cases} \quad \text{and} \quad \Theta_s = \begin{cases} s & (2 \leq s \leq 3), \\ 2^{s-3}s & (\text{otherwise}). \end{cases} \tag{36}
$$

Remark An investigation of the limit behavior as $a \searrow 0$ and $b > 0$ makes clear that $\theta_s = 0$ for $1 < s < 2$ cannot be improved in general.

Proof By homogeneity, it suffices to prove the inequality for $a + b = 1$. Define for $s > 1$,

$$\phi(s) := a^s + b^s + sab.$$

A calculation yields $\phi(2) = \phi(3) = 1$, independently of a and $b = 1 - a$. Furthermore, ϕ is convex in s since

$$\phi''(s) = a^s \ln^2 a + b^s \ln^2 b \geq 0.$$

Hence $\phi(s) \leq 1$ if and only if $2 \leq s \leq 3$. Observe that $a^{s-2} + b^{s-2}$ is concave w.r.t. $a = 1 - b \in (0, 1)$ for $2 \leq s \leq 3$, and convex for all other $s > 1$; the expression attains its extremal value 2^{3-s} at $a = b = \frac{1}{2}$. Hence

$$a^{s-1}b + ab^{s-1} = ab(a^{s-2} + b^{s-2}) \begin{cases} \leq 2^{3-s}ab & (2 \leq s \leq 3), \\ \geq 2^{3-s}ab & (\text{otherwise}). \end{cases}$$

Thus we obtain, for $2 \leq s \leq 3$

$$a^s + b^s + 2^{s-3}s(a^{s-1}b + ab^{s-1}) \leq \phi(s) \leq 1 = (a+b)^s,$$

and with reversed inequalities for $1 < s < 2$ or $s > 3$.

Now let $s > 1$ be fixed and consider for $a \in [0, 1]$

$$f_s(a) := a^s + b^s + s(a^{s-1}b + ab^{s-1}),$$

with $b = 1 - a$. Observe that $f_s(0) = f_s(1) = 1$, and furthermore

$$f_s'(a) = s(s-1)(a^{s-2}b - ab^{s-2}),$$

so that f has $\bar{a} = \frac{1}{2}$ as the only critical point in $(0, 1)$. For $s < 3$, one has $f_s(\bar{a}) = (s+1) \times 2^{1-s} > 1$, so $\bar{a}$ is a maximum point and hence $f_s(a) \geq 1$ for $a \in [0, 1]$; for $s > 3$, it is a minimum point and $f_s(a) \leq 1$. Consequently, for $s \leq 3$,

$$a^s + b^s + s(a^{s-1}b + ab^{s-1}) = f_s(a) \geq 1 = (a+b)^s;$$

the reversed inequality holds for $s \geq 3$. □

Using the upper bound in (36), estimate

$$(p_i v + q_i w)^s \leq p_i^s v^s + q_i^s w^s + \Theta_s(p_i^{s-1}q_i v^{s-1}w + p_i q_i^{s-1}vw^{s-1})$$

under the integral in (35), leading to

$$
\begin{aligned}
\frac{d}{dt} M_s(t) O &\leq \frac{1}{2} \left\langle \int_{\mathbb{R}^2_+} dv\, dw ((p_1^s + p_2^s - 1)v^s + (q_1^s + q_2^s - 1)w^s) f(v) f(w) \right\rangle \\
&\quad + \frac{1}{2} \Theta_s \left\langle \int_{\mathbb{R}^2_+} dv\, dw ((p_1^{s-1} q_1 + p_2^{s-1} q_2)v^{s-1} w \right. \\
&\qquad \left. + (p_1 q_1^{s-1} + p_2 q_2^{s-1})vw^{s-1}) f(v) f(w) \right\rangle \\
&= \mathfrak{S}(s) \cdot \int_{\mathbb{R}_+} v^s f(v)\, dv + \Theta_s \langle p_i^{s-1} q_i + p_i q_i^{s-1} \rangle_+ \\
&\quad \times \int_{\mathbb{R}_+} vf(v)\, dv \int_{\mathbb{R}_+} w^{s-1} f(w)\, dw.
\end{aligned}
\tag{37}
$$

Recall that the total wealth $\int v\, f(v)\, dv = M$ is conserved in time. Further, by Hölder's inequality, it follows that

$$
\int_{\mathbb{R}_+} w^{s-1} f(w)\, dw \leq \left(\int_{\mathbb{R}_+} w^s f(w)\, dw \right)^{1-\frac{1}{s}}.
$$

Hence, we obtain

$$
\frac{d}{dt} \int_{\mathbb{R}_+} v^s f(v)\, dv \leq \mathfrak{S}(s) \int_{\mathbb{R}_+} v^s f(v)\, dv + \Theta_s \cdot \mathfrak{K}(s) \cdot M \cdot \left(\int_{\mathbb{R}_+} v^s f(v)\, dv \right)^{1-\frac{1}{s}},
\tag{38}
$$

where

$$
\mathfrak{K}(s) := \langle p_i^{s-1} q_i + p_i q_i^{s-1} \rangle_+ \leq \langle p_i^s + q_i^s \rangle_+ = \mathfrak{S}(s) + 1.
\tag{39}
$$

In particular, if $\mathfrak{S}(s)$ is a finite number, then so is $\mathfrak{K}(s)$. In this case, the sth moment grows at most exponentially, with rate $\mathfrak{S}(s)$, if it was finite initially. Moreover, if $\mathfrak{S}(s) < 0$, then the sth moment remains uniformly bounded for all times. In fact, an upper bound on the sth moment is determined by the solution of the associated ordinary differential equation

$$
\dot{y} = \mathfrak{S} \cdot y + \Theta_s \cdot \mathfrak{K}(s) \cdot M \cdot y^{1-\frac{1}{s}}
$$

with initial condition $y(0) = M_s(0)$. The solution is explicitly given by

$$
y(t) = \left[(M_s \cdot \exp\{t \cdot \mathfrak{S}(s)\})^{1/s} + \frac{\Theta_s \cdot \mathfrak{K}(s) \cdot M}{\mathfrak{S}(s)} \left(\exp\left\{ \frac{t}{s} \cdot \mathfrak{S}(s) \right\} - 1 \right) \right]^s.
$$

Notice that the first term in the square bracket vanishes for $t \to \infty$ if $\mathfrak{S}(s) < 0$, so that the limiting value depends on the initial condition f_0 only through the total wealth M.

By the same reasoning as above, we construct a bound *from below* on the time-derivative of the integral. For this, we use the lower bound given in the elementary inequality (36). Replacing the respective expressions under the integral, we obtain

$$\frac{d}{dt} M_s(t) \geq \frac{1}{2} \int_{\mathbb{R}_+^2} dv\, dw \big(\langle p_1^s + p_2^s - 1 \rangle v^s + \langle q_1^s + q_2^s - 1 \rangle w^s \big) f(v)\, f(w)$$

$$+ \theta_s \bigg\langle \int_{\mathbb{R}_+^2} dv\, dw \big((p_1^{s-1} q_1 + p_2^{s-1} q_2) v^{s-1} w$$

$$+ (p_1 q_1^{s-1} + p_2 q_2^{s-1}) v w^{s-1} \big) f(v) f(w) \bigg\rangle$$

$$= \mathfrak{S}(s) \cdot \int v^s f(v,t)\, dv + \theta_s \cdot \mathfrak{K}(s) \cdot \int_{\mathbb{R}_+} v f(v)\, dv \int_{\mathbb{R}_+} w^{s-1} f(w)\, dw.$$

We use Hölder's inequality to estimate

$$\int_{\mathbb{R}_+} w^{s-1} f(w)\, dw \geq \left(\int_{\mathbb{R}_+} w f(w)\, dw \right)^{s-1} = M^{s-1}.$$

By Gronwall's inequality, a lower bound is given by

$$M_s(t) \geq M_s(0) \cdot \exp\{t \cdot \mathfrak{S}(s)\} + \frac{\theta_s \cdot \mathfrak{K}(s) \cdot M^s}{\mathfrak{S}(s)} \left(\exp\{t \cdot \mathfrak{S}(s)\} - 1 \right).$$

We conclude that if $\mathfrak{S}(s) > 0$, then the moment M_s diverges exponentially in time.

In the special case that $\mathfrak{S}(s) = 0$, similar but simpler arguments give that the corresponding moment remains either bounded (iff $\mathcal{K}(s) = 0$) or diverges, but only at the algebraic rate t^s. Finally, if $\mathfrak{S}(s) = +\infty$, an easy argument shows that the sth moment of $f(v,t)$ must be infinite for all positive times.

Theorem 3.2 *Consider a solution f to the Boltzmann equation* (20). *Let $s > 1$ be such that $M_s(0) = \int_{\mathbb{R}_+} v^s f_0(v)\, dv < \infty$.*

1. *If $0 < \mathfrak{S}(s) < +\infty$, then the sth moment diverges exponentially fast as $t \to \infty$:*

$$M_s + \frac{\theta_s \mathfrak{K}(s)}{\mathfrak{S}(s)} \cdot M^s + o(1) \leq \frac{\int_{\mathbb{R}_+} v^s f(v,t)\, dv}{\exp\{t \cdot \mathfrak{S}(s)\}} \leq \left[M_s^{1/s} + \frac{\Theta_s \mathfrak{K}(s)}{\mathfrak{S}(s)} \cdot M \right]^s + o(1). \quad (40)$$

2. *If $\mathfrak{S}(s) < 0$, then the sth moment remains uniformly bounded as $t \to \infty$:*

$$\frac{\theta_s \mathfrak{K}(s)}{|\mathfrak{S}(s)|} \cdot M^s + o(1) \leq \int_{\mathbb{R}_+} v^s f(v,t)\, dv \leq \left(\frac{\Theta_s \mathfrak{K}(s)}{|\mathfrak{S}(s)|} \right)^s \cdot M^s + o(1). \quad (41)$$

3. *If $\mathfrak{S}(s) = 0$, then the sth moment either remains bounded or diverges at an algebraic rate:*

$$\left(\frac{\theta_s \mathfrak{K}(s)}{s} \right)^s \cdot M^s + o(1) \leq t^{-s} \cdot \int_{\mathbb{R}_+} v^s f(v,t)\, dv \leq \left(\frac{\Theta_s \mathfrak{K}(s)}{s} \right)^s \cdot M^s + o(1). \quad (42)$$

4. *Finally, if $\mathfrak{S}(s) = +\infty$, then the sth moment is infinite for all $t > 0$.*

3.2 Steady State as the Long-Time Limit

Below, the unique steady state f_∞ to the kinetic equation (20) is obtained as the limit of an arbitrary solution $f(v,t)$ for $t \to \infty$ in a suitable d_s-norm. Since the long-time behavior of

solutions is essentially determined by the quantity $\mathfrak{S}$, we need to start by investigating this function in further detail.

First recall that for an arbitrary non-negative number p, the exponential $s \mapsto p^s$ is convex in $s > 0$. Hence $\mathfrak{S}(s)$, which is the average of convex functions, is convex on its domain. By the dominated convergence theorem, $\mathfrak{S}(s)$ is well-defined at least for $0 < s \leq 1$, but possibly $\mathfrak{S}(s) = +\infty$ for all $s > s_\infty > 1$. Since $\mathfrak{S}(1) = 0$, convexity leaves only three possibilities for the behavior of $\mathfrak{S}$:

1. $\mathfrak{S}(s)$ is non-negative for all $s > 0$.
2. $\mathfrak{S}(s)$ is negative for some $s \in (0, 1)$, and positive for all $s > 1$.
3. $\mathfrak{S}(s)$ is negative for all $1 < s < \bar{s}$, and positive for all $s < 1$ and all $s > \bar{s}$; here either $\bar{s} = +\infty$, or $\mathfrak{S}(\bar{s}) = 0$, or $\mathfrak{S}(s) = +\infty$ for $s > \bar{s}$.

If $\mathfrak{S}$ is differentiable at $s = 1$, then the first case corresponds to $\mathfrak{S}'(1) = 0$, the second to $\mathfrak{S}'(1) > 0$, and the last to $\mathfrak{S}'(1) < 0$. These four cases are now discussed in detail.

In the **first case**, no information about the existence of a long-time limit can be extracted.

In the **second case**, $r := -\mathfrak{S}(s) > 0$ for some $s \in (0, 1)$. Then the solution $f(v, t)$ has the Cauchy-property w.r.t. $t \geq 0$, i.e., for each $\epsilon > 0$, there exists a time $T(\epsilon)$ such that

$$d_s\big(f(t_1), f(t_2)\big) \leq \epsilon \quad \text{for all } t_1, t_2 \geq T(\epsilon). \tag{43}$$

To prove this, let $t_1 > t_2 \geq 0$ be arbitrary, and write $t_1 = t_2 + n\Delta$ with some $\frac{1}{2} \leq \Delta \leq 1$ and $n \in \mathbb{N}$. By the triangle inequality and the contraction estimate from Theorem 2.2,

$$d_s\big(f(t_1), f(t_2)\big) \leq \sum_{k=0}^{n-1} d_s\big(f(t_2 + (k+1)\Delta), f(t_2 + k\Delta)\big)$$

$$\leq \sum_{k=0}^{n-1} e^{-r(t_2 + k\Delta)} d_s\big(f(\Delta), f(0)\big)$$

$$\leq \frac{e^{-rt_2}}{1 - e^{-r\Delta}} d_s\big(f(\Delta), f_0\big) \leq e^{-rt_2} \sup_{1/2 \leq \Delta' \leq 1} \frac{d_s(f(\Delta'), f_0)}{1 - e^{-r\Delta'}}.$$

As $f(t)$ is continuous in the d_s-metric, the last supremum is finite, which implies (43). In addition, the first moment of $f(v, t)$ equals M for all $t \geq 0$. The Cauchy property together with this uniform bound on moments is enough to conclude convergence of $f(v, t)$ to a limit $f_\infty(v)$ in d_s as $t \to +\infty$, see [7]. This limit is a probability density with first moment less or equal to M.

Now let $f'(t)$ be another solution to the Boltzmann equation (20), with first moment equal to M'. Then $f'(t)$ converges to a limit f'_∞ as $t \to +\infty$, with first moment less or equal to M'. On the other hand, as two probability measures always have finite d_s-distance for $s < 1$, it follows from Theorem 2.2 that $d_s(f(t), f'(t)) \to 0$ as $t \to +\infty$. Consequently, $f_\infty = f'_\infty$, and the first moment is less or equal than the minimum of M and M'.

It follows that for an arbitrary initial distribution f_0, the solution $f(t)$ converges to the same limit f_∞. This limit is a probability density with first moment equal to zero, i.e., it is a Dirac distribution centered at $v = 0$. Notice that, although f_∞ possesses finite moments of arbitrary order while $\mathfrak{S}(s) > 0$ for $s > 1$, this is not in contradiction to Theorem 3.2, since also $M = 0$.

The **third case** is the most interesting one. Choose some $s \in (1, 2)$ with $s < \bar{s}$; then $r := -\mathfrak{S}(s) > 0$. Assume that the initial datum of f possesses a moment of order $S > s$. By

the same argument as above, it follows that $f(v, t)$ has the Cauchy property in d_s; notice that $d_s(f(t), f_0)$ is always finite since $s < 2$ and the first moment (mean wealth) is conserved under evolution. Moreover, as we required $s < \bar{s}$, there exists a s' with $s < s' < \min(\bar{s}, S)$; by Theorem 3.2, the moment of order s' remains uniformly bounded. It follows that $f(t)$ converges in d_s to a limit distribution $f_\infty(v)$, which is normalized and has the same first moment as the $f(t)$.

This convergence implies that f_∞ is a steady state for the kinetic equation (20). Indeed, denote by $f_\infty(t)$ the solution to (20) with initial datum f_∞, then Theorem 2.2 gives

$$d_s\big(f_\infty(t), f_\infty\big) \leq d_s\big(f_\infty(t), f(t + T)\big) + d_s\big(f(t + T), f_\infty\big)$$
$$\leq e^{-rt} d_s\big(f_\infty, f(T)\big) + d_s\big(f(t + T), f_\infty\big).$$

The last expression can be made arbitrarily small by choosing T large enough, so that $f_\infty(t) = f_\infty$ for all $t \geq 0$. In fact, f_∞ is the *only* steady state with the respective value of the first moment; for if f'_∞ is another steady state with the same first moment, then $d_s(f_\infty, f'_\infty)$ is finite, and so, invoking Theorem 2.2 again,

$$d_s(f_\infty, f'_\infty) \leq e^{-r} d_s(f_\infty, f'_\infty),$$

which forces $f_\infty = f'_\infty$. Finally, consider a solution f which has arbitrary moments bounded initially. Theorem 3.2 gives a time-uniform bound for moments of order less than $\bar{s}$. Since convergence $f(t) \to f_\infty$ in d_s implies weak* convergence of the associated measures, it follows that f_∞ has finite moments of all orders less than $\bar{s}$. On the other hand, no moment of order larger than $\bar{s}$ can be finite. To see this, simply apply Theorem 3.2 to the steady state solution $f_\infty(t) \equiv f_\infty$ to derive a contradiction.

We summarize the results of this section:

Theorem 3.3 *Let $f(v, t)$ be a weak solution of the Boltzmann equation* (20), *which has initially finite moments up to order $S > 0$. Further assume that $\mathfrak{S}(s) < 0$ for some $s \in (0, S)$. Then f converges exponentially fast in d_s to a steady state f_∞,*

$$d_s(f(t), f_\infty) \leq d_s(f_0, f_\infty)\exp\{-|\mathfrak{S}(s)|t\}. \tag{44}$$

If $s < 1$, then f_∞ is a Dirac distribution centered at $v = 0$, and there are no other steady states. If $s > 1$, then f_∞ has mean wealth equal to M, and it is the only steady state with this mean wealth. Moreover, if $\mathfrak{S}(s') < 0$ exactly for $1 < s' < \bar{s}$, possibly with $\bar{s} = +\infty$, then f_∞ has finite moments of all orders less than $\bar{s}$, while moments of order larger than $\bar{s}$ are infinite.

Remark 3.4 It is possible that $d_s(f(t), f_\infty)$ is infinite in (44); this simply means that also $d_s(f_0, f_\infty)$ is infinite.

3.3 Smoothness of the Steady State

Theorem 3.5 *Assume that there are positive numbers r and δ such that*

$$p_i^r + q_i^r \geq 1 + \delta^r \quad a.s. \tag{45}$$

Assume further that $\mathfrak{S}(\sigma) < 0$ *for some* $1 < \sigma < 2$, *so that a non-trivial steady state* f_∞ *to the kinetic equation* (20) *exists. Then either* f_∞ *is a Dirac distribution centered at* $v = M$, *or* f_∞ *is a smooth function and belongs to the* r*-th Gevrey class, i.e.*

$$\left|\widehat{f_\infty}(\xi)\right| \le \exp\left(-\mu|\xi|^r\right), \quad \text{for } |\xi| \ge \rho,$$

with suitable positive numbers ρ *and* μ.

Remark 3.6 Since the mean wealth is preserved, i.e. $\langle p_i + q_i \rangle_+ = 1$, it follows that $r < 1$. Notice that condition (45) imposes no serious restrictions on the distribution of the interaction coefficients. For instance, it allows any probability measure which is compactly supported in the interior of the first quadrant of the p_i–q_i-plane.

The proof of Theorem 3.5 is done on the level of Fourier transforms, and is divided into a series of lemmas. The first is concerned with the behavior of $\widehat{f_\infty}$ near the origin:

Lemma 3.7 *Under the hypotheses of Theorem 3.5, either* f_∞ *is a Dirac distribution, i.e.* $|\widehat{f}| \equiv 1$, *or there exist positive numbers* ρ *and* κ *s.t.*

$$|\widehat{f_\infty}(\xi)| \le \exp(-\kappa\xi^2), \quad \text{for } |\xi| < \rho. \tag{46}$$

Proof First assume that f_∞ has finite second moment. Then $\widehat{f_\infty}$ is twice differentiable at $\xi = 0$, and

$$\widehat{f_\infty}(\xi) = 1 - iM\xi - \frac{1}{2}Q\xi^2 + o(\xi^2),$$

where M and Q denote the first and second moment of f_∞, respectively. This means that

$$|\widehat{f_\infty}(\xi)|^2 = 1 - (Q - M^2)\xi^2 + o(\xi^2).$$

Now either $Q = M^2$, which implies that f_∞ is concentrated at $\xi = M$, or $\kappa := \frac{1}{4}(Q - M^2) > 0$ makes (46) true for some small $\rho > 0$.

In the case that the second moment of $\widehat{f_\infty}$ is infinite, we write

$$\widehat{f_\infty}(\xi) = \int_{\mathbb{R}} f_\infty(v)e^{-i\xi v}\,dv = 1 - \underbrace{\int_{\mathbb{R}}(1 - \cos(\xi v))\,df_\infty(v)}_{=:H_1(\xi)} - i\underbrace{\int_{\mathbb{R}} \sin(\xi v)\,df_\infty(v)}_{=:H_2(\xi)},$$

so that

$$|\widehat{f_\infty}(\xi)|^2 = 1 - 2H_1(\xi) + H_1(\xi)^2 + H_2(\xi)^2.$$

The elementary inequalities $1 - \cos x \le |x|$ and $|\sin x| \le |x|$ for $x \in \mathbb{R}$ imply

$$H_1(\xi) \le |\xi|\int_{\mathbb{R}} v\,df_\infty(v) = M|\xi|, \qquad |H_2(\xi)| \le |\xi|\int_{\mathbb{R}} v\,df_\infty(v) = M|\xi|.$$

On the other hand, with the elementary inequalities

$$1 - \cos x \ge \frac{1}{3}x^2 \quad \text{for } |x| \le 1, \qquad 1 - \cos x \ge 0 \quad \text{for } x \in \mathbb{R},$$

one obtains for arbitrary $\xi \in \mathbb{R}$

$$H_1 \geq \frac{\xi^2}{3} \int_{|\xi v| < 1} v^2 \, df_\infty(v) = A(\xi)\xi^2.$$

Here $A(\xi)$ is a positive, non-increasing function with $\lim_{\xi \to 0} A(\xi) = +\infty$, since the second moment of f_∞ diverges. Choosing $\rho > 0$ so small that $A(\rho) \geq 2M^2$, it follows that

$$|\hat{f}_\infty(\xi)|^2 \leq 1 - 2M^2\xi^2, \quad \text{for } |\xi| < \rho.$$

Hence (46) holds with $\kappa := \frac{1}{2}M^2$, possibly after diminishing ρ suitably. $\qquad\square$

Inspired by the ideas of Bobylev et al. in [5], and those of Desvillettes et al. in [15], we introduce the fixed point operator

$$\mathbf{R}[\psi](\xi) := \begin{cases} \widehat{f}_\infty(\xi) & \text{if } |\xi| < \rho, \\ \langle \psi(p_i\xi)\psi(q_i\xi) \rangle_+ & \text{if } |\xi| \geq \rho \end{cases}$$

on bounded functions $\psi : \mathbb{R} \to \mathbb{C}$. Notice that $\mathbf{R}$ is closely related to the Fourier transform of the collision kernel. Furthermore, for $\mu > 0$, define K_μ as the set of functions ψ with $\psi(0) = 1$, $\psi'(0) = \widehat{f}'_\infty(0)$, and satisfying the estimates

$$|\psi(\xi)| \leq \exp(-\kappa\xi^2) \quad \text{for } |\xi| < \rho, \qquad |\psi(\xi)| \leq \exp(-\mu|\xi|^r) \quad \text{for } |\xi| \geq \rho. \qquad (47)$$

Here q and ρ are the constants from Lemma 3.7 above. Eventually, we wish to prove that $\widehat{f}_\infty \in K_\mu$ for a suitable $\mu > 0$.

Lemma 3.8 *Under the hypotheses of Theorem 3.5, and assuming that f_∞ is not a Dirac distribution, there exists some $\mu > 0$, depending only on ρ and κ, such that $\mathbf{R}$ maps K_μ into itself.*

Proof By definition of $\mathbf{R}$, it is obvious that $\mathbf{R}[\psi](0) = 1$ and $\mathbf{R}[\psi]'(0) = \widehat{f}'_\infty(0)$. Moreover, Lemma 3.7 assures that the first estimate in (47) holds. Now let $\psi \in K_\mu$ and $\xi \in \mathbb{R}$ with $|\xi| \geq \rho$ be arbitrary. We show that

$$e^{\mu|\xi|^r} \cdot \left| \mathbf{R}[\psi](\xi) \right| \leq \langle e^{\mu|\xi|^r} \cdot |\psi(p_i\xi)| \cdot |\psi(q_i\xi)| \rangle_+ \leq 1. \qquad (48)$$

To this end, we prove that the term inside the expectation value is less/equal one a.s. We distinguish four cases:

1. If both $p_i|\xi| \geq \rho$ and $q_i|\xi| \geq \rho$, then the hypothesis (45) yields

$$e^{\mu|\xi|^r} \cdot |\psi(p_i\xi)| \cdot |\psi(q_i\xi)| \leq e^{\mu|\xi|^r - \mu p_i^r|\xi|^r - \mu q_i^r|\xi|^r} \leq e^{-\mu|\xi|^r \delta^r} \leq 1.$$

2. If both $p_i|\xi| < \rho$ and $q_i|\xi| < \rho$, then we conclude from (45) that $p_i^2 + q_i^2 \geq 2^{1-2/r}$, so

$$e^{\mu|\xi|^r} \cdot |\psi(p_i\xi)| \cdot |\psi(q_i\xi)| \leq e^{\mu|\xi|^r - \kappa p_i^2|\xi|^2 - \kappa q_i^2|\xi|^2} \leq e^{-|\xi|^r(\mu - \kappa \cdot 2^{1-2/r}\rho^{2-r})} \leq 1,$$

provided that $\mu \leq 2^{1-2/r}\rho^{2-r} \cdot \kappa$.

3. Now assume that $p_i|\xi| < \rho$ while $q_i|\xi| \geq \rho$. Using (45) once again, one finds

$$e^{\mu|\xi|^r} \cdot |\psi(p_i\xi)| \cdot |\psi(q_i\xi)| \leq e^{\mu|\xi|^r - \kappa p_i^2|\xi|^2 - \mu q_i^r|\xi|^r} \leq e^{\mu(p_i^r - \delta^r)|\xi|^r - \kappa p_i^2|\xi|^2}.$$

Obviously, the last expression is less than one if $p_i < \delta$. Assume now that $p_i \geq \delta$. Since $|\xi| \geq \rho$, one has $p_i|\xi| \geq \delta\rho$ and hence the exponent can be estimated by

$$\mu p_i^r |\xi|^r - \kappa p_i^2 |\xi|^2 \leq p_i^r |\xi|^r (\mu - \kappa \delta^{2-r} \rho^{2-r}).$$

The last expression is negative provided that $\mu \leq (\delta\rho)^{2-r} \cdot \kappa$.

4. The last case, $p_i|\xi| \geq \rho$ and $q_i|\xi| < \rho$, is treated in the same way and yields the same condition on μ.

In summary, (48) holds if μ is the minimum of $2^{1-2/r} \rho^{2-r} \cdot \kappa$ and $(\delta\rho)^{2-r} \cdot \kappa$. □

Lemma 3.9 *Under the hypotheses of Theorem* 3.5, *and assuming that f_∞ is not a Dirac distribution, $\widehat{f}_\infty \in K_\mu$ with the μ from Lemma* 3.8 *above.*

Proof Let $\psi_0(\xi) = \widehat{f}_\infty(\xi)$ for $|\xi| < \rho$, and $\psi_0(\xi) = 0$ for $|\xi| \geq \rho$. Define inductively the sequence $\psi_{n+1} = \mathbf{R}[\psi_n]$. By Lemma 3.8, each ψ_n belongs to K_μ. Moreover, by the same estimates that lead to Theorem 2.2, one proves that $\mathbf{R}$ acts as a contraction on functions ψ, φ with $|\psi| \leq 1$, $|\varphi| \leq 1$ and $\psi(0) = \varphi(0) = 1$, $\psi'(0) = \varphi'(0) = \widehat{f}_\infty'(0)$. In fact, $|\mathbf{R}[\psi](\xi) - \mathbf{R}[\phi](\xi)| = 0$ for $|\xi| < \rho$, while for $\xi \geq \rho$, one finds

$$\left| \mathbf{R}[\psi](\xi) - \mathbf{R}[\varphi](\xi) \right| \leq \sup_{\xi' \neq 0} \left(\frac{|\psi(\xi') - \varphi(\xi')|}{|\xi'|^\sigma} \right) \cdot \langle p_i^\sigma + q_i^\sigma \rangle_+ \cdot |\xi|^\sigma. \tag{49}$$

The supremum on the right-hand side is finite (recall that $1 < \sigma < 2$). By definition of K_μ, inequality (49) applies in particular to any pair ψ_n, ψ_m, yielding

$$\left| \psi_{n+1}(\xi) - \psi_n(\xi) \right| \leq C \cdot \langle p_i^\sigma + q_i^\sigma \rangle_+^n \cdot |\xi|^\sigma$$

with some finite constant C. By hypothesis, $\mathfrak{S}(\sigma) < 0$, so $0 < \langle p_i^\sigma + q_i^\sigma \rangle_+ < 1$, implying local uniform convergence of the sequence ψ_n to a limit function ψ_∞. Observe that ψ_∞ satisfies the same pointwise estimates (47) as the ψ_n, i.e. $\psi_\infty \in K_\mu$. Moreover, $\psi_\infty(\xi) = \widehat{f}_\infty(\xi)$ for $|\xi| < \rho$. Finally, local uniform convergence in combination with the pointwise estimates (47) is more than enough to invoke Lebesgue's dominated convergence theorem and conclude

$$\psi_\infty(\xi) = \langle \psi_\infty(p_i\xi)\psi_\infty(q_i\xi) \rangle_+, \quad \text{for } |\xi| \geq \rho, \tag{50}$$

i.e. ψ_∞ is a fixed point of the operator $\mathbf{R}$. By (49), we conclude that ψ_∞ is in fact the unique fixed point of $\mathbf{R}$ among the functions ψ with $|\psi| \leq 1$, $\psi(0) = 1$ and $\psi'(0) = \widehat{f}_\infty'(0)$.

Obviously, $\widehat{f}_\infty$ is also a fixed point of $\mathbf{R}$, as (50) is nothing but the equation for the stationary state in Fourier representation, i.e. (28) with $\partial_t \widehat{f} = 0$. So $\widehat{f}_\infty = \psi_\infty \in K_\mu$, which finishes the proof. □

By definition of K_μ, Theorem 3.5 now follows immediately from Lemma 3.9.

4 Shape of the Tails

We are eventually in the situation to study the behavior of the steady state $f_\infty(v)$ for large values of the wealth v.

4.1 Non-pointwise Conservative Trades

In this section we study the formation of tails in models which conserve money only in the mean. For certain probability distributions of the interaction coefficients, these tails are of Pareto type. Recall that the decay behavior of the steady state f_∞ is essentially determined by the function

$$\mathfrak{S}(s) = \frac{1}{2}\langle p_1^s + p_2^s + q_1^s + q_2^s \rangle - 1.$$

Algebraic decay corresponds to $\mathfrak{S}(s)$ being negative on a *finite* interval $(1, \bar{s})$ with $\bar{s} > 1$. In view of (1), we define the Pareto index of the stationary distribution f_∞ as the number $\mu > 0$ such that all moments of f_∞ of order less than μ are finite, while f_∞ possesses no moments of order larger than μ. From Theorem 3.3, it is immediate to conclude that the Pareto index μ coincides with the non-trivial zero $\bar{s} > 1$ of $\mathfrak{S}$.

We focus on various special cases of the trade given by the model (15) in the following. The saving propensity $0 < \lambda < 1$ is fixed, and we assume that the probability variables η and $\tilde{\eta}$ are independent and identically distributed. Under these assumptions,

$$\mathfrak{S}(s) = \langle p_1^s + p_2^s \rangle - 1 = (1 - \lambda)^s - 1 + \int_{-\lambda}^{+\infty} (\lambda + \omega)^s \, d\rho(\omega), \tag{51}$$

where $\rho(s)$ is the probability density of η and $\tilde{\eta}$ on $[-\lambda, +\infty)$.

Example 1 If the distribution of η is so tame that $\eta \leq \lambda - \delta$ a.s. for some $0 < \delta < \lambda$, then

$$p_1, p_2, q_1, q_2 \leq 1 - \delta \quad \text{a.s.} \tag{52}$$

Consequently $\mathfrak{S}(s) \leq 2(1 - \delta)^s - 1$, and this expression converges to -1 as $s \to \infty$. By the basic properties of $\mathfrak{S}$, it follows that $\mathfrak{S}(s) < 0$ for all $s > 1$. This corresponds to a dynamic in which all moments remain uniformly bounded in time; the steady state has no Pareto tails.

Example 2 Assume η only takes values $+\mu$ and $-\mu$, with probability $\frac{1}{2}$ each, where $0 \leq \mu \leq \lambda$. By varying the parameters λ and μ, one encounters the whole variety of possible shapes of the tail of f_∞. If $\lambda + \mu < 1$, then we are back in the trivial scenario (52), where all moments of f_∞ are finite. Hence, assume $\lambda + \mu > 1$. We calculate

$$\mathfrak{S}(s) = (1 - \lambda)^s - 1 + \frac{1}{2}(\lambda - \mu)^s + \frac{1}{2}(\lambda + \mu)^s. \tag{53}$$

The term $(\lambda + \mu)^s$ grows exponentially with respect to s and eventually dominates, making the expression (53) positive for large s. We evaluate numerically the derivative

$$\mathfrak{S}'(1) = (1 - \lambda)\ln(1 - \lambda) + \frac{1}{2}(\lambda - \mu)\ln(\lambda - \mu) + \frac{1}{2}(\lambda + \mu)\ln(\lambda + \mu) \tag{54}$$

to decide on the structure of the tail of f_∞: a Pareto tail is found for $\mathfrak{S}'(1) < 0$, the index of which is given by the non-trivial zero $\bar{s} > 1$ of $\mathfrak{S}$; for $\mathfrak{S}'(1) > 0$, the steady state is a

Dirac distribution at zero. The result is displayed in Fig. 1: region II corresponds to slim tails, region III to Pareto tails, and region IV to the singular steady state. Parameter region I is excluded by the condition $\mu \leq \lambda$.

In the following two examples, η takes arbitrarily large values.

Example 3 Assume a density with exponential decay,

$$d\rho(\omega) = \frac{1}{\lambda e} \exp\left(-\frac{\omega}{\lambda}\right) \mathbf{1}_{(-\lambda,+\infty)}(\omega)\, d\omega.$$

The integral in (51) converges for all $s > 0$. One obtains

$$\mathfrak{S}(s) = (1-\lambda)^s - 1 + \lambda^s \cdot \int_0^\infty t^s e^{-t}\, dt \;=\; (1-\lambda)^s - 1 + \lambda^s \cdot \Gamma(1+s).$$

The Gamma function grows faster than any exponential, so it eventually dominates the expression, causing high moments to diverge in time. The derivative at $s = 1$ reads

$$\frac{d}{ds}\mathfrak{S}(1) = (1-\lambda)\ln(1-\lambda) + \lambda \ln \lambda + (1-\gamma)\lambda. \tag{55}$$

Here $\gamma = 0.577\ldots$ is Euler's constant. The expression (55) is strictly convex in $\lambda \in [0, 1]$, vanishes at $\lambda = 0$, and has a unique positive root $\lambda_0 = 0.877\ldots$. Consequently, Pareto tails form for saving rates $\lambda < \lambda_0$.

Example 4 Finally, take $p > 1$ and define

$$d\rho(\omega) = \frac{1}{p}\lambda^{-p}\left(1 - \frac{1}{p}\right)^p \cdot (\lambda + \omega)^{-(p+1)} \mathbf{1}_{(-\lambda/p,+\infty)}(\omega)\, d\omega.$$

It is clear that the integral in (51) diverges to $+\infty$ when $s \nearrow p$, so that at least the moments higher than p become infinite in the long-time limit. Hence p constitutes an upper bound for the Pareto index. For $s < p$,

$$\mathfrak{S}(s) = (1-\lambda)^s - 1 + \frac{p}{p-s}\left(1 - \frac{1}{p}\right)^s \lambda^s.$$

Numerical investigations indicate that Pareto tails form for $\lambda < \lambda_p$, with $\lambda_p \approx 1$ for $p > 2$. In Fig. 1, the values of λ_p are displayed for $p < 2$.
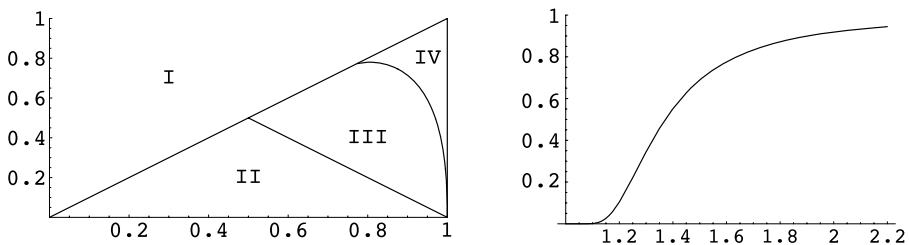
Fig. 1 *Left* Regions for the parameter μ and saving rate λ in Example 2. Pareto tails are found in zone III. *Right* Critical value of the saving rate λ in Example 4. Pareto tails form for λ lying below the curve

4.2 Pointwise Conservative Trades and Their Nonlinear Generalizations

The main result about pointwise conservative trades—satisfying (5) and (7)—is that they cannot form Pareto tails in the long-time limit.

Theorem 4.1 *For each trade of type* (5) *which is pointwise conservative* (7), *exactly one of the following is true*:

1. *Any solution f converges exponentially fast (in any d_s with $s > 1$) to the unique steady state f_∞, which possesses finite moments of all orders.*
2. *Any solution f is constant in time, i.e. $f(t, v) = f_0(v)$ for all $t > 0$.*
3. *Along any solution f, all moments above the first diverge at an algebraic rate, and the unique stationary state is given by the Dirac distribution centered at $v = 0$.*

Proof From the behavior of the functions $s \mapsto p^s + (1 - p)^s$ with $0 \le p \le 1$ one easily concludes that $\mathfrak{S}(s)$ is well-defined and non-positive for all $s \ge 1$. Furthermore, $\mathfrak{S}(s)$ is zero *for some $s > 1$* exactly if it is identically zero *for all s*.

The first case in the theorem is the generic one, where $\mathfrak{S}$ is not identically zero. The claim follows immediately from Theorem 3.3.

If $\mathfrak{S} \equiv 0$, then $p_1 p_2 = q_1 q_2 = 0$ a.s. Consider the quantity $\mathcal{K}$ defined in (39). This quantity is non-negative. Moreover, it is zero *for some $s > 1$* exactly if it is zero *for all s*, and the latter is the case if (almost surely) one of the following is true:

$$p_1 = q_2 = 1 \quad \text{and} \quad p_2 = q_1 = 0, \quad \text{or} \quad p_1 = q_2 = 0 \quad \text{and} \quad p_2 = q_1 = 1. \quad (56)$$

The corresponding dynamics is trivial, i.e. $f(v, t) = f(v)$ for all $t \ge 0$. If (56) does not hold, then $\mathcal{K}(s)$ is positive for all $s > 1$. By Theorem 3.2, all moments of f above the first diverge at algebraic rate. The only possible steady state is the Dirac distribution at $v = 0$. $\qquad \square$

The following three examples illustrate the three cases of qualitative behavior.

Example 5 For a fixed positive number $\lambda > 0$, consider the special (deterministic) case of the trade (15)

$$v^* = \lambda v + (1 - \lambda) w, \qquad w^* = (1 - \lambda) v + \lambda w.$$

Each agent possesses the same global saving propensity λ. It is easily checked that $\mathfrak{S}(s) < 0$ for all $s > 1$, so the associated steady state f_∞ possesses a thin tail, for which all moments are finite. Actually, more can be said: With this radical choice, the model describes a granular gas, for which the Dirac distribution is known to constitute the steady state. Using $\Phi(v) = v^2$ as test function in the weak formulation (20), one learns that the variance $V(t)$ of $f(v, t)$ obeys

$$\frac{d}{dt} V(t) = -2\lambda(1 - \lambda) V(t), \qquad (57)$$

and thus decays to zero at rate $2\lambda(1 - \lambda)$, The solution $f(v, t)$ concentrates in a Dirac distribution centered at $v = M$, the (conserved) mean wealth. *In the long-time limit, all agents end up with the same amount of money.*

Example 6 Consider the limit cases $\lambda = 1$ and $\lambda = 0$ in Example 5. These correspond to *no trading* and *pure exchange*, respectively. One calculates that both $\mathfrak{S}$ and $\mathfrak{K}$ are identically zero here. Note that also the estimate (57) becomes void. These are situations where *any* probability distribution $f(v)$ constitutes a stationary solution of (20).

Example 7 Finally, consider the "Winner takes all" dynamics,

$$v^* = v + w, \qquad w^* = 0,$$

where in every interaction, one agent loses all of his money to the other one. Again, one finds that $\mathfrak{S}$ is identically zero, but now $\mathfrak{K}$ is identically one. Using the indicator function of $[0, \epsilon]$ in the weak formulation (20), we find

$$\frac{d}{dt} m_\epsilon = \frac{1}{2} \int \int_{v+w \leq \epsilon} f(v) f(w) \, dv \, dw + \frac{1}{2} \int \int_{v,w \geq 0} f(v) f(w) \, dv \, dw - m_\epsilon \qquad (58)$$

$$\geq \frac{1}{2} - m_\epsilon, \qquad (59)$$

where $m_\epsilon = \int_0^\epsilon f(v) \, dv$ is the mass in the considered interval. Hence $m_\epsilon(t) \geq \frac{1}{2}(1 - e^{-t}) m_\epsilon(0)$, independently of $\epsilon > 0$. In conclusion, mass concentrates at the point $v = 0$. The following formal argument indicates the rate at which this concentration occurs: pass to the limit $\epsilon \to 0$ in (58), in which $m_0(t)$ denotes the mass concentrated in $v = 0$ at time $t \geq 0$ (the fraction of truely pauper agents). Then

$$\frac{d}{dt} m_0(t) = \frac{1}{2} \big(1 - m_0(t) \big)^2.$$

Assuming $m_0(0) = 0$, the solution to this differential equation is $m_0(t) = t/(t + 2)$, which converges to one as $t \to \infty$. At the same time, the mean wealth M is conserved; Hölder's inequality implies divergence of all moments above the first, exponentially fast in time. *While an increasing majority of agents possesses no money at all, the small fraction of $2/(t + 2)$ agents has become incredibly rich at time $t > 0$.*

In the pertinent literature, a variety of pointwise conservative trades is proposed, which are not of the type considered here. Such *generalized pointwise conservative trades* are defined as binary interactions (5) where the sum of the wealth of the agents is maintained. In contrast to the models considered up to here, the degree of randomness is more general in these models. The probability measure according to which the interaction coefficients p_i and q_i are distributed may depend on the wealth v, w of the interacting agents, and also (in a very sophisticated manner) on time. One example is the *minimum exchange model*, where p_i and q_i are non-linear functions of v/w. Even more complicated is the model with "personalized" saving propensities from [11]; here the time-dependent probability measure for the interaction coefficients evolves with the solution. (This kinetic model is more conveniently posed on an extended phase space, see [32].) Formally, however, these models can still be written in the form of a weak Boltzmann equation,

$$\frac{d}{dt} \int \Phi(v) \, df(v) = \frac{1}{2} \int \int \Big(\big\langle \Phi(p_1 v + q_1 w) + \Phi(p_2 v + q_2 w) \big\rangle_*$$

$$- \Phi(v) - \Phi(w) \Big) df(v) \, df(w), \qquad (60)$$

where $\langle \cdot \rangle_*$ denotes integration against the probability measure, and one has $p_1 + p_2 = q_1 = q_2 = 1$ almost surely. The essential difference to (20) is that the two appearing integrations do not commute. Proving existence and uniqueness of solutions $f(v, t)$ to such kinetic models is beyond the scope of this paper. We shall simply assume that a solution $f(v, t)$ of (60) is given.

Theorem 4.2 *Assume that f is one solution to a generalized pointwise conservative trade. Then the first moment of f is conserved, and higher moments diverge not faster than algebraically. In fact, if the trade rules are such that there exists a $\delta > 0$ for which*

$$\min(v^*, w^*) \geq \delta \cdot w, \tag{61}$$

then all moments of f remain uniformly bounded in time, provided they are finite initially.

Condition (61) expresses that the second agent always saves a tiny portion of his money when trading, and that at least a minimal fraction is passed to the first agent. Solutions to a generalized pointwise conservative trade which obey this sensible rule *do not* develop Pareto tails in the long-time limit. All pointwise conservative trade which do create fat tails for large t can only do so because they violate (61).

Proof This is an extension of the proof for the upper bounds in Theorem 3.2, so we only sketch the necessary changes in the argument. Since $p_1 + p_2 = q_1 + q_2 = 1$ almost surely, one has in the *pointwise* estimate under the integral in (37)

$$p_1^s + p_2^s - 1 \leq 0, \qquad q_1^s + q_2^s - 1 \leq 0, \qquad p_1^{s-1}q_1 + p_1 q_1^{s-1} + p_2^{s-1}q_2 + p_2 q_2^{s-1} \leq 2$$

for all $s > 1$. This is enough to conclude, with exactly the same reasoning, algebraic bounds on the divergence of the moments. If moreover (61) holds, then $\delta \leq q_i \leq 1 - \delta$ for $i = 1, 2$, leading to $q_1^s + q_2^s - 1 \leq \delta^s + (1 - \delta)^s < 1$ a.s., and thus to uniform boundedness of all moments. □

Example 8 Consider model (14) with quenched saving propensities, and assume for simplicity that $\epsilon \equiv \frac{1}{2}$,

$$v^* = \frac{1}{2}(1 + \lambda_A)v + \frac{1}{2}(1 - \lambda_B)w, \qquad w^* = \frac{1}{2}(1 - \lambda_A)v + \frac{1}{2}(1 + \lambda_B)w.$$

The individual saving propensities λ_i are drawn from the interval $[0, 1]$ according to a given probability measure P. If $\lambda < 1 - 2\delta$ holds P-almost surely, then (61) is satisfied, since then $v^* \geq \delta w$ and $w^* \geq \frac{1}{2}w$, and the tail of the wealth distribution remains thin for all times.

On the other hand, allowing saving propensities arbitrarily close to one, the distribution $f(v, t)$ may develop a Pareto tail for $t \to \infty$, see [11] (rather slowly, since this happens on the algebraic time scale). In fact, there is an intimate relation between the density of the probability measure P near $\lambda = 0$ and the shape of the tail of f for large times; basically, tails of arbitrary shape can be produced by choosing P appropriately. This has been formally observed in [33], and the convergence of f will be investigated in detail in [32].

Observe that any small perturbation of the model, i.e., any small change of the probability distribution P near $\lambda = 0$, will change the long-time behavior of the solutions f completely. The Pareto behavior is structurally unstable.

Example 9 A possible generalized pointwise conservative trade with non-constant transaction coefficients is defined by

$$v^* = v + \psi(v/w) \cdot w, \qquad w^* = \big(1 - \psi(v/w)\big) \cdot w,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a given continuous non-linear function satisfying the obvious constraint $0 \le \psi \le 1$. Independent of the actual shape of ψ, the solution $f(v, t)$ to the corresponding kinetic equation will have uniformly bounded moments of all orders if additionally $\delta \le \psi \le 1 - \delta$. A trivial example that satisfies this condition is the *full exchange model* with $\psi \equiv \frac{1}{2}$.

An example which does *not* satisfy this requirement is provided by the *minimum exchange model* where $\psi(r) = \frac{1}{2}\min(1, r)$, or by the *asymmetric exchange model* [40], which interpolates between full and minimum exchange. For the latter, $\psi(r) = \frac{1}{2}\min(1, 1 - \tau(1-r))$ with $\tau \in (0, 1)$, and Pareto tails are observed for $\tau \approx 1$ in numerical simulations.

4.3 Shape of Thin Tails

Finally, we investigate in more detail the situation where

$$\mathfrak{S}(s) < 0 \quad \text{for all } s > 1. \tag{62}$$

Here all initially finite moments of the wealth distribution remain uniformly bounded in time, so that the steady state f_∞ has finite moments of all orders. Still, there is no information available on the actual shape of the "slim" tail of $f_\infty(v)$, apart for that fact that the associated law falls off faster than any negative power of v as $v \to +\infty$. Below, we investigate in more detail the evolution of moments, and obtain a criterion under which the tail is exponentially small.

The evolution of the principal moments $M_n(t)$, with $2 \le n \in \mathbb{N}$, is determined by

$$\frac{d}{dt} M_n(t) = \big\langle (p_i^n + p_i^n - 1) \big\rangle_+ \cdot M_n(t) + \sum_{k=1}^{n-1} \binom{n}{k} \big\langle p_i^k q_i^{n-k} \big\rangle_+ \cdot M_k(t) M_{n-k}(t). \tag{63}$$

Taking into account that the first moment is conserved, $M_1(t) = M$, (63) allows for a recursive computation of the principal moments of the stationary solution,

$$M_n = \frac{\sum_{k=1}^{n-1} \binom{n}{k} \langle p_i^k q_i^{n-k}\rangle_+ M_k M_{n-k}}{1 - \langle p_i^n + q_i^n\rangle_+}, \quad n = 2, 3, \ldots. \tag{64}$$

In particular, starting from $M_1 = M$, one can calculate—one by one—a sequence of bounded constants c_n such that

$$M_n = c_n M^n. \tag{65}$$

Observe that the denominator in (64) equals $-\mathfrak{S}(n)$, and hence is positive by assumption (62).

Theorem 4.3 *Assume* (62). *Then the steady state f_∞ is a Dirac distribution centered at $v = M$ if and only if $p_1 + q_1 = p_2 + q_2 = 1$ almost surely.*

Proof f_∞ is a Dirac distribution iff its variance is zero, i.e. $M_2 = M^2$. By (64), this means that

$$1 = c_2 = \frac{\langle (p_i + q_i)^2 \rangle_+ - \langle p_i^2 + q_i^2 \rangle_+}{1 - \langle p_i^2 + q_i^2 \rangle_+},$$

which is equivalent to $\langle (p_i + q_i)^2 \rangle_+ = 1$. Taking into account that the trade is conservative in the mean, i.e. (10) holds, a standard argument yields further equivalence to $p_i + q_i = 1$ for $i = 1, 2$. □

Under the hypotheses of the theorem, all agents will eventually end up with exactly the same amount of money. In the framework of the model (13) by Chakraborti and Chakrabarti, $c_2 = 1$ is realized exactly iff $\epsilon = 1/2$ with probability one. In the model (15) of Cordier, Pareschi and Toscani, $c_2 = 1$ corresponds exactly to the degenerate situation $\eta \equiv \tilde{\eta} \equiv 0$ almost surely. The corresponding solutions were studied in Example 5.

In the generic case $c_2 > 1$, the variance of the stationary solution is positive, and information on higher moments is needed in order to extract further properties of the steady state. Unfortunately, even if the recursive computation of the moments is explicit, it is quite difficult to extract from (64) their precise growth. We can, however, establish a criterion under which the tail shows exponential decay.

Theorem 4.4 *Let $f_\infty(v)$ be the unique stationary solution to* (20), *corresponding to a trade satisfying condition* (62). *Assume further that for $n \geq \bar{n}$, with a suitable $\bar{n} \in \mathbb{N}$,*

$$\left\langle \sum_{k=0}^{n} p_i^{n-k} q_i^k \right\rangle_+ \leq 1. \tag{66}$$

Then the tail of $f_\infty(v)$ is exponentially small for $v \to \infty$ in the sense that

$$\int_{\mathbb{R}_+} \exp(\beta v) f_\infty(v, t) \, dv < \infty,$$

for a positive constant $\beta = \beta(\bar{n})$.

Proof Since all moments M_n of the steady state are bounded, $M_n = c_n M^n$, it is always possible to estimate the first $\bar{n} - 1$ coefficients c_n from above as follows:

$$c_n \leq n! \alpha^n \quad \text{for } n < \bar{n}.$$

We assume that $\alpha = \alpha(\bar{n})$ is the smallest possible constant. Then, for all $n < \bar{n}$,

$$M_n \leq n! (\alpha M)^n. \tag{67}$$

Bringing together the relations (66) and (67), we obtain from (64)

$$M_{\bar{n}} = \frac{\sum_{k=1}^{\bar{n}-1} \binom{\bar{n}}{k} \langle p_i^k q_i^{\bar{n}-k} \rangle_+ M_k M_{\bar{n}-k}}{1 - \langle p_i^{\bar{n}} q_i^{\bar{n}} \rangle}$$

$$\leq \bar{n}! (\alpha M)^{\bar{n}} \frac{\sum_{k=1}^{\bar{n}-1} \langle p_i^k q_i^{\bar{n}-k} \rangle_+}{1 - \langle p_i^{\bar{n}} + q_i^{\bar{n}} \rangle_+}$$

$$= \bar{n}!(\alpha M)^{\bar{n}} \frac{\sum_{k=0}^{\bar{n}} \langle p_i^k q_i^{\bar{n}-k} \rangle_+ - \langle p_i^{\bar{n}} + q_i^{\bar{n}} \rangle_+}{1 - \langle p_i^{\bar{n}} + q_i^{\bar{n}} \rangle_+} \leq \bar{n}!(\alpha M)^{\bar{n}}.$$

Thus condition (67) holds for $n = \bar{n}$. By induction, we conclude (67) for *all* $n \geq 1$. Hence, for arbitrary positive $\beta < 1/(\alpha M)$, and for any $N \geq 1$,

$$\int_0^{+\infty} \sum_{n=0}^{N} \frac{(\beta v)^n}{n!} f_\infty(v) \, dv = \sum_{n=0}^{N} \frac{\beta^n}{n!} M_n \leq \sum_{n=0}^{N} (\beta \alpha M)^n \leq \frac{1}{1 - \beta \alpha M} < \infty.$$

An application of the monotone convergence theorem finishes the proof:

$$\int_0^{+\infty} \exp(\beta v) f_\infty(v) \, dv = \lim_{N \to \infty} \int_0^{+\infty} \sum_{n=0}^{N} \frac{(\beta v)^n}{n!} f_\infty(v) \, dv < \infty. \tag{68}$$

□

Under the hypotheses of the theorem, the steady state decays at least as rapidly as a Gibbs distribution

$$g(v) = \frac{1}{(\alpha M)} \exp\{-v/(\alpha M)\}.$$

This decay behavior is typical for the steady state corresponding to a non realistic pointwise conservative trade with Jacobian equal to 1. We investigate under which additional assumptions condition (66) is satisfied for the trades (13) of A. Chakraborti and B.K. Chakrabarti, and (15) of S. Cordier, L. Pareschi and G. Toscani, respectively.

Recall that in (13),

$$p_1 - q_1 = q_2 - p_2 = \lambda, \tag{69}$$

which implies $p_1 > q_1$ and $q_2 > p_2$. Therefore

$$\left\langle \sum_{k=0}^{n} p_1^k q_1^{n-k} \right\rangle = \left\langle \frac{p_1^{n+1} - q_1^{n+1}}{p_1 - q_1} \right\rangle = \frac{\langle p_1^{n+1}(1 - (q_1/p_1)^{n+1}) \rangle}{\lambda} \leq \frac{1}{\lambda} \langle p_1^{n+1} \rangle.$$

In analogous way, we find

$$\left\langle \sum_{k=0}^{n} p_2^k q_2^{n-k} \right\rangle \leq \frac{1}{\lambda} \langle q_2^{n+1} \rangle.$$

In order to verify condition (66) it will be enough to prove that, for $n \geq \bar{n}$ sufficiently large, both $\langle p_1^{n+1} \rangle$ and $\langle q_2^{n+1} \rangle$ are less than λ. Assume the random fraction ϵ is absolutely continuous w.r.t. Lebesgue measure $d\epsilon$, and let $g_\lambda(p) \, dp$ be the corresponding measure for p_1. Then, there is a $b_\lambda \in (\lambda, 1)$ such that

$$\int_{b_\lambda}^{1} g_\lambda(p) \, dp \leq \frac{\lambda}{2}.$$

On the other hand,

$$\int_{\lambda}^{b_\lambda} p^n g_\lambda(p) \, dp \leq b_\lambda^n.$$

Now choose $\bar{n}$ large enough so that $b_\lambda^{n_0} \leq \lambda/2$. Then, for all $n \geq \bar{n}$,

$$\langle p_1^n \rangle = \int_\lambda^1 p^n g_\lambda(p)\, dp \leq b_\lambda^n + \frac{\lambda}{2} \leq \lambda.$$

The previous argument allows to choose the exponent n_0 (and hence the constant α) the smaller, the higher the saving rate λ is.

Corollary 4.5 *Assume that the random fraction ϵ in the pointwise conservative trade* (13) *possesses a Lebesgue integrable density function. Then condition* (66) *holds, and the stationary state has an exponentially small tail.*

Example 10 It is worthwhile to investigate the particular case in which ϵ is uniformly distributed on the unit interval. By the considerations above,

$$\left\langle \sum_{k=0}^n p_1^k q_1^{n-k} \right\rangle = \frac{(1-\lambda)^{n+2} + 1 - \lambda^{n+2}}{(n+2)\lambda(1-\lambda)}.$$

The latter expression is smaller than one exactly if

$$\phi_n(\lambda) := \frac{1}{n+2}\left[(1-\lambda)^{n+2} + 1 - \lambda^{n+2}\right] - \lambda(1-\lambda) \leq 0.$$

Obviously, $\phi_n(1) = 0$ and $\phi_n(0) = 2/(n+2) > 0$. A brief inspection of ϕ_n' reveals that ϕ_n is monotonically decreasing in $[0, \frac{1}{2}]$ and monotonically increasing in $[\frac{1}{2}, 1]$. By elementary by tedious estimates, $\phi_n(3/(n+2)) \leq 0$ for $n = 1, 2, \ldots$. In conclusion, $\phi_n(\lambda) \leq 0$ for $\lambda \in (3/(n+2), 1)$. One hence finds that $\bar{n} \geq 3/\lambda - 2$ is a possible choice in (66). In particular, $\bar{n} = 2$ is possible if $\frac{3}{4} \leq \lambda < 1$, in which case $\alpha = 1$ in (67). The corresponding steady state f_∞ of the economic model decays at least as fast as $\frac{1}{M}\exp(-v/M)$.

We turn to investigate the trade (15). As discussed in Sect. 2.1, for η and $\tilde{\eta}$ distributed on the interval $[-\mu, +\mu]$, where $0 < \mu \leq \lambda$, for $\mu + \lambda \leq 1$, we are back in the trivial situation (52) which ensures boundedness of all moments. We prove property (66) under the assumption that $\lambda < 1$ and that η has a Lebesgue integrable density function. By the same argument that lead to Corollary 4.5, there is a (smallest) integer n_0 such that

$$\langle (\lambda + \eta)^k \rangle \leq \frac{\lambda}{2} \quad \text{for all } k \geq n_0.$$

Define further $\bar{n} \geq n_0$ as the smallest integer satisfying

$$(1-\lambda)^{\bar{n}-n_0} \leq \frac{\lambda}{2}.$$

Then, for all $n \geq \bar{n}$, one has

$$\left\langle \sum_{k=0}^n p_1^k q_1^{n-k} \right\rangle = \sum_{k=0}^n \langle (\lambda + \eta)^k \rangle (1-\lambda)^{n-k}$$

$$\leq \sum_{k=0}^{n_0} (1-\lambda)^{n-k} + \sum_{k=n_0}^n \frac{\lambda}{2}(1-\lambda)^{n-k}$$

$$\leq (1-\lambda)^{n-n_0} \sum_{k=0}^{n_0}(1-\lambda)^k + \frac{\lambda}{2} \sum_{k=0}^{n-n_0}(1-\lambda)^k \leq \frac{\lambda}{2}\cdot\frac{1}{\lambda} + \frac{\lambda}{2}\cdot\frac{1}{\lambda} = 1.$$

Corollary 4.6 *Assume that the random variables η and $\tilde{\eta}$ in the conservative trade* (15) *are centered and distributed on an interval $[-\mu, +\mu]$, where $0 < \mu \leq \lambda$, and $\mu + \lambda \leq 1$, with a Lebesgue integrable density function. Then condition* (66) *holds, and the stationary state has an exponentially small tail.*

Example 11 Again, we provide an explicit calculation that leads to an estimate on the index $\bar{n}$. Namely, assume that η and $\tilde{\eta}$ are uniformly distributed in the interval $[-\mu, +\mu]$. For definiteness, assume further that $\frac{1}{2} < \lambda < 1$ (the case $0 < \lambda < \frac{1}{2}$ is analogous). Then

$$\left\langle \sum_{k=0}^{n} p_1^k q_1^{n-k} \right\rangle = \sum_{k=0}^{n}(1-\lambda)^{n-k}\int_{-\mu}^{+\mu}(\lambda+\eta)^k\frac{d\eta}{2\mu}$$

$$= \sum_{k=0}^{n}(1-\lambda)^{n-k}\frac{(\lambda+\mu)^{k+1} - (\lambda-\mu)^{k+1}}{2\mu(k+1)}$$

$$\leq \sum_{k=0}^{n}(1-\lambda)^{n-k}(\lambda+\mu)^k \leq \frac{(\lambda+\mu)^{n+1}}{(\lambda+\mu) - (1-\lambda)}.$$

The last expression is smaller than one for all $n \geq \bar{n}$ where

$$\bar{n} := \frac{\log((\lambda+\mu) - (1-\lambda))}{\log(\lambda+\mu)} - 1.$$

A situation of special interest is where one may choose $\bar{n} = 2$, because then $\alpha = 1$ is allowed in (67), and the steady state decays at least like $M^{-1}\exp(v/M)$. The corresponding sufficient condition in the regime $\lambda \geq \frac{1}{2}$ can be solved w.r.t. μ, and is equivalent to the inequality

$$\mu \leq \frac{1}{2}\left(\sqrt{1 + 6\lambda - 3\lambda^2} - (1+\lambda)\right).$$

5 Conclusions

We investigated the large-time behavior of various kinetic models describing the fluctuation of wealth in conservative economies. The existence of a unique stationary wealth distribution has been proven. It has been shown that the shape of the tail of this stationary distribution depends on the details of the microscopic interaction. More specifically, we have found that both randomness, e.g. in the form of risky investments, and the agents' propensity to save money play a crucial role in the formation of Pareto tails. In particular, Pareto tails are *not* present in pointwise conservative models, but appear in models which are conservative in the mean only, when the effects of randomness and saving are well-balanced. Our analysis has been based on the possibility to apply Fourier transform techniques, and it is reminiscent of previous results for the Boltzmann equation for Maxwell molecules [1, 2]. In order to simplify the analysis, we have restricted ourselves to trades where the random distribution (collision kernel) was independent of the wealth of the interacting agents. Specific econophysical models with more complicated collision kernels will be studied in future work.

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